Introduction to Aerial Robotics
Lecture 2

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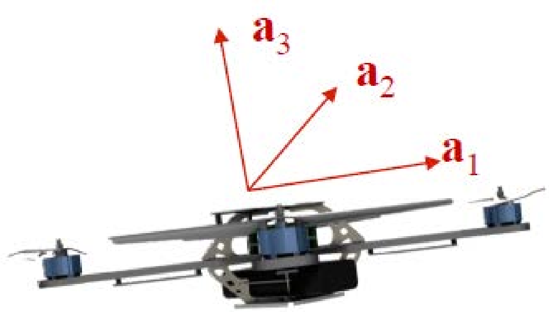
Outline

- Review: Rigid Body Displacement
- Review: Rotational Motions
- Rotation Representations
- Rigid Body Motions
- Rigid Body Velocities
- Quadrotor Dynamics

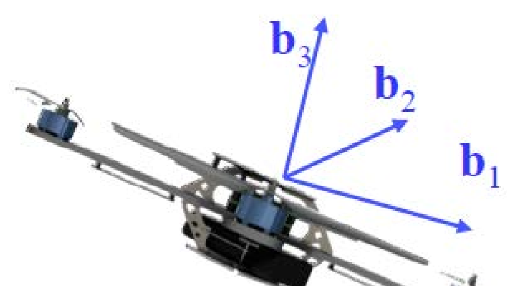
Review: Rigid Body Displacement

Reference Frames

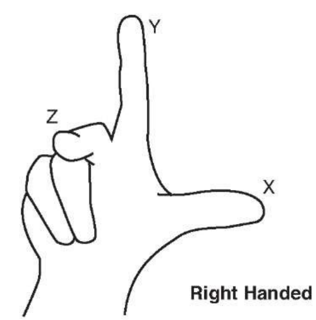
- We associate any position and orientation with a reference frame
  - We use right-handed coordinate frames
  - We can find three linearly independent vectors $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_3$ that are basis vectors for reference frame A
  - We can write any vector as a linear combination of basis vectors in either frame $\mathbf{v} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3$



A



B

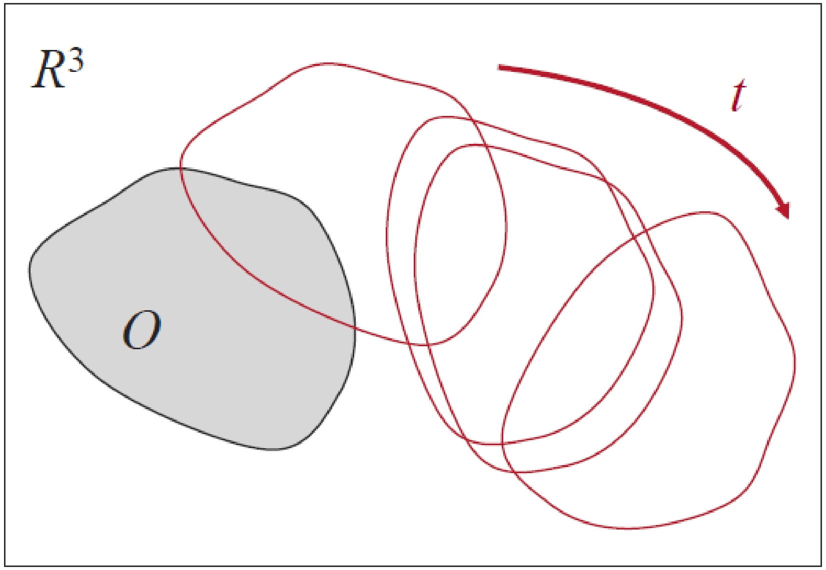


Courtesy Vijay Kumar

Rigid Body Displacement

- Object:

$$g \subset \Re^3$$



- Rigid body displacement
  - map

$$g : O \rightarrow \Re^3$$

- Rigid body motion
  - Continuous family of maps

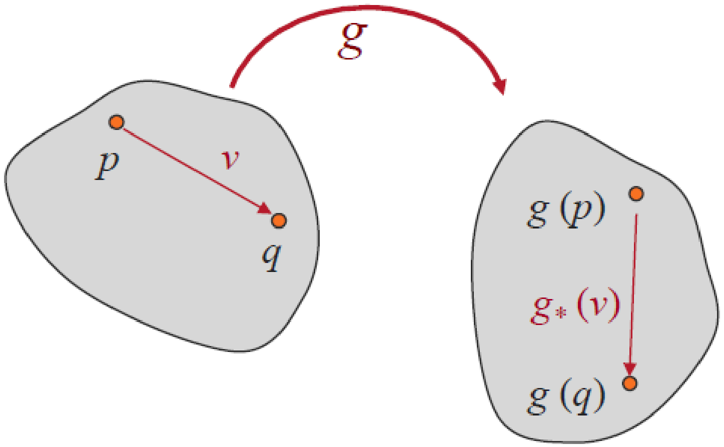
$$g(t) : O \rightarrow \Re^3$$

Courtesy Vijay Kumar

Rigid Body Displacement

- A displacement of a transformation of points
  - Transformation (g) of points induces an action (g*) on vectors

$$g_*(\mathbf{v}) = g(\mathbf{q}) - g(\mathbf{p})$$

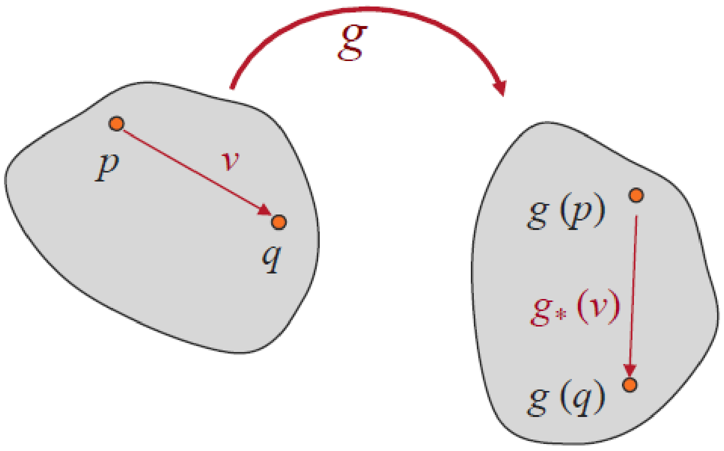


Courtesy Vijay Kumar

Definition of Rigid Body Displacement

- Lengths are preserved

$$\|g(\mathbf{q})-g(\mathbf{p})\|=\|\mathbf{q}-\mathbf{p}\|$$



- Cross products are preserved

$$g_*(\mathbf{v})\times g_*(\mathbf{w})=g_*(\mathbf{v}\times\mathbf{w})$$

Why?

Eliminate internal reflection: $(x, y, z)\rightarrow(x, y, -z)$

Rigid Body Displacement

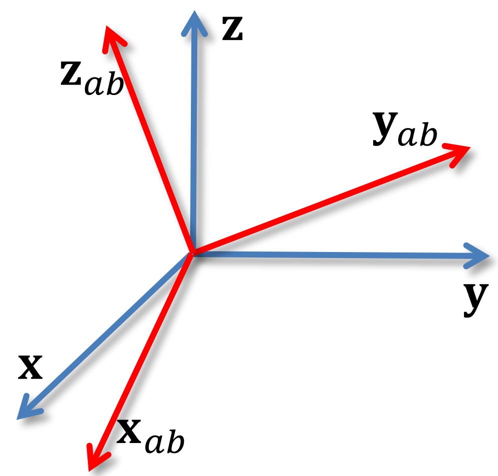
- Rigid body displacements are transformations that satisfy two important properties:
  1. Lengths are preserved
  2. Cross products are preserved

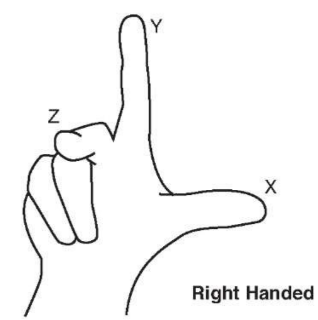
- Rigid body transformations and rigid body displacements are often used interchangeably
  - Transformations generally used to describe relationship between reference frames attached to different rigid bodies.
  - Displacements describe relationships between two positions and orientation of a frame attached to a displaced rigid body

Review: Rotational Motions

Rotation

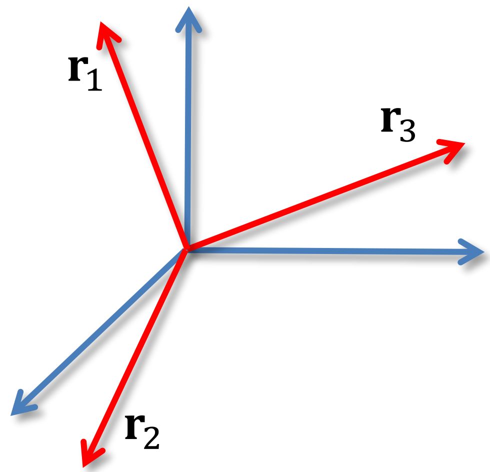
- Coordinate frames are right-handed
- Principle axes of frame *A*:
  - $\mathbf{x} = [1\ 0\ 0]^T$
  - $\mathbf{y} = [0\ 1\ 0]^T$
  - $\mathbf{z} = [0\ 0\ 1]^T$
- Principle axes of frame *B*:
  - $\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab} \subset \mathbb{R}^3$
- The Rotation Matrix:
  - $\mathbf{R}_{ab} = [\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}]$
  - Coordinates of principle axes of *B* related to *A*





Properties of a Rotation Matrix

- Let $\mathbf{R} = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3]$ be a rotation matrix
- Orthogonal:
  - $\mathbf{r}_i{}^T \cdot \mathbf{r}_j = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$
  - $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$
- Special orthogonal:
  - $\det\mathbf{R} = \mathbf{r}_1{}^T \cdot (\mathbf{r}_2 \times \mathbf{r}_3) = \mathbf{r}_1{}^T \cdot \mathbf{r}_1 = 1$
- The set of all rotations forms the Special Orthogonal Group
  - Special orthogonal group
  - 3D rotations: $SO(3)$
  - 2D rotations: $SO(2)$
  - $SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times3} | \mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}, \det\mathbf{R} = 1\}$



Properties of a Rotation Matrix

- $SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} | \mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}, \det\mathbf{R} = 1\}$

- $SO(3)$ is a group under the operation of matrix multiplication
  1. Closure: If $\mathbf{R}_1, \mathbf{R}_2 \in SO(3)$, then $\mathbf{R}_1 \cdot \mathbf{R}_2 \in SO(3)$
  2. Identity: The identity matrix is the identity element
  3. Inverse: If $\mathbf{R} \in SO(3)$, then $\mathbf{R}^{-1} \in SO(3)$
  4. Associativity: $\mathbf{R}_1 \cdot (\mathbf{R}_2 \cdot \mathbf{R}_3) = (\mathbf{R}_1 \cdot \mathbf{R}_2) \cdot \mathbf{R}_3$

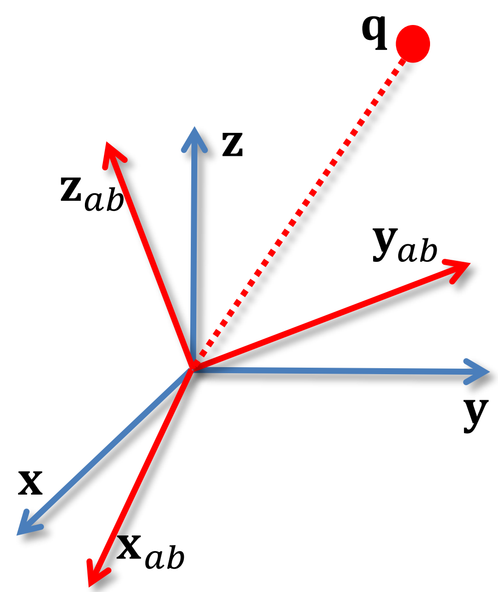
$(G, \cdot)$ is a group if:

