Lecture (8): Complex Integration

6/2/19

Let $f(t) = u(t) + iv(t)$; $a \le t \le b$ be a complex valued f^n where u and v are real valued f^ns.

$f(t)$ is integrable if $u(t)$ & $v(t)$ are integrable.

integrability in the sense $\int_K |f| \, d\mu < \infty$

$K \subset \Omega$ (open in $\mathbb{R}$)

(compact)

Fundamental theorem of calculus applies in $\mathbb{C}$

$$\frac{d}{dt} \int_a^t f(\tau) \, d\tau = f(t)$$

$$\int_a^b f'(t) \, dt = f(b) - f(a).$$

Formulation of integration.

We introduce the notion of complex integration to integration on a curve in the complex plane.

Curve z is parametrized by (t)

$$z(t) = x(t) + iy(t) \; ; \quad a \le t \le b$$

for t in $[a, b]$, $\exists$ a set of ordered points $(x(t), y(t))$ that are ordered image pts. of the interval.

in increasing order of t

The curve $z(t)$ is continuous (differentiable) if $x(t)$ & $y(t)$ are continuous (differentiable) f^ns.

So, first, we need to define curves on $\mathbb{C}$.

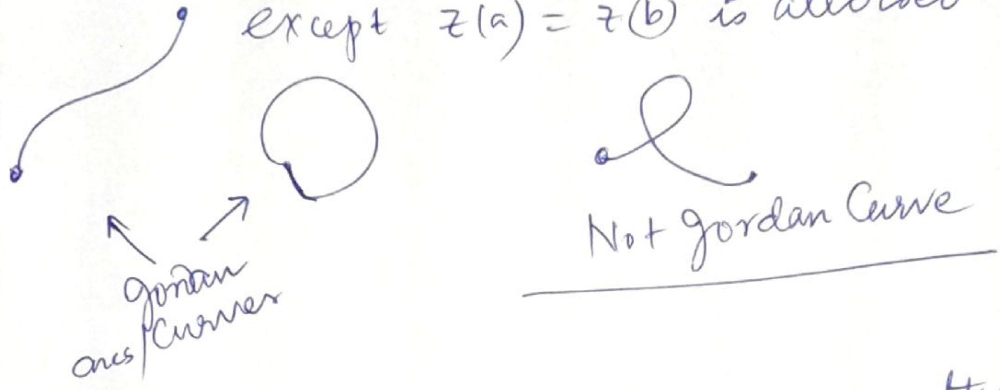
Gordan Curve (Simple Curve, Jordan arc).

A curve/arc C is simple (Jordan arc) if it does not intersect itself.

i.e. $z(t_1) \neq z(t_2)$ if $t_1 \neq t_2$ $\forall t \in [a,b]$

except $z(a) = z(b)$ is allowed.

eg



Convention :- for a closed curve, the direction of integration is taken to be positive if the interior remains to the left of C.

Continuous & piecewise Continuous functions

$f(z)$ is continuous on $\mathbb{C}$ if $f(z(t))$ is continuous for $a \le t \le b$

f is said to be piecewise continuous on $[a,b]$ if $[a,b]$ can be broken up into finite number of subintervals where f is continuous on each of the subintervals.

Smooth arc :- C is a smooth arc for which $z'(t)$ is continuous.

A contour is a piecewise smooth arc.

On a contour, $z(t)$ is continuous & $z'(t)$ is piecewise continuous.

Jordan Contour is a simple closed contour.

Contour integral of a piecewise continuous f^n on a smooth contour C is defined to be

$$\int_C f(z)\,dz = \underbrace{\int_a^b f(z(t))\,z'(t)\,dt}$$ b/c $\frac{dz}{dt} = z'(t)$.

The (value) integral is invariant to choice of parametrization as long as the proper ordering of the parametrization is maintained.

This is basically a "line" integral in the (x,y) plane & hence naturally related to the study of vector calculus in the plane.

* $$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

$dz = dx + i\,dy$

$f(z) = u(x,y) + i\,v(x,y)$

* $$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_C (u\,dx - v\,dy) + i(v\,dx + u\,dy)$$

Symbolically.

