Linear Algebra

Engineering Mathematics In Action

Course Designed by Prof. Amrik Sen
Course Conducted by Prof. Amrik Sen and Prof. Sanjay K. Bose
Plaksha University

FIELD (Definition): A field is a set $\mathbb{F}$ of numbers with the property that if $a, b \in \mathbb{F}$, then $a+b$, $a-b$, ab and $\frac{a}{b}$ are also in $\mathbb{F}$ (assuming, of course, that $b \neq 0$ *in the expression* $\frac{a}{b}$.

e.g. $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are fields of numbers

$\mathbb{N}$ and $\mathbb{Z}$ are **not** fields of numbers!

$\mathbb{Q}$ - Rational Numbers
$\mathbb{R}$ - Real Numbers
$\mathbb{C}$ - Complex Numbers

$\mathbb{N}$: Natural Numbers (positive integers)
$\mathbb{Z}$: Integers

VECTOR SPACES (Definition): A vector space, $\boldsymbol{\mathcal{V}}$ consists of a set $\mathbb{V}$ of vectors, a field $\mathbb{F}$ of scalars, and **two** operations:

i. **Vector Addition:** *if* $v, w \in \mathbb{V}$, *then* $v + w \in \mathbb{V}$

ii. **Scalar Multiplication:** $c \in \mathbb{F}$ *and* $v \in \mathbb{V}$ *produces a new vector* $cv \in \mathbb{V}$

These scalars and vectors also satisfy the following **axioms**

i. ***Associativity*** *of addition:* $(v + u) + w = v + (u + w) \;\; \forall v, u, w \in \mathbb{V}$
ii. ***Associativity*** *of multiplication:* $(ab)u = a(bu)$, *for any* $a, b \in \mathbb{F}, u \in \mathbb{V}$
iii. ***Distributivity:*** $(a + b)u = au + bu \;\; and \;\; a(u + v) = au + av$
$\forall\, a, b \in \mathbb{F},\; u \in \mathbb{V},\; v \in \mathbb{V}$
iv. ***Unitarity***: $1u = u \;\; \forall u \in \mathbb{V}$
v. ***Existence of zero***: $\exists 0 \in \mathbb{V} \;\; s.t. \;\; u + 0 = u \quad \forall u \in \mathbb{V}$
vi. ***Negation***: *For every* $u \in \mathbb{V}, \exists(-u) \in \mathbb{V} \;\; s.t. \;\; u + (-u) = 0 \in \mathbb{V}$

From the notes of Prof. Amrik Sen, Plaksha University

VECTOR SPACES (Examples):

1) Let $\mathbb{V}$ be the set of $n \times 1$ column matrices (vectors), $\mathbb{F}$ be the field of *reals* $\mathbb{R}$, and the laws of vector addition and scalar multiplication are defined as:

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \cdot \\ \cdot \\ x_n + y_n \end{pmatrix} \quad \text{and} \quad c \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \cdot \\ \cdot \\ cx_n \end{pmatrix}.$$

HW: Verify that the above indeed constitutes a vector space!
(Check that the axioms are satisfied.)

VECTOR SPACES (Examples):

2) Let $\mathbb{V}$ be the set of all continuous functions $f: \mathbb{R} \to \mathbb{R}$, let the field of scalars be $\mathbb{R}$, and let the operations be as usually defined.

HW: Verify that the above indeed constitutes a vector space!

3) Let $\mathbb{V}$ be the set of all continuous functions $f: \mathbb{R} \to \mathbb{R}$ that satisfy the equation $f'' = -f$. (Can you think of any function that satisfies this property? Cosine, Sine?)

Let the field of scalars be $\mathbb{R}$. The operations are defined in the usual manner. *Hint: $Suppose\ f_1, f_2 \in \mathbb{V}, c \in \mathbb{R}; then\ (f_1 + f_2)'' = f_1'' + f_2'' = -f_1 - f_2 = -(f_1 + f_2)$; and $(cf_1)'' = cf_1'' = c(-f_1) = -(cf_1)$.*
Are these results consistent with the definition of the vector space?
Also check whether all axioms are compliant?

