Lecture (16): Residue Calculus on $\mathbb{C}$ 1 9/4/19

(16) Part (I)

Let $f(z)$ be analytic in $D := 0 < |z - z_0| < \rho$ & let $z = z_0$ is an isolated singular point of $f(z)$. Laurent expansion of $f(z)$ in D:

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n \quad w/ \ C_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad \text{—— (16.1)}$$

where C is a Jordan contour in D.

<u>Principal part</u> of the series $= \sum_{n=-\infty}^{-1} C_n (z - z_0)^n$

<u>Residue</u> of $f(z)$ at $z_0 = C_{-1} = \text{Res}(f(z); z_0)$

From (16.1) $\oint_C f(z)\,dz = 2\pi i\, C_{-1}$ —— (16.2)

Recall, earlier we had seen that-
$\oint_C f(z)\,dz = 0$ when $f(z)$ is analytic in D.
Now, $\oint_C f(z)\,dz = 2\pi i\, C_{-1}$ when $f(z)$ has one isolated s.p. in D.

Infact, this concept can be generalized to a finite no. of isolated s.p.s & is given by the following theorem known as the <u>Cauchy Residue Theorem</u>.

Thm (16.1) (Cauchy Residue thm)

Let $f(z)$ be analytic inside and on a simple closed contour C, except for a finite no. of isolated s.p.s $z_1, \dots, z_N$ located inside C. Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} a_j = 2\pi i \sum_{j=1}^{N} C_{-1}^{(j)}$$

$$= 2\pi i \sum_{j=1}^{N} \mathrm{Res}(f(z); z_j) \qquad \text{(16-3)}$$

Proof :- This theorem can be proved by the application of "deformation of contour".

Do it yourself as an exercise.

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Examples:

eg (16.1) $\quad I_k = \frac{1}{2\pi i}\oint_{C_0} z^k\,dz$; w/ $C_0 : |z| = 1$

Soln:- for $k = 0, 1, 2, \dots$

We have $I_k = 0$ b/c of Cauchy-Goursat thm.

$k = -1$ $\quad I_{-1} = \frac{1}{2\pi i}\oint_{C_0} \frac{1}{z}\,dz$

$\underset{\text{z=0 is iso. s.p.}}{=} \frac{1}{2\pi i} 2\pi i\, C_{-1} = 1$ b/c Laurent series of $1/z$ is $\dots 0 + 0 + \frac{1}{z} + 0 + 0 + \dots$

for $k = -2, -3, -4, \dots$

$I_k = 0$ b/c $C_{-1} = 0$

$\therefore I_k = \delta_{k,-1}$.

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eg (16.2) Evaluate $I = \frac{1}{2\pi i}\oint_{C_0} z e^{1/z} dz$ w/ $C_0 : |z| = 1$

Soln :- $f(z) = z e^{1/z}$ is analytic everywhere except at $z = 0$.

$$I = \frac{1}{2\pi i}\oint_{C_0} z e^{1/z} dz$$

$$= \frac{1}{2\pi i} 2\pi i \, C_{-1}$$

$$= C_{-1} = \frac{1}{2!} = \frac{1}{2}$$

Recall that $e^{1/z}$ has a full Laurent series

b/c $z e^{1/z}$

Laurent $z\left(\cdots + \frac{1}{2! z^2} + \frac{1}{z} + 1\right)$ $\cancel{+ z + \frac{z^2}{2!} + \cdots}$

$= \left(\cdots + \frac{1}{2! z} + 1 + z\right)$ $\cancel{+ \frac{z^2}{2!} + \cdots}$

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eg (16-3) Evaluate

$$I = \oint_{C_2} \frac{z+2}{z(z+1)} dz \quad \text{w/ } C_2 : |z| = 2.$$

$$\because \frac{z+2}{z(z+1)} = \frac{2}{z} - \frac{1}{z+1}$$

$$I = \oint_{C_2} \frac{2}{z} dz - \oint_{C_2} \frac{1}{z+1} dz$$

$$= 2\pi i \, C_{-1} - 2\pi i \, C_{-1}$$

$$= 2\pi i (2) - 2\pi i (1)$$

$$= 2\pi i$$

$\zeta = z + 1$

$d\zeta = dz$

$\oint_{C_2'} \frac{1}{\zeta} d\zeta$ w/ $C_2' : |\zeta - 1| = 2$

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Note :-

When $f(z)$ has an essential singular point, then expanding as Laurent series & thereby obtaining C_{-1} is the only option. However, if $f(z)$ has a pole, then there is a simpler formula as given below

$$\text{Let } f(z) = \frac{\varphi(z)}{(z-z_0)^m} \quad ; \quad \varphi(z) \text{ is analytic in } B_\epsilon(z_0); \; m \in \mathbb{Z}^+. \qquad \text{(16.4)}$$

