Part (I) :- Application of Derivatives

Motivating question. (Horse Race thm)

If 2 horses start a race at the same time & the race ends in a tie; then can you say anything uniquely special about their velocities during the race?

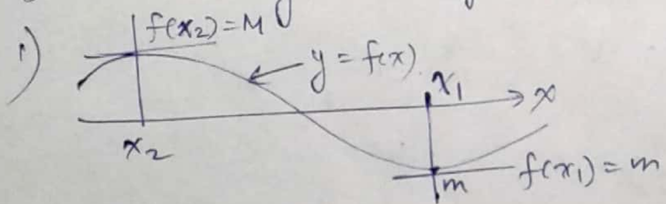
thms :- ① Min-Max thm for continuous f^n's

If $f \in C([a,b])$; then $\exists$ x_1 and x_2 in $[a,b]$ with $f(x_1) = m$ (abs min) and $f(x_2) = M$ (abs max) and $m \leq f(x) \leq M$ for every other x in $[a,b]$.

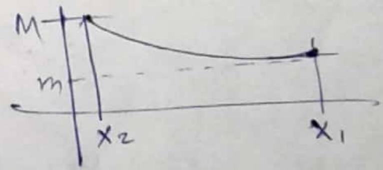
→ same as global max

Intuitively, imagine the following examples

i)



ii)

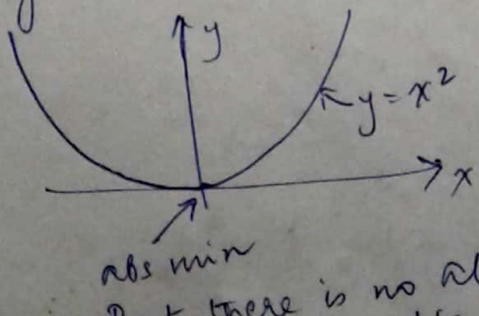


etc...

\# There are 2 key elements in thm ①:-

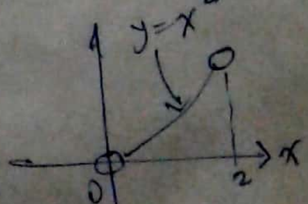
i) The domain under consideration must be closed

eg (a) $y = x^2$ in $(-\infty, \infty)$ has no absolute max



abs min

· But there is no abs max b/c for every $x_2 \neq \infty$, $x_2 > 0$, I can find a $x_3 \neq x_2$ or $x_3 > x_2$ (here) and $x_3 \neq \infty$ s.t. $y(x_3) > y(x_2)$



(b) Similarly, the following curve has no abs max / min

(ii) $f \in C([a,b])$

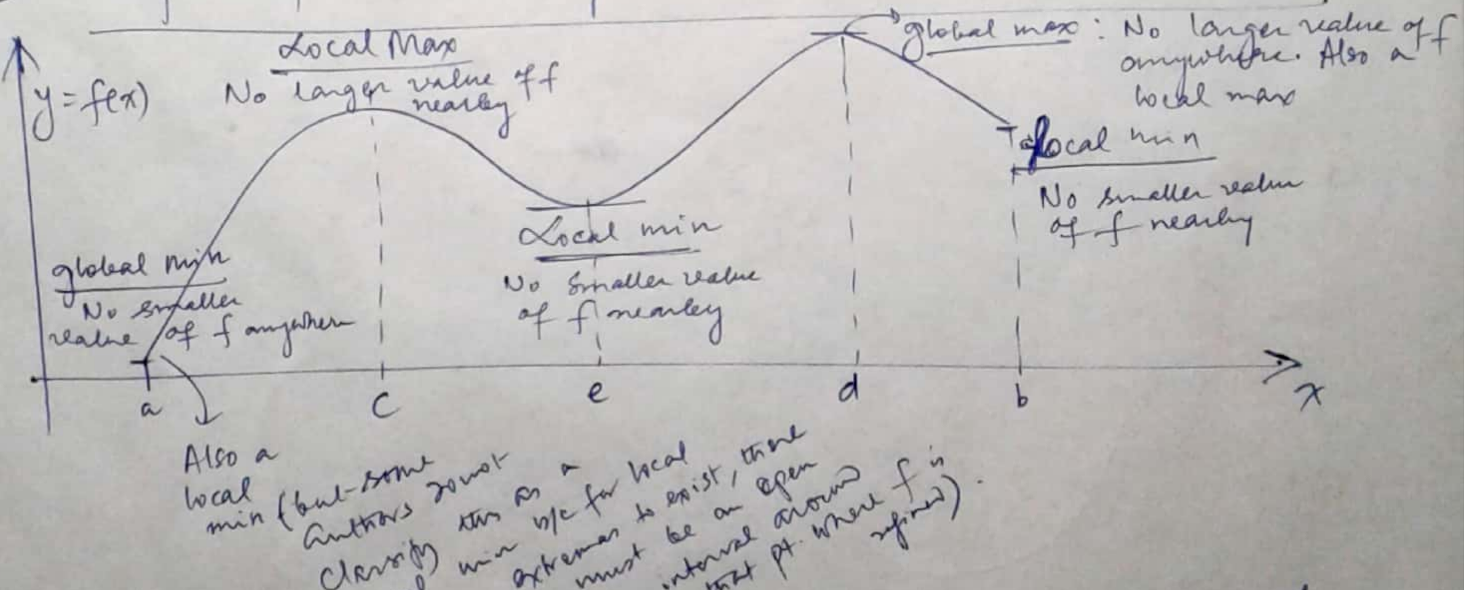
If there is even a single point of discontinuity then the thm does not hold.

eg.

$$y = \begin{cases} x+1 ; & -1 \le x < 0 \\ 0 ; & x = 0 \\ x-1 ; & 0 < x \le 1 \end{cases}$$

y does not have an abs max/min (& the reasoning is the same as in the previous pg.).

② Concept of absolute (global) & local minimum/max^m



* max^n/min pts. are also known as pts. of extrema.

③ Th^m (First Derivative test for local Extreme values)

Let f' be defined at a pt. c in I ($\Rightarrow f$ is cont. at c), then, if f has a local max/min at c, we have
$f'(c) = 0$

(4) Def^n (Critical pt.)

If c is interior pt. in I; then
c is a critical pt. if $f'(c) = 0$ or undefined.

* For finding extreme values of $f \in C([a,b])$

i) Check $f' = 0$
ii) Also check end pts (a and b) separately :-
i.e. evaluate $f(a)$ & $f(b)$ & investigate if these are extreme values.

eg (4.1) Find the absolute max & min values of $f(x) = x^2$ on $[-2,1]$

Soln:- f is differentiable in $[-2,1]$

i) $f'(x) = 2x = 0 \Rightarrow x = 0$ (critical pt.).
ii) Now check for end pts.
$f(-2) = (-2)^2 = 4$
$f(1) = 1$

$\therefore x = -2$ is the pt. where f has global max.
$x = 0$ is global min.

eg (4.2) Find global extrema of $h(x) = x^{2/3}$ on $[-3,3]$

$h'(x) = \frac{2}{3}\frac{1}{x^{1/3}} \neq 0$ anywhere in $[-3,3]$
$h'(0)$ is undefined (or ∞).

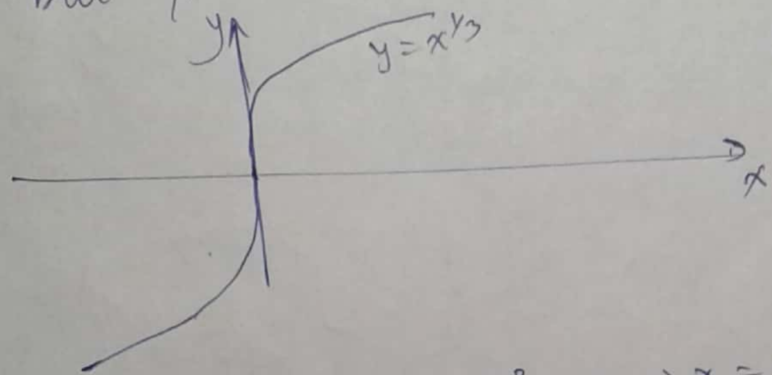
$h(0) = 0$
$h(-2) = (-2)^{2/3} = 4^{1/3}$
$h(3) = 3^{2/3} = 9^{1/3}$
$x = 3$ is abs. max
$x = 0$ is abs. min.

* i) A $f^{n'}$s extrema can only occur at critical pts. or end pts.

BUT → ii) Not every critical pt. / end pt signals the presence of an extreme value.

eg. $y = x^{1/3} = f(x) \Rightarrow f'(x) = \frac{1}{3} x^{-2/3} = \infty$ at $x=0$ (undefined)

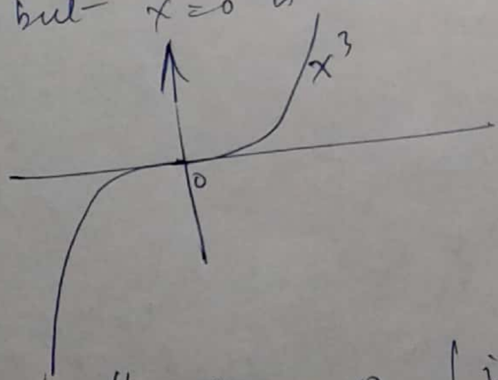
but $x=0$ is not an extrema.



b/c $f'' = -\frac{2}{9} x^{-5/3}$ is undefined

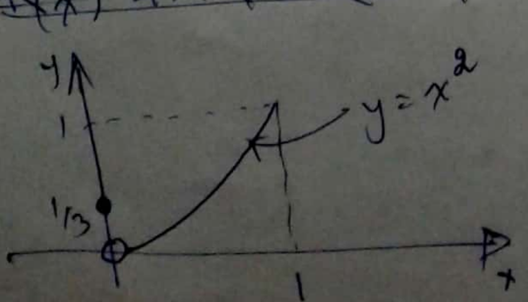
or $f(x) = x^3 \Rightarrow f'(x) = 3x^2 = 0 \Rightarrow x = 0$.

but $x=0$ is not an extrema.



b/c $f'' = 6x = 0$ (it is neither > 0 or < 0)

*



$y = \begin{cases} x^2 & ; x > 0, x \le 1 \\ 1/3 & ; x = 0 \end{cases}$

does not have local extrema at $x = 0$

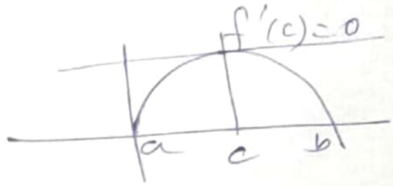
(5) Rolle's thm :-

Let $y = f(x) \in C([a,b])$ and f is differentiable in (a,b).

