Contents

1 Contest Setup

1.1 vimrc

```
set number       " Show line numbers
set mouse=a      " Enable inaction via mouse
set showmatch        " Highlight matching brace
set cursorline       " Show underline
set cursorcolumn     " highlight vertical column

filetype on "enable file detection
syntax on   "syntax highlight

set autoindent       " Auto-indent new lines
set shiftwidth=4     " Number of auto-indent spaces
set smartindent      " Enable smart-indent
set smarttab         " Enable smart-tabs
set tabstop=4    " Number of spaces per Tab

" ---------Optional----------

set undolevels=10000     " Number of undo levels
set scrolloff=5      " Auto scroll

set hlsearch     " Highlight all search results
set smartcase    " Enable smart-case search
set ignorecase   " Always case-insensitive
set incsearch    " Searches for strings incrementally

highlight Comment ctermfg=cyan
set showmode

set encoding=utf-8
set fileencoding=utf-8
scriptencoding=utf-8
```

1.2 bashrc

```
alias g++="g++ -Wall -Wextra -std=c++11 -O2"
```

1.3 Grep Error and Warnings

```
g++ main.cpp 2>&1 | grep -E 'warning|error'
```

1.4 C++ template

```cpp
#include <bits/stdc++.h>

using namespace std;

typedef long long int ll;
typedef pair<int, int> ii;

int main()
{
    return 0;
}
```

1.5 Java template

```java
import java.io.*;
import java.util.*;

public class Main
{
    public static void main(String[] args)
    {
        MyScanner sc = new MyScanner();
        out = new PrintWriter(new BufferedOutputStream(System.out));
        // Start writing your solution here.

        // Stop writing your solution here.
        out.close();
    }

    public static PrintWriter out;

    public static class MyScanner
    {
        BufferedReader br;
        StringTokenizer st;

        public MyScanner()
        {
            br = new BufferedReader(new InputStreamReader(System.in));
        }

        boolean hasNext()
        {
            while (st == null || !st.hasMoreElements()) {
                try {
                    st = new StringTokenizer(br.readLine());
                } catch (Exception e) {
                    return false;
                }
            }
            return true;
        }

        String next()
        {
            if (hasNext())
                return st.nextToken();
            return null;
        }

        int nextInt()
        {
            return Integer.parseInt(next());
        }

        long nextLong()
        {
            return Long.parseLong(next());
        }

        double nextDouble()
        {
            return Double.parseDouble(next());
        }

        String nextLine()
        {
            String str = "";
            try {
                str = br.readLine();
            } catch (IOException e) {
                e.printStackTrace();
            }
            return str;
        }
    }
}
```

1.5.1 Java Issues

1. Random Shuffle before sorting: *Random rnd = new Random(); rnd.nextInt();*
2. Use StringBuilder for large output
3. Java has strict parsing rules. e.g. using sc.nextInt() to read a long will result in RE
4. For class sorting, use code: implements Comparable<Class name>. Or, use code: new Comparator<Interval>() {} at Collections.sort() second argument

2 System Testing

1. Setup bashrc and vimrc
2. Install Java 8, Eclipse 32-bit, g++ compiler
3. Remove Chinese input method
4. Look for compilation parameter and code it into bashrc
5. Test if c++ and java templates work properly on local and judge machine
6. Test "divide by 0" → RE/TLE?
7. Make a complete graph and run Floyd warshall, to test time complexity upper bound

8. Make a linear graph and use DFS to test stack size
9. Print output with extra newline and spaces

3 Reminder

1. 隊友的建議，要認真聽！通常隊友的建議都會突破你盲點
2. Read the problem statements carefully. Input and output specifications and constraints are crucial!
3. Estimate the **time complexity** and **memory complexity** carefully.
4. Time penalty is 20 minutes per WA, **don't rush**!
5. Sample test cases must all be tested and passed before every submission!
6. Test the corner cases, such as 0, 1, -1. Test all edge cases of the input specification.
7. Bus error: the code has $scanf, fgets$ but have nothing to read! Check if you have early termination but didn't handle it properly.
8. Binary search? 數學算式移項合併後查詢?
9. Two Pointer <-> Binary Search
10. Directed graph connectivity -> DFS. Undirected graph -> Union Find
11. Check connectivity of the graph if the problem statement doesn't say anything (just loop over all nodes!)
12. $longlong = int * int$ won't work!
13. Shifting for $longlongint$ should be something like $1LL << 35$
14. For continuous input problems, be sure to read in all input BEFORE terminating and start processing next the input.
15. Don't use anonymous struct
16. 因式分解
17. 有時候，從答案推回來會容易些!
18. 寫出數學式，有時就馬上出現答案了!

4 Topic list

1. enumeration
2. greedy
3. sorting, topological sort
4. binary search
5. 離散化
6. Dynamic programming, 矩陣快速冪
7. Pigeonhole
8. LCA (倍增法, LCA 轉 RMQ)

5 Useful code

5.1 Leap year

```
year % 400 == 0 || (year % 4 == 0 && year % 100 != 0)
```

5.2 Fast Exponentiation $O(log(exp))$

Fermat's little theorem: 若 m 是質數，則 $a^{m-1} \equiv 1 \pmod{m}$

```
ll fast_pow(ll a, ll b, ll M) {
    ll ans = 1;
    ll base = a % M;
    while (b) {
        if (b & 1)
            ans = ans * base % M;
        base = base * base % M;
        b >>= 1;
    }
    return ans;
}
```

5.3 Mod Inverse

Case 1: $gcd(a, m) = 1$: ax + my = gcd(a, m) = 1 (use ext_gcd)

Case 2: $m\ is\ prime$: $a^{m-2} \equiv a^{-1} mod\ m$

5.4 GCD $O(log(a + b))$

注意負數的 case! C++ 是看被除數決定正負號的。

```
ll gcd(ll a, ll b)
{
    return b == 0 ? a : gcd(b, a % b);
}
```

5.5 Extended Euclidean Algorithm GCD $O(log(a + b))$

Bezout identity $ax + by = gcd(a, b)$, where $gcd(a, b)$ is the smallest positive integer that can be written as ax + by, and every integer of the form ax + by is a multiple of $gcd(a, b)$.

```
ll extgcd(ll a, ll b, ll& x, ll&y) {
    if (b == 0) {
        x = 1;
        y = 0;
        return a;
    }
    else {
        ll d = extgcd(b, a % b, y, x);
        y -= (a / b) * x;
        return d;
    }
}
```

5.6 Prime Generator

```
const ll MAX_NUM = 1e6; // 要是合數
bool is_prime[MAX_NUM];
vector<ll> primes;

void init_primes() {
    fill(is_prime, is_prime + MAX_NUM, true);
    is_prime[0] = is_prime[1] = false;
    for (ll i = 2; i < MAX_NUM; i++) {
        if (is_prime[i]) {
            primes.push_back(i);
            for (ll j = i * i; j < MAX_NUM; j += i)
                is_prime[j] = false;
        }
    }
}
```