1. $g_1, g_2 \in G \Rightarrow g_1 \cdot g_2 \in G$
2. $\exists!\ e \in G$, s.t. $g \cdot e = e \cdot g = g, \forall g \in G$
3. $\forall g \in G, \exists!\ g^{-1} \in G$, s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$
4. $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

Courtesy Zexiang Li

Properties of a Rotation Matrix

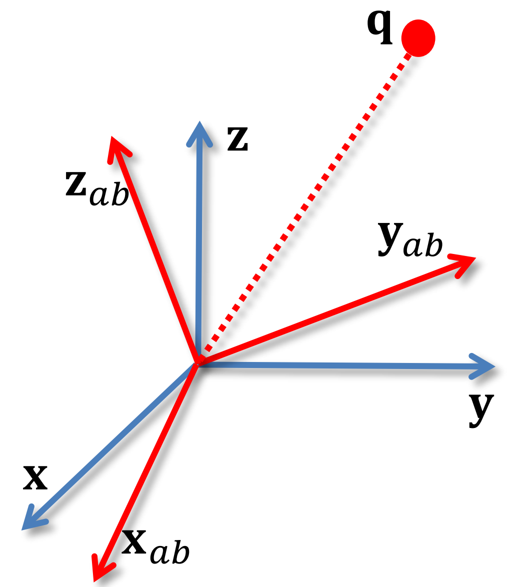
- A transformation that rotates the coordinates of a point from frame *B* to frame *A*
  - Let $\mathbf{q}_b = [x_b, y_b, z_b]^T \in \mathbb{R}^3$ be coordinate of point $\mathbf{q}$ in frame *B*
  - Let $\mathbf{q}_a$ be coordinate of point $\mathbf{q}$ in frame *A*
  - $\mathbf{q}_a = x_b \cdot \mathbf{x}_{ab} + y_b \cdot \mathbf{y}_{ab} + z_b \cdot \mathbf{z}_{ab} = [\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}] \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \mathbf{R}_{ab} \cdot \mathbf{q}_b$



- Composition rule
  - $\mathbf{R}_{ac} = \mathbf{R}_{ab} \cdot \mathbf{R}_{bc}$
  - $\mathbf{q}_a = \mathbf{R}_{ac} \cdot \mathbf{q}_c = \mathbf{R}_{ab} \cdot \mathbf{R}_{bc} \cdot \mathbf{q}_c$

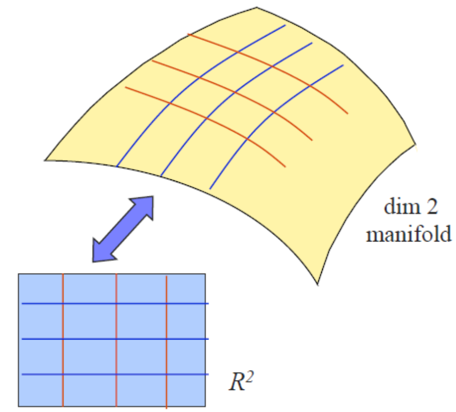
Rotation is Rigid Body Transformation

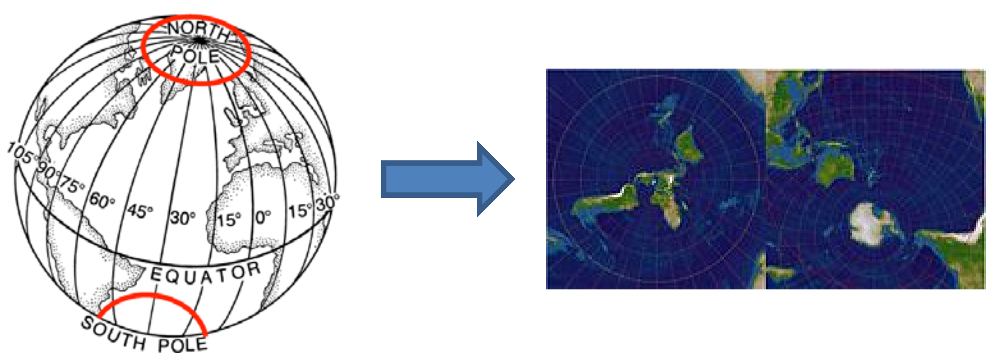
- $\mathbf{R}_{ab} = [\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}]$ preserves:
  - Length:
    - $\|\mathbf{R}_{ab}(\mathbf{p}_b - \mathbf{q}_b)\| = \|\mathbf{p}_b - \mathbf{q}_b\|$
  - Cross product:
    - $\mathbf{R}_{ab}(\mathbf{v} \times \boldsymbol{w}) = (\mathbf{R}_{ab}\mathbf{v}) \times (\mathbf{R}_{ab}\mathbf{w})$
    - Use the fact $\mathbf{R}(\mathbf{v})^{\wedge}\mathbf{R}^T = (\mathbf{R}\mathbf{v})^{\wedge}$ to prove, where $(\mathbf{a})^{\wedge} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$ is the skew-symmetric matrix, and $\mathbf{a} \times \mathbf{b} = (\mathbf{a})^{\wedge}\mathbf{b}$

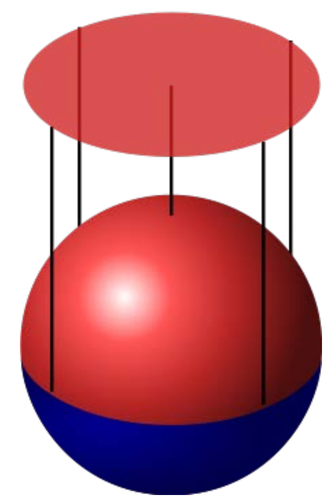


Properties of Rotation

- $SO(3)$ is a continuous group
  - The binary operation (multiplication) is a continuous operation
  - The inverse is a continuous function
- $SO(3)$ is a smooth manifold
  - A manifold of dimension n is a set M which is locally homeomorphic to $\mathbb{R}^n$
  - Sphere is a differentiable manifold that is locally homomorphic to $\mathbb{R}^2$







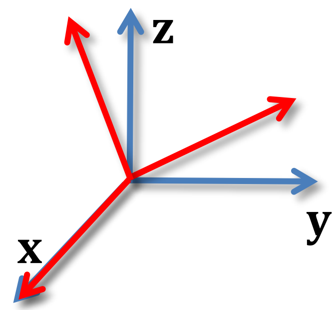
Rotation Representations

Rotation Representations

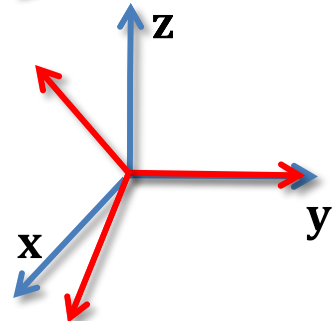
- Rotation matrix
- Euler angle
- Exponential coordinates
- Angle axis parameterization
- Quaternion

Euler Angles

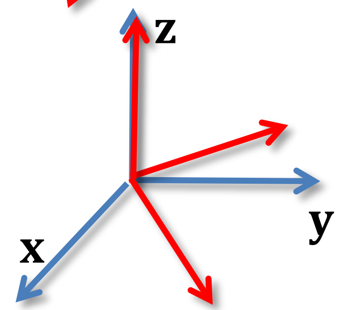
- Elementary rotations:
  - $R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$



  - $R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

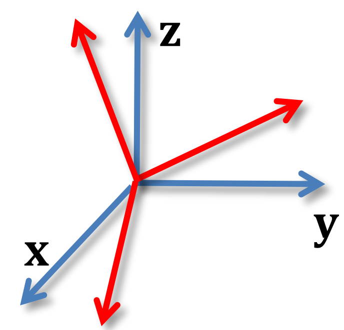


  - $R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Euler Angles

- Any rotation can be described by three successive rotations about linear independent axes
- However, this is an almost 1-1 transform with singularities:
  - $\boldsymbol{R}_z(\psi) \cdot \boldsymbol{R}_x(\phi) \cdot \boldsymbol{R}_y(\theta) \Rightarrow \boldsymbol{R}$
  - $\boldsymbol{R}_z(\psi) \cdot \boldsymbol{R}_x(\phi) \cdot \boldsymbol{R}_y(\theta) \nLeftarrow \boldsymbol{R}$



Euler angles

- Different Euler angle conversions:

Proper Euler angles	Tait-Bryan angles
$X_1Z_2X_3 = \begin{bmatrix} c_2 & -c_3s_2 & s_2s_3 \\ c_1s_2 & c_1c_2c_3 - s_1s_3 & -c_3s_1 - c_1c_2s_3 \\ s_1s_2 & c_1s_3 + c_2c_3s_1 & c_1c_3 - c_2s_1s_3 \end{bmatrix}$	$X_1Z_2Y_3 = \begin{bmatrix} c_2c_3 & -s_2 & c_2s_3 \\ s_1s_3 + c_1c_3s_2 & c_1c_2 & c_1s_2s_3 - c_3s_1 \\ c_3s_1s_2 - c_1s_3 & c_2s_1 & c_1c_3 + s_1s_2s_3 \end{bmatrix}$
$X_1Y_2X_3 = \begin{bmatrix} c_2 & s_2s_3 & c_3s_2 \\ s_1s_2 & c_1c_3 - c_2s_1s_3 & -c_1s_3 - c_2c_3s_1 \\ -c_1s_2 & c_3s_1 + c_1c_2s_3 & c_1c_2c_3 - s_1s_3 \end{bmatrix}$	$X_1Y_2Z_3 = \begin{bmatrix} c_2c_3 & -c_2s_3 & s_2 \\ c_1s_3 + c_3s_1s_2 & c_1c_3 - s_1s_2s_3 & -c_2s_1 \\ s_1s_3 - c_1c_3s_2 & c_3s_1 + c_1s_2s_3 & c_1c_2 \end{bmatrix}$
$Y_1X_2Y_3 = \begin{bmatrix} c_1c_3 - c_2s_1s_3 & s_1s_2 & c_1s_3 + c_2c_3s_1 \\ s_2s_3 & c_2 & -c_3s_2 \\ -c_3s_1 - c_1c_2s_3 & c_1s_2 & c_1c_2c_3 - s_1s_3 \end{bmatrix}$	$Y_1X_2Z_3 = \begin{bmatrix} c_1c_3 + s_1s_2s_3 & c_3s_1s_2 - c_1s_3 & c_2s_1 \\ c_2s_3 & c_2c_3 & -s_2 \\ c_1s_2s_3 - c_3s_1 & c_1c_3s_2 + s_1s_3 & c_1c_2 \end{bmatrix}$
$Y_1Z_2Y_3 = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_1s_2 & c_3s_1 + c_1c_2s_3 \\ c_3s_2 & c_2 & s_2s_3 \\ -c_1s_3 - c_2c_3s_1 & s_1s_2 & c_1c_3 - c_2s_1s_3 \end{bmatrix}$	$Y_1Z_2X_3 = \begin{bmatrix} c_1c_2 & s_1s_3 - c_1c_3s_2 & c_3s_1 + c_1s_2s_3 \\ s_2 & c_2c_3 & -c_2s_3 \\ -c_2s_1 & c_1s_3 + c_3s_1s_2 & c_1c_3 - s_1s_2s_3 \end{bmatrix}$
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$Z_1X_2Z_3 = \begin{bmatrix} c_1c_3 - c_2s_1s_3 & -c_1s_3 - c_2c_3s_1 & s_1s_2 \\ c_3s_1 + c_1c_2s_3 & c_1c_2c_3 - s_1s_3 & -c_1s_2 \\ s_2s_3 & c_3s_2 & c_2 \end{bmatrix}$	$Z_1X_2Y_3 = \begin{bmatrix} c_1c_3 - s_1s_2s_3 & -c_2s_1 & c_1s_3 + c_3s_1s_2 \\ c_3s_1 + c_1s_2s_3 & c_1c_2 & s_1s_3 - c_1c_3s_2 \\ -c_2s_3 & s_2 & c_2c_3 \end{bmatrix}$

