习题 5.4

1. (2)

$$\because P_n = \prod_{k=2}^{n}\left(1-\frac{2}{k(k+1)}\right)$$

$$= \frac{4\times1}{2\times3}\cdot\frac{5\times2}{3\times4}\cdot\frac{6\times3}{4\times5}\cdot\cdots\cdot\frac{(n+1)(n-2)}{(n-1)n}\cdot\frac{(n+2)(n-1)}{n(n+1)}$$

$$= \frac{1}{3}\cdot\frac{n+2}{n}$$

$$\therefore \lim_{n\to\infty} P_n = \frac{1}{3}$$

$$\therefore \prod_{n=2}^{\infty}\left(1-\frac{2}{n(n+1)}\right) = \frac{1}{3}.$$

(4) 考虑级数 $\sum_{n=1}^{\infty} \ln\frac{n}{\sqrt{n^2+1}}$

$$= \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)\ln\left(1+\frac{1}{n^2}\right)$$

$$\because \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} = 1$$

且 $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 收敛

$\therefore \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)\ln\left(1+\frac{1}{n^2}\right)$ 收敛

$\therefore \prod_{n=1}^{\infty}\frac{n}{\sqrt{n^2+1}}$ 收敛.

2. (3)

考虑级数 $\sum_{n=1}^{\infty}\frac{1}{n}\ln\left(1+\frac{1}{n}\right)$

$$\because \lim_{n\to\infty}\frac{\frac{1}{n}\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

$$= 1$$

且 $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 收敛

$\therefore \sum_{n=1}^{\infty}\frac{1}{n}\ln\left(1+\frac{1}{n}\right)$ 收敛

$\therefore \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{\frac{1}{n}}$ 收敛.

习题 5.1

3. (1) $u_n = S_n - S_{n-1} = \frac{2n}{n+1} - \frac{2(n-1)}{n}$

$$= \frac{2}{n(n+1)}$$

(2) $\because \lim\limits_{n\to\infty} S_n = 2$

$\therefore \sum\limits_{n=1}^{\infty} u_n$ 收敛.

5. 证明：[redacted]

令 $S_n = \sum\limits_{k=1}^{n} u_k$, $T_n = \sum\limits_{k=1}^{n} (k+1)(u_{k+1}-u_k)$

则 $T_n = 2(u_2-u_1) + 3(u_3-u_2) + \cdots + (n+1)(u_{n+1}-u_n)$

$$= -u_1 - S_n + (n+1)u_{n+1}$$

$\because \lim\limits_{n\to\infty} nu_n = 0$

$\therefore \lim\limits_{n\to\infty} T_n$ 存在 $\Leftrightarrow \lim\limits_{n\to\infty} S_n$ 存在.

习题 5.2

1. (2) $\because \lim\limits_{n\to\infty} \dfrac{\sin\frac{\pi}{2^n}}{\frac{\pi}{2^n}} = 1$

且 $\sum\limits_{n=1}^{\infty} \frac{\pi}{2^n} = \pi$

$\therefore \sum\limits_{n=1}^{\infty} \sin\frac{\pi}{2^n}$ 收敛.

(4) $\because \lim\limits_{n\to\infty} \dfrac{\frac{\ln n}{n^{\frac{3}{2}}}}{\frac{1}{n^{\frac{4}{3}}}} = \lim\limits_{n\to\infty} \dfrac{\ln n}{n^{\frac{1}{6}}} = 0$

且 $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$ 收敛

$\therefore \sum\limits_{n=1}^{\infty} \frac{\ln n}{n^{\frac{3}{2}}}$ 收敛.

(5) $\because \dfrac{u_n}{u_{n+1}} = \dfrac{\left(\frac{1+n^2}{1+n^3}\right)^2}{\left(\frac{1+(n+1)^2}{1+(n+1)^3}\right)^2}$

$$= \frac{\frac{n^4+2n^2+1}{n^6+2n^3+1}}{\frac{(n+1)^4+2(n+1)^2+1}{(n+1)^6+2(n+1)^3+1}}$$

$$= \frac{n^{10}+6n^9+o(n^9)}{n^{10}+4n^9+o(n^9)}$$

$\therefore n\left(\frac{u_n}{u_{n+1}}-1\right) = n\cdot\frac{2n^9+o(n^9)}{n^{10}+4n^9+o(n^9)}$

$\therefore \lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = 2$

由 Raabe 判别法知 $\sum\limits_{n=1}^{\infty}\left(\frac{1+n^2}{1+n^3}\right)^2$ 收敛.

