Motivate the next section w/ the game of Badminton

<u>Markov Chain</u> is a Stochastic process comprising of events happening in a sequence s.t. probability of an event depends solely on the previous state.

Likewise, a Markov process is the one that satisfies the Markov property.

i.e. if $\{X_n\}_{n\geq 0, n\in I}$ describes a sequence of events;

$$P(X_n = x_n \mid X_{n-1}=x_{n-1}, X_{n-2}=x_{n-2}, \ldots, X_0 = x_0)$$
$$= P(X_n = x_n \mid X_{n-1} = x_{n-1})$$

think carefully; the above statement means that the Markov process indexed by $\{X_n\}_{n\geq 0}$ is characterized by a <u>memoryless property</u>.

Markov process is named after Russian mathematician Andrey Markov.

<u>Formal examples</u>:

<u>Example ①</u>: <u>Gambler's Ruin</u>

Consider a gambling game in which on any turn you win Rs 1 w/ probability $p=0.4$ or lose Rs 1 w/ $p = 1-0.4=0.6$

Suppose you adopt a strategy that you quit playing if your fortune reaches Rs 100. If course if your fortune reaches Rs 0; the casino makes you stop.

Model :- I claim that this situation can be appropriately modeled by a Markov process.

Let X_n = amt. of money you have after n plays.

if $X_n \neq 0$

then $P(X_{n+1} = i+1 \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$

$\overset{\text{why?}}{=} P(X_{n+1} = i+1 \mid X_n = i)$

$= 0.4$

Def^n (Discrete time Markov chain)

X_n is a DTMC w/ transition matrix $p(i,j)$ if for any $j, i, i_{n-1}, \dots i_0$

$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots X_0 = i_0) = p(i,j)$

independent of time "n".

for gambler's ruin

$p(i, i+1) = 0.4$; $p(i, i-1) = 0.6$; $0 < i < 100$

$p(0,0) = 1$; $p(100, 100) = 1$

$$
p(i,j) = \begin{array}{c|cccccc}
 & 0 & 1 & 2 & \cdots & & 100 \\
\hline
0 & 1 & & & & & \\
1 & 0.6 & 0 & 0.4 & \cdots & & 0 \\
2 & 0 & 0.6 & 0 & \cdots & & 0 \\
\vdots & \vdots & & & & & \\
100 & 0 & \cdots & & & & 1
\end{array}
$$

Multi-step Transition probabilities

$p(i,j)$ is probability of going from $i \to j$ in one step

$p^m(i,j) = P(X_{n+m} = j \mid X_n = i)$; $m > 1$

i.e. prob. of going from $i \to j$ in $m > 1$ steps

eg Social mobility problem

$$
p(i,j) = \begin{array}{c|ccc}
 & 1 & 2 & 3 \\
\hline
1 & 0.7 & 0.2 & 0.1 \\
2 & 0.3 & 0.5 & 0.2 \\
3 & 0.2 & 0.4 & 0.4
\end{array}
$$

$X_n :=$ family's social class in n^{th} generation $= \{1, 2, 3\}$

1 = lower class, 2 = middle class, 3 = upper

Q1) Your parents were middle class. What is the probability that you are in upper class but your children are lower class?

Soln :- We need to find

$$P(X_2 = 1, X_1 = 3 \mid X_0 = 2) \; ??$$

$$\overset{\text{Cond}^n \text{ prob}}{=} \frac{P(X_2 = 1, X_1 = 3, X_0 = 2)}{P(X_0 = 2)}$$

$$= \frac{P(X_2 = 1, X_1 = 3, X_0 = 2)}{P(X_1 = 3, X_0 = 2)} \cdot \frac{P(X_1 = 3, X_0 = 2)}{P(X_0 = 2)}$$

$$\overset{\text{Cond}^n \text{ prob}}{=} P(X_2 = 1 \mid X_1 = 3, X_0 = 2) \, P(X_1 = 3 \mid X_0 = 2)$$

$$\underset{\text{Property}}{\overset{\text{Markov}}{=}} P(X_2 = 1 \mid X_1 = 3) \, P(X_1 = 3 \mid X_0 = 2)$$

$$\overset{\text{Notation}}{=} p(3,1) \, p(2,3)$$

$$= 0.2 \times 0.2 = 0.04$$

Q.2) Given you are lower class; what is the probability that your grandchildren are upper class.

Soln. $p^2(1,3)$ i.e. $(1,3)$ entry in p^2 matrix

$$p^2 = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}$$

$$\begin{array}{r} 0.07 \\ +0.08 \\ \hline 0.15 \end{array}$$

Ans 0.15

$$= \begin{pmatrix} \times & \times & 0.07 + 0.04 + 0.04 \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix} = \begin{pmatrix} \text{———} & 0.15 \\ \text{———} & \\ \text{———} & \end{pmatrix}$$

Chapman - Kolmogorov Eqn :

$$p^{m+n}(i,j) = \sum_k p^m(i,k)\, p^n(k,j)$$

Does this make intuitive sense?

