EXERCISE 2 - SOLUTIONS

SGN-1156 Signal Processing Techniques
http://www.cs.tut.fi/courses/SGN-1156
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December 4, 2009

Important: The most important information of Chapter 3 of the book is in tables 3.1, 3.2 and specially 3.3 and 3.4. When computing DTFTs it is often useful to know that:

$$\sum_{n=m}^{k} r^n = \frac{r^{k+1} - r^m}{r-1}$$

Note that I made a mistake when writing the expression above in the whiteboard. Thanks to Junsheng for reporting this. Other useful sums are:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad |r| < 1$$

$$\sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2} \qquad |r| < 1$$

Remember also the relationships:

$$\cos\omega = \frac{e^{j\omega} + e^{-j\omega}}{2}$$

$$\sin\omega = \frac{e^{j\omega} - e^{-j\omega}}{2j}$$

PROBLEM 1: Determine the DTFT of each of the following sequences:

(a) $x_a[n] = \mu[n] - \mu[n-5]$

(b) $x_b[n] = \alpha^n \left(\mu[n] - \mu[n-8]\right) \qquad |\alpha| < 1$

(c) $x_c[n] = (n+1)\alpha^n \mu[n] \qquad |\alpha| < 1$

SOLUTION: There are two ways of computing the DTFT of a sequence. Either you apply directly the formula of the DTFT or you try to express your sequence as a linear combination of elementary sequences for which you know the DTFT.

(a)

Using directly the DTFT formula:

$$X_a(e^{j\omega}) = \sum_{n=0}^{\infty} x_a[n] e^{-j\omega n} = \sum_{n=0}^{4} e^{j\omega n} = \frac{e^{-j\omega 5} - 1}{e^{-j\omega} - 1}$$

Using the fact that the DTFT of the unit step function is:

$$\mu[n] \overset{DTFT}{\rightarrow} G(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$$

then:

$$X_a(e^{j\omega}) = G(e^{j\omega}) - e^{-j\omega 5} G(e^{j\omega}) = \frac{1 - e^{-j\omega 5}}{1 - e^{-j\omega}} - \underbrace{(1 - e^{-j\omega 5}) \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)}_{=0\ \forall \omega}$$

So we finally obtain:

$$X_a(e^{j\omega}) = \frac{1 - e^{-j\omega 5}}{1 - e^{-j\omega}}$$

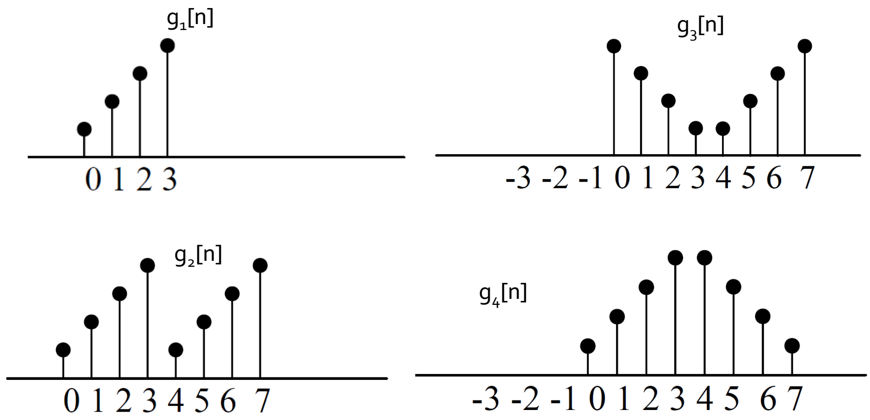
(b)

$$X_b(e^{j\omega}) = \frac{1 - \alpha^8 e^{-j\omega 8}}{1 - \alpha e^{-j\omega}}$$

(c)

$$X_c(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^2}$$

PROBLEM 2: (problem 3.41 from the book) Let $G_1(e^{j\omega})$ denote the discrete-time Fourier transform of the sequence $g_1[n]$ shown in the figure below. Express the DTFTs of $g_2[n]$, $g_3[n]$ and $g_4[n]$ in terms of $G_1(e^{j\omega})$. Do not evaluate $G_1(e^{j\omega})$.



