Discrete-Time Signals and Systems in the z-Domain

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NOTE: This document is a brief summary (a cheatsheet, actually) of Chapter 6 from the textbook *Digital Signal Processing*, by S. K. Mitra.

- The z-transform of a given sequence $g[n]$ is defined as:

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$$

  where $z = \mathrm{Re}(z) + j\mathrm{Im}(z)$ is a complex variable. In general, converges only for an anular region of the z-plane, i.e. the Region of Convergence (ROC) of a z-transform is $R_1 < |z| < R_2$ where $0 \leq R_1 < R_2 \leq \infty$. In some occassions the ROC includes also the points $z = 0$ and $z = \infty$. Of course, you should remember that the modulus of a complex number is given by $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$.

- Notice that when we say that a z-transform, e.g. $G(z)$, diverges for $z = \lambda$ this means that $G(\lambda) = \infty$. Those values $\lambda_1, \lambda_2, ..., \lambda_N$ for which a Z-transform diverges (i.e. $G(\lambda_i) = \infty \; \forall i = 1, 2, ..., N$) are called *poles* of $G(z)$. Indeed, the definition of ROC forces all the poles to be outside the ROC.

- By contrary, those values $\epsilon_1, \epsilon_2, ..., \epsilon_M$ for which $G(z)$ becomes zero (i.e. $G(\epsilon_i) = 0 \; \forall i = 1, 2, ..., N$) are called *zeros* of $G(z)$. Indeed, all the zeros of a Z-transform have to be inside its ROC since $G(\epsilon_i) \neq \infty$.

- Let us compute the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ where $\mu[n]$ is the unit step sequence:

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n \tag{1}$$

  Remember that the sum $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ only if $|r| < 1$. Otherwise the sum diverges (becomes ∞). So this means that Eq. 1 becomes:

Sequence	Z-transform	ROC
$\delta[n]$	1	$\forall z$
$\alpha^n \mu[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-(\alpha)^n \mu[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
$n\alpha^n \mu[n] = n\alpha^n \mu[n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-n\alpha^n \mu[-n-1] = -n\alpha^n \mu[-n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$

Table 1: Z-transforms and ROCs associated with several elementary sequences. Notice that the n factor in $n\alpha^n\mu[n]$ cancels the expression for $n = 0$ which makes such expression equivalent to $n\alpha^n\mu[n-1]$. For the same reason $-n\alpha^n\mu[-n-1] = -n\alpha^n\mu[-n]$.

$$X(z) = \begin{cases} \frac{1}{1-\alpha z^{-1}} & \text{if} \quad |\alpha z^{-1}| < 1 \Leftrightarrow |z| > |\alpha| \\ \infty & \text{if} \quad |\alpha z^{-1}| \geq 1 \Leftrightarrow |z| \leq |\alpha| \end{cases} \tag{2}$$

Instead of expressing the $X(z)$ like above we usually express it by saying that the z-transform of $x[n]$ is $\frac{1}{1-\alpha z^{-1}}$ and its *region of convergence (ROC)* is $|z| > \alpha$. Moreover, you can plot the region of convergence in an *Argand diagram*[1].

- It is very important to realize that a the expression of a z-transform is meaningless without its ROC. This is because different sequences in the time domain might have exactly the same expression in the z-domain but with different ROCs. Therefore knowledge of the ROC is necessary in order to have a 1-to-1 relation between the time-domain and the z-domain, which is what makes inversion of the z-transform possible. For instance the z-transform of the sequence $y[n] = -\alpha^n\mu[-n-1]$ is:

$$Y(z) = \begin{cases} \frac{1}{1-\alpha z^{-1}} & \text{if} \quad |\alpha^{-1} z| < 1 \Leftrightarrow |z| < |\alpha| \\ \infty & \text{if} \quad |\alpha^{-1} z| \geq 1 \Leftrightarrow |z| \geq |\alpha| \end{cases} \tag{3}$$

notice that despite the appearances, $Y(z)$ and $X(z)$ are different because their ROCs are different. The ROC of $X(z)$ are all the points in the z-plane exterior to a circunference of radius $|\alpha|$ (including the point $z = \infty$) whereas the ROC of $Y(z)$ are all the points interior to that circunference (including $z = 0$).

- To make things easier for you in the exam you should memorize the z-transforms of the sequences in Table 1, which you will need all the time when computing inverse Z-transforms.

- You should also memorize the properties of the Z-transform in Table 2, among which the shifting and time-reversal properties are probably the most important

[1]An *Argand diagram* is just a type of cartesian representation of a complex number in which the vertical axis is the imaginary axis and the horizontal axis is the real axis. See Fig. 1 for an example of an Argand diagram.

