习题 4.4

1.(2)

设点 $C(2,1,1)$，$D(2,3,1)$

$$\int_{AB} = \int_{AC} + \int_{CD} + \int_{DB}$$

在 $\overline{AC}$ 上，$1 \le x \le 2$，$y \equiv 1$，$z \equiv 1$

$$\therefore \int_{AC} x\,dx + y\,dy + z\,dz$$

$$= \int_{AC} x\,dx = \int_1^2 x\,dx = \frac{3}{2}$$

同理，在 $\overline{CD}$ 上，

$$\int_{CD} y\,dy = \int_1^3 y\,dy = 4$$

在 $\overline{DB}$ 上，

$$\int_{DB} z\,dz = \int_1^4 z\,dz = \frac{15}{2}$$

$$\therefore \int_{AB} x\,dx + y\,dy + z\,dz$$

$$= \frac{3}{2} + 4 + \frac{15}{2} = 13$$

(3) $$\int_{L^+} \frac{-y\,dx + x\,dy}{x^2+y^2} + b\,dz$$

$$= \int_0^{2\pi} \left(\frac{a^2\sin^2 t + a^2\cos^2 t}{a^2} dt + b^2\,dt \right)$$

$$= \int_0^{2\pi} (1+b^2)\,dt$$

$$= 2\pi(1+b^2)$$

2.(1) $$\int_{L^+} (x^2-y^2)\,dx$$

$$= \int_0^2 (x^2 - x^4)\,dx$$

$$= \frac{1}{3}x^3 \Big|_0^2 - \frac{1}{5}x^5 \Big|_0^2$$

$$= \frac{8}{3} - \frac{32}{5} = -\frac{56}{15}$$

(5) 令 $\begin{cases} x = \cos\theta \\ y = z = \frac{1}{\sqrt{2}}\sin\theta \end{cases}$，$\theta$ 从 0 到 2π

$$dz = \frac{1}{\sqrt{2}}\cos\theta\,d\theta$$

$$\therefore \int_{L^+} xyz\,dz$$

$$= \int_0^{2\pi} \frac{1}{2}\sin^2\theta\cos\theta \cdot \frac{1}{\sqrt{2}}\cos\theta\,d\theta$$

$$= \frac{1}{2\sqrt{2}} \int_0^{2\pi} \sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{1}{8\sqrt{2}} \int_0^{4\pi} \sin^2\varphi\,d\varphi$$

$$= \frac{\sqrt{2}}{16}\pi$$

3.(2) $$\vec{F} = \left(\frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \right)$$

$$\int_{L^+} \vec{F} \cdot d\vec{r}$$

$$= \int_{L^+} \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy$$

令 $\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}$，$\theta$ 从 0 到 2π

$$\therefore \int_{L^+} \vec{F}\,d\vec{r} = \int_0^{2\pi} \left(-\frac{a\sin\theta}{a^2} \cdot a\sin\theta - \frac{a\cos\theta}{a^2} \cdot a\cos\theta \right) d\theta$$

$$= \int_0^{2\pi} -d\theta = -2\pi$$

(4) $$\vec{F} = -Gm \cdot \frac{1}{(x^2+y^2+z^2)^{3/2}} (x, y, z)$$

记：$A(1,1,1)$ $B_1(x_0,1,1)$ $C(x_0,y_0,z_0)$
$B_2(x_0,y_0,1)$

$$\int_{L^+} \vec{F} \cdot d\vec{r} = \int_{AB_1} + \int_{B_1B_2} + \int_{B_2C}$$

$$\int_{AB_1} = -\int_1^{x_0} \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dx$$