(7) $\because \lim\limits_{n\to\infty} \dfrac{\frac{1}{\sqrt{n}}\ln\frac{n+1}{n-1}}{n^{-\frac{3}{2}}}$

$$= \lim_{n\to\infty} \frac{\ln(n+1)-\ln(n-1)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1}-\frac{1}{n-1}}{-\frac{1}{n^2}}$$

$$= 2$$

且 $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ 收敛

$\therefore \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}\ln\frac{n+1}{n-1}$ 收敛.

2. (1)

$\because \lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \dfrac{\frac{2^{n+1}}{(2n+1)!}}{\frac{2^n}{(2n-1)!}} = \lim\limits_{n\to\infty} \dfrac{2}{2n(2n+1)} = 0$

$\therefore \sum\limits_{n=1}^{\infty} \dfrac{2^n}{(2n-1)!}$ 收敛

(3)

$\because \lim\limits_{n\to\infty} \sqrt[n]{u_n} = \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{3^n \cdot n}{n^n}} = \lim\limits_{n\to\infty} \dfrac{3}{n} = 0$

$\therefore \sum\limits_{n=1}^{\infty} \dfrac{3^n \cdot n}{n^n}$ 收敛

3. (2)

$\because \lim\limits_{n\to\infty} \sqrt[n]{u_n} = \lim\limits_{n\to\infty} \dfrac{1}{3} \cdot \left(1+\dfrac{1}{n}\right)^n = \dfrac{e}{3} < 1$

$\therefore \sum\limits_{n=1}^{\infty} \dfrac{1}{3^n} \left(1+\dfrac{1}{n}\right)^{n^2}$ 收敛.

(3) 与例 5.2.4 类似.

① 当 $p>1$ 时, $\lim\limits_{n\to\infty} \dfrac{\frac{1}{n^p (\ln n)^q (\ln\ln n)^r}}{\frac{1}{n^{\frac{1+p}{2}}}} = 0$

$\therefore \sum\limits_{n=3}^{\infty} \dfrac{1}{n^p (\ln n)^q (\ln\ln n)^r}$ 收敛.

② 当 $p<1$ 时, $\lim\limits_{n\to\infty} \dfrac{\frac{1}{n^p (\ln n)^q (\ln\ln n)^r}}{\frac{1}{n^{\frac{1+p}{2}}}} = +\infty$

$\therefore \sum\limits_{n=3}^{\infty} \dfrac{1}{n^p (\ln n)^q (\ln\ln n)^r}$ 发散.

③ 当 $p=1$ 时, 级数与积分 $\displaystyle\int_3^{+\infty} \frac{dx}{x(\ln x)^q (\ln\ln x)^r}$

$\overset{令 t=\ln x}{=} \displaystyle\int_{\ln 3}^{+\infty} \frac{dt}{t^q (\ln t)^r}$

同理, (i) 当 $q>1$ 时, 级数收敛 (ii) $q<1$ 时级数发散

(iii) 当 $q=1$ 时, $\begin{cases} 当 r>1 时收敛 \\ 当 r\le 1 时发散. \end{cases}$

综合得:

① 当 $p>1$ 或 $p=1$ 且 $q>1$

或 $p=1$, $q=1$ 且 $r>1$ 时, 级数收敛.

② 当 $p<1$ 或 $p=1$ 且 $q<1$

或 $p=1$, $q=1$ 且 $r\le 1$ 时, 级数发散.

(6) $\because \lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \dfrac{\ln(n+1)+\ln(n!)}{(n+1)!} \cdot \dfrac{n!}{\ln(n!)}$

$= \lim\limits_{n\to\infty} \left[\dfrac{\ln(n+1)}{n+1} \cdot \dfrac{1}{\ln(n!)} + \dfrac{1}{n+1}\right]$

$= 0$

$\therefore$ 级数收敛.