Euler Angles

- Example: Z-Y-Z Euler angles:
  - Sequence of three rotations about body-fixed axes
  - $\boldsymbol{R} = \boldsymbol{R}_z(\phi) \cdot \boldsymbol{R}_y(\theta) \cdot \boldsymbol{R}_z(\psi)$
  - $\boldsymbol{R} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi c\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$
- If $\sin\theta \neq 0$:
  - $\theta = \mathrm{acos}(r_{33})$
  - $\psi = \mathrm{atan}\,2(\frac{r_{32}}{\sin\theta}, -\frac{r_{31}}{\sin\theta})$
  - $\phi = \mathrm{atan}\,2(\frac{r_{23}}{\sin\theta}, \frac{r_{13}}{\sin\theta})$

Euler Angles

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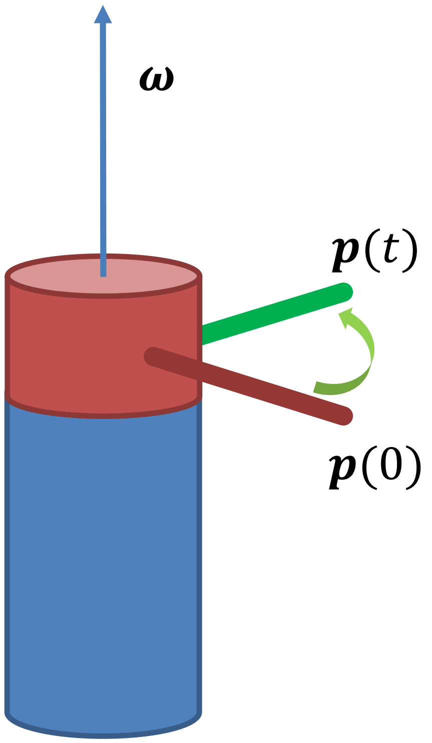
  - $\boldsymbol{R} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi c\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$
- If $\sin\theta = 0$:
  - $\boldsymbol{R} = \begin{bmatrix} c\phi c\psi - s\phi s\psi & -c\phi s\psi - s\phi c\psi & 0 \\ c\phi s\psi + s\phi c\psi & -s\phi s\psi + c\phi c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{R}_z(\phi + \psi)$
  - As long as $\phi + \psi$ is preserved, we have infinite set of Euler angles!

Exponential Coordinates

- Scalar differential equation:
  - $\begin{cases} \dot{x}(t) = ax(t) \\ x(0) = x_0 \end{cases} \Longrightarrow x(t) = e^{at}x_0$
- Matrix differential equation:
  - $\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \Longrightarrow \boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$
  - $e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} + \frac{1}{2}\boldsymbol{A}^2 + \frac{1}{3!}\boldsymbol{A}^3 + \cdots + \frac{1}{n!}\boldsymbol{A}^n + \cdots$

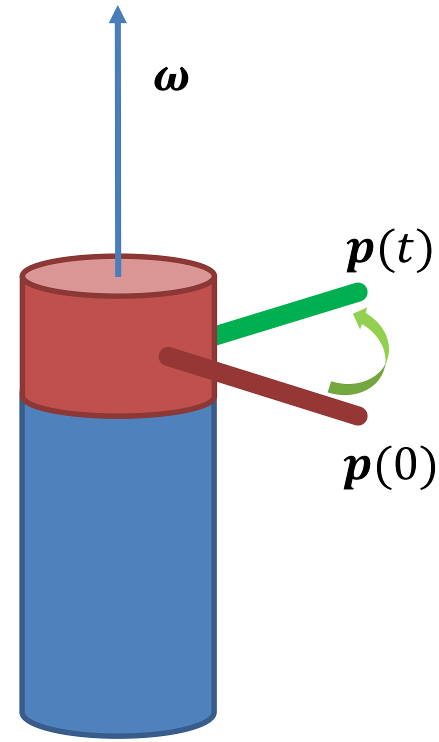
Exponential Coordinates

- Degree-of-freedom of $SO(3)$:
  - $\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$
  - $\boldsymbol{r}_i{}^T \cdot \boldsymbol{r}_j = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \Longrightarrow$ 6 constraints
  - $\boldsymbol{R}$ has only 3 independent parameters
- Consider the motion of a point about a rotating link $\boldsymbol{\omega}$ at constant unit velocity:
  - $\begin{cases} \dot{\boldsymbol{p}}(t) = \boldsymbol{\omega} \times \boldsymbol{p}(t) = \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{p}(t) \\ \boldsymbol{p}(0) = \boldsymbol{p}_0 \end{cases} \Longrightarrow \boldsymbol{p}(t) = e^{\widehat{\boldsymbol{\omega}} t} \boldsymbol{p}_0$



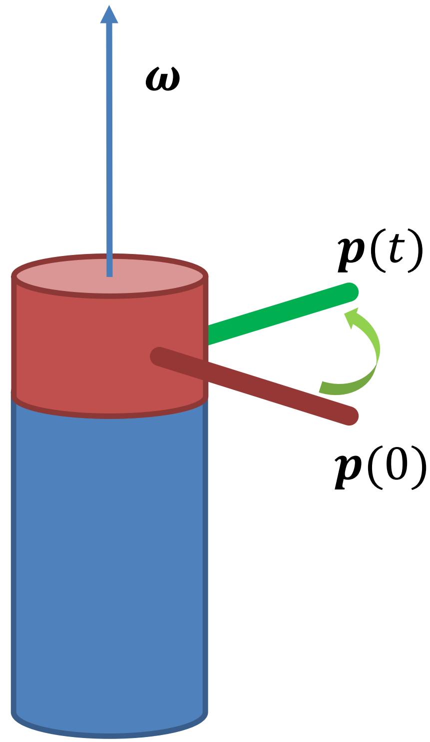
Exponential Coordinates

- Consider the motion of a point about a rotating link $\boldsymbol{\omega}$ at constant unit velocity:
  - $\begin{cases} \dot{\boldsymbol{p}}(t) = \boldsymbol{\omega} \times \boldsymbol{p}(t) = \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{p}(t) \\ \boldsymbol{p}(0) = \boldsymbol{p}_0 \end{cases} \Longrightarrow \boldsymbol{p}(t) = e^{\widehat{\boldsymbol{\omega}}t}\boldsymbol{p}_0$
  - $\widehat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$
- Rotating about ω at unit velocity for θ units:
  - $\boldsymbol{R}(\boldsymbol{\omega}, \theta) = e^{\widehat{\boldsymbol{\omega}}\theta}$



Exponential Coordinates

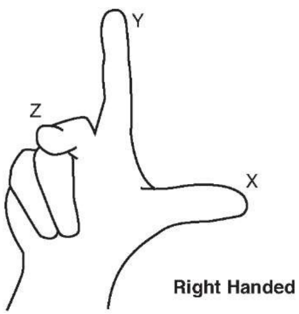
- The vector space of all 3×3 skew-symmetric matrices is denoted as $so(3)$:
  - $so(3) = \{\boldsymbol{S} \in \mathbb{R}^{3\times 3}: \boldsymbol{S}^T = -\boldsymbol{S}\}$
- The exponential map:
  - $\boldsymbol{R}(\boldsymbol{\omega}, \theta) = e^{\widehat{\boldsymbol{\omega}}\theta} = \boldsymbol{I} + \widehat{\boldsymbol{\omega}} \sin\theta + \widehat{\boldsymbol{\omega}}^2(1 - \cos\theta)$
- $e^{\widehat{\boldsymbol{\omega}}\theta} \in SO(3)$
  - $\left[e^{\widehat{\boldsymbol{\omega}}\theta}\right]^{-1} = e^{-\widehat{\boldsymbol{\omega}}\theta} = e^{\widehat{\boldsymbol{\omega}}^T\theta} = \left[e^{\widehat{\boldsymbol{\omega}}\theta}\right]^T$
  - Since $\det e^{\boldsymbol{0}} = 1$, and both determinant and exponential map are continuous functions, we know $\det e^{\widehat{\boldsymbol{\omega}}\theta} = 1$
- The exponential map is onto (many to one)
  - $\theta = 0 \Rightarrow \boldsymbol{\omega}$ can be chosen arbitrary

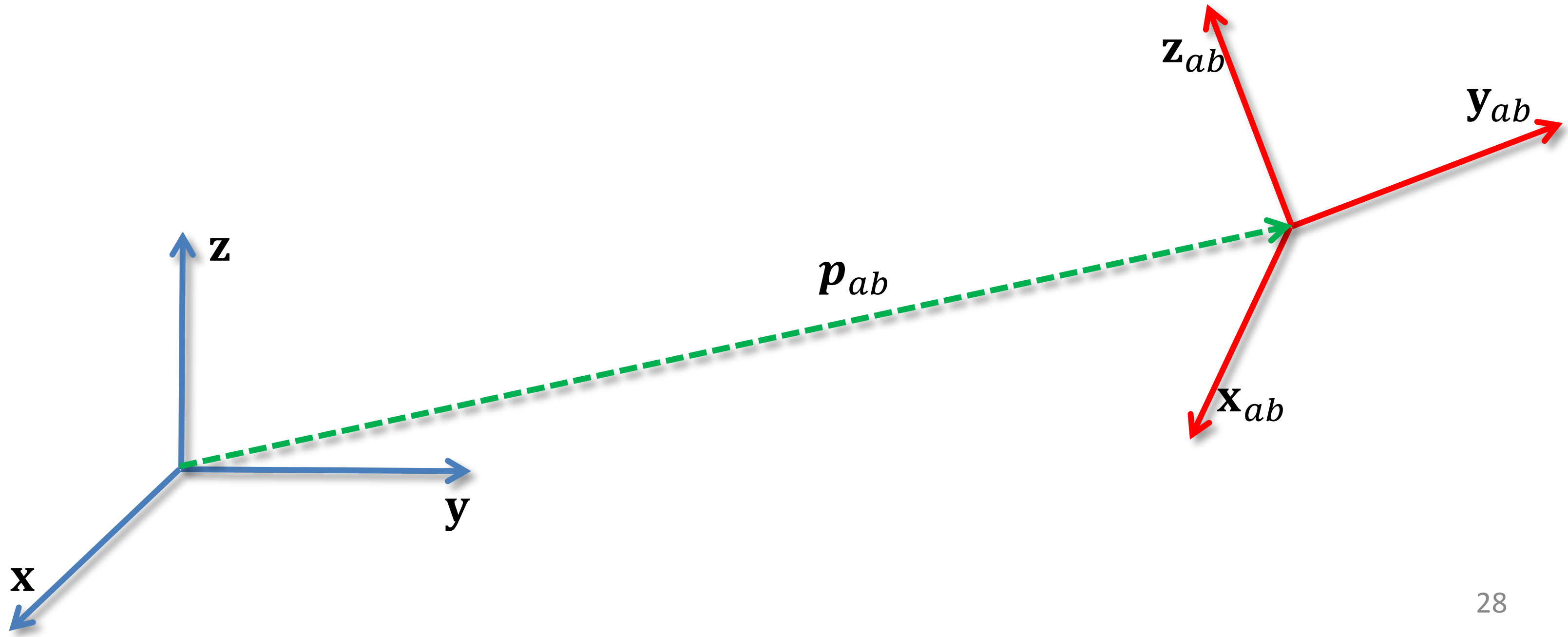


Rigid Body Motions

Rigid Body Motion

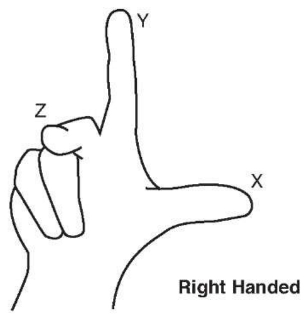
- General rigid body motions that includes both translation and rotation forms the product space of $\mathbb{R}^3$ and $SO(3)$. Denoted as $SE(3)$ – Special Euclidean group.
  - $SE(3) = \{(\boldsymbol{p}, \boldsymbol{R}): \boldsymbol{p} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)\} = \mathbb{R}^3 \times SO(3)$

