1 Chains and Antichains

1.1 Maximality and maximum-, uh, -ness?

To review, the definitions of a chain and antichain:

Definition 1. A *chain* is a totally ordered subset of a poset S; an *antichain* is a subset of a poset S in which any two distinct elements are incomparable.

Now, we have two distinct concepts of a chain/antichain being "as large as possible". One of those concepts is "not easily extendable" – that is to say, a chain or antichain which can't have elements added to make it a larger chain/antichain:

Definition 2. A *maximal* chain (antichain) is one that is not a proper subset of another chain (antichain).

Alternatively, a chain C is *maximal* in a poset $(S, \preceq)$ if no element of $S - C$ is comparable to every element of C; an antichain A is *maximal* if no element of $S - A$ is incomparable to every element of A.

Our other definition of largeness is that of simply being of the greatest size in a poset.

Definition 3. A *maximum* or *longest chain* (*largest antichain*) is one which is of the greatest size possible. The size of the longest chain is known as a poset's *height*. The size of the largest antichain is known as a poset's *width*.

Note, trivially, that a maximum chain/antichain must be maximal: if C is maximum, than there are no chains of size greater than $|C|$; if C were a proper subset of a chain D, then $|D| > |C|$ would contradict the above assertion.

As an example, let us consider the poset $(\{1, 2, 3, 4, 5, 6, 9, 12, 18\}, |)$, recalling that $a \mid b$ is a is a divisor of b.

Then we may note that $\{1, 2, 4, 12\}$ is a maximum chain — there are no chains of length greater than 4. The maximum chain is not unique: $\{1, 3, 6, 18\}$ is also a maximum chain, as is $\{1, 2, 6, 12\}$ and several others. An example of a maximal chain which is *not* maximum is $\{1, 5\}$: this has only two elements, and thus is not maximum, but no more elements of S can be added to it, since 5 is incomparable with everything except 1 and 5. An example of a nonmaximal chain would be $\{1, 3, 18\}$, because it is a subset of a larger chain (in fact, two larger chains, since either 6 or 9 could be included).

Looking at antichains, the largest we could find would have size 4: $\{4, 6, 9, 5\}$ would be an example, and is in fact the unique maximum antichain (uniqueness is not guaranteed; it's just how this example works out). We could find several maximal antichains which are not maximum, such as $\{2, 3, 5\}$ or $\{12, 18\}$, or even $\{1\}$ (which would be maximal because everything is comparable to 1!). And of course, nonmaximal antichains can be found too, such as $\{12, 5\}$, which is a subset of the larger antichain $\{12, 9, 5\}$.

There are two useful simple observations to be made about maximality, one for chains, and one for antichains.

Proposition 1. *A maximal chain in a finite nonempty poset must contain a maximal element of S (and a minimal element).*

Proof. Suppose $C \subset P$ is a maximal chain, Since $(C, \preceq)$ is a finite nonempty poset in its own right, it has some maximal element x; since C is totally ordered, the nonexistence of any y in C such that $x \preceq y$ implies that $y \preceq x$ for all $y \in C$, so x is in fact a greatest element of C (if not necessarily of S). We have two possibilities to address: x may be maximal in S, or it may not. If it is maximal, our condition has been shown. If it is not maximal, there is a $z \in S$ such that $x \preceq z$. Then, for all $y \in C$, $y \preceq x \preceq z$, so $y \preceq z$, so $C \cup \{z\}$ is totally ordered, contradicting C's maximality. The proof for minimal elements proceeds along similar lines. □

Proposition 2. *The set of maximal elements of a finite poset S is a maximal antichain; likewise, the set of minimal elements of a finite poset S is a maximal antichain.*

Proof. Note that this is trivially true if S is empty; henceforth, we will consider the case where S has at elast one element.

Let A be the set of maximal elements of S. We shall first prove that A is an antichain, and then that any augmentation of A by adding an element is *not* an antichain.

Consider distinct elements $x, y \in A$. Since x is a maximal element and $y \neq x$, maximality guarantees $x \not\preceq y$, Likewise, since y is maximal anddistinct from x, $y \not\preceq x$. Thus, x and y are incomparable. Since this is true of arbitrarily chosen distinct elements of A, all distinct elements of A are incomparable and A is an antichain.

