Inverse Z transform: Example 2 (complex poles)

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Given the following system function of a causal LTI system:

$$H(z) = \frac{z^{-3}}{(1 - 2z^{-3})(1 - 0.5z^{-1})} \tag{1}$$

find the impulse response of the system.

SOLUTION:

The denominator of $H(z)$ is a fourth order polynomial, which means that it will have four roots, i.e. our system function has four poles. Obvioulsy, one pole is located in $p_1 = 0.5$. The poles p_2, p_3 and p_4 are the roots of the polynomial $(1 - 2z^{-3})$. Finding the roots of third order polynomial can be tricky but in this case, we can easily find one of its roots (that is pole p_2):

$$1 - 2z^{-3} = 0 \Rightarrow z^{-3} = \frac{1}{2} \Rightarrow p_2 = (2)^{\frac{1}{3}} = \sqrt[3]{2}$$

and therefore, we can factorize the term $1 - 2z^{-3}$ as:

$$(1 - 2z^{-3}) = (1 - \sqrt[3]{2}z^{-1})Q(z)$$

where $Q(z) = \frac{(1-2z^{-3})}{(1-\sqrt[3]{2}z^{-1})}$ must be a second-order polynomial. In order to find $Q(z)$ we use long division:

$$
\begin{array}{r|rrrr}
 & \sqrt[3]{4}z^{-2} & +\sqrt[3]{2}z^{-1} & +1 & \\
\cline{2-5}
-\sqrt[3]{2}z^{-1}+1 & -2z^{-3} & +0z^{-2} & +0z^{-1} & +1 \\
 & -2z^{-3} & +\sqrt[3]{4}z^{-2} & & \\
\cline{2-5}
 & & -\sqrt[3]{4}z^{-2} & +0z^{-1} & +1 \\
 & & -\sqrt[3]{4}z^{-2} & +\sqrt[3]{2}z^{-1} & \\
\cline{3-5}
 & & & -\sqrt[3]{2}z^{-1} & +1 \\
 & & & -\sqrt[3]{2}z^{-1} & +1 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

As expected, the remainder of the division is zero and $Q(z) = \sqrt[3]{4}z^{-2} + \sqrt[3]{2}z^{-1} + 1$ is a second degree polynomial. Now, we can easily find the two roots of $Q(z)$, which will correspond to the two remaining poles p_3 and p_4. By making the variable change $x = z^{-1}$ we find that:

$$
\sqrt[3]{4}x^2 + \sqrt[3]{2}x + 1 = 0 \Rightarrow x = \frac{-\sqrt[3]{2} \pm j\sqrt{3}\sqrt[6]{4}}{2\sqrt[3]{4}}
$$

and then by reversing the variable change ($z = x^{-1}$) we obtain that the two poles that we were looking for are:

$$
\begin{aligned}
p_3 &= \tfrac{2\sqrt[3]{4}}{-\sqrt[3]{2}+j\sqrt{3}\sqrt[6]{4}} = -0.63 - 1.09j \\
p_4 &= \tfrac{2\sqrt[3]{4}}{-\sqrt[3]{2}-j\sqrt{3}\sqrt[6]{4}} = -0.63 + 1.09j
\end{aligned}
$$

Always, when a fractional Z-transform has complex poles they will be in conjugate pairs, i.e. $p_4 = p_3^*$ in this case. So we can write the system function in Eq. 1 as:

$$
H(z) = \frac{z^{-3}}{(1-p_1z^{-1})(1-p_2z^{-1})(1-p_3z^{-1})(1-p_3^*z^{-1})} \qquad ROC \equiv |z| > \sqrt[3]{2}
$$

Because the system is causal, the ROC of $H(z)$ must be the region outside the circumference in which the largest (in absolute value) pole is located. Now, using fractional expansion:

$$H(z) = \frac{A}{1-p_1 z^{-1}} + \frac{B}{1-p_2 z^{-1}} + \frac{C}{1-p_3 z^{-1}} + \frac{C^*}{1-p_3^* z^{-1}}$$

where $p_1 = 0.5$, $p_2 = \sqrt[3]{2} = 1.26$ and $p_3 = -0.63 - 1.09j$ and the residuals are:

$$\begin{array}{rclcl} A & = & \left[(1-p_1 z^{-1})H(z)\right]\Big|_{z=p_1} & = & -0.53 \\ B & = & \left[(1-p_2 z^{-1})H(z)\right]\Big|_{z=p_2} & = & 0.28 \\ C & = & \left[(1-p_3 z^{-1})H(z)\right]\Big|_{z=p_3} & = & 0.13 + 0.04j \end{array} \tag{2}$$

Now we can finally invert $H(z)$ taking into account that all the poles correspond to causal components of the system:

$$h[n] = (0.5)^3 \cdot [A(p_1)^n + B(p_2)^n \mu[n] + C(p_3)^n \mu[n] + C^*(p_3^*)^n \mu[n]] \tag{3}$$

Exercise: try to express the equation above using only real terms. You can do it by writting C and p_3 in polar form.