Introduction to Time Series Analysis.

Time series data $(y_1, y_2, \dots \to y_T)$

or

$$\{y_\ell\}_{\ell=-\infty}^{\infty} = \{\dots, y_{-1}, y_0, y_1, y_2, \dots, y_t, y_{t+1}, y_{t+2}, \dots\}$$

e.g. Gaussian white noise
$y_t = \epsilon_t$ where $\{\epsilon_\ell\}_{\ell=-\infty}^{\infty}$ is a seq. of independent variables each of which is sampled from $N(0, \sigma^2)$

Identity operator :- $\mathbb{1} x_t = x_t$

① Lag operator, $\mathcal{L}$:-

$$\mathcal{L} x_t = x_{t-1}$$

$$\mathcal{L}^2 x_t = \mathcal{L}(\mathcal{L} x_t) = x_{t-2}$$

$$\Rightarrow \mathcal{L}^k x_t = x_{t-k}$$

for $|\phi| < 1$

$$(\mathbb{1} - \phi\mathcal{L})^{-1} = \lim_{j \to \infty} (\mathbb{1} + \phi\mathcal{L} + \phi^2\mathcal{L}^2 + \dots + \phi^j\mathcal{L}^j)$$

e.g. A 2nd order difference eqn. can be written in terms of a lag operator as follows:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \omega_t$$

Or equivalently $(\mathbb{1} - \phi_1\mathcal{L} - \phi_2\mathcal{L}^2)\, y_t = \omega_t$

where it can be shown

$$(\mathbb{1} - \phi_1\mathcal{L} - \phi_2\mathcal{L}^2) = (1 - \lambda_1\mathcal{L})(1 - \lambda_2\mathcal{L}); \quad \lambda_1 + \lambda_2 = \phi_1, \quad \lambda_1\lambda_2 = -\phi_2$$

Lag operators in terms of eigen values (evs) of its matrix

$$\mathbb{F} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}$$

$$(1 - \phi_1 \alpha - \phi_2 \alpha^2) = (1 - \lambda_1 \alpha)(1 - \lambda_2 \alpha)$$

where λ_1 & λ_2 are evs of $\mathbb{F}$ given by

$$|\lambda \mathbb{1} - \mathbb{F}| = 0 \Rightarrow \lambda_{1,2} = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

In general, $\mathbb{F} = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$

for $y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \omega_t$

$$\equiv (1 - \phi_1 \alpha - \phi_2 \alpha^2 - \cdots - \phi_p \alpha^p) y_t = \omega_t$$

where $(1 - \phi_1 \alpha - \phi_2 \alpha^2 - \cdots - \phi_p \alpha^p)$
$= (1 - \lambda_1 \alpha)(1 - \lambda_2 \alpha) \cdots (1 - \lambda_p \alpha)$

Where λ_is are the evs of $\mathbb{F}$

2 Expectations, Stationarity & Ergodicity

R.V. $\equiv Y$

observed samples $= \{y_1, y_2, \ldots, y_T\}$

e.g. Consider a collection of T independent & identically distributed (iid) R.V.

$\{\epsilon_t\}_{t \geq 1}$ w/ $\epsilon_t \sim N(0, \sigma^2)$.

i.e. samples from Gaussian white noise pr.

Let the data be arranged as follows:-

1st realization $\{y_l^{(1)}\}_{l=-\infty}^{\infty}$ ← random sample at t

2nd realization $\{y_l^{(2)}\}_{l=-\infty}^{\infty}$ random sample at t → $\underbrace{\{y_t^{(1)}, y_t^{(2)}, \dots y_t^{(I)}\}}$