Sequence	Z-transform	ROC
$g[n]$	$G(z)$	R_g
$h[n]$	$H(z)$	R_h
$g^*[n]$	$G^*(z^*)$	R_g
$g[-n]$	$G(\frac{1}{z})$	$\frac{1}{R_g}$
$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	includes $R_g \cap R_h$
$g[n-n_0]$	$z^{-n_0}G(z)$	R_g except maybe $z=0$ or $z=\infty$
$\alpha^n g[n]$	$G(\frac{z}{\alpha})$	$\|\alpha\| R_g$
$ng[n]$	$-z\frac{dG(z)}{dz}$	R_g except maybe $z=0$ or $z=\infty$
$g[n] \otimes h[n]$	$G(z)H(z)$	includes $R_g \cap R_h$

Table 2: Most important properties of the Z-transform. R_g denotes the region of the z-plane $R_{g-} < |z| < R_{g+}$ and R_h denotes the region $R_{h-} < |z| < R_{h+}$. Then $1/R_g$ denotes the region $1/R_{g+} < |z| < 1/R_{g-}$.

ones. Do not forget to memorize (or understand) the last column of the table where it says what happens to the ROC under any of these transformations.

- From Table 1 and Table 2 follows the additional properties in Table 3. Using these three tables one can compute the ROC of the Z-transform of many sequences without actually computing the Z-transform of the sequence. This is very often an exam problem. For instance if they ask you to find the ROC of the Z-transform of the sequence:

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & \text{if} \quad 2 \le n \le 7 \\ 0 & \text{otherwise} \end{cases}$$

one can see that the sequence $x[n]$ is of finite duration and therefore its ROC must be the whole z-plane except maybe $z = 0$ and/or $z = \infty$ (Table 3). Moreover, the sequence is different of zero for some positive values of n (that is, $M > 0$ in Table 3) which means that the Z-transform will contain negative powers of z and, therefore, will have one or several poles in $z = 0$, i.e. the point $z = 0$ will be outside the ROC. At the same time, our sequence is always zero for any negative values of n (so $N > 0$ in Table 3) and as a result the Z-transform will not contain positive powers of z which will avoid the existence of poles at $z = \infty$. As a result of all this, the ROC of this sequence is $0 < |z|$.

Consider now the ROC of the sequence $y[n] = 3^n\mu[-n]$. We can rewrite this sequence as $y[n] = (3^{-1})^{-n}\mu[-n]$. Because of Table 1 we know that the ROC of sequence $(3^{-1})^n\mu[n]$ is $|z| > 3^{-1}$. Then, using the time-reversal property in Table 2 we obtain that the ROC of $y[n] = 3^n\mu[-n]$ is $|z| < \frac{1}{3^{-1}} \Leftrightarrow |z| < 3$.

Property of $g[n]$	ROC of $G(z)$
Finite length $g[n] = 0 \; \forall n > M \; \forall n < N$	All z-plane except except $z = 0$ if $M > 0$ except $z = \infty$ if $N < 0$
Causal $g[n] = 0 \; \forall n < 0$	$R_{g-} < \lvert z \rvert$
Anticausal $g[n] = 0 \; \forall n > 0$	$\lvert z \rvert < R_{g+}$
Left-sided $g[n] = 0 \; \forall n > N$ with $N > 0$	$0 < \lvert z \rvert < R_{g+}$
Right-sided $g[n] = 0 \; \forall n < M$ with $M < 0$	$R_{g-} < \lvert z \rvert < \infty$

Table 3: Properties of the ROC of the Z-transform of a sequence depending on the span of the sequence in time-domain.

- A discrete-time LTI system is often characterized by a linear constant coefficient difference equation of the form:

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k] \tag{4}$$

Using the shifting property of the Z-transform (see Table 2) it is obvious that Eq. 4 is represented in the Z-domain by:

$$\left(\sum_{k=0}^{N} d_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} p_k z^{-k}\right) X(z)$$

and then the *transfer function* of the given LTI system is defined as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}} \tag{5}$$

The transfer function of the system can also be written in terms of zeros and poles as:

$$H(z) = \frac{p_0}{d_0} z^{N-M} \frac{\prod_{k=1}^{M}(z-\epsilon_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

where $\epsilon_1, \epsilon_2, ..., \epsilon_k$ are the zeros and $\lambda_1, ..., \lambda_k$ are the poles of $H(z)$. Using $H(z)$ one can characterize an LTI system in many ways. For instance:

  - If the ROC of $H(z)$ includes the unit circle then the system is *stable*.
  - If the ROC of $H(z)$ includes the point $z = \infty$ then the system is *causal*.
  - If the ROC of $H(z)$ includes the unit circle and the point $z = \infty$ then the system is both causal and stable. This is the same as saying that the system is *realizable*.
  - A *realizable* system $H(z)$ is said to be *minimum phase* if its inverse is also realizable, i.e. if all the zeros of $H(z)$ are inside the unit circle.

