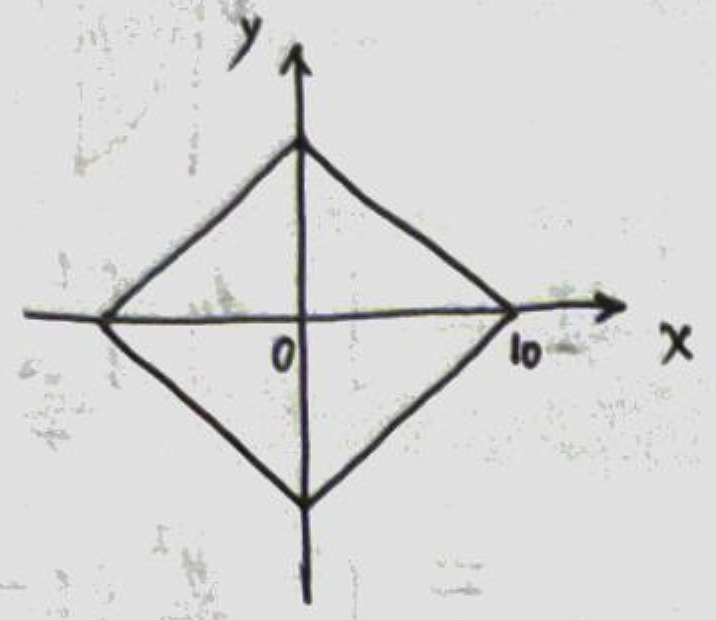
习题 3.2.

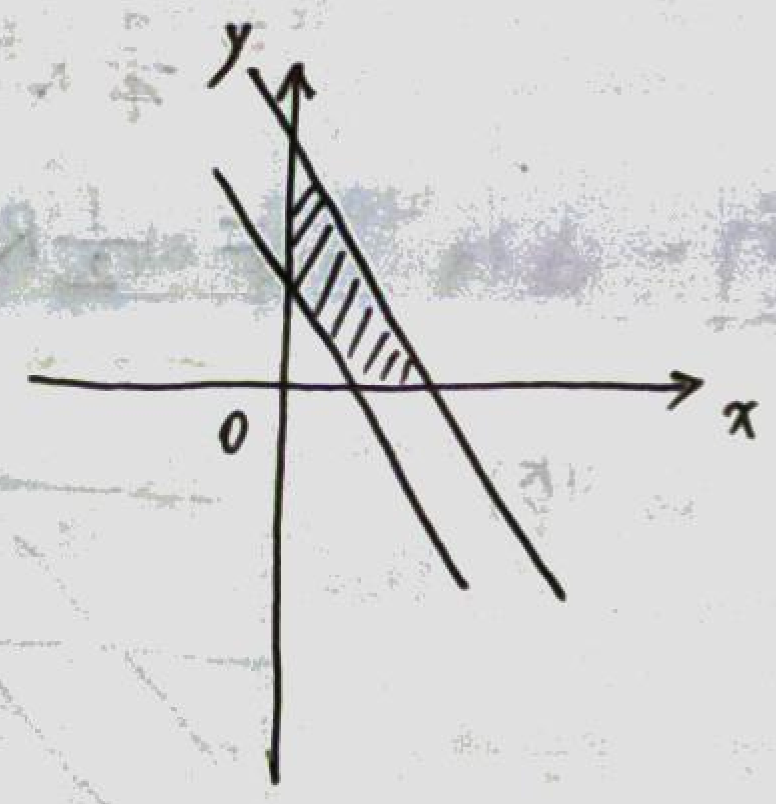
2.



$$\iint_D \frac{1}{102}\,dxdy \leqslant I \leqslant \iint_D \frac{1}{100}\,dxdy$$

即 $\frac{200}{102} \leqslant I \leqslant 2$.

3. (2)



$$I_1 = \iint_D \ln(x+y)\,dxdy < 0$$

$$I_2 = \iint_D xy\,dxdy > 0$$

$\therefore I_1 < I_2$

5. $\because$ f 在 $(0,0)$ 的某个邻域内连续

$\therefore$ $\forall \varepsilon>0$, $\exists \delta>0$ (充分小) s.t. 当 $r<\delta$ 时

$|f(x,y) - f(0,0)| < \varepsilon/\pi$, $\forall (x,y) \in B_0(r)$

$$\therefore \left| I - f(0,0)\pi \right| \leqslant \frac{1}{r^2}\iint_D |f(x,y) - f(0,0)|\,dxdy < \varepsilon$$

$$\therefore \lim_{r\to 0^+} I = f(0,0)\cdot\pi.$$

或者

$$\lim_{r\to0^+} \frac{1}{r^2}\iint_D f(x,y)\,dxdy$$

$$= \lim_{r\to0^+} \frac{1}{r^2}\cdot\left[f(\xi,\eta)\cdot \pi r^2\right]$$

其中 $(\xi,\eta) \in B_0(r)$, r 足够小.

$$= f(0,0)\pi.$$

习题 3.3.

3.

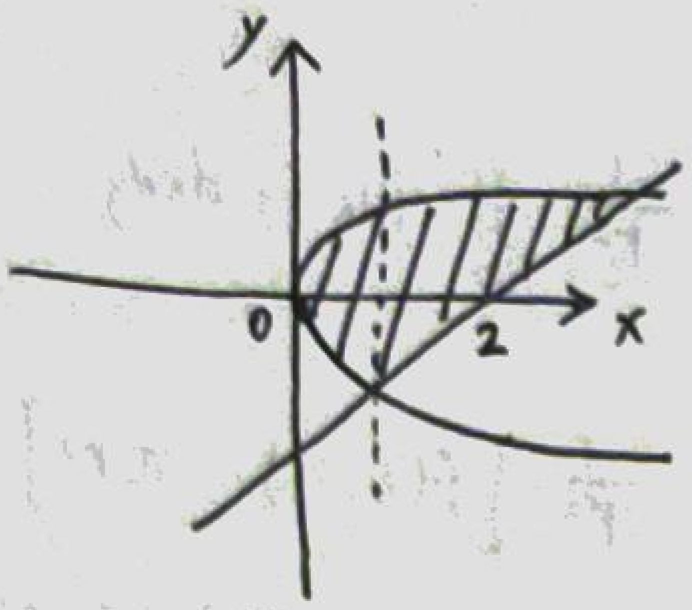
$$\iint_I \frac{\partial^2 f}{\partial x\partial y}\,dx\,dy$$

$$= \int_a^b \left[\int_c^d \frac{\partial^2 f}{\partial x\partial y}\,dy\right]dx$$

$$= \int_a^b \left[f_x(x,d) - f_x(x,c)\right]dx$$

$$= f(b,d) - f(a,d) - f(b,c) + f(a,c).$$

4. (2)

