Web/Networks Structure Mining: Communities

One of the most important structural properties in networks

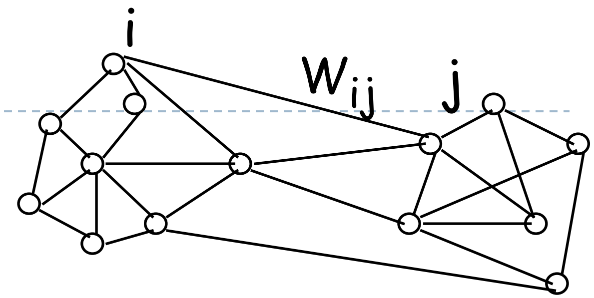
Community Detection Methods

Connection between community detection and clustering

- Agglomerative hierarchical clustering
- Partitional clustering
  - K-means

- Divisive hierarchical algorithm – Girvan and Newman
- Spectral graph cut
- Modularity maximization

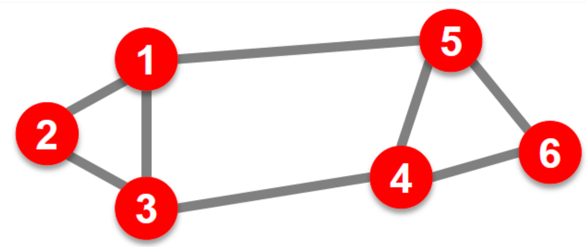
Basic Graph Notation



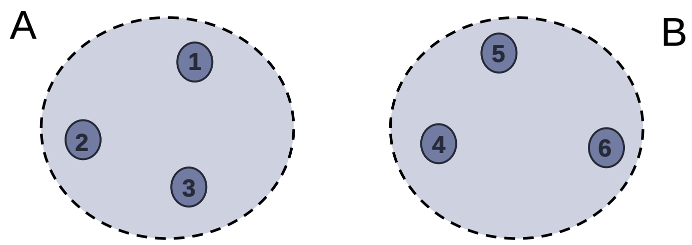
- A graph $G = (V,E)$ consists of vertices V and edges E
- Edges can be weighted or unweighted.
- The adjacency matrix邻接矩阵 A (weight matrix W) describes the graph:
  - $A_{ij}=0$ ($W_{ij}=0$) if vertices i and j are not connected
  - $A_{ij}=1$ (W_{ij}=weight of the edge), if they are connected
- The degree of a vertex is the sum of all adjacent edge weights: $d_i= \Sigma_j A_{ij}$ $(\Sigma_j W_{ij})$
- All vertices which can be reached from each other by a path form a connected component（连通分支）

Graph Partitioning

- Undirected graph G=(V, E):



- Partitioning task:
  - Divide vertices into two disjoint groups (A, B)

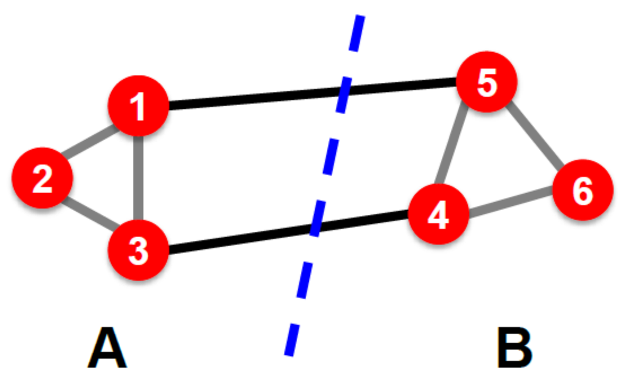


- V=A U B
- Questions:
  - How can we define a "good" partition of G?
  - How can we efficiently identify such a partition?

Graph Partitioning

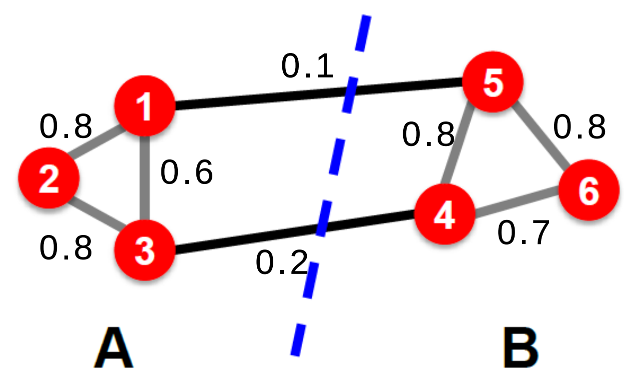
What makes a good partition?

- Maximize the number of within-group connections
- Minimize the number of between-group connections



What makes a good clustering?

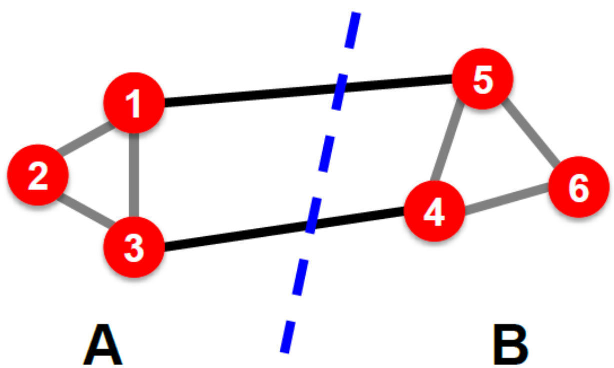
- Points assigned to the same cluster should be highly similar
- Points assigned to different clusters should be highly dissimilar



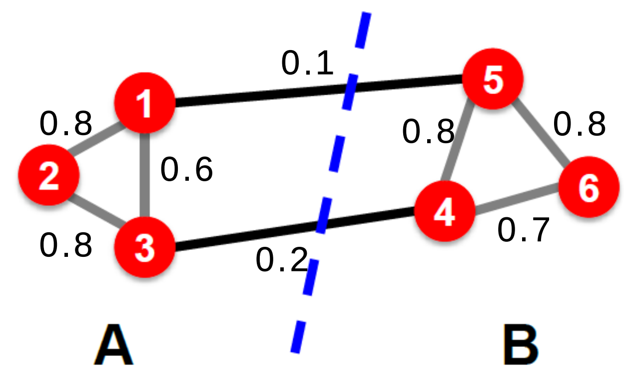
Graph Cuts

- Express partitioning objectives as a function of the "edge cut" of the partition.
- *Cut:* Set of edges with only one vertex in a group.

