HW (Set 1): Solutions

13/10/18

Pg 1

Q.1.

Version (2) (Monte Carlo Roulette I).

Total possibilities = 37

No. of reds = 18

$P(\text{win}) = 18/37$

Note:- In any roll, red → wins or gets your money back, black – lose.

$$P(\text{lose}) \overset{\text{Law of total probability}}{=} P(\text{lose} \mid \text{black})\,P(\text{black}) + P(\text{lose} \mid P1)\,\underbrace{P(P1)}_{\text{leads to } 0}$$

$$= 1 \times \frac{18}{37} + \frac{19}{37} \times \frac{1}{37} = 0.50036523$$

Version (3)

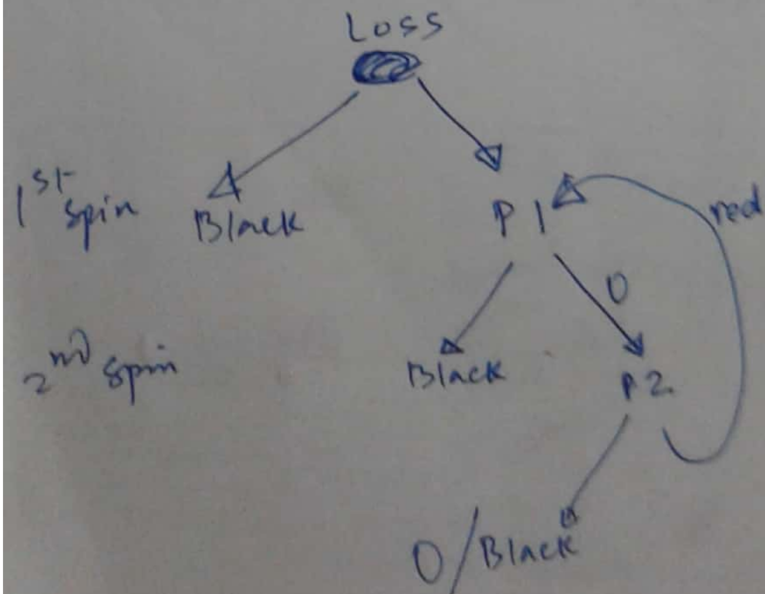
$$P(\text{lose}) = \underbrace{P(\text{lose} \mid \text{black})\,P(\text{black})}_{\text{Same as before}} + P(\text{lose} \mid P1)\,P(P1)$$

2nd spin Black = B2

2nd spin 0 = 02 (sent to P2)

$$\left\{ P(\text{lose } \& \text{ 2nd spin is black} \mid P1) + P(\text{lose } \& \text{ 2nd spin is } 0 \mid P1) \right\} P(P1)$$

$$= \left\{ P(\text{lose} \mid B2, P1) \times P(B2 \mid P1) + P(\text{lose} \mid 02, P1)\,P(02 \mid P1) \right\} \times P(P1)$$



$$= \left\{ 1 \times \frac{18}{37} + \underbrace{\left[\frac{19}{37} + P(\text{lose} \mid P1) \times P(P1 \mid P2) \right]}_{P(\text{lose} \mid P2)} \times \frac{1}{37} \right\} \times \frac{1}{37}$$

$$= \frac{1}{37} \left\{ \frac{18}{37} + \left[\frac{19}{37} + P(\text{lose} \mid P1) \times \frac{18}{37} \right] \frac{1}{37} \right.$$

$$\therefore P(\text{lose}|P1) \times \frac{1}{37} = \frac{1}{37}\left\{\frac{18}{37} + \left[\frac{19}{37} + P(\text{lose}|P1) \times \frac{18}{37}\right]\frac{1}{37}\right\}$$

$$\Rightarrow P(\text{lose}|P1) = \frac{18}{37} + \frac{19}{37} \times \frac{1}{37} + P(\text{lose}|P1)\frac{18}{37\times37}$$

$$\Rightarrow P(\text{lose}|P1)\left\{1 - \frac{18}{37\times37}\right\} = \frac{18\times37 + 19}{37\times37}$$

$$\Rightarrow P(\text{lose}|P1)\left\{\frac{37\times37 - 18}{37\times37}\right\} = \frac{666+19}{37\times37}$$

$$\Rightarrow 1351\ P(\text{lose}|P1) = 685$$

$$\Rightarrow P(\text{lose}|P1) = \frac{685}{1351}$$

$$\therefore P(\text{lose}) = P(\text{lose}|\text{Black})\,P(\text{Black}) + P(\text{lose}|P1)\,P(P1)$$

$$= \frac{18}{37} + \frac{685}{1351} \times \frac{1}{37}$$

$$= 0.50019004$$

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Q2) Baye's thm (Witness reliability)

