Date: $14^{th} - 23^{rd}$ June, 2010

# 1 Multivariate Functions

## 1.1 Basic definitions:

### 1.1.1 Domain of a function:

Let $D$ be a set of $n-$tuples of real numbers $(x_1, x_2, ..., x_n)$. A real valued function, $f : D \to \mathcal{R}$ is a rule that assigns a real number, $y = f(x_1, x_2, ..., x_n)$ to each element in $D$. The set $D$ is the function's **domain**.
eg. Domain of the function, $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ is the entire space of real numbers. Domain of the function, $g(x, y, z) = \frac{1}{x^2+y^2+z^2}$ is all real $x, y, z$ except $(x, y, z) = (0, 0, 0)$.

### 1.1.2 Range of function:

The set of all possible $y$ values taken on by the function $f$ is called the function's **range**.
eg. Range of $f$ in the above example is $[0, \infty)$. Range of $g$ in the above example is $(0, \infty)$.

**Trick question:** Which are the independent and dependent variables?

### 1.1.3 Interior points and Interior of a set:

A point $(x_0, y_0)$ in a region (set) $R$ in the $xy$ plane (equivalently in *space*) is an **interior point** of $R$ if it is the center of a disk that lies entirely in $R$. The set of all interior points is known as the **interior** of the region, $R$.

### 1.1.4 Boundary points and Boundary:

A point $(x_0, y_0)$ is a **boundary point** of $R$ if every disk centered at $(x_0, y_0)$ contains points that lie outside of $R$ as well as points in $R$. *The boundary point itself need not belong to $R$.* The set of all boundary points make up the **boundary** of the region.

### 1.1.5 Open and Closed regions:

A region is **open** if it consists entirely of interior points. eg. $\{(x, y)|x^2 + y^2 < 1\}$.
A region is **closed** if it contains all of its boundary points. eg. $\{(x, y)|x^2 + y^2 \leq 1\}$.

### 1.1.6 Bounded and Unbounded regions:

A region is **bounded** if it lies inside a disk of fixed radius, else it is **unbounded**.

example: The domain of the function $f(x, y) = \sqrt{y - x^2}$ is *closed* and *unbounded*. The parabola $y = x^2$ is the *boundary* of the domain. The points above the parabola make up the domain's *interior*.

### 1.1.7 Level curve, graph, surface, level surface:

The set of points where a function $f$ has a constant value $f(x, y) = c$ is called **level curve** of $f$. The set of all points $(x, y, f(x, y))$ in space, for $(x, y)$ in the domain of $f$, is called the **graph** of $f$. The graph of $f$ is also called the **surface**, $z = f(x, y)$. The set of points $(x, y, z)$ in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of $f$.

**Sample exercise problems:**

1. Find the level curve of the function, $f = \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n$ that passes through the point $(1, 2)$.
   **Soln.** $f(x, y) = \sum_{n=0}^{\infty} (\frac{x}{y})^n$ $@(1, 2) \implies z = \frac{1}{1-\frac{x}{y}} = \frac{y}{y-x}$ $@(1, 2) \implies z = \frac{2}{2-1} = 2 \implies 2 = \frac{y}{y-x} \implies y = 2x$.
2. Find the level surface of the function, $f(x, y, z) = \sqrt{x - y} - \log z$ $@(3, -1, 1)$.
   **Soln.** $f(x, y, z) = \sqrt{x - y} - \log z$ $@(3, -1, 1) \implies w = \sqrt{x - y} - \log z$ $@(3, -1, 1) \implies w = \sqrt{3 - (-1)} - \log 1 = 2 \implies \sqrt{x - y} - \log z = 2$.

## 1.2 Limits and Continuity

**Reading Assignment:** Review definitions and properties from page 917-919, textbook.

### 1.2.1 2-path test for (non) existence of a limit:

If a function $f(x,y)$ has different limits along 2 different paths as $(x,y) \to (x_0,y_0)$, then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

*The notion of **path** should be clear in this context. In calculus 1, when limits where introduced, the **path** almost always was on the real line; here since we are dealing with bi-variate (multivariate) functions, **path** may imply any curve on the $xy-$plane on which the set of points $(x,y)$ may `ride upon` to approach $(x_0,y_0)$. The important thing to know is that only such a curve may be chosen to ensure that the point $(x_0,y_0)$ actually lies on the curve.*

### 1.2.2 Sample review problem:

1. Show

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & ; (x,y) \neq (0,0), \\ 0 & ; (x,y) = (0,0) \end{cases}$$

is **not** continuous at $(0,0)$.