5.7 C++ Reference

```
vector/deque
    ::[]: [idx] -> val // O(1)
    ::erase: [it] -> it
    ::erase: [it s, it t] -> it
    ::resize: [sz, val = 0] -> void
    ::insert: [it, val] -> void // insert before it
    ::insert: [it, cnt, val] -> void // insert before it
    ::insert: [it pos, it from_s, it from_t] -> void // insert before it

set/mulitset
    ::insert: [val] -> pair<it, bool> // bool: if val already exist
    ::erase: [val] -> void
    ::erase: [it] -> void
    ::clear: [] -> void
    ::find: [val] -> it
    ::count: [val] -> sz
    ::lower_bound: [val] -> it
    ::upper_bound: [val] -> it
    ::equal_range: [val] -> pair<it, int>

map/mulitmap
    ::begin/end: [] -> it (*it = pair<key, val>)
    ::[]: [val] -> map_t&
    ::insert: [pair<key, val>] -> pair<it, bool>
    ::erase: [key] -> sz
    ::clear: [] -> void
    ::find: [key] -> it
    ::count: [key] -> sz
    ::lower_bound: [key] -> it
    ::upper_bound: [key] -> it
    ::equal_range: [key] -> it

algorithm
    ::any_of: [it s, it t, unary_func] -> bool // C++11
    ::all_of: [it s, it t, unary_func] -> bool // C++11
    ::none_of: [it s, it t, unary_func] -> bool // C++11
    ::find: [it s, it t, val] -> it
    ::find_if: [it s, it t, unary_func] -> it
    ::count: [it s, it t, val] -> int
    ::count_if: [it s, it t, unary_func] -> int
    ::copy: [it fs, it ft, it ts] -> void // t should be allocated
    ::equal: [it s1, it t1, it s2, it t2] -> bool
    ::remove: [it s, it t, val] -> it (it = new end)
    ::unique: [it s, it t] -> it (it = new end)
    ::random_shuffle: [it s, it t] -> void
    ::lower_bound: [it s, it t, val, binary_func(a, b): a < b] -> it
    ::upper_bound: [it s, it t, val, binary_func(a, b): a < b] -> it
    ::binary_search: [it s, it t, val] -> bool ([s, t) sorted)
    ::merge: [it s1, it t1, it s2, it t2, it o] -> void (o allocated)
    ::includes: [it s1, it t1, it s2, it t2] -> bool (if 2 included in 1)


string::
    ::replace(idx, len, string) -> void
    ::replace(it s1, it t1, it s2, it t2) -> void
string <-> int
    ::stringstream; // remember to clear
    ::sscanf(s.c_str(), "%d", &i);
    ::sprintf(result, "%d", i); string s = result;

numeric
    ::accumulate(it s, it t, val init);

math/cstdlib
    ::atan2(0, -1) -> pi
    ::sqrt(db/ldb) -> db/ldb
    ::fabs(db/ldb) -> db/ldb
    ::abs(int) -> int
    ::ceil(db/ldb) -> db/ldb
    ::floor(db/ldb) -> db/ldb
    ::llabs(ll) -> ll (C++11)
    ::round(db/ldb) -> db/ldb (C99, C++11)
    ::log2(db) -> db (C99)
    ::log2(ldb) -> ldb (C++11)

ctype
    ::toupper(char) -> char (remain same if input is not alpha)
    ::tolower(char) -> char (remain same if input is not alpha)
    ::isupper(char) -> bool
    ::islower(char) -> bool
    ::isalpha(char) -> bool
    ::isdigit(char) -> bool

io printf/scanf
    ::int:                "%d"       /   "%d"
    ::double:             "%lf","f"  /   "%lf"
    ::string:             "%s"       /   "%s"
    ::long long:          "%lld"     /   "%lld"
    ::long double:        "%Lf"      /   "%Lf"
    ::unsigned int:       "%u"       /   "%u"
    ::unsigned long long: "%ull"     /   "%ull"
    ::oct:                "0%o"
    ::hex:                "0x%x"
    ::scientific:         "%e"
    ::width:              "%05d"
    ::precision:          "%.5f"
    ::adjust left:        "%-5d"

io cin/cout
    ::oct:                cout << oct << showbase;
    ::hex:                cout << hex << showbase;
    ::scientific:         cout << scientific;
    ::width:              cout << setw(5);
    ::precision:          cout << fixed << setprecision(5);
    ::adjust left:        cout << setw(5) << left;
```

6 Search

6.1 Ternary Search

6.2 折半完全列舉

能用 vector 就用 vector

6.3 Two-pointer 爬行法（右跑左追）

7 Basic data structure

7.1 1D BIT

```
// BIT is 1-based
const int MAX_N = 20000; //這個記得改!
ll bit[MAX_N + 1];

ll sum(int i) {
    int s = 0;
    while (i > 0) {
        s += bit[i];
        i -= (i & -i);
    }
    return s;
}

void add(int i, ll x) {
    while (i <= MAX_N) {
        bit[i] += x;
        i += (i & -i);
    }
}
```

7.2 2D BIT

```
// BIT is 1-based
const int MAX_N = 20000, MAX_M = 20000; //這個記得改!
ll bit[MAX_N + 1][MAX_M + 1];

ll sum(int a, int b) {
    ll s = 0;
    for (int i = a; i > 0; i -= (i & -i))
        for (int j = b; j > 0; j -= (j & -j))
            s += bit[i][j];
    return s;
}

void add(int a, int b, ll x) {
    // MAX_N, MAX_M 須適時調整!
    for (int i = a; i <= MAX_N; i += (i & -i))
        for (int j = b; j <= MAX_M; j += (j & -j))
            bit[i][j] += x;
}
```

7.3 Union Find

```
#define N 20000 // 記得改
struct UFDS {
    int par[N];

    void init() {
        memset(par, -1, sizeof(par));
    }

    int root(int x) {
        return par[x] < 0 ? x : par[x] = root(par[x]);
    }

    void merge(int x, int y) {
        x = root(x);
        y = root(y);

        if (x != y) {
            if (par[x] > par[y])
                swap(x, y);
            par[x] += par[y];
            par[y] = x;
        }
    }
}
```

7.4 Segment Tree

```
const int MAX_N = 100000;
const int MAX_NN = (1 << 20); // should be bigger than MAX_N

int N;
ll inp[MAX_N];

int NN;
ll seg[2 * MAX_NN - 1];
ll lazy[2 * MAX_NN - 1];
// lazy[u] != 0 : the subtree of u (u not included) is not up-to-date

void seg_gather(int u)
{
    seg[u] = seg[u * 2 + 1] + seg[u * 2 + 2];
}

void seg_push(int u, int l, int m, int r)
{
    if (lazy[u] != 0) {
        seg[u * 2 + 1] += (m - l) * lazy[u];
        seg[u * 2 + 2] += (r - m) * lazy[u];

        lazy[u * 2 + 1] += lazy[u];
        lazy[u * 2 + 2] += lazy[u];
        lazy[u] = 0;
    }
}
```

```cpp

void seg_init()
{
    NN = 1;
    while (NN < N)
        NN *= 2;

    memset(seg, 0, sizeof(seg)); // val that won't affect result
    memset(lazy, 0, sizeof(lazy)); // val that won't affect result
    memcpy(seg + NN - 1, inp, sizeof(ll) * N); // fill in leaves
}

void seg_build(int u)
{
    if (u >= NN - 1) { // leaf
        return;
    }

    seg_build(u * 2 + 1);
    seg_build(u * 2 + 2);
    seg_gather(u);
}

void seg_update(int a, int b, int delta, int u, int l, int r)
{
    if (l >= b || r <= a) {
        return;
    }

    if (a <= l && r <= b) {
        seg[u] += (r - l) * delta;
        lazy[u] += delta;
        return;
    }

    int m = (l + r) / 2;
    seg_push(u, l, m, r);
    seg_update(a, b, delta, u * 2 + 1, l, m);
    seg_update(a, b, delta, u * 2 + 2, m, r);
    seg_gather(u);
}

ll seg_query(int a, int b, int u, int l, int r)
{
    if (l >= b || r <= a) {
        return 0;
    }

    if (a <= l && r <= b) {
        return seg[u];
    }

    int m = (l + r) / 2;
    seg_push(u, l, m, r);
    ll ans = 0;
    ans += seg_query(a, b, u * 2 + 1, l, m);
    ans += seg_query(a, b, u * 2 + 2, m, r);
    seg_gather(u);

    return ans;
}
```