$P(taxi = Black) = 0.8 = P(taxi = B)$

$P(taxi = Yellow) = 0.2 = P(taxi = Y)$

$P(Witness = true \mid taxi\ color) = 8/10 = 0.8$

i.e. $P(Witness\ taxi = Y \mid taxi = Y) = 0.8$

$P(Witness\ taxi = B \mid taxi = B) = 0.8$

for reliability of info we want $taxi = Y \mid witness\ taxi = Y$.

i.e. We need to find $P(taxi = Y \mid witness\ taxi = Y)$

Baye's thm

$$P(taxi = Y \mid witness\ taxi = Y) = \frac{P(taxi = Y, Witness\ taxi = Y)}{P(witness\ taxi = Y)}$$

$$= \frac{P(witness\ taxi = Y \mid taxi = Y)\,P(taxi = Y)}{P(Witness\ taxi = Y)}$$

Law of total prob to compute Denominator

$$= \frac{0.8 \times 0.2}{P(Witness\ taxi = Y \mid taxi = Y) \times P(taxi = Y) + P(witness\ taxi = Y \mid taxi = B) \times P(taxi = B)}$$

$$= \frac{0.8 \times 0.2}{0.8 \times 0.2 + 0.2 \times 0.8} = \frac{1}{2}$$

Conclusion :-

This means it is a 50-50 situation (pure chance) & hence witness report is useless.

Q. 3 (Monty Hall problem).

Prize (P)
Goat 1 (G1)
Goat 2 (G2)

Some assumptions, that you as a Contestant, must make :-

(i) Before the game, 1 catch is placed randomly behind any door i.e. the contestant upon choosing a door has prob = 1/3 of winning.

(ii) The host show knows behind which door(s) the prize is & always opens an empty door. If he has 2 empty doors he can open; he choose one of them at random.

Options :-

Door 1	Door 2	Door 3	
P	G1	G2	← Arrangement 1
P	G2	G1	← Arrangement 2
G1	P	G2	← Arrangement 3
G2	P	G1	← Arrangement 4
G1	G2	P	← Arrangement 5
G2	G1	P	← Arrangement 6

Soln :-

Method (1) :- Elementary probability.

B/c of assumption (i); all 6 arrangements have equal probability = 1/6

1st 2 arrangements - he will lose if he switches.

In 3rd arrangement — host will open door 3; he will win if he switches

Same w/ 4th, 5th & 6th arrangement } → he will win if he switches

⇒ 4 out of 6 cases; he will win if he switches (or 2/3 prob)

i.e. He should switch !!

Method (2) :- Baye's Thm

Let us start w/ arrangement (1) :-

$$P(\text{keep \& win}) = P(\text{prize door } 1 \mid \text{host door } 3)$$

$$\overset{\text{Bayes}}{=} \frac{P(\text{host door } 3 \mid \text{prize door } 1)\,P(\text{prize door } 1)}{P(\text{host door } 3)}$$

$$\overset{\substack{\text{Assumption} \\ \text{(ii) \& (i)}}}{=} \frac{\frac{1}{2} \times \frac{1}{3}}{P(\text{host door } 3 \mid \text{prize door } 1)\,P(\text{prize door } 1) + P(\text{host door } 3 \mid \text{prize door } 2)\,P(\text{prize door } 2) + P(\text{host door } 3 \mid \text{prize door } 3)\,P(\text{prize door } 3)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \times \frac{1}{3}} = \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{3}\left(\frac{1}{2}+1\right)} = \frac{\frac{1}{2}}{3/2}$$

$$\leftarrow \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

i.e. $P(\text{switch \& win}) = 1 - \frac{1}{3} = \frac{2}{3}$ (as in method 1).