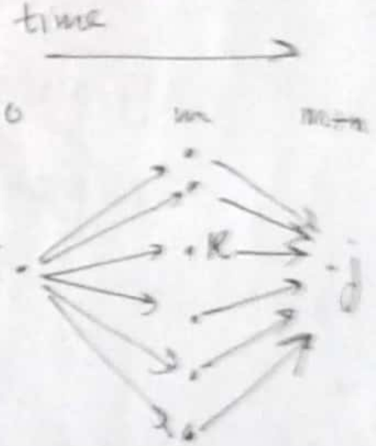
Does this make sense (recall Markov property is time independent of time index)

$$p^{m+n}(i,j) = P(X_{m+n}=j \mid X_0=i)$$

$$= \sum_{k=0}^{\infty} P(X_{m+n}=j, X_m=k \mid X_0=i)$$

Law of total probability

$$= \sum_{k=0}^{\infty} P(X_{m+n}=j \mid X_m=k, X_0=i) \times P(X_m=k \mid X_0=i)$$



$$\overset{\text{Markov}}{\underset{\text{property}}{=}} \sum_{k=0}^{\infty} p^n(k,j)\, p^m(i,k).$$

#

Lecture Notes(2) @ Thapar University

PCL 105 : Statistical methods.

Markov chains contd...

to know the probability distribution at the n^{th} instant of a Markov process, we need

i) Initial probability distribution of the states, &

ii) probability transition matrix.

probability distribution of states

If there are k states $\{s_1, s_2, \dots, s_k\}$

$\vec{\mu}^{(0)} = (\mu_1^{(0)}, \mu_2^{(0)}, \dots, \mu_k^{(0)})$ is the initial distribution;

$= (P(X_0 = s_1), P(X_0 = s_2), \dots, P(X_0 = s_k))$ where the superscript denotes the time index.

Note $\sum_{i=1}^{k} \mu_i^{(0)} = 1$ (b/c of the normalization law of probability).

then, for a Markov chain $(X_0, X_1, \dots)$ w/ state space $\{s_1, \dots, s_k\}$; initial distribution $\vec{\mu}^{(0)}$ & probability transition matrix $\mathbb{P} = (p_{ij})$; the distribution for time $n \geq 0$

$$\vec{\mu}^{(n)} = \vec{\mu}^{(0)} \mathbb{P}^n$$

example (A Simple Weather model)

Consider a simple model that predicts weather on a given day as follows :-

→ the weather stays the same on any given day as the previous day 75% of the time, and

→ 25% of the day it changes.

for simplicity, let us consider that there are only 2 states of the weather s_1 = rainy, s_2 = sunny

Q) What is the long time behavior of the weather distribution given that $\vec{\mu}^{(0)} = (1, 0) = (s_1, s_2)$

Soln :-

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix} \end{array}$$

Let us first calculate a few simple estimates :-

$$\vec{\mu}^{(1)} = \vec{\mu}^{(0)}\mathbb{P} = (1 \quad 0)\begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix} = (0.75 \quad 0.25)$$

$$\vec{\mu}^{(2)} = \vec{\mu}^{(0)}\mathbb{P}^2 = (1 \quad 0)\begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}^2$$

$$= (1 \quad 0)\begin{pmatrix} 0.625 & 0.375 \\ 0.375 & 0.625 \end{pmatrix} = (0.625 \quad 0.375)$$

Note, due to time homogeneity,

$$\vec{\mu}^{(2)} = \vec{\mu}^{(1)}\mathbb{P} = (0.75 \quad 0.25)\begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix} = (0.625 \quad 0.375)$$

Likewise

$$\vec{\mu}^{(n)} = \vec{\mu}^{(n-1)} \mathbb{P}$$
$$= \vec{\mu}^{(0)} \mathbb{P}^n$$

using induction, one can show

$$\vec{\mu}^{(n)} = \left(\tfrac{1}{2}\left(1+\tfrac{1}{2^n}\right), \ \tfrac{1}{2}\left(1-\tfrac{1}{2^n}\right) \right)$$

Long time distribution

$$\vec{\mu}_{eq} = \lim_{n\to\infty} \vec{\mu}^{(n)} = (1/2, 1/2).$$

Check why this makes sense by computing

$$\vec{\mu}_{eq} \mathbb{P} = ?$$

(Later, we will return to stationary distributions!).

example Consider a Markov model of a game of Badminton. For simplicity, let us consider there are only 3 types of shots played by the players, viz {drop (at the net), lift, smash} = {D, L, S}.
We are interested in analyzing a winning Strategy. Depending on a shot played; the return shot by the opponent is as follows:-

Shot	return shot	w/ probability.
Drop	Drop	1/3
Drop	lift	1/3
Drop	Smash	0

Shot	return shot	w/ probability.
lift	drop	1/5
lift	lift	1/5
lift	smash	2/5
smash	lift	2/5
smash	drop	1/5
smash	smash	0

Q1) Identify an appropriate state space.

Q2) Construct a probability transition probability.

Q3) What is the probability of a winning shot given that the final 3 shots in the rally were, respectively

(a) smash, lift, lift

(b) smash, drop, lift

(c) drop, lift, smash

Q4) Given a lifted serve, what is the probability that there is a winner in 3 shots?