If $f(a) = f(b) = 0$; then

$\exists$ atleast one number c in (a,b) s.t.

$$f'(c) = 0.$$



(5.1) Application of Rolle's thm

Horse Race thm

Let the distance betw 2 horses:

$$s(t) = s_1(t) - s_2(t)$$

Let start : $t = t_0$
end : $t = t_f$.

$$s(t_0) = s_1(t_0) - s_2(t_0) = 0$$
$$s(t_f) = s_1(t_f) - s_2(t_f) = 0$$

Then by Rolle's thm,

$\exists\ t_i \in (t_0, t_f)$

s.t. $s'(t_i) = 0$

$$\Rightarrow s_1'(t_i) - s_2'(t_i) = 0$$

$$\Rightarrow s_1'(t_i) = s_2'(t_i)$$

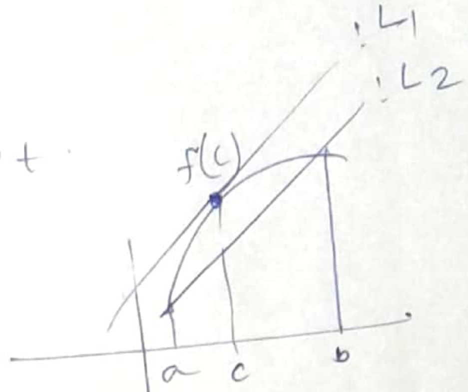
i.e. $$\left.\frac{ds_1}{dt}\right|_{t=t_i} = \left.\frac{ds_2}{dt}\right|_{t=t_i}$$

i.e. Speed of horse (1) at $t = t_i$ = Speed of horse (2) at $t = t_i$

(6) Mean Value Th^m

Let $y \in C([a,b])$;
y' exists in (a,b)

Then $\exists$ atleast one c in (a,b) s.t.



$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

instantaneous change at some interior pt — avg. change over whole interval

$f'(c)$ = slope of L_1

$\frac{f(b)-f(a)}{b-a}$ = slope of L_2

(7) Increasing & Decreasing f^n

(7.1) **Def^n** :- Let f be defined on I
$x_1, x_2 \in I$

1) f is increasing on I if
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

ii) f is decreasing on I if
$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

(7.2) First Derivative test

Let $f \in C([a,b])$ & f' exists in (a,b)

1) $f' > 0$ at each pt. on (a,b); then f is $\nearrow$.

ii) $f' < 0$ at " " " (a,b); then f is $\searrow$.

eg Consider $y = x^2$ on $(-\infty, 0)$. Here $y = f(x)$.
$f'(x) = 2x < 0$ on $(-\infty, 0) \Rightarrow f$ is $\searrow$
$f'(x) = 2x > 0$ on $(0, \infty) \Rightarrow f$ is $\nearrow$ on $(0,\infty)$.

eg

$f(x) = \sin x$
$f'(x) = \cos x$
$|f'(c)| \le 1$ b/c $|\cos x| \le 1$
$\Downarrow$ MVT
$\frac{|\sin b - \sin a|}{|b-a|} \le 1$ $a \ne b$, $c \in (a,b)$

$-1 \le \frac{\sin b - \sin a}{b-a} \le 1$

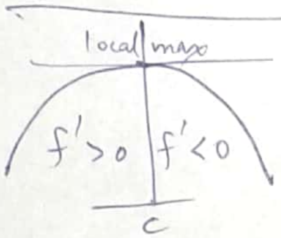
(8) First derivative test for local extrema

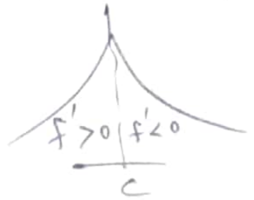
Basic idea is to inspect the change of sign of f' across a critical pt.

$f\in C([a,b])$

(A) At a critical pt, c ($f'(c)=0$).

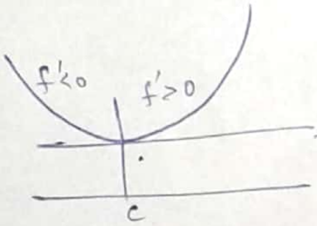
(i)

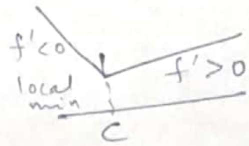




$f'(c)$ is undefined (cusp)

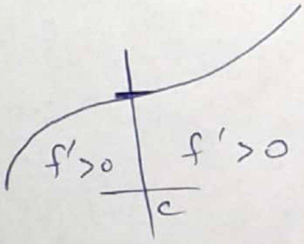
(ii)



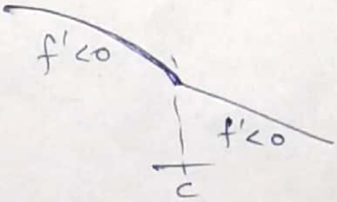


$f'(c)$ undefined (elbow edge)

(iii)

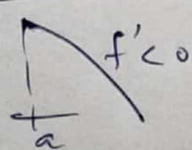


$f'(c)=0$

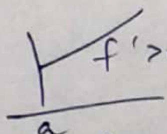


$f'(c)$ is undefined

(B) left bdy pt -

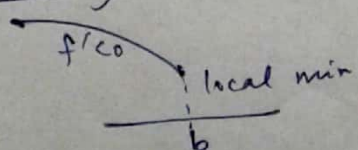


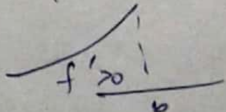
local max



local min

(C) right bdy pt.





Does it have one root?

$f(0)=1>0$

$f(-10)=-1000+\frac{1}{e^{10}}<0$

f changes sign [0, 10] & it is cont. $\Rightarrow$ $\exists$ 1 root exactly

*Q) Show $x^3+e^x=0$ has exactly one real root

Ans) $f(x)=x^3+e^x$; Let f have 2 roots st $f(r_1)=f(r_2)=0$

$f\in C[r_1,r_2]$ & f' exist in (r_1,r_2)

$\Rightarrow\exists$ atleast one $c\in(r_1,r_2)$ s.t. $f'(c)=0$ But $f'(x)=3x^2+e^x>0\ \forall x$

Contradiction $\Rightarrow$ f cannot have 2 roots.

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eg (8.1) Use the first derivative behavior to plot the following f^n.

$$f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$$

Soln:- Clearly, by inspection $f(x) \in C([R])$

$$f'(x) = \cdots = \frac{4(x-1)}{3x^{2/3}} \overset{\text{set}}{=} 0 \text{ (to find Critical pts)}.$$

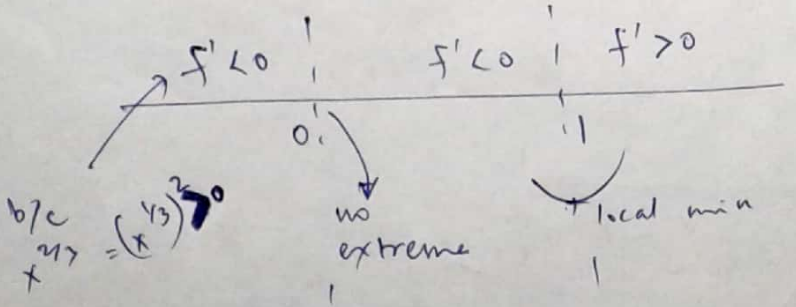
Critical pts. $\begin{cases} x = 1 \\ x = 0 \ (f' \text{ undefined}) \end{cases}$

f' undefined at $x = 0 \Rightarrow$ the f^n f likely has ∞ slope at $x = 0$.

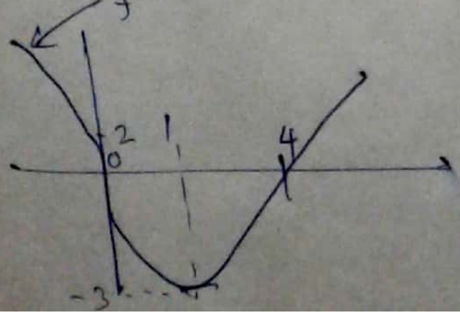
$\because$ there are no "finite" end pts., the critical pts are the likely pts. of extrema

Also note

Behavior of f'



f^n f has -ve slope $\Rightarrow f$ dec | f has -ve slope $\Rightarrow f$ dec | f has +ve slope $\Rightarrow f$ inc.



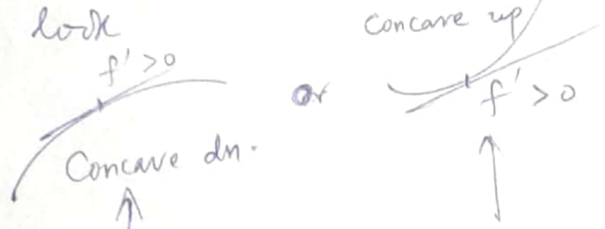
$f(1) = 1 - 4 = -3$

$f(x) = 0$ (gives x-intercept at $x = 4$)

f crosses y-axis at $(0,0) \leftarrow$ plug $x=0$ & find f.

(9) Plotting f^ns w/ f' and f''

We have seen before that if $f' > 0$ the f^n is increasing but it may look



slope (f') dec. w/ inc x

f' inc w/ inc x (going rightward).

Def^n:- Concave up :- $f(x)$ is concave up if f' progressively inc. on an interval w/ inc x.

Concave dn. :- " ... dec ... "

But recall, earlier, we saw a f^n $g(x)$ is inc. if $g' > 0$; (concave up); and $g(x)$ is dec. if $g' < 0$. (concave dn.)

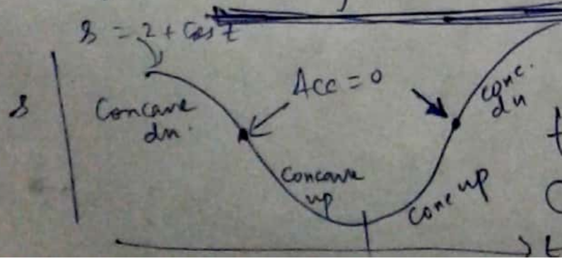
Set $g(x) = f'(x)$ (assuming f is twice differentiable)

so $g'(x) = f''(x) > 0$ (concave up)

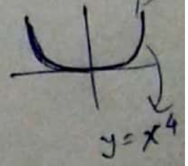
& $g' = f'' < 0$ (concave dn.)

Def^n Inflection pt. :- (Acceleration = 0 pt.)

i.e. $f'' = 0$ ← but not always



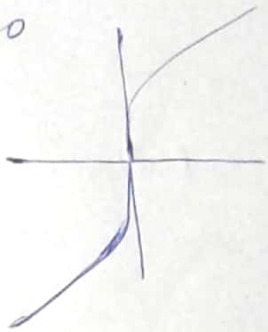
A pt. where the f^n has a tangent line & changes concavity.



eg (9.1) $y = x^{1/3}$

$y' = \frac{1}{3}x^{-2/3}$ ($= \infty$ at $x=0$

$y'' = -\frac{2}{9}x^{-5/3} = \infty$



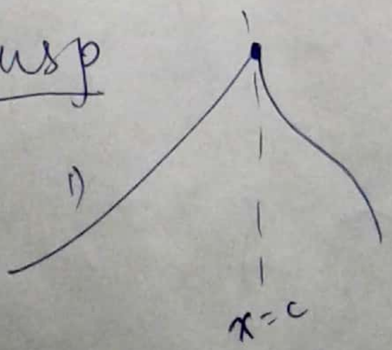
y'' DNE but
y has inflexion pt.
b/c y has a tangent line
concavity changes across 0.
(i.e. $y'' > 0$ to $y'' < 0$ across $x=0$).

