习题4.6

8.(2) 方向向量：$\frac{(dx,dy)}{dl}$

单位外法向量：$\vec{n} = \frac{(dy,-dx)}{dl}$

则左边 $= \oint_{\partial D} v\cdot\frac{\partial u}{\partial \vec{n}}\, dl$

$= \oint_{\partial D} v\cdot\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\cdot\frac{(dy,-dx)}{dl}\,dl$

$= \oint_{\partial D} v\cdot\left(\frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx\right)$

$\overset{Green}{=} \iint_D \left[\frac{\partial}{\partial x}\left(v\cdot\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\cdot\frac{\partial u}{\partial y}\right)\right]dxdy$

$= \iint_D (v\cdot\Delta u + \nabla u\cdot\nabla v)\,dxdy$

9. $\cos(n,i) = \frac{dy}{dl}$, $\cos(n,j) = -\frac{dx}{dl}$

则 $\oint_L [x\cdot\cos(n,i) + y\cos(n,j)]\,dl$

$= \oint_L x\,dy - y\,dx$

$\overset{Green}{=} 2\iint_D dxdy = 2S$.

10. $\frac{\partial}{\partial y}(x^2-y) = \frac{\partial}{\partial x}(-x-\sin^2 y) = -1$

故 $(x^2-y)dx - (x+\sin^2 y)dy = 0$

为一个函数 $u(x,y)$ 的全微分

$\frac{\partial u}{\partial x} = x^2 - y \Rightarrow u = \frac{x^3}{3} - xy + f(y)$

$\frac{\partial u}{\partial y} = -x - \sin^2 y = -x - \frac{1-\cos 2y}{2}$

$= -x + f'(y)$

故 $f'(y) = \frac{1}{2}\cos 2y - \frac{1}{2}$

$f(y) = \frac{1}{4}\sin 2y - \frac{1}{2}y + C$

所以方程的解为：

$\frac{1}{3}x^3 - xy + \frac{1}{4}\sin 2y - \frac{1}{2}y = C$

(3) $\frac{xdx+ydy}{\sqrt{x^2+y^2}} = \frac{ydx-xdy}{x^2}$

注意到

$d(\sqrt{x^2+y^2}) = \frac{xdx+ydy}{\sqrt{x^2+y^2}}$

$d\left(-\frac{y}{x}\right) = \frac{ydx-xdy}{x^2}$

故解为 $\sqrt{x^2+y^2} + \frac{y}{x} = C$

11.(2) $(x+y)dx + (y-x)dy = 0$

$\frac{\partial P}{\partial x} = 1$, $\frac{\partial Q}{\partial y} = -1$

加入积分因子 $\mu(x,y) = \frac{1}{x^2+y^2}$

则 $\frac{xdx+ydy}{x^2+y^2} + \frac{ydx-xdy}{x^2+y^2} = 0$

即 $d\left[\frac{1}{2}\ln(x^2+y^2)\right] + d\left[\arctan\frac{x}{y}\right] = 0$

故解为：$\frac{1}{2}\ln(x^2+y^2) + \arctan\frac{x}{y} = C$

(4) $(x+y)(dx-dy) = dx+dy$

注意到不是全微分

$dx - dy = \frac{dx+dy}{x+y}$

$d[x-y] = d[\ln(x+y)]$

$\therefore \ln(x+y) + y - x = C$

习题 4.7

3.(1) $\oiint\limits_{S^+} x^3 dy\wedge dz + y^3 dz\wedge dx + z^3 dx\wedge dy$

$\underline{\underline{\text{Gauss}}}$ $\iiint\limits_{\Omega} (3x^2+3y^2+3z^2)\, dV$

变换：$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$

则 原式 $= 3\iiint\limits_{\Omega} \rho^4 \sin\varphi\, d\rho\, d\varphi\, d\theta$

$= \frac{12}{5}\pi a^5$

3.(3) $\oiint\limits_{S^+} (x+2y+3z)\, dx\wedge dy + (y+2z)\, dy\wedge dz$

$+ (z^2-1)\, dz\wedge dx$

$\underline{\underline{\text{Gauss}}}$ $\iiint\limits_{\Omega} 3\, dV = \frac{1}{2}$

3.(5) S_ε：以原点为球心，半径为ε为球

S_ε^-：球面内侧

则 $\oiint\limits_{S_\varepsilon^- + S^+} \overset{\text{Gauss}}{=} 0$

则 $\oiint\limits_{S^+} = \oiint\limits_{S_\varepsilon^+} \frac{x dy\wedge dz + y dz\wedge dx + z dx\wedge dy}{\varepsilon^3}$

