Desingularization of function fields

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This paper is meant to be a purely algebraic interpretation of desingularization using an elementary approach (here in terms of basic commutative algebra), with a well-defined algebraic objective of coordinatization in mind.

We will restrict ourselves to the following case, trying to understand it completely, before considering any possible generalizations. Start with $\mathbf{F} = \mathbf{Q}$, the rationals, in characteristic 0 or $\mathbf{F} = \mathbf{F}_p$, the finite field of p elements, in characteristic $p > 0$ (as the default fields used to do exact computations in Computer Algebra Systems); and let $\overline{\mathbf{F}}$ denote the *algebraic closure* in either case. Let $b(x_0, \ldots, x_d)$ be an *irreducible polynomial* in the *multivariate polynomial ring* $\overline{\mathbf{F}}[x_0, \ldots, x_d]$; and assume the ideal of *all* relations among the variables is $I(x_0, \ldots, x_d) := \langle b(x_0, \ldots, x_d) \rangle$ with the *quotient ring*

$$\mathbf{A} := \overline{\mathbf{F}}[x_0, \ldots, x_d] / \langle b(x_0, \ldots, x_d) \rangle$$

then an (*integral*) *domain*. Let its *field of fractions* $Q(\mathbf{A})$ be denoted by $\mathbf{L}$ and be called an *algebraic function field* here. This will be the *universal object* to be studied.

Consider *d-dimensional valuations* $\underline{\nu} : \mathbf{L} \backslash \{0\} \to \mathbf{Z}^d$ as described below. Use them to define coordinates and to frame desingularization as having the clearly defined algebraic objective of finding *coordinate functions* to distinguish valuations and provide d independent, explicit *local parameters* for them.

Desingularization here will produce many domains $\mathbf{A}_l$ of this same flavor, all with the same algebraic function field $\mathbf{L}$. Then birational maps between domains will be viewed as *change-of-basis* maps (meaning identity maps, so merely a renaming of the elements). They will be called *unimodular transformations* here.

Almost everything will be *local* in flavor. That is, the coordinate $coord(f, \underline{\nu})$ will depend only on the rational function f and the valuation $\underline{\nu}$, not on f being defined on some Zariski-open set U. Monomial orders will be local; and the set of initial monomials $init(b)$ will also be local (consisting of monomials of small valuation).

The *non-singular model* will be a directed tree with nodes labelled by the irreducible polynomials b_l defining the domains $\mathbf{A}_l$, all with $Q(\mathbf{A}_l) = \mathbf{L}$ (not unlike that used in SINGULAR's `resolve.lib` [5],[2]), and arcs labelled

by unimodular transformations. Domains that are leafs of the directed tree should be defined by irreducible polynomials b_l that are *reduced* as defined below.

A *local Jacobian criterion* for non-singularity

$$Variety(Radical(\langle Jacobian(b_l)\rangle + \langle b_l\rangle)) = \emptyset$$

will be used in conjunction with $init(b_l)$ and anything known about valuations of the variables $x_{j,l}$ to direct desingularization.

Elements of this variety, restricted by the valuation information, will be viewed as describing failures of the local coordinate functions $x_{j,l}$ for $0 \leq j \leq d$ either to distinguish only one valuation or to provide d explicit local parameters when there is only one valuation.

Unimodular transformations will be used to replace sequences of blowups to simplify the process of desingularization, with care taken always to spilt singularities into disjoint cases.

The first step in this endeavor will be to strip away all of the geometric, topological, and analytical overlay that is found in traditional approaches such as Hartshorne [7], Harris [6], Cutkosky [1], Kollar [8] to expose the algebra that exists. The second step will be to question the algebra that remains, reinventing (or inventing) only what is germane. Since this is an overhaul of existing concepts and terminology, the importance lies more in the new definitions than in the associated theory, in particular relying on d-dimensional valuations as the driving force and coordinatization as the objective. The examples are important, both in motivating the definitions, and in providing benchmarks for comparing this theory to what seems to be in use currently.

It should also be clear that the characteristic plays only a secondary part, in that for most primes p (those not really part of any computation), the desingularization in characteristic p can be gotten by specializing the desingularization over $\mathbf{Q}$, as happens for the author's work on integral closures ([9],[10],[11]).

It should also be clear that dimension $d = 1$ is special in that all valuations are coordinatizable, in stark contrast to what happens when $d > 1$.

1 The algebraic function field L

Before proceeding further, it is necessary to make a basic assumption about the algebraic function field $\mathbf{L}$ that is to be our universal object, namely that it is a field in the algebraic sense that if $f \in \mathbf{L}\backslash\{0\}$, then not only

is $f^{-1} \in \mathbf{L}\backslash\{0\}$, but $f \cdot f^{-1} \equiv 1 \in \mathbf{L}$. Why should this be an obvious requirement? There is little use forcing $\mathbf{A}$ to be a domain so as to have no zero-divisors and hence cancellation, if it field of fractions does not have cancellation as well.

