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Foundation of Algorithms

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Chapter 17 Amortized Analysis

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17.1 Background and Methods

- Background
- Amortized analysis
- Three methods

Incrementing a Binary Counter

- k-bit Binary Counter: $A[0..k\text{-}1]$

$$x = \sum_{i=0}^{k-1} A[i] \cdot 2^i$$

```
INCREMENT(A)
i ← 0
while i < length[A] and A[i] = 1
      do  A[i] ← 0          ⊳ reset a bit
            i ← i + 1
if  i < length[A]
      then  A[i] ← 1        ⊳ set a bit
```

k-bit Binary Counter

Value	A[4]	A[3]	A[2]	A[1]	A[0]	*Cost*
0	0	0	0	0	0	***0***
1	0	0	0	0	1	***1***
2	0	0	0	1	0	***3***
3	0	0	0	1	1	***4***
4	0	0	1	0	0	***7***
5	0	0	1	0	1	***8***
6	0	0	1	1	0	***10***
7	0	0	1	1	1	***11***
8	0	1	0	0	0	***15***
9	0	1	0	0	1	***16***
10	0	1	0	1	0	***18***
11	0	1	0	1	1	***19***

Worst-case analysis

Consider a sequence of n insertions. The worst-case time to execute one insertion is $\Theta(k)$. Therefore, the worst-case time for n insertions is $n \cdot \Theta(k) = \Theta(n \cdot k)$.

WRONG! In fact, the worst-case cost for n insertions is only $\Theta(n) \ll \Theta(n \cdot k)$.

Let's see why.

Note: You'll be correct
If you'd said $O(n \cdot k)$.
But, it's an overestimate.

Tighter analysis

value	A[4]	A[3]	A[2]	A[1]	A[0]	*Cost*
0	0	0	0	0	**0**	***0***
1	0	0	0	**0**	**1**	***1***
2	0	0	0	1	**0**	***3***
3	0	0	**0**	**1**	**1**	***4***
4	0	0	1	0	**0**	***7***
5	0	0	1	**0**	**1**	***8***
6	0	0	1	1	**0**	***10***
7	0	**0**	**1**	**1**	**1**	***11***
8	0	1	0	0	**0**	***15***
9	0	1	0	**0**	**1**	***16***
10	0	1	0	1	**0**	***18***
11	0	1	**0**	**1**	**1**	***19***

Total cost of n operations

A[0] flipped every op	n
A[1] flipped every 2 ops	$n/2$
A[2] flipped every 4 ops	$n/2^2$
A[3] flipped every 8 ops	$n/2^3$
… … … …	…
A[i] flipped every 2^i ops	$n/2^i$

Tighter analysis (cont.)

Cost of n increments $\displaystyle = \sum_{i=1}^{\lfloor \lg n \rfloor} \left\lfloor \frac{n}{2^i} \right\rfloor$

$$< n \sum_{i=1}^{\infty} \frac{1}{2^i} = 2n$$

$$= \Theta(n).$$

Thus, the average cost of each increment operation is $\Theta(n)/n = \Theta(1)$.

Amortized Analysis (平摊分析)

- *Amortized analysis* is a cost analysis technique, which computes *the average cost* required to perform *a sequence of n operations on a data structure*.
- *Background:* Show that although some individual operations may be expensive, on average the cost per operation is small. Often *worst case analysis* is *not tight*.
- *Goal:* The amortized cost of an operation is less than its worst case, so that *average cost in the worst case* for a sequence of n operations is more *tighter*.
- *This average cost* is not based on averaging over a distribution of inputs. Here, *no probability is involved*.

Three Methods

- *Aggregate analysis (聚集分析)* – in worst case, the total amount of time needed for the n operations is computed and divided by n.
- *Accounting (记账方法)* – different operations are assigned different charges. Some operations charged more or less than their actual cost.
- *Potential (势能方法)* – the prepaid work is represented as "potential" energy that can be released to pay for future operations.

Chapter 17 Amortized Analysis

17.2 Aggregate Analysis (聚集分析)

- Basic idea
- Stack example
- Binary counter example

Basic Idea of Aggregate Analysis

- Assume that *n operations together* take worst-case time T(n).
- The *amortized cost* (or average cost) of an operation is T(n)/n.
- Remark
  - *Amortized cost is the same* for any operations, even for several types of operations.
  - Amortized cost *may be more or less than the actual cost* for an operations.