**Soln. :** `(Book's method)` Let us choose a path $y = mx$ and analyze the limit at $(0,0)$.

Note $f(x,y)\Big|_{y=mx} = \frac{2xy}{x^2+y^2}\Big|_{y=mx} = \frac{2m}{1+m^2}$. Therefore,

$$\lim_{(x,y)\to(x_0,y_0)\text{along } y=mx} f(x,y) = \frac{2m}{1+m^2}$$

changes with m, and hence according to the 2-path test, $f(x,y)$ is discontinuous at $(0,0)$. $\square$

`(Alternative method)` Let us a choose a sequence $\{(\frac{1}{k},\frac{1}{k})\}$ that will define our path. Clearly, $\{(\frac{1}{k},\frac{1}{k})\} \to (0,0)$ as $k \to \infty$. And since $f(\frac{1}{k},\frac{1}{k}) = \frac{1}{2}$ for any $k$, the function sequence $\{f(\frac{1}{k},\frac{1}{k})\} \to \frac{1}{2}$. Now lets choose a different sequence $\{(\frac{1}{k},0)\} \to (0,0)$ as $k \to \infty$. But $f(\frac{1}{k},0) = 0$ for any $k$, and so the function sequence $\{f(\frac{1}{k},0)\} \to 0$ as $k \to \infty$. Hence the desired conclusion. $\square$

# 2 Partial Derivatives

**Reading Assignment:** Review page 924-929 from your textbook

## 2.1 Euler's Theorem (Mixed Derivatives):

If $f(x,y)$ and its partial derivatives $f_x, f_y, f_{xy}$ and $f_{yx}$ are defined throughout an open region containing a point $(a,b)$ and are **all continuous** at $(a,b)$, then $f_{xy}(a,b) = f_{yx}(a,b)$.

### 2.1.1 Sample Review Problems

1. Is $f_{xy} = f_{yx}$ always true ?

2. If the limits exist, then is the following always true ?

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y)$$

**Soln. :** `(hint)` Try

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

and use $a = b = 0$.

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**Notation:** $(D_{\hat{u}}f)_{P_0} \implies$ derivative of $f$ at $P_0$ in the direction of $\hat{u}$.

## 4.2 Gradient Vector or Gradient

Gradient of $f(x,y)$ at a point $P_0(x_0,y_0)$ is the vector

$$\nabla f = \left(\frac{\partial f}{\partial x}\right)_{P_0} \hat{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \hat{j}$$

obtained by evaluating the partial derivatives of $f$ at $P_0$.

**Theorem:** If the partial derivatives of $f(x,y)$ are defined at $P_0$; then

$$(D_{\hat{u}}f)_{P_0} = (\nabla f)_{P_0} \cdot \hat{u}$$

## 4.3 Properties of Directional Derivatives

$(D_{\hat{u}}f) = \nabla f \cdot \hat{u} = |\nabla f||\hat{u}|\cos\theta = |\nabla f|\cos\theta$

1. $f$ increases most rapidly when $\cos\theta = 1$ or when $\hat{u}$ is in the direction of $\nabla f$, i.e. f increases most rapidly in the direction of $\nabla f$ at any point $P$ in the domain of $f$.
2. Similarly, $f$ decreases most rapidly in the direction of $-\nabla f$.

### 4.3.1 Sample review problem (example)

Let $f(x,y) = e^{x^2-y^2} \forall (x,y)$ in $R^2$. Note $f : R^2 \to R$ is continuously differentiable (why ?)
Clearly, $\frac{\partial f}{\partial x}(1,1) = 2$, $f_y(1,1) = -2$ and therefore, $\nabla f(1,1) = 2\hat{i} - 2\hat{j}$. Hence, the direction in which the function $f$ is increasing the fastest at $(1,1)$ is given by the unit vector $\hat{v} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

So, remember

1. The direction of $\nabla f(x)$ is the direction of maximal increase of $f$ at $x$.
2. The norm, $||\nabla f(x)||$ is equal to the magnitude of the rate of change of $f$ in its direction of maximal increase.