LINEAR INDEPENDENCE OF VECTORS

Definition (Linearly_dependent_vectors):
Let $\mathcal{V}$ be a vector space and $\mathcal{X} \subset \mathcal{V}$ be a non-empty subset. Then $\mathcal{X}$ is ***linearly dependent*** if there are distinct vectors $v_1, v_2, \ldots, v_k \in \mathcal{X}, and$ $scalars$ $c_1, c_2, \ldots, c_k$ ***(not all of them zero)***, *s.t.* $c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0$.

This is equivalent to saying that *at least one of the vectors* v_i *can be expressed as a linear combination of the others, i.e.* $v_i = \sum_{j \neq i} -(\frac{c_j}{c_i}) v_j$

Definition (Linearly independent vectors):
A subset which is not linearly dependent is said to be ***linearly independent***. Thus a set of distinct vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if an equation of the form $c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0$ always implies that $c_1 = c_2 = \ldots = c_k = 0$.

Geometrical Interpretation of Linear Dependence

Let V_1, V_2, V_3 be the vectors in 3D-Euclidean space $\mathbb{R}^3$ with a common origin. If these vectors form a *linearly dependent* set, then one of them, say V_1, can be expressed as a linear combination of the other two: $V_1 = aV_2 + bV_3$. This implies, by the parallelogram law, that the three vectors are **co-planar.**

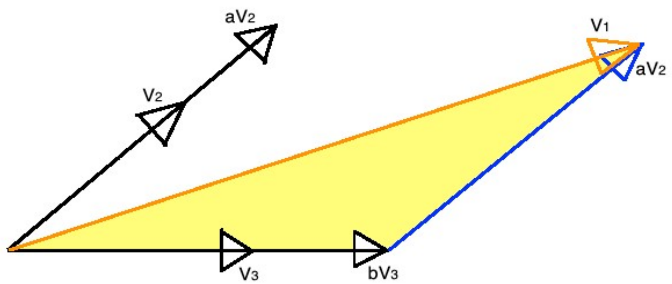


Fig. 1: Linear dependence of vectors is equivalent to coplanar geometry

In fact, **linearly dependent set of vectors with common origin $\Leftrightarrow$ co-planar.**

Can you think of a similar interpretation of vectors in $\mathbb{R}^2$?

Consider what happens when we have three vectors **A**, **B** and **C**, from a common origin, in a 2-dimensional vector space, where –

A = $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ **B** = $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and **C** = $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Can I represent the vector **C** as a linear combination of the vectors **A** and **B** such as **C** = α**A** + β**B** ?

Yes, if we choose α and β as $\alpha = \dfrac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1}$ and $\beta = \dfrac{c_2 a_1 - c_1 a_2}{a_1 b_2 - a_2 b_1}$

Looking at these, we can immediately conclude that this cannot be done if A and B are collinear, because then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$

What can you conclude when either *α* or *β* or both become zero?

Example

Show that these vectors are linearly dependent in $\mathbb{R}^2$

$$\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

We choose scalars c_1, c_2, c_3 such that -

$$c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives

$$\begin{aligned} -c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 + 2c_2 - 4c_3 &= 0 \end{aligned}$$

Since the number of unknowns is more than the number of equations, there will be a non-trivial solution

Therefore, the vectors are Linearly Dependent

Example

Are the polynomials $x+1$, $x+2$, x^2-1 linearly independent in the vector space $P_3(\mathbb{R})$?

Notation: $P_3(\mathbb{R})$ is the set of polynomials of less than degree 3 with real coefficients

We choose scalars c_1, c_2, c_3 such that

$$c_1(x+1)+c_2(x+2)+c_3(x^2-1)=0$$

This gives

$$\begin{aligned} c_1 + 2c_2 - c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

Clearly, this can only have the trivial solution $c_1 = c_2 = c_3 = 0$

Therefore, the polynomials are Linearly Independent

Example

In the vector space $V = P(\mathbb{R})$, consider the subset $S = \{x-1, x^2+1, x^3-x^2-x+3\}$. Is S linearly dependent or linearly independent?

