Modeling and Optimization with OPL

1 Introduction to linear Optimization

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1.1 Modeling

Example: production problem (Lewig Sanstetten)

	machine A (capacity: 64h)	machine B (capacity: 48h)	Sales price
product 1	5,3 h/t	3,9 h/t	2900 €/t
product 2	2,9 h/t	4,8 h/t	3300 €/t
product 3	2,5 h/t	3,1 h/t	2200 €/t

How much of each product should be produced?

What is a model?

- abstract description of a real system
- for example useful for
  - system design
  - performance analysis
  - **decision support**
- guideline: as exact as necessary, as simple as possible

Mathematical optimization models for decision support

Components of a mathematical optimization model[1]:

decision variables: system items which can be set by the decision maker

- in the example: the production quantities of the products

constraints: restrictions which have to be satisfied by the decision variables to get a proper solution

- in the example: the capacity of the machines may not be exceeded

objective function: a function of the decision variables, which the decision maker has to optimize, i.e. minimize or maximise

- in the example: the total revenue

[1] also: mathematical program

Example: production problem – decision variables

x_1 production quantity of product 1

x_2 production quantity of product 2

x_3 production quantity of product 3

Definition: Solution of an optimization model

By assigning specific values to the decision variables, we get a solution for the model.

Example: production problem – constraints

Capacity of machine A may not be exceeded:

- $5,3 \cdot x_1 + 2,9 \cdot x_2 + 2,5 \cdot x_3 \leq 64$

Capacity of machine B may not be exceeded:

- $3,9 \cdot x_1 + 4,8 \cdot x_2 + 3,1 \cdot x_3 \leq 48$

Definition: Feasible Solution of an optimization model

A solution which satisfies all constraints is called a feasible solution. The set of all feasible solutions is called the solution space.

Example: production problem – objective function

Maximize the revenue (in k€):

- $\max U(x_1, x_2, x_3) = 2,9 \cdot x_1 + 3,3 \cdot x_2 + 2,2 \cdot x_3$

Definition: Optimal solution and optimal value of an optimization model

If there is a feasible solution, in which the objective function assumes its maximum resp. its minimum (which might not be the case at all), this solution is an optimal solution and its objective function value is the optimal value.

Example: production problem – complete optimization model

$$
\begin{array}{lll}
\max & 2,9 \cdot x_1 + 3,3 \cdot x_2 + 2,2 \cdot x_3 & \text{(objective function)} \\
\text{s.t.} & 5,3 \cdot x_1 + 2,9 \cdot x_2 + 2,5 \cdot x_3 \leq 64 & \text{(constraint I)} \\
& 3,9 \cdot x_1 + 4,8 \cdot x_2 + 3,1 \cdot x_3 \leq 48 & \text{(constraint II)} \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 & \text{(non negativity constraint)}
\end{array}
$$

1.2 Linear Optimization

Linear functions and constraints

Linear functions

$$f(x_1, \ldots, x_N) = \sum_{n=1}^{N} c_n \cdot x_n$$

Linear constraints

Let f be a linear function:

$$f(x_1, \ldots, x_N) = b$$
$$f(x_1, \ldots, x_N) \leq b$$
$$f(x_1, \ldots, x_N) \geq b$$

Linear optimization model

objective function and constraints linear in the decision variables
$\Longrightarrow$ linear optimization model

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Properties of linear functions

Proportionality Each variable contributes a proportional value to the function.

Independence The value each variable contributes to the function is independent of the manifestation of the other variables.

Typical signs for non-linearity

- variables have other exponents than 1
  - other natural exponents, e.g.: x^2
  - roots, e.g.: $\sqrt{x} = x^{\frac{1}{2}}$
  - variables in the denominator, e.g.: $\frac{1}{x} = x^{-1}$
- variables are multiplied with each other, e.g.: $x_1 \cdot x_2$
- exponential functions, e.g.: 2^x
- absolute values, e.g. $|x|$

Anomaly: constants

While constants are non-linear by definition, they do not interfere with linear optimization models, since they can be eliminated easily.

deterministic vs. stochastic optimization models

deterministic optimization models: all parameters and function values are always exactly known

stochastic optimization models: parameters and function values are subject to random deviations

