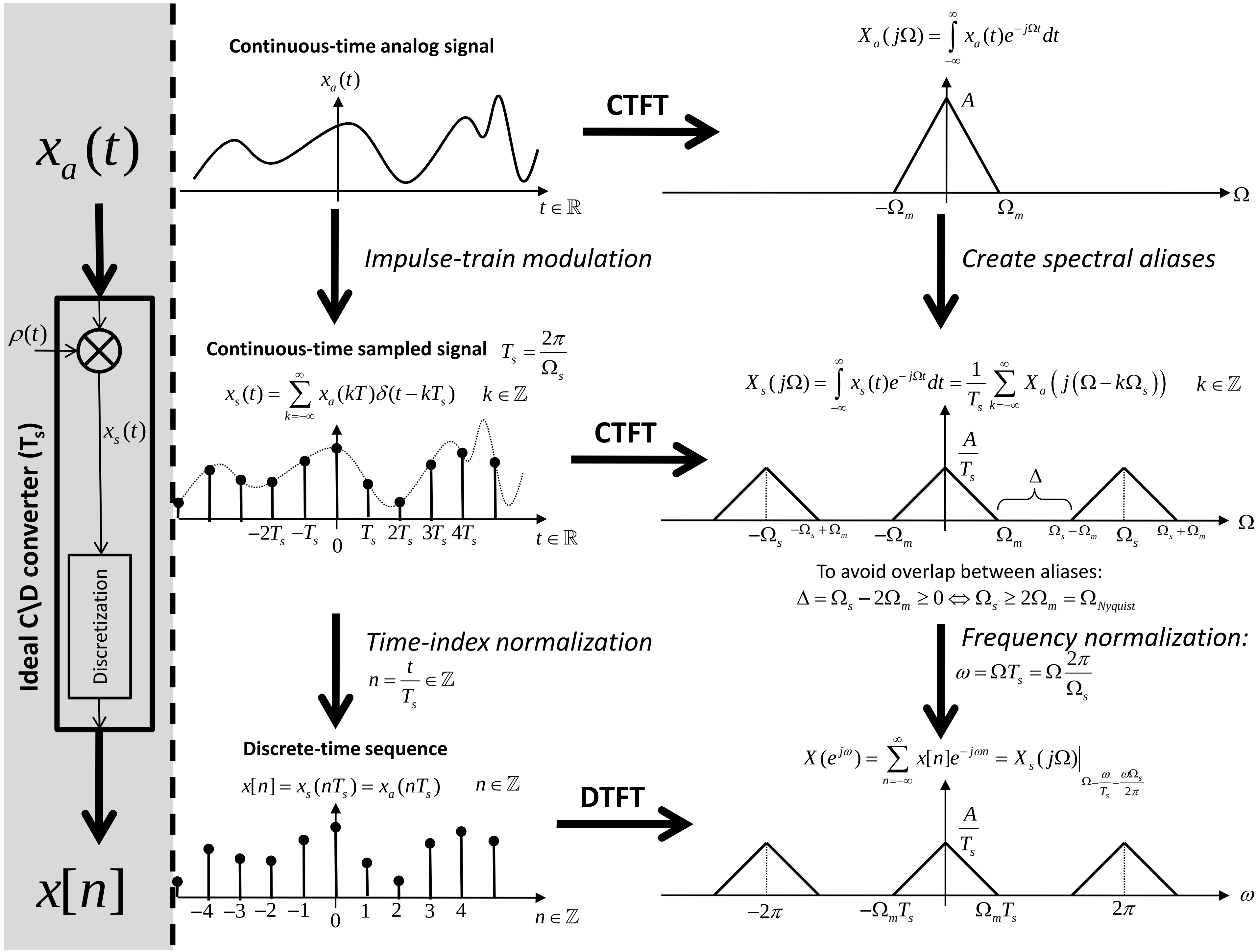