Consider $$a_1(x-1)+a_2(x^2+1)+a_3(x^3-x^2-x+3)=0$$

Equating the coefficients of the powers of x to zero for each term in the LHS, we get -

$$\begin{aligned} -a_1 + a_2 + 3a_3 &= 0 \\ a_1 \qquad\quad - a_3 &= 0 \\ a_2 - a_3 &= 0 \\ a_3 &= 0 \end{aligned}$$

The only solution to this linear homogenous system is the trivial solution, so the vectors in the subset S are **linearly independent**

BASIS OF A VECTOR SPACE (Definition)

Let $\mathbb{X}$ be a non-empty subset of a vector space $\mathcal{V}$. Then $\mathbb{X}$ is called a *basis* of $\mathcal{V}$ if **both** the following are true:

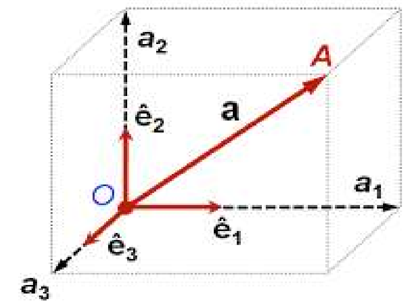
i. $\mathbb{X}$ is linearly independent — cannot generate an element of $\mathbb{X}$ as linear combination of the other elements of $\mathbb{X}$

ii. $\mathbb{X}$ generates $\mathcal{V}$ (i.e. $\mathbb{X}$ *spans* $\mathcal{V}$) — any element of $\mathcal{V}$ can be generated as a linear combination of the elements of $\mathbb{X}$

What is the meaning of "*spans*"?

Technically, it means that every element (vector) in the space $\mathcal{V}$ can be expressed as a linear combination of the elements of the set $\mathbb{X}$.

Examples of Bases



1. Basis of $\mathbb{R}^n$: $\boldsymbol{e_1} = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad \boldsymbol{e_2} = \begin{pmatrix} 0 \\ 1 \\ . \\ . \\ 0 \end{pmatrix}, \quad \dots, \quad \boldsymbol{e_n} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^n$ because (i) they are *linearly independent* (by inspection), and (ii) they *span* $\mathbb{R}^n$ because $c_1\boldsymbol{e_1} + c_2\boldsymbol{e_2} + \cdots c_n\boldsymbol{e_n} = \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix}$ generates any vector in $\mathbb{R}^n$ depending on the values of $c_i \forall i = 1,2,\dots,n$

Examples of Bases (continued):

2. Let $\mathbb{P}_n$ be a vector space of all polynomial functions of degree n or less. The basis of $\mathbb{P}_n$ is $\{1, x, x^2, \ldots., x^n\}$, the set of monomials.
(This is not a unique basis set because $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ also forms a basis where $p_i(x)$ is a polynomial in $\mathbb{P}_n$ of degree i.)

3. Let $\mathbb{M}_{m\times n}(\mathbb{F})$ denote the set of $m \times n$ matrices with entries in $\mathbb{F}$. Then $\mathbb{M}_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$. Vector addition is just matrix addition and scalar multiplication is defined in the obvious way (by multiplying each entry of the matrix by the same scalar). The zero vector is just the zero matrix. One possible choice of basis is the matrices with a single entry equal to 1 and all other entries 0.
(We will study the vector space of matrices in more detail in subsequent lectures!)

PROPERTIES OF BASES:

1. Must every vector space have a basis?

*Ans: Every **non-zero, finitely generated** vector space has a basis!*

2. Does a vector space have a unique basis?

Ans: Usually a vector space will have many bases. e.g., the vector space $\mathbb{R}^2$ *has the basis* $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\}$ *as well as the standard basis* $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$.

3. What is the dimension of a vector space?

Ans: $dim(\mathcal{V}) = no.\ of\ elements\ (vectors)\ in\ the\ basis\ (basis\ set)$.

Can you think of a vector space whose dimension is infinite?

A Few Other Things –

Finitely Generated Vector Space: One where you only need a finite number of elements to generate the vector space using linear combinations, e.g. $\mathbb{R}^2$ needs only (0,1) & (1,0) to generate all vectors in $\mathbb{R}^2$

Infinite Dimensional Vector Space (example): Let P be the vector space of all polynomials in X with rational coefficients. P is infinite dimensional. To see this – If P is given by the span of k polynomials in P, $p_1 \ldots . p_k$ where m is the maximum of the degrees of $p_1 \ldots . p_k$. Then x^{m+1} is a vector which cannot be written as a combination of $p_1 \ldots . p_k$. This is a contradiction so P cannot be finite dimensional.