$$cut(A,B) = \sum_{i \in A, j \in B} w_{ij}$$



cut(A,B) = 2



cut(A,B) = 0.3

Graph Cuts

- The partitioning can be described by a labeling function y that indicates the two joint sets

$$y(u) = \begin{cases} 1 & \text{if } u \in A \\ -1 & \text{if } u \in B \end{cases}$$

$$\begin{aligned} cut(A,B) &= \frac{1}{4} \sum_{i \in A, j \in B} w(i,j)(y(i)-y(j))^2 \\ &= \frac{1}{8} \sum_{i,j} w(i,j)(y(i)-y(j))^2 \\ &= \frac{1}{4} \sum_{i,j} w(i,j)(1-y(i)y(j)) \\ &= \frac{1}{4}(\mathbf{y}^\top D\mathbf{y} - \mathbf{y}^\top W\mathbf{y}) = \frac{1}{4}\mathbf{y}^\top (D-W)\mathbf{y} \end{aligned}$$

$$D = \begin{bmatrix} d_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & d_n \end{bmatrix}$$

D is a diagonal matrix with the degrees of all the vertices on the diagonal

Laplacian matrix *L*

Spectral properties of the Laplacian matrix L

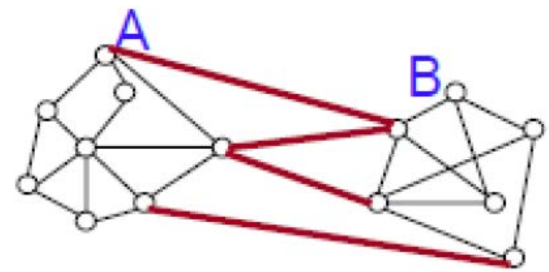
- L is always positive semidefinite（半正定）
- Smallest eigenvalue of L is 0, corresponding eigenvector is $\mathbf{e}$

$$L\mathbf{e} = D\mathbf{e} - W\mathbf{e} = \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_n \end{bmatrix} - \begin{bmatrix} \sum_j w_{1j} \\ \sum_j w_{2j} \\ \dots \\ \sum_j w_{nj} \end{bmatrix} = 0$$

- Thus eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$

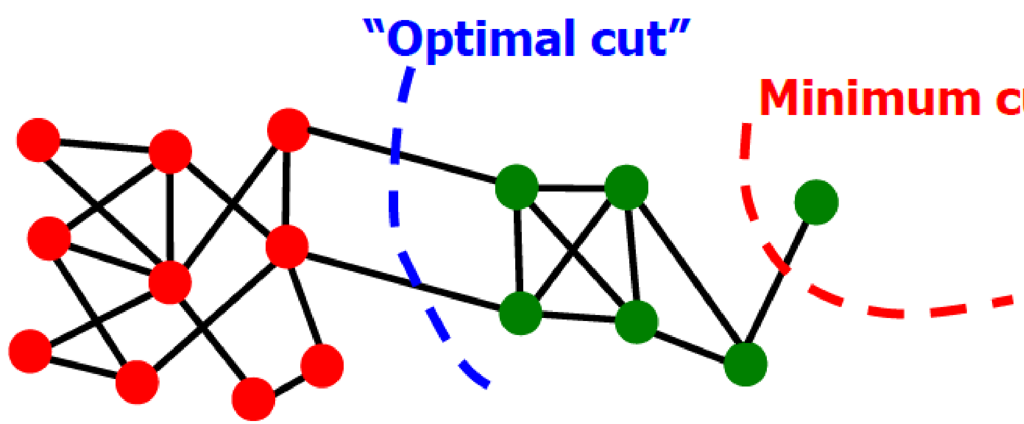
Graph cut criteria

- **Min-cut:** minimize weight of connections between groups



$$\min cut(A,B)$$

  - Easy to solve: O(|V| |E|) algorithm
  - Not satisfactory partition – often isolates（孤立） vertices
    - Only considers external cluster connections
    - Does not consider internal cluster density



Graph cut criteria

Balanced min-cut:

$$\min cut(A,B) \quad \text{subject to} \quad |A| = |B|$$

Ratio cut:

$$RatioCut(A,B) = cut(A,B)\left(\frac{1}{|A|} + \frac{1}{|B|}\right)$$

Normalized cut:

$$Ncut(A,B) = cut(A,B)\left(\frac{1}{vol(A)} + \frac{1}{vol(B)}\right)$$

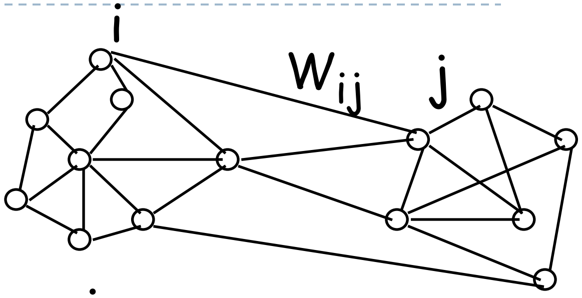
What's the difference between ratio cut and normalized cut in behavior?

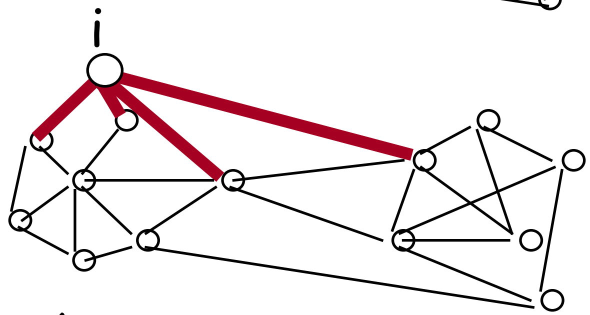
NP-hard to solve!