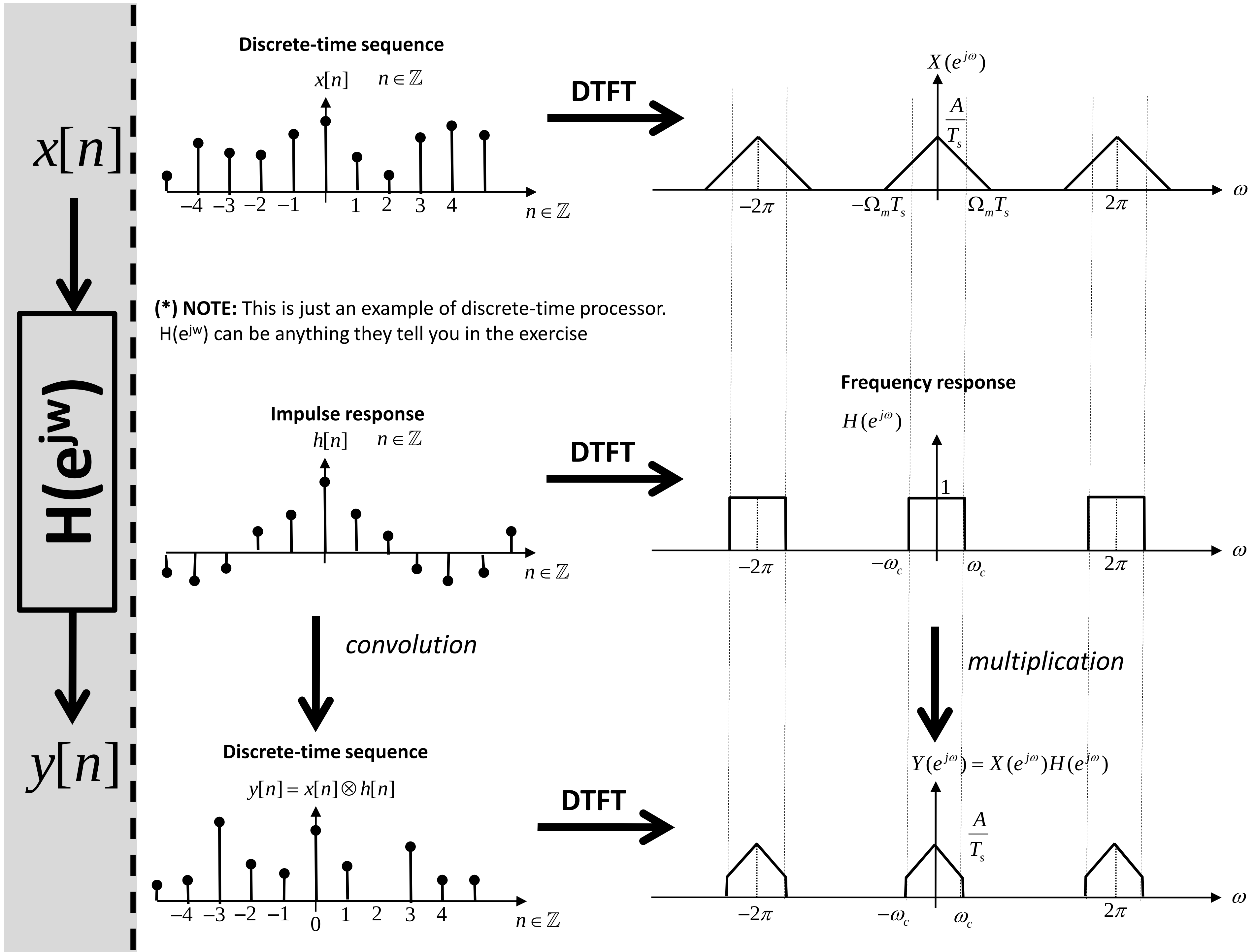