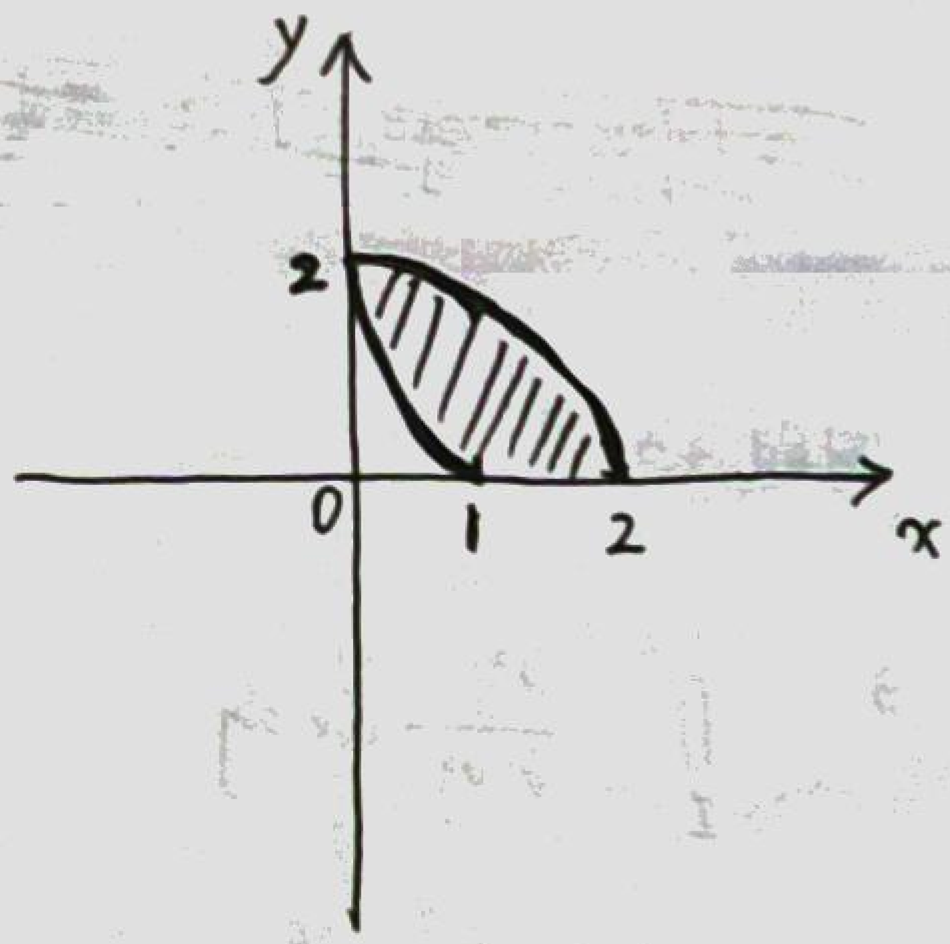


$$\iint_D f(x,y)\,dx\,dy = \int_{-1}^{2} dy \int_{y^2}^{y+2} f(x,y)\,dx$$

$$= \int_0^1 dx \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,dy$$

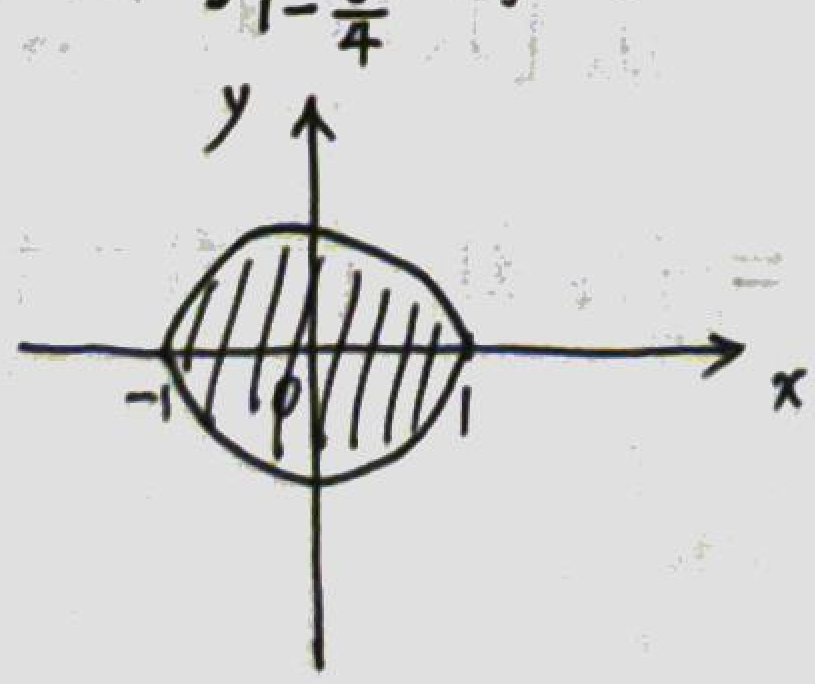
$$+ \int_1^4 dx \int_{x-2}^{\sqrt{x}} f(x,y)\,dy$$

5. (2)



$$I = \int_0^2 dy \int_{1-\frac{y^2}{4}}^{\sqrt{4-y^2}} f(x,y)\,dx$$

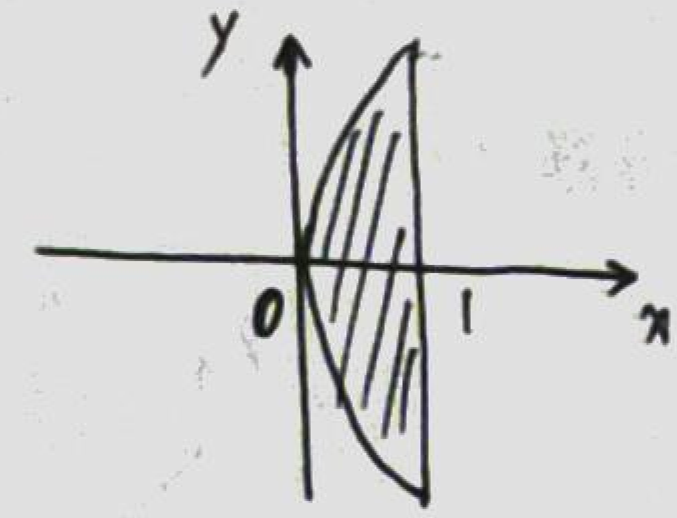
(4)



$$I = \int_{-1}^{0} dy \int_{-\sqrt{y+1}}^{\sqrt{y+1}} f(x,y)\,dx$$

$$+ \int_0^1 dy \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\,dx$$

6. (1)



$$I = \int_{-2}^{2} dy \int_{\frac{y^2}{4}}^{1} xy^2\,dx$$

$$= \int_{-2}^{2} \frac{y^2}{2}\left(1-\frac{y^4}{16}\right) dy$$

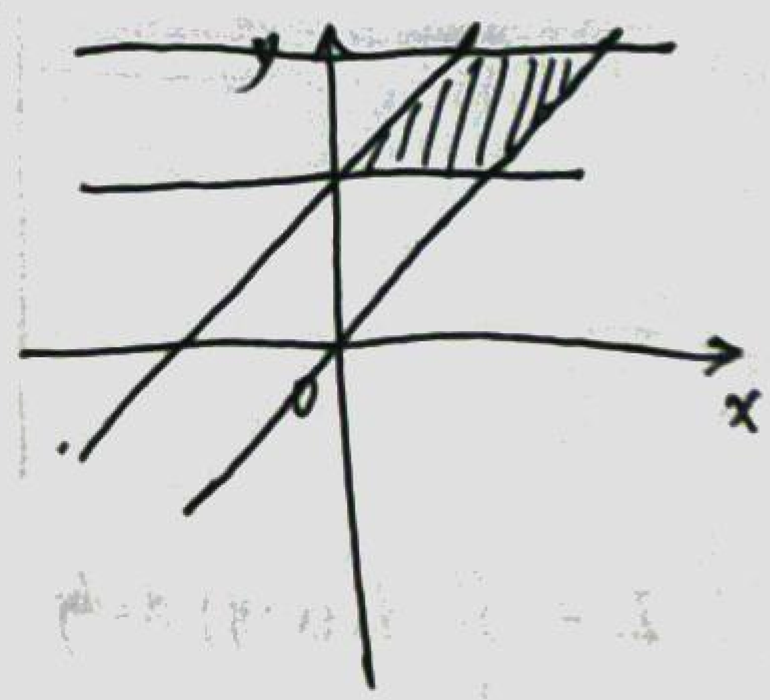
$$= \frac{8}{3} - \frac{256}{224} = \frac{32}{21}.$$

(3) $$I = 4\int_0^{\frac{\pi}{2}} d\theta \int_0^R r^2\sin\theta\cos\theta \cdot r\,dr$$

$$= \frac{2}{4} R^4 \int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta$$

$$= \frac{1}{2} R^4.$$

(5)



$$I = \int_0^1 dx \int_1^{x+1} (x^2+y^2)\,dy$$

$$+ \int_1^3 dx \int_x^{x+1} (x^2+y^2)\,dy$$

$$+ \int_3^4 dx \int_x^4 (x^2+y^2)\,dy$$

$$= \frac{7}{6} + 22 + \frac{37}{3}$$

$$= \frac{71}{2}.$$

(7)

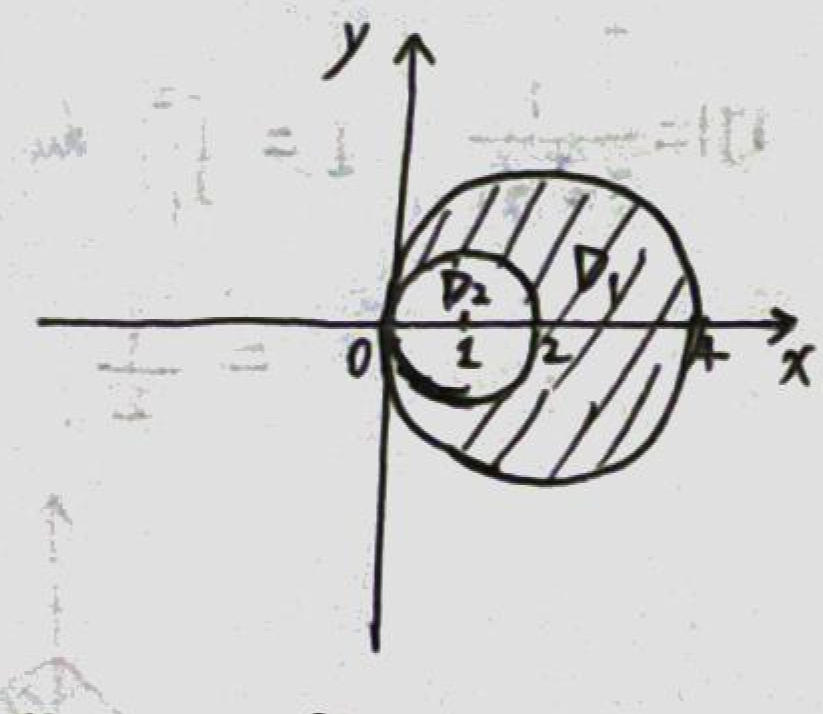
$$I=\int_0^{\pi}dx\int_0^{\pi}\cos(x+y)\,dy$$

$$=\int_0^{\pi}\left[\sin(\pi+x)-\sin x\right]dx$$

$$=2\cos x\Big|_0^{\pi}$$

$$=-4.$$

12. (1)



