

## Definition of Dominant Eigenvalue and Dominant Eigenvector

Let  $\lambda_1, \lambda_2, \dots$ , and  $\lambda_n$  be the eigenvalues of an  $n \times n$  matrix  $A$ .  $\lambda_1$  is called the **dominant eigenvalue** of  $A$  if

$$|\lambda_1| > |\lambda_i|, \quad i = 2, \dots, n.$$

The eigenvectors corresponding to  $\lambda_1$  are called **dominant eigenvectors** of  $A$ .

Not every matrix has a dominant eigenvalue. For instance, the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(with eigenvalues of  $\lambda_1 = 1$  and  $\lambda_2 = -1$ ) has no dominant eigenvalue. Similarly, the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(with eigenvalues of  $\lambda_1 = 2$ ,  $\lambda_2 = 2$ , and  $\lambda_3 = 1$ ) has no dominant eigenvalue.

## ***Finding a Dominant Eigenvalue***

Find the dominant eigenvalue and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

**The characteristic polynomial of A is**

$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$ . Therefore the eigenvalues of A are  $\lambda_1 = -1$  and  $\lambda_2 = -2$ , of which the dominant one is  $\lambda_2 = -2$ .

the dominant eigenvectors of A (those corresponding to  $\lambda_2 = -2$ ) are of the form

$$\mathbf{x} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0.$$



How? Solve  $A\mathbf{x} = -2\mathbf{x}$ , where  $\mathbf{x} = (x_1 \ x_2)^T$

## The Power Method (iterative method)

First we assume that the matrix  $A$  has a dominant eigenvalue with corresponding dominant eigenvectors. Then we choose an initial approximation  $\mathbf{x}_0$  of one of the dominant eigenvectors of  $A$ . This initial approximation must be a *nonzero* vector in  $R^n$ . Finally we form the sequence given by

$$\begin{aligned}\mathbf{x}_1 &= A\mathbf{x}_0 \\ \mathbf{x}_2 &= A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0 \\ \mathbf{x}_3 &= A\mathbf{x}_2 = A(A^2\mathbf{x}_0) = A^3\mathbf{x}_0 \\ &\vdots \\ \mathbf{x}_k &= A\mathbf{x}_{k-1} = A(A^{k-1}\mathbf{x}_0) = A^k\mathbf{x}_0.\end{aligned}$$

For large powers of  $k$ , and by properly scaling this sequence, we will see that we obtain a good approximation of the dominant eigenvector of  $A$ .

## *Approximating a Dominant Eigenvector by the Power Method*

Complete six iterations of the power method to approximate a dominant eigenvector of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

We begin with an initial nonzero approximation of

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We then obtain the following approximations.

	<i>Iteration</i>		<i>Approximation</i>
$\mathbf{x}_1 = A\mathbf{x}_0 =$	$\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix}$	$\Rightarrow$	$-4 \begin{bmatrix} 2.50 \\ 1.00 \end{bmatrix}$
$\mathbf{x}_2 = A\mathbf{x}_1 =$	$\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 28 \\ 10 \end{bmatrix}$	$\Rightarrow$	$10 \begin{bmatrix} 2.80 \\ 1.00 \end{bmatrix}$
$\mathbf{x}_3 = A\mathbf{x}_2 =$	$\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 28 \\ 10 \end{bmatrix} = \begin{bmatrix} -64 \\ -22 \end{bmatrix}$	$\Rightarrow$	$-22 \begin{bmatrix} 2.91 \\ 1.00 \end{bmatrix}$
$\mathbf{x}_4 = A\mathbf{x}_3 =$	$\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -64 \\ -22 \end{bmatrix} = \begin{bmatrix} 136 \\ 46 \end{bmatrix}$	$\Rightarrow$	$46 \begin{bmatrix} 2.96 \\ 1.00 \end{bmatrix}$
$\mathbf{x}_5 = A\mathbf{x}_4 =$	$\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 136 \\ 46 \end{bmatrix} = \begin{bmatrix} -280 \\ -94 \end{bmatrix}$	$\Rightarrow$	$-94 \begin{bmatrix} 2.98 \\ 1.00 \end{bmatrix}$
$\mathbf{x}_6 = A\mathbf{x}_5 =$	$\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} -280 \\ -94 \end{bmatrix} = \begin{bmatrix} 568 \\ 190 \end{bmatrix}$	$\Rightarrow$	$190 \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix}$

If  $\mathbf{x}$  is an eigenvector of a matrix  $A$ , then its corresponding eigenvalue is given by

$$\lambda = \frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}}.$$

This quotient is called the **Rayleigh quotient**.

Why is **this** true?

