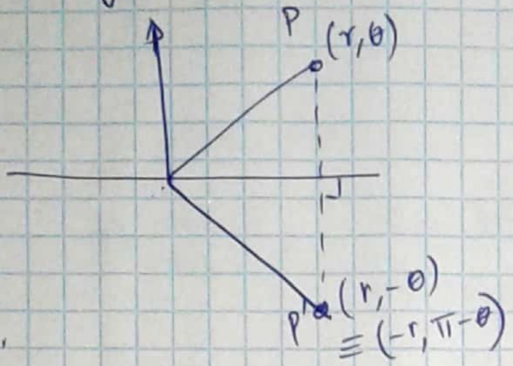
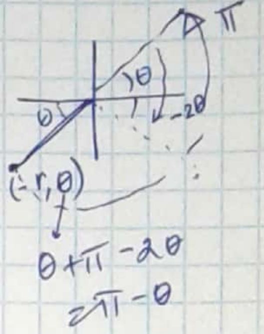
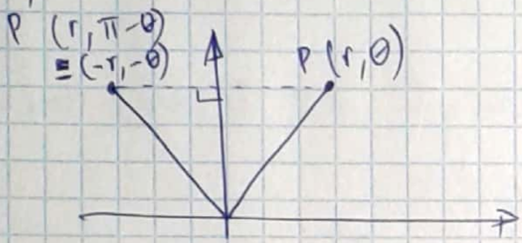
§(9.7) Graphing in polar coordinates

Symmetry

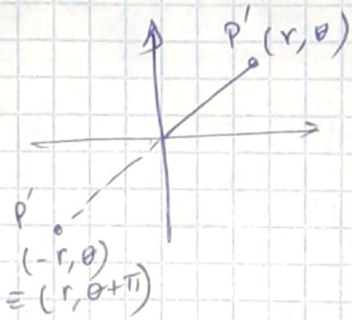


abt. x-axis





abt y-axis



abt origin

for the respective symmetries, if P lies on graph then P' also lies on graph.

Slope

Let us say we have a curve in (r,θ) frame, given by $r = f(\theta)$

Slope of $f(\theta)$ is $\frac{dy}{dx}$

Note, from the polar transformation rule,

$x = r\cos\theta$
$= f(\theta)\cos\theta$ & $y = r\sin\theta = f(\theta)\sin\theta$

Now, $$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{\frac{d}{d\theta}\left(f(\theta)\sin\theta\right)}{\frac{d}{d\theta}\left(f(\theta)\cos\theta\right)}$$ for $\frac{dx}{d\theta} \neq 0$

$$\frac{dy}{dx} = \frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta}$$

$\therefore$ Slope of curve $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \; ; \quad \frac{dx}{d\theta} \neq 0.$$

(x, y) frame

<u>Slope at Origin $O(0,0)$</u>

If $r = f(\theta)$ passes through the origin at $\theta = \theta_0$; then $f(\theta_0) = 0$ (b/c at origin $r = 0$)

$$\therefore \left.\frac{dy}{dx}\right|_{(0,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0 + \cancel{f(\theta_0)}^{\,0}\cos\theta_0}{f'(\theta_0)\cos\theta_0 - \cancel{f(\theta_0)}^{\,0}\sin\theta_0}$$

$$= \frac{\sin\theta_0}{\cos\theta_0} = \tan\theta_0$$

Note slope at origin is referred to as slope at $(0,\theta_0)$ b/c a curve may pass through the origin several times w/ different values of θ_0.

eg.

(1) A cardoid

Plot the curve $r = 1 - \cos\theta$
$= f(\theta)$

(i) Symmetry

$f(-\theta) = 1 - \cos\theta = r \Rightarrow (r, -\theta)$ lies on $r = f(\theta)$ as well

$\Rightarrow$ it is symmetric abt. x-axis

(ii) trace as $\theta = 0 \nearrow \pi$

as θ varies from 0 to π;

$\cos\theta \searrow$ from 1 to -1 & $r \nearrow$ from 0 to 2

Also at $\theta = \pi/2$; $r = 1$

(iii) Periodicity

b/c $\cos\theta$ has period 2π; therefore $r = f(\theta + 2\pi) = f(\theta)$ & the graph repeats itself.