其中 $y \equiv z \equiv 1$

$$\therefore \int_{AB_1} = -\int_1^{x_0} \frac{x\,dx}{(x^2+2)^{3/2}} = -\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{x_0^2+2}}$$

$$\int_{B_1B_2} = -\int_1^{y_0} \frac{y\, dy}{(x^2+y^2+z^2)^{3/2}}$$

其中 $x \equiv x_0,\ z \equiv 1$

$$\therefore \int_{B_1B_2} = -\int_1^{y_0} \frac{y\, dy}{(y^2+x_0^2+1)^{3/2}}$$

$$= -\frac{1}{\sqrt{x_0^2+2}} + \frac{1}{\sqrt{1+x_0^2+y_0^2}}$$

$$\int_{B_2C} = -\int_1^{z_0} \frac{z\, dz}{(x^2+y^2+z^2)^{3/2}}$$

其中 $x \equiv x_0,\ y \equiv y_0$

$$\therefore \int_{B_2C} = -\frac{1}{\sqrt{1+x_0^2+y_0^2}} + \frac{1}{\sqrt{x_0^2+y_0^2+z_0^2}}$$

$$\int_{L_1^+} \vec{F}\cdot d\vec{x} = -\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{x_0^2+y_0^2+z_0^2}}$$

(1) 依题意，$\vec{F} = (-x, -y)$

$$W_1 = \int_{L_1^+} \vec{F}\cdot d\vec{r}$$

令 $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$

$$W_1 = \int_{L_1^+} -x\,dx - y\,dy$$

$$= \int_0^{\frac{\pi}{2}} (a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta)\,d\theta$$

$$= (a^2-b^2)\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta$$

$$= (a^2-b^2)\int_0^1 t\,dt$$

$$= \frac{1}{2}(a^2-b^2)$$

(2) $$W_2 = \oint_{L^+} -x\,dx - y\,dy$$

$$= \int_0^{2\pi} (a^2-b^2)\sin\theta\cos\theta\,d\theta$$

$$= (a^2-b^2)\cdot 0 = 0$$

习题 4.5

1. 将 S^+ 分为两部分

S_1^+：下半球面的外侧

S_2^+：上半球面的外侧

S_1^+：$z_1 = -\sqrt{R^2-x^2-y^2}+R$，$dx\wedge dy = -dxdy$

S_2^+：$z_2 = \sqrt{R^2-x^2-y^2}+R$，$dx\wedge dy = dxdy$

投影区域均为 $D_{xy} = \{(x,y) \mid x^2+y^2 \le R^2\}$

(1) $$\oiint_{S^+} dx\wedge dy$$

$$= \iint_{S_1^+} dx\wedge dy + \iint_{S_2^+} dx\wedge dy$$

$$= \iint_{D_{xy}} -dxdy + \iint_{D_{xy}} dxdy$$

$$= 0$$

(2) $$\oiint_{S^+} z\,dx\wedge dy$$

$$= \iint_{S_1^+} z_1\,dx\wedge dy + \iint_{S_2^+} z_2\,dx\wedge dy$$

$$= \iint_{D_{xy}} -z_1\,dxdy + \iint_{D_{xy}} z_2\,dxdy$$

$$= 2\iint_{D_{xy}} \sqrt{R^2-x^2-y^2}\,dxdy$$

$$= 2\int_0^{2\pi} d\theta \int_0^R \sqrt{R^2-\rho^2}\,\rho\,d\rho \quad \left(令 \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ \rho\in[0,R] \end{cases}\right)$$

$$= 2\int_0^{2\pi} \frac{1}{3}R^3\,d\theta$$

$$= \frac{4}{3}\pi R^3$$

(3) $\oiint\limits_{S^+} z^2 dx\wedge dy$

$= \iint\limits_{S_1^+} z_1^2 dx\wedge dy + \iint\limits_{S_2^+} z_2^2 dx\wedge dy$