Spectral clustering is a relaxation of these

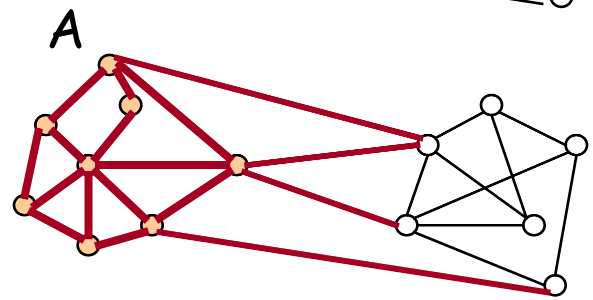
Basic Graph Notation

- $W = [\ W_{ij}\ ]$
- $d_i = \Sigma_{j \in G} W_{ij}$ degree of i

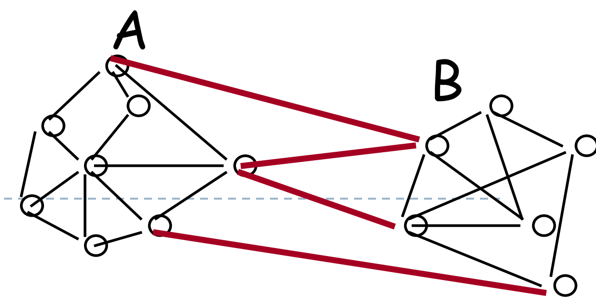




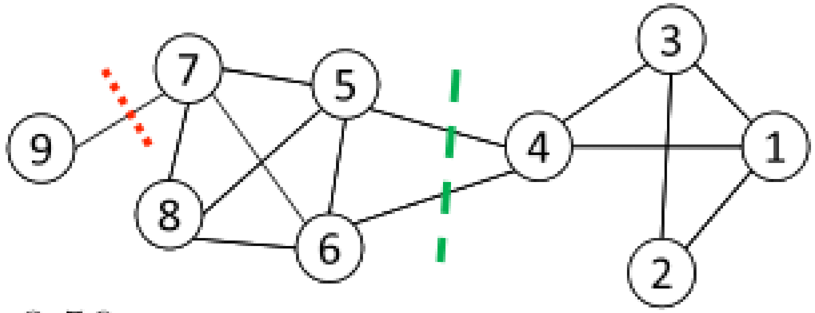
- $\text{Vol}(A) = \Sigma_{i \in A}\, d_i$ degree of $A \subseteq G$



- $\text{cut}(A,B) = \Sigma_{i \in A}\, \Sigma_{j \in B}\, W_{ij}$



Ratio Cut & Normalized Cut Example



For partition in red: π_1

$$\text{Ratio Cut}(\pi_1) = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{8}\right) = 9/16 = 0.56$$

$$\text{Normalized Cut}(\pi_1) = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{27}\right) = 14/27 = 0.52$$

For partition in green: π_2

$$\text{Ratio Cut}(\pi_2) = \frac{1}{2}\left(\frac{2}{4} + \frac{2}{5}\right) = 9/20 = 0.45 < \text{Ratio Cut}(\pi_1)$$

$$\text{Normalized Cut}(\pi_2) = \frac{1}{2}\left(\frac{2}{12} + \frac{2}{16}\right) = 7/48 = 0.15 < \text{Normalized Cut}(\pi_1)$$

Both ratio cut and normalized cut prefer a balanced partition

Derivation of Normalized cut

Let $\beta = \frac{\sum_{i \in A} d(i)}{\sum_i d(i)}$, then

$$
\begin{aligned}
Ncut(A,B) &= \frac{1}{\sum_i d(i)} \left(\frac{1}{\beta} + \frac{1}{1-\beta} \right) cut(A,B) \\
&= \frac{\sum_{i,j} w(i,j)(y(i)-y(j))^2}{8\beta(1-\beta)\sum_i y^2(i)d(i)}
\end{aligned}
$$

and define a new function g:

$$
g(u) = y(u) + (1-2\beta) \quad = \begin{cases} 2(1-\beta) & \text{if } u \in A \\ -2\beta & \text{if } u \in B \end{cases}
$$

Derivation of Normalized cut cont'd

We have $g(i) - g(j) = y(i) - y(j)$

And
$$\begin{aligned}\sum_i g^2(i)d(i) &= \sum_{i\in A} 4(1-\beta)^2 y^2(i)d(i) + \sum_{i\in B} 4\beta^2 y^2(i)d(i) \\ &= 4(1-\beta)^2\beta \sum_i y^2(i)d(i) + 4\beta^2(1-\beta)\sum_i y^2(i)d(i) \\ &= 4\beta(1-\beta)\sum_i y^2(i)d(i)\end{aligned}$$

Therefore

$$\boxed{\begin{aligned}Ncut(A,B) &= \frac{\sum_{i,j} w(i,j)(g(i)-g(j))^2}{2\sum_i g^2(i)d(i)} \\ &= \frac{\mathbf{g}^\top (D-W)\mathbf{g}}{\mathbf{g}^\top D\mathbf{g}}\end{aligned}}$$

Derivation of Normalized cut cont'd

- However, we still have the discrete constraint on g, which causes NP-hardness.
- Relaxation（松弛）： $\mathbf{g}^\top D\mathbf{e} = 0$
- The normalized cut problem can now be written as

$$\min_{\mathbf{g}} \frac{\mathbf{g}^\top (D-W)\mathbf{g}}{\mathbf{g}^\top D\mathbf{g}} \quad \text{s.t.} \quad \mathbf{g}^\top D\mathbf{e} = 0$$

- Or let $\mathbf{g}' = D^{\frac{1}{2}}\mathbf{g}$

Normalized Laplacian matrix L'

$$\min_{\mathbf{g}'} \frac{\mathbf{g}'^\top D^{-\frac{1}{2}}(D-W)D^{-\frac{1}{2}}\mathbf{g}'}{\mathbf{g}'^\top \mathbf{g}'} \quad \text{s.t.} \quad \mathbf{g}'^\top D^{\frac{1}{2}}\mathbf{e} = 0$$

Derivation of Normalized cut cont'd

- Rayleigh-Ritz theorem

$$\min_{\mathbf{x}} \frac{\mathbf{x}^\top S \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \lambda_{min}(S) \quad \text{- smallest eigenvalue of S}$$

Smallest eigenvalue of $D^{-\frac{1}{2}}(D-W)D^{-\frac{1}{2}}$ is 0 with corresponding eigenvector $D^{\frac{1}{2}}\mathbf{e}$ because

$$D^{-\frac{1}{2}}(D-W)D^{-\frac{1}{2}} \cdot D^{\frac{1}{2}}\mathbf{e} = D^{-\frac{1}{2}}(D-W)\mathbf{e} = 0$$

But g' cannot be $D^{\frac{1}{2}}\mathbf{e}$ according to the constraint $\mathbf{g}'^\top D^{\frac{1}{2}}\mathbf{e} = 0$

Therefore, solution g' is the eigenvector of $D^{-\frac{1}{2}}(D-W)D^{-\frac{1}{2}}$ corresponding to **second smallest** eigenvalue, aka second eigenvector

Spectral Graph Theory

- Possible approach
  - Represent a similarity graph as a matrix
  - Apply knowledge from Linear Algebra…

- The *eigenvalues* and *eigenvectors* of a matrix provide global information about its structure.

$$\begin{bmatrix} L_{11} & \dots & L_{1n} \\ \vdots & & \vdots \\ L_{n1} & \dots & L_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- *Spectral Graph Theory*
  - Analyse the "spectrum（光谱）" of matrix representing a graph.
  - *Spectrum* : The eigenvectors of a graph, ordered by the magnitude (strength) of their corresponding eigenvalues.

