

Definition of Dominant Eigenvalue and Dominant Eigenvector

Let $\lambda_1, \lambda_2, \dots$, and λ_n be the eigenvalues of an $n \times n$ matrix A . λ_1 is called the **dominant eigenvalue** of A if

$$|\lambda_1| > |\lambda_i|, \quad i = 2, \dots, n.$$

The eigenvectors corresponding to λ_1 are called **dominant eigenvectors** of A .

Not every matrix has a dominant eigenvalue. For instance, the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(with eigenvalues of $\lambda_1 = 1$ and $\lambda_2 = -1$) has no dominant eigenvalue. Similarly, the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(with eigenvalues of $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 1$) has no dominant eigenvalue.

Finding a Dominant Eigenvalue

Find the dominant eigenvalue and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

The characteristic polynomial of A is

$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$. Therefore the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$, of which the dominant one is $\lambda_2 = -2$.

the dominant eigenvectors of A (those corresponding to $\lambda_2 = -2$) are of the form

$$\mathbf{x} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

How? Solve $Ax = -2x$, where $x = (x_1 \ x_2)^T$

The Power Method (iterative method)

First we assume that the matrix A has a dominant eigenvalue with corresponding dominant eigenvectors. Then we choose an initial approximation \mathbf{x}_0 of one of the dominant eigenvectors of A . This initial approximation must be a *nonzero* vector in R^n . Finally we form the sequence given by

$$\mathbf{x}_1 = A\mathbf{x}_0$$

$$\mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0$$

$$\mathbf{x}_3 = A\mathbf{x}_2 = A(A^2\mathbf{x}_0) = A^3\mathbf{x}_0$$

$$\vdots$$

$$\mathbf{x}_k = A\mathbf{x}_{k-1} = A(A^{k-1}\mathbf{x}_0) = A^k\mathbf{x}_0.$$

For large powers of k , and by properly scaling this sequence, we will see that we obtain a good approximation of the dominant eigenvector of A .

Approximating a Dominant Eigenvector by the Power Method

Complete six iterations of the power method to approximate a dominant eigenvector of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

We begin with an initial nonzero approximation of

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We then obtain the following approximations.

<i>Iteration</i>		<i>Approximation</i>
$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix}$	→	-4 $\begin{bmatrix} 2.50 \\ 1.00 \end{bmatrix}$
$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 28 \\ 10 \end{bmatrix}$	→	10 $\begin{bmatrix} 2.80 \\ 1.00 \end{bmatrix}$
$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 28 \\ 10 \end{bmatrix} = \begin{bmatrix} -64 \\ -22 \end{bmatrix}$	→	-22 $\begin{bmatrix} 2.91 \\ 1.00 \end{bmatrix}$
$\mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -64 \\ -22 \end{bmatrix} = \begin{bmatrix} 136 \\ 46 \end{bmatrix}$	→	46 $\begin{bmatrix} 2.96 \\ 1.00 \end{bmatrix}$
$\mathbf{x}_5 = A\mathbf{x}_4 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 136 \\ 46 \end{bmatrix} = \begin{bmatrix} -280 \\ -94 \end{bmatrix}$	→	-94 $\begin{bmatrix} 2.98 \\ 1.00 \end{bmatrix}$
$\mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -280 \\ -94 \end{bmatrix} = \begin{bmatrix} 568 \\ 190 \end{bmatrix}$	→	190 $\begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix}$

If \mathbf{x} is an eigenvector of a matrix A , then its corresponding eigenvalue is given by

$$\lambda = \frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}}.$$

This quotient is called the **Rayleigh quotient**.

Why is this true?

Since \mathbf{x} is an eigenvector of A , we know that $A\mathbf{x} = \lambda\mathbf{x}$, and we can write

$$\frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{\lambda\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{\lambda(\mathbf{x} \cdot \mathbf{x})}{\mathbf{x} \cdot \mathbf{x}} = \lambda.$$

Approximating a Dominant Eigenvalue

Use the result of Example to approximate the dominant eigenvalue of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

After the sixth iteration of the power method in Example, we had obtained.

$$\mathbf{x}_6 = \begin{bmatrix} 568 \\ 190 \end{bmatrix} \approx 190 \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix}.$$

With $\mathbf{x} = (2.99, 1)$ as our approximation of a dominant eigenvector of A , we use the Rayleigh quotient to obtain an approximation of the dominant eigenvalue of A . First we compute the product $A\mathbf{x}$.

$$A\mathbf{x} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix} = \begin{bmatrix} -6.02 \\ -2.01 \end{bmatrix}$$

Then, since

$$A\mathbf{x} \cdot \mathbf{x} = (-6.02)(2.99) + (-2.01)(1) \approx -20.0$$

and

$$\mathbf{x} \cdot \mathbf{x} = (2.99)(2.99) + (1)(1) \approx 9.94,$$

we compute the Rayleigh quotient to be

$$\lambda = \frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} \approx \frac{-20.0}{9.94} \approx -2.01,$$

which is a good approximation of the dominant eigenvalue $\lambda = -2$.