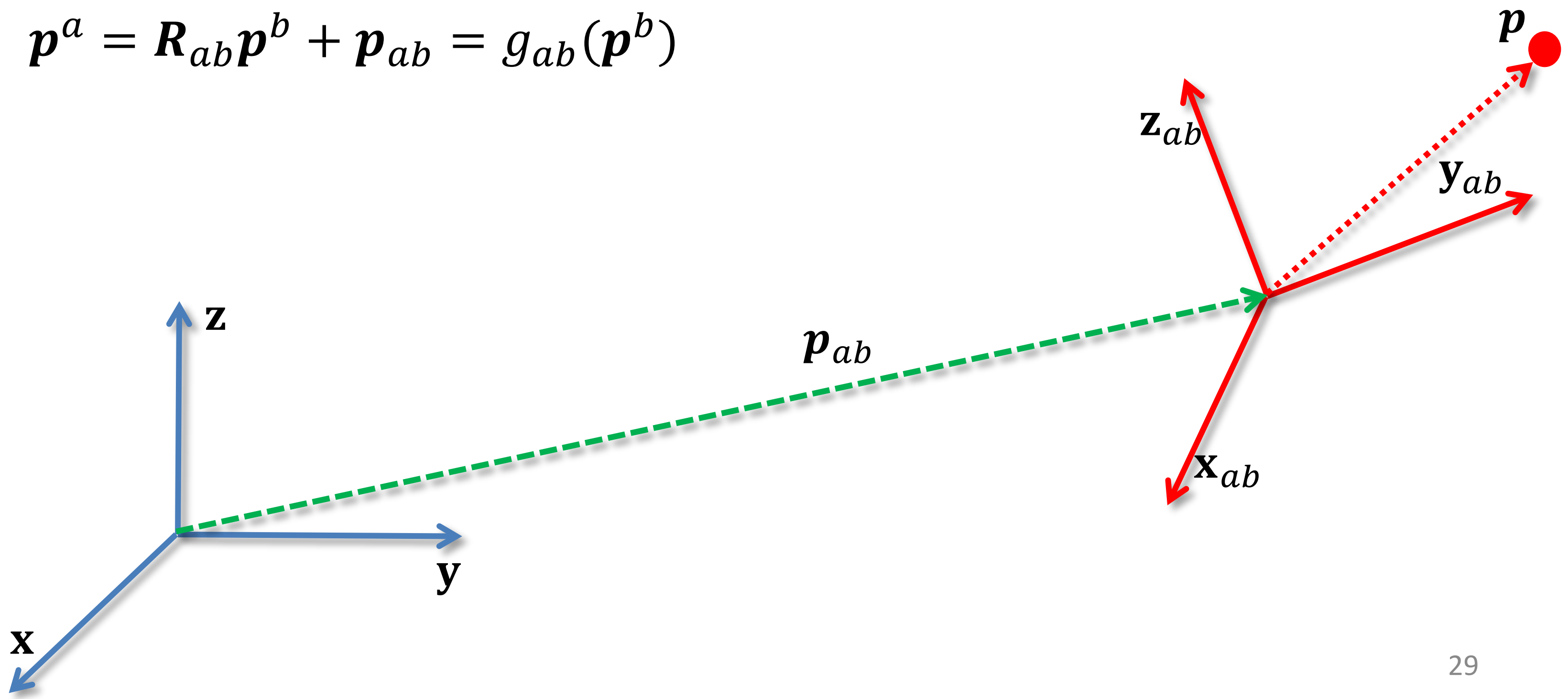




Rigid Body Motion

- Special Euclidean group:
  - $SE(3) = \{(\boldsymbol{p}, \boldsymbol{R}): \boldsymbol{p} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)\} = \mathbb{R}^3 \times SO(3)$
- Transformation of a point between different coordinate frames:
  - $\boldsymbol{p}^a = \boldsymbol{R}_{ab}\boldsymbol{p}^b + \boldsymbol{p}_{ab} = g_{ab}(\boldsymbol{p}^b)$



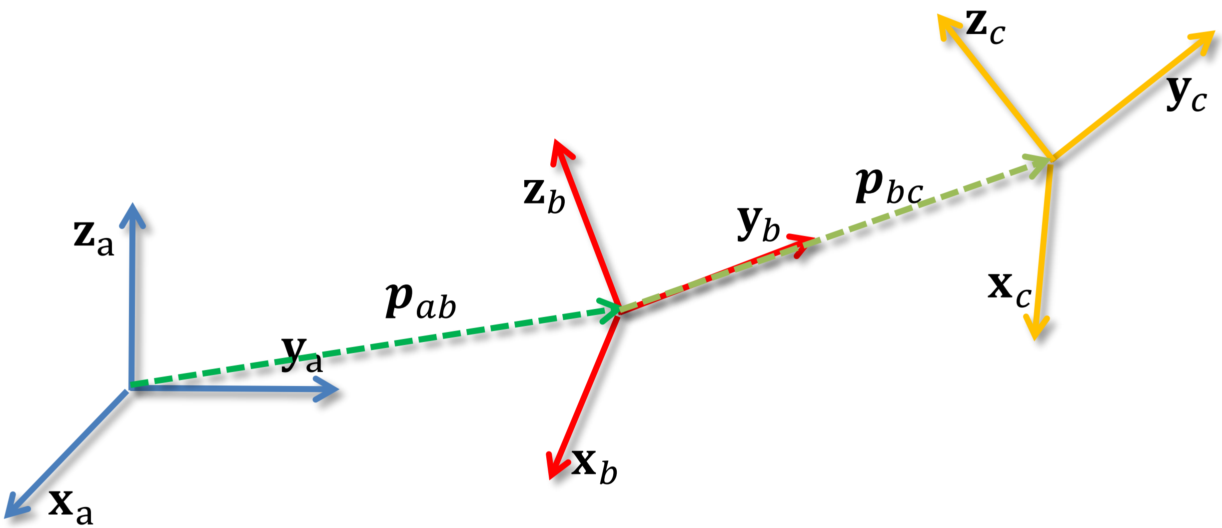


Rigid Body Motion

- Homogeneous coordinates of a point:
  - $\bar{\boldsymbol{p}} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$
- Homogeneous coordinates of a vector:
  - $\bar{\boldsymbol{v}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$
- Homogeneous representation of rigid body motion:
  - $\bar{\boldsymbol{p}}^a = \begin{bmatrix} \boldsymbol{p}^a \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{ab} & \boldsymbol{p}_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}^b \\ 1 \end{bmatrix} = \bar{g}_{ab} \bar{\boldsymbol{p}}^b$

Rigid Body Motion

- Homogeneous representation of rigid body motion:
  - $\bar{g}_{ab} = \begin{bmatrix} \boldsymbol{R}_{ab} & \boldsymbol{p}_{ab} \\ 0 & 1 \end{bmatrix}$
- Composition rule for rigid body motions:
  - $\bar{g}_{ac} = \bar{g}_{ab} \cdot \bar{g}_{bc} = \begin{bmatrix} \boldsymbol{R}_{ab}\boldsymbol{R}_{bc} & \boldsymbol{R}_{ab}\boldsymbol{p}_{bc} + \boldsymbol{p}_{ab} \\ 0 & 1 \end{bmatrix}$
  - Compare with composition of rotational motion: $\boldsymbol{R}_{ac} = \boldsymbol{R}_{ab} \cdot \boldsymbol{R}_{bc}$



Properties of Rigid Body Motion

- $SE(3) = \{(\boldsymbol{p}, \boldsymbol{R}): \boldsymbol{p} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)\} = \mathbb{R}^3 \times SO(3)$

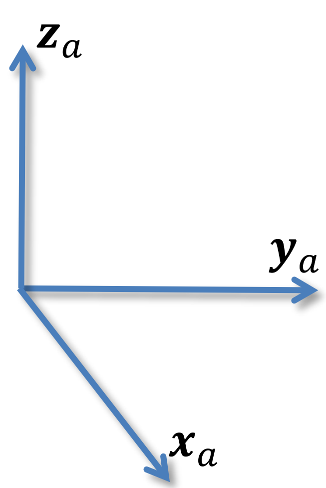
- $SE(3)$ is a group under the operation of matrix multiplication
  - Closure
  - Identity
  - Inverse
  - Associativity
- $g \in SE(3)$ is a rigid body transformation
  - Lengths are preserved
  - Cross products are preserved

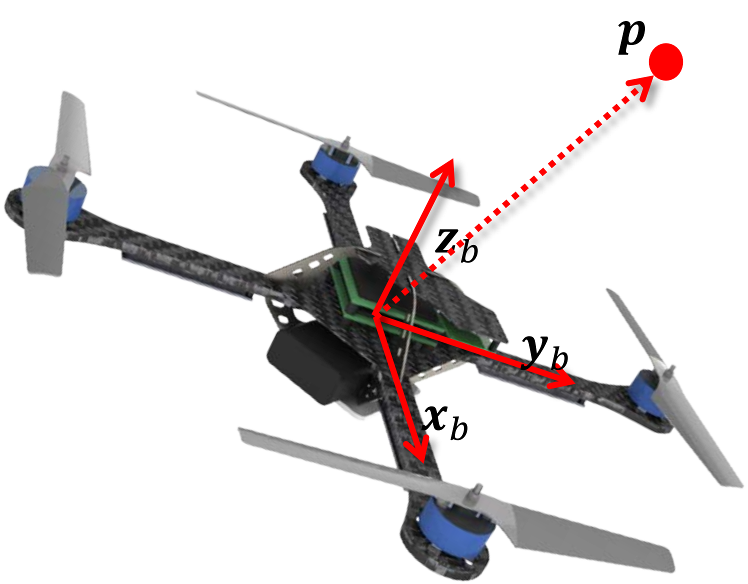
Proof it yourself!

Rigid Body Velocities

Angular Velocity

- Coordinate frames:
  - Frame A: spatial frame
  - Frame B: body frame
- A point attached to the body follows a rotational path in spatial frame:
  - $\boldsymbol{p}^a(t) = \boldsymbol{R}_{ab}\boldsymbol{p}^b$

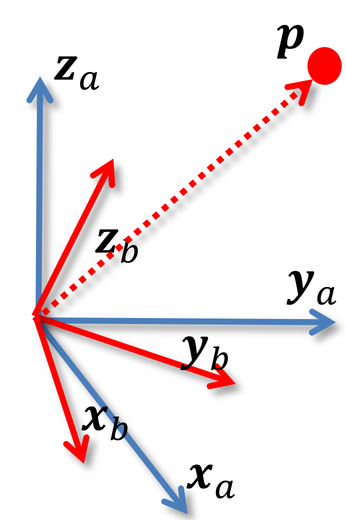




Angular Velocity

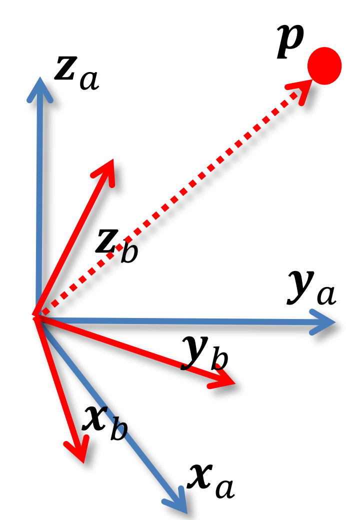
- Coordinate frames:
  - Frame A: spatial frame
  - Frame B: body frame
- A point attached to the body follows a rotational path in spatial frame:
  - $\boldsymbol{p}^a(t) = \boldsymbol{R}_{ab}\boldsymbol{p}^b$
- The velocity of the point in spatial frame:
  - $\boldsymbol{v}_p^a(t) = \frac{d}{dt}\boldsymbol{p}^a(t) = \dot{\boldsymbol{R}}_{ab}(t)\boldsymbol{p}^b$
- This can be rewritten as:
  - $\boldsymbol{v}_p^a(t) = \boxed{\dot{\boldsymbol{R}}_{ab}(t)\boldsymbol{R}_{ab}^{-1}(t)}\boldsymbol{R}_{ab}(t)\boldsymbol{p}^b$

Skew-symmetric matrix. Why?



