

# The Goldbach Conjecture

## Sections

### Summing Primes

*Here we verify the conjecture for small numbers.*

### The Sieve of Eratosthenes

*A fairly fast way to determine if small numbers are prime, given storage.*

# Summing Primes

Here we verify the conjecture for small numbers.

§1. On 7 June 1742, Christian Goldbach wrote a letter from Moscow to Leonhard Euler in Berlin making “eine conjecture hazardiren” that every even number greater than 2 can be written as a sum of two primes.<sup>1</sup> Euler did not know if this was true, and nor does anyone else.

fabrum, nunc hactenus, ut videtur ab eis non est demonstratum,  
 \* namque singulis seriebus tantum numeros unius modi in duo quadrata  
 divisibiles habent, nunc igitur vult ut eius non conjectura  
 hazardarem: sed quod quilibet numerus summa numeris primis  
 compositionem habet, est enim aggregatum quodammodo numerorum  
 primorum, quod ab eis non est demonstratum, sed est conjectura  
 de qua si congeriam omnium unitatum, quod summa  

$$4 = \begin{cases} 1+1+1+1 \\ 1+1+2 \\ 1+3 \end{cases} \quad 5 = \begin{cases} 1+1+1+1+1 \\ 1+1+1+2 \\ 1+1+1+1+1 \end{cases} \quad 6 = \begin{cases} 1+1+1+1+1+1 \\ 1+1+1+2+1 \\ 1+1+1+1+1+1 \end{cases}$$
  
