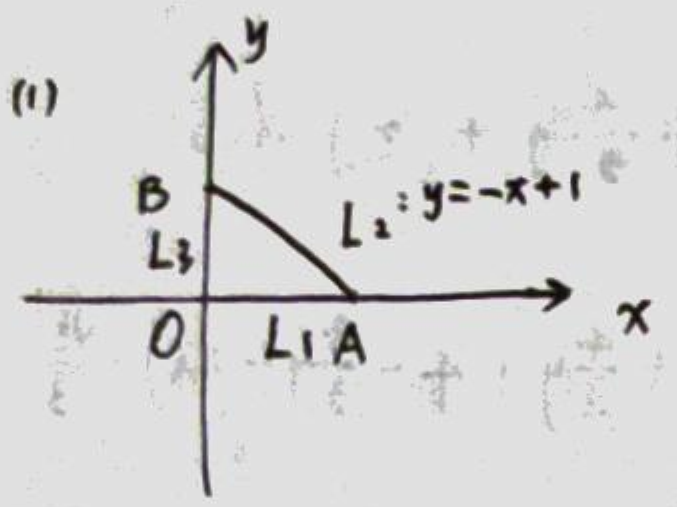
习题 4.2

1. (1)



$$\int_L (x+y)\,dl = \int_{L_1} + \int_{L_2} + \int_{L_3}$$

$$= \int_0^1 x\,dx + \int_0^1 \sqrt{2}\,dx + \int_0^1 y\,dy$$

$$= 1+\sqrt{2}$$

(2) 令 $\begin{cases} x = 1+\cos\theta \\ y = \sin\theta \end{cases}$, $0 \le \theta \le 2\pi$

$$\int_L \sqrt{x^2+y^2}\,dl$$

$$= \int_0^{2\pi} \sqrt{2(1+\cos\theta)} \cdot \sqrt{\sin^2\theta+\cos^2\theta}\,d\theta$$

$$= \int_0^{2\pi} 2\left|\cos\frac{\theta}{2}\right|\,d\theta$$

$$= \int_0^{\pi} 2\cos\frac{\theta}{2}\,d\theta + \int_{\pi}^{2\pi} 2\left(-\cos\frac{\theta}{2}\right)d\theta$$

$$= 4+4 = 8.$$

(4) $$\int_L \left(x^{\frac{4}{3}} + y^{\frac{4}{3}}\right) dl$$

$$= \int_0^{2\pi} a^{\frac{4}{3}} (\cos^4\theta + \sin^4\theta) \cdot \sqrt{a^2(9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta)}\,d\theta$$

$$= 3a^{\frac{7}{3}} \int_0^{2\pi} (\cos^4\theta + \sin^4\theta)\,|\sin\theta\cos\theta|\,d\theta$$

$$= 12a^{\frac{7}{3}} \int_0^{\frac{\pi}{2}} \left(1-\frac{\sin^2 2\theta}{2}\right) \cdot \frac{\sin 2\theta}{2}\,d\theta$$

$$= 4a^{\frac{7}{3}}.$$

2.

(3) $$\int_L xyz\,dl$$

$$= \int_0^1 \frac{\sqrt{2}}{3} t^{\frac{9}{2}} \sqrt{1+2t+t^2}\,dt$$

$$= \frac{\sqrt{2}}{3}\int_0^1 t^{\frac{9}{2}}(t+1)\,dt$$

$$= \frac{\sqrt{2}}{3}\left[\frac{2}{13} + \frac{2}{11}\right] = \frac{16\sqrt{2}}{143}.$$