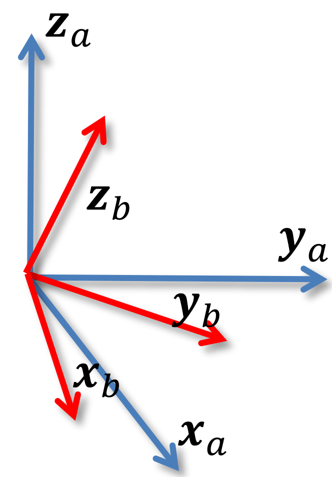
Angular Velocity

- The instantaneous spatial angular velocity $\boldsymbol{\omega}_{ab}^{a}$
  - $\widehat{\boldsymbol{\omega}}_{ab}^{a} = \dot{\boldsymbol{R}}_{ab} \cdot \boldsymbol{R}_{ab}^{-1}$
- The instantaneous body angular velocity $\boldsymbol{\omega}_{ab}^{b}$
  - $\widehat{\boldsymbol{\omega}}_{ab}^{b} = \boldsymbol{R}_{ab}^{-1} \cdot \dot{\boldsymbol{R}}_{ab}$
- Conversion:
  - $\widehat{\boldsymbol{\omega}}_{ab}^{b} = \boldsymbol{R}_{ab}^{-1} \cdot \widehat{\boldsymbol{\omega}}_{ab}^{a} \cdot \boldsymbol{R}_{ab}$
  - $\boldsymbol{\omega}_{ab}^{b} = \boldsymbol{R}_{ab}^{-1} \cdot \boldsymbol{\omega}_{ab}^{a}$
- Velocity induced by rotational motion:
  - $\boldsymbol{v}_{p}^{a} = \widehat{\boldsymbol{\omega}}_{ab}^{a} \cdot \boldsymbol{R}_{ab} \cdot \boldsymbol{p}^{b} = \boldsymbol{\omega}_{ab}^{a} \times \boldsymbol{p}^{a}$
  - $\boldsymbol{v}_{p}^{b} = \boldsymbol{R}_{ab}^{T} \cdot \boldsymbol{v}_{p}^{a} = \boldsymbol{\omega}_{ab}^{b} \times \boldsymbol{p}^{b}$



Angular Velocity

- Numerical Integration
  - $\dot{\boldsymbol{R}} = \boldsymbol{R}\widehat{\boldsymbol{\omega}}^b \Rightarrow \boldsymbol{R}(t+\Delta t) \sim \boldsymbol{R}(t) + \Delta t \cdot \boldsymbol{R}(t)\widehat{\boldsymbol{\omega}}^b$
  - $\dot{\boldsymbol{R}} = \widehat{\boldsymbol{\omega}}^a\boldsymbol{R} \Rightarrow \boldsymbol{R}(t+\Delta t) \sim \boldsymbol{R}(t) + \Delta t \cdot \widehat{\boldsymbol{\omega}}^a\boldsymbol{R}(t)$
- Constant speed rotation
  - $\boldsymbol{R}(t) = \boldsymbol{R}_0 \cdot \exp(\widehat{\boldsymbol{\omega}}_0^b \cdot t)$
  - $\boldsymbol{R}(t) = \exp(\widehat{\boldsymbol{\omega}}_0^a \cdot t) \cdot \boldsymbol{R}_0$



Angular Velocity

- Simple example

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \dot{R} = \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\theta}$$

Angular Velocity

- Simple example

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T \dot{R} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

$$= \dot{R} R^T = \dot{\theta} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} = \widehat{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \dot{\theta}$$

Angular Velocity

- Two rotations

$$R = R_z(\theta)R_x(\phi)$$

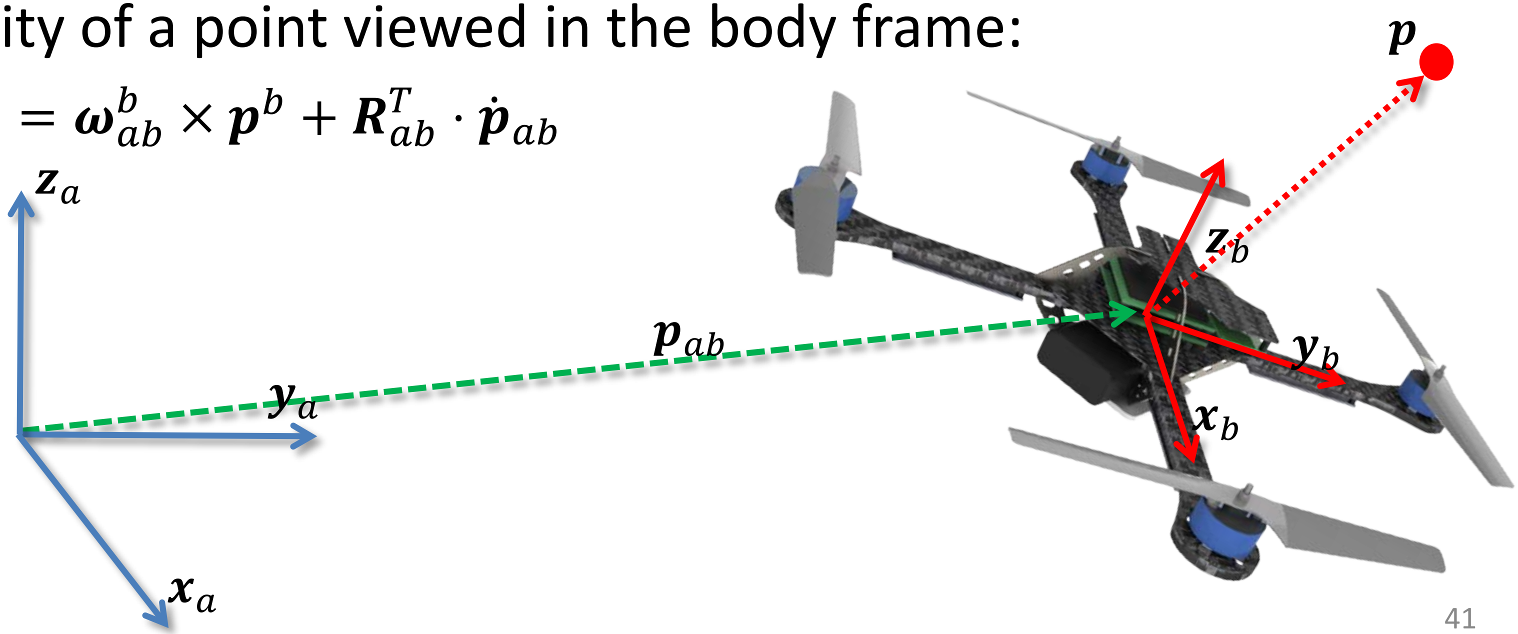
$$\begin{aligned}\hat{\omega}^b = R^T\dot{R} &= (R_zR_x)^T(\dot{R}_zR_x + R_z\dot{R}_x)\\ &= R_x^TR_z^T\dot{R}_zR_x + R_x^T\dot{R}_x\end{aligned}$$

$$\begin{aligned}\hat{\omega}^s = \dot{R}R^T &= (\dot{R}_zR_x + R_z\dot{R}_x)(R_zR_x)^T\\ &= \dot{R}_zR_z^T + R_z\dot{R}_xR_x^TR_z^T\end{aligned}$$

Rigid Body Velocity

For details, refer to Chapter 2.4 of "A Mathematical Introduction to Robotic Manipulation"

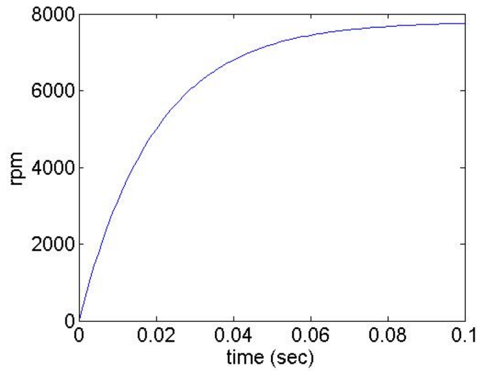
- General rigid body transformation:
  - $g_{ab} = \begin{bmatrix} \boldsymbol{R}_{ab} & \boldsymbol{p}_{ab} \\ 0 & 1 \end{bmatrix}$
- Velocity of a point viewed in the spatial frame:
  - $\boldsymbol{v}_p^a = \boldsymbol{\omega}_{ab}^a \times \boldsymbol{p}^a - \boldsymbol{\omega}_{ab}^a \times \boldsymbol{p}_{ab} + \dot{\boldsymbol{p}}_{ab}$
- Velocity of a point viewed in the body frame:
  - $\boldsymbol{v}_p^b = \boldsymbol{\omega}_{ab}^b \times \boldsymbol{p}^b + \boldsymbol{R}_{ab}^T \cdot \dot{\boldsymbol{p}}_{ab}$