(9) $\because \lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \dfrac{3n+2}{2^{n+1}+2^{-n-1}} \cdot \dfrac{2^n+2^{-n}}{3n-1}$

$= \dfrac{1}{2}$

$\therefore$ 级数收敛.

5. $\because u_n>0$, $\{n u_n\}$ 有界, 可设 $n u_n \le M$, $\forall n$

$\therefore 0 < \dfrac{u_n}{n} = \dfrac{1}{n^2} \cdot n u_n \le \dfrac{M}{n^2}$

$\because \sum\limits_{n=1}^{\infty} \dfrac{M}{n^2}$ 收敛

$\therefore \sum\limits_{n=1}^{\infty} \dfrac{u_n}{n}$ 收敛.

6.

① $\because \max\{v_n, u_n\} \geq u_n > 0$

且 $\sum\limits_{n=1} u_n$ 发散

$\therefore \sum\limits_{n=1}^{\infty} \max\{v_n, u_n\}$ 发散.

② $\min\{v_n, u_n\}$ 收敛性无法判断:

比如: 当 $v_n = u_n$ 时, 显然 $\sum\limits_{n=1}^{\infty} \min\{v_n, u_n\} = \sum\limits_{n=1}^{\infty} v_n$ 发散

而构造 $\{u_n\}$: $1, \frac{1}{2^2}, 1, \frac{1}{3^2}, 1, \frac{1}{4^2}, \cdots, 1, \frac{1}{n^2}, \cdots$

$\{v_n\}$: $\frac{1}{2^2}, 1, \frac{1}{3^2}, 1, \frac{1}{4^2}, 1, \cdots \frac{1}{n^2}, 1, \cdots$

则 $\sum\limits_{n=1}^{\infty} \min\{v_n, u_n\} = 2\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ 收敛.

9. (1) 令 $x = \frac{1}{\sqrt{n}}$, 则 $\frac{1}{\sqrt{n}} - \sqrt{\ln(1+\frac{1}{n})}$

$$= x - \sqrt{\ln(1+x^2)}$$

$$= \frac{x^2 - \ln(1+x^2)}{x + \sqrt{\ln(1+x^2)}}$$

$$\overset{\text{Taylor展开}}{=} \frac{\frac{x^4}{2} + o(x^4)}{x + \sqrt{\ln(1+x^2)}}$$

$$\because \lim_{x \to 0^+} \frac{\frac{x^4}{2} + o(x^4)}{(x + \sqrt{\ln(1+x^2)}) \cdot x^3} = \lim_{x \to 0^+} \frac{\frac{1}{2} + o(1)}{1 + \frac{\sqrt{x^2 - \frac{x^4}{2} + o(x^4)}}{x}}$$

$$= \frac{1}{4}$$

且 $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ 收敛

$\therefore \sum\limits_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \sqrt{\ln\frac{n+1}{n}}\right)$ 收敛.

(2)

$$\because n^{\frac{1}{n^2+1}} - 1 = e^{\frac{\ln n}{n^2+1}} - 1 \sim \frac{\ln n}{n^2+1}$$

$$\therefore \lim_{n \to \infty} \frac{n^{\frac{1}{n^2+1}} - 1}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n \to \infty} n^{\frac{3}{2}} \cdot \frac{\ln n}{n^2+1} \cdot \frac{e^{\frac{\ln n}{n^2+1}} - 1}{\frac{\ln n}{n^2+1}}$$

$$= 0$$

$\because \sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ 收敛

$\therefore$ 原级数收敛.

习题 5.3

4. (3)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1} = \sum_{n=1}^{\infty} \left[(-1)^n + (-1)^{n+1} \frac{1}{n+1}\right]$$

$\because \sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+1}$ 收敛, $\sum\limits_{n=1}^{\infty} (-1)^n$ 发散

$\therefore \sum\limits_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ 发散

(5) $\because \left|\frac{1}{2^n} \sin\frac{n\pi}{4}\right| \leq \frac{1}{2^n}$

且 $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ 收敛

$\therefore \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \sin\frac{n\pi}{4}$ 绝对收敛.