$$\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$$

$$\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$$

So far…

- How to define a "good" partition of the graph?
  - Minimize a given graph cut criterion
- How to efficiently identify such a partition?
  - Approximate using information provided by the eigenvalues and eigenvectors of a graph
- Spectral Clustering

Spectral Clustering

- Ratio Cut

$$\min_{\mathbf{h}} \frac{\mathbf{h}^\top (D - W)\mathbf{h}}{\mathbf{h}^\top \mathbf{h}} \quad \text{s.t.} \quad \mathbf{h}^\top \mathbf{e} = 0$$

- Normalized Cut

$$\min_{\mathbf{g}'} \frac{\mathbf{g}'^\top D^{-\frac{1}{2}}(D - W)D^{-\frac{1}{2}}\mathbf{g}'}{\mathbf{g}'^\top \mathbf{g}'} \quad \text{s.t.} \quad \mathbf{g}'^\top D^{\frac{1}{2}}\mathbf{e} = 0$$

Spectral Clustering Algorithms

- Construct the affinity matrix *W* from data (*A* in community detection setting)

- Compute the second vector of the matrix
  - $D - W$ Ratio Cut
  - $D^{-\frac{1}{2}}(D - W)D^{-\frac{1}{2}}$ Normalized Cut

- Taking the signs of the second eigenvector and cluster the data

Spectral Clustering Algorithms

Three basic stages:

1) Pre-processing
   - Construct a matrix representation of the graph
2) Decomposition
   - Compute eigenvalues and eigenvectors of the matrix
   - Map each point to a lower-dimensional representation based on one or more eigenvectors
3) Grouping
   - Assign points to two or more clusters, based on the new representation

Spectral Clustering Algorithms

- **1) Pre-processing:**
  - Build Laplacian matrix L of the graph

	1	2	3	4	5	6
1	3	-1	-1	0	-1	0
2	-1	2	-1	0	0	0
3	-1	-1	3	-1	0	0
4	0	0	-1	3	-1	-1
5	-1	0	0	-1	3	-1
6	0	0	0	-1	-1	2

- **2) Decomposition:**
  - Find eigenvalues λ and eigenvectors x of the matrix L
  - Map vertices to corresponding components of λ_2

λ=

0.0
1.0
3.0
3.0
4.0
5.0

X =

0.4	0.3	-0.5	-0.2	-0.4	-0.5
0.4	0.6	0.4	-0.4	0.4	0.0
0.4	0.3	0.1	0.6	-0.4	0.5
0.4	-0.3	0.1	0.6	0.4	-0.5
0.4	-0.3	-0.5	-0.2	0.4	0.5
0.4	-0.6	0.4	-0.4	-0.4	0.0

1	0.3
2	0.6
3	0.3
4	-0.3
5	-0.3
6	-0.6

How do we now find the clusters?

Spectral Clustering Algorithms

- **3) Grouping:**
  - Sort components of reduced 1-dimensional vector
  - Identify clusters by splitting the sorted vector in two
- **How to choose a splitting point?**
  - Naïve approaches:
    - Split at **0** or median value
  - More expensive approaches:
    - Attempt to minimize normalized cut in 1-dimension (sweep over ordering of nodes induced by the eigenvector)

1	0.3
2	0.6
3	0.3
4	-0.3
5	-0.3
6	-0.6

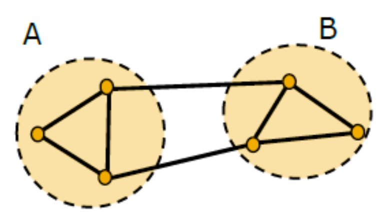
Split at 0:

Cluster A: Positive points

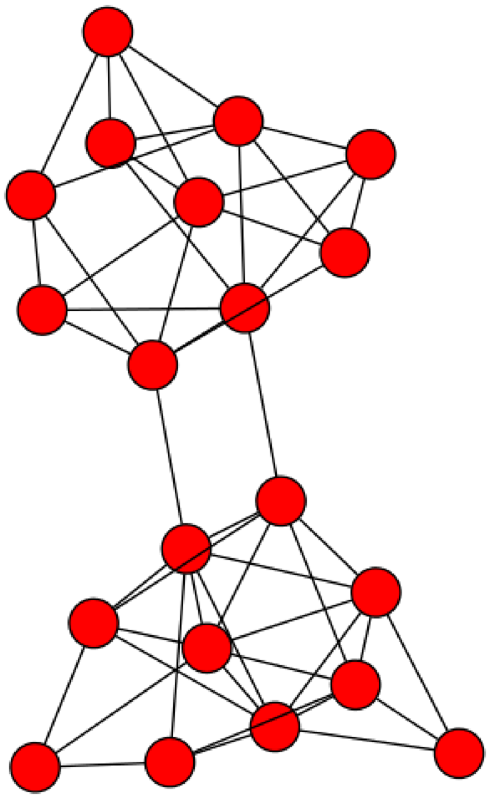
Cluster B: Negative points

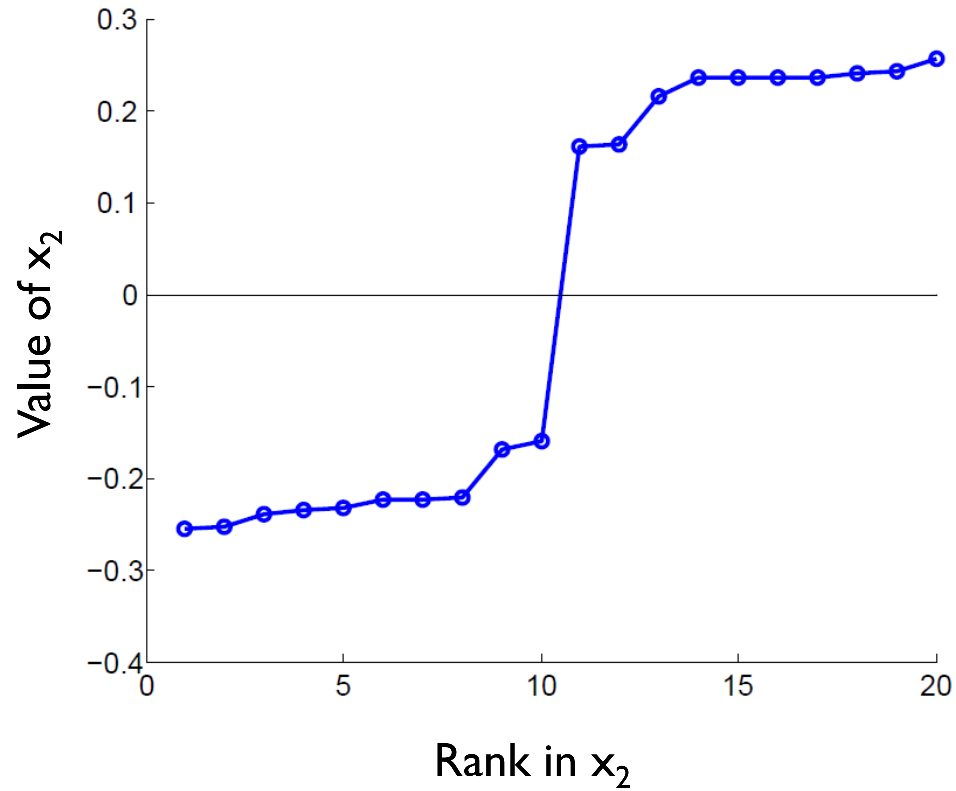
1	0.3
2	0.6
3	0.3

4	-0.3
5	-0.3
6	-0.6

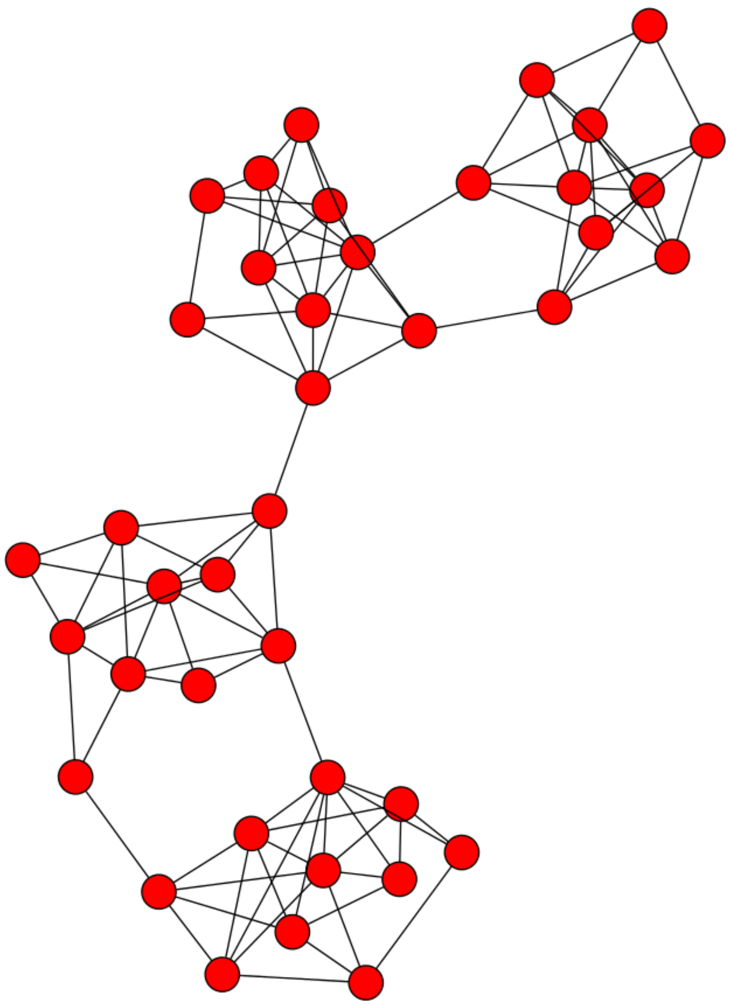


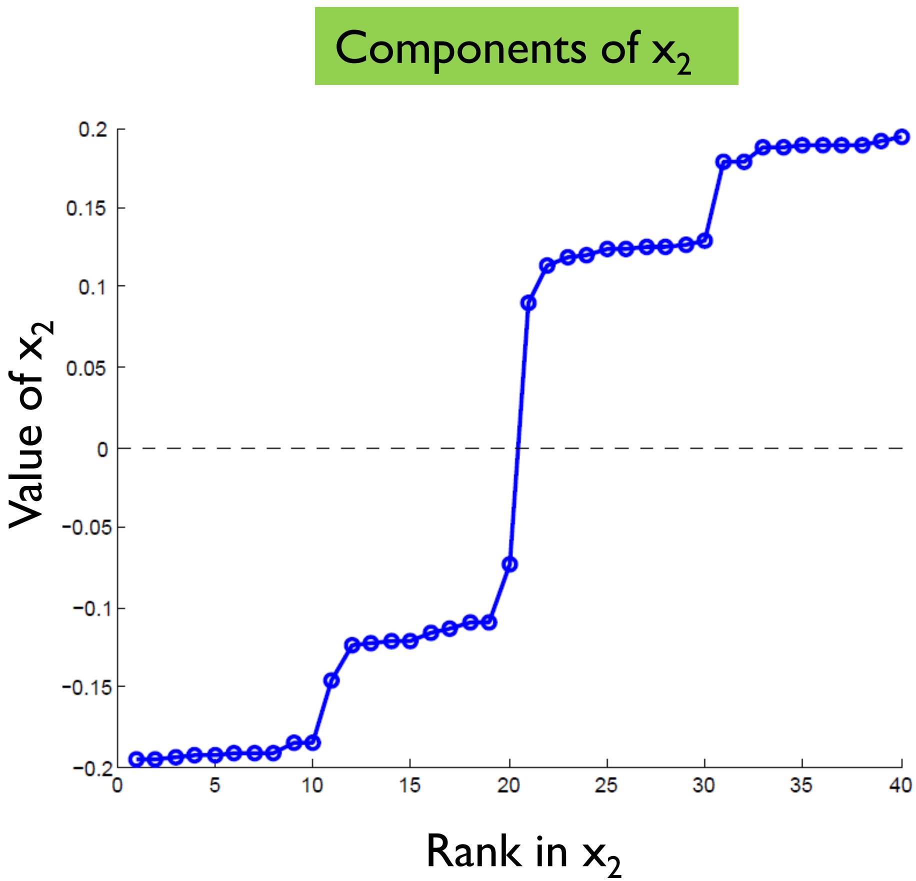