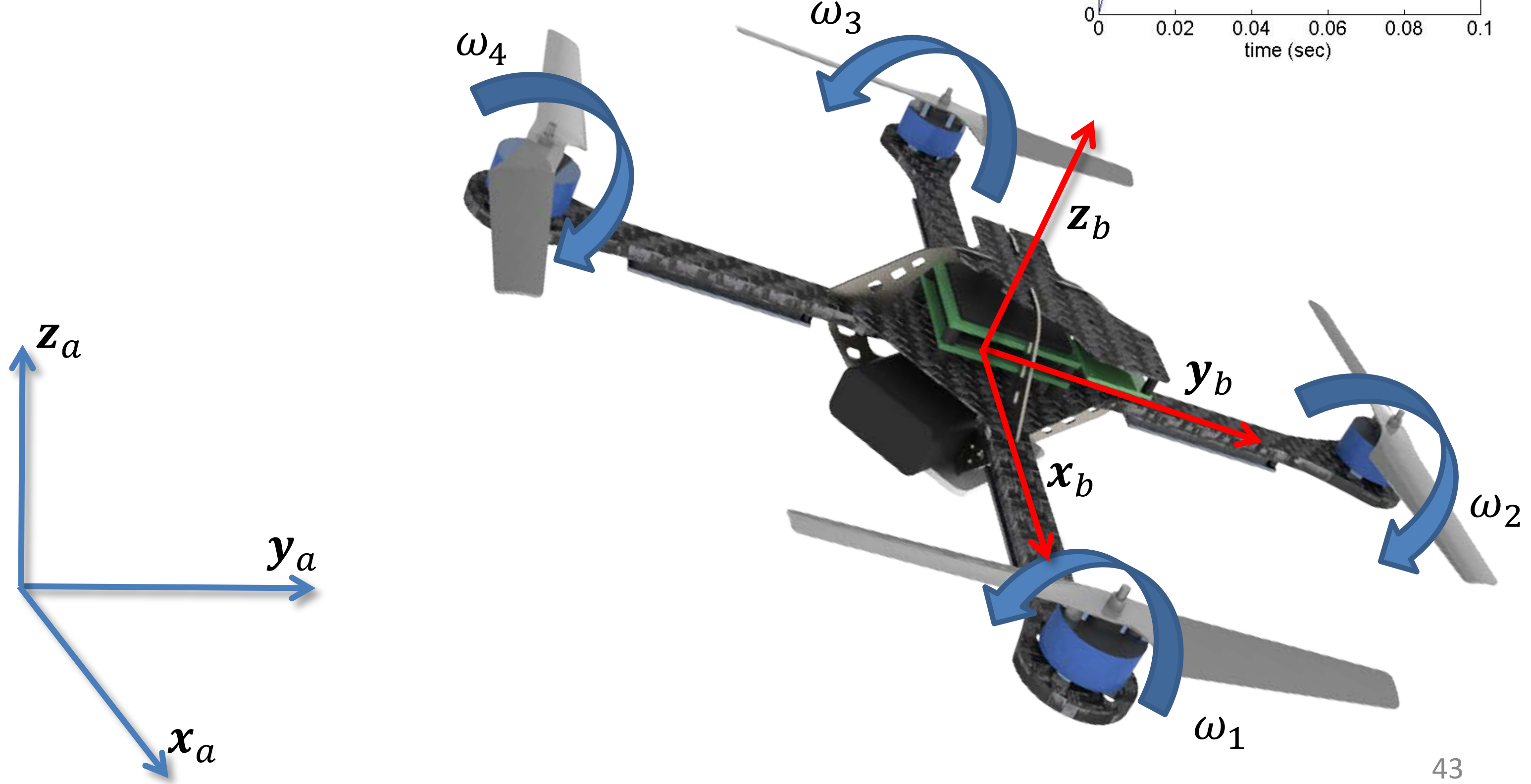


Quadrotor Dynamics

Quadrotor Dynamics

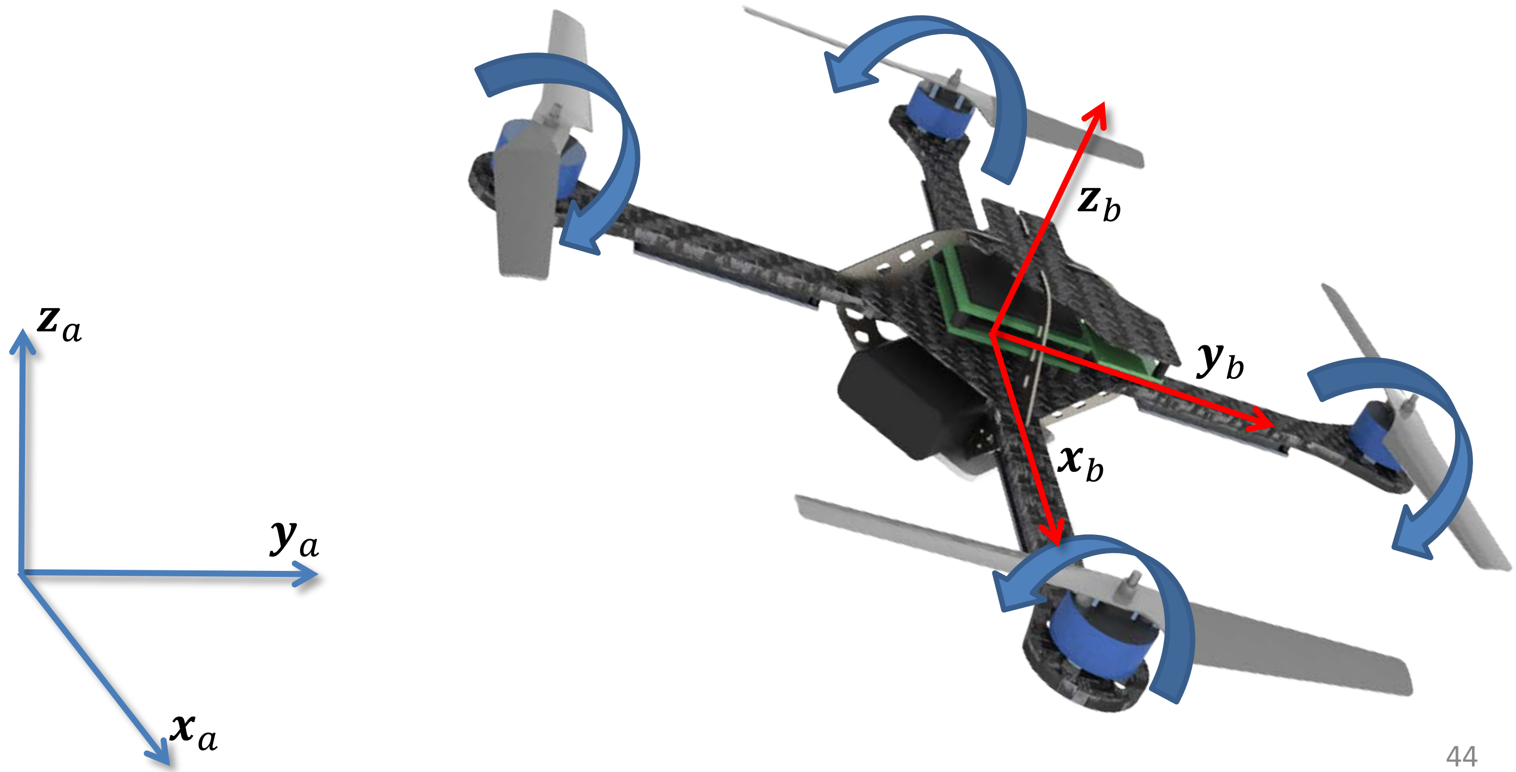
- Motor model: $\dot{\omega}_i = k_m(\omega_i^{des} - \omega_i)$
- Thrust from individual motor: $F_i = k_F \omega_i^2$
- Moment from individual motor: $M_i = k_M \omega_i^2$





Quadrotor Dynamics

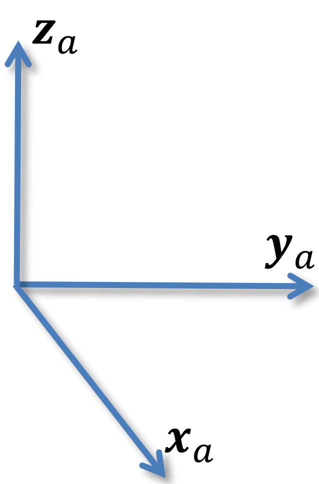
- Z-X-Y Euler Angles: $\boldsymbol{R}_{ab} = \boldsymbol{R}_z(\psi) \cdot \boldsymbol{R}_x(\phi) \cdot \boldsymbol{R}_y(\theta)$
- Sequence of three rotations about body-fixed axes
- What are the singularities?

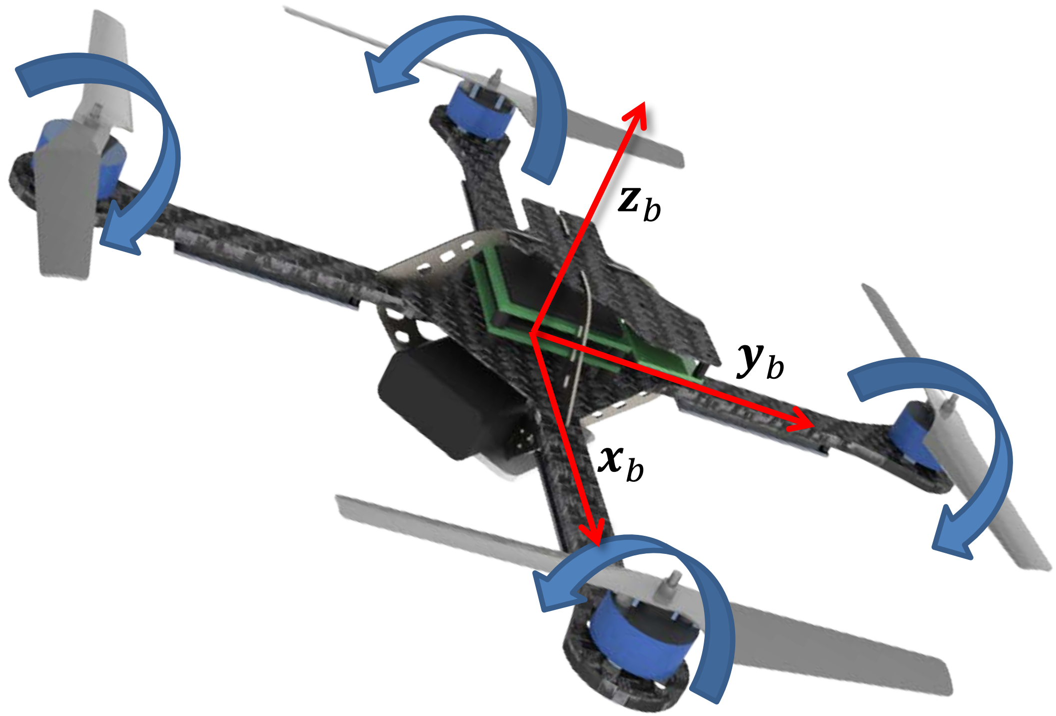


Quadrotor Dynamics

- $$\boldsymbol{R}_{ab} = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\psi c\theta s\phi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$$
- $$\boxed{\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}} = \begin{bmatrix} c\theta & 0 & -c\phi s\theta \\ 0 & 1 & s\phi \\ s\theta & 0 & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

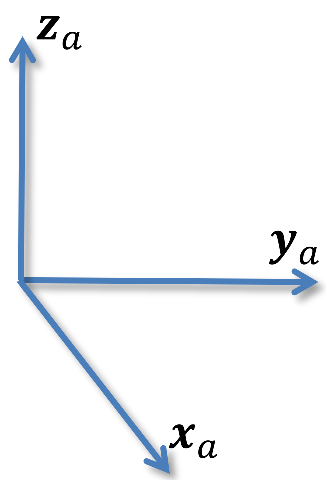
Instantaneous body angular velocity. How to compute?

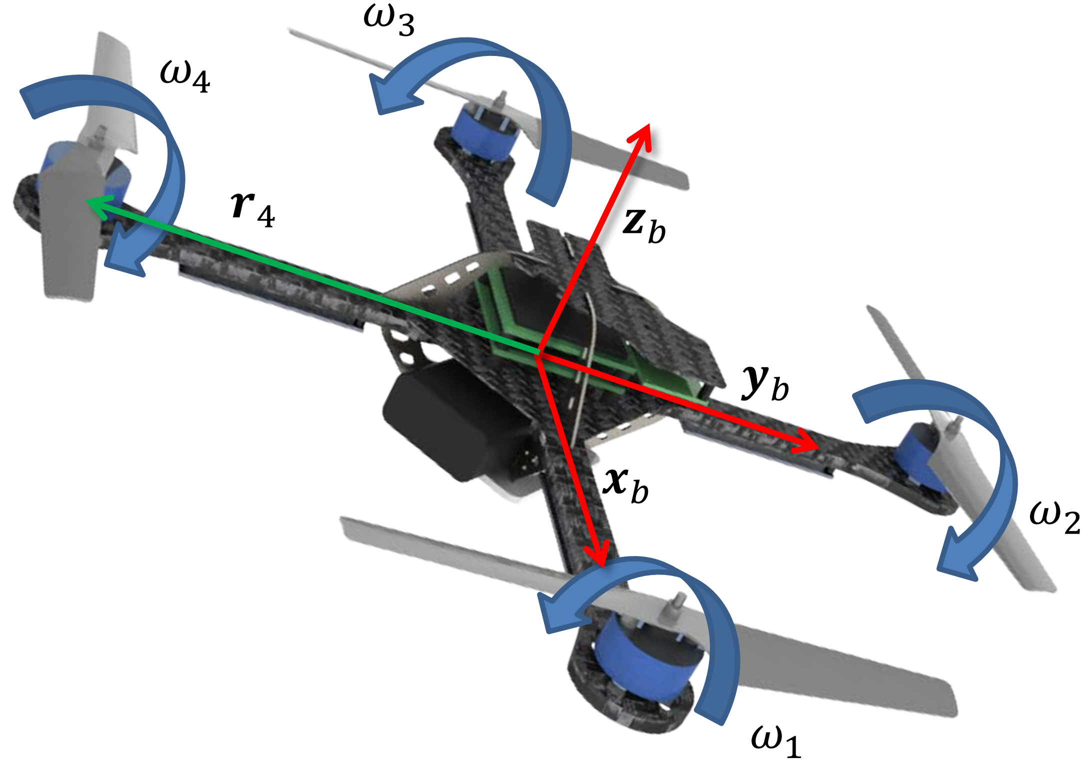




Quadrotor Dynamics

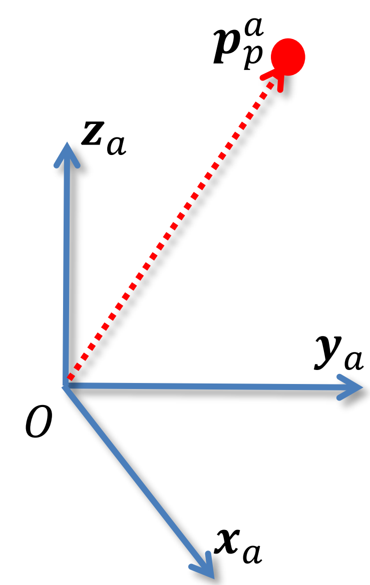
- $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4 - mg\boldsymbol{z}_a$
- $\boldsymbol{M} = \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2 + \boldsymbol{r}_3 \times \boldsymbol{F}_3 + \boldsymbol{r}_4 \times \boldsymbol{F}_4 + \boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3 + \boldsymbol{M}_4$
- $\boldsymbol{F}_i = [0, 0, F_i]^T$
- $\boldsymbol{M}_i = [0, 0, \pm M_i]^T$