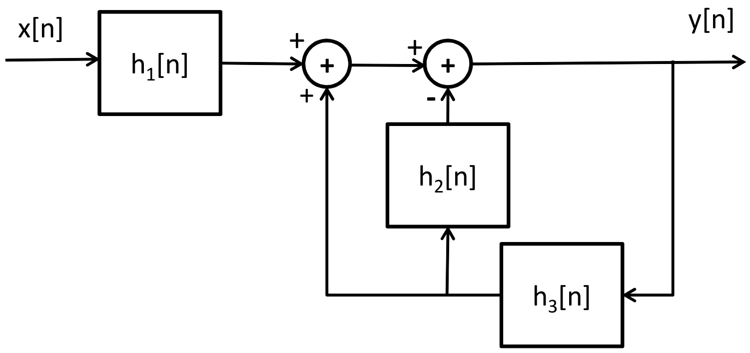
SOLUTION:

$g_2[n] = g_1[n] + g_1[n-4] \Rightarrow G_2 e^{j\omega} = \left(1 + e^{-4j\omega}\right) G_1(e^{j\omega})$

$g_3[n] = g_1[-(n-3)] + g_1[n-4] \Rightarrow G_3 e^{j\omega} = e^{-j3\omega} G_1(e^{-j\omega}) + e^{-j4\omega} G_1(e^{j\omega})$

$g_4[n] = g_1[n] + g_1[-(n-7)] \Rightarrow G_4(e^{j\omega}) = G_1(e^{j\omega}) + e^{-j7\omega} G_1(e^{-j\omega})$

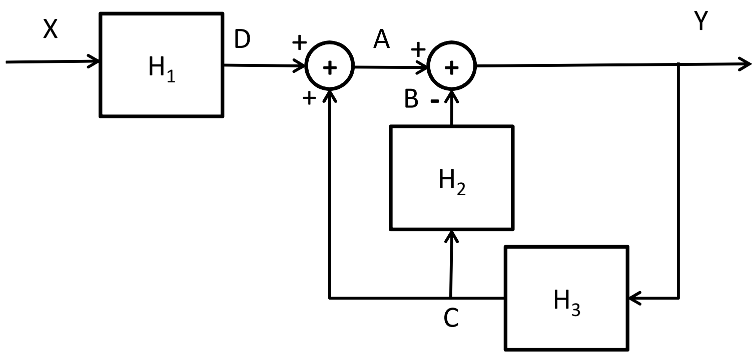
PROBLEM 3: Consider the following interconnection of linear shift-invariant systems:



Express the frequency response of the overall system $H(e^{j\omega})$ in terms of the frequency responses of the subsystems $H_1(e^{j\omega})$, $H_2(e^{j\omega})$, and $H_3(e^{j\omega})$.

SOLUTION:

The first step is to represent the given system in frequency domain and to introduce new intermidiate variables in any interconnection between diagram elements. This is shown below:



For simplicity we have omitted all the terms $(e^{j\omega})$ in the diagram above. Unless otherwise stated uppercase letters will denote Fourier-domain variables and lower-case letters time-domain ones. We can now write all the system equations in Fourier domain:

$$
\begin{aligned}
Y &= A - B && (1)\\
A &= D + C && (2)\\
B &= H_2 \cdot C && (3)\\
C &= H_3 \cdot Y && (4)\\
D &= H_1 \cdot X && (5)
\end{aligned}
$$

The overall frequency response is defined as $H = \frac{Y}{X}$. Therefore, we need to combine the five equations above into a single one that has only X and Y as unknowns. Combining Eqs. 1, 2 and 3 we obtain:

$$Y = D + (1 - H_2)C \tag{6}$$

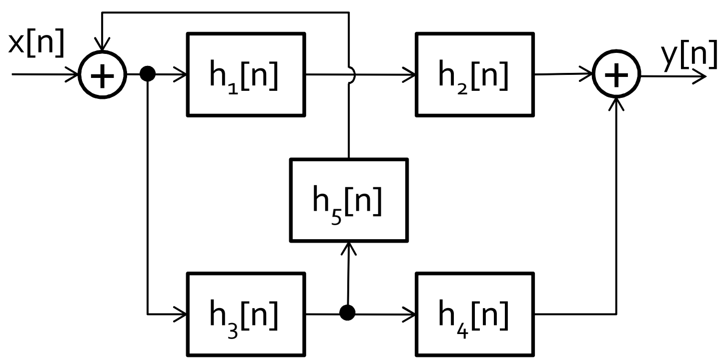
Now combining Eq. 6 with Eqs. 4 and 5 we get to:

$$Y = H_1 \cdot X + (1 - H_2)H_3 Y \tag{7}$$

which has only two unknowns: X and Y. Reorganizing Eq. 7 we finally obtain the overall frequency response of the system:

$$Y = \frac{H_1}{1 + H_3(H_2 - 1)} X \Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{H_1(e^{j\omega})}{1 + H_3(e^{j\omega})(H_2(e^{j\omega}) - 1)} \tag{8}$$

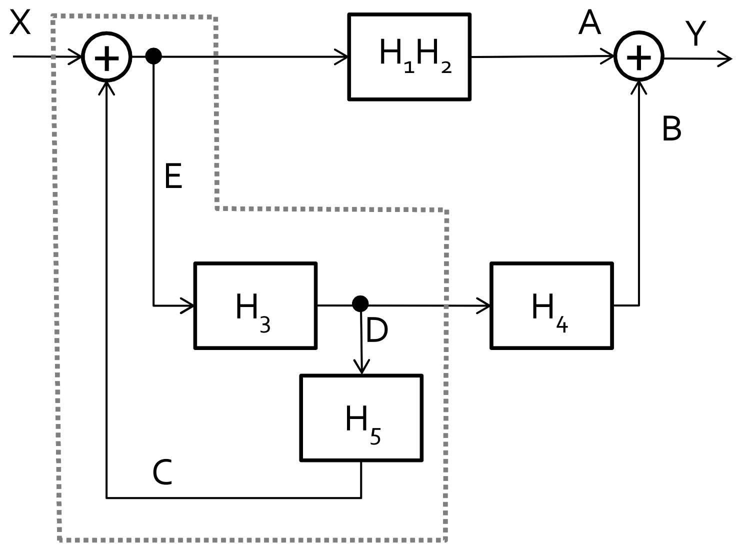
PROBLEM 4. Consider the following interconnection of LTI systems:



Express the frequency response of the overall system $H(e^{j\omega})$ in terms of the frequency responses of the subsystems depicted in the diagram.

SOLUTION:

The first thing that we do is to transform all variables to the DTFT domain (we omit the $e^{j\omega}$ terms), to introduce intermediate variables in every connection between system elements and to simplify as much as possible the diagram:



We can observe that there is a feedback loop involving the first addition operator and systems H_3 and H_5 (surrounded by a dashed line in the diagram above). The best way to proceed is to first compute the frequency response of the sub-system bounded by the dashed line, i.e. to find the relationship between D and X. We can see that the feedback sub-system contains 4 unknowns (X, E, D, C) which means that we will need to write 3 equations:

$$D = H_3E \tag{9}$$
$$C = H_5D \tag{10}$$
$$E = X + C \tag{11}$$

Combining these three equations:

$$D = H_3X + H_3C = H_3X + H_3H_5D \Rightarrow D = \frac{H_3}{1 - H_3H_5}X$$

Then we can also find the relationship between E and the input X:

$$D = H_3E \Rightarrow E = \frac{D}{H_3} = \frac{1}{1 - H_3H_5}X$$

Now we can proceed to determine the overall frequency response of the whole system. Since in the whole system we have 7 unknowns, we need 6 equations to fully determine the output with respect to the input. We already wrote 3 equations above so we just need 3 more:

$$\begin{aligned} Y &= A + B && (12) \\ A &= H_1 H_2 E && (13) \\ B &= H_4 D && (14) \end{aligned}$$