7.5 Sparse Table

```cpp
struct {
    int sp[MAX_LOG_N][MAX_N]; // MAX_LOG_N = ceil(lg(MAX_N))

    void build(int inp[], int n)
    {
        for (int j = 0; j < n; j++)
            sp[0][j] = inp[j];

        for (int i = 1; (1 << i) <= n; i++)
            for (int j = 0; j + (1 << i) <= n; j++)
                sp[i][j] = min(sp[i-1][j],sp[i-1][j+(1 << (i - 1))]);
    }

    int query(int l, int r) // [l, r)
    {
        int k = floor(log2(r - l));
        return min(sp[k][l], sp[k][r - (1 << k)]);
    }
} sptb;
```

8 Tree

8.1 LCA

```cpp
const int MAX_N = 10000;
const int MAX_LOG_N = 14; // (1 << MAX_LOG_N) > MAX_N

int N;
int root;
int dep[MAX_N];
int par[MAX_LOG_N][MAX_N];

vector<int> child[MAX_N];

void dfs(int u, int p, int d) {
    dep[u] = d;
    for (int i = 0; i < int(child[u].size()); i++) {
        int v = child[u][i];
        if (v != p) {
            dfs(v, u, d + 1);
        }
    }
}

void build() {
    // par[0][u] and dep[u]
```

```cpp
    dfs(root, -1, 0);

    // par[i][u]
    for (int i = 0; i + 1 < MAX_LOG_N; i++) {
        for (int u = 0; u < N; u++) {
            if (par[i][u] == -1)
                par[i + 1][u] = -1;
            else
                par[i + 1][u] = par[i][par[i][u]];
        }
    }
}

int lca(int u, int v) {
    if (dep[u] > dep[v]) swap(u, v); // 讓 v 較深
    int diff = dep[v] - dep[u]; // 將 v 上移到與 u 同層
    for (int i = 0; i < MAX_LOG_N; i++) {
        if (diff & (1 << i)) {
            v = par[i][v];
        }
    }

    if (u == v) return u;

    for (int i = MAX_LOG_N - 1; i >= 0; i--) { // 必需倒序
        if (par[i][u] != par[i][v]) {
            u = par[i][u];
            v = par[i][v];
        }
    }
    return par[0][u];
}
```

8.2 Tree Center

```cpp
int diameter = 0, radius[N], deg[N]; // deg = in + out degree
int findRadius()
{
    queue<int> q; // add all leaves in this group
    for (auto i : group)
        if (deg[i] == 1)
            q.push(i);

    int mx = 0;
    while (q.empty() == false) {
        int u = q.front();
        q.pop();

        for (int v : g[u]) {
            deg[v]--;
            if (deg[v] == 1) {
                q.push(v);
                radius[v] = radius[u] + 1;
                mx = max(mx, radius[v]);
            }
        }
    }

    int cnt = 0; // crucial for knowing if there are 2 centers or not
    for (auto j : group)
        if (radius[j] == mx)
            cnt++;

    // add 1 if there are 2 centers (radius, diameter)
    diameter = max(diameter, mx * 2 + (cnt == 2));
    return mx + (cnt == 2);
}
```

8.3 Treap

```cpp
// Remember srand(time(NULL))
struct Treap { // val: bst, pri: heap
    int pri, size, val;
    Treap *lch, *rch;
    Treap() {}
    Treap(int v) {
        pri = rand();
        size = 1;
        val = v;
        lch = rch = NULL;
    }
};

inline int size(Treap* t) {
    return (t ? t->size : 0);
}
// inline void push(Treap* t) {
//     push lazy flag
// }
inline void pull(Treap* t) {
    t->size = 1 + size(t->lch) + size(t->rch);
}

int NN = 0;
Treap pool[30000];

Treap* merge(Treap* a, Treap* b) { // a < b
    if (!a || !b) return (a ? a : b);
    if (a->pri > b->pri) {
        // push(a);
        a->rch = merge(a->rch, b);
        pull(a);
        return a;
    }
    else {
        // push(b);
        b->lch = merge(a, b->lch);
        pull(b);
        return b;
    }
```

```cpp
}

void split(Treap* t, Treap*& a, Treap*& b, int k) {
    if (!t) { a = b = NULL; return; }
    // push(t);
    if (size(t->lch) < k) {
        a = t;
        split(t->rch, a->rch, b, k - size(t->lch) - 1);
        pull(a);
    }
    else {
        b = t;
        split(t->lch, a, b->lch, k);
        pull(b);
    }
}

// get the rank of val
// result is 1-based
int get_rank(Treap* t, int val) {
    if (!t) return 0;
    if (val < t->val)
        return get_rank(t->lch, val);
    else
        return get_rank(t->rch, val) + size(t->lch) + 1;
}

// get kth smallest item
// k is 1-based
Treap* get_kth(Treap*& t, int k) {
    Treap *a, *b, *c, *d;
    split(t, a, b, k - 1);
    split(b, c, d, 1);
    t = merge(a, merge(c, d));
    return c;
}

void insert(Treap*& t, int val) {
    int k = get_rank(t, val);
    Treap *a, *b;
    split(t, a, b, k);
    pool[NN] = Treap(val);
    Treap* n = &pool[NN++];
    t = merge(merge(a, n), b);
}

// Implicit key treap init
void insert() {
    for (int i = 0; i < N; i++) {
        int val; scanf("%d", &val);
        root = merge(root, new_treap(val)); // implicit key(index)
    }
}
```

9 Graph

9.1 Articulation point / Bridge

```cpp
// timer = 1, dfs arrays init to 0, set root carefully!
int timer, dfsTime[N], dfsLow[N], root;
bool articulationPoint[N]; // set<ii> bridge;
void findArticulationPoint(int u, int p)
{
    dfsTime[u] = dfsLow[u] = timer++;

    int child = 0; // root child counter for articulation point
    for(auto v : g[u]) { // vector<int> g[N]; // undirected graph
        if(v == p) // don't go back to parent
            continue;

        if(dfsTime[v] == 0) {
            child++; // root child counter for articulation point
            findArticulationPoint(v, u);
            dfsLow[u] = min(dfsLow[u], dfsLow[v]);

            // <= for articulation point, < for bridge
            if(dfsTime[u] <= dfsLow[v] && root != u)
                articulationPoint[u] = true;
            // special case for articulation point root only
            if(u == root && child >= 2)
                articulationPoint[u] = true;
        } else { // visited before (back edge)
            dfsLow[u] = min(dfsLow[u], dfsTime[v]);
        }
    }
}
```

9.2 CC

9.2.1 BCC vertex

9.2.2 BCC edge

9.2.3 SCC

First of all we run DFS on the graph and sort the vertices in decreasing of their finishing time (we can use a stack).
Then, we start from the vertex with the greatest finishing time, and for each vertex v that is not yet in any SCC, do : for each u that v is reachable by u and u is not yet in any SCC, put it in the SCC of vertex v. The code is quite simple.