$= \iint\limits_{D_{xy}} -z_1^2 dxdy + \iint\limits_{D_{xy}} z_2^2 dxdy$

$= \iint\limits_{D_{xy}} (z_2+z_1)(z_2-z_1) dxdy$

$= \iint\limits_{D_{xy}} 2R\cdot 2\sqrt{R^2-x^2-y^2}\, dxdy$

$= \frac{8}{3}\pi R^4$

3.(2) 由对称性知

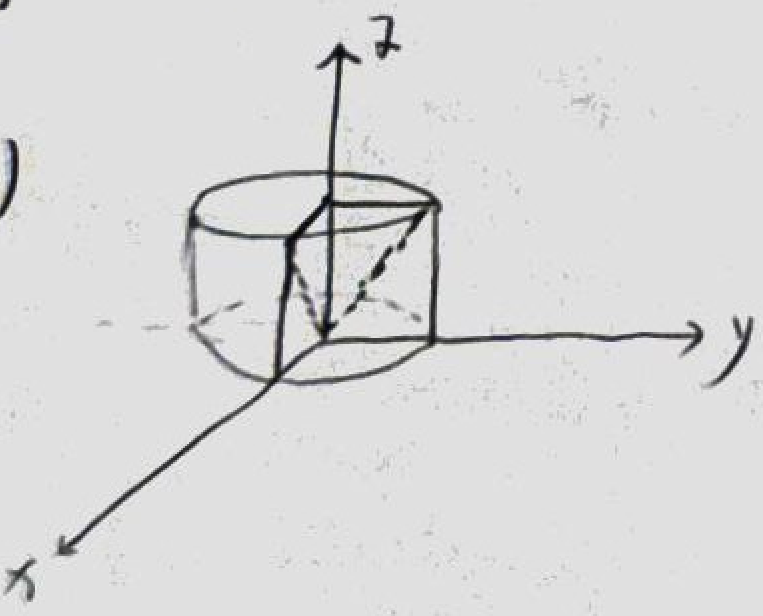
$\iint\limits_{S^+} x^2 dy\wedge dz = \iint\limits_{S^+} y^2 dz\wedge dx = 0$

曲面法向量与z轴垂直

故 $\iint\limits_{S^+} z^2 dx\wedge dy = 0$

综之，原式 $=0$

(4)



曲面由 S_1, S_2, S_3, S_4, S_5 构成

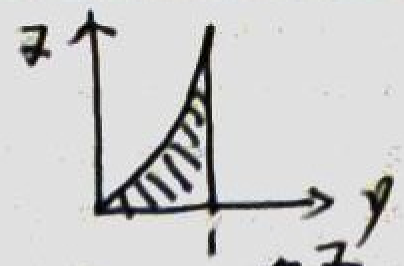
$S_1: \begin{cases} z=0 \\ x^2+y^2\le 1 \\ x>0,\ y>0 \end{cases}$

$dx\wedge dz = dy\wedge dz = 0$

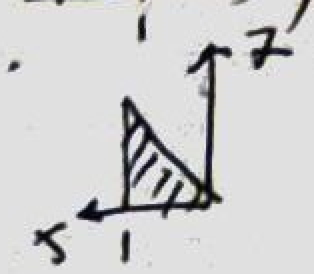
$y^2 z\, dx\wedge dy = 0$ (因为 $z=0$)