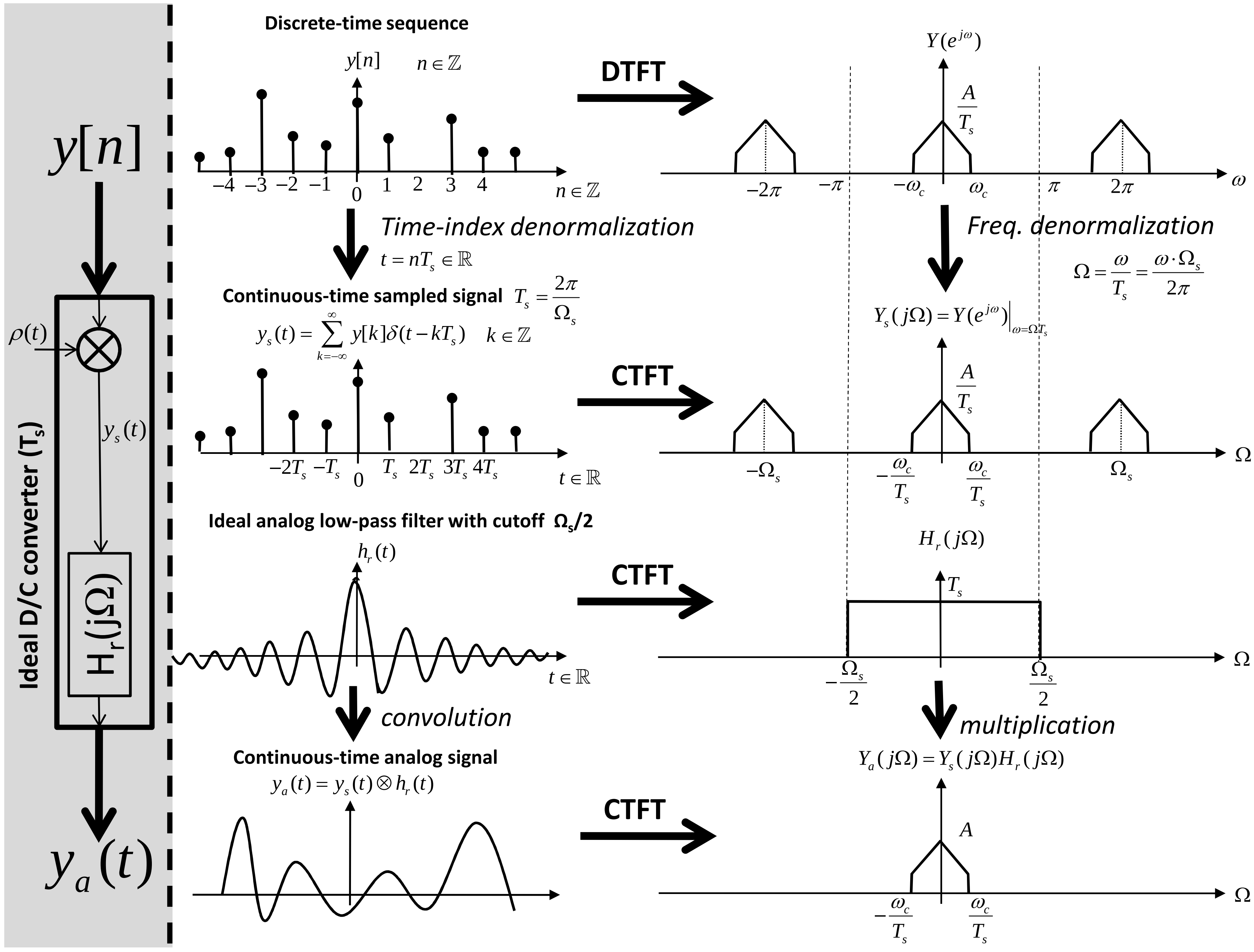