- In a system $H(z)$ is *causal* (the most common case) then the ROC of $H(z)$ is $|z| > \max_k |\lambda_k|$. If a system is causal we can determine if it is stable, realizable and minimum phase by simply examining the positions of the zeros and poles of $H(z)$:
  - If all the poles are within the unit circle, i.e. if $|\lambda_k| < 1 \ \forall k$, then the system will be *stable* and therefore it will also be *realizable*.
  - If all the poles and all the zeros are within the unit circle, i.e. if $|\lambda_k| < 1 \ \forall k$ and $|\epsilon_k| < 1 \ \forall k$ then the system will be *minimum phase* and therefore the inverse system $\frac{1}{H(z)}$ will be realizable (i.e. causal and stable).

- The inverse of the transfer function is the impulse response of the system: $h[n] = Z^{-1}\{H(z)\}$. So it is important to be able to invert rational Z-transforms as the one in Eq. 5. This is one of the most important parts of this chapter. Basically the idea is to manipulate the rational expression $H(z)$ so that it will become just a summation of elementary terms having expressions as the ones given in Table 1. Then, using the linearity property, $H(z)$ can be inverted by inverting each of these elementary terms. An additional document is available in the course web-page where this is explained step-by-step using a practical example. Notice that inverting a Z-transform is almost always necessary in at least one (often more) of the problems of the exam.

- Another problem that often appears in the exam is to compute the linear convolution of two sequences $x[n]$ and $h[n]$ using the Z-transform:

$$y[n] = x[n] \otimes h[n] = Z^{-1}\{X(z)Y(z)\}$$

- An LTI system is fully defined by its system function $H(z)$ but you should be aware that $H(z)$ can be given in different ways:
  1. Using a difference equation like in Eq. 4. In this case the system function can be directly obtained using Eq. 5.
  2. Giving you the output of the system $y[n]$ for a given input sequence $x[n]$. In this case, the system function can be easily obtained as $H(z) = \frac{Y(z)}{X(z)}$.
  3. Giving you directly either the impulse response $h[n]$ or the system function $H(z)$.

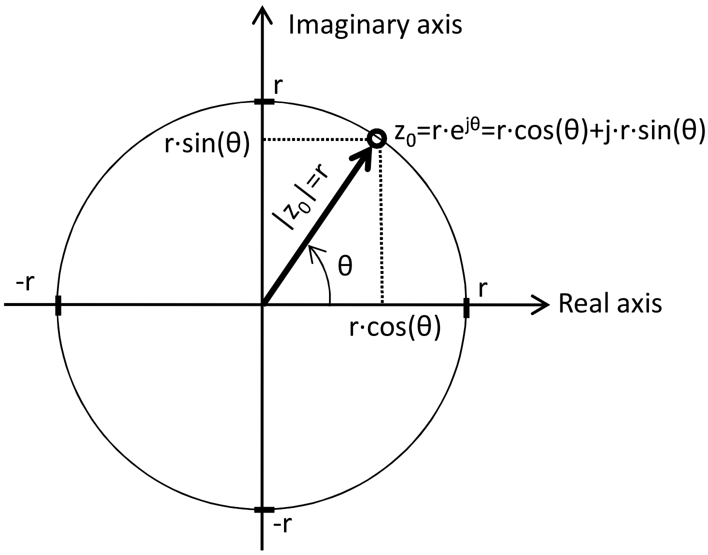

Figure 1: A complex number $z_0 = a + jb = r\cos\theta + jr\sin\theta = re^{j\theta}$ can be represented as a vector in an Argand diagram. The length of the depicted vector is the modulus of the complex number $|z_0| = r$. Obviously, all the complex numbers satisfying $|z| = r$ are located in a circunference of radius r. Then the equation $|z| > r$ describes the area strictly outside that circunference whereas the equation $|z| < r$ describes the area strictly inside that circunference. Similarly the equation $r_0 < |z| < r_1$ with $r_0 < r_1$ describes the anular area between a circunference of radius r_0 and another one of radius r_1.