Now, consider $z_0 \in S - A$; that is, z_0 is a non-maximal element of S. We shall show that $A \cup \{z_0\}$ is not an antichain. Since z_0 is nonmaximal, there is some $z_1 \neq z_0$ such that $z_0 \preceq z_1$. If z_1 is maximal, it is in A; if it is nonmaximal, there is a $z_2 \neq z_1$ such that $z_1 \preceq z_2$. We continue along these lines until we get either a z_k that is maximal, or an infinite ascending sequence $z_0 \preceq z_1 \preceq \cdots$. The latter situation was shown last week to be impossible in a finite poset; thus some z_k is maximal, so $z_k \in A$ and $z_0 \preceq z_1 \preceq \cdots \preceq z_k$ gives $z_0 \preceq z_k$ by transitivity. Since z_0 and z_k are comparable, and $z_k \in A$, $A \cup \{z_0\}$ is not an antichain.

The proof for minimal elements proceeds along similar lines. □

1.2 Limits on width and height; Dilworth's Theorem

One interesting result we see playing with maximum chains and antichains is that they can't both be very small. We can easily build posets with large height and small width (e.g. $(\{1, 2, 3, \ldots, n\}, \leq)$), small height and large width $(\{1, 2, 3, \ldots, n\}, =)$, and even with large height and width $(\{a_1, a_2, a_3, \ldots, b_2, b_3, \ldots b_n\},$ $)$ with $a_i \preceq a_j$ when $i \leq j$, and all other terms incomparable. But keeping both the height and width of an arbitrarily large poset down seem hard (we can kind of do so, with $\{1, 2, 3, n\}^2$ subject to $(a, b) \preceq (c, d)$ if $a \leq c$ and $b = d$ – this has n^2 elements and length and width of n.

We shall show that this is in fact the best we can do.

Proposition 3. *Suppose P is partitioned into a finite set of chains $C_1, C_2, \ldots, C_n$. If A is an antichain, then there is at most one element of A in each C_i; thus $n \geq |A|$.*

Proof. This result is actually quite trivial: suppose two distinct elements x and y of A were both in the same chain C_i. membership in A would force x and y to be incomparable; membership in C_i would force them to be comparable. Thus, $|A \cap C_i| \leq 1$ for all i. □

We thus know that any antichain has size no more than the size of any partition of the underlying poset into chains. However, if we take the largest antichain possible, and the smallest chain-partition possible, this inequality becomes an equality:

Theorem 1 (Dilworth 1948, Galvin 1994)**.** *If A is a largest antichain in a finite poset $(S, \preceq)$, then there is a partition of S into chains $C_1 \cup C_2 \cup \cdots C_n$ such that $n = |A|$. Furthermore, each C_i contains exactly one element of A, and there is no partition of S into fewer than n chains.*

Proof. We will prove this by induction on $|S|$. In the base-cases $|S| = 0$ or $|S| = 1$, $A = S$ with trivial associated chain-partitions. We shall thus proceed with an inductive step on $|S| \geq 2$.

Let x be a maximal element of S. $(S - \{x\}, \preceq)$ is a poset with $|S| - 1$ elements, so by the indictive hypothesis, given that $S - \{x\}$ has width k, $S - \{x\}$ has a partition into chains $C_1 \cup C_2 \cup \cdots C_k$, and some finite number of antichains $A_1, A_2, \ldots, A_r$ of size k. By the proposition above, each intersection $C_i \cap A_j$ consists of at most one element, and since $(C_1 \cup C_2 \cdots \cup C_n) \cap A_j = k$,each intersection consists of exactly one element. Thus, we may denote $C_i \cap A_j = \{a_{ij}\}$. Let $a_i = \max_j a_{ij}$ (note: since all the $a_{ij} \in C_j$, a totally ordered set, all elements are comparable, and a maximum among them is a well-defined concept), and let $A' = \{a_1, a_2, \ldots, a_k\}$.