⋮

I^{th} realization $\{y_l^{(I)}\}_{l=-\infty}^{\infty}$ random sample at t

sample of I realizations of R.V. Y_t

eg. if $Y_t \sim \epsilon_t \sim N(0, \sigma^2)$

$$f_{Y_t}(y_t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y_t^2}{2\sigma^2}}$$

$$E(Y_t) = \int_{-\infty}^{\infty} y_t f_{Y_t}(y_t)\, dy_t$$

$$= \text{prob.} \lim_{I \to \infty} \underbrace{\left(\frac{1}{I}\right) \sum_{i=1}^{\infty} Y_t^{(i)}}_{\text{ensemble avg.}} \quad (2.1)$$

eg. time trend

$$Y_t = \beta t + \epsilon_t$$

$$E(Y_t) = \beta t$$

The variance of the R.V. Y_t denoted by γ_{0t} is defined as

$$\gamma_{0t} = E(Y_t - \mu_t)^2 = \int_{-\infty}^{\infty} (y_t - \mu_t)^2 f_{Y_t}(y_t)\, dy_t$$

j^{th} Auto covariance of Y_t

$$\gamma_{jt} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (y_t - \mu_t)(y_{t-j} - \mu_{t-j}) f_{Y_t, Y_{t-1}, \ldots Y_{t-j}} (y_t, y_{t-1}, \ldots, y_{t-j}) \times dy_t \, dy_{t-1} \cdots dy_{t-j}$$

$$= E\left[(Y_t - \mu_t)(Y_{t-j} - \mu_{t-j})\right]$$

This is akin to

$$Cov(X, Y) = E\left[(X - \mu_X)(Y - \mu_Y)\right]$$

As probability limit of an ensemble average

$$\gamma_{jt} = \underset{I \to \infty}{\text{prob lim}} \left(\frac{1}{I}\right) \sum_{i=1}^{I} \left(Y_t^{(i)} - \mu_t\right)\left(Y_{t-j}^{(i)} - \mu_{t-j}\right) \quad \text{——} (2.2)$$

eg for $Y_t = \mu + \epsilon_t$; $\epsilon_t \sim N(0, \sigma^2)$

$$\gamma_{jt} = E\left[(Y_t - \mu)(Y_{t-j} - \mu)\right]$$
$$= E(\epsilon_t \epsilon_{t-j}) = 0 \quad \forall j \neq 0$$

③ Stationarity

(Def^n)

(i) Co-variance stationary or Weakly stationary

$E(Y_t) = \mu$ (independent of t)

$E[(Y_t - \mu)(Y_{t-j} - \mu)] = \gamma_j$ (independent of t)

$\forall t$ & any j

(Q-2)

"Additionally for any covariance Stationary process $\{Y_t\}$

$$\boxed{\gamma_j = \gamma_{-j} \quad \forall j}$$

(ii) Strictly/Strongly Stationary process.

the joint D^n of $(Y_t, Y_{t+j_1}, Y_{t+j_2}, \ldots, Y_{t+j_n})$ depends only on the intervals separating the dates $(j_1, j_2, \ldots, j_n)$ & not on the date itself (t).

Strong Stnty $\Rightarrow$ Weak Stnty.
(w/ finite 2^{nd} moments)

Weak Stny $\not\Rightarrow$ Strong Stny
if higher moment $E(Y_t^3)$ depend on time.

④ Ergodicity

Q) When can ensemble averages like those defined in eqs (2.1) & (2.2) be replaced by time averages?

$$\bar{y} \equiv \frac{1}{T} \sum_{t=1}^{T} y_t^{(1)}$$

Ans) When $\{Y_t\}$ is an ergodic process

i.e. when $\bar{y} \xrightarrow{p} E(Y_t) \Leftrightarrow \gamma_j \to 0$ sufficiently quickly as $j \to \infty$.

i.e. $\sum_{j=0}^{\infty} |\gamma_j| < \infty \Rightarrow \{Y_t\}$ is ergodic for mean.

* When $\{Y_t\}$ is a stny Gaussian process, $\sum_{j=0}^{\infty}|r_j| < \infty$ is sufficient to ensure ergodicity for all means. Stny & Ergodicity are not always ~~the~~ equivalent although in many applications it turns out so.

(5) White Noise.

$\{\epsilon_t\}_{t=-\infty}^{\infty}$

$$E(\epsilon_t) = 0 \quad \text{— (5.1)}$$

$$E(\epsilon_t^2) = \sigma^2 \quad \text{— (5.2)}$$

& ϵ_s are uncorrelated across time

i.e. $E(\epsilon_t \epsilon_\tau) = 0 \quad \forall\, t \neq \tau$ — (5.3)

→ While noise pr.