Both $\mathbf{A}$ and $\mathbf{L}$ will be said to have *dimension* d, in that $I(x_{i_1}, \ldots, x_{i_d}) = \{0\}$ for some subset of d of the $d+1$ variables, but any $d+1$ elements $z_0, \ldots, z_d \in \mathbf{L}$ have $I(z_0, \ldots, z_d) \neq \{0\}$. (This is meant as a more transparent, descriptive definition than the standard one involving nested sequences of ideals defining *Krull dimension.*)

Note here that for any finite subset of elements of $\mathbf{L}$ there is an ideal of all the induced relations among those elements, but that this is *independent of any computation that produces them.* [That means that exceptional divisors are to be a casualty of this viewpoint. After all, the induced-relation fairy could have put the ideal under our pillow while we were sleeping.]

2 d-dimensional valuations

Valuations are usually defined as 1-dimensional maps, but consider the following d-dimensional generalization instead.

Definition (*d-dimensional valuations*)

The map $\underline{\nu} : \mathbf{L}\backslash\{0\} \to \mathbf{Z}^d$ (think leading, so smallest, exponents in *formal Laurent series* expansions) is a *valuation* of a d-dimensional algebraic function field, if it satisfies:

1. $\underline{\nu}(c) = \underline{0}$ for $c \in \overline{\mathbf{F}}\backslash\{0\}$;
2. $\underline{\nu}(f_1 f_2) = \underline{\nu}(f_1) + \underline{\nu}(f_2)$;
3. if $\nu_i(f_1) < \nu_i(f_2)$, then $\nu_i(f_1 - f_2) = \nu_i(f_1)$;
4. if $\nu_i(f_1) = \nu_i(f_2)$, then there exists $c \in \mathbf{L}$ with $\nu_i(c) = 0$ such that $\nu_i(f_1 - cf_2) > \nu_i(f_1)$ or $f_1 - cf_2 \equiv 0$;
5. there exist d *local parameters* with $\nu_i(t_j) = \delta_{i,j}$ for $1 \leq i, j \leq d$.

3 Coordinates from valuations

To motivate evaluation (so a coordinate value) of a function $f \in \mathbf{L}$ at a valuation, consider that when $d = 1$ if $t, u_1, u_2 \in \mathbf{A}$ with $t(P) = 0$, $u_i(P) =$

$c_i \neq 0$, then for $f := (t^{a_1}u_1)/(t^{a_2}u_2)$ it makes sense for

$$f(P) := \begin{cases} 0/1 \text{ if } a_1 > a_2 \\ c_1/c_2 \text{ if } a_1 = a_2 \\ 1/0 \text{ if } a_1 < a_2 \end{cases}$$

meaning that f has a zero of order $a_1 - a_2$ at P, is a unit at P, or has a pole of order $a_2 - a_1$ at P respectively, rather than just giving up on evaluating f at P when $a_1, a_2 > 0$.

The *coordinate* $coord(f, \underline{\nu})$ will be based on the valuation $\underline{\nu}$ rather than by writing $f = g/h$ and evaluating $g(P), h(P)$ as polynomials.

Definiton (Coordinates)

$$coord(\ ,\underline{\nu})\ :\ \mathbf{L} \to \mathbf{P}^1(\overline{\mathbf{F}})$$

is defined by:

$$coord(f, \underline{\nu}) := \begin{cases} f/1 \text{ for } f \in \overline{\mathbf{F}}; \\ c/1 \text{ if there is a unique } c \in \overline{\mathbf{F}} \text{ such that } \underline{\nu}(f - c) > \underline{\nu}(f); \\ 1/0 \text{ if } \underline{\nu}(f) < \underline{0}; \\ \text{ undefined otherwise.} \end{cases}$$

Definition A function f (element of $\mathbf{L}$) is *regular* at a valuation $\underline{\nu}$ iff $coord(f, \underline{\nu})$ is defined.

[Again, this is supposed to agree with the standard method of thinking of the elements of $\mathbf{L}$ as quotients g/h of elements $g, h \in \mathbf{A}$ and finding coordinate values for both g and h to get one for g/h. But it works as well when $g(P) = 1$ and $h(P) = 0$ as well as sometimes when $g(P) = 0 = h(P)$.]