Example 1: A Stack

- Three operations:
  - push(S, x)
  - pop(S)
  - multipop(S, k) : Pop the stack k times

- Multipop operation

```
MULTIPOP(S, k)
while not STACK-EMPTY(S) and k ≠ 0
      do POP(S)
         k ← k − 1
```

The total cost of MULTIPOP(S, k) is $\min(s, k)$.
The worst-case cost of a MULTIPOP is $O(n)$.

Stack: Regular Cost Analysis

- Consider a sequence of n push(S, x), pop(S) and multipop(S, k) operations on a stack having as many as n items (元素).

- Regualr anlysis:
  - Note that worst-case cost of multipop() is O(n).
  - So, the worst-case cost for n-ops is $O(n^2)$.
  - This is *not tight*.

Stack: Aggregate Analysis

- For a stack is initially empty, Consider a n-sequence of push(), pop() and multipop().

- Aggregate analysis:
  - Each item (元素) can be popped only once for each time it is pushed.
  - So the total number of times pop() can be called, either directly or from multipop, is bounded by the number of pushes.
  - The number of pushes in a sequence of n ops is ≤ n, then the number of all pops (including those from multipop) is O(n).
  - So the total cost of the sequence of n ops is O(n). Therefore, we have O(1) per op on average.

Example 2: A Binary Counter

- A k-bit binary counter A[0..k−1] of bits, where A[0] is the least bit and A[k−1] is the most bit.
  - Value of the counter is $\sum_{i=0}^{k-1} A[i] \cdot 2^i$
  - Initially, counter value is 0. Then, Counts upward from 0.
- Increment operation, add 1:

- Flip all 1's from right to 0 until encountering the first 0.
- Change this 0 to 1 and stop.

INCREMENT(A, k)

$i = 0$
while $i < k$ and $A[i] == 1$
    $A[i] = 0$
    $i = i + 1$
if $i < k$
    $A[i] = 1$

Actual Cost and Regular analysis

- It shows a 8-bit binary counter as its value goes from 0 to 16 by a sequence of 16 Increment operations.
- The average cost per operation is 31/16 < 2.
- However, regular analysis gets O(nk) in the worst case (see 17.1)

Counter value	A[7]	A[6]	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	Total cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26
16	0	0	0	1	0	0	0	0	31

Binary Counter: Aggregate Analysis

- Observations about Increment():
  - No all bits are flipped for each call.
  - In general, A[i] flips only every 2^ith time.
- Thus, A[i] flips only $\lfloor n/2^i \rfloor$ times in a sequence of n Increment ops on an initially 0 counter.
- So the total number of flips is:

$$T(n) = \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor < n \sum_{i=0}^{\infty} \frac{1}{2^i} = 2n.$$

- We have T(n) = O(n). And the amortized cost per operation is O(n)/n = O(1).

Chapter 17 Amortized Analysis

17.1 Background and Methods

17.2 Aggregate Analysis

17.3 Accounting Method

17.4 Potential Method

17.3 Accounting Method

- Basic idea
- Stack example
- Binary counter example

Basic Idea of Accounting Method

- Assign different charges to different operations.
  - Some are charged more than actual cost.
  - Some are charged less than actual cost.
- *Amortized cost = amount we charge.*

- Remark:
  - When amortized cost > actual cost, store the difference on *specific items* in the data structure as *credit* (存款).
  - *Use credit later to pay* for operations whose actual cost > amortized cost.

Credit Rules

- Need credit to *never go negative*.
- Let c_i = actual cost of i-th operation,
  $\hat{c}_i$ = amortized cost of i-th operation.
- For all sequences of n operations, require:

$$\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$$

- *Total credit stored* = $\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i$

Example 1: A Stack

Operation	Actual Cost	Amortized Cost
push	1	2
pop	1	0
multipop	min{n, k}	0

- When pushing an item, pay $2:
  - $1 pays for the push.
  - $1 is prepayment for it being popped by either pop or multipop.
  - Since each item on the stack has $1 credit, the credit can never go negative.
  - The total amortized cost in the worst case is: $2n \in O(n)$
  - It is an upper bound on total actual cost.

Example 2: A Binary Counter

- Charge \$2 to set a bit to 1.
  - \$1 pays for setting a bit to 1.
  - \$1 is prepayment for flipping it back to 0.
  - Have \$1 of credit for every 1 in the counter.
  - Therefore, credit ≥ 0.

- Amortized cost of Increment:
  - Cost of resetting bits to 0 is paid by credit.
  - At most 1 bit is set to 1 in each increment operation.
  - Therefore, amortized cost $\leq$ \$2.
  - For n operations, the total amortized cost in the worst case is $2n \in O(n)$. So, amortized cost for an op is $O(1)$.

