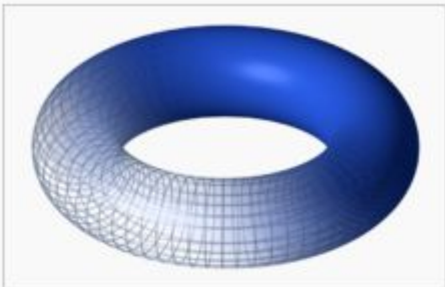
微积分	章节：第四章 4.1	做题人：卢枫



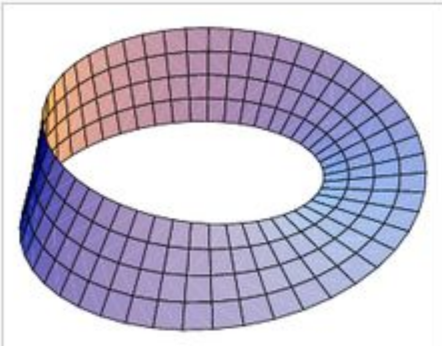
环面是可定向曲面。

环面是可定向曲面，克莱因瓶是不可定向曲面。

克莱因瓶（Klein bottle）

是指一种无定向性的平面，比如 2 维平面，就没有“内部”和“外部”之分。克莱因瓶最初的概念是由德国数学家菲利克斯·克莱因提出的。克莱因瓶和莫比乌斯带非常相像。克莱因瓶的结构非常简单，一个瓶子底部有一个洞，现在延长瓶子的颈部，并且扭曲地进入瓶子内部，然后和底部的洞相连接。

克莱因瓶



莫比乌斯带是不可定向曲面。

微积分	章节：第四章 4.2	做题人：卢枫

1.(1) $L = L_1 \cup L_2 \cup L_3$ 其中 $L_1 = \{(x,y) \mid x=0,\ 0 \le y \le 1\}$

$L_2 = \{(x,y) \mid y=0,\ 0 \le x \le 1\}$

$L_3 = \{(x,y) \mid 0 \le x \le 1,\ y = 1-x\}$

则 $\int_L (x+y)\,dl = \int_{L_1}(x+y)\,dl + \int_{L_2}(x+y)\,dl + \int_{L_3}(x+y)\,dl$

$$= \int_0^1 y\,dy + \int_0^1 x\,dx + \int_0^1 \sqrt{1+1}\,dx = \frac{1}{2} + \frac{1}{2} + \sqrt{2} = \sqrt{2}+1$$

(2) $L: \begin{cases} x = 1+\cos\varphi \\ y = \sin\varphi \end{cases} \quad (-\pi \le \varphi \le \pi)$

$$\int_L \sqrt{x^2+y^2}\,dl = \int_{-\pi}^{\pi} \sqrt{2+2\cos\varphi} \cdot \sqrt{\sin^2\varphi + \cos^2\varphi}\,d\varphi$$

$$= 2\int_{-\pi}^{\pi} \sqrt{\frac{1+\cos\varphi}{2}}\,d\varphi = 4\sin\frac{\varphi}{2}\Big|_{-\pi}^{\pi} = 4-(-4) = 8$$

(3) $\int_L y^2\,dl = \int_0^{2\pi} a^2(1-\cos t)^2 \cdot \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\,dt$

$$= a^3 \int_0^{2\pi} (2-2\cos t)\cdot\sqrt{2-2\cos t}\,dt$$

$$= 8a^3 \int_0^{2\pi} \sin^3\frac{t}{2}\,dt$$

$$= 16a^3 \cdot \left(\frac{1}{3}\cos^3\frac{t}{2} - \cos\frac{t}{2}\right)\Big|_0^{2\pi} = 16a^3 \times \frac{4}{3} = \frac{64}{3}a^3$$

(4) $\int_L (x^{\frac{4}{3}} + y^{\frac{4}{3}})\,dl = \int_0^{2\pi} a^{\frac{4}{3}}(\cos^4 t + \sin^4 t)\sqrt{9a^2(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t)}\,dt$

$$= 3a^{\frac{7}{3}} \int_0^{2\pi} (\cos^4 t + \sin^4 t)\,|\sin t\cos t|\,dt$$

$$= 12a^{\frac{7}{3}} \int_0^{\frac{\pi}{2}} [\cos^5 t + \cos t(1-\cos^2 t)^2]\sin t\,dt$$

$$= 4a^{\frac{7}{3}}$$

2.(1) $r^2 = a^2\cos 2\theta = a^2\cos^2\theta - a^2\sin^2\theta = x^2 - y^2$

故 $\int_L x\sqrt{x^2-y^2}\,dl = r\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a\cos\theta \cdot \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\,d\theta = a^2 r\sin\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \sqrt{2}a^2 r$