Soln: 1) $S = \{D, L, S, W\}$; $W \equiv$ Win

2)

$$\mathbb{P} = \begin{array}{c|cccc} & D & L & S & W \\ \hline D & 1/3 & 1/3 & 0 & 1/3 \\ L & 1/5 & 1/5 & 2/5 & 1/5 \\ S & 1/5 & 2/5 & 0 & 2/5 \\ W & 0 & 0 & 0 & 1 \end{array}$$

Always check that every row adds up to 1 !!

W is an absorbing state (to be formally defined later in the course).

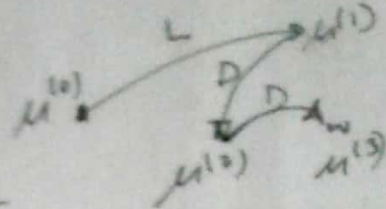
3) (a) $P(X_n = W, X_{n-1} = S, X_{n-2} = L, X_{n-3} = L) = p_{SW}\, p_{LS}\, p_{LL} = \frac{2}{5} \times \frac{2}{5} \times \frac{1}{5} = \frac{4}{125}$

(b) $P(X_n = W, X_{n-1} = S, X_{n-2} = D, X_{n-3} = L) = p_{SW}\, p_{DS}\, p_{LD} = \frac{2}{5} \times 0 \times \frac{1}{5} = 0$

(c) $P(X_n = W, X_{n-1} = D, X_{n-2} = L, X_{n-3} = S) = p\, p\, p = \frac{1}{3} \times \frac{1}{5} \times \frac{2}{5} = \frac{2}{7\ldots}$

Pg 3

Q.4)



$\vec{\mu}^{(0)} = (0, 1, 0, 0)$ i.e. 1st shot

$\vec{\mu}^{(2)} = ?$

2nd shot :- $\vec{\mu}^{(1)} = \vec{\mu}^{(0)} \mathbb{P}$

$$= (0 \quad 1 \quad 0 \quad 0) \begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/5 & 1/5 & 2/5 & 1/5 \\ 1/5 & 2/5 & 0 & 2/5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \left(\frac{1}{5} \quad \frac{1}{5} \quad \frac{2}{5} \quad \frac{1}{5}\right)$$

3rd shot :- $\vec{\mu}^{(2)} = \vec{\mu}^{(0)} \mathbb{P}^2 = (0.1867 \quad 0.2667 \quad 0.08 \quad 0.4667)$

$$= \vec{\mu}^{(1)} \mathbb{P}.$$

Winner :- $\vec{\mu}^{(3)} = \vec{\mu}^{(0)} \mathbb{P}^3 = (0.1316 \quad 0.1476 \quad 0.1067 \quad \underline{\underline{0.6142}})$

$P(\text{Winner after 3 shots}) = 0.6142 \equiv p^3_{LW}$

Also note, that probability of a winning shot increases as the no. of shots in the rally increases. this is intuitively true, b/c as the rally progresses, players tire & the chance of committing an error & hence a winner increases.

Check that $\lim_{n \to \infty} \vec{\mu}^{(n)} = (0 \quad 0 \quad 0 \quad 1)$!

$$P^3 = \begin{pmatrix} 0.1215 & 0.1481 & 0.0711 & 0.6593 \\ 0.1316 & 0.1476 & 0.1067 & 0.6142 \\ 0.1102 & 0.1422 & 0.0587 & 0.6889 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

#.

Concepts covered: 1) Hitting probabilities; 2) Mean Hitting & Absorption times.

(1) Def^n (Hitting probabilities)

Let us consider a Markov chain $\{X_n\}_{n\geq 0, n\in\mathbb{Z}}$ w/ state space S; let $A \subset S$.

T_A := first time the chain hits A starting outside A.

i.e. $T_A := \min\{n \geq 0 \mid X_n \in A\}$ w/ $T_A = 0$ if $X_0 \in A$ & $T_A = \infty$ if $\{n\geq 0 \mid X_n \in A\} = \phi$.

Often we are interested in calculating the following probability.

$g_k(l) = P(X_{T_A} = l \mid X_0 = k)$; i.e. the probability of hitting A through state $l \in A$ starting from $k \in S$.

(k: starting state; l: final state)

Note $\forall\, k \in S\setminus A$; we have $T_A \geq 1$ given that $X_0 = k$

$$g_k(l) = P(X_{T_A} = l \mid X_0 = k) = \sum_{m\in S} P(X_{T_A}=l, X_1 = m \mid X_0 = k) = \sum_{m\in S} P(X_{T_A}=l \mid X_1 = m, X_0 = k)P(X_1 = m \mid X_0 = k)$$

$$\overset{\text{Markov property}}{=} \sum_{m\in S} P(X_{T_A} = l \mid X_1 = m)\underbrace{P(X_1 = m \mid X_0 = k)}_{p_{km}}$$

$$= \sum_{m\in S} p_{km} P(X_{T_A} = l \mid X_1 = m)$$

$$= \sum_{m\in S} p_{km}\, g_m(l) \quad ; k \in S\setminus A,\ l \in A;$$

w/ boundary condition

$$g_k(\ell) = P(X_{T_A} = \ell \mid X_0 = k) = I_{\{k=\ell\}}; \quad k \in A, \ \ell \in S\setminus A$$

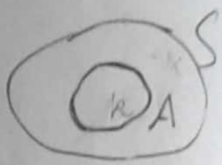
b/c $T_A = 0$ when $X_0 \in A$

This => when $k \neq \ell$

$g_k(\ell) \to 0$??

