# Part-(II) :- Infinite Series

## Ch (8) Thomas & Finney.

(1) **Def^n**

**Infinite Seq** :- An $\infty$ seq. of no.s is a $f^n$ whose Domain is the set of integers $\geq$ some integer $n_0$.
(usually $n_0 = 0$, or 1).

**eg**: $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$ $\quad a_n = (-1)^{n+1}\frac{1}{n}$

$55, 55, 55, \ldots$ $\quad a_n = 55$

seq. $\{(-1)^{n+1}\frac{1}{n}\}$

seq. $\{55\}$

(2) Convergence (& Div.) of seq.

**Def^n** :- $\{a_n\} \longrightarrow L$ (Conv. to L) if for every $\epsilon > 0$, $\exists$ $N \in I$ s.t. $\forall n > N$; we have

$$|a_n - L| < \epsilon.$$

if no such L exists; then $\{a_n\}$ diverges.

$\{a_n\} \to L \equiv a_n \to L \equiv \lim_{n \to \infty} a_n = L$

**eg.** $a_n = \frac{1}{n}$ (Conv.)

(2.1) Choose $\epsilon > 0$ (fix); we need to find N s.t. $\forall n > N$; $\left|\frac{1}{n} - L\right| < \epsilon$ (guess $L = 0$).

$\left|\frac{1}{n}\right| < \epsilon$ or $-\epsilon < \frac{1}{n} < \epsilon$

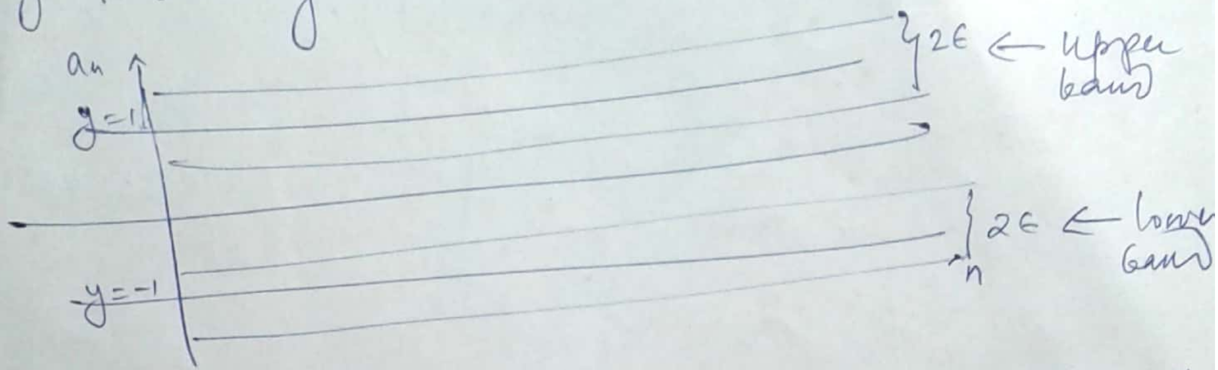
$n > \frac{1}{\epsilon}$

Now, Choose $N = \frac{1}{\epsilon}$ & we will have $\forall n > N$ $|1/n - 0| < \epsilon$.

eg (2.2) Analyse the conv. (or div.) of

$$\left\{ (-1)^{n+1}\left(\frac{n-1}{n}\right) \right\} = \left\{ (-1)^{n+1}\left(1-\frac{1}{n}\right) \right\} = \{a_n\}$$

Choose $\epsilon < 1$ so that the band around $y=1$ and $y=-1$ donot overlap.




potentially, $\because \left(1-\frac{1}{n}\right) \to 1$; the sequence $\{a_n\} \to 1$ or $-1$ depending on $(-1)^{n+1}$

but as soon as for some $n > N$, $\{n, a_n\}$ is trapped in the upper (lower) band; the subsequence $\{n+1, a_{n+1}\}$ will be trapped in the lower (upper) band $\forall n \geq N$.

$\therefore$ $\{a_n\}$ does not converge.

(3)

**Defn (Subsequence)**

If the terms $\{a_{n_i}\}$ appear in another sequence $\{a_n\}$ then $\{a_{n_i}\}$ is a subsequence of $\{a_n\}$.

eg. $\{a_n\} = \left\{(-1)^n \frac{1}{n}\right\}_{n\geq 1} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots\right\}$

$\{a_{n_i}\} = \left\{(-1)^{2n}\frac{1}{2n}\right\}_{n\geq 1} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right\}$

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## (3.1) Properties of Subsequence vis-à-vis Sequence

(1) If $\{a_n\} \to L \Rightarrow \{a_{n_i}\} \to L$

* Sometimes it may be easier to find the limit $L$ by using $a_{n_i}$ if we know $\{a_n\}$ converges.