First, we must prove that A' is itself an antichain. Consider two elements a_i and $a_{i'}$ of A'. By the construction of a_i, these will be some a_{ij} and $a_{i'j'}$ respectively. Then, since $a_{ij} = \max_k a_{ik}$, we know that $a_{ij'} \preceq a_{ij}$. Since $a_{ij'}$ and $a_{i'j'}$ are both in $A_{j'}$, they are incomparable. Thus, $a_{ij} \not\preceq a_{i'j'}$, since if it were so, we would have the transitive comparison $a_{ij'} \preceq a_{ij} \preceq a_{i'j'}$. We may similarly show that $a_{i'j'} \not\preceq a_{ij}$, so a_i and $a_{i'}$ are incomparable, so A' is an antichain.

Now that we have constructed an antichain in $S - \{x\}$, we have two cases to deal with:

Case I: x is incomparable to every element of A'. Then $A = A' \cup \{x\}$ is an antichain. It is clearly a maximum antichain, having size $k + 1$, since if S contained an antichain of size $k + 2$ or larger, then removal of x from the antichain would yield an antichains in $S - \{x\}$ of size $k + 1$, which would be impossible. Thus $A = A' \cup \{x\}$ and the chain-partition $S = C_1 \cup C_2 \cup \cdots \cup C_k \cup \{x\}$ would satisfy the conditions of the theorem.

Case II: x is comparable to some element of A'. There is some $a_i \in A'$ such that a_i and x are comparable. Since x is maximal, $x \not\preceq a_i$, so instead, the comparison between x and a_i is $a_i \preceq x$. By the definition of a_i, a_i is the largest element of C_i lying in *any* antichain of size k in $S - \{x\}$. Now, let us consider $C = \{a_{i1}, a_{i2}, \ldots, a_{ir}\} \cup \{x\}$, with a_{ij} as defined above. Since $\{a_{i1}, a_{i2}, \ldots, a_{ir}\} \subseteq C_i$, it is a chain, and since every $a_{ij} \preceq a_i \preceq x$, C will be a chain as well. Now, let us consider the poset $(S - C, \preceq)$. It contains no antichains of size k, because every antichain A_j of size k in $S - \{x\}$ contained an element a_{ij} of C; however, it contains several antichains of size $k - 1$, since each $A_j - \{a_{ij}\}$ lies in $S - C$. Thus, by the inductive hypothesis, since $S - C$ has a width of $k - 1$, $S - C$ is decomposable into a partition of $k - 1$ chains $C'_1 \cup C'_2 \cup C'_3 \cdots \cup C'_{k-1}$. Then, S has decomposition into k chains $C'_1 \cup C'_2 \cup C'_3 \cdots \cup C'_{k-1} \cup C$. By the proposition above, this means no antichain in S has more than k elements, so $A = A'$ is a maximum antichain. □

We get some rather nice results from this:

Corollary 1. *If $|S| > mn$, then S has either height of at least $m + 1$ or width of at least $n + 1$.*

Proof. Suppose the longest chain in S is of size k, and the largest antichain is of size ℓ. By Dilworth's Theorem, S is partitionable into ℓ chains C_1, C_2, …, C_ℓ. Since they form a partition of S, $|C_1| + |C_2| + \cdots + |C_\ell| = |S|$; since the largest chain in S is of size k, we know each $|C_i| \leq k$, so $|S| = |C_1| + |C_2| + \cdots + |C_\ell| \leq \ell k$. If both $\ell \leq n$ and $k \leq m$, then $|S| \leq \ell k \leq mn$, contradicting the fact that $|S| > mn$. Thus, either $\ell > n$ or $k > m$. ☐

Corollary 2. *Every poset S with n elements has either height or width of size $\lfloor\sqrt{n-1}\rfloor + 1$.*

Proof. This corollary follows trivially from the above corollary; since $|S| = n \geq \lfloor\sqrt{n-1}\rfloor\lfloor\sqrt{n-1}\rfloor$, so S has either height $\lfloor\sqrt{n-1}\rfloor + 1$ or width $\lfloor\sqrt{n-1}\rfloor$. ☐

So you could use this result to, for instance, make some silly number-theoretical statements:

Question 1: *Must a set of 5 natural numbers either contains numbers of the form x, $x \times y$, and $x \times y \times z$, or 3 numbers which are mutually indivisible by each other?*

Answer 1: *Notice that this is* not *true of sets of 4 natural numbers: consider the set* $\{2, 4, 3, 9\}$. *However, any set of 5 natural numbers subjected to the order relation of divisibility is a poset with 5 elements. Since* $5 > 2 \cdot 2$, *a 5-element poset must either contain a 3-element chain (which would be three mutually divisible elements) or a 3-element antichain (which would be 3 mutually indivisible elements).*

You can no doubt find other partial orderings for which Dilworth's Theorem gives curious results.