The purely algebraic goal of *desingularization* here is then to find *coordinate functions* $f_1, \ldots, f_n \in \mathbf{L}$ such that

- the *coordinates* $coord(f_1, \underline{\nu}), \ldots, coord(f_n, \underline{\nu})$ are defined whenever

  $$coord(x_0, \underline{\nu}), \ldots, coord(x_d, \underline{\nu})$$

  are defined;

- that

  $$(coord(x_0, \underline{\nu}), \ldots, coord(x_d, \underline{\nu})) = (coord(x_0, \underline{\mu}), \ldots, coord(x_d, \underline{\mu}))$$

  iff $\underline{\mu} \equiv \underline{\nu}$;

- and that there are d independent, *explicit local parameters* $t_{j,\underline{\nu}}$ with $\nu_i(t_{j,\underline{\nu}}) = \delta_{i,j}$, and each of the form

$$f_{i(j)} - coord(f_{i(j)}, \underline{\nu}) \text{ or } f_{i(j)}^{-1} - coord(f_{i(j)}^{-1}, \underline{\nu}).$$

[Note that when $d = 1$, all functions are *regular* at all valuations. This is in sharp contrast to the case $d > 1$ in which even the simplest function fields do not have this property. That is $\mathbf{L} = \overline{\mathbf{F}}(t_1, t_2)$ has functions such as $t_3 := t_1/t_2$ that can't be regular at $coord(t_1, \underline{\nu}) = 0 = coord(t_2, \underline{\nu})$, and valuations at which t_2 is not regular (since, by symmetry, $\mathbf{L} = \overline{\mathbf{F}}(t_3, t_1)$ and $t_2 = t_1/t_3$).

4 Coordinate systems

The biggest casualty from this algebraic perspective is going to be the coordinate system itself. *Affine coordinates* (elements of $\overline{\mathbf{F}}^n$) will be found to be too restrictive, though having the right flavor. *Projective coordinates* (elements of $\mathbf{P}^n(\overline{\mathbf{F}})$) are not really coordinates in the same sense as affine coordinates; but are shorthand for $n+1$ related sets (affine charts) of affine coordinates. These will be shown (in examples 0 and 1) to be counterproductive to desingularization, so will not be used here for any $n > 1$, except for comparison purposes.

It is standard to have an ordered set of coordinate functions $(x_1, \ldots, x_n)$ to describe a point P by $(x_1(P), \ldots, x_n(P)) \in \mathbf{F}^n$. Most of the time it is probably assumed that given $(a_1, \ldots, a_n) \in \mathbf{F}^n$ there is a unique point P with those coordinates. That is, it is expected to be good enough to give different coordinates to different points, though occasionally it may be used to coordinatize only some points of $\mathbf{F}^n$.

This might be sufficient for affine domains, but for the function field $\mathbf{L}$, since $x_j \in \mathbf{L}\backslash\{0\}$ implies $x_j^{-1} \in \mathbf{L}\backslash\{0\}$, it should follow that $x_j^{-1}(\underline{\nu}) = 0/1$ implies $x_j(\underline{\nu}) = 1/0$. So $coord(x_j(\underline{\nu})) \in \mathbf{P}^1(\overline{\mathbf{F}})$ is natural; hence $(\mathbf{P}^1(\overline{\mathbf{F}}))^n$ is the natural generalization $\mathbf{F}^n$, (not $\mathbf{P}^n(\overline{\mathbf{F}})$). These seem to have no name in the literature (or at least not in a context such as this), so they will be called *multi-homogeneous* coordinates here, for lack of a better name.

Definition (reduction)

The *change-of-basis* defined by

$$\phi(x_d) := y_d g(x_0, \ldots, x_{d-1}), \quad \psi(y_d) := x_d / g(x_0, \ldots x_{d-1})$$

changes a polynomial of the form

$$b(x_0, \ldots, x_d) = \sum_{j=0}^{m} x_d^{m-j} g^j(x_0, \ldots, x_{d-1}) h_j(x_0, \ldots, x_d),$$

into

$$\phi(b) = g^m(x_0, \ldots, x_{d-1}) red(b)(x_0, \ldots, x_{d-1}, y_d)$$

for

$$red(b)(x_0, \ldots, x_{d-1} y_d) := \sum_{j=0}^{m} y_d^{m-j} h_j(x_0, \ldots, x_{d-1}, y_d g(x_0, \ldots, x_{d-1})).$$

Call $red(b)$ a *reduction* of b.

Definition (reduced)

$$b(x_0, \ldots, x_d) := \sum_{\underline{\alpha}} c_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}$$

is *reduced* iff

$$\underline{\nu}(\underline{x}^{\underline{\alpha}}) = \underline{0}$$

for at least one $\underline{x}^{\underline{\alpha}}$ with $c_\alpha \neq 0$.

Also from the example above, there is a clear generalization of solving for a variable x_d in terms of the others.

Lemma 1

If $b := g(x_0, \ldots, x_{d-1}) - x_d h(x_0, \ldots, x_{d-1})$, then x_d is merely a variable name for the function $g(x_0, \ldots, x_{d-1})/h(x_0, \ldots, x_{d-1}) \in \mathbf{L}$. So $\mathbf{L} = \overline{\mathbf{F}}(x_0, \ldots, x_{d-1})$.