9.3 Shortest Path

Time complexity notations: V = vertex, E = edge

9.3.1 Dijkatra (next-to-shortest path)

密集圖別用 priority queue!

```cpp
struct Edge {
    int to, cost;
};
```


8

```
typedef pair<int, int> P; // <d, v>
const int INF = 0x3f3f3f3f;

int N, R;
vector<Edge> g[5000];

int d[5000];
int sd[5000];

int solve() {
    fill(d, d + N, INF);
    fill(sd, sd + N, INF);
    priority_queue< P, vector<P>, greater<P> > pq;

    d[0] = 0;
    pq.push(P(0, 0));

    while (!pq.empty()) {
        P p = pq.top(); pq.pop();
        int v = p.second;

        if (sd[v] < p.first) // 比次短距離還大，沒用，跳過
            continue;

        for (size_t i = 0; i < g[v].size(); i++) {
            Edge& e = g[v][i];
            int nd = p.first + e.cost;
            if (nd < d[e.to]) { // 更新最短距離
                swap(d[e.to], nd);
                pq.push(P(d[e.to], e.to));
            }
            if (d[e.to] < nd && nd < sd[e.to]) { // 更新次短距離
                sd[e.to] = nd;
                pq.push(P(sd[e.to], e.to));
            }
        }
    }

    return sd[N-1];
}
```

9.3.2 SPFA

```
typedef pair<int, int> ii;
vector< ii > g[N];

bool SPFA()
{
    vector<ll> d(n, INT_MAX);
    d[0] = 0; // origin

    queue<int> q;
    vector<bool> inqueue(n, false);
    vector<int> cnt(n, 0);
    q.push(0);
    inqueue[0] = true;
    cnt[0]++;

    while(q.empty() == false) {
        int u = q.front();
        q.pop();
        inqueue[u] = false;

        for(auto i : g[u]) {
            int v = i.first, w = i.second;
            if(d[u] + w < d[v]) {
                d[v] = d[u] + w;
                if(inqueue[v] == false) {
                    q.push(v);
                    inqueue[v] = true;
                    cnt[v]++;

                    if(cnt[v] == n) { // loop!
                        return true;
                    }
                }
            }
        }
    }

    return false;
}
```

9.3.3 Bellman-Ford $O(VE)$

```
vector<pair<ii, int>> edge; // store graph by edge: ((u, v), w)

void BellmanFord()
{
    ll d[n]; // n: total nodes
    fill(d, d + n, INT_MAX);
    d[0] = 0; // src is 0
    bool loop = false;
    for (int i = 0; i < n; i++) {
        // Do n - 1 times. If the n-th time still has relaxation, loop exists
        bool hasChange = false;
        for (int j = 0; j < (int)edge.size(); j++) {
            int u = edge[j].first.first, v = edge[j].first.second, w = edge[j].second;
            if (d[u] != INT_MAX && d[u] + w < d[v]) {
                hasChange = true;
                d[v] = d[u] + w;
            }
        }

        if (i == n - 1 && hasChange == true)
            loop = true;
        else if (hasChange == false)
            break;
    }
```

```
}
```

9.3.4 Floyd-Warshall $O(V^3)$

The graph is stored using adjacency matrix. The initial state is $diagnal = 0$ and $others = INF$. (If INF is int, use long long for the matrix)
If diagonal numbers are negative $\leftarrow$ cycle .

```
for(int k = 0; k < N; k++)
    for(int i = 0; i < N; i++)
        for(int j = 0; j < N; j++)
            dp[i][j] = min(dp[i][j], dp[i][k] + dp[k][j]);
```

9.4 MST

9.4.1 Kruskal

1. Store the graph by $(weight, (from, to))$
2. Sort the graph by $weight$
3. Start from the smallest weight, and keep adding edges that won't form a cycle with the current MST set
4. Early termination condition: $n - 1$ edges has been added, NOT size of the union-find set

9.4.2 Prim

```
priority_queue<ii, vector<ii>, greater<ii>> pq;
pq.push(ii(0, 0)); // push (0, origin)

uf.init(n + 1); // init union find
int ans = 0, earlyTermination = 0;
while (!pq.empty())
{
    ii cur = pq.top();
    pq.pop();

    int u = cur.second;
    if (u != 0 && (uf.root(0) == uf.root(u))) // check loop
        continue;
    uf.merge(0, u);
    ans += cur.first;
    earlyTermination++;
    if (earlyTermination == n) // origin node is dummy node
        break;

    for (int i = 0; i < (int)g[u].size(); i++) {
        int v = g[u][i].first, w = g[u][i].second;

        if (uf.root(0) != uf.root(v)) {
            pq.push(ii(w, v));
        }
    }
}
```

10 Flow

10.1 Max Flow (Dinic)

```
struct Edge {
    int to, cap, rev;
    Edge(int a, int b, int c) {
        to = a;
        cap = b;
        rev = c;
    }
};

const int INF = 0x3f3f3f3f;
const int MAX_V = 20000 + 10;
// vector<Edge> g[MAX_V];
vector< vector<Edge> > g(MAX_V);
int level[MAX_V];
int iter[MAX_V];

inline void add_edge(int u, int v, int cap) {
    g[u].push_back((Edge){v, cap, (int)g[v].size()});
    g[v].push_back((Edge){u, 0, (int)g[u].size() - 1});
}

void bfs(int s) {
    memset(level, -1, sizeof(level));
    queue<int> q;

    level[s] = 0;
    q.push(s);

    while (!q.empty()) {
        int v = q.front(); q.pop();
        for (int i = 0; i < int(g[v].size()); i++) {
            const Edge& e = g[v][i];
            if (e.cap > 0 && level[e.to] < 0) {
                level[e.to] = level[v] + 1;
                q.push(e.to);
            }
        }
    }
}

int dfs(int v, int t, int f) {
    if (v == t) return f;
    for (int& i = iter[v]; i < int(g[v].size()); i++) {
        Edge& e = g[v][i];
        if (e.cap > 0 && level[v] < level[e.to]) {
            int d = dfs(e.to, t, min(f, e.cap));
            if (d > 0) {
                e.cap -= d;
                g[e.to][e.rev].cap += d;
                return d;
            }
```

```
            }
        }
        return 0;
}

int max_flow(int s, int t) { // dinic
    int flow = 0;
    for (;;) {
        bfs(s);
        if (level[t] < 0) return flow;
        memset(iter, 0, sizeof(iter));
        int f;
        while ((f = dfs(s, t, INF)) > 0) {
            flow += f;
        }
    }
}
```