Example





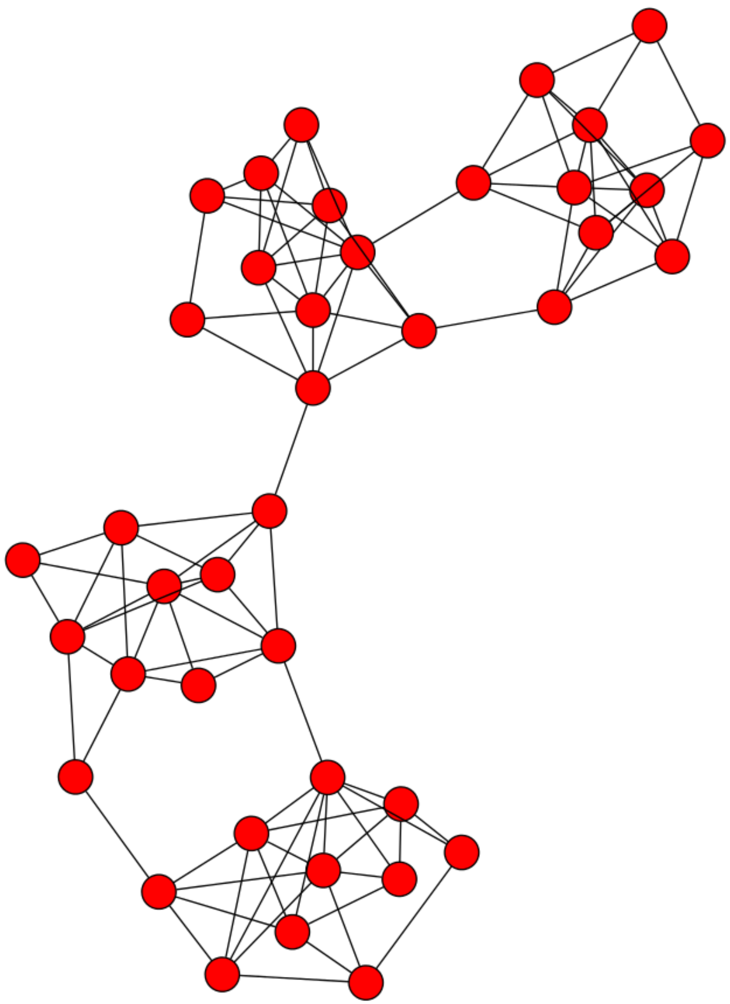
Example



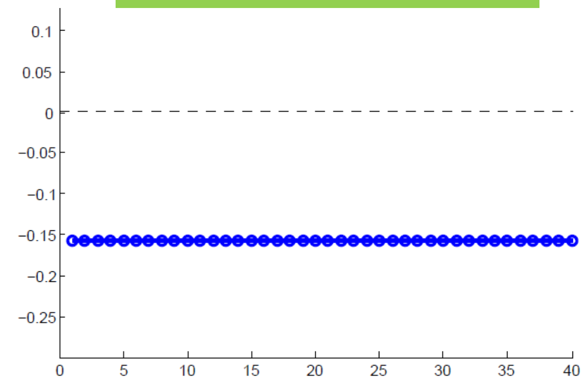


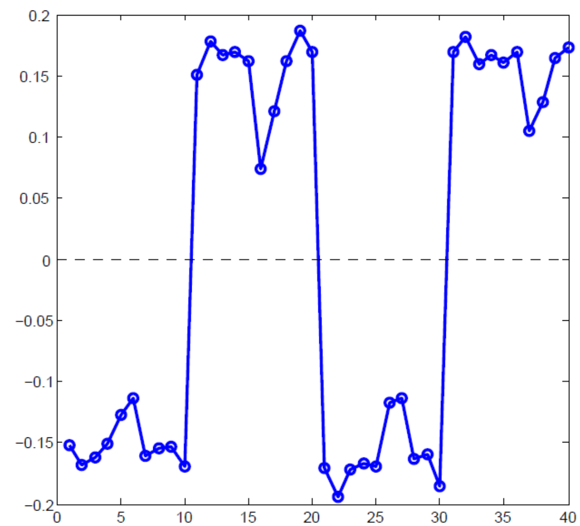
Components of x_2
Value of x_2
Rank in x_2
0.2
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
0
5
10
15
20
25
30
35
40