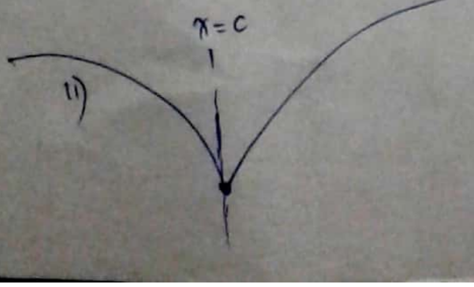
(10) 2^{nd} Derivative test for extrema

local max^m at $x=c$:- $f'(c) = 0$ & $f''(c) < 0$

local min at $x=c$:- $f'(c) = 0$ & $f''(c) > 0$.

(11) Cusp





f has a cusp at $x=c$
if concavity is same on both sides of c
& if either
i) $\lim_{x \to c^-} f'(x) = \infty$ & $\lim_{x \to c^+} f'(x) = -\infty$
OR
ii) $f'(x) \xrightarrow{x \to c^-} -\infty$ & $f'(x) \xrightarrow{x \to c^+} \infty$

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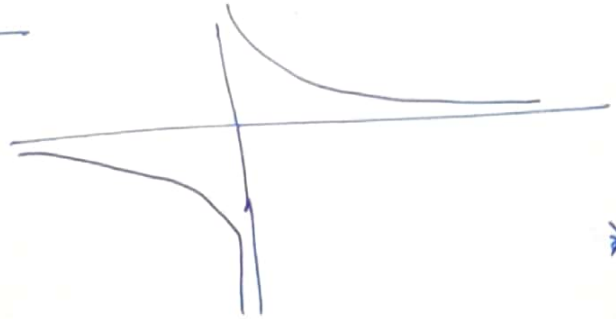
pg (1) ·

§(3·5) Asymptotes & Dominant Terms.

News:-
* On Friday I will assign some problems from ch(3).
* On Friday we will discuss setting up some convenient ot. hrs.
* Does everyone have access to Thomas & Finney 9th edition? Read Sec. covered, Chk course web & syll. regularly.
* Ask who their respective Tut. faculty are?

(1) lim $x \to \pm\infty$.

eg $f(x) = 1/x$



$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

We say the f^n has lim 0 as $x \to \pm\infty$.

Note :- $\lim_{x \to \pm\infty} k = k$ and $\lim_{x \to \pm\infty} \frac{1}{x} = 0$

(2) Th^m

Properties of limits as $x \to \pm\infty$.

If $f(x) \xrightarrow{x \to \pm\infty} L$ and $g(x) \xrightarrow{x \to \pm\infty} M$

(i) $f(x) \pm g(x) \xrightarrow{x \to \pm\infty} L \pm M$

(ii) $f(x) \cdot g(x) \xrightarrow{x \to \pm\infty} LM$

(iii) $\lambda f(x) \xrightarrow{x \to \pm\infty} \lambda L$

(iv) $f/g(x) \xrightarrow{x \to \pm\infty} \frac{L}{M}$; $M \neq 0$.

(v) $m, n \in \mathbb{Z}$ $(f(x))^{m/n} \xrightarrow{x \to \pm\infty} (L)^{m/n}$ provided $L^{m/n} \in \mathbb{R}$

(3) limits of rational f^ns.

eg (3.1)

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2}$$

Div. by highest deg. term

$$\lim_{x\to\infty}\frac{\frac{5x^2+8x-3}{x^2}}{\frac{3x^2+2}{x^2}} = \lim_{x\to\infty}\frac{5+8/x-3/x^2}{3+2/x^2}$$

$$= 5/3$$

eg (3.2)

$$\lim_{x\to-\infty}\frac{11x+2}{2x^3+1} = \lim_{x\to-\infty}\frac{11/x^2+2/x^3}{2-1/x^3}$$

$$= \frac{0}{2} = 0$$

eg (3.3)

$$\lim_{x\to-\infty}\frac{-4x^3+7x}{2x^2-3x-10} = \lim_{x\to-\infty}\frac{-4x+(7/x)}{2-\frac{3}{x}-\frac{10}{x^2}}$$

$$= \infty$$

(4) Horizontal & Vertical Asymptotes

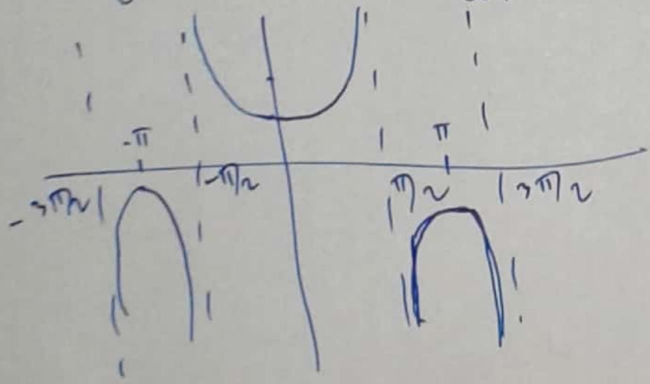
We say that a graph approaches a line asymptotically if the distance bet'n the f^n & the line reduces (to 0) progressively w/ x farther from the origin. In such a case; the line is called an asymptote of the f^n.

Asymptotes

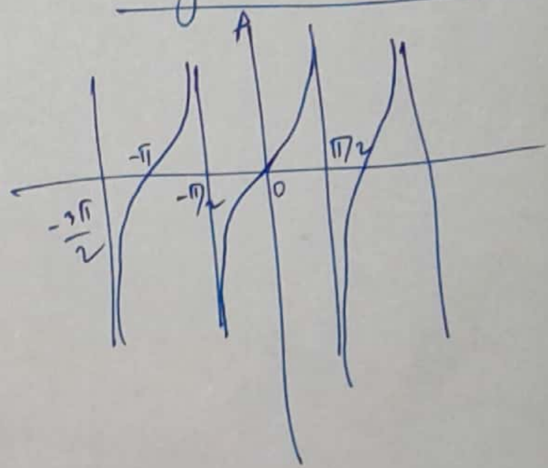
- Horizontal ~~Vertical~~ (H.A.)
  $y = b$ is a H.A. if $f(x)\to b$ as $x\to+\infty$ or $x\to-\infty$
- Vertical ~~Horizontal~~ (V.A.)
  $x = a$ is a V.A. if $f(x)\to\pm\infty$ as $x\to a^+$ or $x\to a^-$

e.g. Vertical Asymptotes

(4.1) $y = \sec x = \dfrac{1}{\cos x}$



(4.2) $y = \tan x = \dfrac{\sin x}{\cos x}$



(4.3) Find the asymptotes of the curve

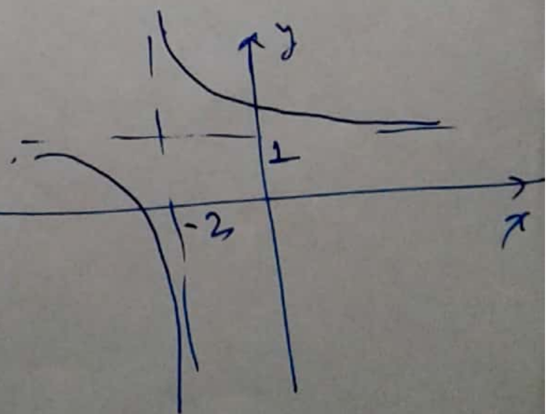
$$y = f(x) = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$$

"Asymptotes" ⇒ $x \to \pm\infty$ is to be investigated for H.A.
$x \to -2$ is to be " for V.A.

$$\lim_{x \to \pm\infty} \frac{x+3}{x+2} = \lim_{x \to \pm\infty} \frac{1 + 3/x}{1 + 2/x} = 1$$

$$\lim_{x \to -2^-} 1 + \frac{1}{x+2} = -\infty$$

$$\lim_{x \to -2^+} 1 + \frac{1}{x+2} = \infty$$



* Rational f^ns reduced to lowest terms have V.A where their denominators are zero.

eg (4.4) Removable Discontinuity.

$$f(x) = \frac{x^3-1}{x^2-1} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1}$$

$f(x)$ has V.A. at $x=-1$ (but not at $x=1$ b/c it is a removable singularity)

Let us find critical pts.

$$f'(x) = \frac{1}{x+1}(2x+1) + (x^2+x+1)\frac{(-1)}{(x+1)^2}$$

$$= \frac{2x+1}{x+1} - \frac{x^2+x+1}{(x+1)^2} = \frac{(2x+1)(x+1)-x^2-x-1}{(x+1)^2}$$

$$= \frac{2x^2+3x+1-x^2-x-1}{(x+1)^2}$$

$$= \frac{x^2+2x}{(x+1)^2}$$

$$= \frac{x(x+2)}{(x+1)(x+1)} = 0$$

$x = 0, -2$.

$$f''(x) = \frac{-1}{(x+1)^2}(2x+1) + \frac{2}{x+1} - \left\{(-2)(x+1)^{-3}(x^2+x+1) + \frac{1}{(x+1)^2}(2x+1)\right\}$$

$$= \frac{-(2x+1)}{(x+1)^2} + \frac{2}{x+1} + \frac{2(x^2+x+1)}{(x+1)^3} - \frac{2x+1}{(x+1)^2}$$

$$= \frac{-2(2x+1)}{(x+1)^2} + \frac{2}{x+1} + \frac{2(x^2+x+1)}{(x+1)^3}$$

$$f''(-2) = \frac{-2[-4+1]}{(-1)^2} + \frac{2}{(-2)+1} + \frac{2(4-2+1)}{(-1)^3}$$

$= 6 - 2 - 6 = -2 < 0 \Rightarrow$ $x=-2$ is local max.

$f''(0) = -2 + 2 + 2 > 0$ $\therefore x = 0$ is local min

$$f''(x) \overset{set}{=} 0$$

$$-\frac{2(2x+1)}{(x+1)^2} + \frac{2}{x+1} + \frac{2(x^2+x+1)}{(x+1)^3} = 0.$$

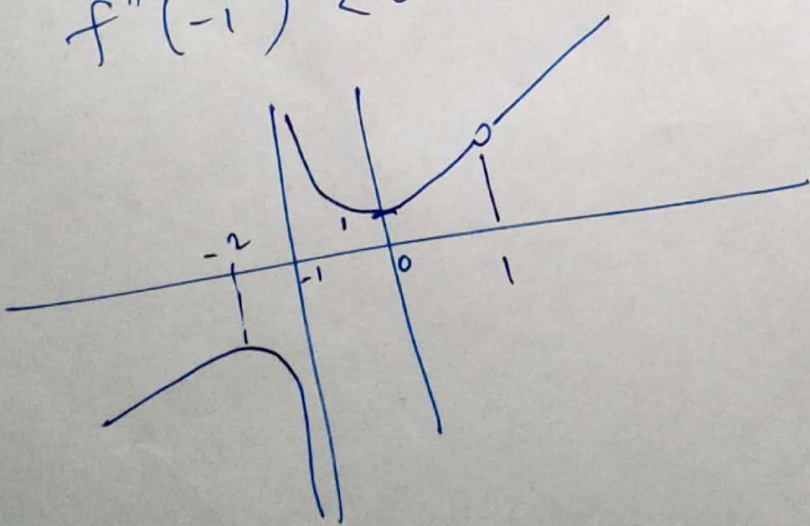
$$\Rightarrow \frac{-2(2x+1)(x+1) + 2(x^2+2x+1) + 2x^2+2x+2}{(x+1)^3} = 0.$$

$$\Rightarrow \frac{(-4x-2)(x+1) + 2x^2+4x+2 + 2x^2+2x+2}{(x+1)^3} = 0$$

$$\Rightarrow -\cancel{4x^2} - 4x - 2x - 2 + \cancel{4x^2} + 6x + 4 = 0$$

$\Rightarrow$ indeterminate.