Q.4. Refer Class Notes.

Q.5 $P(\text{Success}) = p = 0.2$

$X : \text{geom}_0(p)$

$$f_X(x) = \begin{cases} (1-p)^x p & ; x = 0, 1, 2, \ldots \\ 0 & ; \text{o.w.} \end{cases}$$

$$E(X) = \frac{1-p}{p} = \frac{0.8}{0.2} = 4.$$

Q.6. Let $X \sim \text{geom}_0(p)$ (All calculations will be similar for $X \sim \text{geom}_1(p)$)

$$P(X \geq x) = 1 - P(X < x)$$
$$= 1 - P(X \leq x-1)$$
$$= 1 - \left\{ \sum_{m=0}^{x-1} P(X = m) \right\}$$
$$= 1 - \left\{ p \sum_{m=0}^{x-1} (1-p)^m \right\}$$

Geom. series

$$= 1 - p \left\{ \frac{1 - (1-p)^x}{1 - (1-p)} \right\}$$
$$= 1 - 1 + (1-p)^x = (1-p)^x$$

$$P(X \geq i+j \mid X \geq i) = \frac{P(X \geq i+j, X \geq i)}{P(X \geq i)}$$

$$= \frac{P(X \geq i+j)}{(1-p)^i}$$

$$= \frac{(1-p)^{i+j}}{(1-p)^i} = (1-p)^j = P(X \geq j)$$

this means $X \geq i$ is forgotten & the event $X \geq i+j$ is just as likely as $X \geq j$ (as if the info $X \geq i$ was not known to us).

We will again see this kind of behavior when we study Markov chains.

Q.7: $f_X(x) = {}^nC_x\, p^x (1-p)^{n-x}$; given $X \sim Bin(n,p)$

$$= \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

$$\underline{\underline{\lambda = np}} \quad \frac{n!}{(n-x)!\,x!} \frac{\lambda^x}{n^x} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$\lim_{n \to \infty} \frac{n!}{(n-x)!\,n^x} = \lim_{n \to \infty} \frac{n(n-1)\cdots(n-x+1)}{n^x}$$

$$= 1$$

Numerator has $n-(n-x) = x$ terms
$\Rightarrow$ highest order of n is x

$$\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{-x} = \lim_{n \to \infty} \frac{1}{\underbrace{(1-\frac{\lambda}{n})\cdots(1-\frac{\lambda}{n})}_{x \text{ times}}} = 1$$

$\therefore f_X(x) \xrightarrow{n\to\infty} \dfrac{e^{-\lambda}\lambda^x}{x!}$ i.e $X \xrightarrow{n\to\infty} \text{Poisson}(\lambda)$

Q.8. by defⁿ :- $E(C) = \sum_{i=1}^{\infty} i \times P(C=i)$

b/c C runs from 1 to ∞

method (1)

$$R.H.S = \sum_{i=0}^{\infty} P(C>i)$$

$$= P(C>0) + P(C>1) + P(C>2) + \cdots$$

$$= \sum_{i=1}^{\infty} P(C=i) + \sum_{i=2}^{\infty} P(C=i) + \sum_{i=3}^{\infty} P(C=i) + \cdots$$

$$= \left[\sum_{i=1}^{1} P(C=i) + \sum_{i=2}^{\infty} P(C=i)\right] + \left(\sum_{i=2}^{2} P(C=i) + \sum_{i=3}^{\infty} P(C=i)\right) + \left(\sum_{i=3}^{3} P(C=i) + \sum_{i=4}^{\infty} P(C=i)\right)$$

$$= \left[1 \cdot P(C=1) + \sum_{i=2}^{2} P(C=i) + \sum_{i=3}^{\infty} P(C=i)\right] + \cdots + \sum_{i=2}^{2} P(C=i) + \sum_{i=3}^{3} P(C=i) + \sum_{i=4}^{\infty} P(C=i) + \cdots$$