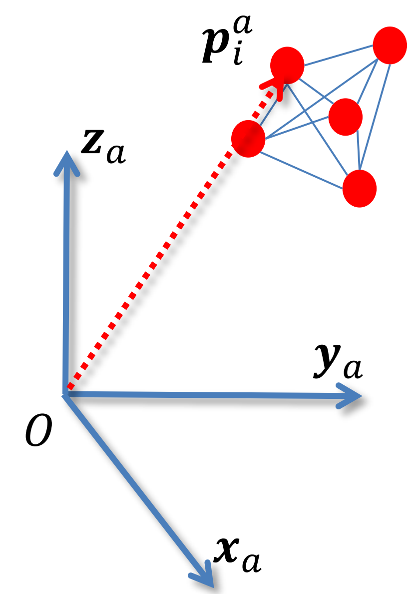
Newton-Euler Equations

- Newton's Second Law for a particle in the inertial frame A:
  - Position vector: $\boldsymbol{p}_p^a$
  - Velocity: $\boldsymbol{v}_p^a = \frac{d\,\boldsymbol{p}_p^a}{dt}$
  - Force acting on the particle with mass m: $\boldsymbol{F} = m \cdot \frac{d\,\boldsymbol{v}_p^a}{dt}$
  - Linear momentum: $\boldsymbol{L}_p^a = m\boldsymbol{v}_p^a$
  - Angular momentum relative to O: $\boldsymbol{H}_p^{ao} = \boldsymbol{p}_p^a \times \boldsymbol{L}_p^a$
- We are interested in the rate of change of linear and angular momentums in A:
  - $\frac{d\,\boldsymbol{L}_p^a}{dt} = \boldsymbol{F}$
  - $\frac{d\,\boldsymbol{H}_p^{ao}}{dt} = ?$



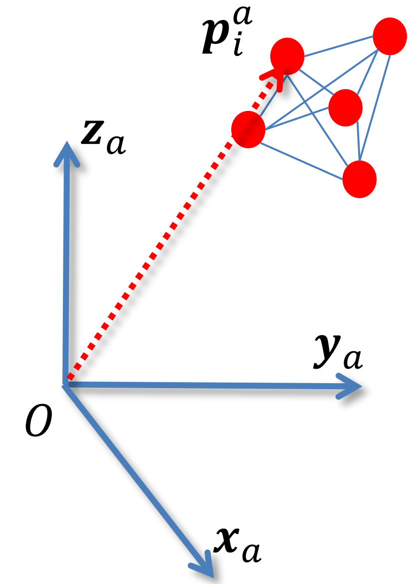
Newton-Euler Equations

- Newton's Second Law for a system of particles in the inertial frame A:
  - Mass m_i at $\boldsymbol{p}_i^a$
  - $\boldsymbol{F}_i = \boldsymbol{F}_{ik}^{int} + \boldsymbol{F}_i^{ext}$ is the net internal and external forces acting on m_i
  - Total mass $m = \sum m_i$
  - Center of mass $\boldsymbol{r}_c = \frac{1}{m} \sum m_i \boldsymbol{p}_i^a$
  - The center of mass of a system of particles S, accelerates in an inertial frame A as if it is a single particle with mass m, acted upon by a force equal to the net external force $\boldsymbol{F} = \sum \boldsymbol{F}_i^{ext}$



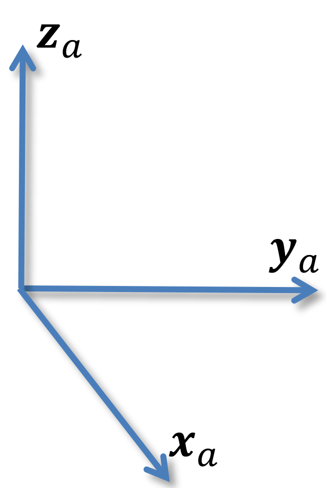
Newton-Euler Equations

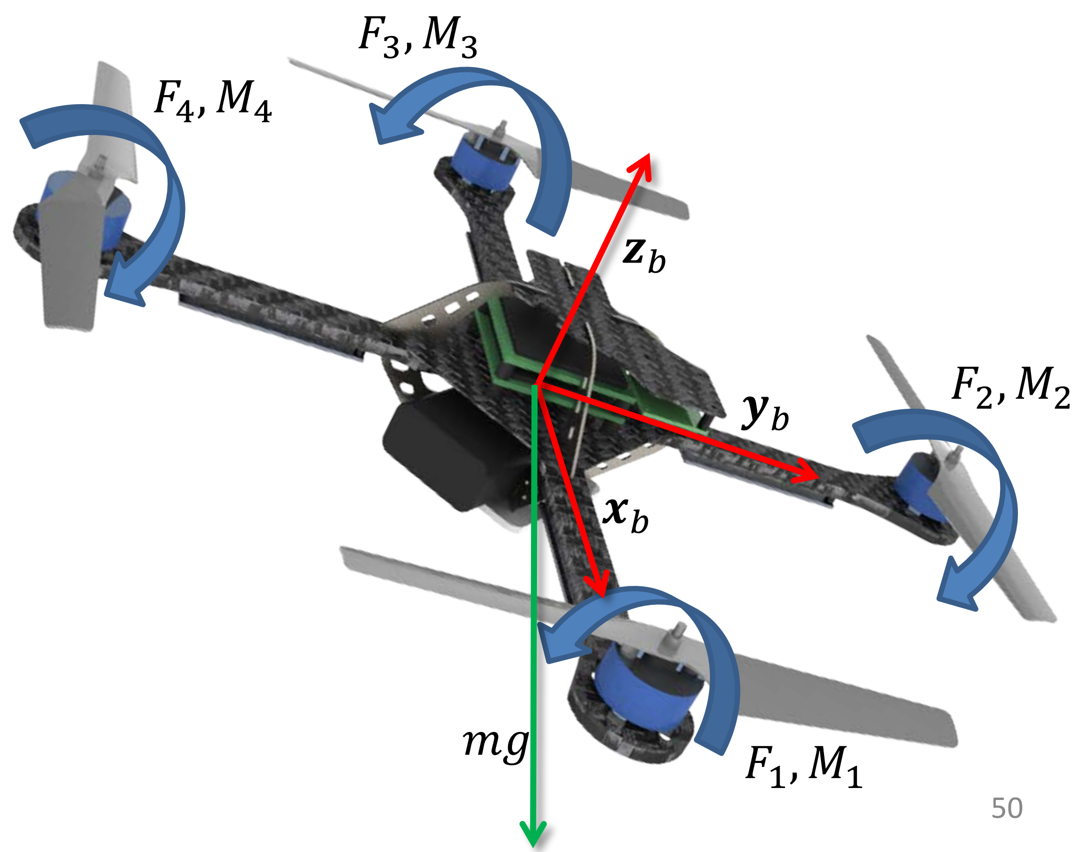
- Linear momentum of the center of mass in frame A:
  - $\boldsymbol{L}_c^a = m \cdot \boldsymbol{v}_c^a$
- Rate of change of linear momentum:
  - $\boldsymbol{F} = m \cdot \frac{d\,\boldsymbol{v}_c^a}{dt} = \frac{d\,\boldsymbol{L}_c^a}{dt}$



Quadrotor Dynamics

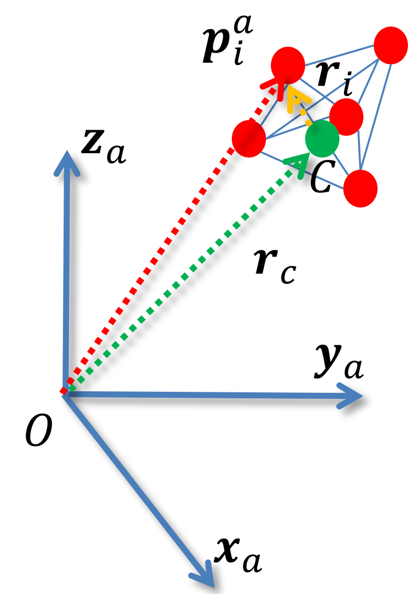
- Newton Equation: $m\ddot{\boldsymbol{r}}^{a} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \boldsymbol{R}_{ab} \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$





Newton-Euler Equations

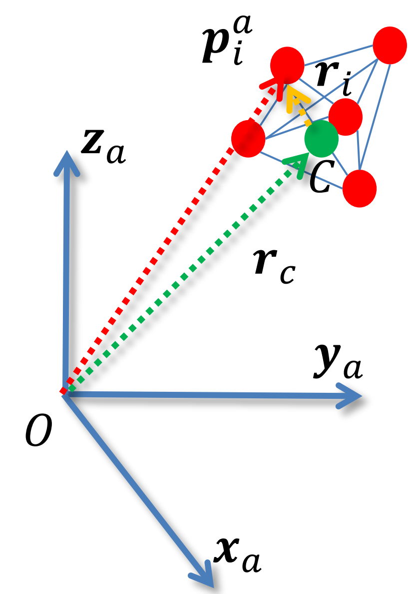
- Angular momentum of a particle in the inertial frame A relative to O:
  - $\boldsymbol{H}_i^{ao} = \boldsymbol{p}_i^a \times m_i \boldsymbol{v}_i^a$
- Angular momentum of a particle in the inertial frame A Relative to C:
  - $\boldsymbol{H}_i^{ac} = \boldsymbol{r}_i \times m_i \boldsymbol{v}_i^a$
- Angular momentum of the system S related to the center of mass C in frame A:
  - $\boldsymbol{I}_S^a$: Moment of inertia tensor calculated in the inertial frame
  - $\boldsymbol{\omega}_S^a$: angular velocity of the system viewed in the inertial frame
  - $\boldsymbol{H}_S^{ac} = \sum \boldsymbol{r}_i \times m_i \, \boldsymbol{v}_i^a = \boldsymbol{I}_S^a \boldsymbol{\omega}_S^a$



Newton-Euler Equations

- Angular momentum of the system S:
  - $\boldsymbol{H}_S^{ac} = \boldsymbol{I}_S^a \cdot \boldsymbol{\omega}_S^a$
- Rate of change of angular momentum is equal to the resultant moment of all external forces and torques acting on the system S related to C:
  - $\frac{d\boldsymbol{H}_S^{ac}}{dt} = \frac{d}{dt}\left(\boxed{\boldsymbol{I}_S^a \cdot \boldsymbol{\omega}_S^a}\right) = \boldsymbol{M}_S^c$

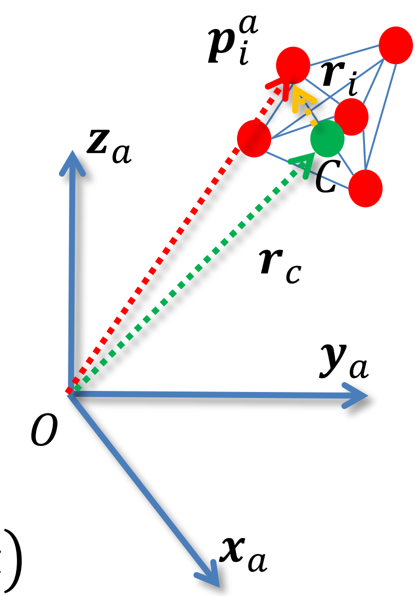
Not very useful. The angular momentum calculated in the inertial frame changes even with constant angular velocity



Switch to an ROTATING reference frame!