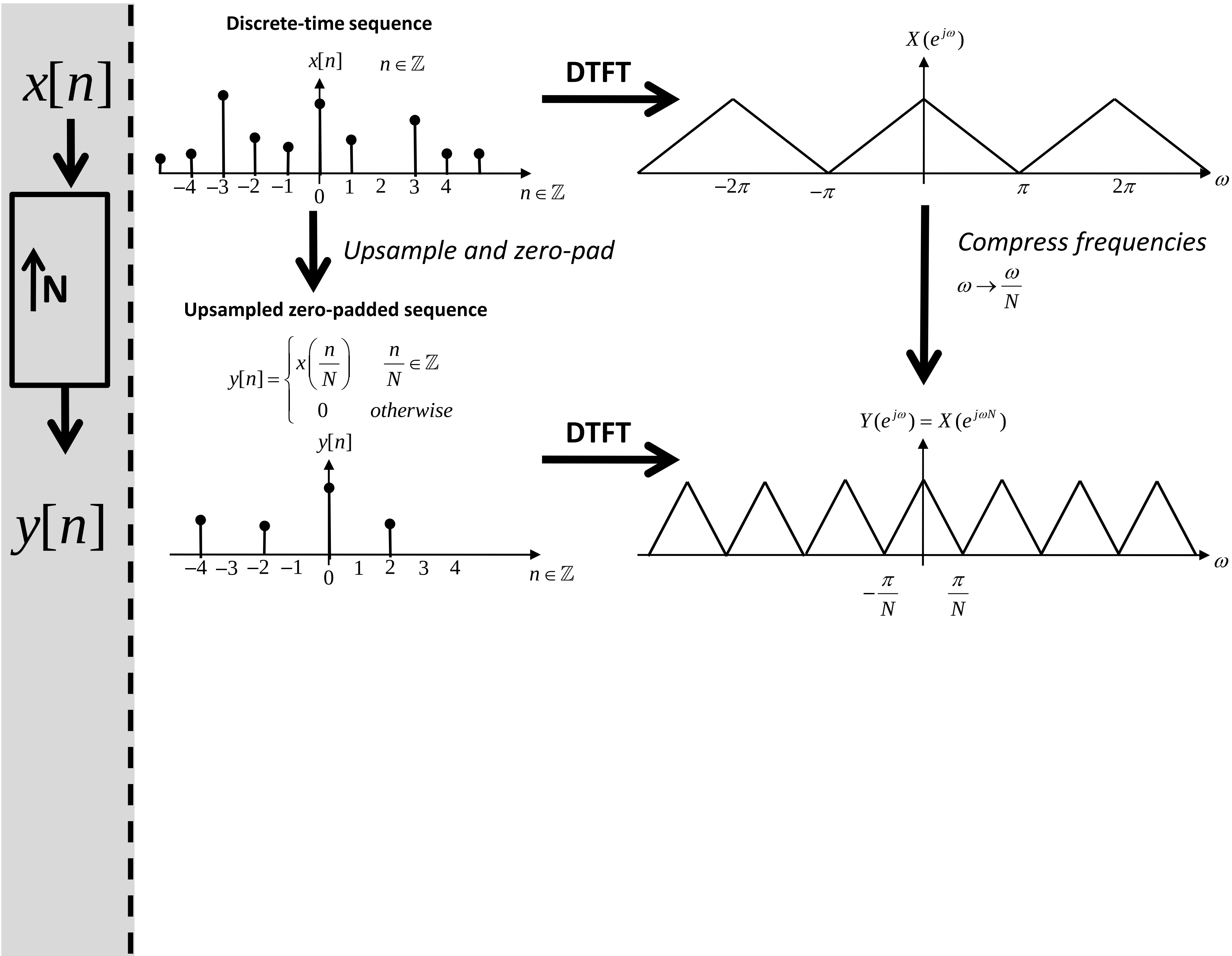