Yes b/c here $k \in A$ & $\ell \in S\setminus A$

$k \neq \ell$ => we are already in "hitzone" but not in target (once we are in hitzone, the pr has terminated)



<u>Defⁿ :-</u>

<u>Absorbing State</u>

$$p_{k\ell} = I_{\{k=\ell\}} \quad \forall\ k, \ell \in A.$$

i.e. $\{X_n\}$ is trapped in $A \subset S$.
(absorbed)

<u>Note :-</u>

$$\boxed{\sum_{\ell \in A} g_k(\ell) + P(T_A = \infty \mid X_0 = k) = 1}$$

② <u>Mean Hitting times</u>
<u>Mean Absorption times.</u>

$$h_k(A) := E(T_A \mid X_0 = k)$$

clearly $h_k(A) = 0 \quad \forall\ k \in A \subset S$.

$\forall\ k \in S\setminus A$;

$$h_k(A) = E(T_A \mid X_0 = k)$$

$$= \sum_{m \in S} E(T_A, X_1 = m \mid X_0 = k) = \sum_{m \in S} E(T_A \mid X_1 = m, X_0 = k) P(X_1 = m \mid X_0 = k)$$

$$\stackrel{\text{Markov prop}}{=} \sum_{m \in A} \underbrace{E(T_A \mid X_1 = m, X_0 = k)}_{1} p_{km} + \sum_{m \in S\setminus A} E(T_A \mid X_0 = m)\, p_{km}$$

$$= \sum_{m \in S} \{1 + h_m(A)\} p_{km} = \sum_{m \in S} p_{km} + \sum_{m \in S} p_{km} h_m(A)$$

$$= 1 + \sum_{m \in S} p_{km} h_m(A)$$

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i.e. Mean hitting time to A

$$\boxed{h_k(A) = 1 + \sum_{m \in S} p_{km} h_m(A)} \; ; \; \forall k \in S \setminus A$$

w/ bdy cond^n

$$h_k(A) = E(T_A \mid X_0 = k) = 0 \; \forall k \in A.$$

③ First Return Times

Def^n (1st return time to state y)

$$T_y^r := \min\{n \geq 1 \mid X_n = y\} \; ; \; y \in S$$

w/ $T_y^r = \infty$ if $X_n \neq y \; \forall n \geq 1$

$\boxed{\text{Note: } T_y^r = T_y \text{ if } X_0 \neq y}$.

Def^n (Mean return time to state y starting at x

$$\mu_x(y) = E(T_y^r \mid X_0 = x) \geq 1$$

$$\begin{aligned}
\mu_x(y) &= E(T_y^r \mid X_0 = x) \\
&\overset{\substack{\text{law of total} \\ \text{Expect}^n}}{=} E(T_y^r \mid X_1 = y, X_0 = x) P(X_1 = y \mid X_0 = x) \\
&\quad + E(T_y^r \mid X_1 \neq y, X_0 = x) P(X_1 \neq y \mid X_0 = x) \\
&= 1 \times P(X_1 = y \mid X_0 = x) + \sum_{\substack{m \in S \\ m \neq y}} \left[1 + E(T_y^r \mid X_1 = m, X_0 = x)\right] \times P(X_1 = m \mid X_0 = x)
\end{aligned}$$

$$\mu_x(y) \overset{\text{Markov}}{\underset{\text{property}}{=}} p_{xy} + \sum_{\substack{m \in S \\ m \neq y}} \left\{ 1 + E\left(T_y^r \mid X_0 = m\right) \right\} p_{xm}$$

$$\mu_x(y) = p_{xy} + \sum_{\substack{m \in S \\ m \neq y}} \left(1 + \mu_m(y)\right) p_{xm} \; ; \; x, y \in S$$

$$= p_{xy} + \sum_{m \neq y} p_{xm} + \sum_{\substack{m \in S \\ m \neq y}} p_{xm} \mu_m(y)$$

$$= \sum_{m \in S} p_{xm} + \sum_{\substack{m \in S \\ m \neq y}} p_{xm} \mu_m(y)$$

$$\boxed{\mu_x(y) = 1 + \sum_{\substack{m \in S \\ m \neq y}} p_{xm} \mu_m(y)} \quad \text{——— } (3.1)$$

Note: - Unlike hitting times, return time problems do not have bdy conditions.

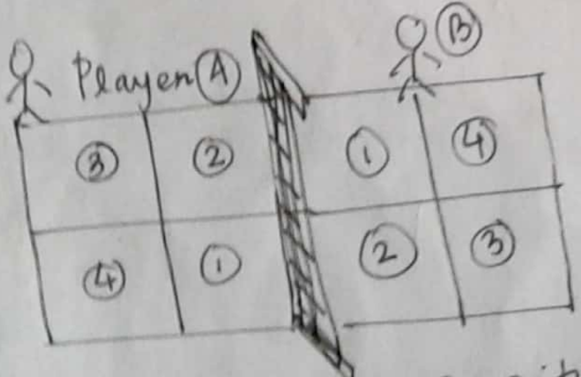
Recall mean hitting time

$$h_m(y) := E\left(T_y \mid X_0 = m\right) \; ; \; m \neq y$$

there

$$\boxed{\mu_y(y) = 1 + \sum_{\substack{m \in S \\ m \neq y}} p_{ym} h_m(y)} \quad - (3.2).$$

example (Badminton: mean return time).



