Fixed point iteration

(1)

* A no. p is a fixed pt. for a given function $g(x)$ if $g(p) = p$.

Relation to root finding problem

given a root finding problem say $f(p) = 0$; we can define f^ns. w/ a fixed pt. at p in many ways

$\to g(x) = x - f(x)$

or

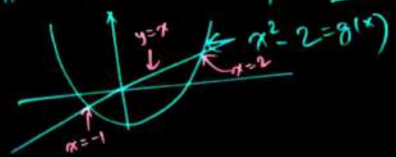
$\to g(x) = x + 7f(x)$

(2)

Root finding problem $\Leftrightarrow$ fixed pt. iteration

may be more convenient & easy to solve !

eg. $g(x) = x^2 - 2$ has 2 fixed pts $x = -1$ & $x = 2$ b/n -2 & 3



$Th^m (2.2)$ (i) If $g(x) \in C[a,b]$ s.t. $g(x) : [a,b] \longrightarrow [a,b]$, then $g(x)$ has a fixed pt. in $[a,b]$ (3)

(ii) Additionally, if $g'(x)$ exists on (a,b) & $\exists$ a constant $k < 1$ w/ $|g'(x)| \le k$ $\forall x \in (a,b)$, then the fixed pt. in $[a,b]$ is unique!

Proof: (i) If $g(a) = a$ or $g(b) = b$; then $g(x)$ has a fixed pt at an end pt. If not, $g(a) > a$ & $g(b) < b$

$p = g(p)$

Define $h(x) = g(x) - x$ on $[a,b]$ which is continuous w/

$h(a) = g(a) - a > 0$ and $h(b) = g(b) - b < 0$

invoke IVT : (4)

$\exists\ p \in (a,b)$ s.t.

$h(p) = g(p) - p = 0 \Rightarrow g(p) = p$. #

(ii) Now,

$|g'(x)| \le k < 1$

$\left.\begin{array}{l} g(p) = p \\ g(q) = q \end{array}\right\}$ p & q distinct (say) in $[a,b]$

MVT : $\exists\ \xi \in (p,q) \subset [a,b]$ s.t.

$$g'(\xi) = \frac{g(p) - g(q)}{p - q}$$

Contradiction

$\Rightarrow |p-q| = |g'(\xi)|\,|p-q| \le k|p-q| < |p-q|$ $\therefore p = q$