(2) $\int_L (x^2+y^2+z^2)\,dl = \int_0^{2\pi} (4+9t^2)\cdot\sqrt{4\sin^2 t + 4\cos^2 t + 9}\,dt$

$$= \sqrt{13}\int_0^{2\pi} (4+9t^2)\,dt = \sqrt{13}(8\pi + 24\pi^3)$$

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(3) $\int_L xyz\,dl = \int_0^1 t\cdot\frac{2}{3}\sqrt{2}t^{\frac{3}{2}}\cdot\frac{1}{2}t^2\sqrt{1+2t+t^2}\,dt$

$= \frac{\sqrt{2}}{3}\int_0^1 t^{\frac{9}{2}}(t+1)\,dt$

$= \frac{\sqrt{2}}{3}\times(\frac{2}{11}+\frac{2}{13}) = \frac{16\sqrt{2}}{143}$

(4) $L = L_1 \cup L_2 \cup L_3$ 其中 $L_1: \begin{cases} x=0 \\ y=2\cos\theta \\ z=2\sin\theta \end{cases}$ $(0\le\theta\le\frac{\pi}{2})$ $L_2: \begin{cases} x=2\cos\theta \\ y=0 \\ z=2\sin\theta \end{cases}$ $(0\le\theta\le\frac{\pi}{2})$ $L_3: \begin{cases} x=2\cos\theta \\ y=2\sin\theta \\ z=0 \end{cases}$ $(0\le\theta\le\frac{\pi}{2})$

则 $\int_L x\,dl = \int_{L_1} x\,dl + \int_{L_2} x\,dl + \int_{L_3} x\,dl$

$= 0 + \int_0^{\frac{\pi}{2}} 2\cos\theta\cdot\sqrt{4\sin^2\theta+4\cos^2\theta}\,d\theta + \int_0^{\frac{\pi}{2}} 2\cos\theta\cdot\sqrt{4\sin^2\theta+4\cos^2\theta}\,d\theta$

$= 2\times 4\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta = 2\times 4 = 8$

3.(1) $\int_L dl = \int_0^1 \sqrt{9+36t^2+36t^4}\,dt = 3\int_0^1(2t^2+1)dt = 5$

(2) $\int_L dl = \int_0^{+\infty}\sqrt{e^{-2t}(-\cos t-\sin t)^2+e^{-2t}(-\sin t+\cos t)^2+e^{-2t}}\,dt$

$= \int_0^{+\infty} e^{-t}\sqrt{1+2\cos t\sin t+1-2\cos t\sin t+1}\,dt$

$= \sqrt{3}\,(-e^{-t})\Big|_0^{+\infty} = \sqrt{3}$

4. $m = \int_L x^2\,dl = \int_{\sqrt{3}}^{\sqrt{15}} x^2\cdot\sqrt{1+(\frac{1}{x})^2}\,dx$

$= \int_{\sqrt{3}}^{\sqrt{15}} x\sqrt{x^2+1}\,dx$

$= \frac{1}{3}(x^2+1)^{\frac{3}{2}}\Big|_{\sqrt{3}}^{\sqrt{15}} = \frac{64}{3}-\frac{8}{3} = \frac{56}{3}$

5. $L: \begin{cases} x=a\cos\theta \\ y=a\sin\theta \end{cases}$ $(0\le\theta\le 2\pi)$

$S = \int_L (a+\frac{x^2}{a})dl = \int_0^{2\pi} a(1+\cos^2\theta)\sqrt{a^2\sin^2\theta+a^2\cos^2\theta}\,d\theta$

$= a^2\int_0^{2\pi}(1+\cos^2\theta)d\theta$

$= a^2\cdot(\frac{3}{2}\theta+\frac{1}{4}\sin 2\theta)\Big|_0^{2\pi} = a^2\times 3\pi = 3\pi a^2$

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6. $M=\int_L dl=\int_0^{\pi}\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,dt=a\int_0^{\pi}\sqrt{2-2\cos t}\,dt=4a$

$M_x=\int_L y\,dl=\int_0^{\pi}a(1-\cos t)\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,dt$