If $\varphi(z_0) \neq 0 \Rightarrow f(z)$ has m^{th} order pole at z_0

then $$C_{-1} = \text{Res}(f(z); z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \varphi(z) \Big|_{z=z_0}$$

$$= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z)\right) \Big|_{z=z_0} \qquad \text{(16.5)}$$

for $m=1$; $f(z) = \frac{N(z)}{D(z)}$ each analytic

$$C_{-1} = \frac{N(z_0)}{D'(z_0)} \qquad \text{(16.6)}$$

eg (16.4) Evaluate

$$I = \frac{1}{2\pi i} \oint_{C_2} \frac{3z+1}{z(z-1)^3} dz;$$

w/ $C_2 : |z| = 2$.

$f(z) = \frac{3z+1}{z(z-1)^3}$ has the form (16.4) near $z=0,1$

$$\text{Res}(f(z); 0) = \frac{(3z+1)/(z-1)^3 \big|_{z=0}}{\frac{d}{dz} z \big|_{z=0}} = -1$$

$$\text{Res}(f(z); 1) = \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}} \frac{3z+1}{z} \Big|_{z=1}$$

$$= \frac{1}{2!} \frac{d^2}{dz^2} \left(\frac{3z+1}{z} \right)_{z=1}$$

$$= 1$$

$$\therefore I = 2\pi i \sum_{j=1}^{2} \text{Res}(f(z); z_j) \quad ; \quad z_1 = 0, \; z_2 = 1$$

$$= 2\pi i (-1+1)$$

$$= 0. \qquad \#.$$

Residue at Infinity (∞)

The concept of residue at ∞ is very useful when we integrate rational f^ns. Rational f^ns have only isolated s.p-s in the extended z-plane & are analytic everywhere else.

Let $z_1, \ldots, z_N$ denote the finite no. of singularities.

$$\text{Res}(f(z), \infty) = \sum_{j=1}^{N} \text{Res}(f(z); z_j).$$

We have 2 methods of calculating L.H.S & hence the R.H.S.

Two ways of Calculating $\operatorname{Res}(f(z);\infty)$.

$$\underbrace{\oint f(z)\,dz}_{C_\infty} = \lim_{R\to\infty} \oint_{C_R} f(z)\,dz = 2\pi i \operatorname{Res}(f(z);\infty)$$

<u>Method (1)</u> :- If $f(z)$ is analytic at ∞ w/ $f(\infty)=0$; then $f(z) = \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$ $\left(\sum_{n=0}^{\infty} a_n z^n = 0\right.$ b/c $\left.f(\infty)=0\right)$

$$\therefore \operatorname{Res}(f(z);\infty) = a_{-1}$$

$$= \lim_{R\to\infty} \frac{1}{2\pi i} \int_0^{2\pi} \left(\frac{a_{-1}}{Re^{i\theta}} + \cdots\right) iRe^{i\theta}\,d\theta \quad \text{——} (16.7)$$

Above holds even if $f(\infty) \neq 0$ as long as $f(z)$ has a Laurent series in $B_\epsilon(\infty)$.

Contd... in Lecture (16) Part (II).

Lecture (16) Contd... 16 Part (II) 14/4/2019.

Residue at ∞.