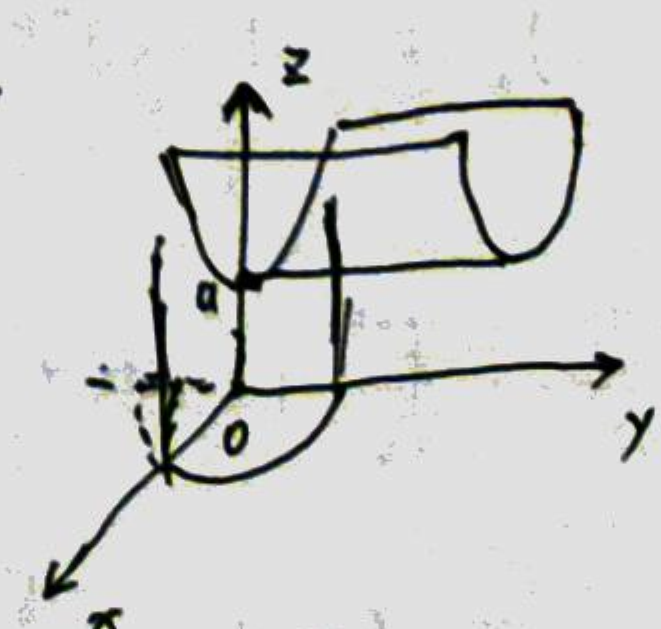
3. (2) $$\int_L dl$$

$$= \int_0^{+\infty} \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2 + e^{-2t}}\,dt$$

$$= \int_0^{+\infty} \sqrt{3}\,e^{-t}\,dt$$

$$= \sqrt{3}.$$

5.



$$S = 4\int_L f(x,y)\,dl$$

$$= 4\int_0^{\frac{\pi}{2}} \left(a+\frac{a^2\cos^2\theta}{a}\right) a\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \left(a^2 + a^2\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 3\pi a^2.$$

习题 4.3

6.

$$\int_L dl = \int_0^{\pi} a\sqrt{(1-\cos t)^2+\sin^2 t}\ dt$$

$$= \int_0^{\pi} a \cdot 2\sin\frac{t}{2}\ dt$$

$$= 4a$$

$$\int_L x\, dl = \int_0^{\pi} 2a^2(t-\sin t)\sin\frac{t}{2}\ dt$$

$$= 2a^2\left(4-\frac{4}{3}\right) = \frac{16a^2}{3}$$

$$\int_L y\, dl = \int_0^{\pi} 2a^2(1-\cos t)\sin\frac{t}{2}\ dt$$

$$= 2a^2\left(2+\frac{2}{3}\right) = \frac{16a^2}{3}$$

$\therefore$ 重心为 $\left(\frac{4a}{3}, \frac{4}{3}a\right)$.

7.

$$J_x = \int_L (y^2+z^2)\ dl$$

$$= \int_0^{2\pi}\left(a^2\sin^2 t + \frac{b^2}{4\pi^2}t^2\right)\cdot\sqrt{a^2+\left(\frac{b}{2\pi}\right)^2}\ dt$$

$$= \left[\pi a^2 + \frac{b^2}{12\pi^2}\cdot(2\pi)^3\right]\cdot\frac{\sqrt{4\pi^2a^2+b^2}}{2\pi}$$

$$= \left(\frac{a^2}{2}+\frac{b^2}{3}\right)\sqrt{4\pi^2a^2+b^2}.$$

1. (2)

$$\iint_S \left(2x+\frac{4}{3}y+z\right) dS$$

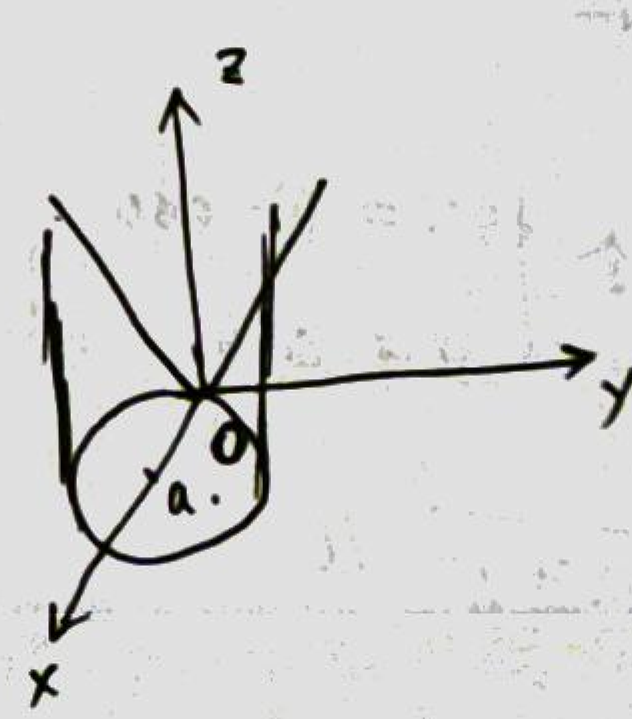
$$= \iint_{D_{xy}}\left(2x+\frac{4}{3}y+4-\frac{4}{3}y-2x\right)\cdot\frac{\sqrt{61}}{3}\ dxdy$$