The best result of desingularization here is that $\mathbf{L} = \overline{\mathbf{F}}(t_1, \ldots, t_d)$, with all the t_j being *global parameters*, and the original variables described by some *global parameterization* (a change-of-basis pair of rational maps (Φ, Ψ)). But any identification of a global parameter reduces the desingularization to a smaller dimensional problem. We'll see below that any homogeneous or even weighted-homogeneous polynomial will produce at least one global parameter.

5 Unimodular transformations

While it is possible to desingularize using only algebraic blowups, this is analogous to relying on subtractions when there are divisions to be used to replace sequences of subtractions.

Reductions defined above are one way to replace sequences of blowups. But another is *unimodular transformations*, defined by unimodular matrices as follows.

Suppose that $\nu_d(x_i) > 0$ for all $0 \le i < m$ and $D := \gcd\{\nu_d(x_i) \ : \ 0 \le i < m\}$. Then the *extended euclidean algorithm* (or equivalently row-reduction over the natural numbers) can produce a *unimodular matrix* M with first column $(\nu_d(x_i)/D \ : \ 0 \le i < m)^T$. Since it is unimodular, M^{-1} has entries in $\mathbf{Z}$. So there is a change of basis defined by

$$\phi(x_i) := \prod_{j=0}^{m-1} y_j^{M_{i,j}}, \qquad \psi(y_j) := \prod_{i=0}^{m-1} x_i^{(M^{-1})_{j,i}},$$

with the other variables left unchanged.

The general result is then:

Lemma If there are weights $wt(x_i)$ such that every monomial $\underline{x^\alpha}$ occurring in b has the same total weight $w := \sum_i \alpha_i wt(x_i)$, then there is a unimodular transformation that produces a global independent parameter and reduces the desingularization dimension.

Proof $\phi(b) = t_0^w f(t_1, \ldots, t_d)$.

Surprisingly many examples in the literature are either *homogeneous* or *weighted-homogeneous*, so have at least one *independent global parameter*.

6 Local non-singular model

To recap, desingularization here produces a directed tree rooted with the root described by the multihomogeneous polynomial b_0^* shorthand for 2^{d+1} affine domains.

All other nodes are labelled by b_k defining the domain A_k. The arcs are labelled with unimodular transformations $(\phi_{k,l}, \psi_{l,k})$, with side conditions that make sure the arcs out of node k represent a partition of the singularities at node k. Further information about what is known about $\nu_d(x_{j,k})$, is kept track of at node k as well. [And it can't hurt to keep track of the composition of unimodular transformations between node the root and node k as well.]

Each arc either separates some valuations or moves toward producing d independent, explicit local parameters at the valuations with the current local coordinates.

The difference between this and what is done in `resolve.lib` in SINGULAR is that

- there is an attempt to have any single valuation correspond to exactly one directed path in the tree,

- exceptional divisors and ambient spaces have no place,
- all 2^{d+1} affine problems are considered instead of just one,
- there is an objective other than just satisfying a Jacobian criterion,
- an algebraic form of normal crossings is used to get a special desingularization.

7 Disjoint cases and reduced form

It is more important to keep track of disjoint cases than to try to operate on geometrical sub-objects. That is, it is not of interest to blow up a line of singularities, but rather to deal with a subcase in which all valuations behave similarly.

So, if $b := x_0^2+x_1^3-1$, then there are disjoint cases for $a_0 := coord(x_0, \nu) = 0$, $a_1 := coord(x_1) = 0$, and $a_0 a_1 \neq 0$, even though the domain is already (affine) non-singular.

1. If $a_0 = 0 = a_1^3 - 1$, then $x_0 = t_1$, $x_1 = a_1 + t_1^2 u_1$.
2. If $a_0^2 - 1 = 0 = a_1$, then $x_1 = t_1$, $x_0 = a_0 + t_1^3 u_1$.
3. If $a_0 a_1 \neq 0$ but $a_0^2 + a_1^3 - 1 = 0$, then $x_0 = a_0 + t_1$, $x_1 = a_1 + t_1 u_1$.

with $\nu_1(t_1) = 1$, $\nu_1(u_1) = 0$. This partition has nothing to do with any geometrical or topological shape of the parts, but everything to do with the valuations of x_0 and x_1.

8 Jacobians

The standard Jacobian criterion for non-singularity at a point P of a variety is

$$rank(Jacobian(I)|_P) = n - d$$

for n the number of (affine) variables and d the dimension, is maybe not so bad if the (affine) coordinates P are given. But when it they are not, determining the variety of all such P for which this criterion fails probably involves computing a Gröbner basis for the ideal

$$I + minors(n - d, Jacobian(I))$$

and then its variety. This could be prohibitive when $n >> d$, but is not so bad when $n = d + 1$.