$x_a(t)$
Ideal C\D converter (T_s)
$\rho(t)$
$x_s(t)$
Discretization
$x[n]$
Continuous-time analog signal
$x_a(t)$
$t \in \mathbb{R}$
CTFT
$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$
A
$-\Omega_m$
Ω_m
Ω
Impulse-train modulation
Create spectral aliases
Continuous-time sampled signal $T_s = \frac{2\pi}{\Omega_s}$
$x_s(t) = \sum_{k=-\infty}^{\infty} x_a(kT)\delta(t - kT_s) \quad k \in \mathbb{Z}$
$-2T_s$ $-T_s$ 0 T_s $2T_s$ $3T_s$ $4T_s$
$t \in \mathbb{R}$
CTFT
$X_s(j\Omega) = \int_{-\infty}^{\infty} x_s(t) e^{-j\Omega t} dt = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - k\Omega_s\right)\right) \quad k \in \mathbb{Z}$
$\frac{A}{T_s}$
Δ
$-\Omega_s$ $-\Omega_s + \Omega_m$ $-\Omega_m$ Ω_m $\Omega_s - \Omega_m$ Ω_s $\Omega_s + \Omega_m$
Ω
To avoid overlap between aliases:
$\Delta = \Omega_s - 2\Omega_m \geq 0 \Leftrightarrow \Omega_s \geq 2\Omega_m = \Omega_{Nyquist}$
Time-index normalization
$n = \frac{t}{T_s} \in \mathbb{Z}$
Frequency normalization:
$\omega = \Omega T_s = \Omega \frac{2\pi}{\Omega_s}$
Discrete-time sequence
$x[n] = x_s(nT_s) = x_a(nT_s) \quad n \in \mathbb{Z}$
-4 -3 -2 -1 0 1 2 3 4
$n \in \mathbb{Z}$
DTFT
$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X_s(j\Omega)\Big|_{\Omega = \frac{\omega}{T_s} = \frac{\omega\Omega_s}{2\pi}}$
$\frac{A}{T_s}$
-2π $-\Omega_m T_s$ $\Omega_m T_s$ 2π
ω