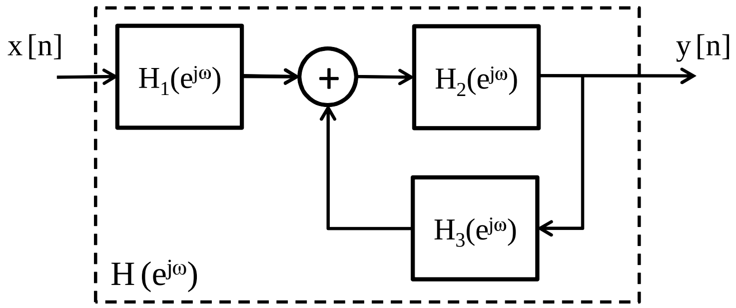
Combining the three equations above:

$$Y = A + B = H_1 H_2 E + H_4 D$$

and using the expressions that we found for E and D with respect to X:

$$Y = H_1 H_2 E + H_4 D = \frac{H_1 H_2 + H_4 H_3}{1 - H_3 H_5} X$$

PROBLEM 5. Consider the interconnection of linear shift-invariant systems in the figure below:



(a) Express the frequency response of the overall system $H(e^{j\omega})$ in terms of the frequency responses of the subsystems $H_1(e^{j\omega})$, $H_2(e^{j\omega})$ and $H_3(e^{j\omega})$.

(b) Determine the frequency response $H(e^{j\omega})$ of the overall system if:

$$\begin{aligned} h_1[n] &= \frac{\sin(\frac{\pi}{3}n)}{\pi n} \\ h_2[n] &= (0.3)^n \mu[n] \\ h_3[n] &= \delta[n-2] \end{aligned}$$

SOLUTION:

The overall frequency response is:

$$Y = \frac{H_1 H_2}{1 - H_2 H_3} X \Rightarrow H = \frac{H_1 H_2}{1 - H_2 H_3}$$

Transforming $h_1[n]$, $h_2[n]$, $h_3[n]$ to the DTFT domain we finally obtain that:

$$H(e^{j\omega}) = \begin{cases} \frac{1}{1-0.3e^{-j\omega}-e^{-j\omega 2}} & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

PROBLEM 6 (problem 3.59 from the book): An LTI IIR discrete-time system is described by the difference equation

$$y[n] + a_1 y[n-1] = b_0 x[n] + b_1 x[n-1]$$

where the input is $x[n]$, the output is $y[n]$, and the constants a_1, b_0 and b_1 are real. Determine the expression for its frequency response. For what values of b_0 and b_1 will the magnitude response be a constant for all values of ω?.

SOLUTION: In order to obtain the frequency response of the system, first we need to transform the system's equation into the frequency domain:

$$Y(e^{j\omega}) + a_1 e^{-j\omega} Y(e^{j\omega}) = b_0 X(e^{j\omega}) + b_1 e^{-j\omega} X(e^{j\omega})$$

so the frequency response of the system is:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b_0 + b_1 e^{-j\omega}}{1 + a_1 e^{-j\omega}}$$

It is difficult to operate with the requirement $|H(e^{j\omega})| = K =$ constant due to the presence of complex quantities inside the absolute value operator. In order to get rid of complex terms the easiest is to solve for the equivalent requirement $|H(e^{j\omega}))|^2 = K^2 =$ constant:

$$
\begin{aligned}
|H(e^{j\omega})|^2 &= H(e^{j\omega})H^*(e^{j\omega}) = \tfrac{(b_0+b_1e^{-j\omega})(b_0+b_1e^{j\omega})}{(1+a_1e^{-j\omega})(1+a_1e^{j\omega})} \\
&= \tfrac{b_0^2+b_1^2+b_0b_1\left(e^{j\omega}+e^{-j\omega}\right)}{1+a_1^2+a_1(e^{j\omega}+e^{j\omega})} = \tfrac{b_0^2+b_1^2+2b_0b_1\cos\omega}{1+a_1^2+2a_1\cos\omega} = K^2 = C
\end{aligned}
$$

Then we have to find the values of a_1, b_0 and b_1 satifying:

$$
\underbrace{b_0^2+b_1^2}_{A}+\underbrace{2b_0b_1}_{B}\cos\omega = \underbrace{C+Ca_1^2}_{A'}+\underbrace{2Ca_1}_{B'}\cos\omega
$$

We can see that the expressions at each side of the equation above corresponds to a scaled sinusoid with non-zero mean. So we can enforce the equality by simply enforcing that the mean of the sinusoids is the same ($A = A'$) and that the scale of both sinusoids is also the same ($B = B'$):

$$
\begin{aligned}
A = A' &\quad\Rightarrow\quad b_0^2+b_1^2 = C+Ca_1^2 \\
B = B' &\quad\Rightarrow\quad b_0b_1 = Ca_1
\end{aligned}
$$