So rather than having a global criterion, compute Jacobians only for the individual domains $\mathbf{A}_l$, each with $n = d + 1$. That is, use

$$Variety(Radical(\langle Jacobian(b_l)\rangle + \langle b_l\rangle))$$

to suggest tuples of current local coordinates that might be bad, center the coordinate variables, and use information about $\nu_d(x_{j,l}) > 0$, $\nu_d(x_{j,l}) \geq 0$, or $\nu_d(x_{j,l}) = 0$ in conjunction with $init(b_l)$ to find minimal choices of $\nu_d(x_{j,l})$, $0 \leq j \leq d$ for which the monomials in $init(b_l)$ satisfy

$$\nu_d(\underline{x}^{\underline{\alpha}}) = \nu_d(\underline{x}^{\underline{\beta}}) \leq \nu_d(\underline{x}^{\underline{\gamma}})$$

for all $\underline{x}^{\underline{\gamma}} \in init(b_l)$.

In particular, it may be that this side information about the $x_{j,l}$ may restrict the variety computed by the Jacobian criterion, possibly even to the point that the variety is already empty.

9 Examples

Example 0 (Coordinate systems, a non-singular example)

Consider the really simple example $b_0 := x_0^3 - x_1$ defining the domain $A := \overline{\mathbf{F}}[x_0, x_1]/\langle x_0^3 - x_1\rangle$ and its field of fractions $\mathbf{L} := Q(A)$. It should be clear that $\mathbf{L} = \overline{\mathbf{F}}(x_0)$ with x_1 merely a variable name for the function $x_0^3 \in \mathbf{L}$.

If the point had been to desingularize only A instead of $\mathbf{L}$, then since $Jacobian(b_0) = [1, -3x_0^2]$ always has rank 1, A would be non-singular. If using homogeneous coordinates and projective space was the correct way to deal with $\mathbf{L}$, then $B_0 := X_1H^2 - X_0^3$ with $Jacobian(B_0) = [H^2, -3X_0^2, 2X_1H]$; would predict a singularity at $(X_0 : X_1 : H)(P) = (0 : 1 : 0)$. This happens because the affine chart with $X_1 = 1$ uses the functions $X_0/X_1 = x_0x_1^{-1}$ and $H/X_1 = x_1^{-1}$ neither of which is a local parameter, whereas $H/X_0 = x_0^{-1}$ should be a local parameter.

The standard projective blowup would produce $U/V := (X_0/X_1)/(H/X_1) = X_0/H$ to get $(X_0, X_1, H, U, V)(P) = ((0 : 1 : 0), (1 : 0))$, with $UH - VX_1 = 0$ and $X_0^2X_1(V^3 - U^2H) = 0$ (but with $(U : V) = (0 : 0)$ not allowed). The other points with coordinates $(X_0, X_1, H)(P) = (a^3 : a : 1)$, now have coordinates $(X_0, X_1, H, U, V)(P) = ((a^3 : a : 1), (a : 1))$.

The alternative suggested here is to rewrite the variables as $x_i := g_i/h_i$ and use what will be called *multi-homogeneous coordinates*. Then $b_0^* :=$

$g_0^3 h_1 - h_0^3 g_1$ has $Jacobian(b_0^*) = [3g_0^2 h_1, -3h_0^2 g_1^2, -h_0^3, g_0^3]$ with rank less than 1 only for $(g_0 : h_0) = (0 : 0)$; meaning the multi-homogeneous variety is empty, so there is no singularity.

But it might be that b_0 is only part of a larger desingularization problem such as that defined by $f_0 := b_0^3 + b_0 x_2 + x_2^5$. Then the change-of-basis defined by $\phi(x_0) := x_0^3 y_0$ produces $\phi(b_0) = x_0^3(y_0 - 1)$ and $\phi(f_0) = x_2^5 - x_0^3(1 - y_0)x_2 - x_0^9(1 - y_0)^3$. The polynomial $1 - y_0$ is a *reduction* of b_0, and $1 - y_0$ is *reduced*.

Example 1(Coordinate systems, singular example)

$$b_0(x_0, x_1) := x_0^3 x_1^3 + x_0^2 + x_1 \in \overline{\mathbf{F}}_2[x_0, x_1]$$

defines an integral domain

$$\mathbf{A} := \overline{\mathbf{F}}_2[x_0, x_1]/\langle b_0(x_0, x_1)\rangle.$$

[Characteristic $p = 2$ is used here whenever possible for at least three reasons. Firstly it is to dispell the belief that characteristic $p > 0$ is different from characteristic 0, since the characteristic should be a factor only when it affects a computation such as factorization. Secondly computations are generally easier in small positive characteristic. Thirdly, we don't wish to introduce any complications arising from having large coefficients with which to deal.]