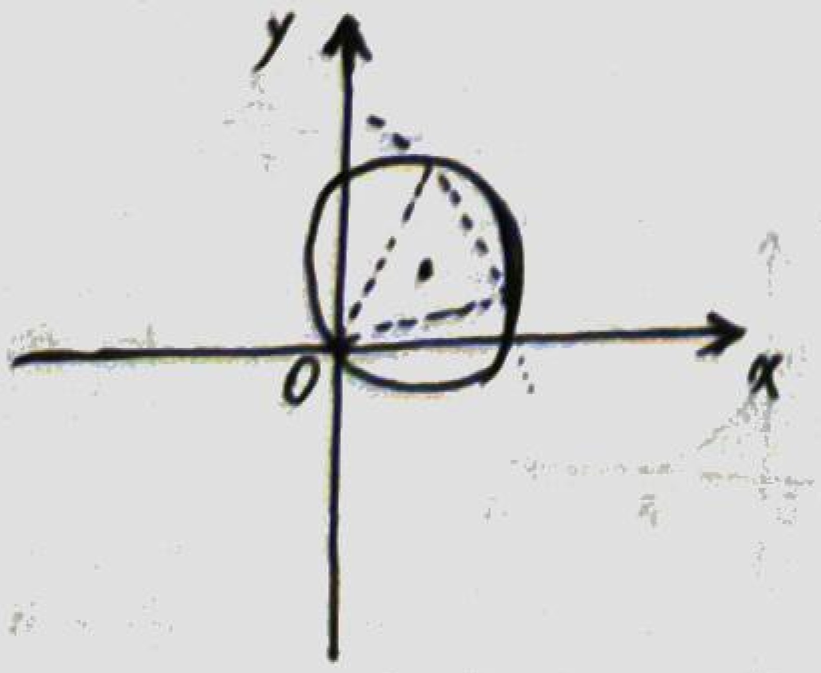
$$I=\iint_{D_1}-\iint_{D_2}$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{4\cos\theta}r^3\,dr-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^3\,dr$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_{2\cos\theta}^{4\cos\theta}r^3\,dr$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{240}{4}\cos^4\theta\,d\theta=\frac{45}{2}\pi.$$

(3)



方法一：

$$I=\int_0^{\sqrt2}dr\int_{\frac{\pi}{4}-\arccos\frac{r}{\sqrt2}}^{\frac{\pi}{4}+\arccos\frac{r}{\sqrt2}}\sqrt2\,r^2\sin\left(\theta+\frac{\pi}{4}\right)d\theta$$

$$=\int_0^{\sqrt2}r^2\sqrt{2-r^2}\,dr$$

$$=\int_0^2\sqrt{t(2-t)}\,dt=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta=\frac{\pi}{2}$$

方法二：

$$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}d\theta\int_0^{\sqrt2\cos(\frac{\pi}{4}-\theta)}\sqrt2\,r^2\sin\left(\theta+\frac{\pi}{4}\right)dr+\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}d\theta\int_0^{\sqrt2\cos(\theta-\frac{\pi}{4})}\sqrt2\,r^2\sin\left(\theta+\frac{\pi}{4}\right)dr$$

$$=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{4}{3}\cos^4\left(\theta-\frac{\pi}{4}\right)d\theta.$$

$$=\frac{4}{3}\cdot\left[\frac{3\theta}{8}+\frac{\sin2(\theta-\frac{\pi}{4})}{4}+\frac{\sin4(\theta-\frac{\pi}{4})}{32}\right]\Bigg|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$=\frac{\pi}{2}$$

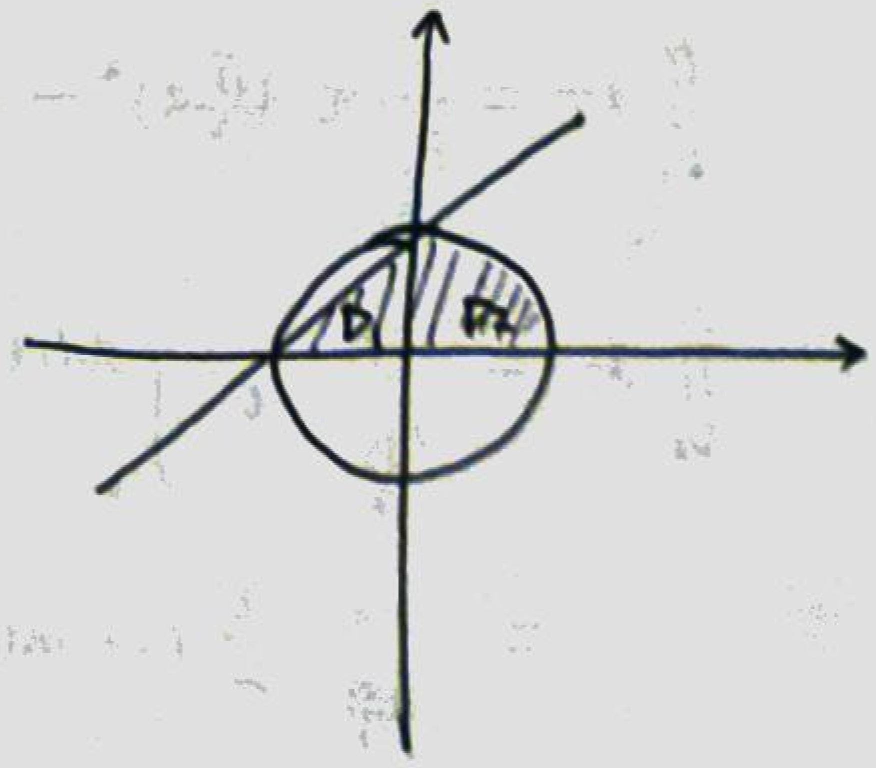
☆ $\boxed{\text{方法三}}$：

令 $\begin{cases}x=\frac{1}{2}+r\cos\theta\\ y=\frac{1}{2}+r\sin\theta\end{cases}$

则 $I=\int_0^{\frac{\sqrt2}{2}}dr\int_0^{2\pi}\left[1+r(\cos\theta+\sin\theta)\right]r\,d\theta$

$$=\int_0^{\frac{\sqrt2}{2}}2\pi r\,dr=\frac{\pi}{2}.$$

(4)



$$\cancel{I=\iint_{D_1}+\iint_{D_2}}$$

$$\cancel{I=\int_0^a dy\int_{y-a}^{\sqrt{a^2-y^2}}(y-x)^2dx}$$

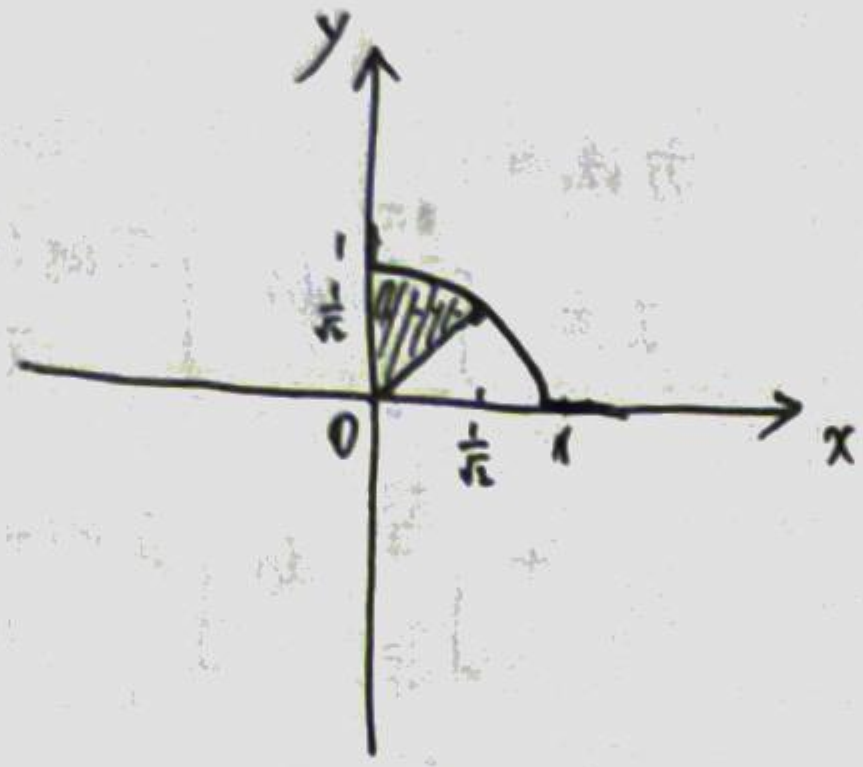
$$I=\iint_{D_1}+\iint_{D_2}$$

$$=\int_{-a}^0dx\int_0^{x+a}(y-x)^2\,dy+\int_0^{\frac{\pi}{2}}d\theta\int_0^a r^3(1-\sin2\theta)\,dr$$

$$=\frac{a^4}{4}+\frac{\pi a^4}{8}-\frac{a^4}{4}$$

$$=\frac{\pi a^4}{8}.$$

(6)

