Sums of some infinite series

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Geometric series. If $a \in \Re$ and $r \in \Re$, a geometric series $\{x_n\} = \{a, ar, ar^2, ..., ar^n, ...\}$ converges if and only if $|r| < 1$ and then its sum is:

$$\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} a\frac{1 - r^{n+1}}{1 - r} = \frac{a}{1 - r} \tag{1}$$

Telescopic series. Given a series $\{a_n\}$ we say that the series $\{x_n\} = \{a_n - a_{n+1}\}$ is a telescopic series. Then, $\{x_n\}$ is convergent if and only if $\lim_{n\to\infty} a_n$ is finite. In that case, the sum of $\{x_n\}$ is:

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_n \tag{2}$$

Arithmetic-geometric series. Given $a, b, r \in \Re$ we say that the series $\{x_n\}$ is arithmetic-geometric if $\{x_n\} = \{a, (a+b)r, (a+2b)r^2, ..., (a+nb)r^n, ...\}$. Such series are convergent if and only if $|r| < 1$ and, in that case, their sum is:

$$\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} (a + nb)r^n = \frac{a(1 - r) + br}{(1 - r)^2} \tag{3}$$

Hypergeometric series. We say that a series $\{x_n\}$ is hypergeometric if:

$$\frac{x_{n+1}}{x_n} = \frac{an + b}{an + c} \tag{4}$$

where $a, b, c \in \Re$ and $a \neq 0$. This type of series converges if and only if $\frac{(c-b)}{a} > 1$ and, in that case, its sum is:

$$\sum_{n=1}^{\infty} x_n = \frac{-x_1 c}{a + b - c} \tag{5}$$