(2) If $\{a_{n_k}\}$ diverges, or 2 diff subseq. $\{a_{n_k}\}$ & $\{a_{n_j}\}$ conv. to diff. limits then $\{a_n\}$ diverges.

## (4) Bounds of Seq.

(4.1) **Def$^n$**:- (non-dec seq)

$\{a_n\}$ s.t. $a_n \leq a_{n+1}$ $\forall n$ is called a non-dec seq.

eg: $\left\{\frac{n}{n+1}\right\}$ b/c $\frac{n}{n+1} < \frac{n+1}{n+2}$

$(n+1)^2 < n^2+2n$ ✓

$n^2+2n+1 < n^2+2n$

(4.2) **Def$^n$**

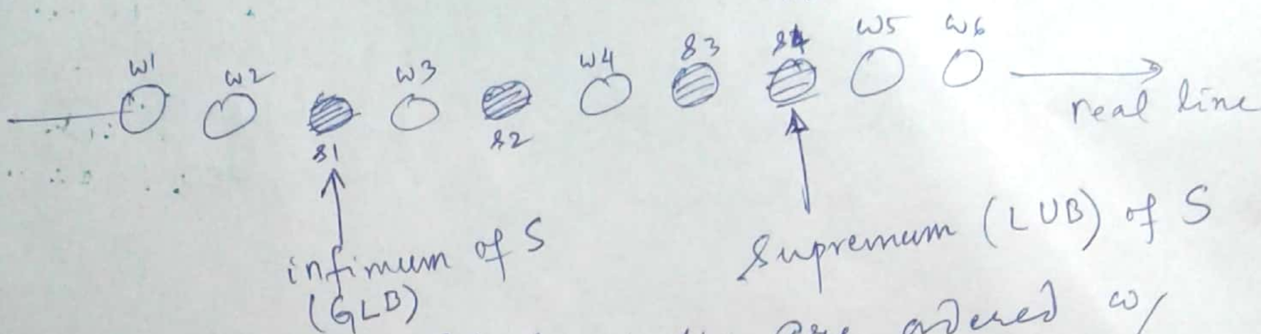
(1) **Upper bound (UB)**

$M$ is an UB of $\{a_n\}$ if $a_n \leq M$ $\forall n$

## (11) Supremum or LUB (Least UB).

M is the supremum of $\{a_n\}$ if

$a_n \leq M \; \forall n$ and $\nexists$ no $K < M$ s.t.
$a_n \leq K \; \forall n$.




from left to right the balls are ordered w/ inc $x \in \mathbb{R}$

$S = \{s_1, s_2, s_3, s_4\}$

$T = \{w_1, w_2, w_3, w_4, w_5, w_6, s_1, s_2, s_3, s_4\}$

$S \subset T$

$s_4$ is $\sup(S)$

$s_1$ is $\inf(S)$

* for finite totally ordered sets ($\inf \equiv \min$).

### Likewise Sup is not always max.

max may never exist for a set

eg $A = \{x \mid x \in \mathbb{R}^-\} = \{x \mid x \in \mathbb{R}, x < 0\}$

A does not have a max; b/c for every negative real chosen (say $x$); $x/2$ is greater than $x$ ($x \in A$).

But $0 \notin A$ is still the supremum of A.

* Sup need not always belong to the set if it does it is the max of the set.

(iii) If $\{a_n\}$ is non-dec & $a_n$ is bdd from above $\Rightarrow \{a_n\}$ always has a sup. || will not prove.

Def: Monotonic Seq: It is a sequence which is either increasing / decreasing

(5) (Thm) :-

Non-Dec. sequence Thm

A non-decreasing seq. of real no.s bdd from above converges iff ($\Leftrightarrow$) it is bdd from above. If a non-decreasing seq converges; it converges to its LUB/Sup.

**Monotonic Seq thm :-** Every bdd, monotonic sequence is convergent!

eg (5.1) Determine if the seq. is non-decreasing & bdd from above.