$f''(-1^+) > 0$ Concave up.
$f''(-1^-) < 0$ Concave dn.



eg 12.1 Graph the f^n

Step I $y = x^4 - 4x^3 + 10$

$\rightarrow y' = 4x^3 - 12x^2 = 4x^2(x-3) \overset{\text{set}}{=} 0$

$\Rightarrow x = 0, 3$ are critical pts.

$\rightarrow y'' = 12x^2 - 24x = 12x(x-2) \overset{\text{set}}{=} 0$

possible inflection pts $x = 0, 2$

Step II Check inc/dec behavior around critical pts.

dec	dec	inc.
$f' < 0$	$f' < 0$	$f' > 0$.

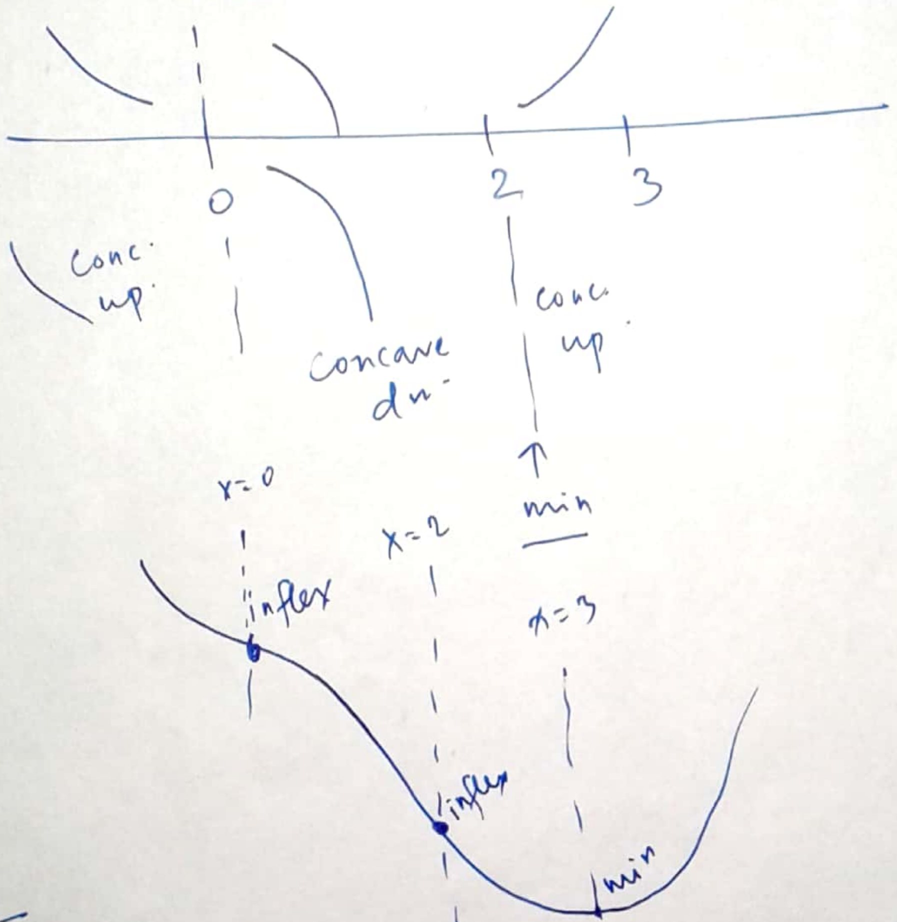
0 ↑ flat; 3 ↑ local min

Step III Concavity around "potential" inflection pts.

$f'' > 0$	$f'' < 0$	$f'' > 0$
Concave up	concave dn	Concave up.

0: inflexion? Yes! 2: inflexion? Yes!

Step 4: overlay sketch of step 2 & step 3.

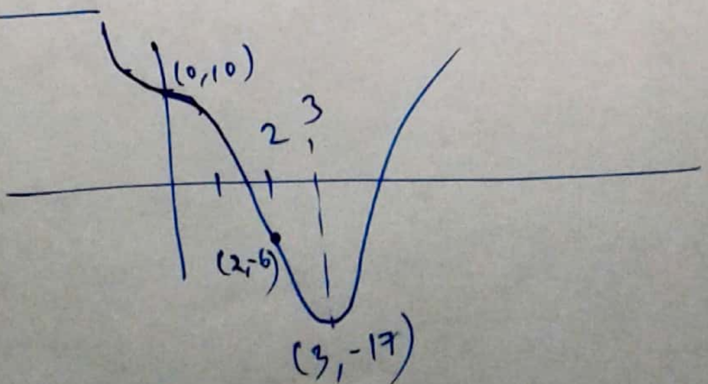


Step 5

Now try out some specific pts.

Not req'd? x-intercepts: $y = 0 \Rightarrow x^4 - 4x^3 + 10 = 0$ [see $y<0$ or $y>0$ for some pts

✓ y-intercept $x = 0 \Rightarrow y = 10$. so $(0, 10)$



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example 12.2

$28 - 6\times 9 + 81$

109
-54
55

Plot $y = 1 - 9x - 6x^2 - x^3$

Step 1

$$y' = -9 - 12x - 3x^2$$
$$= -3(3 + 4x + x^2)$$
$$= -3(x+3)(x+1) \overset{\text{set}}{=} 0 \Rightarrow x = -3, -1$$

Critical pts.

$$y'' = -12 - 6x$$
$$= -6(2 + x) \overset{\text{set}}{=} 0 \Rightarrow x = -2 \text{ (pot. inflexion)}$$

$f' < 0$ (f dec) | -3 | $f' > 0$ (f inc.) | -1 | $f' < 0$ (f dec)

$\boxed{\text{local min}}$ at -3 $\qquad$ $\boxed{\text{local max}}$ at -1

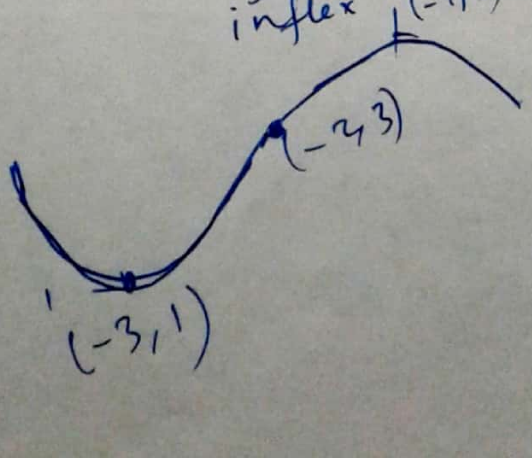
$f(-1) = 1 - 9(-1) - 6(-1)^2 - (-1)^3$
$= 10 - 6 + 1$
$= 5$

$f'' > 0$ Concave up $\qquad$ -2 inflex $\qquad$ $f'' < 0$ Concave dn.

f at local min

$f(-3) = 1 - 9(-3) - 6\times 9 - (-3)^3$
$= 1 + 27 - 54 + 27$
$= 1$

$f(-2) = 1 - 9(-2) - 6(-2)^2 - (-2)^3$
$= 1 + 18 - 24 + 8$
$= 26 - 24 + 1$
$= 3$



eg (12-3)

$$y = (x-2)^3 + 1$$

$$y' = 3(x-2)^2 \overset{set}{=} 0 \Rightarrow x = 2$$

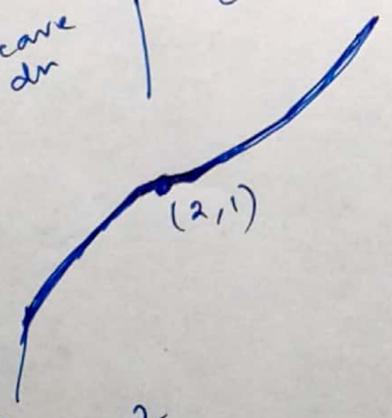
$$y'' = 6(x-2) \overset{set}{=} 0 \Rightarrow x = 2$$

Both values are same.

	2
$y' > 0$	$y' > 0$

no local extrema

$y'' < 0$	$y'' > 0$
concave dn	concave up.



eg (12.4)

$$y = x^4 - 2x^2$$

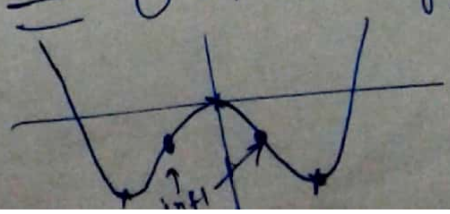
$$y' = 4x^3 - 4x \overset{set}{=} 0 \Rightarrow x = 0, -1, 1$$

$$= 4x(x^2-1)$$

$$= 4x(x-1)(x+1)$$

$$y'' = 12x^2 - 4$$

$$= 4(3x^2-1) \overset{set}{=} 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$



(13) More applications of MVT.

Q) Suppose you did not have a calculator; Is $e^{\pi} > \pi^{e}$ or $e^{\pi} < \pi^{e}$??

<u>Soln</u> :- Whenever you see power law expressions (including $e^{(\,)}$) → always think <u>logarithms</u>!

so say $f(x) = \log(x)$ in $[a,b]$; $a, b > 0$, $a, b \in \mathbb{R}$

$f \in C[a,b]$

f' exists in (a,b)

$\therefore \exists$ a $c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b-a} = \dfrac{1}{c}$

or $\log b - \log a = \dfrac{b-a}{c} = \log(b/a)$

$$\Rightarrow \frac{b-a}{b} < \log(b/a) < \frac{b-a}{a} \quad \text{——(i)}$$

Why?? b/c $a < c < b$!!