Consider the *divisors*

$$((x_0)) = 1 \cdot \nu^{(0)} - 3 \cdot \nu^{(1)} + 2 \cdot \nu^{(2)},$$

$$((x_1)) = 2 \cdot \nu^{(0)} + 1 \cdot \nu^{(1)} - 3 \cdot \nu^{(2)},$$

describing the *zeros* and *poles* of $x_0, x_1 \in \mathbf{L}$.

From the Jacobian $[x_0^2 x_1^3, x_0^3 x_1^2 + 1]$ of rank 1, $\mathbf{A}$ should be (affine) nonsingular, and this would cover all valuations other than $\nu^{(1)}$ and $\nu^{(2)}$.

But over $\mathbf{F}_8 := \mathbf{F}_2[\beta]/\langle 1 + \beta + \beta^3\rangle$ theory says that there should be 24 coordinatizable valuations. Of these, 21 are of the form $coord(x_0, \nu) = \beta^{5i}$ and $coord(x_1, \nu) = \beta^{3i}\delta$ for $1 + \delta + \delta^3 = 0$. For those, it is possible to write either $x_0 = \beta^{5i} + t$, $x_1 = \beta^{3i}\delta + tu$ for t a local parameter and u a local unit or $x_0 = \beta^{5i} + sv$ and $x_1 = \beta^{3i}\delta + s$ for s a local parameter and v a local unit.

At $\nu^{(0)}$ we should probably expect $x_0 = t_0$, $x_1 = t_0^2 u_0$; at ν_1, $x_0 = t_1^{-3} u_1^{-1}$, $x_1 = t_1$; and at ν_2, $x_0 = t_2^2 u_2$, $x_1 = t_2^{-3} u_2^{-1}$ for some local parameters t_i and local units u_i. The multi-homogeneous coordinates at these three are respectively $(0/1, 0/1)$, $(1/0, 0/1)$, and $(0/1, 1/0)$. Note that only $\nu^{(2)}$ did not have an explicit local parameter in terms of x_0 and x_1.

Standardly $x_i = X_i/H$ is used to produce a homogeneous polynomial, here $B(X_0, X_1, H) := X_0^3X_1^3+X_0^2H^4+X_1H^5$. Then a Jacobian computation would predict singularities at $(X_0 : X_1 : H)(P_1) = (0 : 1 : 0)$ and $(X_0 : X_1 : H)(P_2) = (1 : 0 : 0)$, even though x_1 is already a perfectly good local parameter at P_1. Not only that, but a restrictive definition of regular function would give $coord(x_0) = 0/0$ even if $x_0 = X_0/H$ is rewritten as $(X_0H^3 + X_1H^4)/(X_0^2X_1^3)$; whereas its cube $(X_0/H)^3$ could be rewritten as $(X_0H + X_1H^2)/X_1^3$, to get $coord(x_0^3) = 0/1$. That would leave one in an unenviable position of having a function that is not regular but with its cube being regular.

Homogenizing each variable separately as $x_i = g_i/h_i$ gives a multi-homogeneous polynomial $b^*(g_0, h_0, g_1, h_1) := g_0^3g_1^3 + g_0^2h_0h_1^3 + h_0^3g_1h_1^2$. A Jacobian criterion applied to this predicts only one singuality, with multi-homogeneous coordinates $coord(g_0/h_0, g_1/h_1), \underline{\nu}) = (0/1, 1/0)$, so for $\nu = \nu^{(2)}$.

The reason this happens is that projectively the functions in play at $\nu^{(1)}$ are not the original x_0 and x_1 but rather H/X_0 and X_1/X_0, neither being an explicit local parameter. In multihomogeneous coordinates the functions in play at $\nu^{(1)}$ are h_0/g_0 and g_1/h_1, the latter being a local parameter.

[There is a slight trade-off in that while projective coordinates here are shorthand here for $d + 2$ affine sets of coordinates, the proposed multi-homogeneous coordinates are shorthand for 2^{d+1} affine sets. Still, introducing spurious singularities sounds worse than dealing with this; and besides, most of these 2^{d+1} affine sets are non-singular.]