故 $\int_{S_1} \omega = 0$

$S_2: \begin{cases} x=0 \\ 0<z\le y^2 \\ 0<y<1 \end{cases}$

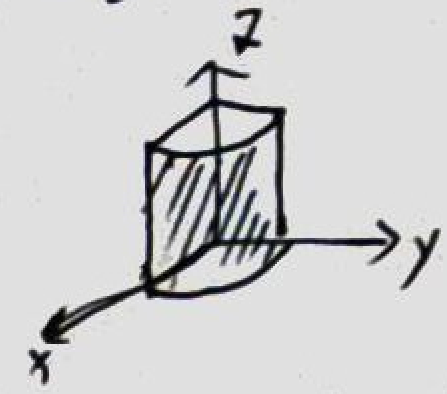


$S_3: \begin{cases} y=0 \\ 0<z\le x^2 \\ 0<x<1 \end{cases}$



同理 $\int_{S_1} \omega = \int_{S_2} \omega = \int_{S_3} \omega = 0$

$S_4: \begin{cases} x^2+y^2=1 \\ x>0,\ y>0 \\ 0<z<1 \end{cases}$

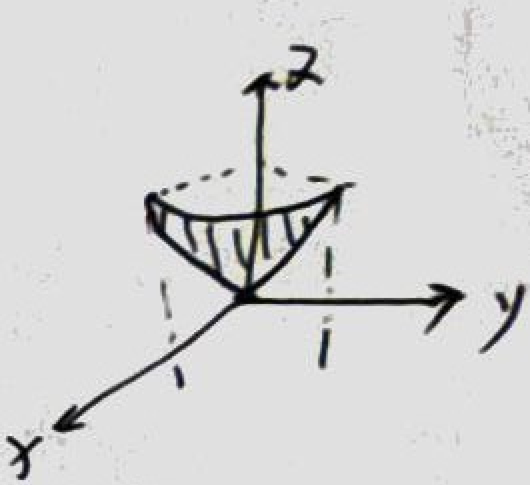


$\int_{S_4} \omega = \int_0^1 z^2 dz \int_0^1 \sqrt{1-y^2}\, dy$

$+ \int_0^1 dz \int_0^1 x^2\sqrt{1-x^2}\, dx$

$= \frac{7}{48}\pi$

$S_5: \begin{cases} z = x^2+y^2 \\ 0<x<1,\quad 0<y<1 \\ 0<z<1 \end{cases}$



$dx\wedge dz = -dxdz$.

$dy\wedge dz = -dydz$

$dx\wedge dy = dxdy$

投影：$D_{xy} = \{(x,y) \mid x^2+y^2\le 1,\ x>0,\ y>0\}$

$\int_{S_5} \omega = \iint\limits_{S_5} y^2 z\, dxdy - xz^2 dydz - x^2 y\, dxdz$

$= \frac{\pi}{24} - 2\iint\limits_{D_{xy}} x^2(x^2+y^2) dxdy - 2\iint\limits_{D_{xy}} x^2y^2 dxdy$

$= \frac{\pi}{24} - \frac{\pi}{16} - \frac{\pi}{48} = -\frac{\pi}{24}$

综之：原式 $= 0 + \frac{7}{48}\pi - \frac{\pi}{24} = \frac{5}{48}\pi$

5、设 S^+ 为球面 $x^2+y^2+z^2=1$ 在第一卦限部分的外表面

流量 $Q = \iint_{S^+} \vec{v}\cdot d\vec{S}$

$\vec{v} = (xy,\ yz,\ xz)$

$\therefore Q = \iint_{S^+} xy\,dy\wedge dz + yz\,dx\wedge dz + xz\,dx\wedge dy$

在 S^+ 上，$dy\wedge dz = dydz$

$dx\wedge dz = dxdz$

$dx\wedge dy = dxdy$

$D_{yz} = \{(y,z) \mid y\geq 0,\ z\geq 0,\ y^2+z^2\leq 1\}$

$D_{xz} = \{(x,z) \mid x\geq 0,\ z\geq 0,\ x^2+z^2\leq 1\}$

$D_{xy} = \{(x,y) \mid x\geq 0,\ y\geq 0,\ x^2+y^2\leq 1\}$

$\therefore Q = \iint_{D_{yz}} xy\,dydz + \iint_{D_{xz}} yz\,dxdz + \iint_{D_{xy}} xz\,dxdy$

$= 3\int_0^1 dy \int_0^{\sqrt{1-y^2}} y\sqrt{1-y^2-z^2}\,dz$

$= 3\int_0^1 dz \int_0^{\sqrt{1-z^2}} y\sqrt{1-y^2-z^2}\,dy$

$= \int_0^{\frac{\pi}{2}} \cos^4 t\,dt = \frac{3}{16}\pi$

7. $\begin{cases} x = u\cos v \\ y = u\sin v \\ z = av \end{cases}$

$\Rightarrow A = \frac{D(y,z)}{D(u,v)} = a\sin v.$

$B = \frac{D(z,x)}{D(u,v)} = -a\cos v$

$C = \frac{D(x,y)}{D(u,v)} = u$

(A,B,C) 与 $\vec{n}_0$ 同向

$\iint_{S^+} = \iint_{D_{uv}} [(x^2+y^2)A + y^2B + z^2C]\,dudv$

$= \iint_{D_{uv}} [u^3 + u^2\sin^2 v\,(a\sin v) + a^2v^2(-a\cos v)]\,dudv$

$= \frac{\pi}{2} - 4a^3\pi$

习题 4.6

1.(2) $\oint_{L^+} (x+y)dx + (x-y)dy$

$= -\iint_D \left(-\frac{\partial(x+y)}{\partial y} + \frac{\partial(x-y)}{\partial x}\right) dxdy$