Let us consider a situation where the badminton court is divided into 4 quadrants. At any given instant in a rally, the shuttle arrives in one of these 4 quadrants according to a Markov process as follows: -

Quadrant (n^th shot)		Quadrant (n+1^th shot)	w/ prob.
(1)	$\longrightarrow$	(1)	0
(1)	$\longrightarrow$	(2)	$1/4$

Quadrant (n^{th} shot)		Quadrant ($n+1^{th}$ shot)	[illegible]
1	$\longrightarrow$	3	1/4
1	$\longrightarrow$	4	1/2
2	$\longrightarrow$	1	1/2
2	$\longrightarrow$	2	0
2	$\longrightarrow$	3	1/4
2	$\longrightarrow$	4	1/4
3	$\longrightarrow$	1	1/4
3	$\longrightarrow$	4	1/2
3	$\longrightarrow$	2	0
3	$\longrightarrow$	3	1/4
4	$\longrightarrow$	2	1/4
4	$\longrightarrow$	3	1/2
4	$\longrightarrow$	1	0
4	$\longrightarrow$	4	1/4

$$\mathbb{P} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1/4 & 1/4 & 1/2 \\ 2 & 1/2 & 0 & 1/4 & 1/4 \\ 3 & 1/4 & 0 & 1/4 & 1/2 \\ 4 & 0 & 1/4 & 1/2 & 1/4 \end{array}$$

Q) Given that in an instant of a game, player Ⓐ serves from quadrant ①; what is the average no. of shots in that rally before the shuttle arrives again in quadrant ① for either of the 2 players.

<u>Soln</u>:-

Let $T_y^r := \min\{n \geq 1 \mid X_n = y\}$;
$\mu_x(y) := E(T_y^r \mid X_0 = x)$

We need to calculate $\mu_1(1)$?

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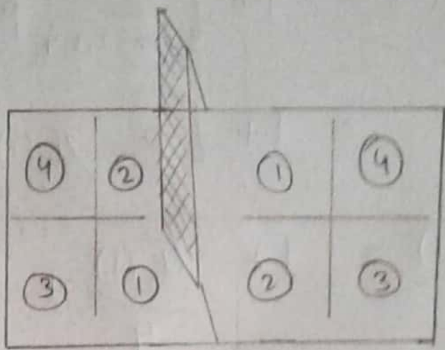
Q. At any given instant in a rally, a shuttle arrives in a particular quadrant as per a Markov process ~

(i) Construct - P

(ii) At a certain instant in a rally, a shot is played from quadrant 1, what is the averg. no. of shots before shuttle arrives in quadrant 1.



n^{th} shot	$(n+1)^{th}$ shot	Prob.
1	1	0
1	2	1/4
1	3	1/4
1	4	1/2
2	1	1/2
2	1	1/2
2	2	0
2	3	1/4
3	1	1/4
3	4	1/2
3	2	0
4	2	1/4
4	3	1/2
4	1	0

Sol. $S = \{1, 2, 3, 4\}$

(i) $$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \\ 0 & 1/4 & 1/2 & 1/4 \end{pmatrix} \end{array}$$

(ii) $\mu_x(y) = 1 + \sum_{\substack{m \in S \\ m \neq y}} p_{xm} \mu_m(y)$

$\mu_1(1) = 1 + \left[\sum_{m=2,3,4} p_{1m} \mu_m(1) \right]$

$= 1 + [p_{12} \mu_2(1)] + [p_{13} \mu_3(1)] + [p_{14} \mu_4(1)]$

$$\boxed{\mu_1(1) = 1 + \frac{\mu_2(1)}{4} + \frac{\mu_3(1)}{4} + \frac{\mu_4(1)}{2}} \quad \text{— (1)}$$

$\mu_2(1) = 1 + \left[\sum_{m=2,3,4} p_{2m} \mu_m(1) \right]$

$= 1 + [p_{22} \mu_2(1)] + [p_{23} \mu_3(1)] + [p_{24} \mu_4(1)]$

$$\boxed{\mu_2(1) = 1 + \frac{\mu_3(1)}{4} + \frac{\mu_4(1)}{4}} \quad \text{— (2)}$$

$\mu_3(1) = 1 + \left[\sum_{m=2,3,4} p_{3m} \mu_m(1) \right]$

$= 1 + [p_{32} \mu_2(1)] + [p_{33} \mu_3(1)] + [p_{34} \mu_4(1)]$

$$\boxed{\mu_3(1) = 1 + \frac{\mu_3(1)}{4} + \frac{\mu_4(1)}{2}} \quad \text{— (3)}$$

$\mu_4(1) = 1 + \sum_{m=2,3,4} p_{4m} \mu_m(1)$

$= 1 + [p_{42} \mu_2(1)] + [p_{43} \mu_3(1)] + [p_{44} \mu_4(1)]$

$$\boxed{\mu_4(1) = 1 + \frac{\mu_2(1)}{4} + \frac{\mu_3(1)}{2} + \frac{\mu_4(1)}{4}} \quad \text{— (4)}$$