VECTOR SPACES: *We will study more about the **vector space of $m \times n$ matrices** over the reals and their mathematical utilities!*
In fact much of this course is a study about matrices and their applications in engineering.

Vector space of $m \times n$ matrices over the reals, $\mathbb{M}_{m\times n}(\mathbb{R})$ and associated vector spaces

1. So what are matrices?

Ans: *Matrices are convenient arrangement of numbers in rows and columns lending a compact structure that are amenable to mathematical laws (laws or rules of matrix algebra) that are a consequence of matrix operations like addition, multiplication, transpose, inverse, etc:*

A+B = B+A Commutative Law of Addition (A+B)+C = A+(B+C) Associative Law of Addition
A+0 = A (AB)C = A(BC) Associative Law of Multiplication
AI = A = IA I: Identity Matrix A(B+C) = AB+AC (A+B)C = AC+BC Distributive Laws
A-B = A+(-1)B (cd)A = c(dA) c(AB) = (cA)B = A(cB)
c(A+B) = cA+cB (c+d)A = cA+dA
$(A+B)^T = A^T + B^T$
$(AB)^T = B^T A^T$ *note the change in order of A and B*

Here, $A, B, C \in \mathbb{M}_{m\times n}(\mathbb{R}), a_{ij} \in \mathbb{R}, c, d \in \mathbb{F}$ *where* $\mathbb{F} \equiv \mathbb{R}$ *in our discussion in this chapter.*

We can generalize a matrix as consisting of appropriately defined sub-matrices (does not have to be just numbers)!

How do you take the transpose of a matrix?

Taking the transpose of a matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{pmatrix}$

m rows and n columns

gives $A^T = \begin{pmatrix} a_{11} & a_{21} & \cdot & \cdot & \cdot & a_{m1} \\ a_{12} & a_{22} & \cdot & \cdot & \cdot & a_{m2} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdot & \cdot & \cdot & a_{mn} \end{pmatrix}$

n rows and m columns

2. The rules of matrix algebra guarantee that $\mathbb{M}_{m\times n}(\mathbb{R})$ is a vector space!

3. Special cases: i) When $n = 1$ in $\mathbb{M}_{m\times n}(\mathbb{R})$, we recover the familiar Euclidean space $\mathbb{R}^m$,
and ii) When $m = 1$, we recover the vector space $\mathbb{R}_n$ of all real n –row vectors.

4. A practical application of matrices: A system of linear equations can be expressed in matrix form and the entire mathematical machinery of matrices can be unleashed to find and analyze the solution(s) of the said system of linear equations.

There is a seamless hierarchy of what are known as ***TENSORS*** *in mathematical parlance, the most simplest tensor being scalars (tensors of rank 0), the next in the hierarchy are vectors (tensors of rank 1), followed by matrices (tensors of rank 2), etc. It must be noted that not all matrices are tensors, but all tensors of rank 2 are definitely matrices!*

Example 4.1: Consider the two set of linear equations –

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

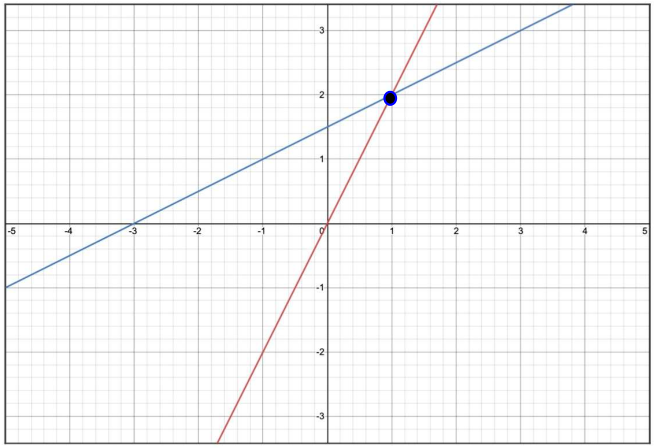
We begin by sketching out the respective straight lines

$$\boldsymbol{2x - y = 0} \text{ and } \boldsymbol{-x + 2y = 3}.$$

The solution of this system is the *intersection point* of these two straight lines, which is $x = 1, y = 2$.