$=4a^2\int_0^{\pi}\sin^3\frac{t}{2}\,dt=\frac{16}{3}a^2$

$M_y=\int_L x\,dl=\int_0^{\pi}a(t-\sin t)\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,dt$

$=2a^2\int_0^{\pi}(t\sin\frac{t}{2}-\sin t\sin\frac{t}{2})dt$

$=2a^2\left(\int_0^{\pi}t\sin\frac{t}{2}dt-\int_0^{\pi}2\sin^2\frac{t}{2}\cos\frac{t}{2}dt\right)$

$=2a^2\times(4-\frac{4}{3})=\frac{16}{3}a^2$

故 $\bar{x}=\frac{M_y}{M}=\frac{4}{3}a$ $\quad \bar{y}=\frac{M_x}{M}=\frac{4}{3}a$ $\quad$ 故重心坐标为 $(\frac{4}{3}a,\frac{4}{3}a)$

7. $J_x=\int_L(y^2+z^2)dl=\int_0^{2\pi}(a^2\sin^2 t+\frac{b^2}{4\pi^2}t^2)\sqrt{a^2\sin^2 t+a^2\cos^2 t+\frac{b^2}{4\pi^2}}\,dt$

$=\sqrt{a^2+\frac{b^2}{4\pi^2}}\int_0^{2\pi}(a^2\sin^2 t+\frac{b^2}{4\pi^2}t^2)dt$

$=\sqrt{a^2+\frac{b^2}{4\pi^2}}\times(a^2\pi+\frac{2b^2}{3}\pi)$

$=\sqrt{4\pi^2a^2+b^2}\,(\frac{a^2}{2}+\frac{b^2}{3})$

8. $L:\begin{cases}x=\cos\theta\\ y=-1+\sin\theta\end{cases}\quad(-\frac{3}{2}\pi\le\theta\le\frac{\pi}{2})$

$M_x=\int_L y\sqrt{x^2+y^2}\,dl=\int_{-\frac{3}{2}\pi}^{\frac{\pi}{2}}(\sin\theta-1)\sqrt{\sin^2\theta+\cos^2\theta}\sqrt{\cos^2\theta+(\sin\theta-1)^2}\,d\theta$

$=\sqrt{2}\int_{-\frac{3}{2}\pi}^{\frac{\pi}{2}}(\sin\theta-1)\left|\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right|d\theta\quad(\cos\frac{\theta}{2}\ge\sin\frac{\theta}{2},\ \because -\frac{3}{2}\pi\le\theta\le\frac{\pi}{2})$

$=\sqrt{2}\int_{-\frac{3}{2}\pi}^{\frac{\pi}{2}}(2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}-2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}+\sin\frac{\theta}{2}-\cos\frac{\theta}{2})d\theta$

$=\sqrt{2}\times(-\frac{4}{3}\cos^3\frac{\theta}{2}-\frac{4}{3}\sin^3\frac{\theta}{2}-2\cos\frac{\theta}{2}-2\sin\frac{\theta}{2})\Big|_{-\frac{3}{2}\pi}^{\frac{\pi}{2}}=\sqrt{2}\times(-\frac{8\sqrt{2}}{3})=-\frac{16}{3}$

$M=\int_L\sqrt{x^2+y^2}\,dl=\sqrt{2}\int_{-\frac{3}{2}\pi}^{\frac{\pi}{2}}(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})d\theta=4\sqrt{2}\cdot\sqrt{2}=8$

微积分	章节：第四章　4.3	做题人：[illegible]

1. (1) $\iint_S (x+y+z)dS = \iint_{D_{\theta\varphi}} a(\sin\theta\cos\varphi+\sin\theta\sin\varphi+\cos\theta)\cdot a^2\sin\theta\, d\theta$

$= a^3\int_0^{2\pi} d\varphi \int_0^{\pi}(\sin^2\theta(\cos\varphi+\sin\varphi)+\sin\theta\cos\theta)d\theta$

$= a^3\int_0^{2\pi}[\frac{1}{2}+\frac{\pi}{4}(\cos\varphi+\sin\varphi)]d\varphi$

$= \pi a^3$

(2) $\iint_S (2x+\frac{4}{3}y+z)dS = \iint_{D_{xy}} 4\cdot\sqrt{1+4+\frac{16}{9}}\,dxdy \qquad (z=4-2x-\frac{4}{3}y)$