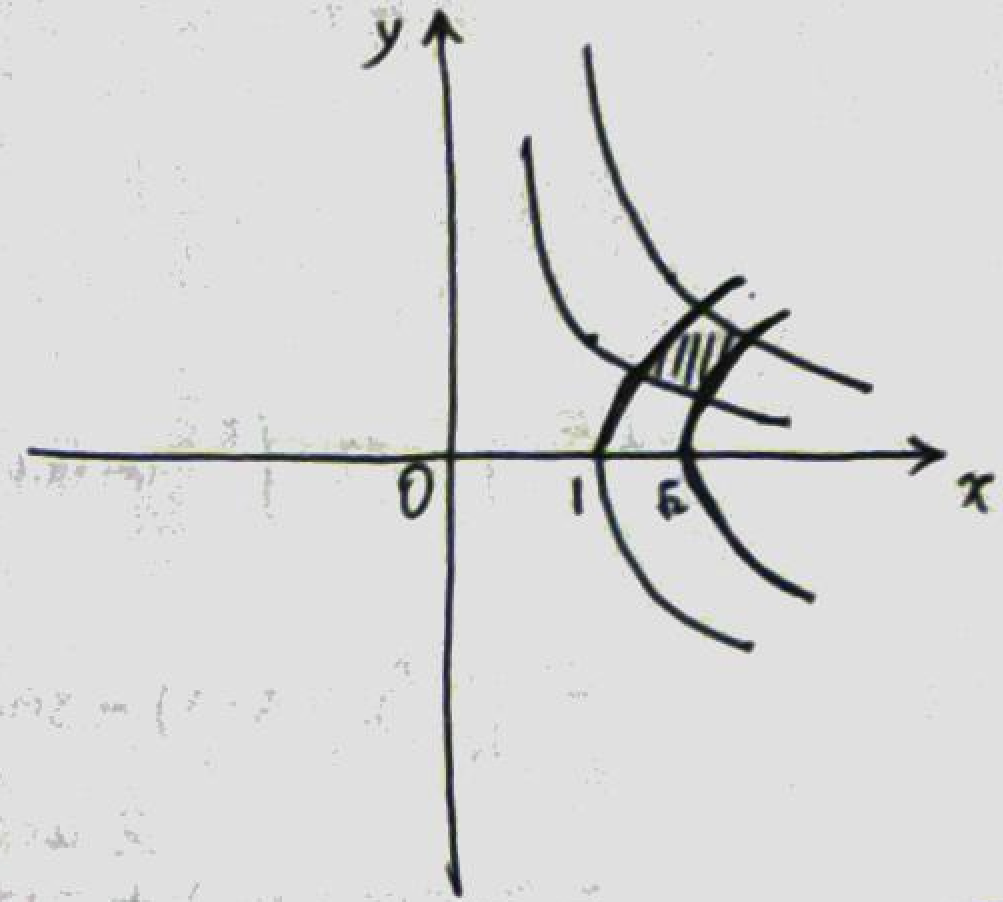


$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^1 e^{-r^2} \cdot r\, dr$$

$$= \frac{\pi}{4} \left(-e^{-r^2} \cdot \frac{1}{2}\right)\Big|_0^1$$

$$= \frac{\pi}{8} (1 - e^{-1})$$

(2)

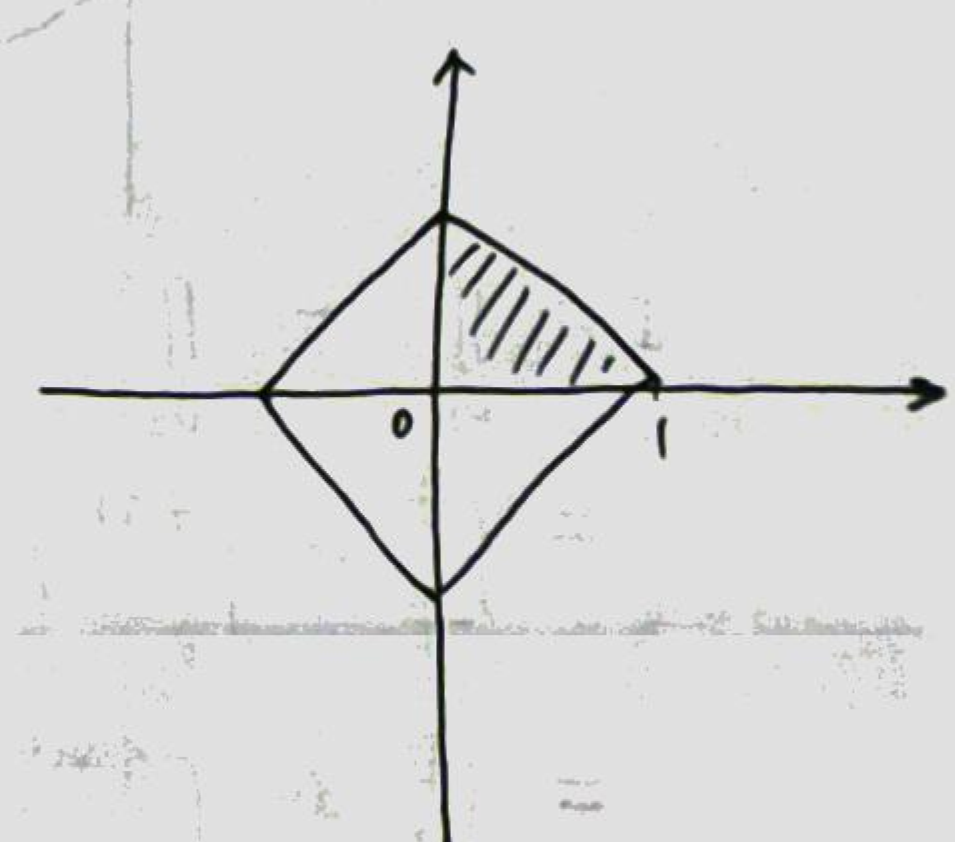


令 $\begin{cases} u = xy \\ v = x^2 - y^2 \end{cases} \Rightarrow \begin{cases} x^2 = \frac{v + \sqrt{v^2 + 4u^2}}{2} \\ y^2 = \frac{-v + \sqrt{v^2 + 4u^2}}{2} \end{cases}$

$|J| = \frac{1}{2(x^2+y^2)}$ $\therefore$ $I = \int_1^2 du \int_1^2 \frac{1}{2}\, dv$

$$= \frac{1}{2}.$$

(3)

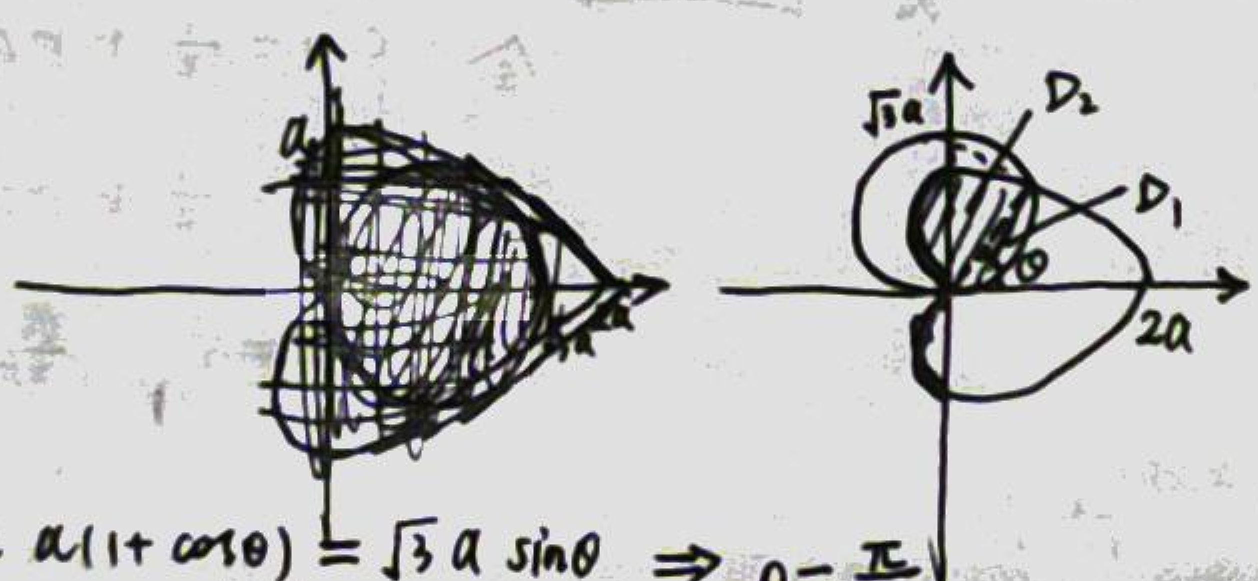


$$I = 4 \int_0^1 dx \int_0^{1-x} (x^2 + y^2)\, dy$$

$$= 4 \int_0^1 \left(-\frac{4}{3}x^3 + 2x^2 - x + \frac{1}{3}\right) dx$$

$$= \frac{2}{3}.$$

13. (2)