If we express this system of linear equations in matrix-vector notation, then we have $A\boldsymbol{x} = \boldsymbol{b}$, where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is the *coefficient matrix*, $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

The system (i) corresponds to the *row picture*, and the solution strategy presented above lends a geometrical interpretation of this *row picture*.



Points which follow 2x-y=0 lie on the red line. Points which follow –x+2y=3 lie on the blue line.
Our solution is the point which lies on both the lines, i.e. their intersection point.
Now think of what happens when there are -
(a) No intersection between the two lines (except at infinity) or
(b) Infinitely many such points.

There is an alternative (*and sometimes more useful*) geometrical picture, the *column picture*, which lends a different interpretation of the situation at hand.

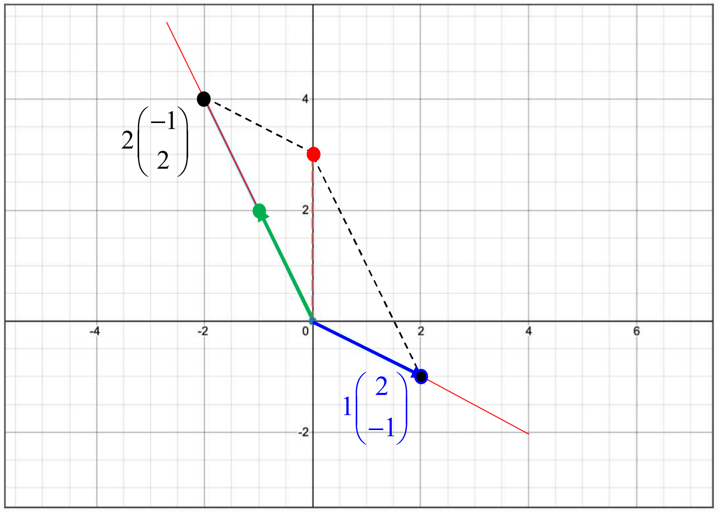
By careful inspection, we notice that the earlier system of equations can be re-written as follows:

$$x\begin{pmatrix}2\\-1\end{pmatrix} + y\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}$$

Here the system of equations is expressed in terms of a **linear combination** of the column vectors of A.

If we treat the columns as vectors in 2D Euclidean space, and consider the correct solutions (say we somehow know that $x = 1 \text{ and } y = 2$), then we have the following –

*"The resultant of adding the two vectors (properly scaled, i.e. weighted) on the left-hand side is identically equal to the vector **b** on the right-hand side."*



Suggest doing this by taking the projection on one vector, along the direction of the other

Throughout our study about linear algebra, the idea of *linear combination* and *column vectors of a matrix* will play a very important role in terms of the mathematical machinery as well as interpretation of the physical picture.

Of course, the question arises whether we can always solve this system as follows:

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} \qquad \text{for any given 2D vector } \boldsymbol{b}?$$

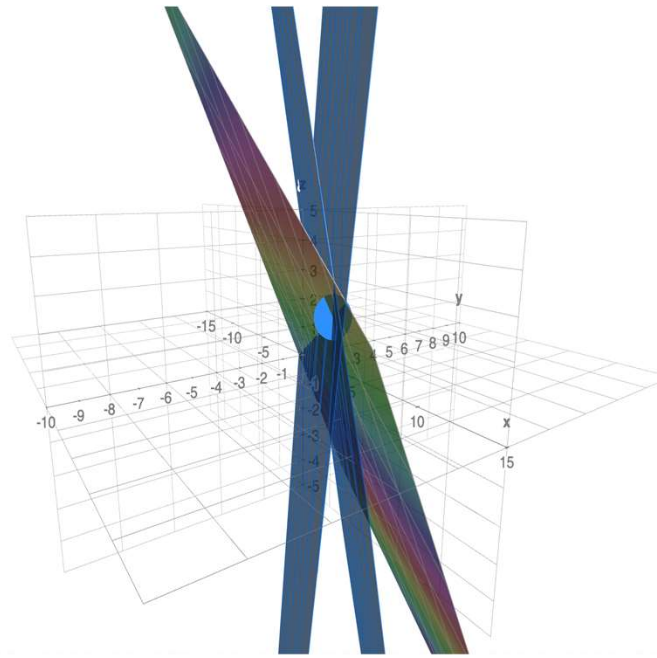
This will be possible only if A is invertible, i.e., A^{-1} exists! In that case, the columns of A will span the entire 2D Euclidean plane (and $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ will be the solution for any 2D vector $\boldsymbol{b}$), i.e. there will be one combination of the columns which will give $\boldsymbol{b}$.