$= \frac{4\sqrt{61}}{3}\iint_{D_{xy}} dxdy$

$= 4\sqrt{61}$

(3) $S=S_1\cup S_2\cup S_3\cup S_4$　其中 $S_1=\{(x,y,z)\mid x=0,\ 0\le y\le 1,\ 0\le z\le 1-y\}$

$S_2=\{(x,y,z)\mid y=0,\ 0\le x\le 1,\ 0\le z\le 1-x\}$

$S_3=\{(x,y,z)\mid z=0,\ 0\le x\le 1,\ 0\le y\le 1-x\}$

$S_4=\{(x,y,z)\mid z=1-x-y,\ 0\le x\le 1,\ 0\le y\le 1-x\}$

$\iint_S \frac{dS}{(1+x+y)^2} = \iint_{S_1}\frac{dS}{(1+x+y)^2}+\iint_{S_2}\frac{dS}{(1+x+y)^2}+\iint_{S_3}\frac{dS}{(1+x+y)^2}+\iint_{S_4}\frac{dS}{(1+x+y)^2}$

$= \int_0^1 dy\int_0^{1-y}\frac{1}{(1+y)^2}dz + \int_0^1 dx\int_0^{1-x}\frac{1}{(1+x)^2}dz + \int_0^1 dx\int_0^{1-x}\frac{1}{(1+x+y)^2}dy$

$+\int_0^1 dx\int_0^{1-x}\frac{1}{(1+x+y)^2}\cdot\sqrt{1+1+1}\,dy$

$= -\frac{1}{2}-\frac{1}{2}+\ln 2-\frac{1}{2}+\sqrt{3}\ln 2-\frac{\sqrt{3}}{2}$

$= (\sqrt{3}+1)(\ln 2-\frac{\sqrt{3}}{2})$

(4) $\iint_S (xy+yz+xz)dS = \iint_{D_{xy}} [xy+(x+y)\sqrt{x^2+y^2}]\cdot\sqrt{2}\,dxdy$

$= \iint_{D_{\rho\theta}} [\rho^2\sin\theta\cos\theta+\rho^2(\sin\theta+\cos\theta)]\sqrt{2}\rho\, d\rho\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_0^{2a\cos\theta}\sqrt{2}\rho^3(\cos\theta\sin\theta+\cos\theta+\sin\theta)d\theta$

$= 4\sqrt{2}a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^5\theta\sin\theta+\cos^4\theta\sin\theta+\cos^5\theta)d\theta = 4\sqrt{2}a^4\times 2\times(1-\frac{2}{3}+\frac{1}{5}) = \frac{64\sqrt{2}}{15}a^4$

微积分	章节：第四章 4.3	做题人：卢枫

(5) $\iint\limits_S x\,dS = \iint\limits_{D_{uv}} u\cos v\cdot\sqrt{a^2\sin^2 v+a^2\cos^2 v+u^2}\,du\,dv$

$$= \int_0^{2\pi}\cos v\,dv\cdot\int_0^r u\sqrt{a^2+u^2}\,du = 0$$

2. (此题看成曲线积分)

$L:\begin{cases} x=\frac{1}{2}a+\frac{1}{2}a\cos\theta \\ y=\frac{1}{2}a\sin\theta \end{cases}$ $(0\leq\theta\leq 2\pi)$ 联立 $\begin{cases} x^2+y^2=ax \\ x^2+y^2+z^2=a^2 \end{cases}$ 得 $z=\sqrt{a^2-ax}$

(只考虑 $z\geq 0$)

由对称性得，$z<0$ 的面积与 $z\geq 0$ 时相同.

故 $S = 2\int_L\sqrt{a^2-ax}\,dl = 2\int_0^{2\pi}\sqrt{a^2-\frac{1}{2}a^2-\frac{1}{2}a^2\cos\theta}\cdot\sqrt{\frac{1}{4}a^2\sin^2\theta+\frac{1}{4}a^2\cos^2\theta}\,d\theta$

$$= a^2\cdot\left(-2\cos\frac{\theta}{2}\right)\Big|_0^{2\pi} = 4a^2$$

3. $M = \iint\limits_S \sigma\,dS = \iint\limits_{D_{xy}} \frac{x^2+y^2}{2}\cdot\sqrt{1+x^2+y^2}\,dx\,dy$ $\left(D_{xy}=\{(x,y)\mid 0\leq x^2+y^2\leq 2\}\right)$