$r = a(1+\cos\theta) = \sqrt{3}\, a \sin\theta \Rightarrow \theta = \frac{\pi}{3}$

$$S = \iint_{D_1} d\sigma + \iint_{D_2} d\sigma$$

$$\iint_{D_1} d\sigma = \frac{1}{3}\pi \cdot \left(\frac{\sqrt{3}}{2}a\right)^2 - \frac{\sqrt{3}}{4}a \cdot \frac{3}{4}a = \frac{\pi}{4}a^2 - \frac{3\sqrt{3}}{16}a^2$$

$$\iint_{D_2} d\sigma = \int_{\frac{\pi}{3}}^{\pi} d\theta \int_0^{a(1+\cos\theta)} r\, dr$$

$$= \int_{\frac{\pi}{3}}^{\pi} \frac{a^2}{2}(1+\cos\theta)^2 d\theta = \frac{\pi}{2}a^2 - \frac{9\sqrt{3}}{16}a^2$$

$$\therefore I = \frac{3a^2}{4}(\pi - \sqrt{3}).$$

14. (1) 令 $\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \Rightarrow \begin{cases} 2 \le u \le 4 \\ 1 \le v \le 3 \end{cases}$

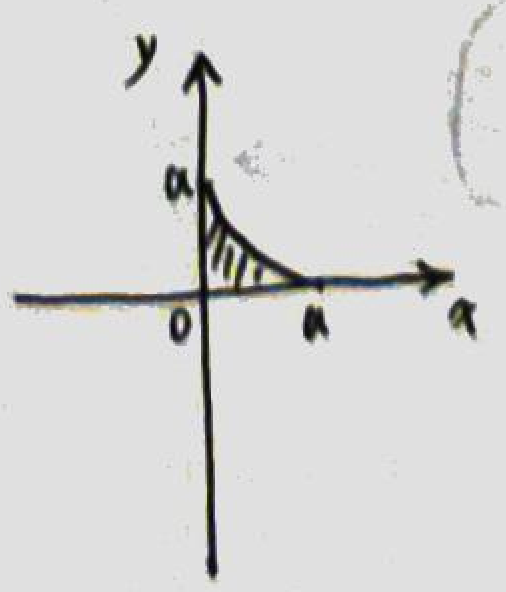
$|J| = \frac{1}{2v}$

$$\therefore I = \int_1^3 dv \int_2^4 u^2 \cdot \frac{1}{2v}\, du$$

$$= \frac{28}{3} \int_1^3 \frac{1}{v}\, dv$$

$$= \frac{28}{3} \ln 3.$$

15. (2)



$$S = \iint_D 1\, d\sigma$$

$$= \int_0^a dx \int_0^{(\sqrt{a}-\sqrt{x})^2} dy$$

$$= \int_0^a (a + x - 2\sqrt{ax})\, dx$$

$$= \frac{1}{6}a^2.$$

16. (2)

令 $\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases}$ $|J| = \frac{1}{2v}$

$$\therefore I = \iint_D f(xy)\, dx\, dy$$

$$= \int_1^2 du \int_1^4 f(u) \cdot \frac{1}{2v}\, dv$$

$$= \ln 2 \int_1^2 f(u)\, du.$$

习题 3.4.

5. (1)

$$\iiint_\Omega xy^2z^3\,dxdydz$$

$$= \iint_{D_{xy}} dxdy \int_0^{xy} xy^2z^3\,dz$$

$$= \iint_{D_{xy}} \frac{1}{4}x^5y^6\,dxdy$$

$$= \int_0^1 dx \int_0^x \frac{1}{4}x^5y^6\,dy$$

$$= \int_0^1 \frac{1}{28}x^{12}\,dx$$

$$= \frac{1}{364}.$$

(4) 由对称性知:

$$\iiint_\Omega x\,dxdydz = 0$$

$$\iiint_\Omega |x|\,dxdydz = \iiint_\Omega |y|\,dxdydz = \iiint_\Omega |z|\,dxdydz$$

$$\therefore I = 2\iiint_\Omega \cancel{|x|+|y|+|z|}\ |x|\,dxdydz$$

$$= 16\iiint_{\Omega_1} x\,dxdydz \qquad \Omega_1 = \{(x,y,z): x+y+z\le 1,\ x,y,z\ge 0\}$$

$$= 16\iint_{D_{xy}} dxdy \int_0^{1-x-y} x\,dz$$

$$= 16\iint_{D_{xy}} x(1-x-y)\,dxdy$$

$$= 16\int_0^1 dx \int_0^{1-x} x(1-x-y)\,dy$$

$$= 16\int_0^1 \frac{x}{2}(1-x)^2\,dx$$

$$= \frac{2}{3}.$$

(5)

$$\iiint_\Omega \frac{\sin z}{z}\,dxdydz$$

$$\cancel{= \iint_{D_{xy}} dxdy \int_{\sqrt{x^2+y^2}}^4 \frac{\sin z}{z}\,dz}$$

$$= 2\iint_{D_{yz}} dydz \int_0^{\sqrt{z^2-y^2}} \frac{\sin z}{z}\,dx$$

$$= 2\iint_{D_{yz}} \frac{\sin z}{z}\sqrt{z^2-y^2}\,dydz$$

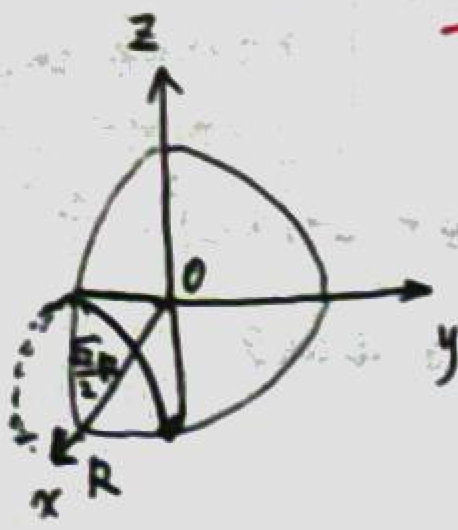
$$= 2\int_0^4 dz \int_{-z}^{z} \frac{\sin z}{z}\sqrt{z^2-y^2}\,dy$$

$$= \pi\int_0^4 \frac{\sin z}{z}\cdot z^2\,dz$$

$$= \pi(\sin 4 - 4\cos 4).$$

---

7. (2)



$$\iiint_\Omega (x^2+y^2+z^2)\,dxdydz$$

$$\boxed{= \iint_{D_{yz}} dydz \int_{\sqrt{y^2+z^2}}^{\sqrt{R^2-y^2-z^2}} (x^2+y^2+z^2)\,dx \qquad \cancel{= \iint_{D_{yz}} (x^2+y^2+z^2)}}$$

$$= \iint_{D_{yz}} \frac{(R^2+2y^2+2z^2)}{3}\sqrt{R^2-y^2-z^2} - \frac{4}{3}(y^2+z^2)^{\frac{3}{2}}\,dydz$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\sqrt{2}}{2}R} \left(\frac{(R^2+2r^2)}{3}\sqrt{R^2-r^2} - \frac{4}{3}r^3\right) r\,dr$$

$$= 2\pi\cdot\left[\frac{4-\sqrt{2}}{36}R^5 - \frac{\sqrt{2}}{36}R^5 + \frac{8-\sqrt{2}}{90}R^5 - \frac{4}{15}\cdot\frac{\sqrt{2}}{8}R^5\right]$$

$$= \frac{2-\sqrt{2}}{5}\pi R^5$$

☆: 球坐标: $0\le \varphi \le 2\pi$, $0\le\theta<$

(3) $\iiint\limits_{\Omega} \dfrac{z}{x^2+y^2}\,dx\,dy\,dz$

$= \iint\limits_{D_{xy}} dx\,dy \int_0^{x^2+y^2} \dfrac{z}{x^2+y^2}\,dz$

$= \iint\limits_{D_{xy}} \dfrac{x^2+y^2}{2}\,dx\,dy$

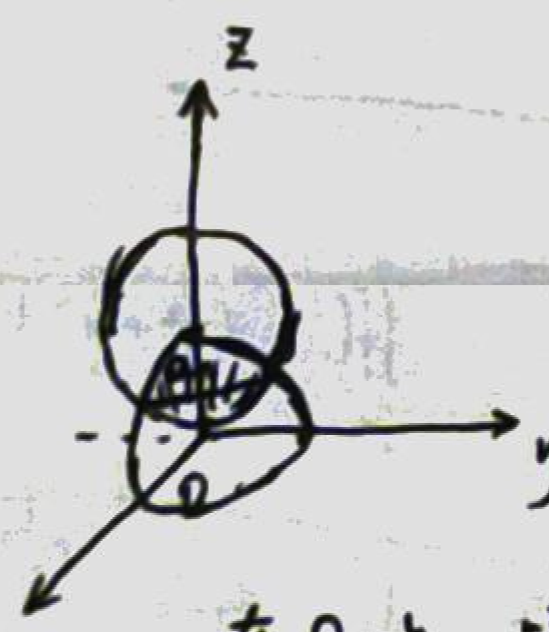
$= \int_0^1 dy \int_0^{1-y} \dfrac{x^2+y^2}{2}\,dx$

$= \int_0^1 \left[\dfrac{y^2}{2}(1-y) + \dfrac{(1-y)^3}{6}\right] dy$

$= \int_0^1 \left(-\dfrac{2}{3}y^3 + y^2 + \dfrac{1}{6} - \dfrac{1}{2}y\right) dy$

$= \dfrac{1}{12}$.