## 4.4 Real world application: (a little more physics will do nobody any harm ☺)

**Law of Conservation of Energy:** By now, "Sum of potential and kinetic energy is constant "should be more familiar to you than any kind of rock music ever composed by *The Beatles* ! ☺(`Oh ya ! The B's are one of my favorites`).
Lets prove this famous result from Physics using the vector calculus tools that we have just learnt ! But let's first begin with a few definitions.
**Potential energy:** If $\mathcal{F}$ is a vector **field** and if $\exists$ a differentiable function $\phi$ s.t. $\mathcal{F} = -\nabla\phi$; then $\phi$ is called the potential energy of the vector field, $\mathcal{F}$ and $\mathcal{F}$ is called *conservative* for the following reasons: Suppose that a particle of mass, $m$ moves along a differentiable curve $\vec{r}(t)$ in $U$, where $U$ is any open set in $R^n$; and it obeys *Newton's Laws*: $\mathcal{F}(\vec{r}(t)) = m\vec{r}''(t)\forall t$ where $\vec{r}(t)$ is defined.

Let us know formally state the *law of conservation of energy* as "If $\mathcal{F} = -grad \quad \phi = -\nabla\phi$; then the sum of potential and kinetic energy is constant."
**Proof:** We have to prove: $\phi(\vec{r}(t)) + \frac{1}{2}m\vec{r}'(t)^2$ is constant; let us differentiate this sum and apply chain rule $\nabla\phi(\vec{r}(t)).\vec{r}'(t) + m\vec{r}'(t).\vec{r}''(t) = \nabla\phi(\vec{r}(t)).\vec{r}'(t) + \Big(-\nabla\phi(\vec{r}(t))\Big).\vec{r}'(t) = 0$; where the second last equality is obtained by $m\vec{r}''(t) = \mathcal{F}(\vec{r}(t)) = -\nabla\phi(\vec{r}(t))$. Hence proved ! $\square$

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### 5.2 Increments and Distance

To estimate the change in the value of $f$ when we move a small distance $ds$ from a point $P_0$ in a particular direction $\hat{u}$; we use

$$df = \left( (\nabla f)_{P_0} \cdot \hat{u} \right)(ds)$$

## 6 More on Directional Derivatives and Gradients

### 6.1 Why should the definition of Directional Derivative in $(4.1) in Handout 5 make sense?

Recall, the *Taylor series* for a function, $f$ of 2 variables can be expressed as

$$f(x + \Delta x, y + \Delta y) = f(x, y) + \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y} + ...$$

Consider some function $z = f(x, y)$. Geometrically this represents a surface as shown in figure(3). Let $(x, y)$ be the co-ordinates of a point $P$ in the $xy$-plane. The height of the surface above this point is represented by the length of $PQ$; i.e. $PQ = z = f(x, y)$. Suppose now we take a short step in the $xy$-plane to a new point $P'$ with coordinates $(x + \Delta x, y + \Delta y)$. The height of the surface above this point is $P'Q' = f(x + \Delta x, y + \Delta y)$. Let $\Delta s$ be the length of the step $\Delta s = PP'$.

We next ask how much the function $f$ has changed as a result of taking this step. Clearly this change is the difference in the two heights $PQ$ and $P'Q'$, and

$$P'Q' - PQ = \Delta f = f(x + \Delta x, y + \Delta y) - f(x, y)$$

Applying the Taylor series formula stated above, we get

$$\Delta f = f(x, y) + \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y} + ... - f(x, y) = \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y} + ...$$

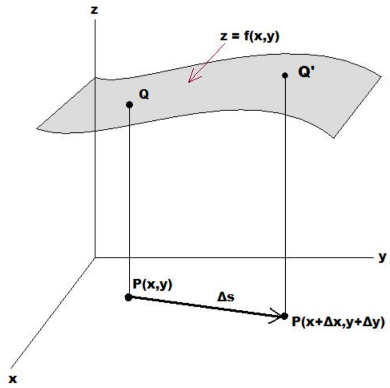


Figure 3: Directional derivative

We now recast this expression by what at first may seem an unnecessary elaboration of the notation. Let $\vec{\Delta s}$ be a vector that has magnitude $\Delta s$ and points from $P$ to $P'$. Clearly, $\vec{\Delta s} = \Delta x\hat{i} + \Delta y\hat{j}$. But the gradient of $f$ is $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$. It follows at once that