This should the lead us to ask when is A invertible? i.e., ***when will the columns of A span the entire 2D Euclidean plane?***
The answer should be obvious by inspecting the above 2D graph: ***whenever the columns of A are linearly independent.***

Note that *spanning a plane* here is analogous to *obtaining any vector in the plane by a linear combination of vectors.*

Example 4.2: Consider the following system of linear equations-

$$\begin{aligned} 2x + 8y + 4z &= 2 \\ 2x + 5y + z &= 5 \\ 4x + 10y - z &= 1 \end{aligned} \qquad \text{(ii)}$$



This system can be expressed in matrix form as: $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{pmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{pmatrix}, \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}.$$

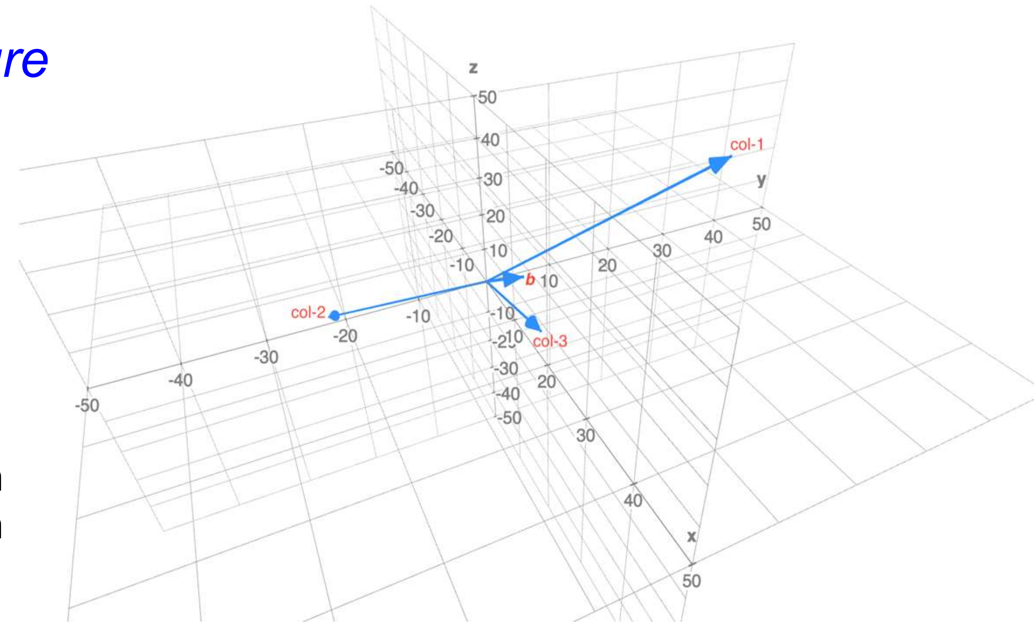
Here A is called the *coefficient matrix.*

The three planes defined by the system of equations (ii) intersect at a point $x = 11, y = -4, z = 3$ which is the solution.