$\underline{\underline{\text{Gauss}}}$ $\frac{3}{\varepsilon^3}\iiint\limits_{\Omega} dV = 4\pi$

4.(2) $\Phi = \oiint\limits_{S^+} (x-y+z)\, dy\wedge dz + (y-z+x)\, dz\wedge dx$

$+ (z-x+y)\, dx\wedge dy$

$= 3\iiint\limits_{\Omega} dx = 4\pi abc$

5.(1) $\oint\limits_{L^+} y\, dx + z\, dy + x\, dz$

$\underline{\underline{\text{Stokes}}}$ $\iint\limits_{\Sigma} -dy\wedge dz - dz\wedge dx - dx\wedge dy$

$= -\sqrt{3}\iint\limits_{\Sigma} dS = -\sqrt{3}\pi R^2$

(3) $\int\limits_{L^+} y^2 dx + z^2 dy + x^2 dz$

$\underline{\underline{\text{Stokes}}}$ $\iint\limits_{\Sigma} -2z\, dy\wedge dz - 2x\, dz\wedge dx - 2y\, dx\wedge dy$

$= -\iint\limits_{D_{yz}} 2z\, dy\wedge dz - \iint\limits_{D_{zx}} 2x\, dz\, dx$

$- \iint\limits_{D_{xy}} 2y\, dx\, dy$

$= -\frac{1}{3}(a^2c + b^2a + c^2b)$

6.(2) 记 $P = \frac{y}{(x+z)^2+y^2}$

$Q = \frac{-(x+z)}{(x+z)^2+y^2}$

$R = \frac{y}{(x+z)^2+y^2}$

$\frac{\partial R}{\partial y} = \frac{-y^2+(x+z)^2}{((x+z)^2+y^2)^2} = \frac{\partial Q}{\partial z}$

$\frac{\partial P}{\partial z} = -\frac{2y(x+z)}{((x+z)^2+y^2)^2} = \frac{\partial R}{\partial x}$

$\frac{\partial Q}{\partial x} = \frac{-y^2+(x+z)^2}{((x+z)^2+y^2)^2} = \frac{\partial P}{\partial y}$

所以原式是某三元函数 $u(x,y,z)$ 的全微分

解为：$u(x,y,z)=\arctan\left(\frac{x+z}{y}\right)=C$

7.(1) 记 $P=y+z$，$Q=z+x$

$R=x+y$

$\frac{\partial R}{\partial y}=1=\frac{\partial Q}{\partial z}$，$\frac{\partial P}{\partial z}=1=\frac{\partial R}{\partial x}$

$\frac{\partial Q}{\partial x}=1=\frac{\partial P}{\partial y}$

故积分与路径无关

原式 $=(xy+yz+zx)\Big|_{(0,0,0)}^{(1,2,1)}$

$=5$

(2) 记 $P=y^2z$，$Q=2xyz$

$R=xy^2$

$\frac{\partial R}{\partial y}=2xy=\frac{\partial Q}{\partial z}$

$\frac{\partial P}{\partial z}=y^2=\frac{\partial R}{\partial x}$

$\frac{\partial Q}{\partial x}=2yz=\frac{\partial P}{\partial y}$

故积分与路径无关。

原式 $=(xy^2z)\Big|_{(1,-1,1)}^{(1,1,-1)}$

$=-2$

12. (1) 记 $P=y\cos(xy)$

$Q=x\cos(xy)$

$R=\sin z$

$\frac{\partial R}{\partial y}=0=\frac{\partial Q}{\partial z}$，$\frac{\partial P}{\partial z}=0=\frac{\partial R}{\partial x}$

$\frac{\partial Q}{\partial x}=\cos(xy)-xy\sin(xy)=\frac{\partial P}{\partial y}$

故该向量场为保守场

则：$\int_{(4,0,1)}^{(2,1,-1)} v\,dr$

$=(\sin xy-\cos z)\Big|_{(4,0,1)}^{(2,1,-1)}$

$=\sin 2$

(2) 记 $P=6xy+z^2$

$Q=3x^2-z$

$R=3xz^2-y$

$\frac{\partial R}{\partial y}=-1=\frac{\partial Q}{\partial z}$

$\frac{\partial P}{\partial z}=2z$，$\frac{\partial R}{\partial x}=3z^2$

故 v 不是保守场。