$$\Delta f = (\vec{\Delta s}) \cdot (\nabla f) + ...$$

Let $\hat{u}$ be a unit vector in the direction of $\vec{\Delta s}$. Then $\vec{\Delta s} = \Delta s\hat{u}$ and $\Delta f = (\hat{u} \cdot \nabla f)\Delta s + ...$ so that $\frac{\Delta f}{\Delta s} = \hat{u} \cdot \nabla f + ...$We now take the limit of this equation to get

$$\frac{df}{ds} := \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} = \hat{u} \cdot \nabla f \tag{1}$$

Equation(1) may be interpreted as the *rate of change of* $f(x, y)$ in the direction of $\vec{\Delta s}$($\equiv$ in the direction of $\hat{u}$). Redrawing figure(4) and passing a plane through $P$ and $P'$ parallel to the $z$-axis in figure(4), we see that it cuts the surface $z = f(x, y)$ in a curve, $C$. The quantity $\frac{df}{ds}$ defined in equation(1) is the slope of this curve at the point $Q$.

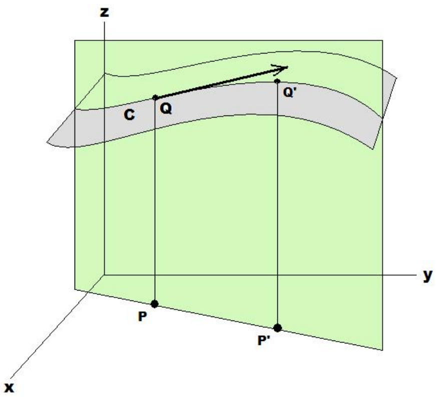


Figure 4: Directional derivative

## 6.2 Interpretation of the Directional derivative through an example

Let us consider an inverted right circular cone whose axis coincides with the $z$-axis,

$$z = f(x, y) = (x^2 + y^2)^{1/2} \tag{2}$$

We seek the directional derivative of this function at some point $x = a$ and $y = b$ and in the direction specified by $\hat{u} = \cos\theta\hat{i} + \sin\theta\hat{j}$, figure(6).

The gradient of $f$ is $\nabla f = f_x\hat{i} + f_y\hat{j} = \frac{x\hat{i}+y\hat{j}}{z}$ and therefore,

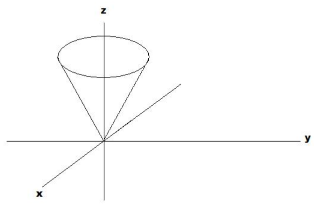


Figure 5: Right circular cone

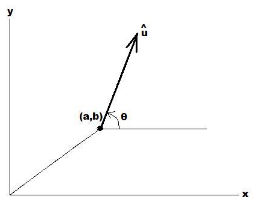


Figure 6: arbitrary direction, $\hat{u}$

$$\left(\frac{df}{ds}\right)_{x=a,y=b} = \hat{u}\cdot(\nabla f)_{x=a,y=b} = \frac{a\cos\theta + b\sin\theta}{\sqrt{a^2+b^2}} \tag{3}$$

Now, let us consider two cases.

1. Let $\theta$ be chosen such that $\hat{u}$ is in the radial direction in the $xy$-plane as shown in figure(9c)
   This means $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$ and so

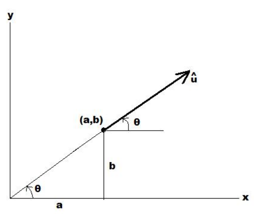


Figure 7: radial direction, $\hat{u}$

$$\frac{df}{ds} = \frac{a}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\frac{b}{\sqrt{a^2+b^2}} = 1$$

   The significance of this result is shown in figure(8)

2. Now, we choose $\theta$ s.t. $\hat{u}$ is $\perp$ to the $\hat{u}$ in case(1), figure(9)

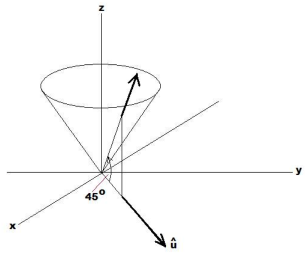


Figure 8: fastest rate of increase corresponding to radial $\hat{u}$

We then have $\cos\theta = \frac{-b}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$ and so

$$\frac{df}{ds} = \frac{a}{\sqrt{a^2+b^2}}\frac{-b}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}} = 0$$