This is the *Row Picture.*

Now let us examine the *Column Picture*

$$x\begin{bmatrix}2\\2\\4\end{bmatrix}+y\begin{bmatrix}8\\5\\10\end{bmatrix}+z\begin{bmatrix}4\\1\\-1\end{bmatrix}=\begin{bmatrix}2\\5\\1\end{bmatrix}$$



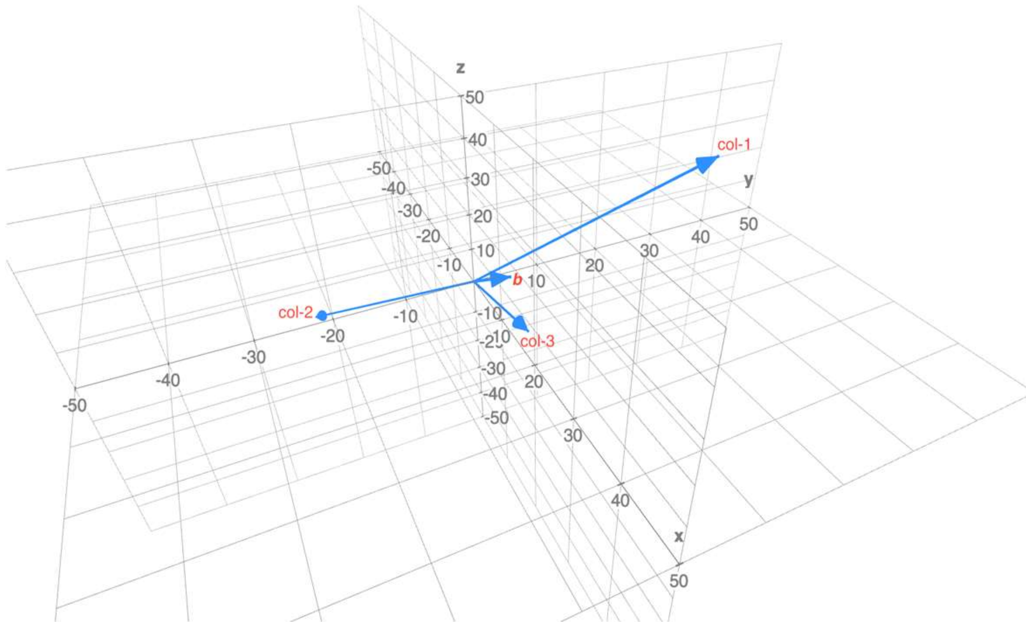
Like in the previous example, a given 3D-vector $\boldsymbol{b}$ can be obtained by the linear combination of the column vectors of A with the appropriate coefficients (x, y, z).

The appropriate coefficients on the l.h.s. form the solution set.

Augmented Matrix $\tilde{A}$, *where* $\tilde{A}=\begin{pmatrix}2 & 8 & 4 & 2\\2 & 5 & 1 & 5\\4 & 10 & -1 & 1\end{pmatrix}$

We will find this to be useful later

This *column picture* and the *linear combination of the columns* allow us to determine, geometrically, the conditions when a unique solution can be attained for any 3D-vector $\boldsymbol{b}$.



The answer is, once again, when A is invertible, i.e. when the columns of A are linearly independent (or equivalently when the column vectors of A are **not** co-planar), *i.e. in that case, the columns vectors of A span the 3-D volume and a proper combination of them will give* ***b***

If the column vectors of A were co-planar, then a linear combination of them will not span the entire 3D plane and a unique solution will not be possible.

A Peek into the Future - "Motivating the **Reduced Row Echelon Form**"

Note that the route to obtain the solution involves finding the inverse of A. This may be difficult especially if the number of unknown variables (and thereby the number of equations) are large.

Computing the inverse of a large matrix becomes simpler by transforming the original coefficient matrix into what is called a **Reduced Row-Echelon Form**
This form lies at the heart of several numerical techniques to solve systems of linear equations and also characterizes several important features of the coefficient matrix.

In any case, as a prelude to what is to come soon, the solution to this system of linear equations can be obtained by transforming the **Augmented Matrix** $\tilde{A}$, *where* $\tilde{A} = \left(\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right)$, into its **Reduced Row-Echelon Form** $rref(\tilde{A}) = \begin{pmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$.

From this, the solutions can be directly obtained as $x = 11, y = -4, z = 3$. We will study this next.
We will study subsequently why this turns out to be the case. The point here is that the matrix structure and its rules (laws of matrix algebra) are useful to solve such systems of linear equations.
We will later do a lab project to further understand the power of this technique to solve engineering problems.