2nd method (to find an equivalent formulation so that Res at ∞ can be calculated as Res at 0)

Alternatively, take $z = \frac{1}{t}$ which gives $dz = -\frac{1}{t^2}dt$

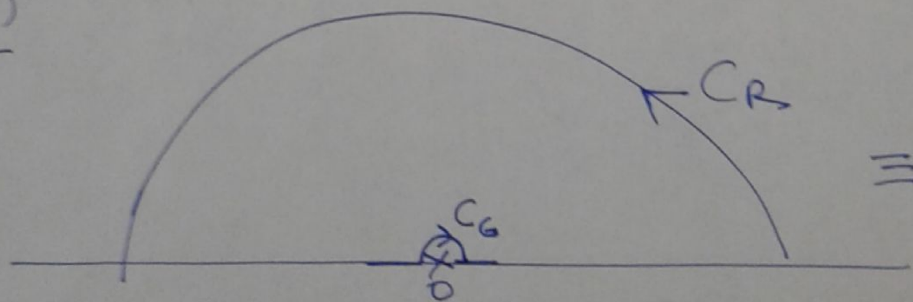
& $Re^{i\theta} \to \frac{1}{R}e^{-i\theta}$
$\equiv \epsilon e^{-i\theta}$

$$\Rightarrow \text{Res}(f(z);\infty) = \frac{1}{2\pi i}\oint_{C_\infty} f(z)\,dz = \frac{1}{2\pi i}\oint_{-C_\epsilon}\left(-\frac{1}{t^2}\right)f(1/t)\,dt$$

$$= \frac{1}{2\pi i}\oint_{C_\epsilon}\frac{1}{t^2}f\left(\frac{1}{t}\right)dt$$

Q) Why did the orientation of the contour integral (direction) change from counterclockwise (in the case of $\oint_{C_\infty}(\cdot)dz$) to clockwise (in the case of $\oint_{-C_\epsilon}(\cdot)dt$)?

Ans)



$\equiv$ C_R is counter-clockwise while C_ϵ is clockwise!

$$\therefore \boxed{\text{Res}(f(z);\infty) \equiv \text{Res}\left(\frac{1}{t^2}f(t);0\right)} \quad \text{— (16.8)}$$

i.e. $\left.\begin{array}{l}\text{Coeff. of } \frac{1}{z} \text{ in} \\ \text{the expansion of } f(z) \\ \text{at } z=\infty\end{array}\right\} \equiv \left\{\begin{array}{l}\text{Coeff. of } \frac{1}{t} \text{ in the} \\ \text{expansion of } \frac{1}{t^2}f(t) \\ \text{about } t=0.\end{array}\right.$

Further, when $f(\infty)=0$

$$f(z) = \cdots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z}$$ is the appropriate Laurent expansion of $f(z)$ about $z=\infty$ b/c this is the "only" way $f(\infty)$ can be equal to 0 (if the $a_0, a_1, a_2, \ldots \equiv 0$).

$$z f(z) = \cdots + \frac{a_{-2}}{z} + a_{-1}$$

$$\boxed{\lim_{z\to\infty} z f(z) = a_{-1} = \mathrm{Res}(f(z);\infty)} \quad (16.9)$$

So when $f(\infty)=0$; this is yet another way to find $\mathrm{Res}(f(z);\infty)$.

<u>eg. (16.5)</u> Evaluate $I = \frac{1}{2\pi i}\oint_C \frac{a^2-z^2}{a^2+z^2}\frac{dz}{z}$; C is Jordan contour enclosing $z=0, \pm ia$

<u>Soln</u>:- $f(z) = \dfrac{a^2-z^2}{z(a^2+z^2)}$ has isolated s.p.s at $z=0$, $z=\pm ia$; i.e. $\{z_j\} = \{0, ia, -ia\}$

$$f(\infty) = \left.\frac{a^2/z^2 - 1}{z(a^2/z^2+1)}\right|_{z=\infty} = 0$$

$$\Rightarrow f(z) \xrightarrow[\text{expansion about } z=\infty]{\text{Laurent}} \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$$

$$\therefore \mathrm{Res}(f(z);\infty) = \lim_{z\to\infty} z f(z) = \lim_{z\to\infty} z\,\frac{1}{z}\frac{(a^2/z^2-1)}{(a^2/z^2+1)} = -1$$

Now by Cauchy Residue th^m

$$I = 2\pi i \sum_{j=1}^{3} \mathrm{Res}(f(z); z_j) = \mathrm{Res}(f(z),\infty) = -1 \#$$

Winding Number of a contour C.

$$W(z_j) = \frac{1}{2\pi i}\oint_C \frac{dz}{(z-z_j)} = \frac{1}{2\pi i}(i2\pi n) = n$$ is called the winding no. of the curve C around $z = z_j$.

b/c $$\oint_C \frac{dz}{z-z_j} = \int_0^{(2\pi)n} \frac{r i e^{i\theta} d\theta}{r e^{i\theta}} = i2\pi n$$

by taking $z - z_j = re^{i\theta}$

Note the upper limit of the integral is $2\pi n$ instead of 2π b/c (we are) the curve C is looping around (winding) n times abt. z_j.

