

7. Definition (Linear transformations):

A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a *linear transformation* if $\exists A \in \mathbb{M}_{m \times n}(\mathbb{R})$ such that $T(\mathbf{x}) = \mathbf{Ax}$, $\forall \mathbf{x} \in \mathbb{R}^n$.

e.g. The rotation matrix $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ is a linear transformation which rotates a vector in \mathbb{R}^2 by θ .

Ques: Given $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, how do we find A ?

Ans: $A = \begin{pmatrix} | & | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & & | \end{pmatrix}$ where \mathbf{e}_i is the i^{th} standard basis element of \mathbb{R}^n .

A square matrix is *invertible* if its linear transformation is invertible.

Theorem: A $n \times n$ matrix A is invertible $\iff rref(A) = I_n \equiv \text{rank}(A) = n$.

Finding inverse of a matrix: $A \in \mathbb{M}_{n \times n}(\mathbb{R})$. In order to find A^{-1} , form the augmented matrix $\tilde{A} = (A \quad | \quad I_n)$ and compute $rref(\tilde{A})$.

- If $rref(\tilde{A})$ is of the form $(I_n \quad | \quad B)$, then $A^{-1} = B$.
- If $rref(\tilde{A})$ is of another form, then A is not invertible.

$$(AB)^{-1} = B^{-1}A^{-1}.$$

8. Definition (Image or range of a matrix/linear transformation):

$Im(A) = Im(T)$ is the *span* of the column vectors of A .

Q) Find a basis of the image of $A = \begin{pmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{pmatrix} = \begin{pmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{pmatrix}$ and determine $\dim(Im(A))$.

Ans) To find the basis of $Im(A)$, we need to identify the redundant columns of A from amongst all the column vectors of A . By inspection of A , it will be hard to tell which of the columns of A are redundant (linearly dependent on the others). So we will transform A to $B = rref(A)$.

$$B = rref(A) = \begin{pmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} | & | & | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \\ | & | & | & | & | \end{pmatrix}.$$

The redundant columns of B correspond to the redundant columns of A . The redundant columns of B are also easy to spot: *They are the columns that do not contain a leading 1*, namely, $\mathbf{b}_2 = 2\mathbf{b}_1$, $\mathbf{b}_4 = 3\mathbf{b}_1 - 4\mathbf{b}_3$, and $\mathbf{b}_5 = -4\mathbf{b}_1 + 5\mathbf{b}_3$. Thus the redundant columns of A are $\mathbf{a}_2 = 2\mathbf{a}_1$, $\mathbf{a}_4 = 3\mathbf{a}_1 - 4\mathbf{a}_3$, and $\mathbf{a}_5 = -4\mathbf{a}_1 + 5\mathbf{a}_3$. And the non-redundant columns of A are \mathbf{a}_1 and \mathbf{a}_3 , they form a basis of image of A . Therefore, a basis of image of A is

$$\begin{pmatrix} 1 \\ -1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \\ 1 \end{pmatrix}$$

$$\dim(Im(A)) = 2.$$

9. **Definition (Kernel of T (or equivalently the null space of A , $\text{Null}(A)$):** The set of all $x \in \mathbb{R}^n$ s.t. $T(x) = Ax = \mathbf{0}$.

Q) Find a basis of the kernel of A (equivalently, $\text{Null}(A)$) and determine $\dim(\text{Ker}(A)) = \dim(\text{null}(A))$.

Ans) Most importantly $\text{Ker}(A) = \text{Ker}(\text{rref}(A)) = \text{Ker}(B)$. So we might as well solve for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ s.t. $B\mathbf{x} = \mathbf{0}$. This is done by considering the augmented matrix $\tilde{B} = (B \mid \mathbf{0})$ from which we have the following:

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + 3x_4 - 4x_5 &= 0 \\ 0x_1 + 0x_2 + x_3 - 4x_4 + 5x_5 &= 0 \end{aligned}$$

or equivalently,

$$\begin{aligned} x_1 &= -2x_2 - 3x_4 + 4x_5 \\ x_3 &= 4x_4 - 5x_5 \end{aligned}$$

whence $x_2 = \alpha$, $x_4 = \beta$, $x_5 = \gamma$ are set arbitrarily. Therefore,

$$\mathbf{x} = \begin{pmatrix} -2\alpha - 3\beta + 4\gamma \\ \alpha \\ 4\beta - 5\gamma \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -2\alpha & -3\beta & +4\gamma \\ \alpha & 4\beta & -5\gamma \\ 4\beta - 5\gamma & \beta & \gamma \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} + \gamma \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}.$$

The $\text{Null}(A)$ is spanned by these basis vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix}$ and $\dim(\text{Null}(A)) = 3$.

Exercise problem: Find the basis for the null space of the matrix $A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 2 & 1 & 3 & 2 \\ 1 & 5 & 3 & -2 \end{pmatrix}$ and determine its dimension?

Answer: $\begin{pmatrix} -4/3 \\ -1/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4/3 \\ 2/3 \\ 0 \\ 1 \end{pmatrix}$ and the dimension of null space of A is 2.

10. **Theorem:** $A \in \mathbb{M}_{m \times n}(\mathbb{R})$. Then $\text{Ker}(A) = \{\mathbf{0}\} \iff \text{rank}(A) = n$.

For a square matrix the statement is true when A is invertible

(cf. remark under point 7 above: When A is invertible, $\text{rref}(A) = I_n \implies$ no. of pivots = $n = \text{rank}(A)$ by def. Further,

$A\mathbf{x} = \mathbf{0}$ can be solved by considering the augmented matrix $\text{rref}(A \mid \mathbf{0}) = (I_n \mid \mathbf{0})$ which gives us

$x_1 = 0, x_2 = 0, x_3 = 0$. which gives $\text{Ker}(A) = \{\mathbf{0}\}$. The converse is obvious.

11. **Theorem (Rank-nullity theorem):** For any $m \times n$ matrix A , the following is known as the *fundamental theorem of linear algebra*:

$$\dim(\text{Null}(A)) + \dim(\text{Im}(A)) = n$$

or equivalently,

$$(\text{nullity of } A) + (\text{rank of } A) = n$$