* $$\int_C f = \sum_{j=1}^{n} \int_{C_i} f$$

Thm (8.1) Application of Fundamental thm. of Calculus on $\mathbb{C}$ to ~~path independence of def. integrals~~.
prove

Let $F(z)$ be analytic s.t. $F'(z)$ ($= f(z)$) is continuous in a domain D. Then for a contour C lying in D w/ endpoints z_1 and z_2

$$\int_C f(z)\,dz = \int_C F'(z)\,dz = F(z_2) - F(z_1)$$

Corollary :- $\oint_C F'(z)\,dz = 0 = \oint_C f(z)\,dz$

Later on, we will show this in the simplest form of Cauchy integral theorem.

Proof :-

$$\int_C F'(z)\,dz = \int_a^b F'(z(t))\,z'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\Big(F(z(t))\Big)\,dt$$ from pg① of lecture(8)

$$= F(z(b)) - F(z(a))$$

$$= F(z_2) - F(z_1)$$

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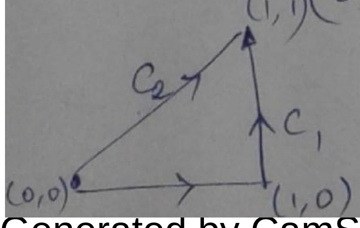
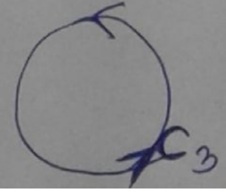
eg (8.1) Evaluate $\int_C \bar{z}\,dz$ for

(a) $C = C_1$ is a contour from $z = 0$ to $z = 1$ to $z = 1+i$

(b) $C = C_2$ is a line from $z = 0$ to $z = 1+i$

(c) $C = C_3$ is the unit circle $|z| = 1$





Soln :-

(a) $\displaystyle\int_{C_1} \bar{z}\,dz = \int_{C_1} (x-iy)(dx+i\,dy)$

$\displaystyle = \int_{x=0}^{1} x\,dx + \int_{y=0}^{1} (1-iy)(i\,dy) = \frac{1}{2} + i\left(y - \frac{iy^2}{2}\right)\Big|_0^1$

$= 1+i$.

Note in the integral from $z=0$ to $z=1$; $y=0 \Rightarrow dy=0$

Likewise for the path $z=1$ to $z=1+i$; $x=1$
$\Rightarrow dx = 0$.

(b) $\displaystyle\int_{C_2} \bar{z}\,dz = \int_{x=0}^{1} (x-ix)(dx+i\,dx) = (1-i)(1+i)\int_0^1 x\,dx = 1$

Note C_2 is the line $y=x \Rightarrow dy = dx$

$\because$ $\bar{z}$ is not analytic, we see that

$\displaystyle\int_{C_1} \bar{z}\,dz \neq \int_{C_2} \bar{z}\,dz$.

need not be equal.

(c) $\displaystyle\int_{C_3} \bar{z}\,dz = \int_0^{2\pi} e^{-i\theta}\, i e^{i\theta}\, d\theta = 2\pi i$

Note $z = e^{i\theta}$
$\bar{z} = e^{-i\theta}$
$dz = ie^{i\theta}d\theta$ on $|r|=1$ (unit circle) #.

eg (8.2) Evaluate $\int_C z\,dz$ along the three contours defined in eg (8.1) above.

Soln :- Thm (8.1) $\Rightarrow \int_{C_1} z\,dz = \int_{C_2} z\,dz$

$$= \frac{1}{2}\int_{C_2} \frac{dz^2}{dz}\,dz$$

b/c z^2 is analytic

$$= \frac{1}{2}\int_{C_2} \frac{d}{dz} z^2\,dz$$

$$= \frac{1}{2}\, z^2 \Big|_{0,0}^{1,1}$$

$$= \frac{1}{2}(1+i)^2 = i$$

$$\int_{C_3} z\,dz = 0$$

b/c C_3 is closed Jordan curve.