In (i)

set $a = n$
$b = m$; $m > n \geq e$

If $n \geq e$; the R.H.S. ineq in (i)

$$\Rightarrow \log\left(\frac{m}{n}\right) < \frac{m-n}{n}$$

$$\Rightarrow \log\left(\frac{m}{n}\right)^n < m-n$$

$$\Rightarrow e^{\log\left(\frac{m}{n}\right)^n} < e^{m-n}$$

$$\Rightarrow \left(\frac{m}{n}\right)^n < e^{m-n} \leq n^{m-n}$$

$$\Rightarrow \frac{m^n}{n^n} < \frac{n^m}{n^n}$$

$$\Rightarrow m^n < n^m \quad \text{—(iii)}$$

set $m = \pi$
$n = e$
$\pi > e$ ✓

We have $\boxed{\pi^e < e^\pi}$

If $e \geq m$, $n > 0$

L.H.S. ineq of (i) gives.

$$m^{(m-n)/m} \leq e^{(m-n)/m} < m/n$$

NOT Reqd

$$\Rightarrow m^{m-n} < (m/n)^m \text{ and } m^{n-m} > \left(\frac{n}{m}\right)^m$$

$$\Rightarrow m^n > n^m$$

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eg(3) HW :- graph (upon analysing) $f(x) = 4x^{1/3} - x^{4/3}$

Soln :- $f(x) = 4x^{1/3} - x^{4/3} = x^{1/3}(4-x)$

Roots of $f(x)$ are @ $x=0, 4$ (x-intercepts).

$f(0)=0 \Rightarrow$ curve passes through origin.

$f'(x) = \frac{4}{3}\frac{1}{\sqrt[3]{x^2}}(1-x) = 0 \Rightarrow x = 1$ &

$x=0$ (undefined)

$f''(x) = -\frac{4}{9}\frac{1}{\sqrt[3]{x^5}}(2+x) = 0 \Rightarrow x = -2 \leftarrow$ inflexion pt

$x=0$ (undefined)

$f''(-2)$

	$x<0$	$0<x<4$	$x>4$
$f(x)$	$-$	$+$	$-$

	$x<0$	$0<x<1$	$x>1$
$f'(x)$	$+$	$+$	$-$

	$x<-2$	$-2<x<0$	$x>0$
$f''(x)$	$-$	$+$	$-$

Also $\lim_{x\to\infty} x^{1/3}(4-x) = -\infty$

$\lim_{x\to-\infty} x^{1/3}(4-x) = -\infty$

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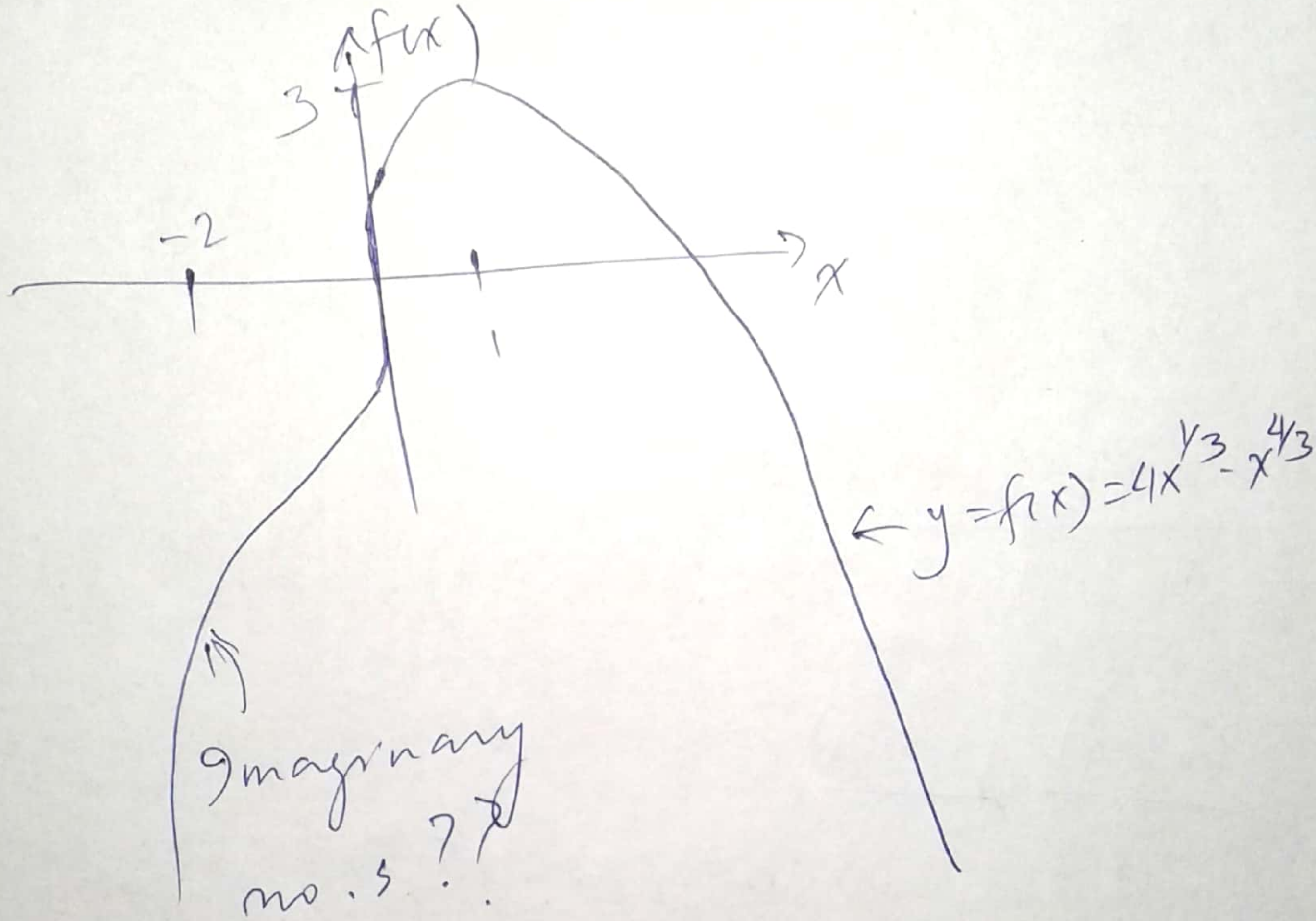

f(x)
3
-2
1
x
← y = f(x) = 4x^1/3 - x^4/3
9magrnary
no.s ??

Examples on plotting f^n's w/ asymptotes

eg (1) $g(x) = \dfrac{1}{1 + \sin x}$

Step (1) :- ∵ We have an expression in the denominator, let us identify points where the f^n $g(x)$ is not defined ($g(x)$'s domain).

$$1 + \sin x = 0 \Rightarrow \sin x = -1 \Rightarrow x = \left(\frac{3\pi}{2} + 2n\pi\right)$$

i.e. Domain of $g(x) = \{x \mid x \neq \frac{3\pi}{2} + 2n\pi\}$

Step (2) :- Intercepts.

y-intercept at $x = 0 \Rightarrow g(0) = \frac{1}{1+0} = 1$. i.e. $(0,1)$ is the y-intercept.

x-intercept :- Is $g(x) = 0$? (ever)

Note $\sin x$ takes a value bet'n -1 & 1 so $(1 + \sin x)$ can never be so large as to make $y = 0$

Step (3) :- Periodicity

We have a sine f^n which is periodic in $[0, 2\pi] \Rightarrow g(x + 2\pi) = g(x)$ & hence we only need to consider $[0, 2\pi]$ initially to draw the f^n & will replicate the behavior in the remaining domain.

Step (4) :- (Asymptotes) NO H.A. b/c $\sin x$ oscillates bet'n $+1$ & -1.

V.A. as $\sin x \to -1$ i.e. $x \to 3\pi/2$.

Step (5) :- Critical pts at $f'(x) = 0$.

$$f'(x) = -\frac{1}{(1+\sin x)^2}\cos x = 0.$$

i.e. $x = \pi/2, 3\pi/2$

But caution $3\pi/2$ is not in domain of $g(x)$ b/c of step ①.

<u>Step ⑥</u> :- $$f''(x) = +\frac{1}{(1+\sin x)^2}\sin x - \frac{\cos x\,(-2)\cos x}{(1+\sin x)^3}$$

$$= \frac{\sin x}{(1+\sin x)^2} + \frac{2\cos^2 x}{(1+\sin x)^3}$$

$$= \frac{\sin x + \sin^2 x + 2\cos^2 x}{(1+\sin x)^3}$$

$$= \frac{1+\sin x+\cos^2 x}{(1+\sin x)^3}$$

$$= \cancel{\frac{1+\sin x + \frac{1+\cos 2x}{2}}{(1+\sin x)^3}}$$

$$= \frac{1}{(1+\sin x)^2} + \frac{\cos^2 x}{(1+\sin x)^3}$$

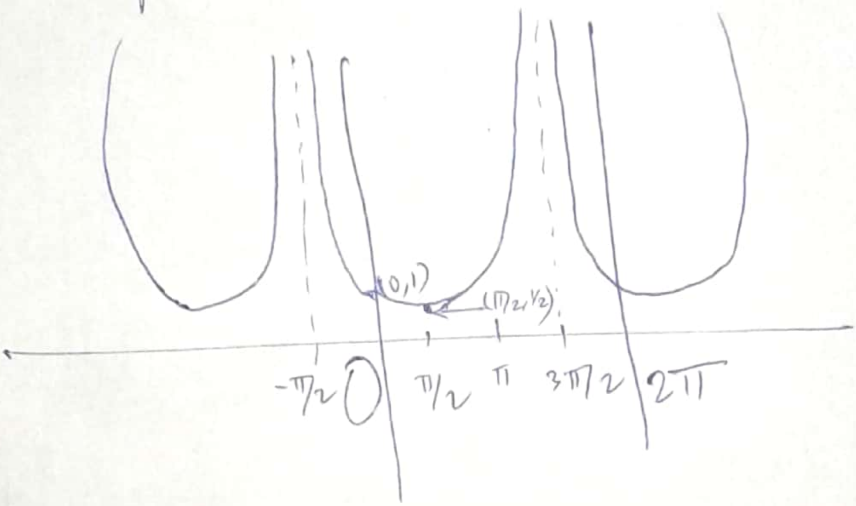
$$= \frac{1}{(1+\sin x)^2}\left\{1 + \frac{\cos^2 x}{1+\sin x}\right\}$$

$\because$ Each term is >0 for $x \in [0, 2\pi]$

$\Rightarrow f''(x) > 0 \quad \forall x \in [0, 2\pi]$

$\Rightarrow$ <u>$f(x)$ is concave up. & $x = \pi/2$ is local minimum</u>

Step ⑦ Gather all info together from Steps 1 to 6 & plot $f(x)$ -



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eg ② Investigate $f(x) = \frac{(x-2)^3}{x^2}$ & plot it.

Step ① :- (Symmetry) $f(-x) = \frac{-(x+2)^3}{x^2} \neq f(x)$
$\neq -f(x)$

$\therefore$ No Symmetry (☹)

Step ② :- (Periodicity)
Nah! (☹)

Step ③ :- Intercepts.

$f(0) = \infty$ (so no y - intercept, potentially V.A. at $x = 0$).

$f(x) = 0$ @ $x = 2$ (x - intercept).

Step ④ :- Assymptotes ① V.A. @ $x = 0$.