$$\cancel{= \int_0^2 dx\int_0^{3-\frac{3}{2}x}\frac{4\sqrt{61}}{3}\ dxdy}$$

$$= \frac{4\sqrt{61}}{3}\iint_{D_{xy}} dxdy$$

$$= \frac{4\sqrt{61}}{3}\times 3 = 4\sqrt{61}.$$

(4).



由对称性可知 $\iint_S xy = 0$

$D_{xy} = \{(x,y): x^2+y^2=2ax\}$

$$\iint_S (y+x)z\ dS$$

$$= \iint_{D_{xy}}(y+x)\sqrt{x^2+y^2}\cdot\sqrt{2}\ dxdy$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_0^{2a\cos\theta} r(\sin\theta+\cos\theta)\cdot r\cdot\sqrt{2}\,r\ dr$$

$$= 4\sqrt{2}a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta(\sin\theta+\cos\theta)\ d\theta$$

$$= 4\sqrt{2}a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2\theta)^2\ d\sin\theta$$

$$= \frac{64\sqrt{2}}{15}a^4.$$

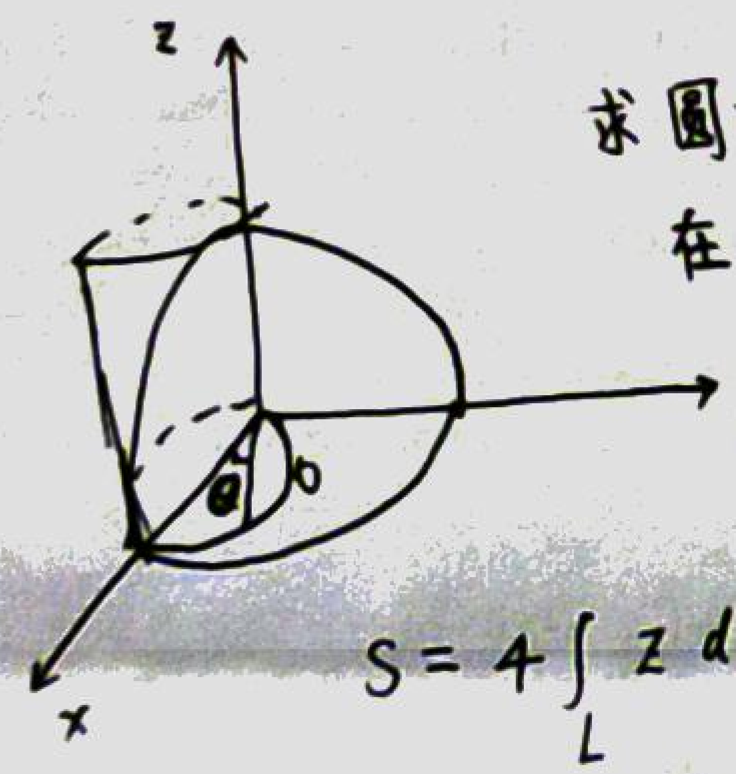
(5) $\iint\limits_S x\,dS$

$$= \iint\limits_{D_{uv}} (u\cos v)\cdot\sqrt{u^2+a^2}\,du\,dv$$

$$= \int_0^r du\int_0^{2\pi} u\sqrt{u^2+a^2}\cos v\,dv$$

$$= 0.$$

2.



求圆柱面面积，

在Oxy平面上投影是一条曲线.