Chapter 17 Amortized Analysis

17.4 Potential Method

- Basic idea
- Stack example
- Binary counter example

Basic Idea of Potential Method

- Idea: like the accounting method, but think of the credit as *potential stored with the entire data structure*.
  - *Accounting method* stores credit with specific items.
  - Can release potential to pay for future operations.
- It is the most flexible among the amortized analysis methods.

Understanding Potential (1)

- Framework:
  - Start with an initial data structure D_0.
  - Operation i transforms D_{i-1} to D_i.
  - The cost of operation i is c_i.
  - Define a *potential function* $\Phi: \{D_i\} \to R$, such that $\Phi(D_0) = 0$ and $\Phi(D_i) \geq 0$ for all i
  - The *amortized cost* $\hat{c}_i$ with respect to Φ is defined to be $\hat{c}_i = c_i + \underbrace{\Phi(D_i) - \Phi(D_{i-1})}_{\textit{potential difference } \Delta\Phi_i}$
- In practice, $\Phi(D_0) = 0$, $\Phi(D_i) \geq 0$ for all i. So,
  - the *amortized cost* is always *an upper bound on actual cost*.
  - work is stored *in the entire data structure* for later use.

Understanding Potential（2）

- The total amortized cost of n operations is

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} \left(c_i + \Phi(D_i) - \Phi(D_{i-1}) \right)$$

$$= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

$$\geq \sum_{i=1}^{n} c_i$$

since $\Phi(D_n) \geq 0$ and $\Phi(D_0) = 0$.

Example 1: A Stack

- Define potential function Φ on a stack = number of items on the stack.
- Let D_0 = empty, then $\Phi(D_0) = 0$.
- Since the number of items on a stack is always ≥ 0, $\Phi(D_i) \geq \Phi(D_0) = 0$.

operation	actual cost	$\Delta\Phi$	amortized cost
PUSH	1	$(s+1)-s=1$ where $s = \#$ of objects initially	$1+1=2$
POP	1	$(s-1)-s=-1$	$1-1=0$
MULTIPOP	$k' = \min(k, s)$	$(s-k')-s=-k'$	$k'-k'=0$

- So, the total amortized cost of a sequence of n operations in the worst case is 2n = O(n).

Example 1: A Stack (cont.)

operation	actual cost	$\Delta\Phi$	amortized cost
PUSH	1	$(s+1)-s=1$ where $s=\#$ of objects initially	$1+1=2$
POP	1	$(s-1)-s=-1$	$1-1=0$
MULTIPOP	$k'=\min(k,s)$	$(s-k')-s=-k'$	$k'-k'=0$

Push: $\hat{c}_i = c_i + \phi(D_i) - \phi(D_{i-1})$

$= 1 + j - (j-1)$

$= 2$

Pop: $\hat{c}_i = c_i + \phi(D_i) - \phi(D_{i-1})$

$= 1 + (j-1) - j$

$= 0$

Multi-pop: $\hat{c}_i = c_i + \phi(D_i) - \phi(D_{i-1})$

$= k' + (j-k') - j$ $\quad$ **k'=min(|S|,k)**

$= 0$

Example 2: A Binary Counter

$\Phi = b_i = \#$ of 1's after ith INCREMENT

Suppose ith operation resets t_i bits to 0.

$c_i \le t_i + 1$ (resets t_i bits, sets ≤ 1 bit to 1)

- If $b_i = 0$, the ith operation reset all k bits and didn't set one, so $b_{i-1} = t_i = k \Rightarrow b_i = b_{i-1} - t_i$.
- If $b_i > 0$, the ith operation reset t_i bits, set one, so $b_i = b_{i-1} - t_i + 1$.
- Either way, $b_i \le b_{i-1} - t_i + 1$.
- Therefore,

$$\begin{aligned} \Delta\Phi(D_i) &\le (b_{i-1} - t_i + 1) - b_{i-1} \\ &= 1 - t_i \,. \end{aligned}$$

$$\begin{aligned} \hat{c}_i &= c_i + \Delta\Phi(D_i) \\ &\le (t_i + 1) + (1 - t_i) \\ &= 2 \,. \end{aligned}$$

If counter starts at 0, $\Phi(D_0) = 0$.

Therefore, amortized cost of n operations $= O(n)$.

Example 2: A Binary Counter (cont.)

General Case

The potential method gives us an easy way to analyze the counter even when it does not start at 0. There are initially b_0 1's and after n INCREMENT operations there are b_n 1's.

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} 2 - b_n + b_0$$

$$= 2n - b_n + b_0$$

No matter what initial value the counter contains, the actual cost has an upper bound of $O(n)$.

End of Ch17