作业

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第一题

(1)

原本的式子是 $P(error|x) = \min(P(\omega_1|x), P(\omega_2|x))$，现改为 $2P(\omega_1|x)P(\omega_2|x)$。

设 $P(\omega_1|x) = t, t \in [0, 1]$，由于 $P(\omega_1|x) + P(\omega_2|x) = 1$，故 $P(\omega_2|x) = 1 - t$。

$\therefore 2P(\omega_1|x)P(\omega_2|x) = 2t(1-t), \min(P(\omega_1|x), P(\omega_2|x)) = \min(t, 1-t)$

当 $0.5 \le t \le 1$ 时，$2t \ge 1, 2t(1-t) \ge 1 \times (1-t) = 1-t = \min(t, 1-t)$。

当 $0 \le t < 0.5$ 时，$2(1-t) \ge 1, 2t(1-t) \ge 1 \times t = t = \min(t, 1-t)$

综上所述，$2P(\omega_|x)P(\omega_2|x) \ge P(error|x)$

$$\therefore \int_{-\infty}^{\infty} 2P(\omega_1|x)P(\omega_2|x)P(x)dx \ge \int_{-\infty}^{\infty} \min(P(\omega_1|x), P(\omega_2|x))P(x)dx$$

(2)

不妨令 $\forall x, P(\omega_1|x) = P(\omega_2|x) = \frac{1}{2}$，那么按照原来的式子计算

$$P(error) = \int_{-\infty}^{\infty} \min(P(\omega_1|x), P(\omega_2|x))P(x)dx = \int_{-\infty}^{\infty} \frac{1}{2}P(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} P(x)dx = \frac{1}{2}$$

按照新式子计算的话

$$P(error) = \int_{-\infty}^{\infty} \alpha P(\omega_1|x)P(\omega_2|x)P(x)dx = \int_{-\infty}^{\infty} \frac{\alpha}{4}P(x)dx = \frac{\alpha}{4}\int_{-\infty}^{\infty} P(x)dx = \frac{\alpha}{4}$$

因为 $\alpha < 2$，故 $\frac{\alpha}{4} < \frac{1}{2}$，故这种算法无法得到误差上界。

(3)

类似 (1) 问，设 $P(\omega_1|x)=t, t\in[0,1]$，由于 $P(\omega_1|x)+P(\omega_2|x)=1$，故 $P(\omega_2|x)=1-t$。

由于 $t\in[0,1]$，故 $0\le t\le 1, 0\le 1-t\le 1$，从而得出 $t(1-t)\le t, t(1-t)\le 1-t$，即 $t(1-t)\le \min(t,1-t)$

$$\therefore \int_{-\infty}^{\infty} P(\omega_1|x)P(\omega_2|x)P(x)dx \le \int_{-\infty}^{\infty} \min(P(\omega_1|x),P(\omega_2|x))P(x)dx$$

(4)

不妨令 $\forall x, P(\omega_1|x)=\frac{1+\frac{1}{\beta}}{2}, P(\omega_2|x)=1-P(\omega_1|x)$。

这样的话，$\frac{1}{\beta}<P(\omega_1|x)<1, \beta P(\omega_1|x)>\beta\frac{1}{\beta}=1$，故 $\beta P(\omega_1|x)P(\omega_2|x)>P(\omega_2|x)\ge \min(P(\omega_1|x),P(\omega_2|x))$

$$\therefore \int_{-\infty}^{\infty} \beta P(\omega_1|x)P(\omega_2|x)P(x)dx > \int_{-\infty}^{\infty} \min(P(\omega_1|x),P(\omega_2|x))P(x)dx$$

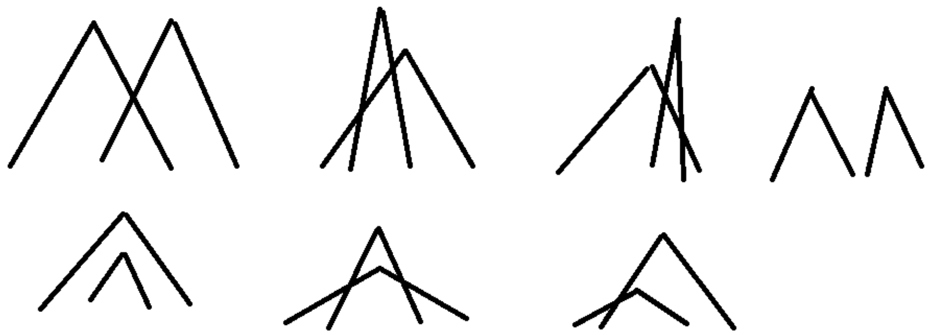
因此这种算法无法得到误差下界。

第二题

(1)