(或投影在Oxz平面上是1/4圆)

$$\begin{cases} x = a\cos^2\theta \\ y = a\sin\theta\cos\theta \end{cases}$$

$$S = 4\int_L z\,dl$$

由对称性，

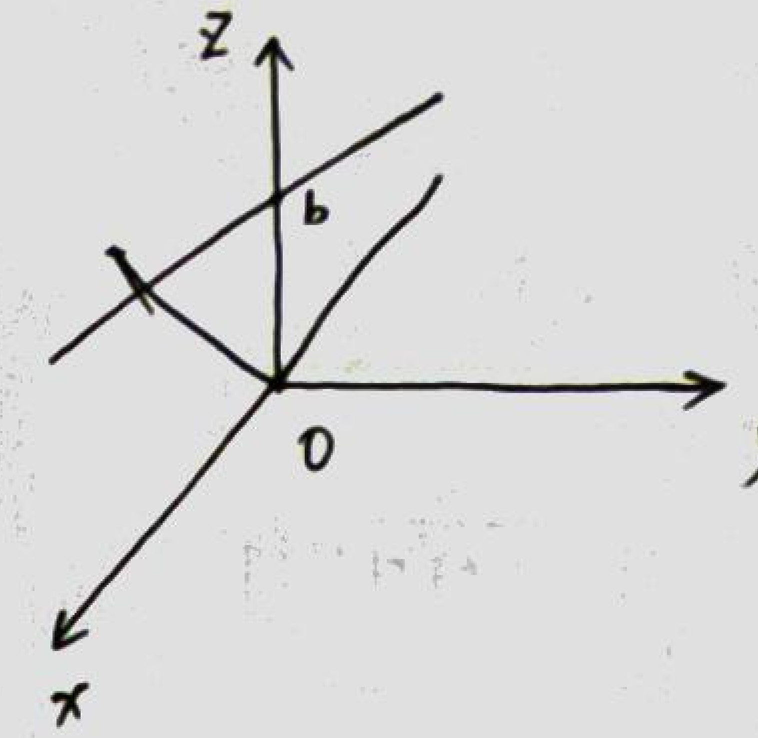
$$= 4\int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2\cos^2\theta}\cdot\sqrt{(dx)^2+(dy)^2}$$

$$= 4a^2\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta$$

$$= 4a^2$$

$$\boxed{\begin{aligned} S &= 4\iint\limits_S 1\,dS \\ &= 4\iint\limits_{D_{xy}}\sqrt{1+\frac{x^2+y^2}{a^2-x^2-y^2}}\,dx\,dy \\ &= 4\iint\limits_{D_{xy}}\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy \\ &= 4\int_0^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}\frac{a}{\sqrt{a^2-r^2}}\,r\,dr \\ &= 4\int_0^{\frac{\pi}{2}} a^2(1-\sin\theta)\,d\theta \\ &= 4a^2\left(\frac{\pi}{2}-1\right) \end{aligned}}$$

5.



$$J = \sigma_0\iint\limits_S y^2+(z-b)^2\,dS$$

$$= \sigma_0\iint\limits_{D_{xy}}\left[y^2+\left(\frac{b}{a}\sqrt{x^2+y^2}-b\right)^2\right]\sqrt{1+\frac{b^2}{a^2}}\,dx\,dy$$

$$= \frac{\sigma_0\sqrt{a^2+b^2}}{a}\int_0^{2\pi}d\theta\int_0^a\left(r^2\sin^2\theta+\left(\frac{b}{a}r-b\right)^2\right)r\,dr$$

$$= a\sigma_0\pi\sqrt{a^2+b^2}\left(\frac{1}{4}a^2+\frac{1}{6}b^2\right)$$

6. (1) $\iint\limits_S ds = \frac{1}{8}\cdot 4\pi a^2 = \frac{\pi a^2}{2}$

$$\iint\limits_S x\,dS = \iint\limits_{D_{xy}} x\cdot\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}}d\theta\int_0^a r\cos\theta\cdot\frac{a}{\sqrt{a^2-r^2}}\,r\,dr$$