In this example a *global non-singular model* is not so unwieldy, as $g_2/h_2 := h_0h_1/g_0g_1$ produces an ideal of relations that reduces to

$$g_0^2g_1^2h_2 + g_0^2h_1^2g_2 + h_0^2g_1h_1g_2;$$

$$h_0g_1^2g_2^2 + g_0g_1^2h_2^2 + g_0h_1^2g_2h_2;$$

$$g_0^2h_1g_2^2 + h_0^2g_1g_2^2 + g_0h_0g_1h_2^2;$$

$$g_0g_1g_2 - h_0h_1h_2.$$

The initial multi-homogeneous polynomial

$$b_0^* := g_0^3g_1^3 + g_0^2h_0h_1^3 + h_0^3g_1h_1^2$$

is shorthand for the $2^2 = 4$ affine polynomials:

$$b_1 := g_0^3g_1^3+g_0^2+g_1;\ b_2 := g_0^3+g_0^2h_1^3+h_1^2;\ b_3 := g_1^3+h_0+h_0^3g_1;\ b_4 := 1+h_0h_1^3+h_0^3h_1.$$

Of these, only b_2 has an affine singularity. A unimodular transformation (change-of-basis) $\phi(g_0/h_0) = t^2u$, $\phi(g_1/h_1) = t^{-3}u^{-1}$, for $t := h_2/g_2 := g_0/h_0g_1/h_1$, and $u := (g_0/h_0)^3(g_1/h_1)^2$ to get $b_5 := 1 + u + t^7u^3$, describes the resolution of the singularity of b_2.

Example 2 (Narasimhan, found as exercise 7.35.1 in Cutkosky [1] and restricted there to char 2)

$$b_0(x_0, x_1, x_2, x_3) := x_0^2 + x_1^3x_3 + x_1x_2^3 + x_2x_3^7$$

has *weights* $wt(x_0, x_1, x_2, x_3) = (32, 19, 15, 7)$ (given in the exercise) to make all terms have weight 64. So there is a *unimodular transformation*

$$\phi_1(x_0) := y_0^{32}y_1^4y_2,\ \phi_1(x_1) := y_0^{19}y_1^2y_3,\ \phi_1(x_2) := y_0^{15}y_1^2,\ \phi_1(x_3) := y_0^7y_1,$$

$$\psi_1(y_0) := x_2/x_3^2,\ \psi_1(y_1) := x_3^{15}/x_2^7,\ \psi_1(y_2) := x_0x_3^4/x_2^4\,\psi_1(y_3) := x_1x_3^8/x_2^5$$

such that

$$\phi_1(b_0) = y_0^{64}y_1^7b_1,\ \text{for } b_1(y_1, y_2, y_3) := (y_1y_2^2 + y_3^3 + y_1y_3 + y_1^2).$$

One *default blowup* of this is

$$\phi_2(y_0) := z_0,\ \phi_2(y_1) := z_3z_1,\ \phi_2(y_2) := z_3z_2,\ \phi_2(y_3) := z_3,$$

$$\psi_2(z_0) := y_0,\ \psi_2(z_1) := y_1/y_3,\ \psi_2(z_2) := y_1/y_3,\ \psi_2(z_3) := y_3$$

such that

$$\phi_2(b_1) = z_3^2b_2,\ \text{for } b_2(z_1, z_2, z_3) := z_3(z_1z_2^2 + 1) + (z_1 + z_1^2).$$

But then $z_3 := -(z_1(1 + z_1)/(1 + z_1z_2^2)$. So $\mathbf{L} = \overline{\mathbf{F}}(z_0, z_1, z_2)$ with *full parameterization*

$$\Phi(x_0) := -z_0^{32}z_1^9\left((1 + z_1)/(1 + z_1z_2^2)\right)^5 z_2,\ \Phi(x_1) := -z_0^{19}z_1^5\left((1 + z_1)/(1 + z_1z_2^2)\right)^3,$$

$$\Phi(x_2) := z_0^{15}z_1^4\left((1 + z_1)/(1 + z_1z_2^2)\right)^2,\ \Phi(x_3) := -z_0^7z_1^2\left((1 + z_1)/(1 + z_1z_2^2)\right).$$

Example 3 (Hauser, found as exercise 7.35.2 in Cutkosky [1] and again only in char 2)

$$b_0(x_0, x_1, x_2) := x_0^2 + x_2(x_1^4 + x_1^2x_2^3 + x_2^6)$$

happens to be weighted as well, with $weight(x_0, x_1, x_2) = (7, 3, 2)$, though this is not given. A characteristic-independent version of this is

$$b_0(x_0, x_1, x_2) := (x_0 + x_1x_2^2)^2 - x_2(x_1^2 + x_2^3)^2.$$

This has

$$x_2 = \left(\frac{x_0 + x_1 x_2^2}{x_1^2 + x_2^3} \right)^2$$

.