(2)

from eqⁿ (3), and eqⁿ (2)

$$\mu_4(1) = \left[\frac{3}{4}\mu_3(1) - 1\right]2 = \frac{3}{2}\mu_3(1) - 2$$

and $$\mu_2(1) = 1 + \frac{\mu_3(1)}{4} + \frac{3}{8}\mu_3(1) - \frac{2}{4}$$

$$= \frac{1}{2} + \frac{5}{8}\mu_3(1)$$

from eqⁿ (4),
put the values,

$$\frac{3}{4}\left[\frac{3}{2}\mu_3(1) - 2\right] = 1 + \frac{1}{4}\left[\frac{1}{2} + \frac{5}{8}\mu_3(1)\right] + \frac{\mu_3(1)}{2}$$

$$\frac{9}{8}\mu_3(1) - \frac{6}{4} = 1 + \frac{1}{8} + \frac{5}{32}\mu_3(1) + \frac{\mu_3(1)}{2}$$

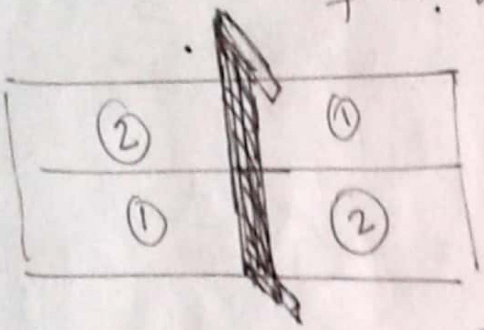
$$\boxed{\mu_3(1) = \frac{28}{5}}$$

$$\boxed{\mu_4(1) = 32/5} \; ; \; \boxed{\mu_2(1) = 4}$$

$$\therefore \quad \mu_1(1) = 1 + \frac{4}{4} + \frac{28}{5 \times 4} + \frac{1}{2}\left(\frac{32}{5}\right)$$

$$= 33/5 = \boxed{6.6}$$

A simpler version of the above problem



2 stage M.C. built from $\mathbb{P}$

$$\mathbb{P}' = \begin{array}{c|cccc} & 11 & 12 & 21 & 22 \\ \hline 11 & 2/3 & 1/3 & 0 & 0 \\ 12 & 0 & 0 & 1/3 & 2/3 \\ 21 & 2/3 & 1/3 & 0 & 0 \\ 22 & 0 & 0 & 1/3 & 2/3 \end{array}$$

$$\mathbb{P} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \end{array}$$

Q) Given a valid service (1,1), find (A) $\mu_{11}(22)$ (B) $\mu_{11}(11)$ (C) $\mu_{11}(21)$ (d) $\mu_{11}(12)$?

Q) given a service from ①; when does the shuttle return to quadrant ① on an avg. on either side?

Soln: $\mu_1(1) = 1 + p_{12}\,\mu_2(1) = 1 + \frac{1}{3}\mu_2(1)$

$\mu_2(1) = 1 + p_{22}\,\mu_2(1) = 1 + \frac{2}{3}\mu_2(1)$

$\Rightarrow \frac{1}{3}\mu_2(1) = 1 \Rightarrow \mu_2(1) = 3$

$\mu_1(1) = 1 + \frac{1}{3} \times 3 = 2$

i.e. on an avg. every 2^{nd} shot in a rally returns to quadrant ① given a serve from quadrant ①.

Advanced Concepts of Markov Chains

Classification of States

1 (Def^n) Communicating States

A state $j \in S$ is <u>accessible</u> from $i \in S$; i.e.

$$(i) \longmapsto (j)$$

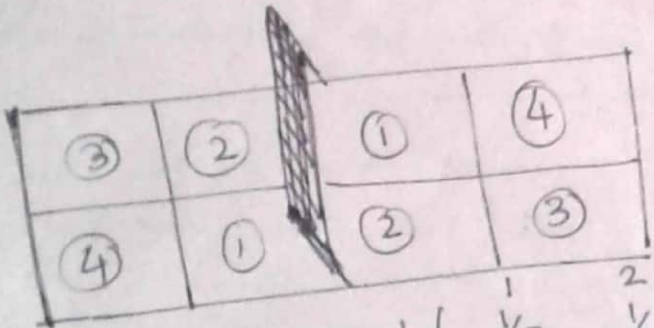
if $\exists$ a finite integer $n \geq 0$ s.t.

$$p_{ij}^{n} \equiv (\mathbb{P}^n)_{i,j} := P(X_n = j \mid X_0 = i) > 0$$

Note, since $\mathbb{P}^0 = I$ (Identity matrix), $(i) \longmapsto (i)$ even if $p_{ii} = 0$.