$$= \frac{\pi}{4}a^3$$

$$\iint\limits_S y\,dS = \frac{\pi a^3}{4}$$

$$\iint\limits_S z\,dS = \iint\limits_{D_{xy}}\sqrt{a^2-(x^2+y^2)}\cdot\frac{a}{\sqrt{a^2-(x^2+y^2)}}\,dx\,dy$$

$$= \frac{\pi a^3}{4}$$

∴ 球面在第一卦限部分重心为 $\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right)$

(2) 在上半球面，$\iint\limits_S z\,dS = \pi a^3$. $\iint\limits_S dS = 2\pi a^3$

由对称性知上半球面重心为 $\left(0,0,\frac{a}{2}\right)$

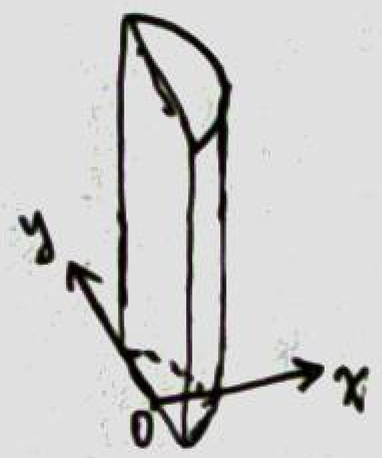
9.

$$\iint_S 1\, dS$$

$$= \iint_{D_{xy}} \sqrt{x^2+y^2+1}\,dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^a r\sqrt{r^2+1}\,dr$$

$$= \frac{2\pi}{3}\left[(1+a^2)^{\frac{3}{2}}-1\right]$$



补充：习题3.5 第7题.

由对称性知在y轴方向压力 $F_y=0$.

$$\text{而 } F_x = \int_0^h \rho g l\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos\theta\, d\theta\right) dl$$

$$= \int_0^h 2\rho g a l\, dl$$

$$= \rho g a h^2.$$

11. 证明：S 应改为 $\{x^2+y^2+z^2=1\}$. 不改动则结果为

$$2\pi a\int_{-a}^{a} f(\sqrt{a^2+b^2+c^2}\,t)\,dt$$

令

~~则 $\iint_S f(ax+by+cz)\,dS$~~

令 $u=\dfrac{ax+by+cz}{\sqrt{a^2+b^2+c^2}}$

$$= \frac{1}{\sqrt{a^2+b^2+c^2}}(a,b,c)\,(x,y,z)^T$$

(Schmidt正交化)

令 $\alpha_1 = \frac{1}{\sqrt{a^2+b^2+c^2}}(a,b,c)$，则可作向量 α_2，α_3 使

$A=(\alpha_1,\alpha_2,\alpha_3)^T$ 是正交矩阵

令 $v=\alpha_2(x,y,z)^T$，$w=\alpha_3(x,y,z)^T$

则 $u^2+v^2+w^2=(u,v,w)(u,v,w)^T=(x,y,z)(\alpha_1,\alpha_2,\alpha_3)^T(\alpha_1,\alpha_2,\alpha_3)(x,y,z)^T$

$$=(x^2+y^2+z^2)=1$$

$$\therefore I=\iint_S f(ax+by+cz)\,dS = \iint_{\{u^2+v^2+w^2=1\}} f(\sqrt{a^2+b^2+c^2}\,u)\cdot 1\, dS_1$$

令 $\begin{cases} v=\sin\theta\cos\varphi \\ w=\sin\theta\sin\varphi & 0\le\varphi\le 2\pi \\ u=\cos\theta & 0\le\theta\le\pi \end{cases}$

$$\therefore I=\int_0^{2\pi} d\varphi\int_0^{\pi} f(\sqrt{a^2+b^2+c^2}\cos\theta)\cdot\sin\theta\, d\theta$$

令 $t=\cos\theta$

$$= 2\pi\int_1^{-1} f(\sqrt{a^2+b^2+c^2}\,t)\,(-1)\,dt$$

$$= 2\pi\int_{-1}^{1} f(\sqrt{a^2+b^2+c^2}\,t)\,dt.$$