Example



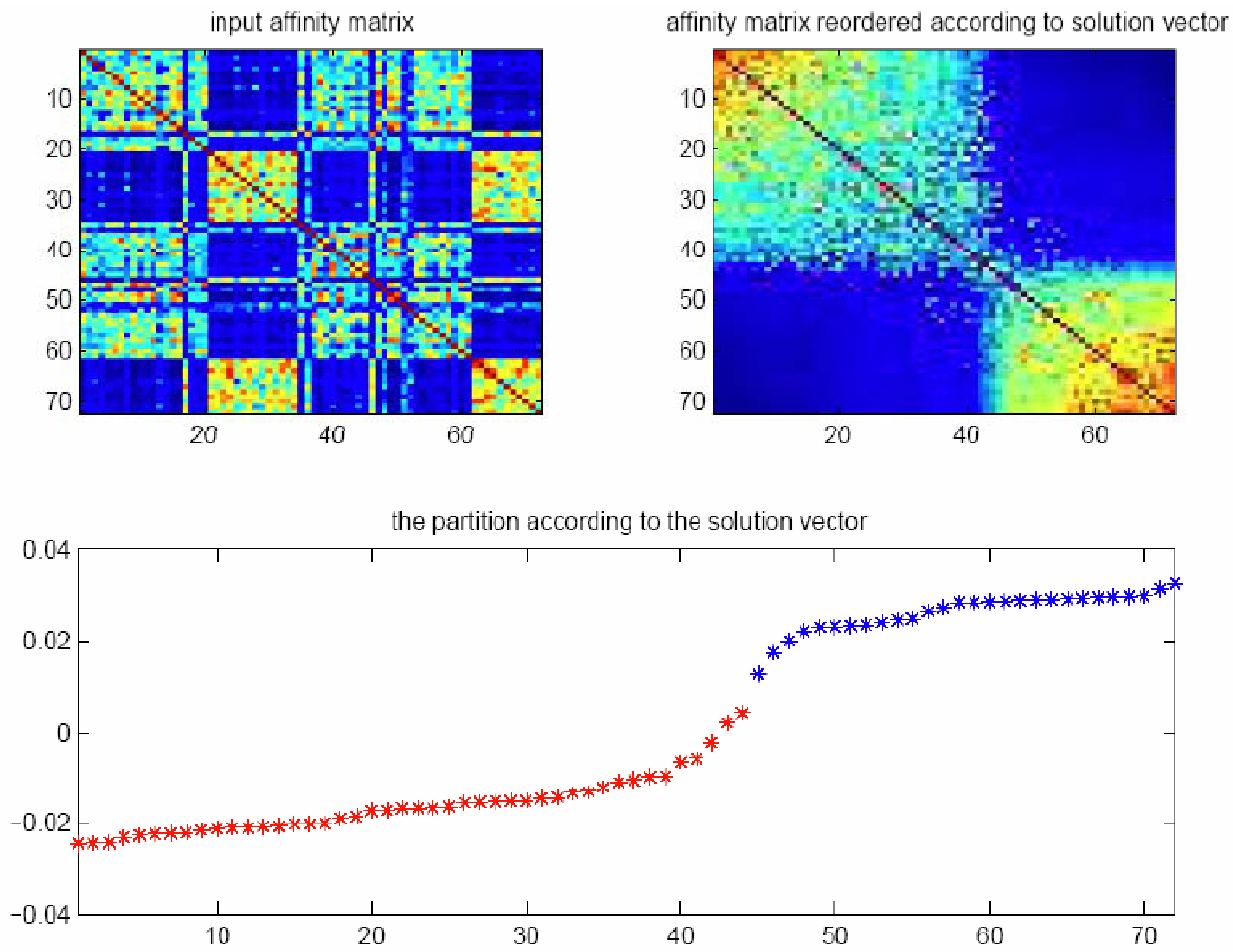
Components of x_1





Components of x_3

Example



input affinity matrix
10
20
30
40
50
60
70
20
40
60
affinity matrix reordered according to solution vector
10
20
30
40
50
60
70
20
40
60
the partition according to the solution vector
0.04
0.02
0
-0.02
-0.04
10
20
30
40
50
60
70

K-way Spectral Clustering

- How do we partition a graph into *k* clusters?

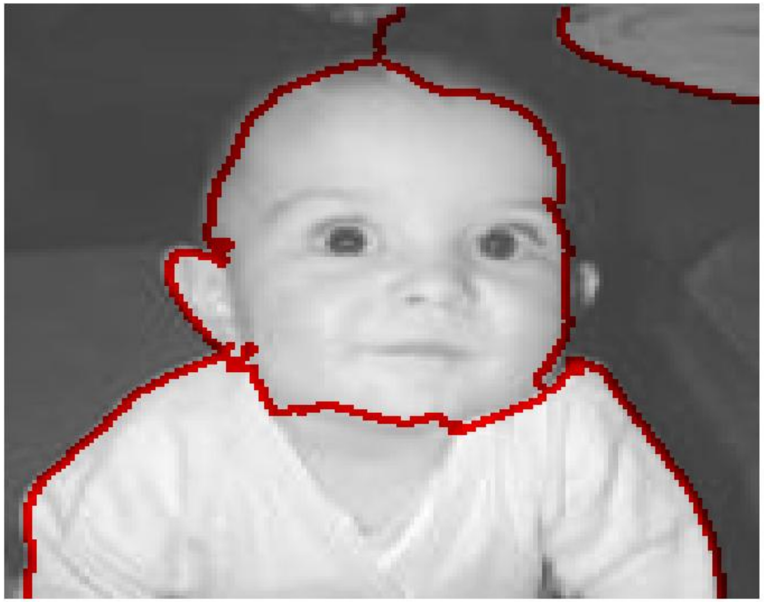
- Two basic approaches:
  - Recursive bi-partitioning
    - Recursively apply bi-partitioning algorithm in a hierarchical divisive manner
    - Disadvantages: Inefficient, unstable
  - Cluster multiple eigenvectors
    - Build a reduced space from multiple eigenvectors
    - Commonly used in recent papers
    - A preferable approach…

K-way Spectral Clustering Algorithm

Three basic stages:

1) Pre-processing
   - Construct a matrix representation of the graph (*nxn* matrix)
2) Decomposition
   - Compute eigenvalues and eigenvectors of the matrix
   - Map each point to a lower-dimensional representation based on smallest *k* eigenvectors (*V*: *nxk* matrix)
3) Grouping
   - Assign points to *k* clusters, by running *k*-means on the new representation (kmeans(V, k))

Normalized Cut in Image Segmentation



Comparison with K-means

Normalized Cuts

K-means Segmentation

Spectral Clustering Summary

- Algorithms that cluster points using eigenvectors of matrices derived from the data
- Useful in hard non-convex clustering problems
- Obtain data representation in the low-dimensional space that can be easily clustered
- Empirically very successful

Community Detection Methods

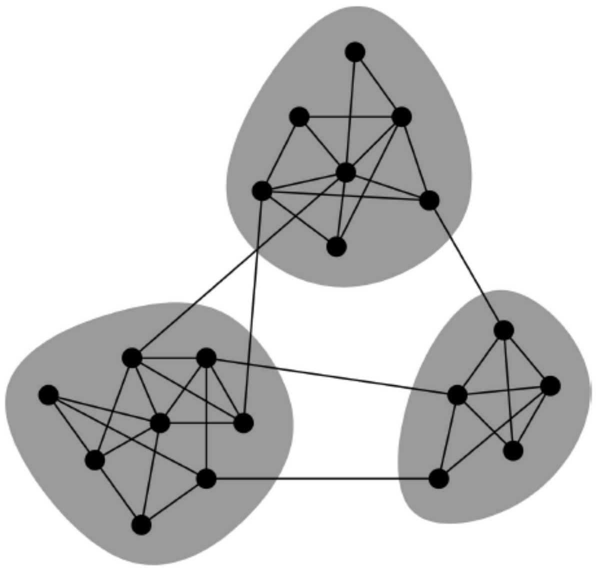
Connection between community detection and clustering

- Agglomerative hierarchical clustering
- Partitional clustering
  - K-means

- Divisive hierarchical algorithm – Girvan and Newman
- Spectral graph cut
- Modularity maximization

Modularity

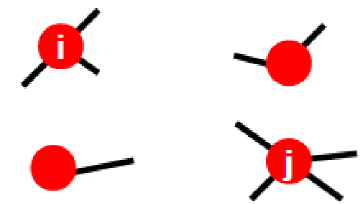
- Communities: set of tightly connected nodes
- Define: Modularity Q
  - A measure of how well a network is partitioned into communities
  - Given a partitioning of the network into groups



$$Q \propto \sum_{s \in S} [(\#\text{ edges within group } s) - (\text{expected }\#\text{ edges within group } s)]$$