Sometimes (5.3) is replaced by a slightly stronger condⁿ :-

$\epsilon_t, \epsilon_\tau$ are independent $\forall\, t \neq \tau$

if additionally $\epsilon_t \sim N(0, \sigma^2)$

$\Rightarrow \{\epsilon_t\}$ is Gaussian White noise pr.

Moving Average processes-

MA(1)
$\uparrow$ 1st order

$\{\epsilon_t\}$ is white noise

$$Y_t = \mu + \underbrace{\epsilon_t + \theta\epsilon_{t-1}}_{\text{behaves/models weighted avg. of last 2 terms of } \epsilon_t} ; \quad \mu, \theta \text{ are Constants}$$

$$E(Y_t) = \mu + E(\epsilon_t)^{\nearrow 0} + \theta E(\epsilon_{t-1})^{\nearrow 0} = \mu$$

$$Var(Y_t) = Var(\mu) + Var(\epsilon_t) + \theta^2 Var(\epsilon_t)$$
$$E[(Y_t-\mu)^2] = 0 + \sigma^2 + \theta^2\sigma^2$$
$$= (1+\theta^2)\sigma^2 \quad \text{Not a } f^n \text{ of time}$$

$Var(X) = E(X^2) - (E(X))^2$

1st Auto Covariance
$$= E[(Y_t-\mu)(Y_{t-1}-\mu)]$$
$$= E[(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-1} + \theta\epsilon_{t-2})]$$
$$= E[\epsilon_t\epsilon_{t-1} + \theta\epsilon_{t-1}^2 + \theta^2\epsilon_{t-1}\epsilon_{t-2} + \theta\epsilon_t\epsilon_{t-2}]$$
$$= 0 + \theta E(\epsilon_{t-1}^2) + 0 + 0$$
$$= \theta\, Var(\epsilon_{t-1})$$
$$= \theta\sigma^2 \quad \text{Not a } f^n \text{ of time.}$$

Higher Autocovariance
$$E[(Y_t-\mu)(Y_{t-j}-\mu)] = 0 \quad \forall j > 1$$

$\Rightarrow$ MA(1) is Covariance (weakly) stny irrespective of value of θ. —①

Moreover $\sum_{j=0}^{\infty}|\gamma_j| = |\gamma_0| + \sum_{j=1}^{\infty}|\gamma_j|$
$$= (1+\theta^2)\sigma^2 + |\gamma_1| + 0$$
$$= (1+\theta^2)\sigma^2 + \theta\sigma^2 < \infty \quad ②$$

(8)

① & ② $\Rightarrow$ if $\{\epsilon_t\}$ is Gaussian white noise $\forall$ $j \geq 1$
then MA(1) is ergodic.

j^{th} Auto-correlation of covariance – Stny process

$$\rho_j = \frac{\gamma_j}{\gamma_0} \quad \text{b/c} \quad Corr(Y_t, Y_{t-j}) = \frac{Cov(Y_t, Y_{t-j})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-j})}} = \frac{\gamma_j}{\sqrt{\gamma_0}\sqrt{\gamma_0}} = \frac{\gamma_j}{\gamma_0} = \rho_j$$

Note $|\rho_j| \leq 1 \;\forall j$ by Cauchy-Schwartz inequality.

for MA(1): $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta}{1+\theta^2}$

$\rho_j = 0 \;\forall j > 1$

MA(q) process :-

$$Y_t = \mu + \epsilon_t + \theta\epsilon_{t-1} + \theta^2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}$$

Where $\{\epsilon_t\}$ is a white noise process

$(\theta_1, \theta_2, \ldots, \theta_q) \in q$ real no.s.

$E(Y_t) = \mu$

$\gamma_0 = E[(Y_t - \mu)^2] = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma^2$

$$\gamma_j = \begin{cases} (\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \cdots + \theta_q\theta_{q-j})\sigma^2 & ; j = 1, 2, \ldots, q \\ 0 & ; j > q. \end{cases}$$