(6) $\because \left|\frac{1}{n(\ln n)^3}\cos\frac{n\pi}{4}\right| \le \frac{1}{n(\ln n)^3}$

且 $\int_2^{+\infty}\frac{dx}{x(\ln x)^3} = \frac{1}{(\ln x)^2}\left(-\frac{1}{2}\right)\Big|_2^{+\infty} = \frac{1}{2}(\ln 2)^{-2}$

$\therefore \sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$ 收敛

$\therefore \sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}\cos\frac{n\pi}{4}$ 绝对收敛.

(☆: 这里 n 应该从 2 开始)

(8) ① $\sum_{n=1}^{\infty}(-1)^n$ 的部分和数列有界. 且

$\left\{\frac{1}{n-\ln n}\right\}$ 单调递减趋于 0

由 Leibniz 判别法知

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n-\ln n}$ 收敛.

② $\because \left|\frac{(-1)^n}{n-\ln n}\right| = \frac{1}{n-\ln n} > \frac{1}{n}$

且 $\sum_{n=1}^{\infty}\frac{1}{n}$ 发散

$\therefore \sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n-\ln n}\right|$ 发散

综合知 $\sum_{n=1}^{\infty}\frac{(-1)^n}{n-\ln n}$ 条件收敛

(9) ① $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1}-\sqrt{n}) = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$

由 Leibniz 判别法知 级数收敛.

② $\because \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{3\sqrt{n}}$

且 $\sum_{n=1}^{\infty}\frac{1}{3\sqrt{n}}$ 发散

$\therefore \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$ 发散

综合知 原级数条件收敛.

(11) $\because \sum_{n=2}^{\infty}\frac{|(-1)^n|}{\sqrt{n+(-1)^n}} = \frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{4}}+\cdots$

$\therefore$ 原级数不绝对收敛

(☆: 不能把两项组合在一起后由其收敛推出原级数收敛)

$\because \frac{(-1)^n}{\sqrt{n+(-1)^n}} = \frac{(-1)^n}{\sqrt{n}}\left(1+\frac{(-1)^n}{n}\right)^{-\frac{1}{2}}$

$= \frac{(-1)^n}{\sqrt{n}}\left[1-\frac{(-1)^n}{2n}+o\left(\frac{(-1)^n}{n}\right)\right]$

$= \frac{(-1)^n}{\sqrt{n}} - \frac{1}{2n^{\frac{3}{2}}} + \frac{(-1)^n}{\sqrt{n}}o\left(\frac{(-1)^n}{n}\right)$

$\because \sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ 条件收敛, $\sum_{n=2}^{\infty}\frac{1}{2n^{\frac{3}{2}}}$ 绝对收敛

$\left|\frac{(-1)^n}{\sqrt{n}}o\left(\frac{(-1)^n}{n}\right)\right| \le \frac{1}{n^{\frac{3}{2}}}$ (当 n 足够大时)

即 $\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}}o\left(\frac{(-1)^n}{n}\right)$ 绝对收敛. $\therefore$ 原级数条件收敛.

6.

① $\because (a_n+b_n)^2 \le 2(a_n^2+b_n^2)$

且 $\sum_{n=1}^{\infty}2(a_n^2+b_n^2)$ 收敛

$\therefore \sum_{n=1}^{\infty}(a_n+b_n)^2$ 收敛.

② $\because \frac{|a_n|}{n} \le a_n^2+\frac{1}{4n^2}$

且 $\sum_{n=1}^{\infty}a_n^2$ 与 $\sum_{n=1}^{\infty}\frac{1}{4n^2}$ 均收敛

$\therefore \sum_{n=1}^{\infty}\frac{a_n}{n}$ 绝对收敛.

9.

$\because \{u_n\}$ 单调递减 且为正项数列

$\therefore \lim_{n\to\infty}u_n$ 存在且 ≥ 0

若 $\lim_{n\to\infty}u_n = 0$. 则由 Leibniz 判别法知

$\sum_{n=1}^{\infty}(-1)^n u_n$ 收敛, 矛盾.

故 $\lim_{n\to\infty}u_n = c > 0$.

$\therefore \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{1+u_n}\right)^n} = \lim_{n\to\infty}\frac{1}{1+u_n} = \frac{1}{1+c} < 1$

$\therefore \sum_{n=1}^{\infty}\left(\frac{1}{1+u_n}\right)^n$ 收敛.