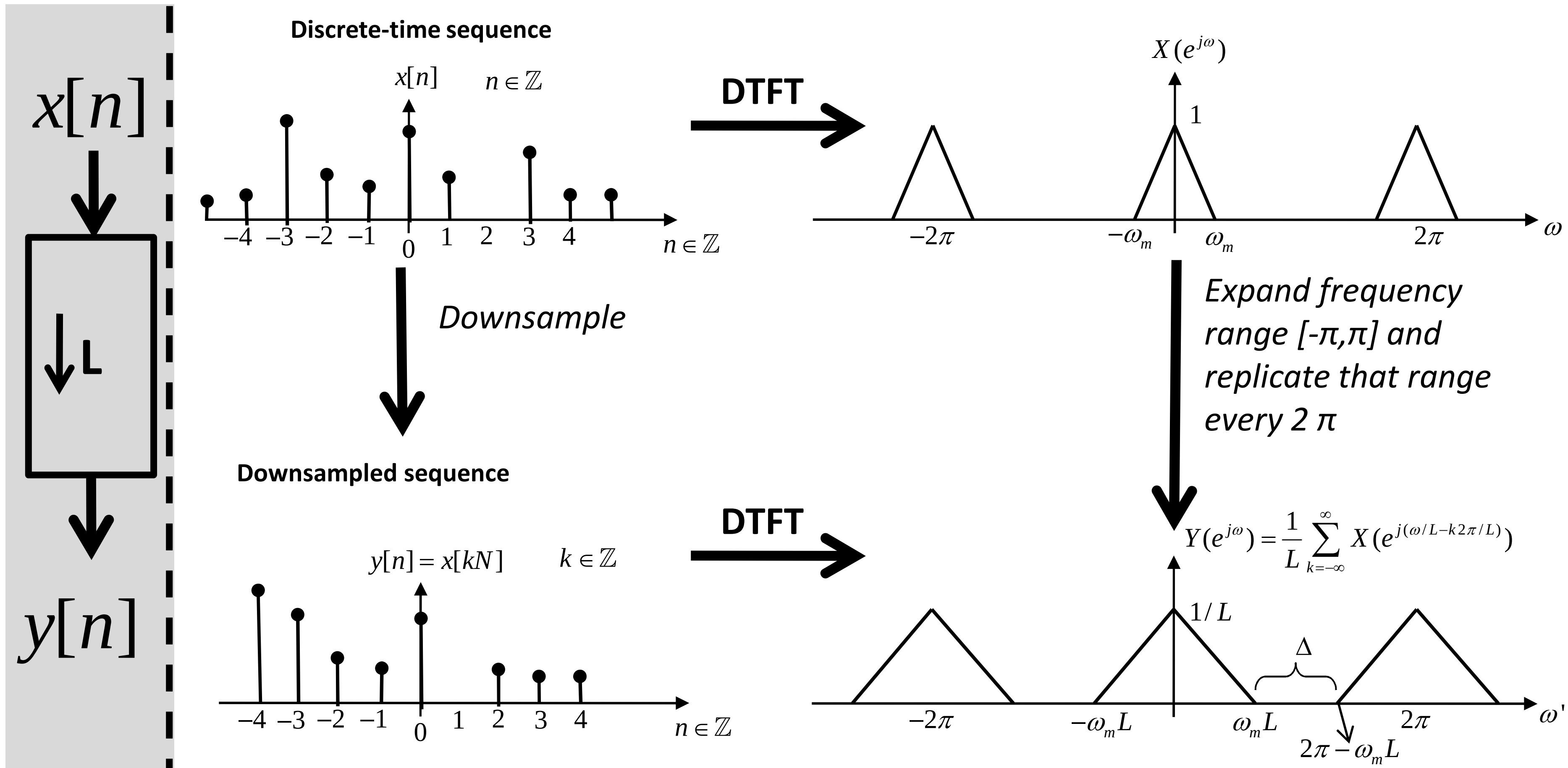

x[n]
H(e^jw)
y[n]
Discrete-time sequence
x[n] n ∈ ℤ
−4 −3 −2 −1 1 2 3 4
n ∈ ℤ
DTFT
X(e^jω)
A/T_s
−2π
−Ω_m T_s
Ω_m T_s
2π
ω
(*) NOTE: This is just an example of discrete-time processor.
H(e^jw) can be anything they tell you in the exercise
Impulse response
h[n] n ∈ ℤ
n ∈ ℤ
DTFT
Frequency response
H(e^jω)
1
−2π
−ω_c
ω_c
2π
ω
convolution
multiplication
Discrete-time sequence
y[n] = x[n] ⊗ h[n]
−4 −3 −2 −1 0 1 2 3 4
n ∈ ℤ
DTFT
Y(e^jω) = X(e^jω)H(e^jω)
A/T_s
−2π
−ω_c
ω_c
2π
ω



Ideal D/C converter (T_s)
$y[n]$
$\rho(t)$
$y_s(t)$
$H_r(j\Omega)$
$y_a(t)$
Discrete-time sequence
$y[n]$ $n \in \mathbb{Z}$
-4 -3 -2 -1 0 1 2 3 4 $n \in \mathbb{Z}$
DTFT
$Y(e^{j\omega})$
$\frac{A}{T_s}$
-2π $-\pi$ $-\omega_c$ ω_c π 2π ω
Time-index denormalization
$t = nT_s \in \mathbb{R}$
Freq. denormalization
$\Omega = \frac{\omega}{T_s} = \frac{\omega \cdot \Omega_s}{2\pi}$
Continuous-time sampled signal $T_s = \frac{2\pi}{\Omega_s}$
$y_s(t) = \sum_{k=-\infty}^{\infty} y[k]\delta(t-kT_s)$ $k \in \mathbb{Z}$
$Y_s(j\Omega) = Y(e^{j\omega})\big|_{\omega=\Omega T_s}$
CTFT
$\frac{A}{T_s}$
$-2T_s$ $-T_s$ 0 T_s $2T_s$ $3T_s$ $4T_s$ $t \in \mathbb{R}$
$-\Omega_s$ $-\frac{\omega_c}{T_s}$ $\frac{\omega_c}{T_s}$ Ω_s Ω
Ideal analog low-pass filter with cutoff $\Omega_s/2$
$h_r(t)$
$t \in \mathbb{R}$
CTFT
$H_r(j\Omega)$
T_s
$-\frac{\Omega_s}{2}$ $\frac{\Omega_s}{2}$ Ω
convolution
multiplication
Continuous-time analog signal
$y_a(t) = y_s(t) \otimes h_r(t)$
CTFT
$Y_a(j\Omega) = Y_s(j\Omega)H_r(j\Omega)$
A
$-\frac{\omega_c}{T_s}$ $\frac{\omega_c}{T_s}$ Ω