MA(∞) :-

$$Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$$

MA(∞) is cov-stny if $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ but-

we will often use the slightly stricter condition of absolute-summability i.e. $\sum_{j=0}^{\infty} |\psi_j| < \infty$. (Note $\sum_{j=0}^{\infty} |\psi_j| < \infty \Rightarrow \sum_{j=0}^{\infty} \psi_j^2 < \infty$)

for MA(∞) w/ $\sum_{j=0}^{\infty} |\psi_j| < \infty$:-

$$E(Y_t) = \lim_{T \to \infty} E(\mu + \psi_0 \epsilon_t + \psi_1 \epsilon_{t-1} + \dots + \psi_T \epsilon_{t-T})$$

$$= \mu$$

$$\gamma_0 = E(Y_t - \mu)^2 = \dots = \lim_{T \to \infty} (\psi_0^2 + \psi_1^2 + \dots \psi_T^2)\sigma^2$$

$$\gamma_j = E[(Y_t - \mu)(Y_{t-j} - \mu)]$$

$$= \sigma^2(\psi_j \psi_0 + \psi_{j+1}\psi_1 + \dots + \psi_{j+m}\psi_m + \dots)$$

if coeff. are absolutely summable $\Rightarrow \sum_{j=0}^{\infty} |\gamma_j| < \infty$

$\therefore$ MA(∞) with $\sum_{j=0}^{\infty} |\psi_j| < \infty$ is ergodic for mean

& if $\{\epsilon_t\}$ is Gaussian; then MA(∞) is ergodic for all ~~[illegible]~~ moments.

(7) Auto regressive processes

AR(1):—

$$Y_t = c + \phi Y_{t-1} + \epsilon_t \quad \text{where } \epsilon_t \text{ is white noise} \qquad (7.1)$$

i.e. $(1-\phi L)Y_t = c + \epsilon_t$

It can be shown that when $|\phi| \geq 1$;

$\nexists$ a cov-stny process Y_t w/ $\gamma_0 < \infty$ that satisfies eq (7.1).

if $|\phi| < 1$; then there is a cov-stny process Y_t satisfying (7.1) which is obtained by the stable soln to (7.1)

which is $Y_t = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots$;

$$w_t = c + \epsilon_t$$

i.e. $Y_t = (c+\epsilon_t) + \phi(c+\epsilon_{t-1}) + \phi^2(c+\epsilon_{t-2}) + \cdots$

$$= \frac{c}{1-\phi} + \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \cdots$$

i.e. AR(1) can be viewed as a MA(∞) pr. w/ $\boxed{\mu = \frac{c}{1-\phi}}$ & $\psi_j = \phi^j$

so $\sum_{j=0}^{\infty} |\psi_j| = \sum_{j=0}^{\infty} |\phi|^j = \frac{1}{1-|\phi|} < \infty$ for $|\phi| < 1$.

In what follows, we will always assume $|\phi| < 1$ unless otherwise stated.

(1) contd...

$$\gamma_0 = E[(Y_t - \mu)^2]$$
$$= \frac{\sigma^2}{1-\phi^2}$$
$$\gamma_j = E[(Y_t - \mu)(Y_{t-j} - \mu)] = \frac{\phi^j}{1-\phi^2}\sigma^2$$

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \phi^j$$

AR(2) process.

$$\left.\begin{aligned} Y_t &= c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t \\ \text{i.e. } (1 - \phi_1 L - \phi_2 L^2) Y_t &= c + \epsilon_t \end{aligned}\right\} \text{ --- } (7.2)$$

If the roots of $(1-\phi_1 z - \phi_2 z^2) = 0$ lie outside the unit circle; then (7.2) is stable

$\Rightarrow$ then AR(2) is cov-stny and

$$\psi(L) = (1 - \phi_1 L - \phi_2 L^2)^{-1} = \psi_0 + \psi_1 L + \psi_2 L^2 + \cdots$$

& ψ_j can be found from the (1,1) element of the matrix F raised to j^{th} power ($\psi_j = f_{11}^{(j)}$)

see pg. 12 in Hamilton.