(1) $a_n = \frac{3n+1}{n+1}$

non decreasing $\Rightarrow a_{n+1} \geq a_n \ \forall n$

$$\Rightarrow \frac{3(n+1)+1}{n+2} \geq \frac{3n+1}{n+1}$$

$$\Rightarrow \frac{3n+4}{n+2} \geq \frac{3n+1}{n+1}$$

$$\Rightarrow (n+1)(3n+4) \geq (3n+1)(n+2)$$

$$\Rightarrow 3n^2+7n+4 \geq 3n^2+7n+2$$

$$\Rightarrow 4 \geq 2 \text{ always true}$$

Also $\frac{3n+1}{n+1} = \frac{n}{n+1} + \frac{2n+1}{n+1}$

$= \frac{n}{n+1} + \frac{n}{n+1} + \frac{n+1}{n+1}$

$= 1 + 2\frac{n}{n+1}$

$< 1+2 = 3$ (b/c $\frac{n}{n+1} < 1$)

$\therefore$ 3 is an upper bd.

eg(5.2) find if the sequence converges / diverges.

$$a_n = \{(-1)^n + 1\}\left\{\frac{n+1}{n}\right\}$$

$$a_n = \begin{cases} 0 \; ; & n \in \text{odd} \\ 2\left(\frac{n+1}{n}\right) \; ; & n \in \text{even} \\ \to 2 & \end{cases}$$

$\therefore \{a_n\}$ diverges.

⑥ thm's for Calculating limits of sequence.

(6.1) thm :- Let $\{a_n\}$ & $\{b_n\}$ be seq. of real no.s
Also, $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$; $A, B \in \mathbb{R}$

then
(i) $(a_n + b_n) \xrightarrow{n\to\infty} A+B$
(ii) $(a_n - b_n) \longrightarrow A-B$
(iii) $a_n \cdot b_n \longrightarrow A \cdot B$
(iv) $k \cdot b_n \longrightarrow kB$ ($k$ const).
(v) $\frac{a_n}{b_n} \longrightarrow A/B$

(6.2) Sandwich thm for sequences.

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of real no.s.
If $a_n \leq b_n \leq c_n \; \forall n > N$ and if $a_n \to L$, $c_n \to L$
then $b_n \to L$ as $n \to \infty$.

Corollary :- If $|b_n| \leq c_n$ & $c_n \to 0$
then $b_n \to 0$ b/c $-c_n \leq b_n \leq c_n$.

eg. b/c $\left|\frac{\cos n}{n}\right| = \frac{|\cos n|}{n} \leq \frac{1}{n} \to 0 \Rightarrow \frac{\cos n}{n} \to 0$

(6.3) Thm :- (Continuous $f^n$. Thm for sequences)

Let $\{a_n\} \in \mathbb{R}$ & if $a_n \to L$ and $f$ is continuous at $L$ & defined $\forall a_n$;

then $f(a_n) \to f(L)$.

eg: Consider $\sqrt{\frac{n+1}{n}}$

We know $a_n = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1$

Consider $f(x) = \sqrt{x} = x^{1/2}$ is continuous at $x = 1$ (by inspec)

$\therefore f(a_n) = f\left(\frac{n+1}{n}\right)$

$= \sqrt{\frac{n+1}{n}} \to f(1) = 1$

(6.4) Thm :- Let $f(x)$ be defined $\forall x \geq n_0$; and $\{a_n\} \in \mathbb{R}$ s.t. $a_n = f(n)$ $\forall n \geq n_0$

then; $f(x) \xrightarrow{x \to \infty} L \Rightarrow a_n \xrightarrow{n \to \infty} L$

eg Show $\lim_{n \to \infty} \frac{\log n}{n} = 0$

$f(x) = \frac{\log x}{x}$ $\forall x \geq 1$

$\lim_{x \to \infty} f(x) \overset{\text{l'hopitals}}{=} \frac{1/x}{1} = \frac{1}{x} = 0$.

$a_n = \frac{\log n}{n}$ $\forall n \geq 1$

$= f(n)$

$\therefore a_n \to 0$ by thm (6.4).

## (6.5) Some well known limits

i) $\lim_{n \to \infty} \frac{\log n}{n} = 0$.

ii) $\lim_{n \to \infty} \sqrt[n]{n} = 1$ (why take $y = x^{1/x}$, $\log y = \frac{\log x}{x} \xrightarrow{x \to \infty} 0$ $\Rightarrow \log(y) \to 0$ whence $y \to 1$).

iii) $\lim_{n \to \infty} x^{1/n} = 1$ $(x > 0)$

iv) $\lim_{n \to \infty} x^n = 0$ $(|x| < 1)$

v) $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ (Any $x$)

vi) $\lim_{n \to \infty} \frac{x^n}{n!} = 0$ (Any $x$).

