Euler's Method

The object of Euler's method is to obtain approximations to the well-posed initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

A continuous approximation to the solution $y(t)$ will not be obtained; instead, approximations to y will be generated at various values, called **mesh points**, in the interval $[a, b]$.

We first make the stipulation that the mesh points are equally distributed throughout the interval $[a, b]$. This condition is ensured by choosing a positive integer N and selecting the mesh points

$$t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \ldots, N.$$

The common distance between the points $h = (b - a)/N = t_{i+1} - t_i$ is called the **step size**.

Taylor's expansion of y about t(i)

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i),$$

for some number ξ_i in (t_i, t_{i+1}). Because $h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i),$$

and, because $y(t)$ satisfies the differential equation (5.6),

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i).$$

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \ldots, N$, by deleting the remainder term. Thus Euler's method is

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1. \end{aligned}$$

Ques) Find approximate solution to the following ODE by Euler's method

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$$

at $t = 2$. Here we will simply illustrate the steps in the technique when we have $h = 0.5$.

Solution:-

For this problem $f(t, y) = y - t^2 + 1$, so

$$w_0 = y(0) = 0.5;$$

$$w_1 = w_0 + 0.5\left(w_0 - (0.0)^2 + 1\right) = 0.5 + 0.5(1.5) = 1.25;$$

$$w_2 = w_1 + 0.5\left(w_1 - (0.5)^2 + 1\right) = 1.25 + 0.5(2.0) = 2.25;$$

$$w_3 = w_2 + 0.5\left(w_2 - (1.0)^2 + 1\right) = 2.25 + 0.5(2.25) = 3.375;$$

and

$$y(2) \approx w_4 = w_3 + 0.5\left(w_3 - (1.5)^2 + 1\right) = 3.375 + 0.5(2.125) = 4.4375.$$

Ques: Euler's method was used in the first illustration with $h = 0.5$ to approximate the solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Use Algorithm with $N = 10$ to determine approximations, and compare these with the exact values given by $y(t) = (t + 1)^2 - 0.5e^t$.

Solution With $N = 10$ we have $h = 0.2$, $t_i = 0.2i$, $w_0 = 0.5$, and

$$w_{i+1} = w_i + h(w_i - t_i^2 + 1) = w_i + 0.2[w_i - 0.04i^2 + 1] = 1.2w_i - 0.008i^2 + 0.2,$$

for $i = 0, 1, \ldots, 9$. So

$$w_1 = 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8; \quad w_2 = 1.2(0.8) - 0.008(1)^2 + 0.2 = 1.152;$$

and so on. Table shows the comparison between the approximate values at t_i and the actual values.

Table

t_i	w_i	$y_i = y(t_i)$	$\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

Higher-order Taylor Methods

Suppose the solution $y(t)$ to the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

has $(n+1)$ continuous derivatives. If we expand the solution, $y(t)$, in terms of its nth Taylor polynomial about t_i and evaluate at t_{i+1}, we obtain

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i),$$

for some ξ_i in (t_i, t_{i+1}).

Successive differentiation of the solution, $y(t)$, gives

$$y'(t) = f(t, y(t)), \quad y''(t) = f'(t, y(t)), \qquad \text{and, generally,} \quad y^{(k)}(t) = f^{(k-1)}(t, y(t)).$$

Substituting these results into Eq. (5.15) gives

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}f'(t_i, y(t_i)) + \cdots$$
$$+ \frac{h^n}{n!}f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)).$$

The difference-equation method corresponding to equation is obtained by deleting the remainder term involving ξ_i.

Taylor method of order n

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1,$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i).$$

Euler's method is Taylor's method of order one.

Example 1 Apply Taylor's method of orders **(a)** two and **(b)** four with $N = 10$ to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solution **(a)** For the method of order two we need the first derivative of $f(t, y(t)) = y(t) - t^2 + 1$ with respect to the variable t. Because $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t,$$

so

$$T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) = w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i)$$

$$= \left(1 + \frac{h}{2}\right)(w_i - t_i^2 + 1) - ht_i$$

Because $N = 10$ we have $h = 0.2$, and $t_i = 0.2i$ for each $i = 1, 2, \ldots, 10$. Thus the second-order method becomes

$$w_0 = 0.5,$$

$$w_{i+1} = w_i + h\left[\left(1 + \frac{h}{2}\right)(w_i - t_i^2 + 1) - ht_i\right]$$