&

It can be shown that $\psi(L) = \frac{1}{(1-\phi_1-\phi_2)}$

$$\sum_{j=0}^{\infty} |\psi_j| < \infty$$

from which it follows $\mu = \frac{c}{1-\phi_1-\phi_2}$

We will not derive the results for the
but simply state :-

$$\gamma_j = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} \quad \text{for } j = 1, 2, \ldots$$

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2} \quad ; j = 1, 2, \ldots$$

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2$$
$$= \ldots$$
$$= \frac{(1-\phi_2)\sigma^2}{(1+\phi_2)\left[(1-\phi_2)^2 - \phi_1^2\right]}$$

<u>AR(p)</u>

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \epsilon_t$$

$$\mu = \frac{c}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

$$\gamma_j = \begin{cases} \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} + \cdots + \phi_p \gamma_{j-p} & ; j = 1, 2, \ldots \\ \phi_1 \gamma_1 + \phi_2 \gamma_2 + \cdots + \phi_p \gamma_p & ; j = 0 \end{cases}$$

Additionally $\gamma_j = \gamma_{-j}$

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2} + \cdots + \phi_p \rho_{j-p} \quad ; j = 1, 2, \ldots$$

is the Yule-Walker eqns.

Mixed pr.

ARMA(p,q)

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

equivalently

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) Y_t = c + (1 + \theta_1 L + \theta_2 L^2 + \cdots \theta_q L^q) \epsilon_t \quad (\phi\text{-}1)$$

provided that the roots of $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0$ lie outside the unit circle; both sides can be divided by $(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)$ to obtain

$$Y_t = \mu + \psi(L) \epsilon_t \; ; \text{ where}$$

$$\psi(L) = \frac{(1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)}{(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)} \; ,$$

$$\sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$\mu = \frac{c}{(1 - \phi_1 - \phi_2 - \cdots - \phi_p)}$$

The stny of ARMA(p,q) process depends entirely on the AR parameters $(\phi_1, \phi_2, \ldots, \phi_p)$.

The calculations for γ_j are complicated.

** MA(1) is an AR(∞) process!

Practise problems

Q) Consider the MA(2) process

$$Y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} ; \quad \theta_1 = 2/5$$
$$\theta_2 = -1/5$$

Calculate all the auto-correlation f^n's. ϵ_t is white noise pr.

Soln:-
$$\gamma_j = E[(Y_t - \mu)(Y_{t-j} - \mu)]$$
$$= E(Y_t Y_{t-j}) - \mu E(Y_{t-j}) - \mu E(Y_t) - \mu^2$$
$$\equiv E[(\epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2})(\epsilon_{t-j} + \theta_1 \epsilon_{t-j-1} + \theta_2 \epsilon_{t-j-2})]$$
$$= E[\underline{\underline{\epsilon_t \epsilon_{t-j}}}] + \theta_1 E(\epsilon_t \epsilon_{t-j-1}) + \theta_2 E(\epsilon_t \epsilon_{t-j-2})$$
$$+ \theta_1 E(\epsilon_{t-1}\epsilon_{t-j}) + \theta_1^2 E\underline{\underline{(\epsilon_{t-1}\epsilon_{t-j-1})}}$$
$$+ \theta_1\theta_2 E(\epsilon_{t-1}\epsilon_{t-j-2})$$
$$+ \theta_2 E(\epsilon_{t-2}\epsilon_{t-j})$$
$$+ \theta_2\theta_1 E(\epsilon_{t-2}\epsilon_{t-j-1})$$
$$+ \theta_2^2 E\underline{\underline{(\epsilon_{t-2}\epsilon_{t-j-2})}}$$

$\because E(\epsilon_t^2) = E(\epsilon_{t-1}^2)$

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2)\sigma^2$$

$$\gamma_1 = 0 + 0 + 0 + \theta_1 \sigma^2 + 0 + 0 + 0 + \theta_1\theta_2\sigma^2$$

$$\gamma_1 = \theta_1(1+\theta_2)\sigma^2$$

Likewise, $\gamma_2 = \theta_2 \sigma^2$; and $\gamma_j = 0 \;\forall j > 2$

$$\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1(1+\theta_2)\sigma^2}{(1+\theta_1^2+\theta_2^2)\sigma^2} = \frac{\frac{2}{5}\left(1-\frac{1}{5}\right)}{1+\frac{4}{25}+\frac{1}{25}} = \frac{\frac{2}{5}\times\frac{4}{5}}{\frac{30}{25}} = \frac{8}{30} = \frac{4}{15}$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\theta_2}{1+\theta_1^2+\theta_2^2} = \frac{-1/5}{1+\frac{4}{25}+\frac{1}{25}} = -\frac{1}{6}$$

$$\rho_j = 0 \quad \forall j > 2$$

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