Null Model: Configuration Model

- Given real *G* on *n* nodes and *m* edges, construct rewired network *G'*



  - Same degree distribution but random connections
  - Consider *G'* as a multigraph
  - The expected number of edges between nodes *i* and *j* of degrees d_i and d_j equals to: $d_i \cdot \frac{d_j}{2m} = \frac{d_i d_j}{2m}$
    - The expected number of edges in (multigraph) *G'*

$$= \frac{1}{2}\sum_{i\in V}\sum_{j\in V}\frac{d_i d_j}{2m} = \frac{1}{2}\cdot\frac{1}{2m}\sum_{i\in V} d_i\left(\sum_{j\in V} d_j\right)$$

$$= \frac{1}{4m} 2m \cdot 2m = m$$

Modularity

- Modularity of partitioning *S* of graph *G*:

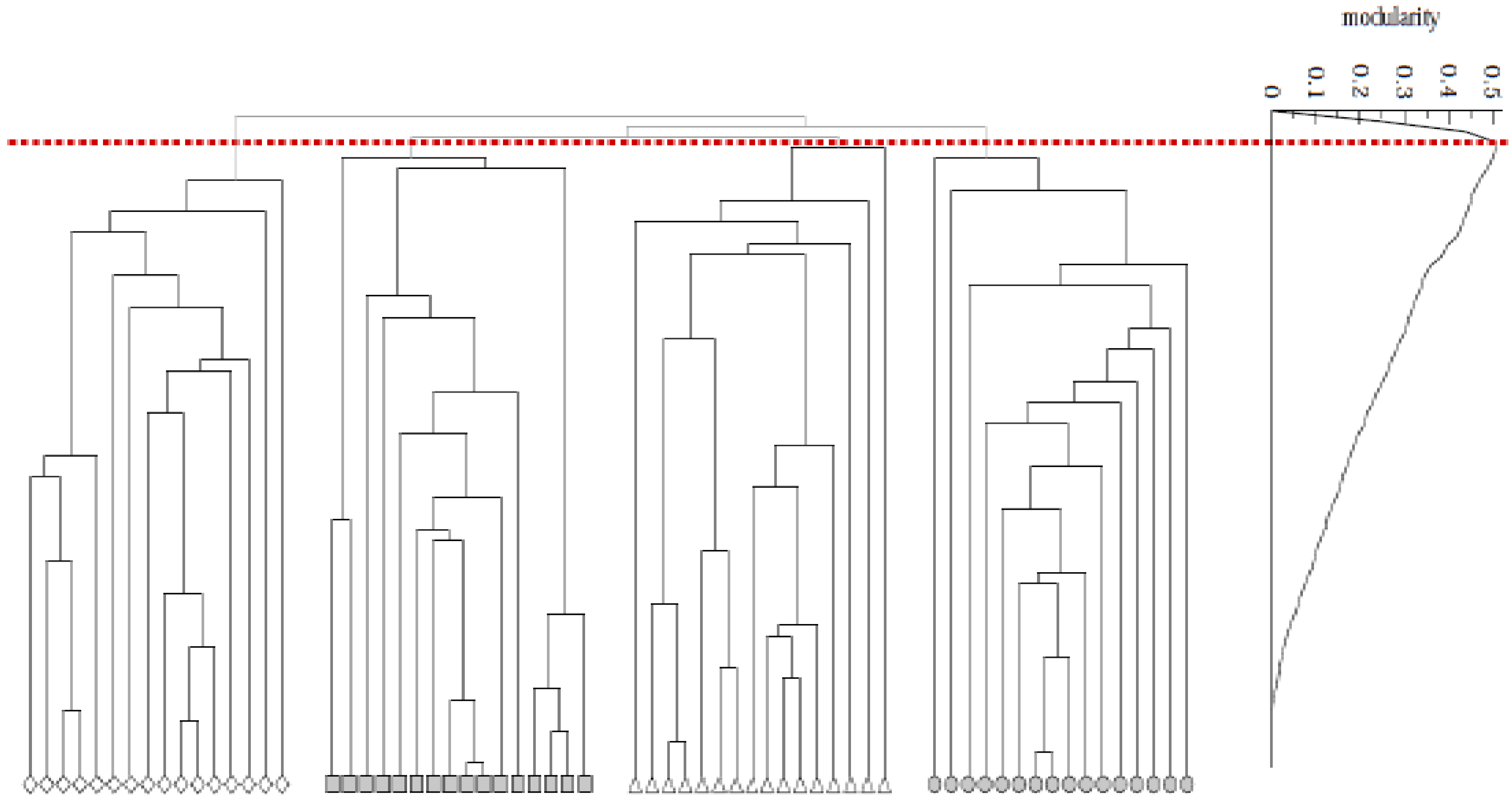
$$Q \propto \sum_{s \in S} [(\#\text{ edges within group } s) - (\text{expected }\#\text{ edges within group } s)]$$

$$Q(G, S) = \frac{1}{2m} \sum_{s \in S} \sum_{i \in s} \sum_{j \in s} (A_{ij} - \frac{d_i d_j}{2m})$$

- Modularity values take range [-1, 1]
  - It is positive if the number of edges within groups exceeds the expected number
  - 0.3<Q<0.7 means significant community structure

Modularity: Number of Clusters

- Modularity is useful for selecting the number of clusters:



Modularity Maximization

- Modularity Q tells us whether S represents any significant community structure

- Let's find S that maximizes modularity itself

Modularity Maximization

- Simple case: split the graph into 2 communities
- The partitioning can be described by a labeling function y that indicates the two joint sets

$$y(u) = \begin{cases} 1 & \text{if } u \in A \\ -1 & \text{if } u \in B \end{cases}$$

$$\begin{aligned} Q(G,S) &= \frac{1}{2m} \sum_{i,j \in V} \left(A_{i,j} - \frac{d_i d_j}{2m} \right) \frac{(y_i y_j + 1)}{2} \\ &= \frac{1}{4m} \sum_{i,j \in V} \left(A_{i,j} - \frac{d_i d_j}{2m} \right) y_i y_j \\ &= \frac{1}{4m} \mathbf{y}^\top (A - \frac{\mathbf{d}\mathbf{d}^\top}{2m}) \mathbf{y} \end{aligned}$$

Modularity Matrix

- Modularity matrix:

$$B = A - \mathbf{d}\mathbf{d}^\top/2m \quad (B_{ij} = A_{ij} - d_i d_j/2m)$$

- Similar to spectral clustering, modularity maximization can be reformulated as

$$\max_{\mathbf{y}} \frac{\mathbf{y}^\top B \mathbf{y}}{\mathbf{y}^\top \mathbf{y}}$$

- Optimal solution: top (biggest) eigenvector(s) of the modularity matrix

- Apply k-means as a post-processing step to obtain k-way community partition

References

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