

31/1/2022

Agenda items

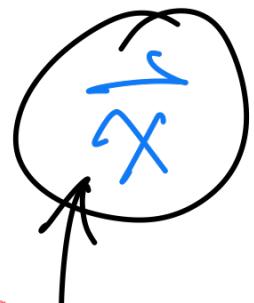
Module 2
Lec. Set 2

- Eigenvalues (evs) and Eigenvectors (EVs) of a matrix
- Meaning of evs and EVs
- Diagonalizable matrices & Similarity transformations
- Analytical (pen-paper) method of finding evs.
- Computational method of finding evs. of a matrix (power method, etc)

Defⁿ(evs and EVs) :

may be \mathbb{R}
or \mathbb{C}

Let $A \in M_{n \times n}(\mathbb{F})$



is an EV of A if

$$\lambda \vec{x} \in \mathbb{F}^n \quad A\vec{x} = \lambda \vec{x}$$

λ cannot
be 0!

Constant (either \mathbb{R}
or \mathbb{C})

\vec{x} is an ev of A associated
w/ the EV λ !

Meaning of the eq. $A\vec{x} = \lambda\vec{x}$

(i) Algebraic meaning

$A\vec{x} = \lambda\vec{x}$ can also be written as

$$(A - \lambda I)\vec{x} = \vec{0}$$

recall we
had left out 0
from the def' of
EV!

i.e. the $(EV\vec{x} + \vec{0})$ form
the null sp. of the
matrix $\underline{(A - \lambda I)}$

Why?

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\lambda\vec{x}$

$\lambda I\vec{x}$

this is a subspace of $(A - \lambda I)$
that has a spl. name
"EIGENSPACE" of λ w.r.t. A !

in search of meaning of $A\vec{x} = \lambda \vec{x}$ - - -

Consider an example

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$$

$$(A - \lambda I) = \begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}$$

Solving $A\vec{x} = \lambda \vec{x}$ is equivalent to solving
the system of linear eqs.

b/c EV can't be 0!

$$\begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} (2-\lambda)x_1 - x_2 = 0 \\ 2x_1 + (4-\lambda)x_2 = 0 \end{cases}$$

When does this System of linear
eqns. have a non-trivial soln. ??

Ans:-

$$\text{Ker}(A - \lambda I) \neq \{\vec{0}\}$$

Characteristic eqn.

this is true only if

$(A - \lambda I)$ is non-invertible

i.e. $\det(A - \lambda I) = |A - \lambda I| = 0$

i.e. $\begin{vmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = 3+i \\ \lambda_2 = 3-i \end{cases}$

EV w.r.t. λ_1 : $x_1 = k(-1+i)/2$
Some arb. const. $x_2 = k$

Various features of an invertible matrix

$B \in M_{n \times n}$

- ✓ 1) B is invertible
- 2) $B\vec{x} = \vec{b}$ has a unique soln. $\vec{x} \neq \vec{0} \in \mathbb{R}^n$
- 3) $\text{rref}(B) = I_n$
- 4) $\text{rank}(B) = n$
- 5) $\text{im}(B) = \mathbb{R}^n$
- ✓ 6) $\text{Ker}(B) = \{\vec{0}\}$

all the above statements are equivalent

A slight digression - - -

Q) Why $\text{null}(B) = \{0\} \Leftrightarrow B$ is ~~not~~ invertible?
finite dim. vector sp.

Ans) $T: U \rightarrow V$ is invertible if
and only if T is one to one
& onto

$$\Rightarrow \dim(U) = \dim(V)$$

Rank nullity thm : $\text{null}(T) + \underbrace{\text{rank}(T)}_{\dim(V)} = \dim(U)$

$$\{0\}$$

#

by inspecting
im & ker
of T

HW/exercise problem

(Q) Consider $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

- i) Find the characteristic polynomial.
- ii) Find the EVs of A.
- iii) Find the EVs of A.

Ans :- EVs : $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

ii) Geometrical meaning of

$$A\vec{x} = \lambda \vec{x} \dots$$

when λ is real . . .

$A\vec{x}$ is parallel to \vec{x}

i.e. the "EV" \vec{x} either gets stretched or compressed longitudinally when acted upon by the matrix

A.

Coming Soon !

- * Diagonalizable matrices
- * Similarity transformation
- * Application of evs & EVs in
Solutions to ODEs