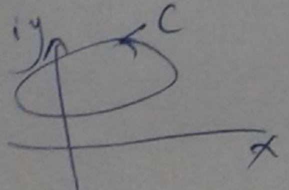
eg (8.3) Evaluate $\int_C \frac{1}{z}\,dz$ for

(a) any simple closed contour C not enclosing the origin

(b) any simple closed contour enclosing the origin.

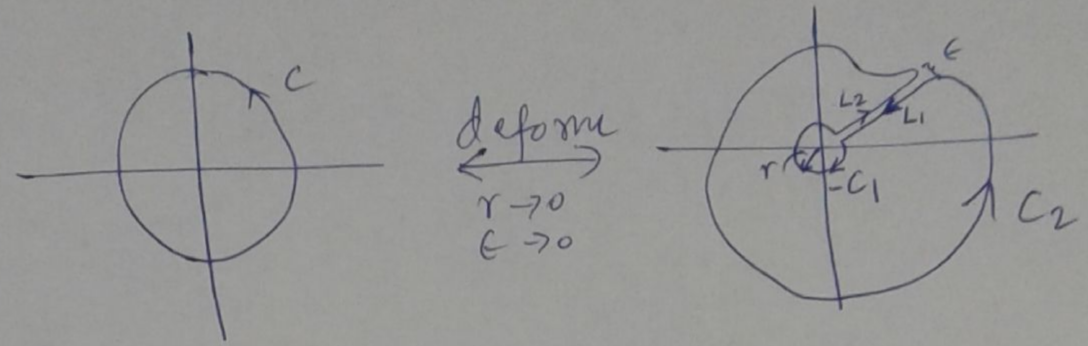
Soln :- B/c $\frac{1}{z}$ is analytic $\forall\, z \neq 0$

(a)



$$\int \frac{1}{z}\,dz = \int \frac{d}{dz}(\log z)\,dz = (\log z)_C = 0$$

(b) Now C encloses $z=0$.



$$\lim_{\substack{r\to 0\\ \epsilon \to 0}} \underbrace{\left(C_2 + L_1 + L_2 - C_1\right)}_{\text{say } C_{\epsilon, r}} = C$$

$$\int_C \frac{1}{z}\,dz = \int_{\lim_{\substack{r\to 0\\ \epsilon\to 0}} C_{\epsilon, r}} \frac{1}{z}\,dz = 0 \text{ from part(a)}$$

$$\lim_{\epsilon, r \to 0} \int_{C_2} \frac{1}{z}\,dz + \int_{L_1} \frac{1}{z}\,dz + \int_{L_2} \frac{1}{z}\,dz - \int_{C_1} \frac{1}{z}\,dz = 0.$$

$$\Rightarrow \int_C \frac{1}{z}\,dz + \int_{L_1} \frac{1}{z}\,dz - \int_{L_1} \frac{1}{z}\,dz = \lim_{\epsilon, r \to 0} \int_{C_1} \frac{1}{z}\,dz.$$

$$= \lim_{\epsilon, r \to 0} \int_{C_1} \frac{1}{re^{i\theta}}\, r i e^{i\theta}\, d\theta$$

$$\underset{\substack{\theta \text{ goes}\\ \text{from } 0 \text{ to}\\ 2\pi}}{\overset{\text{as } \epsilon \to 0}{=}} \int_0^{2\pi} i\, d\theta = 2\pi i$$

$$\therefore \int_C \frac{1}{z}\,dz = 2\pi i$$

(if C encloses $(0,0)$).

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Th^m (8.2) Let $f(z)$ be continuous on a Contour C. Then

$$\left|\int_C f(z)\,dz\right| \le ML$$

Where $|f| \le M$ on C and L is length of C.

Proof:-

$$I = \left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$$

If we break up an integral into summation over little strips & thereafter apply Δ-ineq

$$\le \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

$$= M\int_a^b |z'(t)|\,dt$$

$$= M\int_a^b \sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

$$= M\int_a^b \underbrace{ds}_{\text{infinitesimal arc length}}$$

$$= ML$$

i.e. $I \le ML$.

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