Putting both equations together:

$$
b_0 = C\frac{a_1}{b_1} \Rightarrow \frac{C^2a_1^2}{b_1^2}+b_1^2 = C+Ca_1^2 \Rightarrow b_1^4 - C(1+a_1^2)b_1^2 + C^2a_1^2 = 0
$$

The quadratic equation above has two solutions:

$$
b_1^2 = \frac{C(1+a_1^2) \pm \sqrt{C^2(1+a_1^2)^2 - 4C^2a_1^2}}{2}
$$

Operating we get that the solution is either $b_1 = \pm K$ or $b_1 = \pm Ka_1$. We first try the former solution to check if it fulfills our requirement:

$$
|H(e^{j\omega})| = \left|\frac{\pm Ka_1 \pm Ke^{-j\omega}}{1+a_1e^{-j\omega}}\right| = K\left|\frac{a_1+e^{-j\omega}}{1+a_1e^{-j\omega}}\right|
$$

Clearly, the expression above is not equal to a constant in general so this is not a valid solution of the problem. We try now the solution $b_1 = \pm Ka_1$:

$$|H(e^{j\omega})| = \left| \frac{\pm K \pm K a_1 e^{-j\omega}}{1 + a_1 e^{-j\omega}} \right| = K \left| \frac{1 + a_1 e^{-j\omega}}{1 + a_1 e^{-j\omega}} \right| = K$$

And therefore $b_1 = \pm K a_1$ is the solution that we need. The reason for one of the solutions that we found to be invalid is that we substituted the original requirement $|H(e^{j\omega})| = K$ for the alternative requirement $|H(e^{j\omega})|^2 = K^2$. This had the effect of introducing an spurious solution since the latter equality is fulfilled not only when $|H(e^{j\omega})| = K$ but also when $|H(e^{j\omega})| = -K$. Obviously, the latter solution is not valid.

PROBLEM 7: Consider the system defined by the difference equation

$$y[n] = ay[n-1] + bx[n] + x[n-1]$$

where a and b are real, and $|a| < 1$. Find the relationship between a and b that must exist if the frequency response is to have a constant magnitude for all ω, that is $|H(e^{j\omega})| = 1$.

SOLUTION: In order to obtain the frequency response of the system, first we need to transform the system's equation into the frequency domain:

$$Y(e^{j\omega}) = ae^{-j\omega}Y(e^{j\omega}) + bX(e^{j\omega}) + e^{-j\omega}X(e^{j\omega})$$

so the frequency response of the system is:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b + e^{-j\omega}}{1 - a \cdot e^{-j\omega}}$$

If the equality $|H(e^{j\omega})| = 1$ hold then it must also hold the equality $|H(e^{j\omega})|^2 = 1$. Solving this latter equality is easier because the expression on the left side has only real terms inside the absolute value operator. Operating a bit:

$$|H(e^{j\omega})|^2 = 1 \Longrightarrow H(e^{j\omega}) \cdot H^*(e^{j\omega}) = 1$$

substituting the value of $H(e^{j\omega})$ into the expression above we obtain:

$$H(e^{j\omega}) \cdot H^*(e^{j\omega}) = \frac{(b + e^{-j\omega}) \cdot (b + e^{j\omega})}{(1 - a \cdot e^{-j\omega}) \cdot (1 - a \cdot e^{j\omega})} = \frac{1 + b^2 + b \cdot (e^{j\omega} + e^{-j\omega})}{1 + a^2 - a \cdot (e^{j\omega} + e^{-j\omega})} = 1$$

So, the numerator and denominator of the fraction above must be equal, which translates into the two equations:

$$\begin{aligned} 1 + b^2 &= 1 + a^2 \\ b &= -a \end{aligned}$$

Clearly, if and only if $b = -a$ the two equations are fullfiled and $|H(e^{j\omega})|^2 = 1$. Since we have only a single solution to the squared version of our original condition we can conclude that this solution has to be also the solution to our original constraint. That is $|H(e^{j\omega})| = 1$ if and only if $b = -a$.