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eg. (6.6) Do the following seq. $\{a_n\}$ conv./diverge?

i) $a_n = \frac{n^2 - 2n + 1}{n - 1} = n - 1 \longrightarrow \infty$ as $n \to \infty$

$\therefore$ Div.

ii) $a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right) \to 6$ (using iv above).

$\therefore$ Conv.

# (7) Infinite Series.

Can we write a summation series

like $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

as a sequence, $s_n$

Let's try :-

Partial Sum

| | $s_n$ |
|---|---|
| $s_1 = 1$ | $2 - 1$ |
| $s_2 = 1 + \frac{1}{2}$ | $2 - \frac{1}{2}$ |
| $s_3 = 1 + \frac{1}{2} + \frac{1}{4}$ | $2 - \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4}$ |
| $\vdots$ | |
| $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$ | $2 - \frac{1}{2^{n-1}}$ |

Note $s_n = 2 - \frac{1}{2^{n-1}} \longrightarrow 2$ as $n \to \infty$ b/c $\frac{1}{2^n} = \left(\frac{1}{2}\right)^n \to 0$

$\therefore$ The sum of the above $\infty$ series is 2

(7.1) Def$^n$ :- Let $a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$

be an $\infty$ series.

Then the sequence $\{s_n\}$ s.t.

$s_1 = a_1$

$s_2 = a_1 + a_2$

$\vdots$

$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_n$

is the seq. of partial sum; w/ $s_n$ being the $n^{th}$ partial sum.

(7.2) Def$^n$ :-

→ If $s_n \to L \Rightarrow \sum_{n=1}^{\infty} a_n = L$

(i.e. $\{s_n\}$ conv. to $L$)

→ If $s_n$ does not conv. $\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent

eg (7.3) Geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

$$s_n := a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + \cdots + ar^n$$

$$-$$

$$s_n(1-r) = a - ar^n = a(1-r^n)$$

$$s_n = \frac{a(1-r^n)}{(1-r)} \longrightarrow \frac{a}{1-r} \quad \left(\begin{array}{l} r^n \to 0 \\ |r|<1 \\ n \to \infty \end{array}\right).$$

$\longrightarrow$ div. if $|r| \geq 1$

Note :-

If $r = 1$: $s_n = a + a(1) + a(1)^2 + \cdots + a(1^{n-1})$
$= na \to \infty$

$r = -1$: $s_n = \begin{cases} a ; & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$

∴ the geometric series $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ ; $|r|<1$

eg (7.4) You drop a ball from 6 m above a flat surface. Each time the ball hits the surface after falling $h$ meters; it rebounds to a height $\left(\frac{2}{3}h\right)$. Find the total distance the ball travels up & dn.

Soln:-


Always draw a pic for word problems when possible!

$$S = 6 + 2\left\{\left(\frac{2}{3}\right)\times 6 + 6\times\left(\frac{2}{3}\right)^2 + \cdots\right\}$$

$$= 6 + 2\times\frac{2}{3}\left\{\underbrace{6 + 6\left(\frac{2}{3}\right) + \cdots}_{a + ar + \cdots}\right\}$$

$$= 6 + \frac{4}{3}\left\{\frac{6}{1-\frac{2}{3}}\right\}$$

$$= 6 + \frac{8}{\frac{1}{3}} = 30 \text{ m}.$$

## (7.5) Telescoping Series.

Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$ax^3+bx^2+cx+d=0$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) + \cdots$$

$S_k = 1 - \frac{1}{k+1}$ (all other terms cancel out).

$S_k \to 1$ as $k \to \infty$; so the series conv. to 1.

i.e. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

## (8) Thm (Series convergences)

e.g. harmonic series

(i) If $\sum_{n=1}^{\infty} a_n$ converges; then $a_n \to 0$ (Converse ~~reverse~~ is NOT true)

(ii) If $\lim_{n\to\infty} a_n$ { fails to exist or different from 0 }; then $\sum_{n=1}^{\infty} a_n$ diverges.

(9) Just like the results for limits; we have for series

$\sum a_n = A$ & $\sum b_n = B$; then

i) $\sum(a_n \pm b_n) = A \pm B$

ii) $\sum k a_n = kA$; $k$ const.