10.2 Min Cost Flow

```
#define st first
#define nd second

typedef pair<double, int> pii;
const double INF = 1e10;

struct Edge {
    int to, cap;
    double cost;
    int rev;
};

const int MAX_V = 2 * 100 + 10;
int V;
vector<Edge> g[MAX_V];
double h[MAX_V];
double d[MAX_V];
int prevv[MAX_V];
int preve[MAX_V];
// int match[MAX_V];

void add_edge(int u, int v, int cap, double cost) {
    g[u].push_back((Edge){v, cap, cost, (int)g[v].size()});
    g[v].push_back((Edge){u, 0, -cost, (int)g[u].size() - 1});
}

double min_cost_flow(int s, int t, int f) {
    double res = 0;
    fill(h, h + V, 0);
    fill(match, match + V, -1);
    while (f > 0) {
        // dijkstra 找最小成本增廣路徑
        // without h will reduce to SPFA = O(V*E)
        fill(d, d + V, INF);
        priority_queue< pii, vector<pii>, greater<pii> > pq;

        d[s] = 0;
        pq.push(pii(d[s], s));

        while (!pq.empty()) {
            pii p = pq.top(); pq.pop();
            int v = p.nd;
            if (d[v] < p.st) continue;
            for (size_t i = 0; i < g[v].size(); i++) {
                const Edge& e = g[v][i];
                if (e.cap > 0 && d[e.to] > d[v] + e.cost + h[v] - h[e.to]) {
                    d[e.to] = d[v] + e.cost + h[v] - h[e.to];
                    prevv[e.to] = v;
                    preve[e.to] = i;
                    pq.push(pii(d[e.to], e.to));
                }
            }
        }

        // 找不到增廣路徑
        if (d[t] == INF) return -1;

        // 維護 h[v]
        for (int v = 0; v < V; v++)
            h[v] += d[v];

        // 找瓶頸
        int bn = f;
        for (int v = t; v != s; v = prevv[v])
            bn = min(bn, g[prevv[v]][preve[v]].cap);

        // // find match
        // for (int v = prevv[t]; v != s; v = prevv[prevv[v]]) {
        //     int u = prevv[v];
        //     match[v] = u;
        //     match[u] = v;
        // }

        // 更新剩餘圖
        f -= bn;
        res += bn * h[t]; // SPFA: res += bn * d[t]
        for (int v = t; v != s; v = prevv[v]) {
            Edge& e = g[prevv[v]][preve[v]];
            e.cap -= bn;
            g[v][e.rev].cap += bn;
        }
    }
    return res;
}
```

10.3 Bipartite Matching

```cpp
const int MAX_V = ...;
int V;
vector<int> g[MAX_V];
int match[MAX_V];
bool used[MAX_V];

void add_edge(int u, int v) {
    g[u].push_back(v);
    g[v].push_back(u);
}

// 回傳有無找到從 v 出發的增廣路徑
//（首尾都為未匹配點的交錯路徑）
// [待確認] 每次遞迴都找一個未匹配點 v 及匹配點 u
bool dfs(int v) {
    used[v] = true;
    for (size_t i = 0; i < g[v].size(); i++) {
        int u = g[v][i], w = match[u];
        // 尚未配對或可從 w 找到增廣路徑（即路徑繼續增長）
        if (w < 0 || (!used[w] && dfs(w))) {
            // 交錯配對
            match[v] = u;
            match[u] = v;
            return true;
        }
    }
    return false;
}

int bipartite_matching() { // 匈牙利演算法
    int res = 0;
    memset(match, -1, sizeof(match));
    for (int v = 0; v < V; v++) {
        if (match[v] == -1) {
            memset(used, false, sizeof(used));
            if (dfs(v)) {
                res++;
            }
        }
    }
    return res;
}
```

11 String

11.1 Rolling Hash

1. Use two rolling hashes if needed.
2. The prime for pre-calculation can be 137 and 257, for modulo can be $1e9 + 7$ and $0xdefaced$

```cpp
#define N 1000100
#define B 137
#define M 1000000007

typedef long long ll;

char inp[N];
int len;
ll p[N], h[N];

void init()
{ // build polynomial table and hash value
    p[0] = 1; // b to the ith power
    for (int i = 1; i <= len; i++) {
        h[i] = (h[i - 1] * B % M + inp[i - 1]) % M; // hash value
        p[i] = p[i - 1] * B % M;
    }
}

ll get_hash(int l, int r) // [l, r] of the inp string array
{
    return ((h[r + 1] - (h[l] * p[r - l + 1])) % M + M) % M;
}
```

11.2 KMP

```cpp
void fail()
{
    int len = strlen(pat);

    f[0] = 0;
    int j = 0;
    for (int i = 1; i < len; i++) {
        while (j != 0 && pat[i] != pat[j])
            j = f[j - 1];

        if (pat[i] == pat[j])
            j++;

        f[i] = j;
    }
}

int match()
{
    int res = 0;
    int j = 0, plen = strlen(pat), tlen = strlen(text);

    for (int i = 0; i < tlen; i++) {
        while (j != 0 && text[i] != pat[j])
            j = f[j - 1];

        if (text[i] == pat[j]) {
            if (j == plen - 1) { // find match
                res++;
                j = f[j];
            } else {
                j++;
            }
        }
    }
```

```
    return res;
}
```

11.3 Z Algorithm

```
int len = strlen(inp), z[len];
z[0] = 0; // initial

int l = 0, r = 0; // z box bound [l, r]
for (int i = 1; i < len; i++)
{
    if (i > r) { // i not in z box
        l = r = i; // z box contains itself only
        while (r < len && inp[r - l] == inp[r])
            r++;
        z[i] = r - l;
        r--;
    } else { // i in z box
        if (z[i - l] + i < r) // over shoot R bound
            z[i] = z[i - l];
        else {
            l = i;
            while (r < len && inp[r - l] == inp[r])
                r++;
            z[i] = r - l;
            r--;
        }
    }
}
```

11.4 Trie

注意 count 的擺放位置，視題意可以擺在迴圈外

```
struct Node {
    int cnt;
    Node* nxt[2];
    Node() {
        cnt = 0;
        fill(nxt, nxt + 2, nullptr);
    }
};

const int MAX_Q = 200000;
int Q;

int NN = 0;
Node data[MAX_Q * 30];
Node* root = &data[NN++];

void insert(Node* u, int x) {
    for (int i = 30; i >= 0; i--) {
        int t = ((x >> i) & 1);
        if (u->nxt[t] == nullptr) {
            u->nxt[t] = &data[NN++];
        }

        u = u->nxt[t];
        u->cnt++;
    }
}

void remove(Node* u, int x) {
    for (int i = 30; i >= 0; i--) {
        int t = ((x >> i) & 1);
        u = u->nxt[t];
        u->cnt--;
    }
}

int query(Node* u, int x) {
    int res = 0;
    for (int i = 30; i >= 0; i--) {
        int t = ((x >> i) & 1);
        // if it is possible to go the another branch
        // then the result of this bit is 1
        if (u->nxt[t ^ 1] != nullptr && u->nxt[t ^ 1]->cnt > 0) {
            u = u->nxt[t ^ 1];
            res |= (1 << i);
        }
        else {
            u = u->nxt[t];
        }
    }
    return res;
}
```