$$= \iint\limits_{D_{\rho\theta}} \frac{\rho^2}{2}\cdot\sqrt{1+\rho^2}\,\rho\,d\rho\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\frac{1}{2}\rho^3\sqrt{1+\rho^2}\,d\rho$$

$$= 2\pi\cdot\frac{1}{4}\cdot\left[\frac{2}{5}(1+\rho^2)^{\frac{5}{2}}-\frac{2}{3}(1+\rho^2)^{\frac{3}{2}}\right]\Big|_0^{\sqrt{2}}$$

$$= \frac{12\sqrt{3}+2}{15}\pi$$

4. 令该直径在 z 轴上。

$M = \iint\limits_S \sqrt{x^2+y^2}\,dS = \iint\limits_{D_{\theta\varphi}} a\sin\theta\cdot a^2\sin\theta\,d\theta\,d\varphi$

$$= \int_0^{2\pi}d\varphi\int_0^{\pi}a^3\sin^2\theta\,d\theta = 2\pi\cdot\frac{\pi}{2}\cdot a^3 = \pi^2a^3$$

5. 直线为经过点 $(0,0,b)$，平行于 x 轴的直线. $z=\frac{b}{a}\sqrt{x^2+y^2}$ $(0\leq z\leq b)$

$J = \iint\limits_{D_{xy}}(y^2+(z-b)^2)\,\sigma_0\sqrt{1+\frac{b^2x^2}{a^2(x^2+y^2)}+\frac{b^2y^2}{a^2(x^2+y^2)}}\,dx\,dy$

$$= \sqrt{1+\frac{b^2}{a^2}}\,\sigma_0\int_0^{2\pi}d\theta\int_0^a\left[\rho^2\sin^2\theta+b^2\left(\frac{\rho^2}{a^2}-\frac{2\rho}{a}+1\right)\right]\rho\,d\rho$$

$$= \sqrt{1+\frac{b^2}{a^2}}\,\sigma_0\int_0^{2\pi}\left(\frac{a^4}{4}\sin^2\theta+\frac{1}{12}a^2b^2\right)d\theta = a\sqrt{a^2+b^2}\,\pi\sigma_0\left(\frac{a^2}{4}+\frac{b^2}{6}\right)$$

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6. 第一卦限： $M_1=\frac{1}{2}\pi a^2$ $z=\sqrt{a^2-x^2-y^2}$

$$M_{yz_1}=\iint_S x\,dS=\iint_{D_{xy}} ax\frac{1}{\sqrt{a^2-x^2-y^2}}dxdy=\int_0^{\frac{\pi}{2}}d\theta\int_0^a\frac{a\rho^2\cos\theta}{\sqrt{a^2-\rho^2}}d\rho$$

$$=a\cdot\frac{\pi}{4}a^2=\frac{\pi}{4}a^3$$

故 $\bar{x}=\frac{M_{yz_1}}{M}=\frac{a}{2}$ 由对称性得 $\bar{y}=\bar{z}=\bar{x}$ 故重心坐标 $(\frac{a}{2},\frac{a}{2},\frac{a}{2})$.

上半球面， $M_2=2\pi a^2$ 由对称性得，重心在z轴上。

$$M_{xy}=\iint_S z\,dS=\iint_{D_{xy}}\sqrt{a^2-x^2-y^2}\cdot\frac{a}{\sqrt{a^2-x^2-y^2}}dxdy=a\cdot\pi a^2=\pi a^3$$

$\bar{z}=\frac{M_{xy}}{M_2}=\frac{a}{2}$ 故重心坐标 $(0,0,\frac{a}{2})$

7. 由对称性得 $\bar{y}=0$ ，由例4.3.3得 $M=\sqrt{2}\pi a^2$ (书上图4.3.3是错的!!!)

$$M_{yz}=\iint_S x\,dS=\iint_{D_{\rho\theta}}\rho\cos\theta\cdot\sqrt{2}\cdot\rho\,d\rho\,d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}\sqrt{2}\rho^2\cos\theta\,d\rho$$

$$=\frac{8\sqrt{2}a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,d\theta=\frac{8\sqrt{2}}{3}a^3\cdot\frac{3}{8}\pi=\sqrt{2}\pi a^3$$

$$M_{xy}=\iint_S z\,dS=\iint_{D_{\rho\theta}}\sqrt{2}\rho^2d\rho\,d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}\sqrt{2}\rho^2d\rho$$

$$=\frac{8\sqrt{2}}{3}a^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\theta\,d\theta=\frac{8\sqrt{2}}{3}a^3\cdot\frac{4}{3}=\frac{32\sqrt{2}}{9}a^3$$