 Rursus sequitur non parva observatio, quod demonstravit mihi  
 Leonhardus Euler:  
 Si  $x$  sit functio ipsius  $x$ , cuius modi ut facta  $v = c$ , numero cu-  
 que, determinari possit  $x$  per  $c$ , et reliquis constantibus in functi-  
 one expressis, poterit etiam determinari valor ipsius  $x$ , in æ-  
 quatione  $v^{n+1} = (2v+1)(v+1) \cdot \frac{v^{n+1}-1}{v-1} - \frac{(v+1)(v-1)}{v-1} \cdot \frac{v^{n+1}-1}{v-1}$  dicitur,  $v=1$   
 Si accipiamus curvas cuius abscissa sit  $x$ , applicata deo sit  
 summa serie  $\frac{x^n}{n \cdot 2^{n-1}}$  posita  $x$ , pro exponente terminorum, hoc est,  
 applicata  $= \frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 2^2} + \frac{x^3}{3 \cdot 2^3} + \frac{x^4}{4 \cdot 2^4} + \text{etc.}$  dicitur, si fuerit  
 abscissa  $= 1$ , applicatum fore  $= \frac{1}{2} = \frac{1}{2}$ : sed hoc apparet  $= \frac{1}{2}$   
 $\frac{1}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \frac{1}{2048} + \frac{1}{4096} + \frac{1}{8192} + \frac{1}{16384} + \frac{1}{32768} + \frac{1}{65536} + \frac{1}{131072} + \frac{1}{262144} + \frac{1}{524288} + \frac{1}{1048576} + \frac{1}{2097152} + \frac{1}{4194304} + \frac{1}{8388608} + \frac{1}{16777216} + \frac{1}{33554432} + \frac{1}{67108864} + \frac{1}{134217728} + \frac{1}{268435456} + \frac{1}{536870912} + \frac{1}{1073741824} + \frac{1}{2147483648} + \frac{1}{4294967296} + \frac{1}{8589934592} + \frac{1}{17179869184} + \frac{1}{34359738368} + \frac{1}{68719476736} + \frac{1}{137438953472} + \frac{1}{274877906944} + \frac{1}{549755813888} + \frac{1}{1099511627776} + \frac{1}{2199023255552} + \frac{1}{4398046511104} + \frac{1}{8796093022208} + \frac{1}{17592186044416} + \frac{1}{35184372088832} + \frac{1}{70368744177664} + \frac{1}{140737488355328} + \frac{1}{281474976710656} + \frac{1}{562949953421312} + \frac{1}{1125899906842624} + \frac{1}{2251799813685248} + \frac{1}{4503599627370496} + \frac{1}{9007199254740992} + \frac{1}{18014398509481984} + \frac{1}{36028797018963968} + \frac{1}{72057594037927936} + \frac{1}{144115188075855872} + \frac{1}{288230376151711744} + \frac{1}{576460752303423488} + \frac{1}{1152921504606846976} + \frac{1}{2305843009213693952} + \frac{1}{4611686018427387904} + \frac{1}{9223372036854775808} + \frac{1}{18446744073709551616} + \frac{1}{36893488147419103232} + \frac{1}{73786976294838206464} + \frac{1}{147573952589676412928} + \frac{1}{295147905179352825856} + \frac{1}{590295810358705651712} + \frac{1}{1180591620717411303424} + \frac{1}{2361183241434822606848} + \frac{1}{4722366482869645213696} + \frac{1}{9444732965739290427392} + \frac{1}{18889465931478580854784} + \frac{1}{37778931862957161709568} + \frac{1}{75557863725914323419136} + \frac{1}{151115727451828646838272} + \frac{1}{302231454903657293676544} + \frac{1}{604462909807314587353088} + \frac{1}{1208925819614629174706176} + \frac{1}{2417851639229258349412352} + \frac{1}{4835703278458516698824704} + \frac{1}{9671406556917033397649408} + \frac{1}{19342813113834066795298816} + \frac{1}{38685626227668133590597632} + \frac{1}{77371252455336267181195264} + \frac{1}{154742504910672534362390528} + \frac{1}{309485009821345068724781056} + \frac{1}{618970019642690137449562112} + \frac{1}{1237940039285380274899124224} + \frac{1}{2475880078570760549798248448} + \frac{1}{4951760157141521099596496896} + \frac{1}{9903520314283042199192993792} + \frac{1}{19807040628566084398385987584} + \frac{1}{39614081257132168796771975168} + \frac{1}{79228162514264337593543950336} + \frac{1}{158456325028528675187087900672} + \frac{1}{316912650057057350374175801344} + \frac{1}{633825300114114700748351602688} + \frac{1}{1267650600228229401496703205376} + \frac{1}{2535301200456458802993406410752} + \frac{1}{5070602400912917605986812821504} + \frac{1}{10141204801825835211973625643008} + \frac{1}{20282409603651670423947251286016} + \frac{1}{40564819207303340847894502572032} + \frac{1}{81129638414606681695789005144064} + \frac{1}{162259276829213363391578010288128} + \frac{1}{324518553658426726783156020576256} + \frac{1}{649037107316853453566312041152512} + \frac{1}{1298074214633706907132624082305024} + \frac{1}{2596148429267413814265248164610048} + \frac{1}{5192296858534827628530496329220096} + \frac{1}{10384593717069655257060992658440192} + \frac{1}{20769187434139310514121985316880384} + \frac{1}{41538374868278621028243970633760768} + \frac{1}{83076749736557242056487941267521536} + \frac{1}{166153499473114484112975882535043072} + \frac{1}{332306998946228968225951765070086144} + \frac{1}{664613997892457936451903530140172288} + \frac{1}{1329227995784915872903807060280344576} + \frac{1}{2658455991569831745807614120560689152} + \frac{1}{5316911983139663491615228241121378304} + \frac{1}{10633823966279326983230456482242756608} + \frac{1}{21267647932558653966460912964485513216} + \frac{1}{42535295865117307932921825928971026432} + \frac{1}{85070591730234615865843651857942052864} + \frac{1}{170141183460469231731687303715884105728} + \frac{1}{340282366920938463463374607431768211456} + \frac{1}{680564733841876926926749214863536422912} + \frac{1}{1361129467683753853853498429727072845824} + \frac{1}{2722258935367507707706996859454145691648} + \frac{1}{5444517870735015415413993718908291383296} + \frac{1}{10889035741470030830827987437816582766592} + \frac{1}{21778071482940061661655974875633165533184} + \frac{1}{43556142965880123323311949751266331066368} + \frac{1}{87112285931760246646623899502532662132736} + \frac{1}{174224571863520493293247799