Deg N > Deg D by 1 (possible oblique asymp.)

$$\frac{(x-2)^3}{x^2} = \frac{x^3 - 8 - 6x(x-2)}{x^2}$$

$$= \frac{x^3 - 6x^2 + 12x - 8}{x^2} = x - 6 + \frac{12}{x} - \frac{8}{x^2}$$

$\cancel{= x - 6 + \frac{4}{x^2} \to x \text{ as } x \to \infty}$

$\longrightarrow x - 6$ as $x \to \infty$

$\therefore y = x - 6$ is Oblique Asymptote

Step ⑦ :-

Analyse signs of $f(x)$, $f'(x)$ & $f''(x)$

Recall V.A. at $x=0$

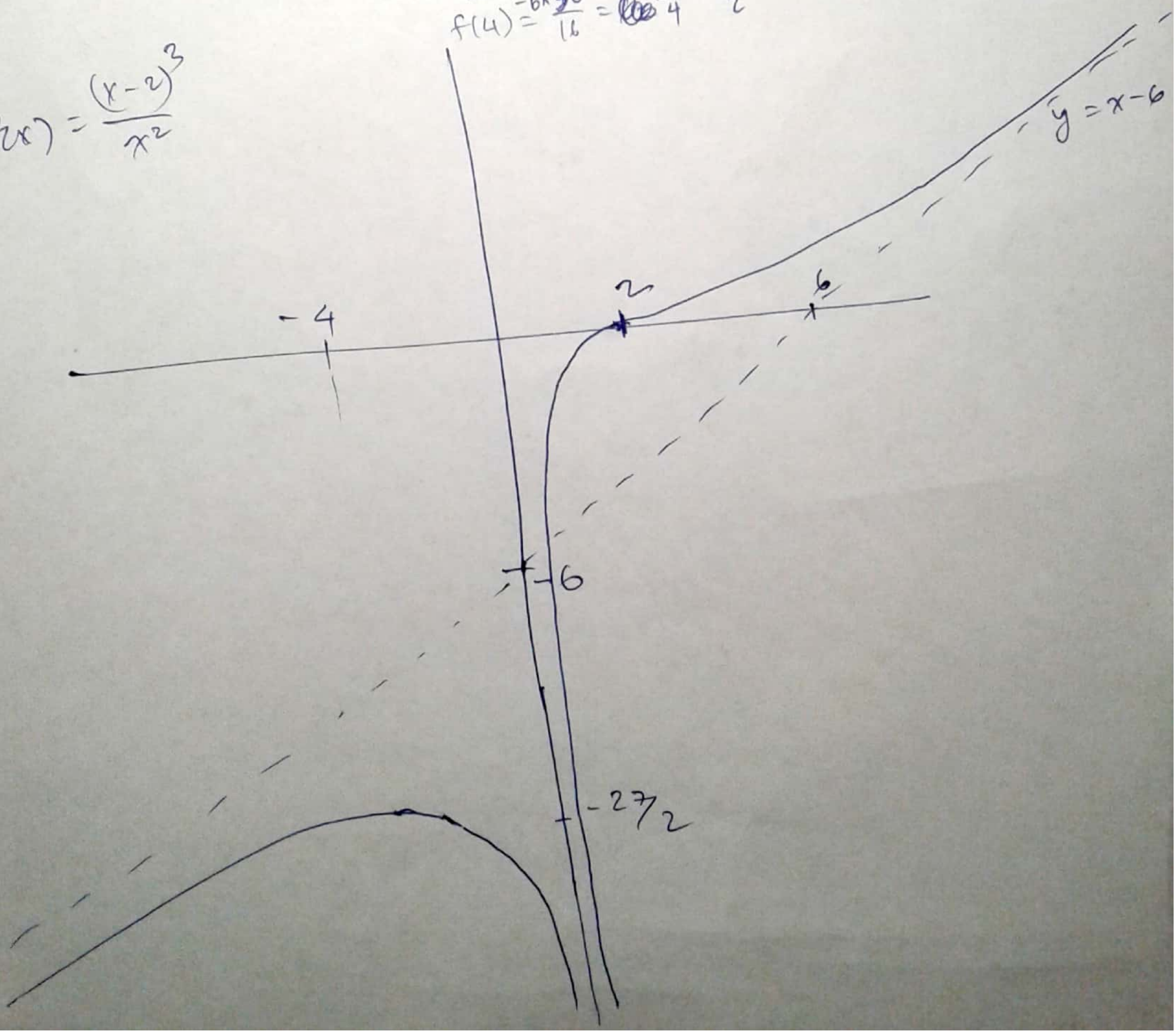
	Interval			
	$x<-4$	$-4<x<0$	$0<x<2$	$x>2$
$f(x)$	$-$	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$+$	$+$
$f''(x)$	$-$	$-$	$-$	$+$

$x=-4$ local max $f(-4) = \frac{-6\times 36}{16} = \frac{-54}{4} = \frac{-27}{2}$

$x=0$ V.A.

$x=2$ is an inflexion pt. $f(2)=0$.

$f(x) = \frac{(x-2)^3}{x^2}$



Plotting w/ oblique asymptotes

$$y(x) = e^x + \frac{\cos x}{x^3}$$

$$= \frac{x^3 e^x + \frac{1}{2}(e^{ix} + e^{-ix})}{x^3}$$

$$= \frac{2x^3 e^x + (e^{ix} + e^{-ix})}{2x^3}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = +\cos\theta - i\sin\theta$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\cos\theta +$

$e^{ix}(1 + e^{-2ix})$

Q) $$y(x) = \frac{2x^3 e^x + e^{ix}(1 + e^{-2ix})}{2x^3}$$

$$= e^x + \frac{e^{ix} + e^{-ix}}{2x^3}$$

$$= e^x + \frac{\cos x}{x^3}$$

Step ①:- Symm (NO)

Step ⑪: Periodicity (NO)

Step ⑪:- y intercept at $x = 0$

But as $x \to 0^{\pm}$; $\frac{\cos x}{x^3} \to \pm\infty$ (so no y-intercept)

& $e^x \to 1$

$\Rightarrow y(x) \to \pm\infty$ as $x \to 0^{\pm}$

$\Rightarrow x = 0$ is a V.A

Also note as $x \to \pm\infty$ $\frac{\cos x}{x^3} \to 0$ (Show by Sandwich thm)

as $x \to +\infty$; $e^x \to \infty$

x-intercept { but as $x \to -\infty$; $e^x \to 0$

so (x-intercept at $x = -\infty$)!

Step (iv) Dominant term as $x \to \infty$

$y(x) \simeq e^x$ as $x \to \infty$

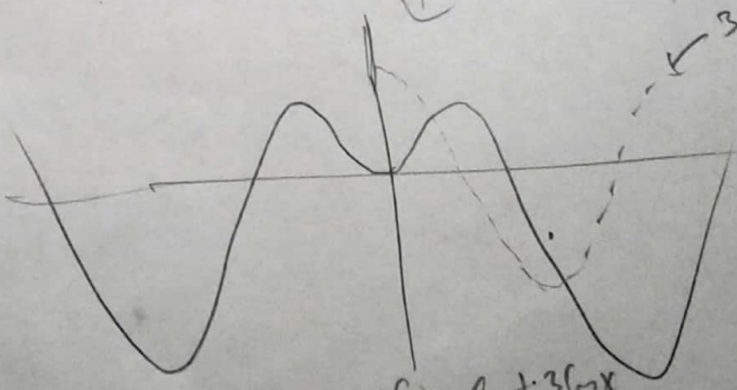
so y is asymptotically close to e^x.

Step (v) $y'(x) = e^x + \frac{1}{x^3}(-\sin x) + \cos(x)\frac{(-3)}{x^4} \overset{\text{set}}{=} 0$

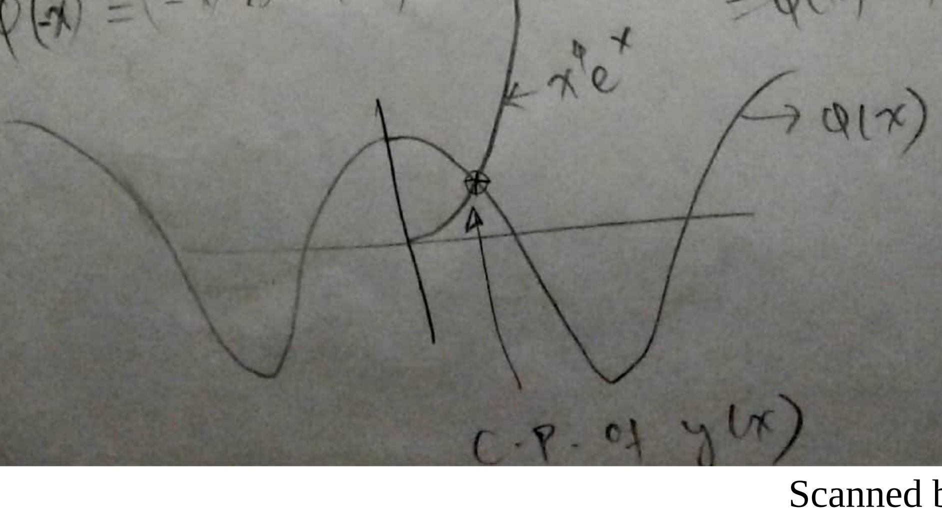
$\Rightarrow e^x - \frac{\sin x}{x^3} - \frac{3\cos x}{x^4} = 0$

$\Rightarrow x^4 e^x - x\sin x - 3\cos x = 0$

$\Rightarrow x^4 e^x = \underbrace{x\sin x + 3\cos x}_{\varphi(x)}$



$\varphi(-x) = (-x)\sin(-x) + 3\cos(-x) = x\sin x + 3\cos x$
$= \varphi(x) \Rightarrow \varphi$ is even & symm abt y-axis



$$y''(x) = e^x + \frac{3\sin x}{x^4} - \frac{\cos x}{x^3} + \frac{12\cos x}{x^5} + \frac{3\sin x}{x^4}$$

$$= e^x + \frac{6\sin x}{x^4} - \frac{\cos x}{x^3} + \frac{12\cos x}{x^5}$$

$\sin x$ will go -ve at $x = \pi^+$

$\cos x$ will go -ve at $x = \pi/2^+$

$$\underbrace{e^{\pi/2}}_{4.81}, \ \frac{6\sin \pi/2}{(\pi/2)^4}, \ -\frac{\cos \pi/2}{(\pi/2)^3} > 0$$

$(\pi/2)^4 = 6.1$ $\quad (\pi/2)^3 = 3.87$ $\quad (\pi/2)^5 = 9.56$

$e^{\pi} = 23.14$

$$\frac{-12}{x^5} \le \frac{12\cos x}{x^5}$$

Clearly the dominance of e^{π} for $x > \pi/2$ will make $y''(x) > 0 \ \forall x > 0$

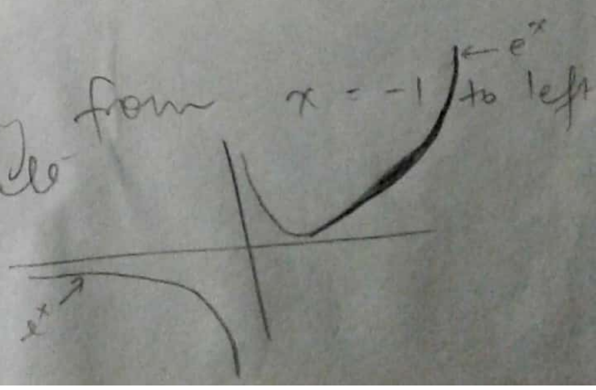
$\Rightarrow y$ is concave up & the c.p. of y at $x > 0$ is minima.