So $\phi_1(x_2) = y_2^2$ for $\psi_1(y_2) = (x_0 + x_1 x_2^2)/(x_1^2 + x_2^3)$ gives

$$\phi_1(b_0) = b_1^2, \text{ for } b_1(y_0, y_1, y_2) := y_0 + y_1 y_2^4 - y_2(y_1^2 + y_2^6);$$

which, in turn, means that $y_0 = y_1 y_2^4 - y_2(y_1^2 + y_2^6)$. So $\mathbf{L} = \overline{\mathbf{F}}(y_1, y_2)$ with full parameterization

$$\Phi(x_0) = y_0 = y_2^7 - y_2^4 y_1 + y_2 y_1^2,$$

$$\Phi(x_1) = y_1,$$

$$\Phi(x_2) = y_2^2.$$

A slightly better version in terms of weights is gotten by setting $y_1 := y_2^3 y_3$ to get

$$\Phi(x_0) = y_0 = y_2^7(1 - y_3 + y_3^2),$$

$$\Phi(x_1) = y_2^3 y_3,$$

$$\Phi(x_2) = y_2^2.$$

Example 4

$$b_0(x_0, x_1, x_2, x_3) := x_0^2 + x_1^2 + x_2^{2m+1} x_3^{2n+1}, \ m \geq n$$

is inspired by example 6.2 in Kollar [8]. Kollar suggests the red herring of choosing between blowing up $(x_0, x_1, x_2) = (0, 0, 0)$ or $(x_0, x_1, x_3) = (0, 0, 0)$. when it is better to consider three cases $(x_0, x_1, x_2, x_3) = (0, 0, 0, 0)$ $(x_0, x_1, x_2, x_3) = (0, 0, 0, a)$ or $(x_0, x_1, x_2, x_3) = (0, 0, a, 0)$ for $a \neq 0$. [This is a geometric blindspot, wherein one thinks blowups are to be applied to a geometric sub-object rather than a difference of such. Algebraically, the singularity at the origin is different from the rest, so if default algebraic blowups were employed, they would probably be dealing with three disjoint cases, the origin and two lines with the origin removed as opposed to two cases, one for each line.]

This has reduction $\phi_1(x_0) := y_2^m y_3^n y_0$, $\phi_1(x_1) := y_2^m y_3^n y_1$, $\phi_1(x_2) := y_2$, and $\phi_1(x_3) := y_3$. This produces

$$\phi_1(b_0) = y_2^{2m} y_3^{2n} b_1, \quad b_1(y_0, y_1, y_2, y_3) := y_0^2 + y_1^2 + y_2 y_3.$$

Then $y_3 := -(y_0^2 + y_1^2)/y_2$. So $\mathbf{L} = \overline{\mathbf{F}}(y_0, y_1, y_2)$ with full parameterization

$$\Phi(x_0) := (-1)^n y_2^{m-n}(y_0^2 + y_1^2)^n y_0,$$

$$\Phi_1(x_1) := (-1)^n y_2^{m-n}(y_0^2 + y_1^2)^n y_1,$$

$$\Phi_1(x_2) := y_2,$$

$$\Phi_1(x_3) := -(y_0^2 + y_1^2)/y_2.$$

Example 5 Let $b_0 := x_2^3 + x_1^5 + x_0^7$. Then $\nu_2(x_2, x_1, x_0) = (35, 21, 15)$,

$$M = \begin{pmatrix} 35 & 7 & 2 \\ 21 & 4 & 1 \\ 15 & 3 & 1 \end{pmatrix},\ M^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 6 & -5 & -7 \\ -3 & 0 & 7 \end{pmatrix},$$

and

$$\phi(x_2) := y_0^{35} y_1^7 y_2^2,\ \phi(x_1) := y_0^{21} y_1^4 y_2,\ \phi(y_0) := y_0^{15} y_1^3 y_2$$

$$\psi(y_0) := x_1 x_2 / x_0,\ \psi(y_1) := x_0^6/(x_1^5 x_2^7),\ \psi(y_2) := x_2^7/x_0^3$$

$$\phi(b_0) = y_0^{105} y_1^{20} y_2^5 (y_1 y_2 + 1 + y_1 y_2^2)$$

In this example the valuations make b_0 a *weighted-homogeneous polynomial*, which results in y_0 being a *global independent parameter*, reducing the desingularization to a lower dimension. Here $\mathbf{A} = \overline{\mathbf{F}}[y_0, y_1, y_2]/\langle 1 + y_1(y_2 + y_2^2)\rangle$. Here, applying lemma 1 gives $y_1 = -1/(y_2 + y_2^2)$, so $\mathbf{L} = \overline{\mathbf{F}}(y_0, y_2)$, with global parameterization

$$\Phi(x_2, x_1, x_0) := \left(\frac{-y_0^{35}}{y_2^5(1+y_2)^7},\ \frac{y_0^{21}}{y_2^3(1+y_2)^4},\ \frac{-y_0^{15}}{y_2^2(1+y_2)^3} \right).$$

[Note that a general algorithm applying geometric blowups, namely `resolve` and `presentTree` in SINGULAR's `resolve.lib` produces 365 affine charts after 723 blowups for this easy desingularization, which each affine chart the result of a sequence of between 4 and 19 blowups.]

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