11.5 Suffix Array

12 Matrix

12.1 Gauss Jordan

```
typedef long long ll;
typedef vector<ll> vec;
typedef vector<vec> mat;

vec gauss_jordan(mat A) {
    int n = A.size(), m = A[0].size();
    for (int i = 0; i < n; i++) {
        // float: find j s.t. A[j][i] is max
        // mod: find min j s.t. A[j][i] is not 0
        int pivot = i;
        for (int j = i; j < n; j++) {
            // if (fabs(A[j][i]) > fabs(A[pivot])) {
            //     pivot = j;
            // }
            if (A[pivot][i] != 0) {
                pivot = j;
```

```
                break;
            }
        }

        swap(A[i], A[pivot]);
        if (A[i][i] == 0) { // if (fabs(A[i][i]) < eps)
            // 無解或無限多組解
            // 可改成 continue, 全部做完後再判
            return vec();
        }

        ll divi = inv(A[i][i]);
        for (int j = i; j < m; j++) {
            // A[i][j] /= A[i][i];
            A[i][j] = (A[i][j] * divi) % MOD;
        }

        for (int j = 0; j < n; j++) {
            if (j != i) {
                for (int k = i + 1; k < m; k++) {
                    // A[j][k] -= A[j][i] * A[i][k];
                    ll p = (A[j][i] * A[i][k]) % MOD;
                    A[j][k] = (A[j][k] - p + MOD) % MOD;
                }
            }
        }
    }

    vec x(n);
    for (int i = 0; i < n; i++)
        x[i] = A[i][m - 1];
    return x;
}
```

12.2 Determinant

```
typedef long long ll;
typedef vector<ll> vec;
typedef vector<vec> mat;

ll determinant(mat m) { // square matrix
    const int n = m.size();
    ll det = 1;
    for (int i = 0; i < n; i++) {
        for (int j = i + 1; j < n; j++) {
            int a = i, b = j;
            while (m[b][i]) {
                ll q = m[a][i] / m[b][i];
                for (int k = 0; k < n; k++)
                    m[a][k] = m[a][k] - m[b][k] * q;
                swap(a, b);
            }

            if (a != i) {
                swap(m[i], m[j]);
                det = -det;
            }
        }

        if (m[i][i] == 0)
            return 0;
        else
            det *= m[i][i];
    }
    return det;
}
```

13 Geometry

1. Keep things in integers as much as possible!
2. Try not to divide
3. If you have decimals, if they are fixed precision, you can usually just multiply all the input and use integers instead

13.1 EPS

$= 0$: $fabs \leq eps$
< 0: $< -eps$
> 0: $> +eps$

13.2 Template

```
// if the points are given in doubles form, change the code accordingly

typedef long long ll;

typedef pair<ll, ll> pt;  // points are stored using long long
typedef pair<pt, pt> seg; // segments are a pair of points

#define x first
#define y second

#define EPS 1e-9

pt operator+(pt a, pt b)
{
    return pt(a.x + b.x, a.y + b.y);
}

pt operator-(pt a, pt b)
{
    return pt(a.x - b.x, a.y - b.y);
}

pt operator*(pt a, int d)
{
    return pt(a.x * d, a.y * d);
}

ll cross(pt a, pt b)
{
```

```cpp
    return a.x * b.y - a.y * b.x;
}

int ccw(pt a, pt b, pt c)
{
    ll res = cross(b - a, c - a);
    if (res > 0) // left turn
        return 1;
    else if (res == 0) // straight
        return 0;
    else // right turn
        return -1;
}

double dist(pt a, pt b)
{
    double dx = a.x - b.x;
    double dy = a.y - b.y;
    return sqrt(dx * dx + dy * dy);
}

bool zero(double x)
{
    return fabs(x) <= EPS;
}

bool overlap(seg a, seg b)
{
    return ccw(a.x, a.y, b.x) == 0 && ccw(a.x, a.y, b.y) == 0;
}

bool intersect(seg a, seg b)
{
    if (overlap(a, b) == true) { // non-proper intersection
        double d = 0;
        d = max(d, dist(a.x, a.y));
        d = max(d, dist(a.x, b.x));
        d = max(d, dist(a.x, b.y));
        d = max(d, dist(a.y, b.x));
        d = max(d, dist(a.y, b.y));
        d = max(d, dist(b.x, b.y));

        // d > dist(a.x, a.y) + dist(b.x, b.y)
        if (d - (dist(a.x, a.y) + dist(b.x, b.y)) > EPS)
            return false;
        return true;
    }
    //                    |
    // Equal sign for -----| case
    // non qeual sign => proper intersection
    if (ccw(a.x, a.y, b.x) * ccw(a.x, a.y, b.y) <= 0 &&
        ccw(b.x, b.y, a.x) * ccw(b.x, b.y, a.y) <= 0)
        return true;
    return false;
}

double area(vector<pt> pts)
{
    double res = 0;
    int n = pts.size();
    for (int i = 0; i < n; i++)
        res += (pts[i].y + pts[(i + 1) % n].y) * (pts[(i + 1) % n].x - pts[i].x);
    return res / 2.0;
}

vector<pt> halfHull(vector<pt> &points)
{
    vector<pt> res;

    for (int i = 0; i < (int)points.size(); i++) {
        while ((int)res.size() >= 2 &&
                ccw(res[res.size() - 2], res[res.size() - 1], points[i]) < 0)
            res.pop_back(); // res.size() - 2 can't be assign before size() >= 2
        // check, bitch

        res.push_back(points[i]);
    }

    return res;
}

vector<pt> convexHull(vector<pt> &points)
{
    vector<pt> upper, lower;

    // make upper hull
    sort(points.begin(), points.end());

    upper = halfHull(points);
    // make lower hull
    reverse(points.begin(), points.end());
    lower = halfHull(points);

    // merge hulls
    if ((int)upper.size() > 0) // yes sir~
        upper.pop_back();
    if ((int)lower.size() > 0)
        lower.pop_back();

    vector<pt> res(upper.begin(), upper.end());
    res.insert(res.end(), lower.begin(), lower.end());

    return res;
}

bool completelyInside(vector<pt> &outer, vector<pt> &inner)
{
    int even = 0, odd = 0;
    for (int i = 0; i < (int)inner.size(); i++) {
```

```
        // y = slope * x + offset
        int cntIntersection = 0;
        ll slope = rand() % INT_MAX + 1;
        ll offset = inner[i].y - slope * inner[i].x;

        ll farx = 111111 * (slope >= 0 ? 1 : -1);
        ll fary = farx * slope + offset;
        seg a = seg(pt(inner[i].x, inner[i].y), pt(farx, fary));
        for (int j = 0; j < (int)outer.size(); j++) {
            seg b = seg(outer[j], outer[(j + 1) % (int)outer.size()]);

            if ((b.x.x * slope + offset == b.x.y) ||
                (b.y.x * slope + offset == b.y.y)) { // on-line
                i--;
                break;
            }

            if (intersect(a, b) == true)
                cntIntersection++;
        }

        if (cntIntersection % 2 == 0) // outside
            even++;
        else
            odd++;
    }

    return odd == (int)inner.size();
}

// srand(time(NULL))
// rand()
```

