Elastic Collision of Particles

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November 23, 2015

1 Elastic Collision in 1 Dimension

Conservation of energy:

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$$

Conservation of momentum:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

This gives a system with 2 equations and 2 unknowns. Solving this for the end velocities gives:

$$v_1' = \frac{v_1(m_1 - m_2) + 2m_2 v_2}{m_1 + m_2}$$
$$v_2' = \frac{v_2(m_2 - m_1) + 2m_1 v_1}{m_1 + m_2}$$

An alternative way is it to express the final velocities with the old velocity and the change in velocity Δv:

$$v_1' = v_1 + \Delta v_1$$
$$v_2' = v_2 + \Delta v_2$$

The change in velocity Δv is:

$$\Delta v_1 = \frac{v_1(m_1 - m_2) + 2m_2 v_2}{m_1 + m_2} - v_1$$
$$\Delta v_2 = \frac{v_2(m_2 - m_1) + 2m_1 v_1}{m_1 + m_2} - v_2$$

After some algebraic transformations:

$$\Delta v_1 = \frac{2m_2 \cdot (v_2 - v_1)}{m_1 + m_2} \tag{1}$$

$$\Delta v_2 = \frac{-2m_1 \cdot (v_2 - v_1)}{m_1 + m_2} \tag{2}$$

2 Elastic Collision in N Dimensions

Let P_1 and P_2 be the N dimensional center points of the colliding particles. The collision normal $\vec{n}$ is:

$$\vec{n} = \frac{P_1 - P_2}{|P_1 - P_2|}$$

All change in velocity/momentum is along the collision normal. So the resulting velocities can be rewritten to:

$$\vec{v_1}' = \vec{v_1} + \Delta v_1 \cdot \vec{n}$$

$$\vec{v_2}' = \vec{v_2} + \Delta v_2 \cdot \vec{n}$$

Δv_1 and Δv_2 are both scalar values and behave like one dimensional collisions projected onto $\vec{n}$. The projection from a vector onto another vector of unit length is the dot product between them. Substituting the velocities in the one dimensional equations (**??**) and (**??**) with the projection onto $\vec{n}$ gives us the change in velocity in N Dimensions:

$$\Delta v_1 = \frac{2m_2 \cdot (dot(\vec{v_2}, \vec{n}) - dot(\vec{v_1}, \vec{n}))}{m_1 + m_2}$$

$$\Delta v_2 = \frac{-2m_1 \cdot (dot(\vec{v_2}, \vec{n}) - dot(\vec{v_1}, \vec{n}))}{m_1 + m_2}$$

$$\Delta v_1 = \frac{2m_2 \cdot dot(\vec{v_2} - \vec{v_1}, \vec{n})}{m_1 + m_2}$$

$$\Delta v_2 = \frac{-2m_1 \cdot dot(\vec{v_2} - \vec{v_1}, \vec{n})}{m_1 + m_2}$$

The final velocities are:

$$\vec{v_1}' = \vec{v_1} + \frac{2m_2 \cdot dot(\vec{v_2} - \vec{v_1}, \vec{n})}{m_1 + m_2} \cdot \vec{n}$$

$$\vec{v_2}' = \vec{v_2} - \frac{2m_1 \cdot dot(\vec{v_2} - \vec{v_1}, \vec{n})}{m_1 + m_2} \cdot \vec{n}$$

Alternatively the collision can be expressed in change in momentum:

$$p_1 = m \cdot v_1$$

$$\Delta p_1 = m_1 \cdot \Delta v_1$$

$$\Delta p_2 = m_2 \cdot \Delta v_2$$

$$\Delta p_1 = -\Delta p_2$$

$$\Delta p_1 = \frac{2m_2 \cdot dot(\vec{v_2} - \vec{v_1}, \vec{n})}{m_1 + m_2} \cdot m_1$$

$$\Delta p_1 = \frac{2 \cdot dot(\vec{v_2} - \vec{v_1}, \vec{n})}{\frac{1}{m_1} + \frac{1}{m_2}} = -\Delta p_2$$

The final velocities using the change in momentum:

$$\vec{v_1}' = \vec{v_1} + \frac{2 \cdot dot(\vec{v_2} - \vec{v_1}, \vec{n})}{\frac{1}{m_1} + \frac{1}{m_2}} \cdot \frac{1}{m_1} \cdot \vec{n}$$

$$\vec{v_2}' = \vec{v_2} - \frac{2 \cdot dot(\vec{v_2} - \vec{v_1}, \vec{n})}{\frac{1}{m_1} + \frac{1}{m_2}} \cdot \frac{1}{m_2} \cdot \vec{n}$$