1.3 Extensions and Variations of Dilworth's Theorem

Dilworth's original paper actually covers a very specific infinite case: a poset of finite width k, even if the poset is not itself finite, can bepartitioned into k (possibly infinite) chains.

Erdős posed the question of whether Dilworth's theorem is true for infinite-width posets: that is, if $(S, \preceq)$ does not have finite width, is there necessarily an antichain whose elements can be put into one-to-one correspondence with the chains in a chain-partition of S? Surprisingly, the answer is "no".

Proposition 4. *The poset $(\mathbb{Z}^+ \times \mathbb{Z}^+, \preceq)$ with $(a, b) \preceq (c, d)$ iff $a \leq c$ and $b \leq d$ has no infinite antichains and no partitions into a finite number of chains.*

Proof. Let A be an antichain in the aforementioned poset. Since (a, b) and (a, c) are always comparable, every element of A must have a distinct first coordinate. Since the positive integers are a well-ordered set, the set of first coordinates of elements of A must have a least element; we may then order the elements of $A = \{(x_0, y_0), (x_1, y_1), \ldots,\}$ such that $x_0 < x_1 < x_2 < x_3 < \cdots$. Note that if any $y_i \leq y_{i+1}$, it would follow that $(x_i, y_i) \preceq (x_{i+1}, y_{i+1})$; then A would not be an antichain. Thus, it is also the case that $y_0 > y_1 > y_2 > y_3 > \cdots$. However, a strictly descending sequence of positive integers cannot be infinite, so A must be finite.

On the other hand, suppose $\mathbb{Z}^+ \times \mathbb{Z}^+$ is partitionable into a finite union of chains $C_1, C_2, \ldots, C_n$. Consider the elements $(1, n+1), (2, n), (3, n), \ldots, (n+1, 1)$. These elements are mutually incomparable, so no more than one of them can be in each chain. There are $n+1$ of them, so at least one of them is not in any of the C_i. ☐

In addition, there is a surprisingly simple statement which is something of a dual to Dilworth's theorem: partitioning S into antichains has a particular relationship with chains in S:

Proposition 5. *Suppose P is partitioned into a finite set of antichains $A_1, A_2, \ldots, A_n$. If C is a chain, then there is at most one element of C in each A_i; thus $n \geq |C|$.*

Proof. This result is actually quite trivial: suppose two distinct elements x and y of C were both in the same antichain A_i. Membership in C would force x and y to be incomparable; membership in A_i would force them to be comparable. Thus, $|C \cap A_i| \leq 1$ for all i. □

Theorem 2 (Dual of Dilworth (folklore))**.** *If C is a largest chain in a finite poset $(S, \preceq)$, then there is a partition of S into antichains $A_1 \cup A_2 \cup \cdots A_n$ such that $n = |C|$. Furthermore, each A_i contains exactly one element of C, and there is no partition of S into fewer than n antichains.*

Proof. We prove this by induction on the height n of the poset. If $n = 0$, then S is empty and the theorem is vacuously true. If $n = 1$, S has no chains of length 2, so all elements of S are incomparable and $S = A_1$ is a trivial decomposition into one antichain. Based on these base cases, we will make use of the inductive hypothesis for $n \geq 2$.

In a previous proposition on maximal antichains, we saw that the set of maximal elements of S is an antichain. Let us denote this set as A_n. Another earlier proposition informed us that every maximal chain contains a maximal element; since longest chains are maximal, every longest chain of S contains an element of A_n. Thus, since every chain of length n in S intersects A_n, there are no chains of length n in $S - A_n$. There will be chains of length $n-1$, since every longest chain in S only intersects A_n in one element. Thus $S - A_n$ has height $n-1$, so by the inductive hypothesis is partitionable into antichains $A_1, A_2, \ldots, A_{n-1}$. Together with A_n these form a partition of S. □