The Power Method with Scaling

Calculate seven iterations of the power method with *scaling* to approximate a dominant eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}.$$

Use $\mathbf{x}_0 = (1, 1, 1)$ as the initial approximation.

One iteration of the power method produces

$$A\mathbf{x}_0 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix},$$

and by scaling we obtain the approximation

$$\mathbf{x}_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix}.$$

A second iteration yields

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 1.00 \\ 1.00 \\ 2.20 \end{bmatrix}$$

and

$$\mathbf{x}_2 = \frac{1}{2.20} \begin{bmatrix} 1.00 \\ 1.00 \\ 2.20 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.45 \\ 1.00 \end{bmatrix}.$$

Continuing this process, we obtain the sequence of approximations shown in Table.

TABLE 1

\mathbf{x}_0	\mathbf{x}_1	\mathbf{x}_2	\mathbf{x}_3	\mathbf{x}_4	\mathbf{x}_5	\mathbf{x}_6	\mathbf{x}_7
$\begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.45 \\ 0.45 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.48 \\ 0.55 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.51 \\ 0.51 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.49 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}$

From Table 1 we approximate a dominant eigenvector of A to be

$$\mathbf{x} = \begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}.$$

Using the Rayleigh quotient, we approximate the dominant eigenvalue of A to be $\lambda = 3$. (For this example you can check that the approximations of \mathbf{x} and λ are exact.)

R E M A R K : Note that the *scaling factors* used to obtain the vectors in Table 10.6,

$$\begin{array}{ccccccc} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 & \mathbf{x}_6 & \mathbf{x}_7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5.00 & 2.20 & 2.82 & 3.13 & 3.02 & 2.99 & 3.00, \end{array}$$

are approaching the dominant eigenvalue $\lambda = 3$.

Convergence of the Power Method

If A is an $n \times n$ diagonalizable matrix with a dominant eigenvalue, then there exists a nonzero vector \mathbf{x}_0 such that the sequence of vectors given by

$$A\mathbf{x}_0, A^2\mathbf{x}_0, A^3\mathbf{x}_0, A^4\mathbf{x}_0, \dots, A^k\mathbf{x}_0, \dots$$

approaches a multiple of the dominant eigenvector of A .

necessary condition

Inverse power method

A simple change allows us to compute the **smallest** eigenvalue (in magnitude). Let us assume now that A has eigenvalues

$$|\lambda_1| \geq |\lambda_2| \cdots > |\lambda_n|.$$

Then A^{-1} has eigenvalues λ_j^{-1} satisfying

$$|\lambda_n^{-1}| > |\lambda_2^{-1}| \geq \cdots \geq |\lambda_n^{-1}|.$$

Thus if we apply the power method to A^{-1} , the algorithm will give $1/\lambda_n$, yielding the smallest eigenvalue of A (after taking the reciprocal at the end).

Note that in practice, instead of computing A^{-1} , we first compute an LU factorization of A , and then solve

$$Ax^{(k+1)} = x^{(k)}$$

at each step, which only takes $O(n^2)$ operations after the initial work.

Now suppose instead we want to find the eigenvalue closest to a number μ . Notice that the matrix $(A - \mu I)^{-1}$ has eigenvalues

$$\frac{1}{\lambda_j - \mu}, \quad j = 1, \dots, n.$$

The eigenvalue of largest magnitude will be $1/(\lambda_{j_0} - \mu)$ where λ_{j_0} is the closest eigenvalue to μ (assuming there is only one). This leads to the **inverse power method** (sometimes called **inverse iteration**):

Inverse power method: To find the eigenvalue of A closest to μ ,

- 1) Apply the power method to $(A - \mu I)^{-1}$, solving

$$(A - \mu I)\mathbf{x}_k = \mathbf{x}_{k-1}$$

at each step using some linear system solver (e.g. LU factorization).

- 2) Compute λ from the output $1/(\lambda - \mu)$.

Note that if μ is fixed, the LU factorization only needs to be computed once!