5. Reduced Row-Echelon Form or rref (Definition):

A matrix is said to be in rref if it satisfies **all** the following conditions

i. If a row has non-zero entries, then the first non-zero entry is a 1, known as the *leading 1* (or *pivot*) in this row.
ii. If a column has a leading 1, then all the other entries in that columns are 0.
iii. If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

The third condition implies that rows of 0's, if any, appear at the bottom of the matrix.

6. Types of elementary row operations (in order to obtain the rref):

i. Divide a row by a non-zero scalar.
ii. Subtract a multiple of a row from another row.
iii. Swap two rows.

Question: Why can we do these operations?

Answer: Since the rows are the rows of the corresponding system of linear equations, these operations are such that doing them will not affect the solution to the equations.

We will later see that points 5 and 6 above form the core of the powerful GAUSS-JORDAN ELIMINATION approach to solve systems of linear equations.

Example:

System of Equations

Ax=b

$$\left| \begin{array}{c} x+y+z=3 \\ 2x-3y-z=-8 \\ -x+2y+2z=3 \end{array} \right| \quad A=\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ -1 & 2 & 2 \end{pmatrix} \quad \widetilde{A}=\begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{pmatrix}$$

Augmented Matrix

Augmented Matrix

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -1 & -8 \\ -1 & 2 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -6 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

Row Echelon Form

Row Echelon Form

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

Reduced Row Echelon Form

Solution $x=1, \quad y=4, \quad z=-2$

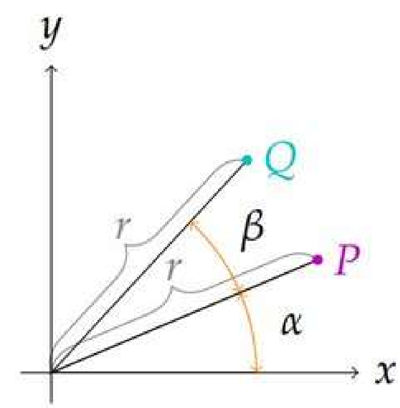
7. Linear Transformations (Definition):

A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a *linear transformation* if $\exists A \in \mathbb{M}_{m\times n}(\mathbb{R})$ such that
$T(\boldsymbol{x}) = \boldsymbol{Ax}, \forall \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ *m*X1 ← *m*X*n* *n*X1

This is a mapping from *n*-dimensional space to *m*-dimensional space

Example: The rotation matrix $\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$ is a linear transformation which rotates a vector in $\mathbb{R}^2$by θ.

We shall derive the rotation matrix that counter-clockwise rotates a 2D point $P=(x,y)$, β radians around the origin, to yield the rotated point Q. Let r and α be the polar coordinates of P. This means



$$P=(x,y)=r(\cos(\alpha),\sin(\alpha))$$

In order to rotate this point β radians, we simply add β to α. That is,

$$Q=r(\cos(\alpha+\beta),\sin(\alpha+\beta))$$

Applying the trigonometric addition formulas gives

$$\begin{aligned}
Q&=r(\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta),\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta))\\
&=(\cos(\beta)r\cos(\alpha)-\sin(\beta)r\sin(\alpha),\sin(\beta)r\cos(\alpha)+\cos(\beta)r\sin(\alpha))\\
&=(\cos(\beta)x-\sin(\beta)y,\sin(\beta)x+\cos(\beta)y)\\
&=\begin{pmatrix}\cos(\beta) & -\sin(\beta)\\ \sin(\beta) & \cos(\beta)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\cos(\beta) & -\sin(\beta)\\ \sin(\beta) & \cos(\beta)\end{pmatrix}P
\end{aligned}$$

To conclude, applying the blue matrix to P gives us the rotated point Q.

Ques: Given $T: \mathbb{R}^n \to \mathbb{R}^m$, how do we find A?

………. n columns ………….

m rows

Ans:

$$A = \begin{pmatrix} | & | & & & | \\ T(\boldsymbol{e_1}) & T(\boldsymbol{e_2}) & \cdot & \cdot & T(\boldsymbol{e_n}) \\ | & | & & & | \end{pmatrix}$$

where $\boldsymbol{e_i}$ is the i^{th} standard basis element of $\mathbb{R}^n$.