005065324265472} + \frac{1}{348449143727040986586495598010130648530944} + \frac{1}{696898287454081973172991196020261297061888} + \frac{1}{1393796574908163946345982392040522594123776} + \frac{1}{2787593149816327892691964784081045188247552} + \frac{1}{5575186299632655785383929568162090376495104} + \frac{1}{11150372599265311570767859136324180752990208} + \frac{1}{22300745198530623141535718272648361505980416} + \frac{1}{44601490397061246283071436545296723011960832} + \frac{1}{89202980794122492566142873090593446023921664} + \frac{1}{178405961588244985132285746181186892047843328} + \frac{1}{356811923176489970264571492362373784095686656} + \frac{1}{713623846352979940529142984724747568191373312} + \frac{1}{1427247692705959881058285969449495136382746624} + \frac{1}{2854495385411919762116571938898990272765493248} + \frac{1}{5708990770823839524233143877797980545530986496} + \frac{1}{11417981541647679048466287755595961091061972992} + \frac{1}{22835963083295358096932575511191922182123945984} + \frac{1}{45671926166590716193865151022383844364247891968} + \frac{1}{91343852333181432387730302044767688728495783936} + \frac{1}{182687704666362864775460604089535377456991567872} + \frac{1}{365375409332725729550921208179070754913983135744} + \frac{1}{730750818665451459101842416358141509827966271488} + \frac{1}{1461501637330902918203684832716283019655932542976} + \frac{1}{2923003274661805836407369665432566039311865085952} + \frac{1}{5846006549323611672814739330865132078623730171904} + \frac{1}{11692013098647223345629478661730264157247460343808} + \frac{1}{23384026197294446691258957323460528314494920687616} + \frac{1}{46768052394588893382517914646921056628989841375232} + \frac{1}{93536104789177786765035829293842113257979682750464} + \frac{1}{187072209578355573530071658587684226515959365500928} + \frac{1}{374144419156711147060143317175368453031918731001856} + \frac{1}{748288838313422294120286634350736906063837462003712} + \frac{1}{1496577676626844588240573268701473812127674924007424} + \frac{1}{2993155353253689176481146537402947624255349848014848} + \frac{1}{5986310706507378352962293074805895248510699696029696} + \frac{1}{11972621413014756705924586149611790497021399392059392} + \frac{1}{23945242826029513411849172299223580994042798784118784} + \frac{1}{47890485652059026823698344598447161988085597568237568} + \frac{1}{95780971304118053647396689196894323976171195136475136} + \frac{1}{191561942608236107294793378393788647952342390272950272} + \frac{1}{383123885216472214589586756787577295904684780545900544} + \frac{1}{766247770432944429179173513575154591809369561091801088} + \frac{1}{1532495540865888858358347027150309183618739122183602176} + \frac{1}{3064991081731777716716694054300618367237478244367204352} + \frac{1}{6129982163463555433433388108601236734474956488734408704} + \frac{1}{12259964326927110866866776217202473468949912977468817408} + \frac{1}{24519928653854221733733552434404946937899825954937634816} + \frac{1}{49039857307708443467467104868809893875799651909875269632} + \frac{1}{98079714615416886934934209737619787751599303819750539264} + 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\frac{1}{13164036458569648337239753460458804039861886925068638906788872192} + \frac{1}{26328072917139296674479506920917608079723773850137277813577744384} + \frac{1}{52656145834278593348959013841835216159447547700274555627155488768} + \frac{1}{105312291668557186697918027683670432318895095400549111254310977536} + \frac{1}{210624583337114373395836055367340864637790190801098222508621955072} + \frac{1}{421249166674228746791672110734681729275580381602196445017243910144} + \frac{1}{842498333348457493583344221469363458551160763204392890034487820288} + \frac{1}{1684996666696914987166688442938726917102321526408785780068975640576} + \frac{1}{3369993333393829974333376885877453834204643052817571560137951281152} + \frac{1}{6739986666787659948666753771754907668409286105635143120275902562304} + \frac{1}{13479973333575319897333507543509815336818572211270286240551805124608} + \frac{1}{26959946667150639794667015087019630673637144422540572481103610249216} + 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\frac{1}{220855883097298041197912187592864814478435487109452369765200775161577472} + \frac{1}{441711766194596082395824375185729628956870974218904739530401550323154944} + \frac{1}{883423532389192164791648750371459257913741948437809479060803100646309888} + \frac{1}{1766847064778384329583297500742918515827483896875618958121606201292619776} + \frac{1}{3533694129556768659166595001485837031654967793751237916243212402585239552} + \frac{1}{7067388259113537318333190002971674063309935587502475832486424805170479104} + \frac{1}{14134776518227074636666380005943348126619871175004951664972849610340958208} + \frac{1}{28269553036454149273332760011886696253239742350009903329945699220681916416} + \frac{1}{56539106072908298546665520023773392506479484700019806659891398441363832832} + \frac{1}{113078212145816597093331040047546785012958969400039613319782796882727665664} + \frac{1}{226156424291633194186662080095093570025917938800079226639565593765455331328} + \frac{1}{452312848583266388373324160190187140051835877600158453279131187530910662656} + \frac{1}{904625697166532776746648320380374280103671755200316906558262375061821325312} + \frac{1}{1809251394333065553493296640760748560207343510400633813116524750123642650624} + \frac{1}{3618502788666131106986593281521497120414687020801267626233049500247285301248} + \frac{1}{723700557733226221397318656$