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# More examples on Sequences & Series (Convergence)



**eg ①** Recall the th$^m$: If $a_n \to L$ & $f(x)$ is continuous at $L$; then $f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$

Use it to find $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$.

Soln:- $\because$ $\sin(x)$ is continuous everywhere & $\frac{1}{n} \to 0$

we have $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin(0)$
$= 0$ #

**eg ②** Application of Sandwich th$^m$

Discuss the convergence of the sequence

$a_n = \frac{n!}{n^n}$ where $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$

Soln:- Both numerator & denominator approach $\infty$ as $n \to \infty$ but here we have no corresponding f$^n$ for use w/ L'hopital's rule (& the th$^m$ in eg ①) b/c $x!$ is not defined for $x \notin I$.

So lets write down a few terms:

$a_1 = 1$

$a_2 = \frac{1 \cdot 2}{2 \cdot 2}$

$a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdots n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n}\right)$$
$$\leq \frac{1}{n} \cdot 1$$ b/c stuff w/in ( ) is $\leq 1$

$\therefore$ $0 < a_n \leq \frac{1}{n}$

Apply sandwich th$^m$ to conclude $a_n \to 0$. #

eg ③ Application of "Non-decreasing sequence th^m / Montonic Sequence th^m".

Q) Investigate the sequence $\{a_n\}$ defined by the recurrence relation

$$a_1 = 2, \; a_{n+1} = \frac{1}{2}(a_n + 6), \quad n = 1, 2, \ldots$$

Soln:- Let us first list a few terms to begin with to get a feel for the seq.

$a_1 = 2$
$a_2 = \frac{1}{2}(2+6) = 4$
$a_3 = 5$
$a_4 = 5.5$
$a_5 = 5.75$
$a_6 = 5.875$
$a_7 = 5.9375$

Seems like $a_n \to 6$.
& $a_n \nearrow 6$

to claim this we need to show (1) $a_n \nearrow$
(2) $a_n$ is bdd.

(1) $a_n$ is $\nearrow$ (use mathematical induction)

$n = 1$ ; $a_2 = 4 > a_1 = 2$ (true)

$n = k$ ; $a_{k+1} > a_k$ (assume)

$n = k+1$ :- to be shown that $a_{k+2} > a_{k+1}$

$\because a_{k+1} > a_k \Rightarrow a_{k+1} + 6 > a_k + 6$

$\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$

$a_{k+2} > a_{k+1}$

$\Rightarrow$ for $n = k+1$ ; $a_k$ is $\nearrow$ . ✓

(2) <u>$a_k$ is bdd.</u>

We know $a_k$ is $\nearrow$ & $a_1 = 2$

So clearly $a_n$ is bdd from below!

<u>bdd from above.</u>

i) $n=1$ $\quad a_1 < 6$ is true.

ii) $n=k$ $\quad a_k < 6$ (assume)

iii) $n=k+1$ $\quad a_k + 6 < 12$

$\frac{1}{2}(a_k+6) < \frac{1}{2}(12) = 6$.

$a_{k+1} < 6$

$\Rightarrow a_k < 6 \ \forall k$.

(1) & (2) $\Rightarrow$ According to Monotone Seq thm that ~~the~~ seq. converges to its limit.

Now we need to find that limit, L?

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n+6)$$

$$= \frac{1}{2}\left(\lim_{n\to\infty} a_n + 6\right)$$

$$= \frac{1}{2}(L+6)$$

$\Rightarrow 2L = L + 6 \Rightarrow L = 6$ (as guessed) #.

<u>eg(4)</u>. $\lim_{n\to\infty} \tan h\, n = ?$

$\tan h x$ is continuous & defined $\forall x$

& $\lim_{x\to\infty} \tan h x = 1 \Rightarrow$ thm (6.4) in notes that $\lim_{n\to\infty} \tan h n = 1$. #

eg ⑤ w/ any series $\sum a_n$, we associate two sequences:-

(i) $\{s_n\}$ ; i.e. the sequence of its partial sums.

(ii) $\{a_n\}$ ; seq. of its terms

Q) Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges

Soln:- $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{5n^2+4} = \lim_{n \to \infty} \frac{1}{5 + 4/n^2}$

$= \frac{1}{5} \neq 0$

$\Rightarrow$ thm on series convergence that $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges!

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NOTE:-

(i) If $\sum a_n$ is convergent; then the limit of $s_n \to s$ is the sum of the series

(ii) If $\sum a_n$ converges then $a_n \to 0$.
(Converse Not true eg harmonic series)

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