(5)



令 $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$

在 Ω_2 上，$r^2\sin^2\theta + (2-r\cos\theta)^2 \le 4$

$\Rightarrow r \le 4\cos\theta$

$\iiint\limits_{\Omega} xyz\,dx\,dy\,dz$

$= \iiint\limits_{\Omega_1} xyz\,dx\,dy\,dz + \iiint\limits_{\Omega_2} xyz\,dx\,dy\,dz$

$= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{3}} d\theta \int_0^2 r^3\sin^2\theta\cos\theta\sin\varphi\cos\varphi \cdot r^2\sin\theta\,dr$

$+ \int_0^{\frac{\pi}{2}} d\varphi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int_0^{4\cos\theta} r^2\sin^2\theta\sin\varphi\cos\varphi \cdot r\cos\theta \cdot r^2\sin\theta\,dr$

$= \dfrac{3}{4} + \dfrac{4^5}{3}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^3\theta\cos^7\theta\,d\theta = \dfrac{3}{4} + \dfrac{4^5}{3}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\cos^2\theta-1)\cos^7\theta\,d(\cos\theta) = \dfrac{53}{60}$

[redacted]

(6) $\iiint\limits_{\Omega} (x^2+y^2+z^2)\,dx\,dy\,dz$

$= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\frac{1}{2}} \left[r^2\sin^2\theta + \left(\frac{1}{2}+r\cos\theta\right)^2\right] r^2\sin\theta\,dr$

$= \dfrac{\pi}{15}$. 令 $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = \frac{1}{2} + r\cos\theta. \end{cases}$

8. (1) 令 $\begin{cases} x = ar\sin\theta\cos\varphi \\ y = br\sin\theta\sin\varphi \\ z = cr\cos\theta \end{cases}$

$\therefore \iiint\limits_{\Omega} \sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}}\,dx\,dy\,dz$

$= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 \sqrt{1-r^2}\cdot abc\,r^2\sin\theta\,dr$

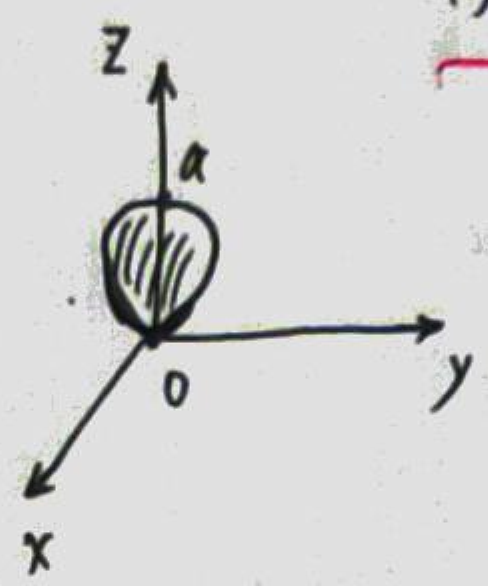
令 $r = \sin t$

$= 4\pi abc \int_0^{\frac{\pi}{2}} \cos^2 t\sin^2 t\,dt$

$= 4\pi abc \left(\dfrac{t}{8} - \dfrac{\sin 4t}{32}\right)\Big|_0^{\frac{\pi}{2}}$

$= \dfrac{abc\,\pi^2}{4}$

9. (1)



当 $z=6-x^2-y^2=\sqrt{x^2+y^2}$ 时

$z=2$

$\therefore\ V=\iiint\limits_{\Omega} dxdydz$

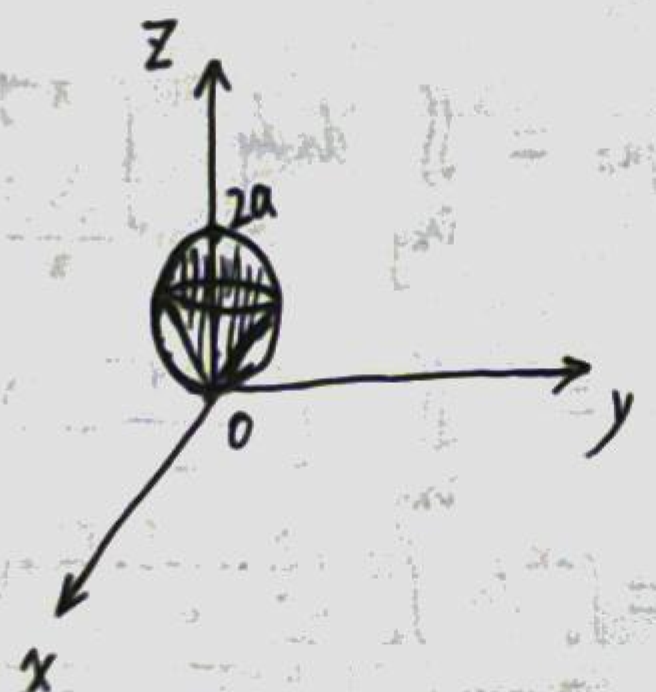
$=\iint\limits_{D_{xy}} dxdy \int_{\sqrt{x^2+y^2}}^{6-x^2-y^2} dz$

$=\iint\limits_{D_{xy}} \left[6-x^2-y^2-\sqrt{x^2+y^2}\right] dxdy$

$=\int_0^{2\pi} d\theta \int_0^2 (6-r^2-r)\, r\, dr$

$=\frac{32\pi}{3}$.

(2)



当 $x^2+y^2=z^2=2az-z^2$ 时, $z=a$

$\therefore\ V=\frac{1}{2}\cdot V_{球}+V_{锥}$

$=\frac{1}{2}\cdot\frac{4}{3}\pi a^3+\frac{1}{3}\cdot\pi a^2\cdot a$

$=\pi a^3$.

或★ $V=\iiint\limits_{\Omega} dxdydz$

$=\iint\limits_{D_{xy}} dxdy \int_{\sqrt{x^2+y^2}}^{a+\sqrt{a^2-x^2-y^2}} dz$

$=\int_0^{2\pi} d\theta \int_0^a \left[a+\sqrt{a^2-r^2}-r\right] r\, dr$

$=2\pi\cdot\left[\frac{a^3}{2}-\frac{a^3}{3}+\frac{a^3}{3}\right]=\pi a^3$.

(5) 令 $\begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases}$

当 $r^4=a^3 r\cos\theta$ 时

$r=a\sqrt[3]{\cos\theta}$

$\therefore\ V=\iiint\limits_{\Omega} dxdydz$

$=\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^{a\cos^{\frac{1}{3}}\theta} r^2\sin\theta\, dr$

$=\frac{\pi a^3}{3}$.

★ 判断 θ 的范围时, 根据 $r^4=a^3 r\cos\theta$ 可知当 $r\to 0^+$ 时, $\cos\theta\to 0$, 即图形在原点的切平面正好是 Oxy 平面.

(7) 令 $\begin{cases} u=a_{11}x+a_{12}y+a_{13}z \\ v=a_{21}x+a_{22}y+a_{23}z \\ w=a_{31}x+a_{32}y+a_{33}z \end{cases}$

$\therefore\ |J|=\frac{1}{|\det A|}$

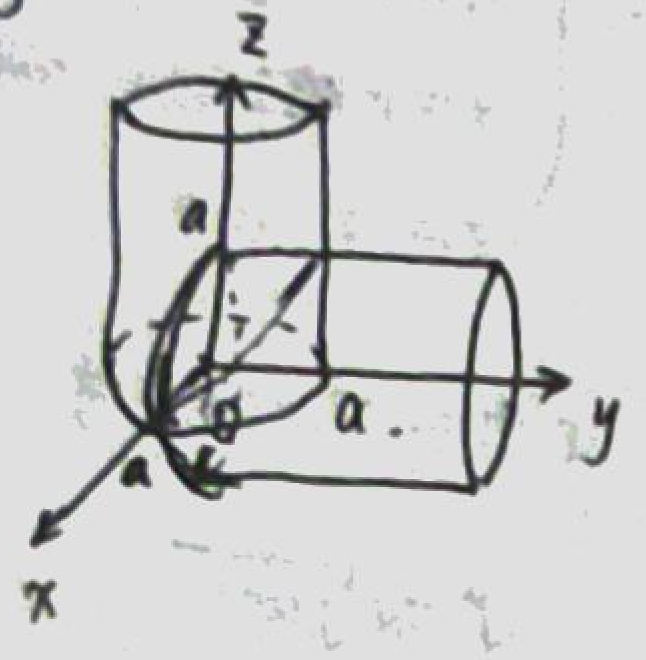
$\therefore\ V=\iiint\limits_{\Omega} dxdydz$

$=\iiint\limits_{\Omega'} \frac{1}{|\det A|}\, dudvdw$

$=\frac{4}{3}\pi r^3\cdot\frac{1}{|\det A|}$.

习题 35

1. (2)



该曲面在 Oxy 平面上的投影区域为: $D_{xy}=\{(x,y):\ x^2+y^2\le a^2\}$.