故 $\bar{x}=\frac{M_{yz}}{M}=a$ $\bar{z}=\frac{M_{xy}}{M}=\frac{32}{9\pi}a$ 故重心坐标 $(a,0,\frac{32}{9\pi}a)$

8. 联立 $\begin{cases}z=\sqrt{x^2+y^2}\\ z^2=2x\end{cases}$ 得 L: $x^2+y^2=2x$ 令 $\begin{cases}x=1+\cos\theta\\ y=\sin\theta\end{cases}$ $(-\pi\le\theta\le\pi)$

则 $S=\int_L\sqrt{x^2+y^2}\,dl=\int_{-\pi}^{\pi}\sqrt{2+2\cos\theta}\,d\theta$

$$=2\int_{-\pi}^{\pi}\cos\frac{\theta}{2}d\theta$$

$$=4\sin\frac{\theta}{2}\Big|_{-\pi}^{\pi}=4\times2=8$$

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9. $S=\iint_S dS=\iint_{D_{xy}}\sqrt{1+x^2+y^2}\,dxdy=\iint_{D_{\rho\theta}}\sqrt{1+\rho^2}\,\rho\,d\rho\,d\theta$

$=\int_0^{2\pi}d\theta\int_0^a\rho\sqrt{1+\rho^2}\,d\rho=2\pi\times\frac{1}{3}(1+\rho^2)^{\frac{3}{2}}\Big|_0^a$

$=\frac{2\pi}{3}\left[(1+a^2)^{\frac{3}{2}}-1\right]$

10. $S:\begin{cases}x=a\sin\theta\cos\varphi\\ y=b\sin\theta\sin\varphi\\ z=c\cos\theta\end{cases}$

$A=\begin{vmatrix}\frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi}\\ \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}b\cos\theta\sin\varphi & b\cos\varphi\sin\theta\\ -c\sin\theta & 0\end{vmatrix}=bc\sin^2\theta\cos\varphi$

$B=\begin{vmatrix}\frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi}\\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}-c\sin\theta & 0\\ a\cos\theta\cos\varphi & -a\sin\theta\sin\varphi\end{vmatrix}=ac\sin^2\theta\sin\varphi$

$C=\begin{vmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}a\cos\theta\cos\varphi & -a\sin\theta\sin\varphi\\ b\cos\theta\sin\varphi & b\sin\theta\cos\varphi\end{vmatrix}=ab\cos\theta\sin\theta$

$L(x,y,z)=\dfrac{2\left|\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right|}{\sqrt{\frac{4x^2}{a^4}+\frac{4y^2}{b^4}+\frac{4z^2}{c^4}}}=\dfrac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}=\dfrac{abc\sin\theta}{\sqrt{A^2+B^2+C^2}}$

故 $\iint_S L(x,y,z)\,dS=\int_0^{2\pi}d\varphi\int_0^{\pi}abc\sin\theta\,d\theta=4\pi abc$

11. 证明：将坐标系 $Oxyz$ 保持原点不动而旋转得到新坐标系 $Ouvt$，

其中将平面 $ax+by+cz=0$ 作为平面 uOv，且 Ot 轴与其垂直，

则 $t=\dfrac{ax+by+cz}{\sqrt{a^2+b^2+c^2}}$ （点到面的距离）

球面 S：$u^2+v^2+t^2=1$ 　　则 $\iint_S f(ax+by+cz)\,dS=\iint_{S'}f(\sqrt{a^2+b^2+c^2}\,t)\,dS'$

令 $\begin{cases}u=\sqrt{1-t^2}\cos\varphi\\ v=\sqrt{1-t^2}\sin\varphi\\ t=t\end{cases}$ $(-1\le t\le 1,\ 0\le\varphi\le 2\pi)$

则 $dS'=\sqrt{A^2+B^2+C^2}\,dtd\varphi=dtd\varphi$

则 $\iint_S f(ax+by+cz)\,dS=\iint_{S'}f(\sqrt{a^2+b^2+c^2}\,t)\,dS'$

$=\int_0^{2\pi}d\varphi\int_{-1}^{1}f(\sqrt{a^2+b^2+c^2}\,t)\,dt$

$=2\pi\int_{-1}^{1}f(\sqrt{a^2+b^2+c^2}\,t)\,dt$ 　　证毕。