If $(i) \longmapsto (j)$ and $(j) \longmapsto (i) \Rightarrow (i) \longleftrightarrow (j)$ i.e. i & j <u>Communicate</u>.

e.g.



$$\text{Given. } \mathbb{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 2/5 & 1/5 & 0 & 2/5 \\ 1/4 & 1/4 & 1/2 & 0 \end{pmatrix} \end{array}$$

In this model: i) $(3) \longleftrightarrow (3)$ even though $p_{33} = 0$

ii) $(2) \longleftrightarrow (3)$ even though $p_{23} = 0$ b/c $p_{24} > 0$ & $p_{43} > 0$ hence $\exists$ n s.t. $(\mathbb{P}^n)_{23} > 0$.

infact $(\mathbb{P}^2)_{23} = 0.4167 > 0$

The binary relation "$\leftrightarrow$" satisfies the following:

i) Reflexivity

$\forall i \in S$; $i \leftrightarrow i$

ii) Symmetry

$\forall i, j \in S$; $i \leftrightarrow j \equiv j \leftrightarrow i$

iii) Transitivity

$\forall i, j, k \in S$ s.t. $i \leftrightarrow j$ & $j \leftrightarrow k \Rightarrow i \leftrightarrow k$

i), ii) & iii) $\Rightarrow$ $\leftrightarrow$ is an equivalence relation & it induces a partition of S into disjoint subsets $A_1, \ldots, A_m$ s.t. $S = \bigcup_{i=1}^{m} A_i$, &

(a) $i \leftrightarrow k$ $\forall i, j \in A_q$,

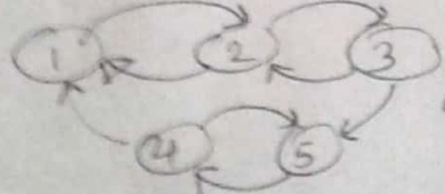
(b) $i \not\leftrightarrow k$ whenever $i \in A_p$ and $j \in A_q$ w/ $p \neq q$.

② Def^n (Irreducible & reducible Markov chain).

Irreducible M.C :- regardless of the present state, we can reach any other state in finite time i.e.

$$\forall i, j \in S.\ \exists n \text{ s.t. } \mathbb{P}(X_n = j \mid X_0 = i) > 0$$

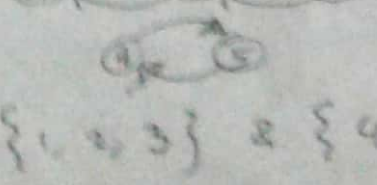
e.g.



$$\sim \mathbb{P} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1/2 & 0 & 1/2 & 0 & 0 \\ 3 & 0 & 2/3 & 0 & 0 & 1/3 \\ 4 & 1/3 & 0 & 0 & 0 & 2/3 \\ 5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Reducible M.C. :-

eg (i)



$\{1, 2, 3\}$ & $\{4, 5\}$ are disjoint classes.

eg (ii)

$q_{ij} \equiv p^{n}_{ij} := P(T^{r}_{j} < \infty \mid X_0 = i) = P(X_n = j \mid X_0 = i)$ for some $n \geq 1$ — (3.1)

i.e. probability of return to state (j) in a finite time starting from state (i).

Mean no. of returns.

$R_j := \sum_{n=1}^{\infty} I_{\{X_n = j\}}$ is the no. of visits to state (j) by the chain $\{X_n\}_{n \in I, \, n \geq 0}$

$$E(R_j \mid X_0 = i) = \sum_{m=0}^{\infty} m \, P(R_j = m \mid X_0 = i)$$

$$= \sum_{m=1}^{\infty} m \, q_{ij} \, q_{jj}^{m-1} (1 - q_{jj})$$

prob. of jumping from (i) → (j) in some $n \leq m$ steps (1st visit)

never again visits (j) after m visits

(m-1) remaining visits to (j) w/ prob q_{jj}^{m-1}

$$= (1 - q_{jj}) q_{ij} \sum_{m=1}^{\infty} m \, q_{jj}^{m-1}$$

$$= (1 - q_{jj}) q_{ij} \frac{1}{(1 - q_{jj})^2} = \frac{q_{ij}}{(1 - q_{jj})}$$ — (3.2)

Identity

$$\sum_{m=1}^{\infty} m r^{m-1} = \frac{1}{(1-r)^2}$$

b/c $S = \sum_{m=1}^{\infty} r^{m-1} = \frac{1}{1-r}$

$\frac{\partial S}{\partial r} = \frac{1}{(1-r)^2} = \sum_{m=1}^{\infty} m r^{m-1}$

④ Def^n

Recurrent states

(i) $\in S$ is recurrent if

$$q_{ii} := P(T^{r}_{i} < \infty \mid X_0 = i) = P(X_n = i \mid X_0 = i) \text{ for some } n \geq 1$$

$$= \mathbb{P}^{n}_{ii} = 1.$$

Also,

i) State (i) is recurrent iff $E(R_i \mid X_0 = i) = \infty$

ii) State (i) is recurrent iff $P(R_i = \infty \mid X_0 = i) = 1$

iii) Th^m:- State (i) $\in S$ is recurrent iff $\sum^{\infty} (P^n)_{ii} = \infty$

⑤ (Def^n) Transient states

I) A state $i \in S$ is transient when it is **not** recurrent
i.e.
$$P(R_i = \infty \mid X_0 = i) < 1 \text{ or } \equiv P(R_i = \infty \mid X_0 = i) = 0$$

Alternatively,

II) $i \in S$ is transient when
$$P(R_i < \infty \mid X_0 = i) > 0 \text{ or } \equiv P(R_i < \infty \mid X_0 = i) = 1$$

i.e. the no. of returns to state $i \in S$ is finite w/ a non-zero probability which is necessarily equal to 1

III) $i \in S$ is transient iff.
$$q_{ii} := P(T_i^r < \infty \mid X_0 = i) < 1 \text{ or}$$
$$P(T_i^r = \infty \mid X_0 = i) > 0$$

IV) $i \in S$ is transient iff
$$E(R_i \mid X_0 = i) < \infty$$

V) $i \in S$ is transient iff:
$$\sum_{n=1}^{\infty} (P^n)_{ii} < \infty$$ i.e. the above series converges.