The meaning of this result is illustrated in figure(9)


Figure 9: zero rate of increase corresponding to transverse $\hat{u}$ (*locked in circular loop, hence no increase in* $z = f(x, y)$)

**Note that the direction of $\nabla f$ at $(a, b)$ is obtained from $\nabla f = \frac{a\hat{i}+b\hat{j}}{\sqrt{a^2+b^2}}$ which is the same direction as $\hat{u}$ (case 1) in figure(8) and is also the direction of the fastest rate of change of f(x,y) at that point. However, if the direction of $\hat{u}$ was chosen as in case(2) i.e. $\perp$ to the radial direction; then the rate of change of $f$ is $0$, i.e. we cannot go up the surface of the cone but keep making circles at the same altitude, refer figure(9).**

So as was stated before in section(4.3), *the gradient of a scalar function $F(x, y, z)$ is a vector that is in the direction in which $F$ undergoes the greatest rate of increase and that has the magnitude equal to the rate of increase in that direction.*

Recall, the electric field at a point is equal to the negative gradient of the electric potential there, i.e. $\vec{E} = -\nabla\phi$. Why does this make sense ? Since, $\nabla\phi$ is a vector in the direction of *increasing* $\phi$, the force on the *positive* charge $q$ is $\vec{F} = q\vec{E} = -q\nabla\phi$, which is in the direction of *decreasing* $\phi$. Thus, the negative sign ensures that a positive charge moves `downhill` from a higher to a lower potential.

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Date: $22^{nd} - 23^{rd}$ June, 2010

# 1 Extreme points and Saddle points

## 1.1 Local extrema

For a bi-variate function $f(x, y)$, we look for points where the surface $z = f(x, y)$ has a *horizontal tangent plane*; at such points we then look for local maxima, local minima and saddle points (think of inflection points in 1D).

### 1.1.1 Definition:

Let $f(x, y)$ be defined on a region $R$ containing the point $(a, b)$. Then

1. $f(a, b)$ is a **local maximum** of $f$ if $f(a, b) \geq f(x, y) \forall$ domain points $(x, y)$ in an open disk centered at $(a, b)$.
2. $f(a, b)$ is a **local miniimum** of $f$ if $f(a, b) \leq f(x, y) \forall$ domain points $(x, y)$ in an open disk centered at $(a, b)$.

Thus, local maxima correspond to mountain peaks on the surface $z = f(x, y)$ and local minima correspond to valley bottoms. At such points, the tangent planes, if they exist, are horizontal.

### 1.1.2 Theorem: (First derivative test for local extrema)

If $f(x, y)$ has a local maximum or minimum at an interior point $(a, b)$ of its domain, and if the first partial derivatives exist there; then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

**Warning:** The theorem does not apply to

1. boundary points of a function's domain, where it is possible for a function to have extreme values along with non-zero derivatives,
2. points where either $f_x$ or $f_y$ fails to exist.

### 1.1.3 Definition:

An interior point of the domain of a function $f(x, y)$ where both $f_x$ and $f_y$ are zero or where one or both of $f_x$ and $f_y$ do not exist is called a **critical point** of $f$.

**Note:**

1. All extreme values of $f$ occur at critical points and/or boundary points.
2. Not every critical point gives rise to a local extremum. (may be a saddle point)

### 1.1.4 Definition:

A differentiable function $f(x, y)$ has a **saddle point** at a critical point $(a, b)$ if in every open disk centered at $(a, b)$ there are domain points $(x, y)$ where $f(x, y) > f(a, b)$ **and** domain points $(x, y)$ where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a **saddle point** of the surface.

### 1.1.5 More on Saddle point

In mathematics, a saddle point is a point in the domain of a function which is a stationary point but not a local extremum. The name derives from the fact that in two dimensions the surface resembles a saddle that curves up in one direction, and curves down in a different direction (like a horse saddle or a mountain pass). In terms of contour lines, a saddle point can be recognized, in general, by a contour that appears to intersect itself. For example, two hills separated by a high pass will show up a saddle point, at the top of the pass, like a figure-eight contour line.

