

Rate of convergence of iterative methods (1)

Def:- $\{p_n\}_{n \geq 0} \rightarrow p$ w/ $p_n \neq p \forall n$.

If \exists +ve constants λ and α s.t.

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda ;$$

then $\{p_n\}_{n \geq 0} \rightarrow p$ with order of convergence α

$\alpha = 1 \Rightarrow$ lin. convergence. and asymptotic error const λ
 $\alpha = 2 \Rightarrow$ quadratic convergence.
etc ...

* The fixed pt. iteration scheme

$p_n = g(p_{n-1})$ converges linearly, when
 $g'(p) \neq 0$

ref. pg. 77 & notes
 where g have
 shown this
 numerically.

* for fixed pt. iteration (as above),

we have $g'(p) = 0$ but $g''(p) \neq 0$ & bdd.
 from above; then convergence is Quadratic

(again see pg. 77 &
 numerical example from
 prev. lec. set).

(3)

Multiple roots.

Defⁿ:- A sol. p of $f(x) = 0$ is a zero of
multiplicity m of f if for $x \neq p$ we can

write

$$f(x) = (x - p)^m q(x) ; \quad \lim_{x \rightarrow p} q(x) \neq 0$$

* If f is a polynomial f^n ; then
 Above defⁿ translates to

$$f(x) = (x - p)^m q(x) ; \quad q(p) \neq 0$$

where $q(x)$ is
 a poly. $\cdot f^n$.

(4)

Q) How to easily identify f 's w/ simple roots ($m=1$) & multiple roots ($m > 1$)?

Ans) (i) $f \in C^1[a, b]$ has a simple zero at p in $(a, b) \Leftrightarrow f(p) = 0$ but $f'(p) \neq 0$

(ii) $f \in C^m[a, b]$ has a zero of multiplicity m at p in $(a, b) \Leftrightarrow$

$0 = f(p) = f'(p) = f''(p) = \dots = f^{(m-1)}(p);$
but $f^{(m)}(p) \neq 0$.

*if p is a root
then Newton's
method converges
Quadratically*

Q) What about order of convergence
 of Newton's method when applied
 to f^s . w) multiplicity $m > 1$? (5)

Ans) $\mu(x) := \frac{f(x)}{f'(x)}$; $f(x) = (x - p)^m q(x)$

$$= \frac{(x - p)^m q(x)}{m(x - p)^{m-1} q'(x) + (x - p)^m q''(x)}$$

$$= (x - p) \frac{q(x)}{mq(x) + (x - p)q'(x)}$$

$\Psi(x)$

$\mu(x)$ also has a zero at p ; but $\Psi(p) = \frac{1}{m} \neq 0$
 $\Rightarrow p$ is a simple root of $\mu(x)$.

Now apply Newton's method to $\mu(x)$ b/c μ has a simple zero for which Newton's method is quadratically convergent! (6)

$$g(x) = x - \frac{\mu(x)}{\mu'(x)}$$
$$= x - \frac{f(x) f'(x)}{(f'(x))^2 - f(x) f''(x)}$$

↑
Calculating this
will incur numerical
cost!

Q) What is the order of conv. of the Secant method?

Hint :- Assume for secant method, following is true $|p_{n+1} - p| \approx C |p_n - p| |p_{n-1} - p|$

for sufficiently
large n

Ans :- $e_n := p_n - p$

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda > 0 \Rightarrow \text{for suff. large } n$$

$$|e_{n+1}| \approx \lambda |e_n|^\alpha$$

$$\therefore |e_n| \approx \lambda |e_{n-1}|^\alpha \text{ and } |e_{n-1}| = \frac{1}{\lambda^\alpha} |e_n|^{1/\alpha}$$

$$\therefore \lambda |e_n|^\alpha \approx |e_{n+1}| \approx C |e_n|^{1/\alpha} |e_n|$$

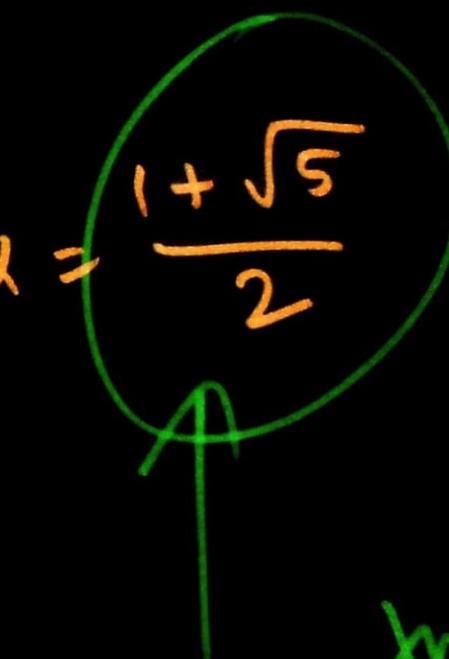
8)

$$\text{so } |\text{en}|^\alpha \approx C \lambda^{-\frac{1}{\alpha}-1} |\text{en}|^{1+\frac{1}{\alpha}}$$

this means

Since powers of $|\text{en}|$ on both sides
must match

$$\alpha = 1 + \frac{1}{\alpha} \Rightarrow \alpha = \frac{1 + \sqrt{5}}{2}$$



$$\frac{1 + \sqrt{5}}{2}$$

this is the
golden ratio!