$$= 1 \cdot P(C=1) + 2P(C=2) + 3P(C=3) + \cdots = \sum i\, P(C=i)$$

Re-writing

$$E(C) = \sum_{i=0}^{\infty} P(C > i) = \sum_{i=1}^{\infty} i \cdot P(C = i)$$

$$= \sum_{i=0}^{\infty} \left[1 - P(C \le i)\right]$$

$$= \sum_{i=0}^{\infty} \left[1 - \sum_{j=1}^{i} P(C = j)\right]$$

$$= \sum_{i=0}^{\infty} \left[1 - \sum_{j=1}^{i} (1-p)^{j-1} p\right]$$

finite Geom series

$$= \sum_{i=0}^{\infty} \left[1 - p \times \frac{1 - (1-p)^i}{1 - (1-p)}\right]$$

infinite Geom. series

$$= \sum_{i=0}^{\infty} (1-p)^i = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

$$= \frac{1}{0.4}$$

$$= \frac{10}{4} = 5/2$$

Method (2)

Using Law of total expectation.

Let A be the event that the system fails in the 1st hr

Recall squirrel ~~problem~~ !!

$$E(C) = E(C|A) P(A) + E(C|A') P(A')$$

$$= 1 \times P(A) + \{1 + E(C)\} P(A')$$

$$E(C) = p + (1 + E(C))(1-p)$$
$$= \cancel{p} + 1 + E(C) - \cancel{p} - pE(C).$$
$$= 1 + E(C)(1-p)$$

$$\Rightarrow E(C)[1-(1-p)] = 1$$
$$\Rightarrow E(C) = \frac{1}{p} = \frac{1}{0.4}$$
$$= \frac{10}{4}$$
$$= 5/2 \text{ hrs.}$$

Q. 9. (Power of Indicator RVs; useful even if we don't know the probability D^n of a random phenomenon?)

Let Y be the no. of men who get their own hats back.

$$Y_i = \begin{cases} 1 ; & i^{th} \text{ man gets his hat back} \\ 0 ; & i^{th} \text{ man does NOT get his hat back.} \end{cases}$$

Then $Y = \sum_{i=1}^{100} Y_i$

$$E(Y) = \sum_{i=1}^{100} E(Y_i)$$
$$= \sum_{i=1}^{100} P(Y_i = 1) = \sum_{i=1}^{100} \frac{1}{100}$$
$$= 100 \times \frac{1}{100}$$
$$= 1.$$

Class Notes

Q 10

Let us consider the coupons received in the corresponding visit as follows.

1st visit, 2nd visit, 3rd

$\underbrace{\text{blue}}_{X_0}$ $\underbrace{\text{green}}_{X_1}$ $\underbrace{\text{green red}}_{X_2}$ $\underbrace{\text{blue orange}}_{X_3}$ $\underbrace{\text{blue orange gray}}_{X_4}$

Let us partition the above outcomes into zones; s.t. a new zone terminates upon receiving a "new" type of coupon.

This way we have a seq. $X_0 \; X_1 \; X_2 \; \cdots \; X_{10}$ each of varying length

here eg: $X_0 = 1$

$X_1 = 1$

$X_2 = 2$

$X_3 = 2$

$X_4 = 3$

$\vdots$

Let $T = X_0 + X_1 + X_2 + \cdots + X_{10}$

We need to find $E(T)$.

Note X_k is the length of k^{th} game

At the beginning of k^{th} game; we already have $k < 10$ different types of coupons

When we have k types, each visit contains a type w/ probability $k/10$

$\Rightarrow$ Each visit contains a new type w/ probability $1 - \frac{k}{10} = \frac{10-k}{10} = p_{success} = p_{suc}$

$\therefore$ Expected no. of visits until we get the $(k+1)^{th}$ new type is the same as expected/mean time to failure w/ probability $\frac{10-k}{10}$ (Q-8)

$$\Rightarrow E(X_k) = \frac{1}{\frac{10-k}{10}} = \frac{10}{10-k}$$

$$E(T) = \sum_{i=1}^{10} E(X_i) = \frac{10}{10-0} + \frac{10}{10-1} + \frac{10}{10-2} + \dots \frac{10}{1}$$

$$= 10 \left\{ \frac{1}{10} + \frac{1}{9} + \frac{1}{8} + \dots \frac{1}{2} + 1 \right\}$$

$$= 10 \left(H_{10} \right)$$

$H_{10} \rightarrow$ Harmonic Series truncated at 10^{th} term

more true for large no. of types $>>10$ $\rightarrow \sim 10 \log_e 10$