(iv) Slope at origin

$$\left.\frac{dy}{dx}\right|_{(0,0)} = \tan 0 = 0 \Rightarrow$$ tangent at origin, as $r = f(\theta)$ leaves origin Counter-Clkwise, is horizontal

$$\left.\frac{dy}{dx}\right|_{(0,2\pi)} = \tan(2\pi) = 0 \Rightarrow$$ $r = f(\theta)$ returns to origin w/ a flat/horizontal tangent line.

eg ② Graph the curve $r^2 = 4\cos\theta$

soln:- $r^2 = 4\cos\theta = f(\theta)$

① Symmetry:- $f(-\theta) = 4\cos\theta = f(\theta) \Rightarrow (r, -\theta)$ passes through curve

$\Rightarrow$ symmetry abt x-axis

$f(-\theta) = 4\cos\theta = (-r)^2 = r^2 \Rightarrow (-r, -\theta)$ passes through curve

$\Rightarrow$ symmetry abt y-axis

Additionally $(-r, \theta)$ also passes through curve

$\Rightarrow$ symmetry abt origin

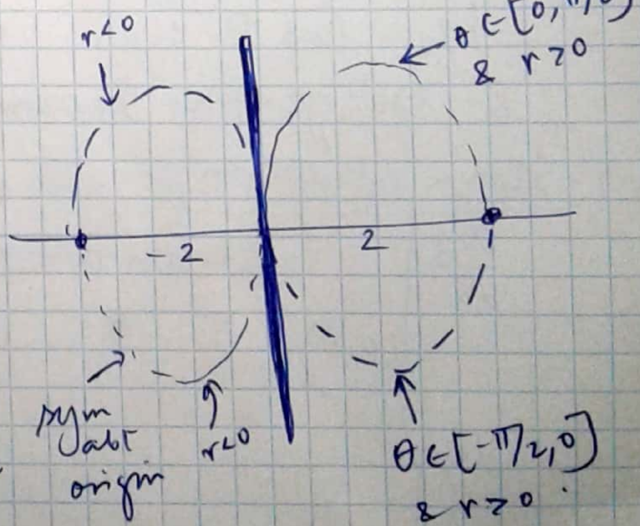
(ii) Periodicity

2π

(iii) Origin & trace as $\theta = 0 \nearrow$

$r = \pm 2\sqrt{\cos\theta}$

θ	$\cos\theta$	r
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	± 1.9
$\pm\pi/4$	$1/\sqrt{2}$	± 1.7
$\pm\pi/3$	$1/2$	± 1.4
$\pm\pi/2$	0	0 $\leftarrow$ origin



(iv) Slope

$$\left.\frac{dy}{dx}\right|_{(0,\pi/2)} = \tan\pi/2 = \infty$$

Scanned by CamScanner

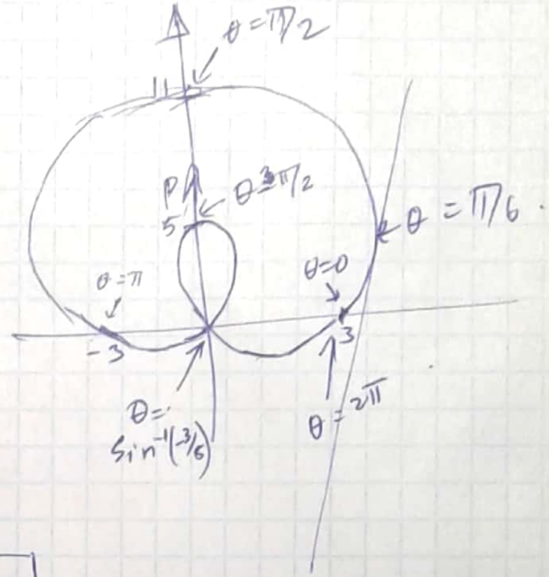
Q (3) Draw $r = 3 + 8\sin\theta = f(\theta)$ & find the eqn. of the tangent line at $\theta = \pi/6$ & plot it.

Soln:-

(i) Sym $f(-\theta) = 3 - 8\sin\theta \neq r$

$f(\pi-\theta) = 3 + 8\sin(\pi-\theta) = 3 + 8\sin\theta = r$

$\Rightarrow$ Symmetry about y-axis.



