Finite-Length Discrete Transforms

Germán Gómez-Herrero, `http://germangh.com`

February 3, 2013

NOTE: This document is a brief summary (a cheatsheet, actually) of Chapter 5 from the textbook *Digital Signal Processing*, by S. K. Mitra.

- The N-point DFT of a sequence $x[n]$ is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad 0 \leq k \leq N-1 \tag{1}$$

  where $W_N = e^{-j\frac{2\pi}{N}}$

- The N-point inverse DFT (IDFT) of a sequence $X[k]$ is defined as:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad 0 \leq n \leq N-1 \tag{2}$$

- The N-point DFT of a sequence $x[n]$ of length $L < N$ can be obtained by zero-padding the sequence $x[n]$ until it has the desired length of N samples. That is:

$$\mathrm{DFT}_N\{x[n]\} = \mathrm{DFT}_N\{x_{zp}[n]\}$$

  where

$$x_{zp}[n] = \begin{cases} x[n] & \text{if} \quad 0 \leq n \leq L-1 \\ 0 & \text{if} \quad L \leq n \leq N \end{cases}$$

- Whenever computing DTFs or IDFTs it is useful to remember that:

$$\sum_{n=0}^{N-1} W_N^{An} = \begin{cases} N & \text{if} \quad A = rN \quad \text{with} \quad r \quad \text{an integer} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

- The DFT can also be expressed in matrix form like:

$$\mathbf{X} = \mathbf{D}_N \mathbf{x} \tag{4}$$

where

$$\begin{aligned} \mathbf{X} &= [X[0], X[1], ..., X[N-1]]^T \\ \mathbf{x} &= [x[0], x[1], ..., x[N-1]]^T \end{aligned}$$

and $\mathbf{D}_N$ is the $N \times N$ DFT matrix given by:

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \tag{5}$$

- Equivalently, the inverse DFT (IDFT) can be expressed in matrix form:

$$\mathbf{x} = \mathbf{D}_N^{-1} \mathbf{X} \tag{6}$$

where:

$$\mathbf{D}_N^{-1} = \frac{1}{N}\mathbf{D}_N^* = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix} \tag{7}$$

- An important operator when working with DTFs is the *modulo* operator. This operator is denoted by $< a >_b = c$, which reads as "*c is equal to a modulo b*". This operator works differently depending whether a is a positive or negative integer.

$$< a >_b = \begin{cases} a - \left\lfloor \frac{a}{b} \right\rfloor b & \text{if} \quad a > 0 \\ \left\lceil \frac{|a|}{b} \right\rceil b + a & \text{if} \quad a < 0 \end{cases}$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the *floor* and *ceiling* functions, respectively. For instance:

$$\begin{aligned} < 10 >_4 &= 10 - \left\lfloor \frac{10}{4} \right\rfloor \cdot 4 = 10 - 2 \cdot 4 = 2 \\ < -5 >_3 &= \left\lceil \frac{|-5|}{3} \right\rceil \cdot 3 + (-5) = 2 \cdot 3 + (-5) = 1 \end{aligned}$$

- The *circular* convolution can be defined using the *modulo* operator:

$$g[n] \overset{N}{\otimes} h[n] = \sum_{m=0}^{N-1} g[m]h[< n-m >_N] \qquad 0 \leq n \leq N-1 \tag{8}$$

- The DFT has several symmetry properties that can greatly simplify the computation of the DFT:

Length-N sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[< -k >_N]$
$\text{Re}\{x[n]\}$	$X_{ccs}[k] = \frac{1}{2}[X[k] + X^*[< -k >_N]]$
$j \cdot \text{Im}\{x[n]\}$	$X_{cca}[k] = \frac{1}{2}[X[k] - X^*[< -k >_N]]$
$x_{ccs}[n] = \frac{1}{2}[x[n] + x^*[< -n >_N]]$	$\text{Re}\{X[k]\}$
$x_{cca}[n] = \frac{1}{2}[x[n] - x^*[< -n >_N]]$	$j \cdot \text{Im}\{X[k]\}$

- Furthermore, if $x[n]$ is a length-N real sequence then the corresponding N-point DFT has the additional symmetry properties:

$$\begin{array}{ccc} X[k] & = & X^*[< -k >_N] \\ \text{Re}\{X[k]\} & = & \text{Re}\{X[< -k >_N]\} \\ \text{Im}\{X[k]\} & = & -\text{Im}\{X[< -k >_N]\} \\ |X[k]| & = & |X[< -k >_N]| \\ \arg\{X[k]\} & = & -\arg\{X[< -k >_N]\} \end{array}$$

- The following theorems are also very useful when computing the DFT of a length-N sequence:

Length-N sequence	N-point DFT
$g[n]$	$G[k]$
$h[n]$	$H[K]$
$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
$g[< n - n_0 >_N]$	$W_N^{kn_0} G[k]$
$W_N^{-k_0 n} g[n]$	$G[< k - k_0 >_N]$
$g[n] \overset{N}{\otimes} h[n]$	$G[k]H[k]$
$g[n]h[n]$	$\frac{1}{N} G[k] \overset{N}{\otimes} H[k]$

- The Parseval's relation can be also useful, specially when we want to compute the energy of a sequence $g[n]$ that has a DFT $G[k]$:

$$\sum_{n=0}^{N-1} |g[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |G[k]|^2$$

- The circular convolution of two sequences $g[n]$ and $h[n]$ can be computed using the DFT:

$$\left.\begin{array}{ccc} g[n] & \stackrel{DFT}{\rightarrow} & G[k] \\ h[n] & \stackrel{DFT}{\rightarrow} & H[k] \end{array}\right\} \Longrightarrow g[n] \overset{N}{\otimes} h[n] = \text{IDFT}\left\{G[k]H[k]\right\}$$

- The linear convolution of a length-L_x sequence $x[n]$ and a length-L_y sequence $y[n]$ can also be computed using the DFT:

$$x[n] \otimes y[n] = x[n] \overset{N}{\otimes} y[n] = \text{IDFT}\left\{\text{DFT}_N\left\{x[n]\right\}\text{DFT}_N\left\{x[n]\right\}\right\}$$

where $N = L_x + L_y - 1$.

- The DFTs of two length-N real sequences $g[n]$ and $h[n]$ can be computed using the DFT of a single complex sequence $x[n] = g[n] + jh[n]$:

$$\left.\begin{array}{ccc} g[n] & \stackrel{DFT}{\rightarrow} & G[k] \\ h[n] & \stackrel{DFT}{\rightarrow} & H[k] \\ x[n] & \stackrel{DFT}{\rightarrow} & X[k] \end{array}\right\} \Longrightarrow \left\{\begin{array}{ccc} G[k] & = & \frac{1}{2}\left[X[k] + X^*[< -k >_N]\right] \\ H[k] & = & \frac{1}{2j}\left[X[k] - X^*[< -k >_N]\right] \end{array}\right.$$