Linear optimization models are deterministic in general.

continuous vs. integer optimization models

continuous optimization models: the values of the decision variables are continuous (real) values

integer optimization models: the values of the decision variables can only be integer

Types of linear optimization models by possible values for decision variables:

- continuous decision variables $\Longrightarrow$ (continuous) linear optimization model
- integer decision variables $\Longrightarrow$ integer linear optimization model
- continuous and integer decision variables $\Longrightarrow$ mixed integer linear optimization model

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Solution structure of linear optimization models

Possible structure of solutions:

- there is exactly one optimal solution
- there is an unlimited number of optimal solutions
- there is no optimal solution
  - the solution space is empty
  - the solution space is unbound and the objective function approaches infinity

Well known solution methods for linear optimization models

Solution methods for (continuous) linear optimization models

- Dantzig's simplex method
- Karmarkar's inner point method

Solution methods for (mixed) integer linear optimization models

- Branch and bound method
- Cutting planes methods

1.3 Model and Model Instance

Optimization model of example “Lewig Sanstetten”

Model: production problem

$$\begin{aligned}
\max \quad & 2,9 \cdot x_1 + 3,3 \cdot x_2 + 2,2 \cdot x_3 \\
s.t. \quad & 5,3 \cdot x_1 + 2,9 \cdot x_2 + 2,5 \cdot x_3 \leq 64 \\
& 3,9 \cdot x_1 + 4,8 \cdot x_2 + 3,1 \cdot x_3 \leq 48 \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0
\end{aligned}$$

Substitution of data with parameters

Model: production problem

$$
\begin{aligned}
\max \quad & p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3 \\
s.t. \quad & v_{A1} \cdot x_1 + v_{A2} \cdot x_2 + v_{A3} \cdot x_3 \leq c_A \\
& v_{B1} \cdot x_1 + v_{B2} \cdot x_2 + v_{B3} \cdot x_3 \leq c_B \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0
\end{aligned}
$$

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Index sets as a type of parameter

Version 1: maximal index as parameter

Let $I \in \mathbb{N}$ by the number of products, so e.g. the objective functions becomes:

$$\sum_{i=1}^{I} p_i \cdot x_i$$

Version 2: Index set as parameter

Let I be the set of products, so e.g. the objective functions becomes:

$$\sum_{i \in I} p_i \cdot x_i$$

Summing up over parameterized index sets

Model: production problem

$$
\begin{aligned}
\max \quad & \sum_{i\in I} p_i \cdot x_i \\
s.t. \quad & \sum_{i\in I} v_{Ai} \cdot x_i \leq c_A \\
& \sum_{i\in I} v_{Bi} \cdot x_i \leq c_B \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0
\end{aligned}
$$

Application of the universal quantifier ("for all" quantifier)

$$\sum_{i \in I} v_{Ai} \cdot x_i \leq c_A$$

$$\sum_{i \in I} v_{Bi} \cdot x_i \leq c_B$$

↓ index set R of resources ↓

$$\sum_{i \in I} v_{ri} \cdot x_i \leq c_r \qquad \forall r \in R$$

General model

Model: production problem

$$\begin{aligned} \max \quad & \sum_{i \in I} p_i \cdot x_i & \\ s.t. \quad & \sum_{i \in I} v_{ri} \cdot x_i \leq c_r & \forall r \in R \\ & x_i \geq 0 & \forall i \in I \end{aligned}$$

Model: production problem

Index Sets:

I set of products
R set of resources

Parameters:

p_i sales price of product $i \in I$
c_r capacity of resource $r \in R$
v_{ri} capacity consumption of product $i \in I$ on resource $r \in R$

Decision variables:

x_i production quantity of product $i \in I$

Model description:

$$\begin{aligned} \max \quad & \sum_{i \in I} p_i \cdot x_i \\ s.t. \quad & \sum_{i \in I} v_{ri} \cdot x_i \leq c_r && \forall r \in R \quad \text{(I)} \\ & x_i \geq 0 && \forall i \in I \end{aligned}$$

Model vs. model instanz

Model

The general description of the problem structure with general index sets and parameters.

Model instance

A specific problem, in which specific values are assigned to the index sets and parameters.