如果我们作出 $P(\omega_i|x)$ 的图像，就会得到三条折线，那么最优决策点 x_1^*, x_2^* 必然是折线的交点。

但是折线的交点有很多种情况：

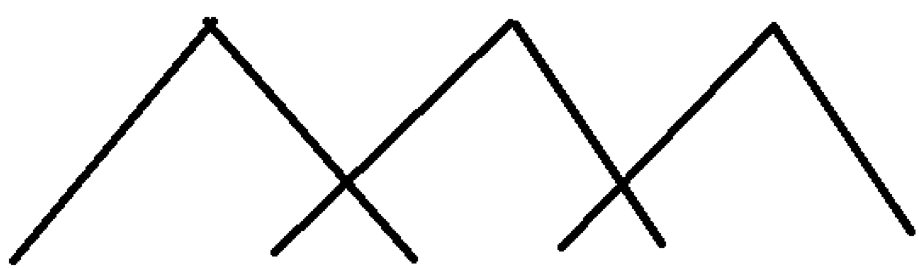


如上图，这么多种情况会给我们的讨论带来极大的麻烦，他们有的甚至会导致最优决策点不止两个。

为了方便起见，我们希望所有的情况都是形如第一种的。为此，我们给 δ 增加一些约束条件，让函数的分布曲线唯一：

$$\mu_1-\delta_1<\mu_1<\mu_2-\delta_2<\mu_1+\delta_1<\mu_2<\mu_3-\delta_3<\mu_2+\delta_2<\mu_3<\mu_3+\delta_3 \tag{1}$$

与之对应的，函数图像将长这样：



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这是一种比较简单的情形，我们只用求出 ω_1 右半部分折线与 ω_2 左半部分折线的交点和 ω_2 右半部分折线与 ω_3 左半部分折线的交点即可。

根据 x_1^*, x_2^* 的定义，此时显然有 $\mu_2-\delta_2 \le x_1^* \le \mu_1+\delta_1, \mu_3-\delta_3 \le x_2^* \le \mu_2+\delta_2$。

$P(\omega_i|x)=P(x|\omega_i)P(\omega_i)=T(\mu_i,\delta_i)P(\omega_i)=\frac{(\delta_i-|x-\mu_i|)}{\delta_i^2}P(\omega_i)$

故

$$\begin{cases}\frac{\delta_1-x_1^*+\mu_1}{\delta_1^2}P(\omega_1)=\frac{\delta_2-\mu_2+x_1^*}{\delta_2^2}P(\omega_2)\\ \frac{\delta_2-x_2^*+\mu_2}{\delta_2^2}P(\omega_2)=\frac{\delta_3-\mu_3+x_2^*}{\delta_3^2}P(\omega_3)\end{cases}$$

解得

$$\begin{cases}x_1^*=[\frac{\delta_1+\mu_1}{\delta_1^2}P(\omega_1)-\frac{\delta_2-\mu_2}{\delta_2^2}P(\omega_2)]/[\frac{P(\omega_1)}{\delta_1^2}+\frac{P(\omega_2}{\delta_2^2}]\\ x_2^*=[\frac{\delta_2+\mu_2}{\delta_2^2}P(\omega_2)-\frac{\delta_3-\mu_3}{\delta_3^2}P(\omega_3)]/[\frac{P(\omega_2)}{\delta_2^2}+\frac{P(\omega_3}{\delta_3^2}]\end{cases}$$

(2)