$= \iint_D (1-1)\,dxdy = 0$

(4) 记 $P(x,y) = e^x(1-\cos y)$

$Q(x,y) = e^x(\sin y - y)$

原式 $= \oint_{L^+} Pdx + Qdy$

$\frac{\partial P}{\partial y} = e^y \sin y$，$\frac{\partial Q}{\partial x} = e^x(\sin y - y)$

原式 $= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

$= \iint_D -e^x\cdot y\,dxdy$

$= -\int_0^\pi dx \int_0^{\sin x} e^x y\,dy$

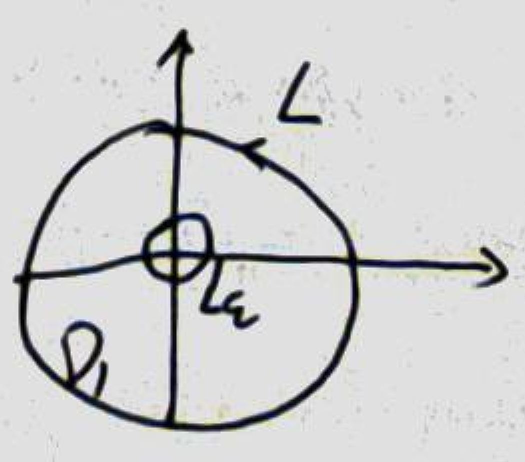
$= \frac{1}{5}(1-e^\pi)$

(2) $P(x,y) = \frac{x+y}{x^2+y^2}$

$Q(x,y) = \frac{y-x}{x^2+y^2}$

$\frac{\partial P}{\partial y} = \frac{x^2-2xy-y^2}{(x^2+y^2)^2}$

$\frac{\partial Q}{\partial x} = \frac{x^2-2xy-y^2}{(x^2+y^2)^2}$



L_ε 为包含原点的小圆

$\int_{L^+} + \int_{L_\varepsilon^-} = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

$= 0$

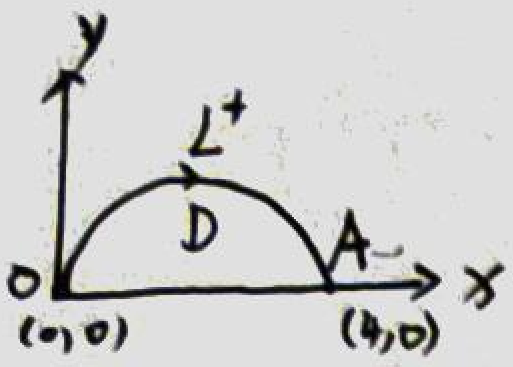
$\therefore \int_{L^+} = -\int_{L_\varepsilon^-} = \int_{L_\varepsilon^+}$

$= \int_0^{2\pi} \frac{[-\sin^2 t + \sin t\cos t - \cos^2 t + \sin t\cos t]\varepsilon^2}{\varepsilon^2} dt$

$= -2\pi$

(4) $\int_{L^+} = -2\pi$

3.(1)



如图，$\int_{L^+} = \left(\int_{L^+} + \int_{AO}\right) + \int_{OA}$

$P(x,y) = 1 + xe^{2y}$

$Q(x,y) = x^2 e^{2y} - y^2$

$\frac{\partial P}{\partial y} = 2xe^{2y}$，$\frac{\partial Q}{\partial x} = 2xe^{2y}$