14 Math

14.1 Euclid's formula (Pythagorean Triples)

$a = p^2 - q^2$
$b = 2pq$ (always even)
$c = p^2 + q^2$

14.2 Difference between two consecutive numbers' square is odd

$(k+1)^2 - k^2 = 2k + 1$

14.3 Summation

$\sum_{k=1}^{n} 1 = n$
$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

14.4 FFT

```
typedef unsigned int ui;
typedef long double ldb;
const ldb pi = atan2(0, -1);

struct Complex {
    ldb real, imag;
    Complex(): real(0.0), imag(0.0) {;}
    Complex(ldb a, ldb b) : real(a), imag(b) {;}
    Complex conj() const {
        return Complex(real, -imag);
    }
    Complex operator + (const Complex& c) const {
        return Complex(real + c.real, imag + c.imag);
    }
    Complex operator - (const Complex& c) const {
        return Complex(real - c.real, imag - c.imag);
    }
    Complex operator * (const Complex& c) const {
        return Complex(real*c.real - imag*c.imag, real*c.imag + imag*c.real);
    }
    Complex operator / (ldb x) const {
        return Complex(real / x, imag / x);
    }
    Complex operator / (const Complex& c) const {
        return *this * c.conj() / (c.real * c.real + c.imag * c.imag);
    }
};

inline ui rev_bit(ui x, int len){
    x = ((x & 0x55555555u) << 1)  | ((x & 0xAAAAAAAAu) >> 1);
    x = ((x & 0x33333333u) << 2)  | ((x & 0xCCCCCCCCu) >> 2);
    x = ((x & 0x0F0F0F0Fu) << 4)  | ((x & 0xF0F0F0F0u) >> 4);
    x = ((x & 0x00FF00FFu) << 8)  | ((x & 0xFF00FF00u) >> 8);
    x = ((x & 0x0000FFFFu) << 16) | ((x & 0xFFFF0000u) >> 16);
    return x >> (32 - len);
}

// flag = -1 if ifft else +1
void fft(vector<Complex>& a, int flag = +1) {
    int n = a.size(); // n should be power of 2

    int len = __builtin_ctz(n);
    for (int i = 0; i < n; i++) {
        int rev = rev_bit(i, len);

        if (i < rev)
            swap(a[i], a[rev]);
    }

    for (int m = 2; m <= n; m <<= 1) { // width of each item
        auto wm = Complex(cos(2 * pi / m), flag * sin(2 * pi / m));
        for (int k = 0; k < n; k += m) { // start idx of each item
            auto w = Complex(1, 0);
```

```
            for (int j = 0; j < m / 2; j++) { // iterate half
                Complex t = w * a[k + j + m / 2];
                Complex u = a[k + j];
                a[k + j] = u + t;
                a[k + j + m / 2] = u - t;
                w = w * wm;
            }
        }
    }

    if (flag == -1) { // if it's ifft
        for (int i = 0; i < n; i++)
            a[i].real /= n;
    }
}

vector<int> mul(const vector<int>& a, const vector<int>& b) {
    int n = int(a.size()) + int(b.size()) - 1;
    int nn = 1;
    while (nn < n)
        nn <<= 1;

    vector<Complex> fa(nn, Complex(0, 0));
    vector<Complex> fb(nn, Complex(0, 0));
    for (int i = 0; i < int(a.size()); i++)
        fa[i] = Complex(a[i], 0);
    for (int i = 0; i < int(b.size()); i++)
        fb[i] = Complex(b[i], 0);

    fft(fa, +1);
    fft(fb, +1);
    for (int i = 0; i < nn; i++) {
        fa[i] = fa[i] * fb[i];
    }
    fft(fa, -1);

    vector<int> c;
    for(int i = 0; i < nn; i++) {
        int val = int(fa[i].real + 0.5);
        if (val) {
            while (int(c.size()) <= i)
                c.push_back(0);
            c[i] = 1;
        }
    }

    return c;
}
```

14.5 Combination

14.5.1 Pascal triangle

```
#define N 210
ll C[N][N];

void Combination() {
    for(ll i=0; i<N; i++) {
        C[i][0] = 1;
        C[i][i] = 1;
    }

    for(ll i=2; i<N; i++) {
        for(ll j=1; j<=i; j++) {
            C[i][j] = (C[i-1][j] + C[i-1][j-1])%M; // if needed, mod it
        }
    }
}
```

14.5.2 線性

```
ll binomialCoeff(ll n, ll k)
{
    ll res = 1;

    if ( k > n - k ) // Since C(n, k) = C(n, n-k)
        k = n - k;

    for (int i = 0; i < k; ++i) // n...n-k / 1...k
    {
        res *= (n - i);
        res /= (i + 1);
    }

    return res;
}
```

14.6 重複組合

14.7 Chinese remainder theorem

```
typedef long long ll;

struct Item {
    ll m, r;
};

ll extgcd(ll a, ll b, ll &x, ll &y)
{
    if (b == 0) {
        x = 1;
        y = 0;
        return a;
    } else {
        ll d = extgcd(b, a % b, y, x);
        y -= (a / b) * x;
        return d;
    }
}
```

```
Item extcrt(const vector<Item> &v)
{
    ll m1 = v[0].m, r1 = v[0].r, x, y;

    for (int i = 1; i < int(v.size()); i++) {
        ll m2 = v[i].m, r2 = v[i].r;
        ll g = extgcd(m1, m2, x, y); // now x = (m/g)^(-1)

        if ((r2 - r1) % g != 0)
            return {-1, -1};

        ll k = (r2 - r1) / g * x % (m2 / g);
        k = (k + m2 / g) % (m2 / g); // for the case k is negative

        ll m = m1 * m2 / g;
        ll r = (m1 * k + r1) % m;

        m1 = m;
        r1 = (r + m) % m; // for the case r is negative
    }

    return (Item) {
        m1, r1
    };
}
```

14.8 2-Circle relations

$d =$ 圓心距, R, r 為半徑 $(R \geq r)$
內切: $d = R - r$
外切: $d = R + r$
內離: $d < R - r$
外離: $d > R + r$
相交: $d < R + r$ 且 $d > R - r$

14.9 Fun Facts

1. 如果 $\frac{b}{a}$ 是最簡分數, 則 $1 - \frac{b}{a}$ 也是
2.

15 Dynamic Programming - Problems collection

好題收集

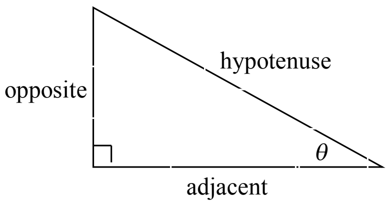
Trig Cheat Sheet

Definition of the Trig Functions

Right triangle definition

For this definition we assume that

$0 < \theta < \frac{\pi}{2}$ or $0° < \theta < 90°$.