$$= w_i + 0.2\left[\left(1 + \frac{0.2}{2}\right)(w_i - 0.04i^2 + 1) - 0.04i\right]$$

$$= 1.22w_i - 0.0088i^2 - 0.008i + 0.22.$$

The first two steps give the approximations

$$y(0.2) \approx w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.22 = 0.83;$$

$$y(0.4) \approx w_2 = 1.22(0.83) - 0.0088(0.2)^2 - 0.008(0.2) + 0.22 = 1.2158$$

All the approximations and their errors are shown in Table

Table

t_i	Taylor Order 2 w_i	Error $\lvert y(t_i) - w_i \rvert$
0.0	0.500000	0
0.2	0.830000	0.000701
0.4	1.215800	0.001712
0.6	1.652076	0.003135
0.8	2.132333	0.005103
1.0	2.648646	0.007787
1.2	3.191348	0.011407
1.4	3.748645	0.016245
1.6	4.306146	0.022663
1.8	4.846299	0.031122
2.0	5.347684	0.042212

(b) For Taylor's method of order four we need the first three derivatives of $f(t, y(t))$ with respect to t. Again using $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = y - t^2 + 1 - 2t,$$

$$f''(t, y(t)) = \frac{d}{dt}(y - t^2 + 1 - 2t) = y' - 2t - 2$$

$$= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1,$$

and

$$f'''(t, y(t)) = \frac{d}{dt}(y - t^2 - 2t - 1) = y' - 2t - 2 = y - t^2 - 2t - 1,$$

so

$$\begin{aligned}
T^{(4)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \frac{h^2}{6}f''(t_i, w_i) + \frac{h^3}{24}f'''(t_i, w_i) \\
&= w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i) + \frac{h^2}{6}(w_i - t_i^2 - 2t_i - 1) \\
&\quad + \frac{h^3}{24}(w_i - t_i^2 - 2t_i - 1) \\
&= \left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(w_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)(ht_i) \\
&\quad + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}.
\end{aligned}$$

Hence Taylor's method of order four is

$$\begin{aligned}
w_0 &= 0.5, \\
w_{i+1} &= w_i + h\Bigg[\left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(w_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)ht_i \\
&\quad + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}\Bigg],
\end{aligned}$$

for $i = 0, 1, \ldots, N - 1$.

Because $N = 10$ and $h = 0.2$ the method becomes

$$\begin{aligned}
w_{i+1} &= w_i + 0.2\Bigg[\left(1 + \frac{0.2}{2} + \frac{0.04}{6} + \frac{0.008}{24}\right)(w_i - 0.04i^2) \\
&\quad - \left(1 + \frac{0.2}{3} + \frac{0.04}{12}\right)(0.04i) + 1 + \frac{0.2}{2} - \frac{0.04}{6} - \frac{0.008}{24}\Bigg] \\
&= 1.2214w_i - 0.008856i^2 - 0.00856i + 0.2186,
\end{aligned}$$

for each $i = 0, 1, \ldots, 9$. The first two steps give the approximations

$$y(0.2) \approx w_1 = 1.2214(0.5) - 0.008856(0)^2 - 0.00856(0) + 0.2186 = 0.8293;$$

$$y(0.4) \approx w_2 = 1.2214(0.8293) - 0.008856(0.2)^2 - 0.00856(0.2) + 0.2186 = 1.214091$$

Table

t_i	Taylor Order 4 w_i	Error $\lvert y(t_i) - w_i \rvert$
0.0	0.500000	0
0.2	0.829300	0.000001
0.4	1.214091	0.000003
0.6	1.648947	0.000006
0.8	2.127240	0.000010
1.0	2.640874	0.000015
1.2	3.179964	0.000023
1.4	3.732432	0.000032
1.6	4.283529	0.000045
1.8	4.815238	0.000062
2.0	5.305555	0.000083

The results from Table indicate the Taylor's method of order 4 results are quite accurate at the nodes 0.2, 0.4, etc. But suppose we need to determine an approximation to an intermediate point in the table, for example, at $t = 1.25$. If we use linear interpolation on the Taylor method of order four approximations at $t = 1.2$ and $t = 1.4$, we have

$$y(1.25) \approx \left(\frac{1.25 - 1.4}{1.2 - 1.4}\right)3.1799640 + \left(\frac{1.25 - 1.2}{1.4 - 1.2}\right)3.7324321 = 3.3180810.$$

The true value is y(1.25)=3.3173285, error = 0.0007525!