$x[n]$
$\uparrow$N
$y[n]$
Discrete-time sequence
$x[n]$ $n \in \mathbb{Z}$
-4 -3 -2 -1 0 1 2 3 4 $n \in \mathbb{Z}$
DTFT
$X(e^{j\omega})$
-2π $-\pi$ π 2π ω
Upsample and zero-pad
Compress frequencies
$\omega \to \frac{\omega}{N}$
Upsampled zero-padded sequence
$y[n] = \begin{cases} x\left(\frac{n}{N}\right) & \frac{n}{N} \in \mathbb{Z} \\ 0 & otherwise \end{cases}$
$y[n]$
-4 -3 -2 -1 0 1 2 3 4 $n \in \mathbb{Z}$
DTFT
$Y(e^{j\omega}) = X(e^{j\omega N})$
$-\frac{\pi}{N}$ $\frac{\pi}{N}$ ω

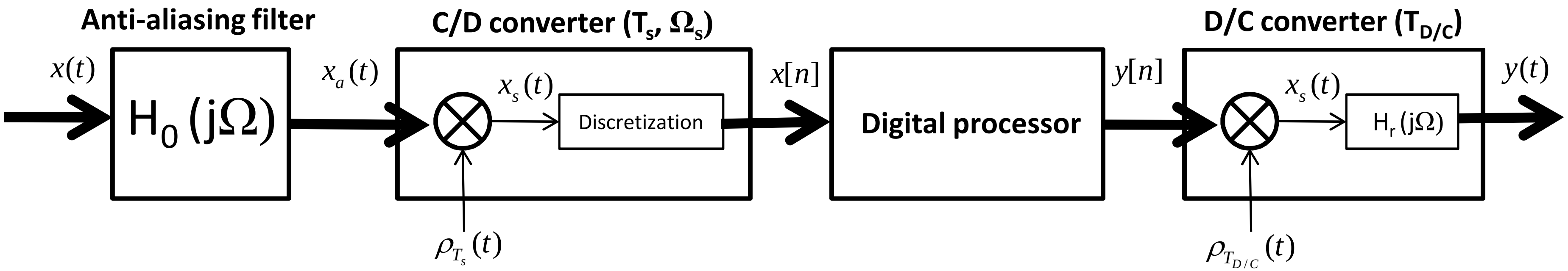


To avoid overlap between the spectral aliases:

$$\Delta = (2\pi - \omega_m L) - \omega_m L > 0 \Leftrightarrow L < \frac{\pi}{\omega_m}$$

About the reconstruction filter...

- A hybrid digital-analog system for processing **baseband signals** has usually the following basic components:



- The anti-aliasing filter is a low-pass filter used in order to guarantee that the maximum frequency in $x_a(t)$ satisfies that $\Omega_{\max} < \Omega_s / 2$. If this block is absent and for certain input signal $\Omega_{\max} \geq \Omega_s / 2$ we will have aliasing and perfect reconstruction will not be possible. The shape of $H_0(j\Omega)$ is sketched below.

- The reconstruction filter in the C/D converter is exactly the same as the anti-aliasing filter but with a gain T_s.

- The C/D converter might use an impulse train $\rho_{T_{D/C}}(t)$ with a different period ($T_{D/C}$) than the period (T_s) of the impulse train used in the C/D converter $\rho_{T_s}(t)$. However, the cutoff filter of the reconstruction filter is defined by the sampling rate used in the C/D converter and not by the period $T_{D/C}$ used in the D/C converter.

- **NOTE:** An alternative way of defining the cut-off of the reconstruction filter is to use $\Omega_c = \Omega_{max}$ where Ω_{max} is the maximum frequency in $x_a(t)$. However, this makes the definition of the reconstruction filter dependent on the input to the system.