当 $z=\sqrt{a^2-x^2}$ 时,

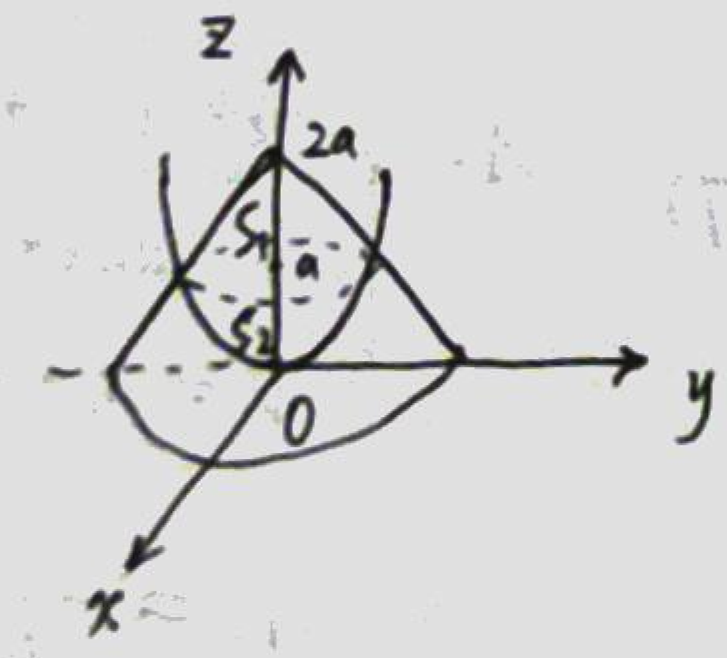
$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\frac{a}{\sqrt{a^2-x^2}}$$

$$\therefore\ S=8\iint_{D_{xy}}\frac{a}{\sqrt{a^2-x^2}}\,dx\,dy$$

$$=8\int_0^a dx\int_0^{\sqrt{a^2-x^2}}\frac{a}{\sqrt{a^2-x^2}}\,dy$$

$$=8a^2.$$

(4)



当 $z=\frac{x^2+y^2}{a}$ 时, $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{4x^2}{a^2}+\frac{4y^2}{a^2}}$

当 $z=2a-\sqrt{x^2+y^2}$ 时, $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{2}$.

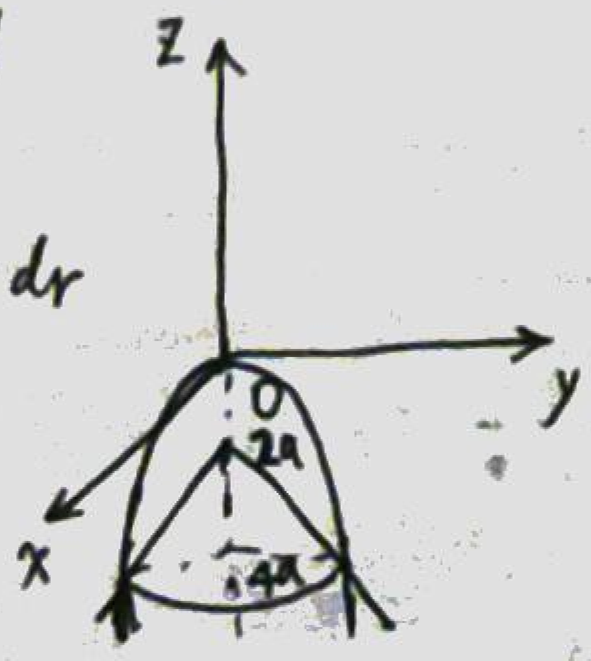
① $a>0$ 时. $S=S_1+S_2$

$$=\iint_{D_{xy}}\sqrt{2}\,dx\,dy+\iint_{D_{xy}}\sqrt{1+\frac{4(x^2+y^2)}{a^2}}\,dx\,dy$$

$$=\sqrt{2}\pi a^2+\int_0^{2\pi}d\theta\int_0^a\sqrt{1+\frac{4r^2}{a^2}}\cdot r\,dr$$

$$=\pi a^2\left(\sqrt{2}+\frac{5\sqrt{5}-1}{6}\right)$$

② $a<0$ 时.



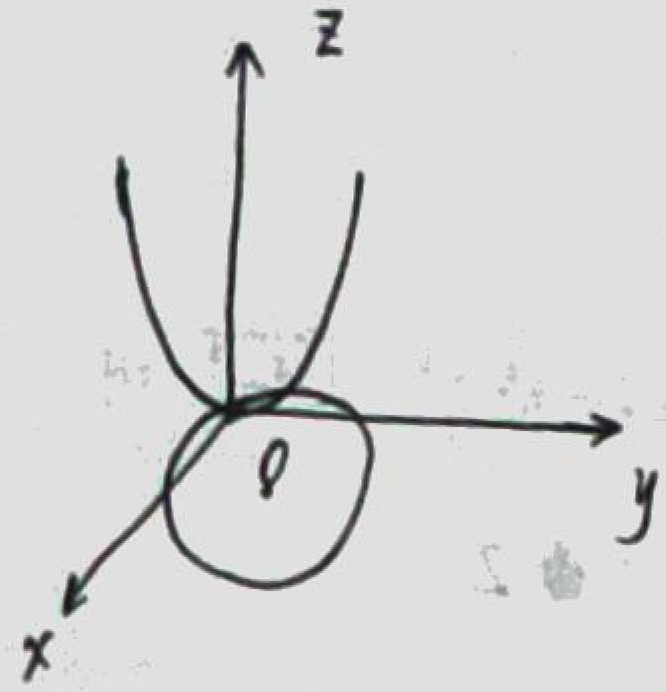
$S=S_1+S_2$

$$=\iint_{D_{xy}}\sqrt{2}\,dx\,dy+\iint_{D_{xy}}\sqrt{1+\frac{4(x^2+y^2)}{a^2}}\,dx\,dy$$

$$=4\sqrt{2}\pi a^2+\int_0^{2\pi}d\theta\int_0^{-2a}\sqrt{1+\frac{4r^2}{a^2}}\,r\,dr$$

$$=\pi a^2\left[4\sqrt{2}+\frac{17\sqrt{17}-1}{6}\right]$$

2. (2)



$$V=\iiint_\Omega dx\,dy\,dz=\iint_{D_{xy}}dx\,dy\int_{\frac{x^2+y^2}{2}}^{x+y}dz$$

$$=\iint_{D_{xy}}\left(x+y-\frac{x^2+y^2}{2}\right)dx\,dy$$

令 $\begin{cases}x=1+r\cos\theta\\ y=1+r\sin\theta\end{cases}$ $=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\left(1-\frac{r^2}{2}\right)r\,dr$

$$=\pi.$$

$$M_{xy}=\iiint_\Omega z\,dx\,dy\,dz=\iint_{D_{xy}}dx\,dy\int_{\frac{x^2+y^2}{2}}^{x+y}z\,dz$$

$$=\iint_{D_{xy}}\left[\frac{(x+y)^2}{2}-\frac{(x^2+y^2)^2}{8}\right]dx\,dy$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\left[\frac{3}{2}-\frac{r^4}{8}-\frac{r^2}{2}+\left(r-\frac{r^3}{2}\right)(\sin\theta+\cos\theta)\right]r\,dr$$

$$=\frac{5\pi}{3}.$$

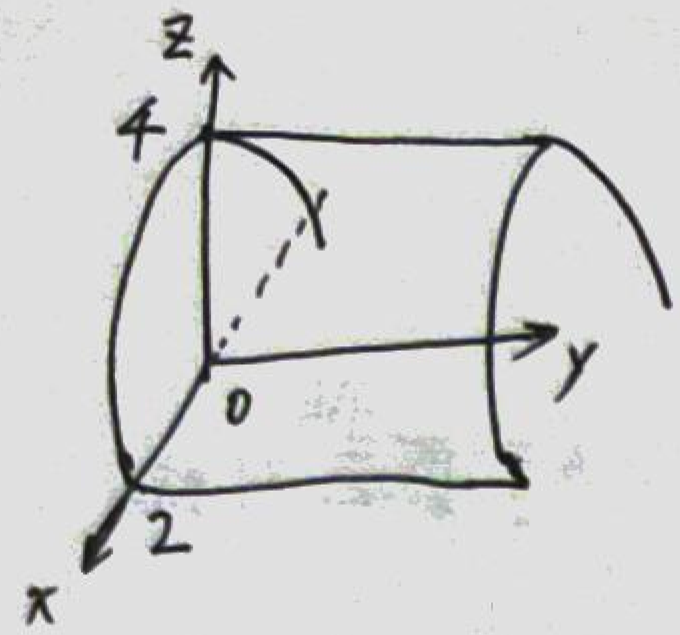
$$M_{yz}=\iiint_\Omega x\,dx\,dy\,dz=\iint_{D_{xy}}dx\,dy\int_{\frac{x^2+y^2}{2}}^{x+y}x\,dz=\pi$$

$$M_{yz}=M_{xz}=\pi$$

$$\left(1,1,\frac{5}{3}\right)$$

$$\therefore\ \bar{x}=\bar{y}=1,\quad \bar{z}=\frac{5}{3}$$

(3)



$$V = \iiint_{\Omega} dx\,dy\,dz$$

$$= \iint_{D_{xz}} dx\,dz \int_0^6 dy$$

$$= 6\int_0^2 dx \int_0^{4-x^2} dz$$

$$= 32.$$

$$M_{xy} = \iiint_{\Omega} z\,dx\,dy\,dz$$

$$= 6\int_0^2 dx \int_0^{4-x^2} z\,dz$$

$$= \frac{256}{5}$$

$$M_{yz} = \iiint_{\Omega} x\,dx\,dy\,dz$$

$$= 6\int_0^2 dx \int_0^{4-x^2} x\,dz$$

$$= 24$$

$$M_{xz} = \iiint_{\Omega} y\,dx\,dy\,dz$$

$$= \iint_{D_{xz}} dx\,dz \int_0^6 y\,dy$$

$$= 18 \iint_{D_{xy}} dx\,dz$$

$$= 96$$

$$\therefore\ \bar{x} = \frac{24}{32} = \frac{3}{4}$$

$$\bar{y} = \frac{96}{32} = 3$$

$$\bar{z} = \frac{8}{5}.$$

3. (2)

由于 Ω 与 ρ 关于 z 轴对称，故 $\bar{x}=\bar{y}=0$.