Since  $\mathbf{x}$  is an eigenvector of  $A$ , we know that  $A\mathbf{x} = \lambda\mathbf{x}$ , and we can write

$$\frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{\lambda\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{\lambda(\mathbf{x} \cdot \mathbf{x})}{\mathbf{x} \cdot \mathbf{x}} = \lambda.$$

### *Approximating a Dominant Eigenvalue*

Use the result of Example 1 to approximate the dominant eigenvalue of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

After the sixth iteration of the power method in Example 1, we had obtained.

$$\mathbf{x}_6 = \begin{bmatrix} 568 \\ 190 \end{bmatrix} \approx 190 \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix}.$$

With  $\mathbf{x} = (2.99, 1)$  as our approximation of a dominant eigenvector of  $A$ , we use the Rayleigh quotient to obtain an approximation of the dominant eigenvalue of  $A$ . First we compute the product  $A\mathbf{x}$ .

$$A\mathbf{x} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix} = \begin{bmatrix} -6.02 \\ -2.01 \end{bmatrix}$$

Then, since

$$A\mathbf{x} \cdot \mathbf{x} = (-6.02)(2.99) + (-2.01)(1) \approx -20.0$$

and

$$\mathbf{x} \cdot \mathbf{x} = (2.99)(2.99) + (1)(1) \approx 9.94,$$

we compute the Rayleigh quotient to be

$$\lambda = \frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} \approx \frac{-20.0}{9.94} \approx -2.01,$$

which is a good approximation of the dominant eigenvalue  $\lambda = -2$ .

### ***The Power Method with Scaling***

Calculate seven iterations of the power method with *scaling* to approximate a dominant eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}.$$

Use  $\mathbf{x}_0 = (1, 1, 1)$  as the initial approximation.

One iteration of the power method produces

$$A\mathbf{x}_0 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix},$$

and by scaling we obtain the approximation

$$\mathbf{x}_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix}.$$



A second iteration yields

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 1.00 \\ 1.00 \\ 2.20 \end{bmatrix}$$

and

$$\mathbf{x}_2 = \frac{1}{2.20} \begin{bmatrix} 1.00 \\ 1.00 \\ 2.20 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.45 \\ 1.00 \end{bmatrix}.$$

Continuing this process, we obtain the sequence of approximations shown in Table.

TABLE 10.6

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$
$\begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.45 \\ 0.45 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.48 \\ 0.55 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.51 \\ 0.51 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.49 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}$

From Table 10.6 we approximate a dominant eigenvector of  $A$  to be

$$\mathbf{x} = \begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}.$$

Using the Rayleigh quotient, we approximate the dominant eigenvalue of  $A$  to be  $\lambda = 3$ . (For this example you can check that the approximations of  $\mathbf{x}$  and  $\lambda$  are exact.)

REMARK: Note that the *scaling factors* used to obtain the vectors in Table 10.6,

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$
↓	↓	↓	↓	↓	↓	↓
5.00	2.20	2.82	3.13	3.02	2.99	3.00,

are approaching the dominant eigenvalue  $\lambda = 3$ .

## Convergence of the Power Method

necessary condition



If  $A$  is an  $n \times n$  diagonalizable matrix with a dominant eigenvalue, then there exists a nonzero vector  $\mathbf{x}_0$  such that the sequence of vectors given by

$$A\mathbf{x}_0, \quad A^2\mathbf{x}_0, \quad A^3\mathbf{x}_0, \quad A^4\mathbf{x}_0, \quad \dots, \quad A^k\mathbf{x}_0, \quad \dots$$

approaches a multiple of the dominant eigenvector of  $A$ .

# Inverse power method

A simple change allows us to compute the **smallest** eigenvalue (in magnitude). Let us assume now that  $A$  has eigenvalues

$$|\lambda_1| \geq |\lambda_2| \cdots > |\lambda_n|.$$

Then  $A^{-1}$  has eigenvalues  $\lambda_j^{-1}$  satisfying

$$|\lambda_n^{-1}| > |\lambda_2^{-1}| \geq \cdots \geq |\lambda_1^{-1}|.$$

Thus if we apply the power method to  $A^{-1}$ , the algorithm will give  $1/\lambda_n$ , yielding the smallest eigenvalue of  $A$  (after taking the reciprocal at the end).

Note that in practice, instead of computing  $A^{-1}$ , we first compute an  $LU$  factorization of  $A$ , and then solve

$$Ax^{(k+1)} = x^{(k)}$$

at each step, which only takes  $O(n^2)$  operations after the initial work.

Now suppose instead we want to find the eigenvalue closest to a number  $\mu$ . Notice that the matrix  $(A - \mu I)^{-1}$  has eigenvalues

$$\frac{1}{\lambda_j - \mu}, \quad j = 1, \dots, n.$$

The eigenvalue of largest magnitude will be  $1/(\lambda_{j_0} - \mu)$  where  $\lambda_{j_0}$  is the closest eigenvalue to  $\mu$  (assuming there is only one). This leads to the **inverse power method** (sometimes called **inverse iteration**):

**Inverse power method:** To find the eigenvalue of  $A$  closest to  $\mu$ ,

1) Apply the power method to  $(A - \mu I)^{-1}$ , solving

$$(A - \mu I)\mathbf{x}_k = \mathbf{x}_{k-1}$$

at each step using some linear system solver (e.g. LU factorization).

2) Compute  $\lambda$  from the output  $1/(\lambda - \mu)$ .

Note that if  $\mu$  is fixed, the LU factorization only needs to be computed once.!