Step (iv) $\quad x < 0$

$\frac{12\cos x}{x^5}$ is dominating term near $x \approx 0$

& $\frac{12\cos x}{x^5} < 0 \Rightarrow y''(x) < 0$

but farther away from $x = -1$ to left e^x will dominate



§(3.6) Optimization problems.

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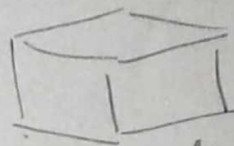
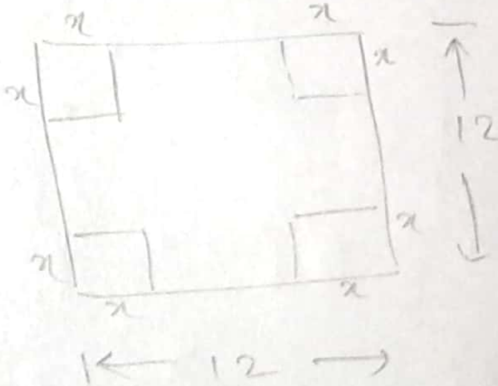
pg (3)

→ Depending on the situation, we define a cost f^n or objective f^n & try to optimize (max/min) the parameters (variables)

eg ① Metal fabrication

An open-top box  is to be fabricated by cutting small congruent squares from the corners of a 12 cm by 12 cm sheet of tin & bending up the sides. How large should the squares cut from the corners to make the box hold as much as possible.



Let edge cut be x cm by x cm

$V(x)$ = length × breadth × ht
$= (12-2x) \times (12-2x) \times x$
$= (12-2x) \cdot$

This is our cost/obj f^n which we need to maximize for the box to hold max amt.

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left\{x(12-2x)^2\right\} = x(2)(12-2x)(-2) + (12-2x)^2 \overset{\text{set}}{=} 0$$

$\Rightarrow -4x(12-2x) + (12-2x)^2$

$\Rightarrow -48x + 8x^2 + 144 + 4x^2 - 48x = 0$

$\Rightarrow 12x^2 - 96x + 144 = 0$

$\Rightarrow x^2 - 8x + 12 = 0$

$\Rightarrow x^2 - 6x - 2x + 12 = 0$

$\Rightarrow x(x-6) - 2(x-6) = 0$

$\Rightarrow x = 2$ or 6

$$\frac{\partial v}{\partial x} = -4x(12-2x) + (12-2x)^2 = -48x + 8x^2 + 144 + 4x^2 - 48x$$

$$\frac{\partial^2 v}{\partial x^2} = 16x - 96$$
$$= 16(x-6)$$

$$\left.\frac{\partial^2 v}{\partial x^2}\right|_{x=6} = 0$$

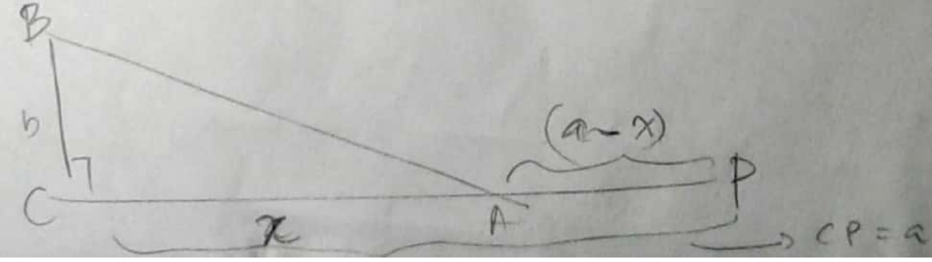
$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=2} = 16(-4) < 0$$

$\Rightarrow x = 2$ maximizes $V(x)$.

---

eg ② Triathlon problem – these are problems that take the form when a person/thing needs to travel in one direction at a certain speed, & in another direction at a different speed.

Q) A man who can row at a speed of 4 mph & run at a speed of 6 mph wishes to reach the pt. P from a boat at pt. B as shown in the figure below in the least amt. of time as possible. Find the distance AP the man must run on the beach



obj / Cost f^n $t(x) = \dfrac{\sqrt{x^2+b^2}}{4} + \dfrac{a-x}{6}$

$$t'(x) = \frac{1}{4}\frac{1}{\cancel{2}}(x^2+b^2)^{-1/2}(\cancel{2}x) - \frac{1}{6} = 0$$

$$\Rightarrow \frac{(x^2+b^2)^{-1/2}}{2} = \frac{1}{3}$$

$$\Rightarrow \frac{x}{\sqrt{x^2+b^2}} = 2/3$$

$$\Rightarrow \frac{9x^2}{4} = x^2+b^2$$

$$\Rightarrow 5x^2 = 4b^2$$

$$x^2 = \frac{4b^2}{5}$$

$$x = \frac{2b}{\sqrt{5}}$$

$x = 2b/\sqrt{5}$

Dist he runs $\left(a - \dfrac{2b}{\sqrt{5}}\right)$

Dist he rows $= \sqrt{b^2+x^2} = \sqrt{b^2+\dfrac{4b^2}{5}}$ (independent of a)!

Is this strange

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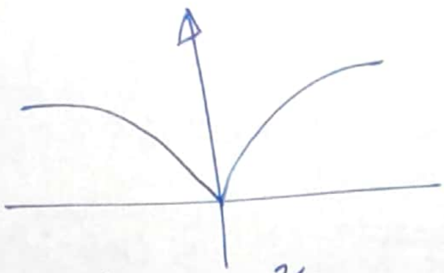
pg (1)

eg. of cusps

(i) $f(x) = x^{2/3}$



$f(x) = x^{2/3}$

f has a vertical cusp at (0, 0)

$\because f'(x) = \frac{2}{3} x^{-1/3}$

$f'(x) \to -\infty$ as $x \to 0^-$

$f'(x) \to +\infty$ as $x \to 0^+$

(ii) $g(x) = 2 - (x-1)^{2/5}$

g has a vertical cusp at (1, 2)

$\because g'(x) = -\frac{2}{5}(x-1)^{-3/5}$ &

$g'(x) \to \infty$ as $x \to 0^-$

$g'(x) \to -\infty$ as $x \to 0^+$

Q) plot $\sqrt{|x-2|} - \sqrt[3]{|x+2|} = f(x)$

for $x > 2$ ~~$-2 < x < 2$~~ ; $f(x) = \sqrt{(x-2)} - \sqrt[3]{(x+2)} = (x-2)^{1/2} - (x+2)^{1/3}$

$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x-2}} - \frac{1}{3}\frac{1}{(x+2)^{2/3}}$

$\lim_{x \to 2^+} f'(x) = +\infty$ (1^{st} term dominates).

for $-2 < x < 2$; $f(x) = \sqrt{(2-x)} - (x+2)^{1/3}$

$f'(x) = \frac{-1}{2}\frac{1}{\sqrt{(2-x)}} - \frac{1}{3}\frac{1}{(x+2)^{2/3}}$

$\lim_{x \to 2^-} f'(x) = -\infty$

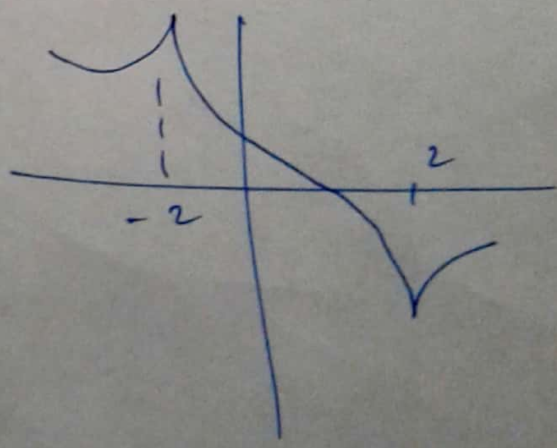
Similarly

$x < -2$

$\lim_{x \to -2^-} f'(x) = +\infty$

& $x \in (-2, 2)$

$\lim_{x \to -2^+} f'(x) = -\infty$



Removable Discontinuity.

(i) $f(x) = x^3 \cos\left(\frac{1}{x}\right)$.

$$\lim_{x \to 0} x^3 \cos\left(\frac{1}{x}\right)$$

$$= \lim_{y \to \infty} \frac{\cos y}{y^3} = 0$$ by Sandwich thm.

Is $f(0)$ defined ??

No b/c $\cos(1/x)$ is not defined at $x=0$.

$\Rightarrow f(x)$ is not continuous at $x=0$.

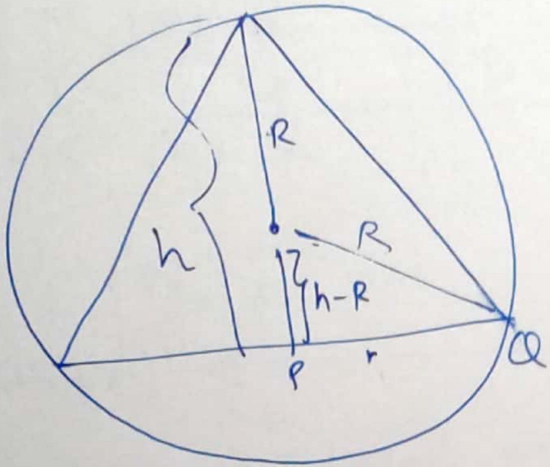
But if we defined $f(x) = \begin{cases} x^3 \cos(1/x) & ; x \neq 0 \\ 0 \text{ or } \lim_{x \to 0} x^3 \cos 1/x & ; x = 0 \end{cases}$

Thus $x=0$ is a removable discontinuity for $f(x) = x^3 \cos 1/x$.

(ii) Similarly $\varphi(x) = \frac{\sin x}{x}$ is not defined/continuous at $x=0$.
Set $\varphi(0) := 1$ & remove discontinuity at $x=0$. Also note $\sin x \approx x$ near 0!

optimization

Q1) What is the volume of the largest cone that can be inscribed inside a sphere of radius R.



$$r^2 = R^2 - (h-R)^2$$
$$= R^2 - h^2 - R^2 + 2hR$$
$$= 2hR - h^2$$
$$= h(2R-h)$$

$2R - \frac{4R}{3}$

$\frac{2R}{3}$

$$V(\text{cone}) = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi h^2(2R-h)$$
$$= \frac{\pi}{3}\left\{2Rh^2 - h^3\right\}$$

$$\frac{\partial v}{\partial h} = \frac{\pi}{3}\left\{4Rh - 3h^2\right\} = 0 \Rightarrow$$

$h(4R-3h) = 0$

$h = \frac{4R}{3}$.

$$\therefore V(\text{cone}) = \frac{1}{3}\pi h^2(2R-h) = \frac{\pi}{3}\,\frac{16R^2}{9}\,\frac{2R}{3} = \frac{32\pi R^3}{81}$$

Plotting f^n.