(ii) Period:- 2π

(iii) trace $\theta = 0 \nearrow 2\pi$

$r = 0 = 3 + 8\sin\theta$

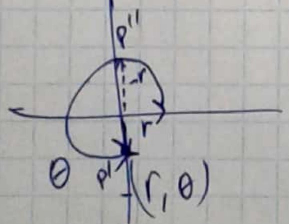
$\Rightarrow \theta = \sin^{-1}(-3/8)$

$\theta = 3\pi/2$

$r = 3 + 8\sin(3\pi/2)$

$= 3 - 8 = -5 \Rightarrow \boxed{-r = 5}$

Pt P on the graph is interesting; it seems on +ve y-axis



When pt is at $P'(r,\theta)$

$P''(-r,\theta)$

To find eqn. of tangent line at $\theta = \pi/6$.

$\frac{df}{d\theta} = 8\cos\theta = 8 \times 0.866$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \left.\frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\right|_{\theta=\pi/6} = \frac{(6.9282)\frac{1}{2} + 6.0622}{(6.9282)\times 0.866 - 3.5}$$

$= 3.808 =$ slope of tangent

at $\theta = \pi/6$; $r = 7$

$x = r\cos\theta = 7\cos(\pi/6) = 6.0622$; $y = 7\sin(\pi/6) = 3.5$

$\therefore$ Eqn. of tangent line

$$(y - y_1) = m(x - x_1)$$

$$\Rightarrow y - 3.5 = 3.8708(x - 6.0622)$$

$$\Rightarrow y = 3.8708x + 3.5 - 23.1018$$

$$= 3.8708x - 19.6018$$

#

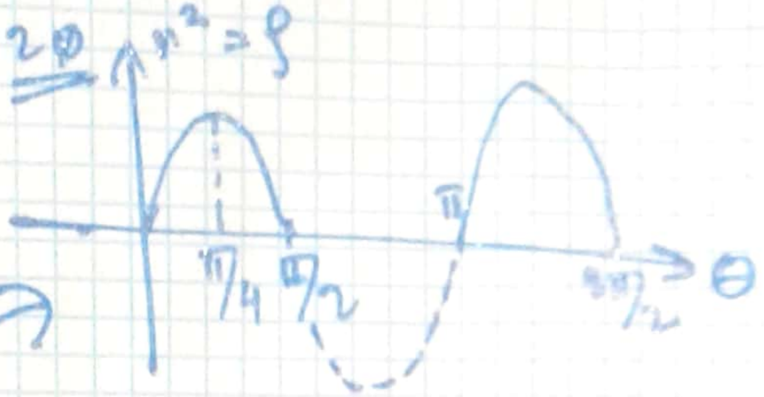
Scanned by CamScanner

pg ① 23/8/18.

graph the polar curve.

$$r^2 = \sin 2\theta$$

$\sin 2\theta$ $r^2 = \rho$



Step ①

B/c $\sin 2\theta$ can take -ve & +ve values $\Rightarrow$ r^2 can take -ve & +ve values

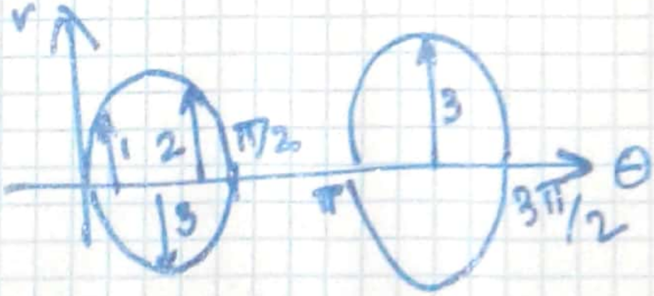
$\therefore$ We will plot $\rho = r^2 = \sin 2\theta$ in (ρ, θ) plane in Cartesian frame

News! –

① Inflexion pts (consider tangent line)

⑪ $(y, x) = (r\sin\theta, r\cos\theta) \equiv (r, \theta)$ but $\neq (r, \theta)$