$$\begin{aligned}R=&\int_{R_1}[\lambda_{11}P(\omega_1)P(x|\omega_1)+\lambda_{12}P(\omega_2)P(x|\omega_2)+\lambda_{13}P(\omega_3)P(x|\omega_3)]dx\\ &+\int_{R_2}[\lambda_{21}P(\omega_1)P(x|\omega_1)+\lambda_{22}P(\omega_2)P(x|\omega_2)+\lambda_{23}P(\omega_3)P(x|\omega_3)]dx\\ &+\int_{R_3}[\lambda_{31}P(\omega_1)P(x|\omega_1)+\lambda_{32}P(\omega_2)P(x|\omega_2)+\lambda_{33}P(\omega_3)P(x|\omega_3)]dx\\ &=P(\omega_1)[\lambda_{11}\int_{R_1}P(x|\omega_1)dx+\lambda_{21}\int_{R_2}P(x|\omega_1)dx+\lambda_{31}\int_{R_3}P(x|\omega_1)dx]\\ &+P(\omega_2)[\lambda_{12}\int_{R_1}P(x|\omega_2)dx+\lambda_{22}\int_{R_2}P(x|\omega_2)dx+\lambda_{32}\int_{R_3}P(x|\omega_2)dx]\\ &+(1-P(\omega_1)-P(\omega_2)[\lambda_{13}\int_{R_1}P(x|\omega_3)dx+\lambda_{23}\int_{R_2}P(x|\omega_3)dx+\lambda_{33}\int_{R_3}P(x|\omega_3)dx]\\ &=P(\omega_1)A+P(\omega_2)B+C\end{aligned}$$

其中：

$$\begin{aligned}A=&[\lambda_{11}\int_{R_1}P(x|\omega_1)dx+\lambda_{21}\int_{R_2}P(x|\omega_1)dx+\lambda_{31}\int_{R_3}P(x|\omega_1)dx]\\ &-[\lambda_{13}\int_{R_1}P(x|\omega_3)dx+\lambda_{23}\int_{R_2}P(x|\omega_3)dx+\lambda_{33}\int_{R_3}P(x|\omega_3)dx]\\ B=&[\lambda_{12}\int_{R_1}P(x|\omega_2)dx+\lambda_{22}\int_{R_2}P(x|\omega_2)dx+\lambda_{32}\int_{R_3}P(x|\omega_2)dx]\\ &-[\lambda_{13}\int_{R_1}P(x|\omega_3)dx+\lambda_{23}\int_{R_2}P(x|\omega_3)dx+\lambda_{33}\int_{R_3}P(x|\omega_3)dx]\\ C=&[\lambda_{13}\int_{R_1}P(x|\omega_3)dx+\lambda_{23}\int_{R_2}P(x|\omega_3)dx+\lambda_{33}\int_{R_3}P(x|\omega_3)dx]\end{aligned}$$

不妨设 $S_{ij}=\int_{R_i}P(x|\omega_j)dx$，那么有

$$
\begin{aligned}
A=&[\lambda_{11}S_{11}+\lambda_{21}S_{21}+\lambda_{31}S_{31}]-[\lambda_{13}S_{13}+\lambda_{23}S_{23}+\lambda_{33}S_{33}]\\
B=&[\lambda_{12}S_{12}+\lambda_{22}S_{22}+\lambda_{32}S_{32}]-[\lambda_{13}S_{13}+\lambda_{23}S_{23}+\lambda_{33}S_{33}]\\
C=&[\lambda_{13}S_{13}+\lambda_{23}S_{23}+\lambda_{33}S_{33}]
\end{aligned}
$$

我们仍然采用 (1) 式的条件，来计算 S_{ij}，从图中可以看出，$\mu_2-\delta_2\le x_1^*\le\mu_1+\delta_1,\mu_3-\delta_3\le x_2^*\le\mu_2+\delta_2$

$$
\begin{cases}
S_{21}=\int_{R_2}P(x|\omega_1)dx=\int_{x_1^*}^{\mu_1+\delta_1}\frac{\delta_1-x+\mu_1}{\delta_1^2}dx=\frac{(\mu_1+\delta_1-x_1^*)^2}{2\delta_1^2}\\
S_{31}=0\\
S_{11}=1-S_{21}-S_{31}=1-\frac{(\mu_1+\delta_1-x_1^*)^2}{2\delta_1^2}\\
S_{23}=\int_{R_2}P(x|\omega_3)dx=\int_{\mu_3-\delta_3}^{x_2^*}\frac{\delta_3-\mu_3+x}{\delta_3^2}dx=\frac{(x_2^*-\mu_3+\delta_3)^2}{2\delta_3^2}\\
S_{13}=0\\
S_{33}=1-S_{23}-S_{13}=1-\frac{(x_2^*-\mu_3+\delta_3)^2}{2\delta_3^2}\\
S_{12}=\int_{R_1}P(x|\omega_2)dx=\int_{\mu_2-\delta_2}^{x_1^*}\frac{\delta_2-x+\mu_2}{\delta_2^2}dx=\frac{(x_1^*-\mu_2+\delta_2)^2}{2\delta_2^2}\\
S_{32}=\int_{R_3}P(x|\omega_2)dx=\int_{x_2^*}^{\mu_2+\delta_2}\frac{\delta_2-x+\mu_2}{\delta_2^2}dx=\frac{(\mu_2+\delta_2-x_2^*)^2}{2\delta_2^2}\\
S_{22}=1-S_{12}-S_{32}=1-\frac{(x_1^*-\mu_2+\delta_2)^2}{2\delta_2^2}-\frac{(\mu_2+\delta_2-x_2^*)^2}{2\delta_2^2}
\end{cases}
$$