§2. Computer verification has been made up to around  $10^{18}$ , but by rather better methods than the one we use here. We will only go up to:

```
define RANGE 100
```

```
#include <stdio.h>
```

```
int main(int argc, char *argv[]) {
```

```
    for (int i=4; i<RANGE; i=i+2)
```

*stepping in twos to stay even*

```
        ⟨Solve Goldbach's conjecture for i 2.1⟩ ;
```

```
}
```

§2.1. This ought to print:

```
$ goldbach/Tangled/goldbach
```

```
4 = 2+2
```

```
6 = 3+3
```

```
8 = 3+5
```

```
10 = 3+7 = 5+5
```

```
12 = 5+7
```

```
14 = 3+11 = 7+7
```

```
...
```

We'll print each different pair of primes adding up to  $i$ . We only check in the range  $2 \leq j \leq i/2$  to avoid counting pairs twice over (thus  $8 = 3 + 5 = 5 + 3$ , but that's hardly two different ways).

⟨Solve Goldbach's conjecture for i 2.1⟩  $\equiv$

```
printf("%d", i);
```

```
for (int j=2; j<=i/2; j++)
```

```
    if ((isprime(j)) && (isprime(i-j)))
```

```
        printf(" = %d+%d", j, i-j);
```

```
printf("\n");
```

This code is used in §2.

# The Sieve of Eratosthenes

*A fairly fast way to determine if small numbers are prime, given storage.*

---

S1 Storage; S2 Primality

---

§1. This technique, still essentially the best sieve for finding prime numbers, is attributed to Eratosthenes of Cyrene and dates from the 200s BC. Since composite numbers are exactly those numbers which are multiples of something, the idea is to remove everything which is a multiple: whatever is left, must be prime.

This is very fast (and can be done more quickly than the implementation below), but (a) uses storage to hold the sieve, and (b) has to start right back at 2 - so it can't efficiently test just, say, the eight-digit numbers for primality.

```
int still_in_sieve[RANGE + 1];
int sieve_performed = FALSE;
```

§2. We provide this as a function which determines whether a number is prime:

```
define TRUE 1
define FALSE 0

int isprime(int n) {
    if (n <= 1) return FALSE;
    if (n > RANGE) { printf("Out of range!\n"); return FALSE; }
    if (!sieve_performed) <Perform the sieve 2.1> ;
    return still_in_sieve[n];
}
```

§2.1. We save a little time by noting that if a number up to RANGE is composite then one of its factors must be smaller than the square root of RANGE. Thus, in a sieve of size 10000, one only needs to remove multiples of 2 up to 100, for example.

<Perform the sieve 2.1>  $\equiv$

```
<Start with all numbers from 2 upwards in the sieve 2.1.1> ;
for (int n=2; n*n <= RANGE; n++)
    if (still_in_sieve[n])
        <Shake out multiples of n 2.1.2> ;
sieve_performed = TRUE;
```

This code is used in §2.

§2.1.1.

<Start with all numbers from 2 upwards in the sieve 2.1.1>  $\equiv$

```
still_in_sieve[1] = FALSE;
for (int n=2; n <= RANGE; n++) still_in_sieve[n] = TRUE;
```

This code is used in §2.1.

§2.1.2.

<Shake out multiples of n 2.1.2>  $\equiv$

```
for (int m= n*n; m <= RANGE; m += n) still_in_sieve[m] = FALSE;
```

This code is used in §2.1.