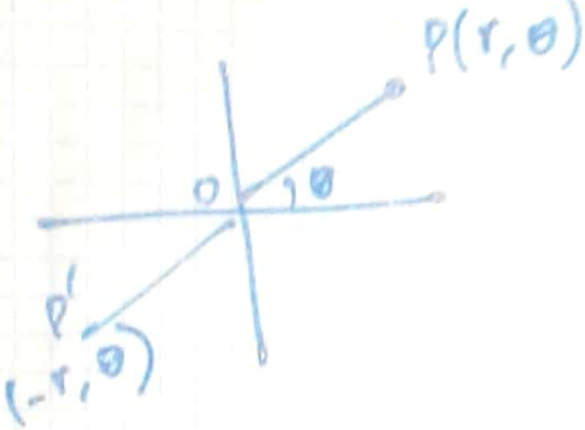
Step ⑪ $r = \pm\sqrt{\sin 2\theta}$ (Note $\because$ r is a position vector, it is a real qty & hence we donot plot the imag. part)

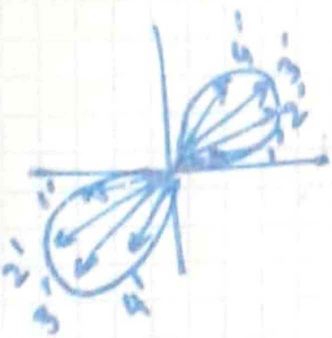


following +ve r & θ from $0 \nearrow \pi/2$ & $\pi \nearrow 3\pi/2$

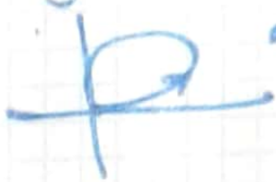
tep (III) for $r < 0$

Recall





So the lower lobe $(-r, \theta)$ is traced as the locus of diametrically opposite trace of pts.



as $\theta \nearrow 0$ to $\pi/2$

Step (IV) :- Step (III) & Step (II) ⇒ a "double" covering of the lobes of $r = f(\theta)$.

$$\left.\frac{dy}{d\theta}\right|_{\text{origin } \theta_0 = 0} = \tan\theta_0 = 0$$

$$\left.\frac{dy}{d\theta}\right|_{\text{origin } \theta_0 = \pi/2} = \tan \pi/2 = \infty .$$

Finding pts. where graphs intersect.

* Solving eqns. of 2 curves simultaneously may NOT identify all their pts. of intersection. The only sure way to identify all the pts. of intersection is to graph the eqns.

Deceptive Co-ordinates.

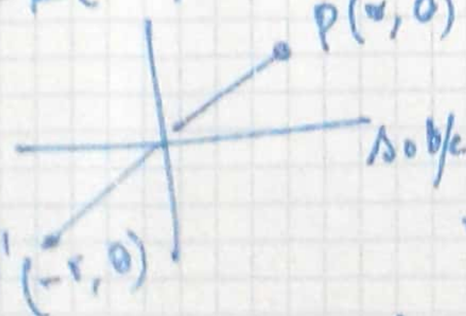
eg① Show $(2, \pi/2)$ lies on $r = 2\cos 2\theta = f(\theta)$.

Soln. Note $2\cos 2\cdot\pi/2 = 2\cos\pi = -2$

So it seems $(+2, \pi/2)$ does not lie on $r = f(\theta)$

But- Caution!

Note



$P(r,\theta) \equiv P(-r, \theta+\pi)$

So b/c $\cos 2(\pi+\theta)$; $\theta = \pi/2$

$= \cos 2(\pi + \pi/2)$

$= \cos(2\pi + \pi) = \cos\pi = -1$

$\Rightarrow r = 2\cos 2\theta\big|_{\theta=\pi/2} = -2$

i.e. $(-2, \pi + \pi/2)$ satisfies $r = f(\theta)$

$\Rightarrow (2, \pi/2)$ is on curve.

This is b/c $(-2, \pi + \pi/2) \equiv (-r, \theta + \pi)$ is the same pt. as $(2, \pi/2) = (r, \theta)$

eg (11) Determine if $(1, 3\pi/4)$ lies on $r = \sin 2\theta$?