⑥ Th^m:-

Let $\{X_n\}_{n \geq 0, n \in I}$ be a Markov Chain w/ finite state space $S \Rightarrow \{X_n\}_{n \geq 0, n \in I}$ has atleast one recurrent state.

Positive recurrent

A recurrent state $i \in S$ is said to be positive recurrent if $\mu_i(i) := E(T_i^r \mid X_0 = i) < \infty$;

& is null recurrent if

$$\mu_i(i) := E(T_i^r \mid X_0 = i) = \infty$$

Thm's (i) If S is finite; then all recurrent states of a M.C. $\{X_n\}_{\substack{n\geq 0\\ n\in I}}$ are positive recurrent.

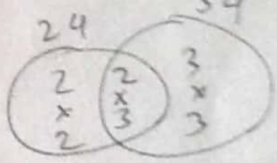
(ii) If $\{X_n\}_{\substack{n\geq 0\\ n\in I}}$ is an irreducible MC w/ finite state space; then all states are positive recurrent.

eg: Badminton model in pg(1) of this lecture note set?

How to find gcd & lcm?

$$\begin{array}{r|l} 2 & 24 \\ 2 & 12 \\ 2 & 6 \\ 3 & 3 \\ & 1 \end{array} \qquad \begin{array}{r|l} 2 & 54 \\ 3 & 27 \\ 3 & 9 \\ 3 & 3 \\ & 1 \end{array}$$

$54 = 3\times3\times3\times2$
$24 = 3\times2\times2\times2$



$gcd = 2\times3 = 6$
$lcm = 2\times2\times2\times3\times3\times3 = 24\times9 = 216$

(8) Periodicity & Aperiodicity

Def^n) Period: The period of the state $i \in S$ is the greatest common divisor (gcd) of the set

$$\{n \geq 1 \mid (\mathbb{P}^n)_{ii} > 0\}$$

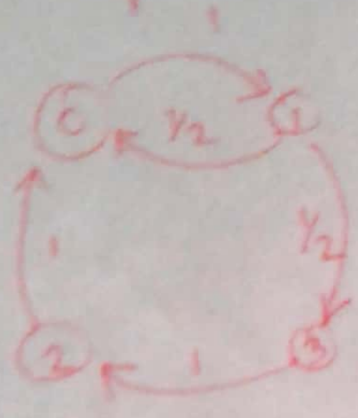
A state having period $=1$ is said to be aperiodic which is the case in particular if $p_{ii} > 0$.

A recurrent state $i \in S$ is said to be ergodic if it is both positive recurrent & aperiodic.

If $(\mathbb{P}^n)_{ii} = 0 \;\forall\, n \geq 1$ then the period of $i = 0$ & in this case, the state (i) is also transient.

If the set $\{n \geq 1 \mid P_{ii}^n > 0\}$ contains elements that are co-prime w.r.t. each other; then the state i is aperiodic.

e.g. (i)



Since,

$\{n \geq 1 \mid P_{00}^n > 0\} = \{2, 4, 6, 8, 10, \cdots\}$.

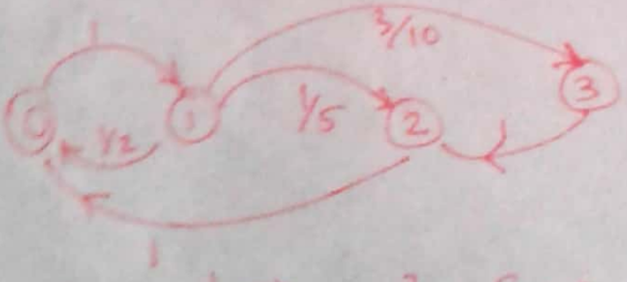
$\{n \geq 1 \mid P_{11}^n > 0\} = \{2, 4, 6, 8, 10, \cdots\}$.

$\{n \geq 1 \mid P_{22}^n > 0\} = \{4, 6, 8, 10, 12, \cdots\}$.

$\{n \geq 1 \mid P_{33}^n > 0\} = \{4, 6, 8, 10, 12, \cdots\}$.

$\therefore$ All states have period = 2.

eg (ii)



$\{n \geq 1 \mid P_{00}^n > 0\} = \{2, 3, 4, 5, 6, 7 \cdots\}$.

$\{n \geq 1 \mid P_{11}^n > 0\} = \{2, 3, 4, 5, 6, 7, \cdots\}$

$\{n \geq 1 \mid P_{22}^n > 0\} = \{3, 4, 5, 6, 7, 8 \cdots\}$.

$\{n \geq 1 \mid P_{33}^n > 0\} = \{4, 6, 7, 8, 9, 10, \cdots\}$.

$\therefore$ All states have period = 1 (i.e. are aperiodic) b/c gcd of the set = 1.