In the most general terms, a saddle point for a smooth function (whose graph is a curve, surface or hypersurface) is a stationary point such that the curve/surface/etc. in the neighborhood of that point is not entirely on any side of the tangent space at that point. In one dimension, a saddle point is a point which is both a stationary point and a point of inflection. Since it is a point of inflection, it is not a local extremum.

In dynamical systems, a saddle point is a periodic point whose stable and unstable manifolds have a dimension which is not zero. If the dynamic is given by a differentiable map f then a point is hyperbolic if and only if the differential of n (where n is the period of the point) has no eigenvalue on the (complex) unit circle when computed at the point.

In a two-player zero sum game defined on a continuous space, the equilibrium point is a saddle point. A saddle point is an element of the matrix which is both the smallest element in its column and the largest element in its row. For a second-order linear autonomous systems, a critical point is a saddle point if the characteristic equation has one positive and one negative real eigenvalue.

Figure (1) shows a typical saddle point.

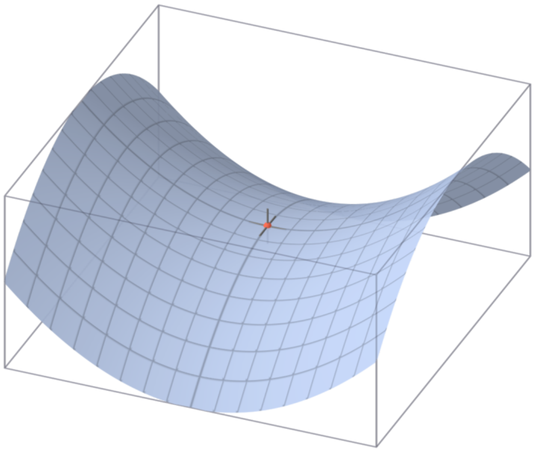

Figure 1: A saddle point on the graph of $z = x^2 - y^2$

#### 1.1.6 Theorem: (second derivative test for local extrema)

Let $f(x, y)$ and its first and second partial derivatives be continuous throughout a disk centered at $(a, b)$ and that $f_x(a, b) = f_y(a, b) = 0$; also define the **discriminant** of $f$ as

$$\Im = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

then

1. $f$ has a **local maximum** at $(a, b)$ if $f_{xx} < 0$ and $\Im > 0$ at $(a, b)$.
2. $f$ has a **local minimum** at $(a, b)$ if $f_{xx} > 0$ and $\Im > 0$ at $(a, b)$.
3. $f$ has a **saddle point** at $(a, b)$ if $\Im < 0$ at $(a, b)$.
4. Inconclusive at $(a, b)$ if $\Im = 0$ at $(a, b)$.

## 1.2 Absolute maxima and minima on closed bounded regions

The following steps must be executed.

1. *List the interior critical points of* $f$ in the region $R$and evaluate $f$ at these points.
2. *List the boundary points of* $R$ *where* $f$ has local maxima and minima and evaluate $f$ at these points.
3. Pick the absolute maxima and minima from the list of candidate points obtained in steps 1 and 2 above.

## 1.3 Sample review problem:

`Ques:` Near Kate's lake in Frodo Baggins National Park, the elevation of the solid ground (in feet) can be described by the function $f(x, y) = 8000 - 30x^2y^2 + 30x^2 + 30y^2$.

1. Determine the coordinate location, $(x, y, z)$ corresponding to the bottom of the Kate's lake.
2. Determine the coordinate location(s), $(x, y, z)$ corresponding to potential drainage points of Kate's lake.
3. What is the maximum possible depth of Kate's lake ? State clearly if information provided is insufficient to calculate it's depth.

**Soln:**

1. Set $\nabla f = 0$ to obtain $(0,0)$ and $(\pm 1, \pm 1)$ as critical points. Then check $\Im(0,0) = 60^2 > 0$ and $f_{xx} = 60 > 0 \implies (0,0,8000)$ is the minimum and hence the location of the bottom of the lake !

2. $(\pm 1, \pm 1)$ are the **drainage (i.e. saddle) points** since $\Im(\pm 1, \pm 1) = -4(60^2) < 0$.

3. The maximum depth of the lake is $f(\pm 1, \pm 1) - f(0,0) = 8030 - 8000 = 30 ft$.

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