S	A
T	C

$$\sin 2 \times \frac{3\pi}{4} = \sin \frac{3\pi}{2} = \sin\left(\pi + \frac{\pi}{2}\right)$$
$$= -\sin \pi/2$$
$$= -1$$

It seems $(1, 3\pi/4)$ may not be on $r = \sin 2\theta$!

But wait!

Let's check if $(-r, \pi + \theta) \equiv \left(-1, \pi + \frac{3\pi}{4}\right)$ satisfies $r = \sin 2\theta$.

$$\sin 2\theta = \sin 2(\pi + 3\pi/4) = \sin\left(2\pi + \frac{3\pi}{2}\right)$$
$$= \sin \frac{3\pi}{2}$$
$$= -1$$
$$r = \sin 2\theta = -1$$

i.e. $(-1, \pi + 3\pi/4)$ satisfies $r = \sin 2\theta$

$\Rightarrow (1, 3\pi/4)$ is also on curve $r = \sin 2\theta$!

Elusive Intersection pts.

Recall! Simply solving $r_1 = f_1(\theta)$ & $r = f_2(\theta)$ simultaneously may NOT suffice!

eg $r^2 = 4\cos\theta$ & $r = 1 - \cos\theta$ (What are their pts. of intersection?)

Sub. $\cos\theta = r^2/4$ in $r = 1 - \cos\theta$

$$= 1 - \frac{r^2}{4}$$

$$\Rightarrow r^2 + 4r - 4 = 0$$

$$\Rightarrow r = -2 \pm 2\sqrt{2}$$

for $r = -2 - 2\sqrt{2}$

$|r| > 2$ & we know for $r = 1 - \cos\theta$

$|r| \le 2$

so $r = -2 - 2\sqrt{2}$ is NOT their pt. of intersection.

for $r = -2 + 2\sqrt{2}$

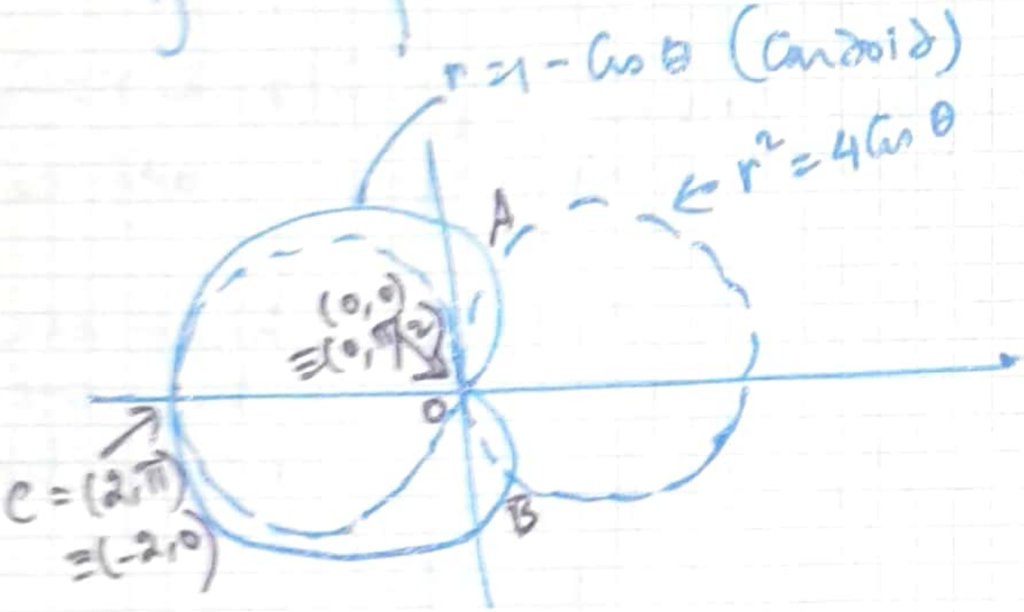
we have $\cos\theta = 1 - r = 1 + 2 - 2\sqrt{2}$

$\theta = \cos^{-1}(3 - 2\sqrt{2})$

$\approx \pm 80^\circ$

$\therefore$ We have identified two intersection pts $(2\sqrt{2} - 2, \pm 80^\circ)$.

Now if we plot the two curves



Two additional pts. of intersections at C & O (in addition to A & B identified earlier).

#.