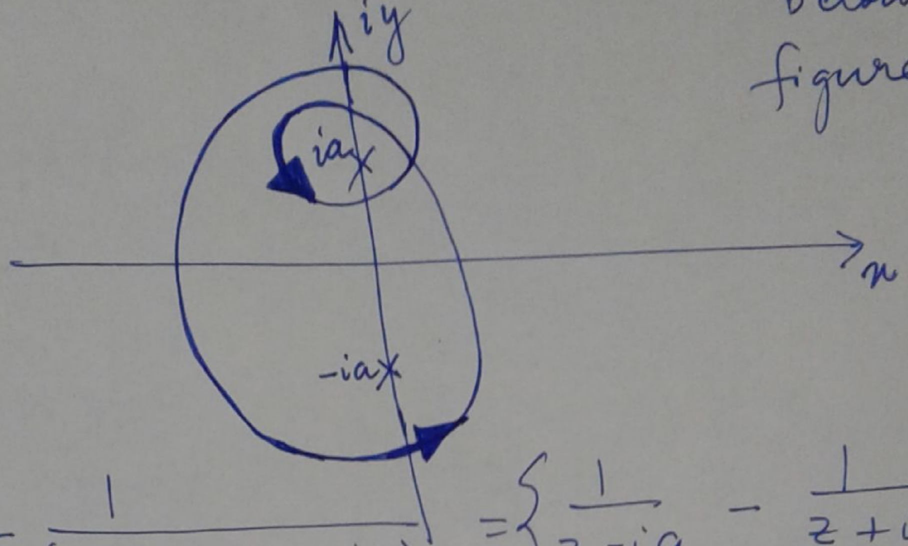
So, Cauchy's Residue th^m can be generalized for the case of a curve that winds around $W(z_j)$ times about z_j as follows :-

$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} W(z_j)\,\mathrm{Res}\,(f(z); z_j)} \quad (16.10)$$

this is Generalized Cauchy Residue th^m!

Application of Winding no. & Generalized Cauchy Residue thm.

eg.(16.6) Evaluate $I = \oint_C \frac{dz}{z^2+a^2}$; $a>0$ & C is as shown below in the figure.



Let $f(z) = \frac{1}{(z+ia)(z-ia)} = \left\{\frac{1}{z-ia} - \frac{1}{z+ia}\right\}\frac{1}{2ai}$

$$I = \oint_C \frac{dz}{z^2+a^2} = \frac{1}{2ai}\left\{\oint_C \underbrace{\frac{1}{z-ia}}_{f_1(z)}dz - \oint_C \underbrace{\frac{1}{z+ia}}_{f_2(z)}dz\right\}$$

Gen. Cauchy Res. thm

$$= \frac{1}{2ai}\left\{2\pi i\sum_{j=1}^{1} W(z_j)\,\text{Res}(f_1(z); z_j) - 2\pi i\sum_{j=1}^{1} W(z_j)\,\text{Res}(f_2(z); z_j)\right\}$$

b/c $W(z_j)$ for $z_j = ia$ is $= 2$ by inspecting the fig. above

$$= \frac{\pi}{a}\left\{2(1) - 1(1)\right\} = \pi/a.$$

The residues above were calculated as follows:

$$\text{Res}\left(\frac{1}{z-ia}; ia\right) = \frac{N(ia)}{D'(ia)} = \frac{1}{1} = 1$$

$$\text{Res}\left(\frac{1}{z+ia}; -ia\right) = \text{Res}\left(\frac{1}{z-(-ia)}; -ia\right) = \text{coeff. of } (z-(-ia))^{-1} = 1.$$

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