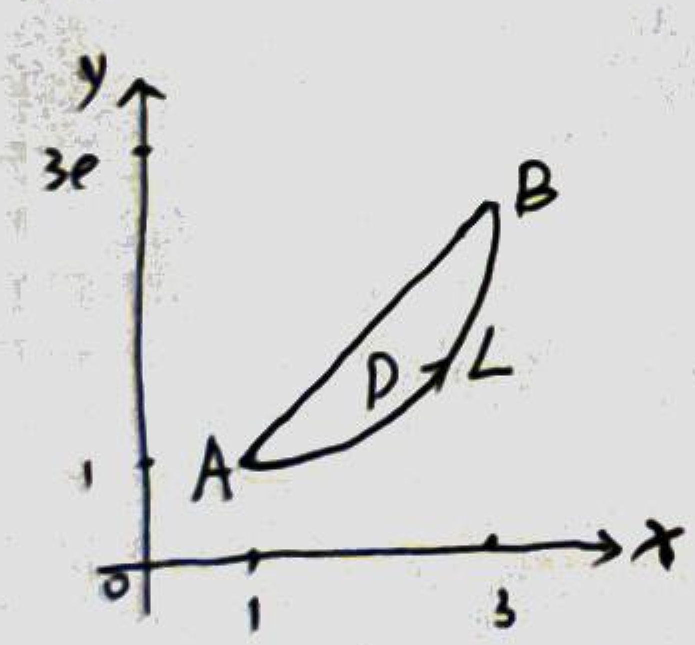
在图中区域D中，$\frac{\partial P}{\partial y}$，$\frac{\partial Q}{\partial x}$ 连续

$$\int_{L^+} + \int_{AO} = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = 0$$

∴ $\int_{L^+} = \int_{OA}$，x从0到4，$y \equiv 0$，$dy = 0$

∴ $\int_{L^+} = \int_0^4 (1+x)\,dx = 12$

(3)



如图，$\int_{L^+} = \left(\int_{L^+} + \int_{BA}\right) + \int_{AB}$

$P(x,y) = \ln\frac{y}{x} - 1$

$Q(x,y) = \frac{x}{y}$

$\frac{\partial P}{\partial y} = \frac{1}{y}$，$\frac{\partial Q}{\partial x} = \frac{1}{y}$

在区域D中，$\frac{\partial P}{\partial y}$，$\frac{\partial Q}{\partial x}$ 均连续

$$\int_{L^+} + \int_{BA} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = 0$$

∴ $\int_{L^+} = \int_{AB}$

$AB: \begin{cases} x = 2t+1 \\ y = (3e-1)t+1 \end{cases}$，$t$从0到1

$dx = 2dt$，$dy = (3e-1)dt$

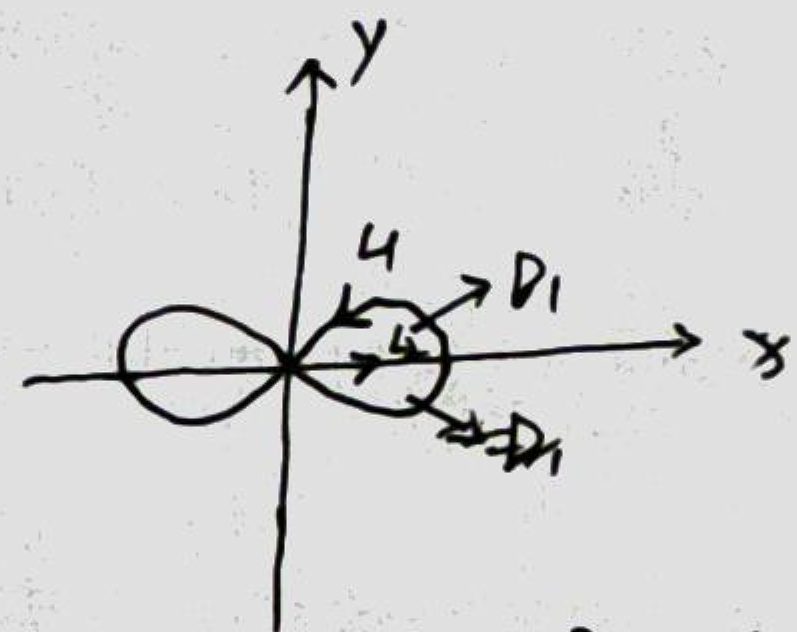
$$\int_{L^+} \left(\ln\frac{y}{x} - 1\right)dx + \frac{x}{y}dy$$