由于要求的是极小化极大决策，故这里满足

$$
\lambda_{ij}=\begin{cases}1 & i=j\\ 0 & i\ne j\end{cases}
$$

代入后有方程组

$$
\begin{cases}
\frac{\mu_1+\delta_1-x_1^*}{\delta_1}=\frac{x_2^*-\mu_3+\delta_3}{\delta_3}\\
\frac{(x_1^*-\mu_2+\delta_2)^2+(\mu_2+\delta_2-x_2^*)^2}{\delta_2^2}=\frac{(x_2^*-\mu_3+\delta_3)^2}{\delta_3^2}
\end{cases}
$$

解此二元方程组即可。

联立可解得 x_1^*,x_2^*。

(3)

三个折线的定义域区间分别是 $[-1,1],[0,1],[0,2]$，根据 x_1^*,x_2^* 的定义显然有 $0\le x_1^*\le x_2^*\le 1$，可以由此计算 S_{i1} 和 S_{i3}

$$\begin{cases} S_{21} = \int_{x_1^*}^{x_2^*} \frac{1-(x-0)}{1^2} dx = \frac{(x_2^*-x_1^*)(2-x_1^*-x_2^*)}{2} \\ S_{31} = \int_{x_2^*}^{1} \frac{1-(x-0)}{1^2} dx = \frac{(1-x_2^*)^2}{2} \\ S_{11} = 1 - S_{21} - S_{31} = 1 - \frac{(x_1^*-1)^2}{2} \\ S_{13} = \int_{0}^{x_1^*} \frac{1-(1-x_1^*)}{1^2} dx = \frac{(x_1^*)^2}{2} \\ S_{23} = \int_{x_1^*}^{x_2^*} \frac{1-(1-x)}{1^2} dx = \frac{(x_2^*)^2-(x_1^*)^2}{2} \\ S_{33} = 1 - S_{13} - S_{23} = 1 - \frac{(x_2^*)^2}{2} \end{cases}$$

为求极小化极大决策，取

$$\lambda_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

由于 $S_{21} + S_{31} - S_{13} - S_{23} = 0$，故 $(x_1^*)^2 - (x_2^*)^2 - 2x_1^* + 1 = 0$，解得 $(x_1^* - 1)^2 = (x_2^*)^2$

考虑到 $0 \leq x_1^* \leq x_2^* \leq 1$，故有 $0 \leq x_1^* \leq 0.5 \leq x_2^* \leq 1$, 且$x_1^* + x_2^* = 1$。

从而可以推出

$$\begin{cases} S_{12} = \int_{0}^{x_1^*} \frac{0.5-(0.5-x)}{0.5^2} dx = 2(x_1^*)^2 \\ S_{32} = \int_{x_2^*}^{1} \frac{0.5-(x-0.5)}{0.5^2} dx = 2(1-x_2^*)^2 \\ S_{22} = 1 - S_{12} - S_{32} = 1 - 2(x_1^*)^2 - 2(x_2^*-1)^2 \end{cases}$$

由于 $S_{12} + S_{32} - S_{13} - S_{23} = 0, x_2^* = 1 - x_1^*, 0 \leq x_1^* \leq 0.5$，解得

$$\begin{cases} x_1^* = \frac{-2+4\sqrt{2}}{14} \\ x_2^* = \frac{16-4\sqrt{2}}{14} \end{cases}$$

(4)

极小化极大风险为 $[\lambda_{13}S_{13} + \lambda_{23}S_{23} + \lambda_{33}S_{33}] = S_{13} + S_{23} = \frac{(x_1^*)^2}{2} + \frac{(x_2^*)^2-(x_1^*)^2}{2} = \frac{(x_2^*)^2}{2}$。

由于 $x_2^* = \frac{16-4\sqrt{2}}{14}$，故极小化极大风险为 0.2729。