EXERCISE 4 - SOLUTIONS

SGN-1156 Signal Processing Techniques
http://www.cs.tut.fi/courses/SGN-1156
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PROBLEM 1 (problem 5.21 from the book): Let $x[n]$, $0 \leq n \leq N-1$ be a length-N sequence with an N-point DFT $X[k]$, $0 \leq k \leq N-1$. Determine the N-point DFTs of the following length-N sequences in terms of $X[k]$:

(a) $w[n] = \alpha x[\langle n - m_1 \rangle_N] + \beta x[\langle n - m_2 \rangle_N]$, where m_1 and m_2 are positive integers less than N.

(b) $g[n] = \begin{cases} x[n] & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

(c) $y[n] = x[n] \overset{N}{\otimes} x[n]$.

SOLUTION:

(a)

Using the circular time-shifting property from Table 5.3 of the book:

$$W[k] = \left(\alpha W_N^{m_1 k} + \beta W_N^{m_2 k}\right) X[k]$$

(b)

$$g[n] = \left(\frac{1}{2} + (-1)^n\right) x[n] = \left(\frac{1}{2} + W_N^{\frac{Nn}{2}}\right) x[n]$$

So using the circular frequency-shifting property from Table 5.3 of the book:

$$G[k] = \frac{1}{2}X[k] + \frac{1}{2}X\left[\langle k - \frac{N}{2}\rangle_N\right]$$

(c)

Using the N-point circular convolution property from Table 5.3 of the book:

$$Y[k] = X[k]X[k] = X^2[k]$$

PROBLEM 2 (problem 5.23 from the book): Let $x[n]$, $0 \leq n \leq N-1$ be a length-N sequence with an N-point DFT $X[k]$, $0 \leq k \leq N-1$. Determine the N-point inverse DFTs of the following length-N DFTs in terms of $x[n]$:

(a) $W[k] = \alpha X[\langle k - m_1\rangle_N] + \beta x[\langle k - m_2\rangle_N]$, where m_1 and m_2 are positive integers less than N.

(b) $G[k] = \begin{cases} X[k] & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases}$

(c) $Y[k] = X[k] \overset{N}{\otimes} X[k]$.

SOLUTION:

(a)

Using the circular frequency-shifting property from Table 5.3 of the book:

$$w[n] = \left(\alpha W_N^{-m_1 n} + \beta W_N^{-m_2 n}\right) x[n]$$

(b)

$$G[k] = \left(\frac{1}{2} + (-1)^n\right) X[k] = \left(\frac{1}{2} + W_N^{\frac{Nk}{2}}\right) X[k]$$

So using the circular time-shifting property from Table 5.3 of the book:

$$g[n] = \frac{1}{2}x[n] + \frac{1}{2}x[\langle n - \frac{N}{2}\rangle_N]$$

(c)

Using the modulation property from Table 5.3 of the book:

$$y[n] = Nx[n]x[n] = Nx^2[n]$$

PROBLEM 3 (problem 5.42 from the book): A 126-point DFT $X[k]$ of a real-valued sequence $x[n]$ has the following DFT samples: $X[0] = 12.8 + j\alpha$, $X[13] = -3.7 + j2.2$, $X[k_1] = 9.1 - j5.4$, $X[k_2] = 6.3 + j2.3$, $X[51] = -j1.7$, $X[63] = 13 + j\beta$, $X[k_3] = \gamma + j1.7$, $X[79] = 6.3 + j\delta$, $X[108] = \epsilon + j5.4$, $X[k_4] = -3.7 - j2.2$. The remaining DFT samples are assumed to be equal to zero.

(a) Determine the values of the indices k_1, k_2, k_3 and k_4.

(b) Determine the values of α, β, γ, δ, and ϵ.

(c) What is the DC value of $\{x[n]\}$?

(d) Determine the expression for $\{x[n]\}$ without computing the IDFT.

(e) What is the energy of $\{x[n]\}$?

SOLUTION:

(a)

Since $x[n]$ is real-valued we have the symmetry property $X[k] = X^*[\langle -k\rangle_N]$. This means that if $X[k_1] = 9.1 - j5.4$ then $X[\langle -k_1\rangle_N] = 9.1 + j5.4$. From the given values of $X[k]$ it is clear that $\langle -k_1\rangle_N = N - k_1 = 108 \Rightarrow k_1 = 126 - 108 = 18$. Similarly we get:

$$\langle -k_2\rangle_N = N - k_2 = 79 \Rightarrow k_2 = 126 - 79 = 47$$

$$\langle -k_3 \rangle_N = N - k_3 = 51 \Rightarrow k_3 = 126 - 51 = 75$$

$$\langle -k_4 \rangle_N = N - k_4 = 13 \Rightarrow k_4 = 126 - 13 = 113$$

(b)

Again, using the symmetry $X[k] = X^*[\langle -k \rangle_N]$ we have that $X[0] = X^*[0] \Rightarrow \mathrm{Im}\{X[0]\} = 0 \Leftrightarrow \alpha = 0$. Similarly:

$$\begin{aligned}
X[63] &= 13 + j\beta = X^*[\langle -63 \rangle_N] = X^*[N - 63] = X^*[63] \Leftrightarrow \mathrm{Im}\{X[63]\} = 0 \Leftrightarrow \beta = 0 \\
X[75] &= \gamma + j1.7 = X^*[\langle -75 \rangle_N] = X^*[N - 75] = X^*[51] = j1.7 \Leftrightarrow \gamma = 0 \\
X[79] &= 6.3 + j\delta = X^*[\langle -79 \rangle_N] = X^*[N - 79] = X^*[47] = 6.3 - j2.3 \Leftrightarrow \delta = -2.3 \\
X[108] &= \epsilon + j5.4 = X^*[\langle -108 \rangle_N] = X^*[N - 108] = X^*[18] = 9.1 + j5.4 \Leftrightarrow \epsilon = 9.1
\end{aligned}$$

(c)

The DC value is $X[0] = 12.8$

(d)

$$\begin{aligned}
x[n] &= \tfrac{1}{126} \textstyle\sum_{k=0}^{125} X[k] W_{126}^{-kn} = \tfrac{1}{126} \big(X[0] + X[63] W_{126}^{-63n} + 2\mathrm{Re}\{X[13]\} W_N^{-13n} + \\
&\quad 2\mathrm{Re}\{X[18]\} W_N^{-18n} + 2\mathrm{Re}\{X[47]\} W_N^{-47n} + 2\mathrm{Re}\{X[75]\} W_N^{-75n} + \\
&\quad + 2\mathrm{Re}\{X[113]\} W_N^{-113n} \big)
\end{aligned}$$

(e)

$$E_x = \sum_{n=1}^{125} |x[n]|^2 = \frac{1}{126} \sum_{k=0}^{125} |X[k]|^2 = 5.767$$