$$= \int_0^1 \left(\ln\frac{(3e-1)t+1}{2t+1} - 1\right)\cdot 2dt + \int_0^1 \frac{2t+1}{(3e-1)t+1}\cdot(3e-1)dt$$

$$= 2\int_0^1 \left(\ln[(3e-1)t+1] - \ln(2t+1) - 1\right)dt + (3e-1)\int_0^1 \frac{2t+1}{(3e-1)t+1}dt$$

$$= 3$$

4.(2)



$$S_{D_1} = \iint_{D_1} dxdy = \frac{1}{2}\int_{L_1} -ydx + xdy + \frac{1}{2}\int_{L_2} -ydx + xdy$$

参数变换：$\begin{cases} x = a\sqrt{\cos 2\theta}\cos\theta \\ y = a\sqrt{\cos 2\theta}\sin\theta \end{cases}$

$dx = -a\frac{\sin 3\theta}{\sqrt{\cos 2\theta}}d\theta$

$dy = a\frac{\cos 3\theta}{\sqrt{\cos 2\theta}}d\theta$

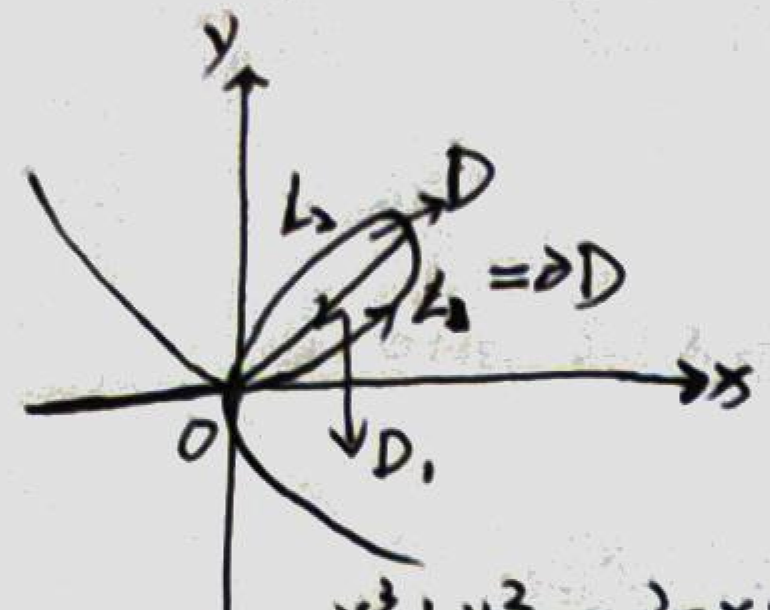
∴ $\int_{L_1} -ydx + xdy$

$$= \left|\int_0^{\frac{\pi}{4}} a^2\sin\theta\sin 3\theta + a^2\cos\theta\cos 3\theta\, d\theta\right|$$

$= \frac{a^2}{2}$，$\int_{L_2} -ydx + xdy = 0$

$S_{D_1} = \frac{a^2}{4}$，$S = 4S_{D_1} = a^2$

(3)



$x^3+y^3=3axy$

$$S_D=\iint_D dxdy=\frac{1}{2}\oint_{L^+}-ydx+xdy$$

用参数方程表示笛卡尔叶形线

$$\begin{cases} x=\frac{3at}{1+t^3} \\ y=\frac{3at^2}{1+t^3} \end{cases}，其中 t=\tan\theta$$

$$S_D=2S_{D_1}$$

$$S_{D_1}=\frac{1}{2}\int_{L_1^+}-ydx+xdy+\frac{1}{2}\int_{L_2^+}-ydx+xdy$$

L_1上：$dx=\frac{a(3-6t^3)}{(1+t^3)^2}dt$

$$dy=-\frac{3at(-2+t^3)}{(1+t^3)^2}dt$$

$$\int_{L_1^+}-ydx+xdy$$

$$=\int_0^1\left[-\frac{3at^2}{1+t^3}\cdot\frac{a(3-6t^3)}{(1+t^3)^2}-\frac{3at}{1+t^3}\cdot\frac{3at(-2+t^3)}{(1+t^3)^2}\right]dt$$

