NATIONAL UNIVERSITY OF SINGAPORE

MA2001 — LINEAR ALGEBRA

2021 – 2022 SEMESTER II

28 April 2022, 9:00 am – 11:00 am

INSTRUCTIONS TO CANDIDATES

1. Use A4 size paper and pen (blue or black ink) to write your answers. You **CANNOT** use electronic devices (e.g., iPad) to write answers.
2. Write down your student number clearly on the top left of every page of the answers.
3. Write on one side of the paper only. Start a new question in a new page. Write the question number and page number on the top right corner of each page (e.g., Q1P1, Q1P2, . . . , Q4P1, Q4P2).
4. This exam paper contains **EIGHT (8)** questions. Answer **ALL** questions.
5. The total mark for this paper is **ONE HUNDRED (100)**.
6. This is an **OPEN BOOK** exam. You may use any hardcopy or softcopy in your computer or in LumiNUS. But you cannot use any software for computation and cannot use Internet except LumiNUS.
7. You may use non-graphing calculators. However, you should lay out systematically the various steps in the calculations.
8. Join the Zoom conference and turn on the video setting at all time during the exam. Adjust your camera such that your face, upper body (including your hands), as well as your computer screen are captured on Zoom.
9. You may go for a short toilet break (not more than 5 minutes) during the exam.
10. At the end of the exam,
    - (i) Scan or take pictures of your work (make sure the images can be read clearly).
    - (ii) Merge all your images into one PDF file in correct order.
    - (iii) Name the PDF file by “Student Number” (e.g., A1234567X.pdf).
    - (iv) Upload your PDF into the LumiNUS Folder “Final Exam – Submission”.
    - (v) Review your submission to ensure that it is successful.
    - (vi) You have 15 minutes for submission after the exam. After the folder is closed, no submission will be accepted.

Question 1 [10 marks]

Consider the following linear system

$$\begin{cases} x + 2y + z = a \\ 4x + 5y + 6z = 2a \\ - ay + 2z = b. \end{cases}$$

Determine the conditions on the constants a, b such that the linear system has: (i) exactly one solution; (ii) no solution; (iii) infinitely many solutions.

Question 2 [15 marks]

Consider the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & -3 & 0 \\ 2 & -1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}.$$

(i) Find the determinant of $\boldsymbol{A}$.

(ii) Find the inverse of $\boldsymbol{A}$.

(iii) Let $\boldsymbol{B}$ be another square matrix of order 4 such that $\boldsymbol{B}^T\boldsymbol{A}\boldsymbol{B} = \boldsymbol{A}$. Show that $\boldsymbol{B}$ is also invertible.

Question 3 [10 marks]

Consider the matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Find all eigenvalues of $\boldsymbol{A}$.

(ii) For each of the eigenvalues, find a basis for the associated eigenspace.

(iii) Find an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

Question 4 [15 marks]

Let $z = ax^2 + by + c$ be a function in variables x, y for some fixed real numbers a, b, c. Suppose that we have measured four data points for the values (x, y, z) as follows

$$(1,2,0), \quad (0,1,1), \quad (-1,0,1), \quad (1,1,1).$$

Use least squares method to find a best approximation of the function such that the sum of the squares of the errors of the function is minimal.

Question 5 [15 marks]

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are vectors in $\mathbb{R}^3$, and let $T = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be such that $\boldsymbol{u}_1 = \frac{2}{3}\boldsymbol{v}_1 + \frac{1}{3}\boldsymbol{v}_2 + \frac{2}{3}\boldsymbol{v}_3$, $\boldsymbol{u}_2 = -\frac{2}{3}\boldsymbol{v}_1 + \frac{2}{3}\boldsymbol{v}_2 + \frac{1}{3}\boldsymbol{v}_3$, $\boldsymbol{u}_3 = \frac{1}{3}\boldsymbol{v}_1 + \frac{2}{3}\boldsymbol{v}_2 - \frac{2}{3}\boldsymbol{v}_3$.

(i) Prove that $\operatorname{span}(S) = \operatorname{span}(T)$.

(ii) Prove that if S is an orthonormal set, then T is also an orthonormal set.

Question 6 [10 marks]

Let $\boldsymbol{A}$ be a square matrix of order n such that

$$\operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{A} - \boldsymbol{I}) = n.$$

Prove that $\boldsymbol{A}^2 = \boldsymbol{A}$.

Question 7 [10 marks]

Let $\boldsymbol{A}$ be an invertible matrix of order n with n distinct eigenvalues. Suppose $\boldsymbol{B}$ is a square matrix such that $\boldsymbol{AB} = \boldsymbol{BA}^{-1}$. Prove that $\boldsymbol{B}^2$ is diagonalizable.

Question 8 [15 marks]

Determine whether each of the following statements is true or false. You need to **justify** your answers.

(a) Let $\boldsymbol{A}$ be an orthogonal matrix of order n. Then $\boldsymbol{A}$ is the transition matrix from some orthonormal basis S of $\mathbb{R}^n$ to some orthonormal basis T of $\mathbb{R}^n$.

(b) The mapping $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = xyz$ is a linear transformation.

(c) Let $\boldsymbol{A}$ be an $m \times n$ matrix. If there is an $m \times m$ matrix $\boldsymbol{B}$ such that $\boldsymbol{BA}$ is full rank, then $\boldsymbol{A}$ is also full rank.

(d) Let $\boldsymbol{A}$ be a square matrix. If $\boldsymbol{A}$ is both nilpotent and diagonalizable, then $\boldsymbol{A}$ is the zero matrix. (Recall that $\boldsymbol{A}$ is nilpotent if $\boldsymbol{A}^m = \boldsymbol{0}$ for some positive integer m.)

(e) Let $\boldsymbol{A}$ be a square matrix of order n. If $\boldsymbol{A}$ has n distinct eigenvectors, then $\boldsymbol{A}$ is diagonalizable.

End of Paper