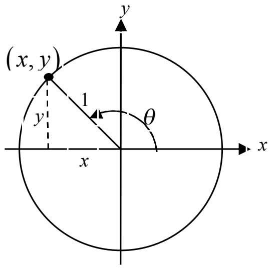
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Unit circle definition

For this definition θ is any angle.



$$\sin\theta = \frac{y}{1} = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{x}{1} = x \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Facts and Properties

Domain

The domain is all the values of θ that can be plugged into the function.

$\sin\theta$, θ can be any angle

$\cos\theta$, θ can be any angle

$\tan\theta$, $\theta \neq \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

$\csc\theta$, $\theta \neq n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

$\sec\theta$, $\theta \neq \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

$\cot\theta$, $\theta \neq n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

Range

The range is all possible values to get out of the function.

$$-1 \le \sin\theta \le 1 \qquad \csc\theta \ge 1 \text{ and } \csc\theta \le -1$$

$$-1 \le \cos\theta \le 1 \qquad \sec\theta \ge 1 \text{ and } \sec\theta \le -1$$

$$-\infty < \tan\theta < \infty \qquad -\infty < \cot\theta < \infty$$

Period

The period of a function is the number, T, such that $f(\theta + T) = f(\theta)$. So, if ω is a fixed number and θ is any angle we have the following periods.

$$\sin(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \rightarrow T = \frac{\pi}{\omega}$$

$$\csc(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\sec(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\cot(\omega\theta) \rightarrow T = \frac{\pi}{\omega}$$

Formulas and Identities

Tangent and Cotangent Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Even/Odd Formulas

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Periodic Formulas

If n is an integer.

$$\sin(\theta + 2\pi n) = \sin\theta \qquad \csc(\theta + 2\pi n) = \csc\theta$$

$$\cos(\theta + 2\pi n) = \cos\theta \qquad \sec(\theta + 2\pi n) = \sec\theta$$

$$\tan(\theta + \pi n) = \tan\theta \qquad \cot(\theta + \pi n) = \cot\theta$$

Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180} \quad \text{and} \quad x = \frac{180t}{\pi}$$

Half Angle Formulas (alternate form)

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}} \qquad \sin^2\theta = \frac{1}{2}\left(1 - \cos(2\theta)\right)$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}} \qquad \cos^2\theta = \frac{1}{2}\left(1 + \cos(2\theta)\right)$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} \qquad \tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Product to Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right]$$

Sum to Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

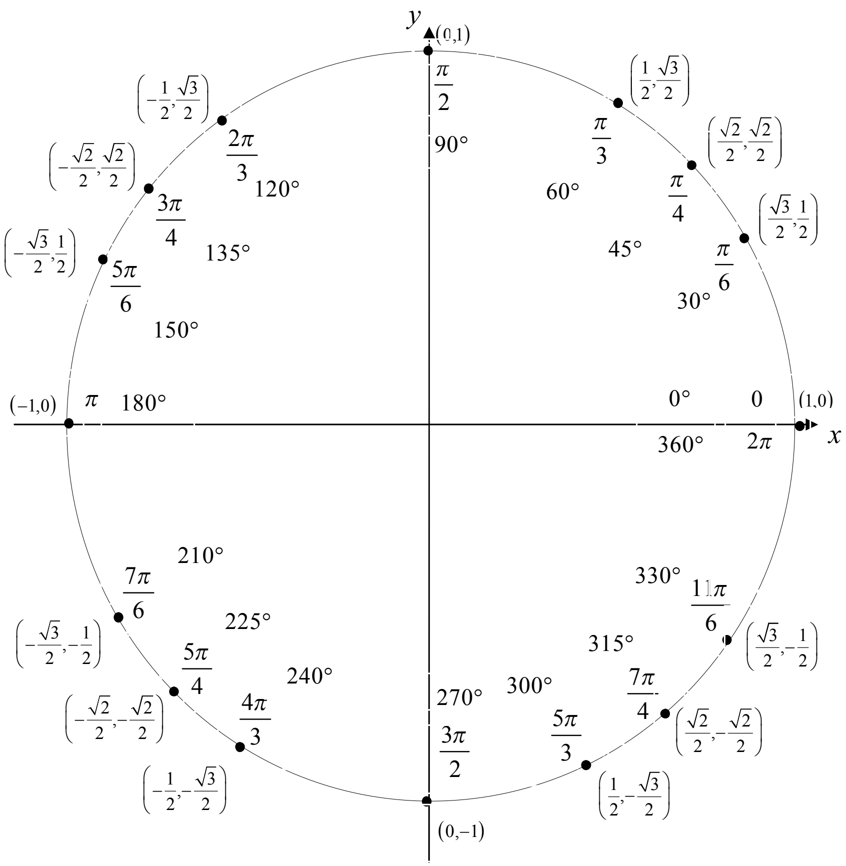
Cofunction Formulas

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

Unit Circle



For any ordered pair on the unit circle (x,y) : $\cos\theta = x$ and $\sin\theta = y$

Example

$$\cos\left(\frac{5\pi}{3}\right)=\frac{1}{2} \qquad \sin\left(\frac{5\pi}{3}\right)=-\frac{\sqrt{3}}{2}$$

Inverse Trig Functions

Definition

$y=\sin^{-1}x$ is equivalent to $x=\sin y$

$y=\cos^{-1}x$ is equivalent to $x=\cos y$

$y=\tan^{-1}x$ is equivalent to $x=\tan y$

Domain and Range

Function	Domain	Range
$y=\sin^{-1}x$	$-1\le x\le 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$
$y=\cos^{-1}x$	$-1\le x\le 1$	$0\le y\le\pi$
$y=\tan^{-1}x$	$-\infty< x<\infty$	$-\frac{\pi}{2}< y<\frac{\pi}{2}$

Inverse Properties

$\cos\left(\cos^{-1}(x)\right)=x \qquad \cos^{-1}\left(\cos(\theta)\right)=\theta$

$\sin\left(\sin^{-1}(x)\right)=x \qquad \sin^{-1}\left(\sin(\theta)\right)=\theta$

$\tan\left(\tan^{-1}(x)\right)=x \qquad \tan^{-1}\left(\tan(\theta)\right)=\theta$

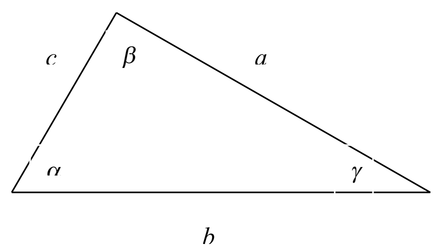
Alternate Notation

$\sin^{-1}x=\arcsin x$

$\cos^{-1}x=\arccos x$

$\tan^{-1}x=\arctan x$

Law of Sines, Cosines and Tangents



Law of Sines

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

Law of Cosines

$$a^2=b^2+c^2-2bc\cos\alpha$$

$$b^2=a^2+c^2-2ac\cos\beta$$

$$c^2=a^2+b^2-2ab\cos\gamma$$

Mollweide's Formula

$$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}$$

Law of Tangents

$$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$$

$$\frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$$

$$\frac{a-c}{a+c}=\frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$