$$=\int_0^1 9a^2\cdot\frac{t^2(1+t^3)}{(1+t^3)^3}dt=9a^2\int_0^1\frac{t^2}{(1+t^3)^2}dt$$

$$=3a^2\int_0^1\frac{dt^3}{(1+t^3)^2}=3a^2\cdot\left(-\frac{1}{1+x}\right)\Big|_0^1$$

$$=\frac{3}{2}a^2$$

L_2：$x=y \quad dx=dy$

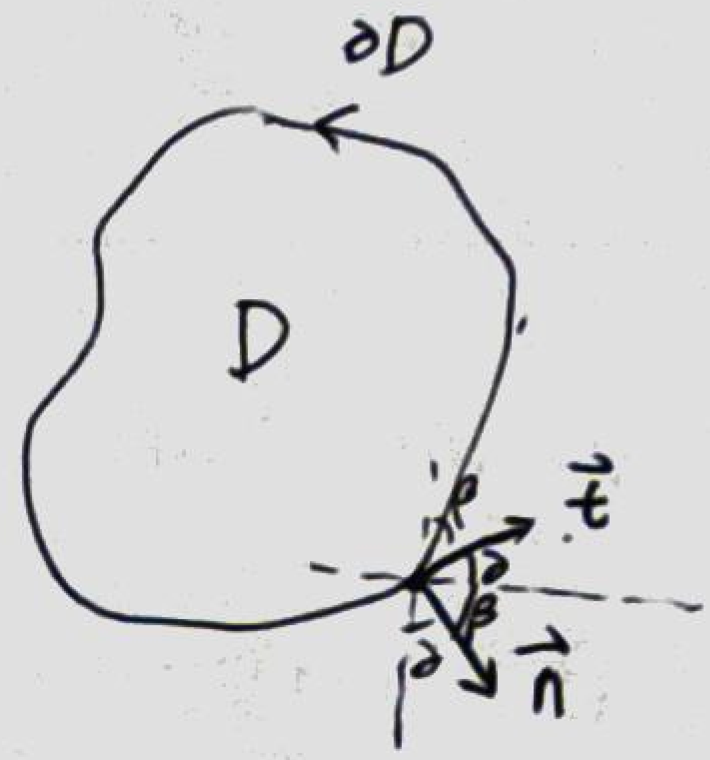
$$\therefore \int_{L_2^+}-ydx+xdy$$

$$=\int_{L_2^+}-xdx+xdx=0$$

$$\therefore S_{D_1}=\frac{3}{4}a^2$$

$$S_D=\frac{3}{2}a^2$$

7.



设 $Q=\frac{\partial f}{\partial x}$，$P=-\frac{\partial f}{\partial y}$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$

$$\iint_D\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)dxdy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

$$=\oint_{\partial D}Pdx+Qdy=\oint_{\partial D}-\frac{\partial f}{\partial y}dx+\frac{\partial f}{\partial x}dy$$

$$\cancel{dt=\frac{dx}{\cos\alpha}=\frac{dy}{\cos\beta}}$$

$$\cancel{n\perp t,\quad dn=\frac{dx}{\cos\beta}=-\frac{dy}{\cos\alpha}}$$

$$\frac{\partial f}{\partial n}=\frac{\partial f}{\partial x}\cos(n,x)+\frac{\partial f}{\partial y}\cos(n,y)$$

$$\cos(n,x)dl=\cos\beta\, dl=dy$$

$$\cos(n,y)dl=-\cos\alpha\, dl=-dx$$

$$\therefore \oint_{\partial D}\frac{\partial f}{\partial n}dl=\oint_{\partial D}\left(\frac{\partial f}{\partial x}\cos(\vec{n},x)+\frac{\partial f}{\partial y}\cos(\vec{n},y)\right)d\vec{l}$$

$$=\oint_{\partial D}-\frac{\partial f}{\partial y}dx+\frac{\partial f}{\partial x}dy$$

$$=\iint_D\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)dxdy$$