Q 1)(a) Plot the graph of a f^n f which has only one pt. of discontinuity on its domain $[-4,\infty)$ & that satisfies

$\lim_{x \to \infty} f(x) = -2$

$\lim_{x \to 0^-} f(x) = \infty$

$f(0) = 2$

$f''(x) < 0 ; x \in (-4,-1)$

$f''(x) > 0 ; x \in (-1,0)$

$f''(x) < 0 ; x \in (0,4)$

$f''(x) > 0 ; x \in (4,\infty)$

$f'(x) < 0 ; x \in (-4,-1)$

$f'(x) > 0 ; x \in (-1,0)$

$f'(x) > 0 ; x \in (0,2)$

$f'(x) < 0 ; x \in (2,\infty)$

(b) Find all inflection pts; for each inflexion pt., determine if it is possible ~~that~~ $f''(x) = 0$.

ans) (a) V.A at $x=0$

H.A at $y=-2$

Interval	$(-4,-1)$	$(-1,0)$	$(0,2)$	$(2,4)$	$(4,\infty)$
Monotonicity	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$
Concavity	Dn	Up	Dn	Dn	Up.

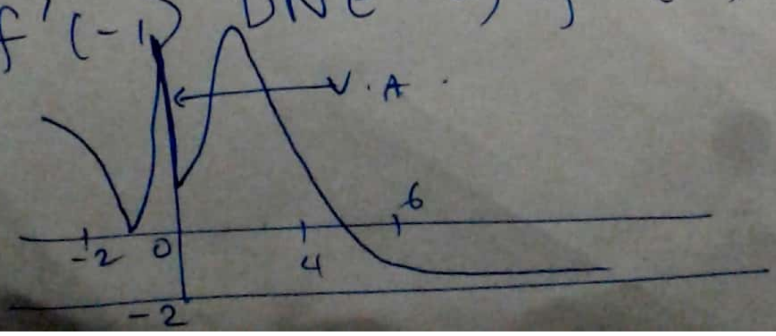
(b) 2 inflexion pts. $x=-1, 4$

$x=-1$ is local min

If $f''(-1)=0 \Rightarrow f'(-1)$ exist & $f'(-1)=0$

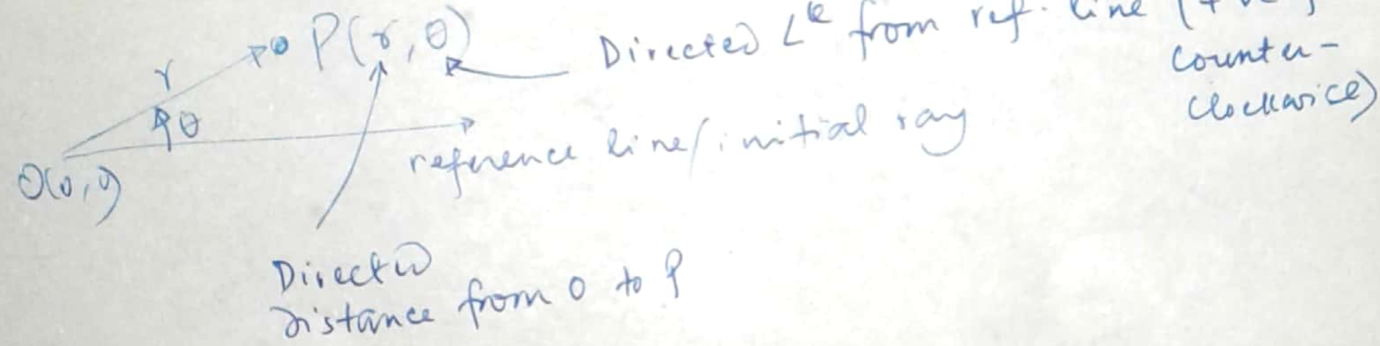
$\Rightarrow$ at $x=-1$; f is above the tangent line at $x=-1$ which contradicts the fact that f crosses its tangent line at each inflection pt.

$\Rightarrow f'(-1)$ DNE $\Rightarrow f''(-1)$ DNE

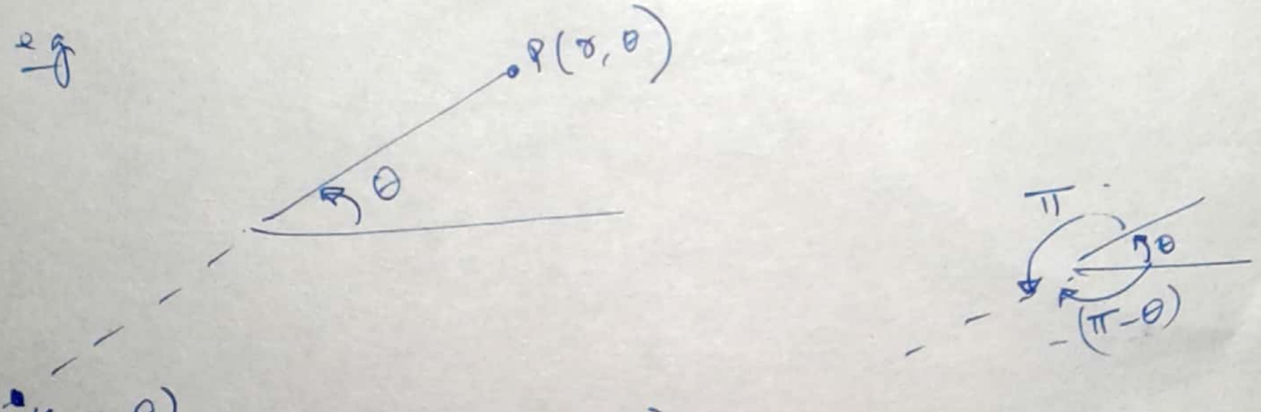


P(9.6) Polar Co-ordinates

Pg ① 20/8/18



eg



$P'(-r, \theta)$

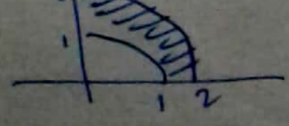
$\equiv P'(r, \pi+\theta) \equiv P'(r, -\pi+\theta)$

Elementary graphs in polar-co-ordinates

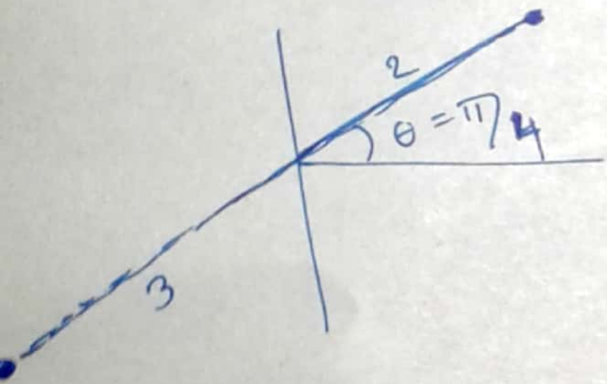
Eqn.	Graph -
$r = a$	Circle of radius $\|a\|$ centred at origin
$\theta = \theta_0$	Line through (0, 0) at an angle θ_0 from ref. line

eg Graph the sets of pts. whose polar co-ordinates satisfy the following cond's :-

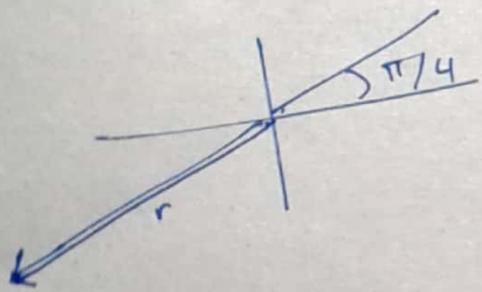
(i) $1 \le r \le 2$; $0 \le \theta \le \pi/2$



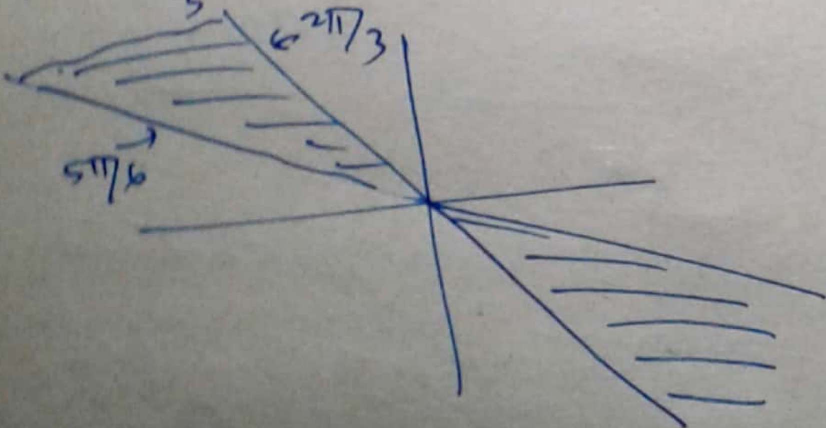
(ii) $-3 \le r \le 2$; $\theta = \pi/4$



(iii) $r \le 0$; $\theta = \pi/4$



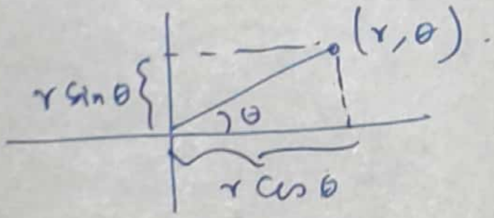
(iv) $\frac{2\pi}{3} \le \theta \le 5\pi/6$



Open ended graph b/c no restriction on r.

Cartesian vs Polar co-ordinates.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



→ this gives us

$$x^2 + y^2 = r^2$$

and

$$\tan\theta = \frac{y}{x}$$

Eqns in polar & Cartesian frames.

Polar	Cartesian
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$\sqrt{x^2+y^2} = 1 + 2x$ $\Rightarrow y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

eg. Find a polar eqn. for the circle

$$x^2 + (y-3)^2 = 9$$

$$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta + 9 - 6r\sin\theta = 9$$

$$\Rightarrow r^2 - 6r\sin\theta = 0$$

$$\Rightarrow r = 0 \text{ or } \underline{r = 6\sin\theta}$$

→ this is suff. as $r = 0$ when $\theta = 0$

eg Give the cartesian equivalent of polar curves

i) $r\cos\theta = -4$

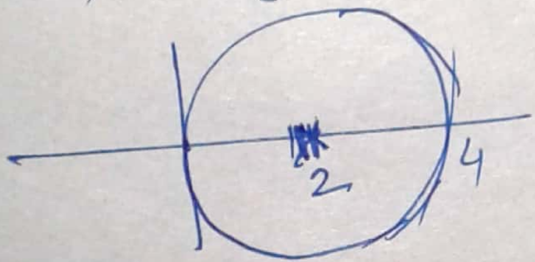
$\Rightarrow x = -4$

ii) $r^2 = 4r\cos\theta$

$\Rightarrow x^2 + y^2 = 4x$

$\Rightarrow x^2 - 4x + (2)^2 + y^2 = (2)^2$

$\Rightarrow (x-2)^2 + y^2 = 4$



(iii) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

$\Rightarrow 2x - y = 4$

$\Rightarrow y = 2x - 4$