当 $r^2 \le 2ar\cos\theta$ 时，$r \le 2a\cos\theta$.

$$V = \iiint_{\Omega} \rho\,dx\,dy\,dz$$

$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} \frac{1}{r}\cdot r^2\sin\theta\,dr$$

$$= 2\pi \int_0^{\frac{\pi}{2}} 2a^2\cos^2\theta\sin\theta\,d\theta.$$

$$= \frac{4\pi a^2}{3}$$

$$M_{xy} = \iiint_{\Omega} z\rho\,dx\,dy\,dz$$

$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} r\sin\theta\cdot r\cos\theta\,dr$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{8a^3}{3}\sin\theta\cos^4\theta\,d\theta$$

$$= \frac{16\pi a^3}{15}$$

$$\therefore\ \bar{z} = \frac{M_{xy}}{V} = \frac{4a}{5}$$

(3)

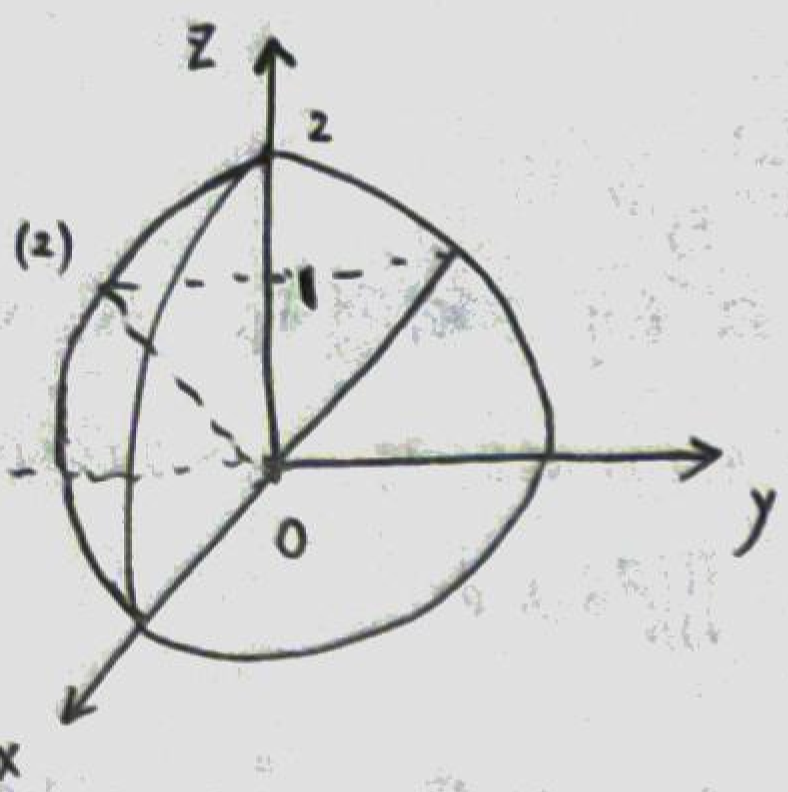
$$m = \iint_{D_{xy}} \rho\,dx\,dy$$

$$= \int_0^{2\pi} d\theta \int_2^4 \frac{2}{r}\cdot r\,dr$$

$$= 8\pi.$$

☆ 圆环内、外径指直径.

4. (2)



设密度为1.

$$J_z = \iiint_\Omega (x^2+y^2)\,dxdydz$$

$$= \iint_{D_{xy}} dxdy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} (x^2+y^2)\,dz$$

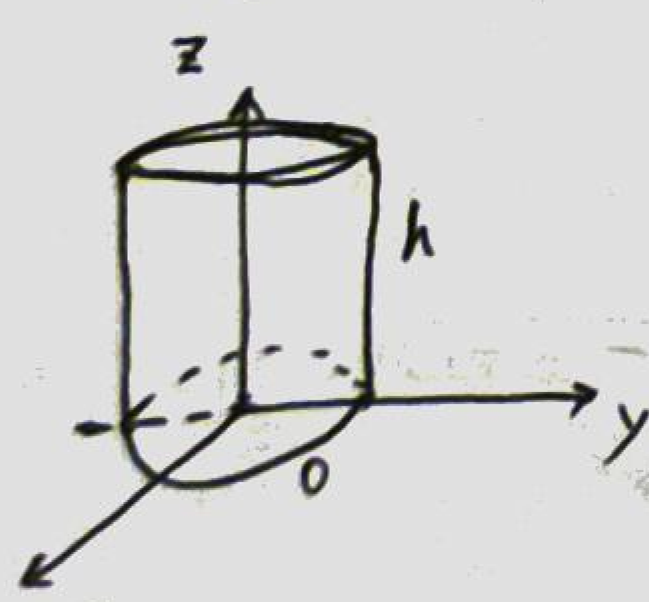
$$= \iint_{D_{xy}} (x^2+y^2)\left(\sqrt{2-x^2-y^2}-\sqrt{x^2+y^2}\right)dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^2\left(\sqrt{2-r^2}-r\right) r\,dr$$

$$= \frac{(16\sqrt{2}-14)\pi}{15} - \frac{2\pi}{5}$$

$$= \frac{(16\sqrt{2}-20)\pi}{15}$$

7.



$$F = \int_0^h \rho g l \cdot \pi a\, dl$$

$$= \pi\rho g a \int_0^h l\, dl$$

$$= \frac{1}{2}\pi\rho g a h^2.$$

9.

由于 Ω 关于轴对称

$\therefore$ 对P点的引力在x轴、y轴方向的分量 $F_x = F_y = 0$.

$$F_z = \iiint_\Omega km \frac{(z-a)\frac{M}{\frac{4}{3}\pi R^3}}{\left[x^2+y^2+(z-a)^2\right]^{\frac{3}{2}}}\,dxdydz$$

令 $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$

$$\therefore F_z = km\rho\int_0^{2\pi} d\varphi \int_0^\pi d\theta \int_0^R \frac{(r\cos\theta - a)}{\left[r^2-2ar\cos\theta+a^2\right]^{\frac{3}{2}}}\cdot r^2\sin\theta\, dr$$

$$F_z = km\rho \iint_{D_{xy}} dxdy \int_{-\sqrt{R^2-(x^2+y^2)}}^{\sqrt{R^2-(x^2+y^2)}} \frac{(z-a)}{\left[x^2+y^2+(z-a)^2\right]^{\frac{3}{2}}}\,dz$$

$$= km\rho \iint_{D_{xy}} -\frac{1}{\left[x^2+y^2+(z-a)^2\right]^{\frac{1}{2}}}\Bigg|_{-\sqrt{R^2-(x^2+y^2)}}^{\sqrt{R^2-(x^2+y^2)}} dxdy$$

$$= km\rho \iint_{D_{xy}} \sqrt{R^2+2a\sqrt{R^2-(x^2+y^2)}+a^2} - \sqrt{R^2-2a\sqrt{R^2-(x^2+y^2)}+a^2}\,dxdy$$

$$= km\rho \int_0^{2\pi} d\theta \int_0^R \left(\sqrt{R^2+2a\sqrt{R^2-r^2}+a^2} - \sqrt{R^2-2a\sqrt{R^2-r^2}+a^2}\right) r\,dr$$

$$F_z = \frac{3kmM}{4\pi R^3}\int_{-R}^{R}(z-a)\,dz \iint_D \frac{dxdy}{\left[x^2+y^2+(z-a)^2\right]^{\frac{3}{2}}}$$

其中 $D = \{(x,y): x^2+y^2 \le R^2 - z^2\}$

$$\therefore F_z = \frac{3kmM}{4\pi R^3}\int_{-R}^{R}(z-a)\,dz \int_0^{2\pi} d\theta \int_0^{\sqrt{R^2-z^2}} \frac{r}{\left[r^2+(z-a)^2\right]^{\frac{3}{2}}}\,dr$$

$$= \frac{3kmM}{4\pi R^3}2\pi \int_{-R}^{R}\left[-1-\frac{z-a}{\sqrt{R^2-2az+a^2}}\right]dz$$

$$= \frac{3kmM}{2R^3}\int_{-R}^{R}\left[\frac{1}{2a}\sqrt{R^2-2az+a^2} + \left(\frac{a}{2}-\frac{R^2}{2a}\right)\frac{1}{\sqrt{R^2-2az+a^2}} - 1\right]dz$$

$$= \frac{3kmM}{2R^3}\cdot\left[\left(\frac{R^3}{3a^2}+R\right)+\left(R-\frac{R^3}{a^2}\right)-2R\right]$$

$$= \frac{3kmM}{2R^3}\cdot\left(-\frac{2R^3}{3a^2}\right)$$

$$= -\frac{kmM}{a^2}$$ （负号表示沿z轴负方向）.