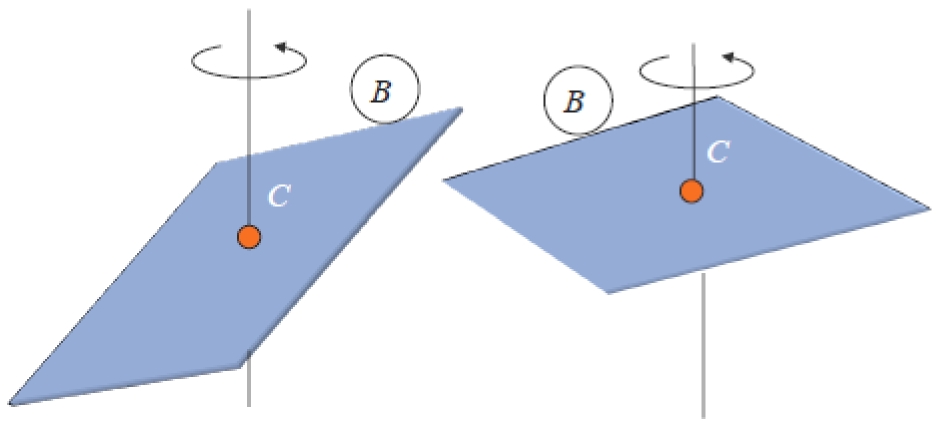
Newton-Euler Equations

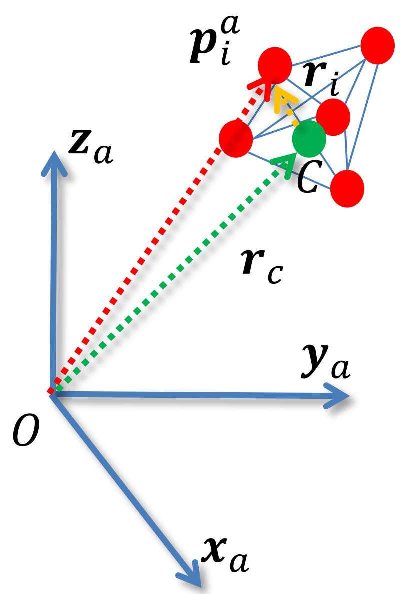
- Let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be the unit basis vectors of the rotating reference frame. The derivative of a unit vector in the rotating frame about the axis $\boldsymbol{\omega}$:
  - $\dot{\boldsymbol{u}} = \boldsymbol{\omega} \times \boldsymbol{u}$
- Consider the vector function:
  - $\boldsymbol{f}(t) = f_x(t) \cdot \boldsymbol{i} + f_y(t) \cdot \boldsymbol{j} + f_z(t) \cdot \boldsymbol{k}$
- Time derivative in rotating reference frame:
  - $\dot{\boldsymbol{f}} = \dot{f}_x \cdot \boldsymbol{i} + \dot{f}_y \cdot \boldsymbol{j} + \dot{f}_z \cdot \boldsymbol{k} + \dot{\boldsymbol{i}} \cdot f_x + \dot{\boldsymbol{j}} \cdot f_y + \dot{\boldsymbol{k}} \cdot f_z = \left(\dot{f}_x \cdot \boldsymbol{i} + \dot{f}_y \cdot \boldsymbol{j} + \dot{f}_z \cdot \boldsymbol{k}\right) + \boldsymbol{\omega} \times \left(f_x \cdot \boldsymbol{i} + f_y \cdot \boldsymbol{j} + f_z \cdot \boldsymbol{k}\right)$
  - $\dot{\boldsymbol{f}} = \dot{\boldsymbol{f}}|_{rot} + \boldsymbol{\omega} \times \boldsymbol{f}$



Newton-Euler Equations

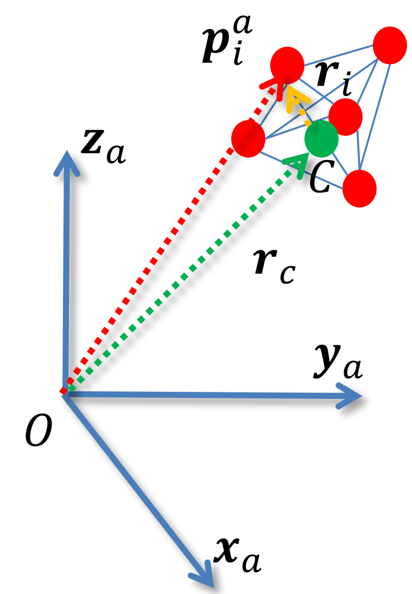
- Align the moment of inertia tensor with the rotating reference frame $\boldsymbol{I}_s^b$, such that it becomes constant and diagonal.





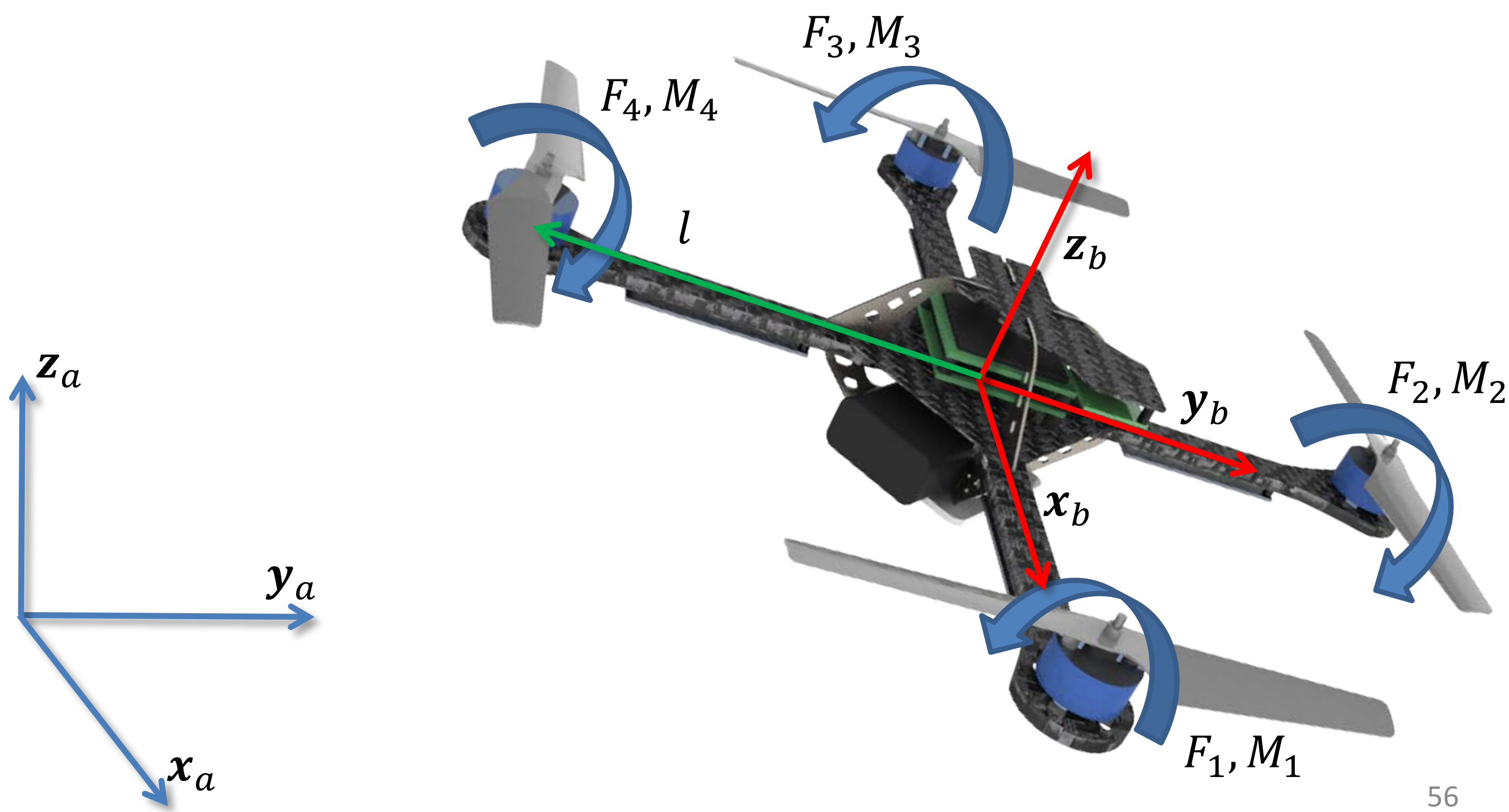
Newton-Euler Equations

- Angular momentum of the system S:
  - $\boldsymbol{H}_S^{ac} = \boldsymbol{I}_S^a \cdot \boldsymbol{\omega}_S^a$
- Rate of change of angular momentum in the inertial frame:
  - $\frac{d\boldsymbol{H}_S^{ac}}{dt} = \boldsymbol{M}_S^c$
- Rate of change of angular momentum in the rotating reference frame, where $\boldsymbol{I}_S^b$ is a constant:
  - $\boldsymbol{H}_S^{bc} = \boldsymbol{I}_S^b \boldsymbol{\omega}_S^b$
  - $\frac{d\boldsymbol{H}_S^{bc}}{dt} + \boldsymbol{\omega}_S^b \times \boldsymbol{H}_S^{bc} = \boldsymbol{M}_S^c$
- The Euler equation of motion:
  - $\boldsymbol{I}_S^b \dot{\boldsymbol{\omega}}_S^b + \boldsymbol{\omega}_S^b \times \boldsymbol{I}_S^b \boldsymbol{\omega}_S^b = \boldsymbol{M}_S^c$



Quadrotor Dynamics

- Euler Equation: $\boldsymbol{I} \cdot \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \boldsymbol{I} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} l(F_2 - F_4) \\ l(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}$

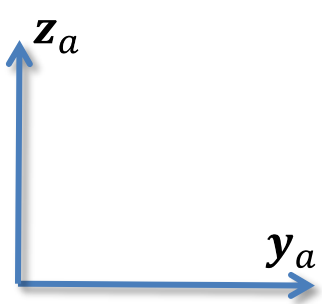


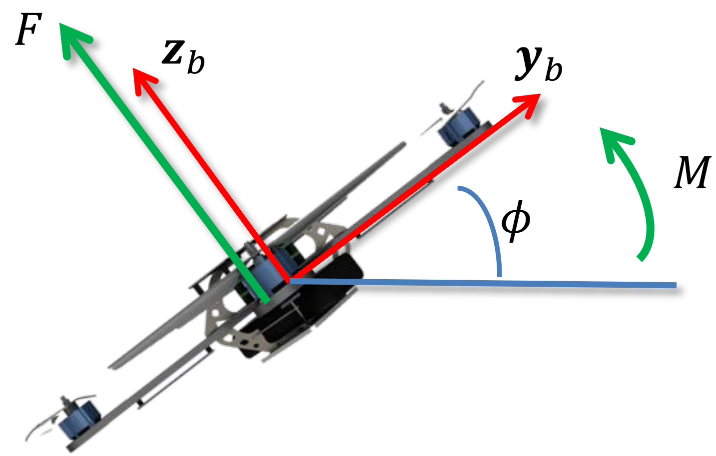
Quadrotor Dynamics

- Motor model: $\dot{\omega}_i = k_m(\omega_i^{des} - \omega_i)$
- Thrust from individual motor: $F_i = k_F \omega_i^2$
- Moment from individual motor: $M_i = k_M \omega_i^2$
- Newton Equation: $m\ddot{\boldsymbol{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \boldsymbol{R} \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$
- Euler Equation: $\boldsymbol{I} \cdot \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \boldsymbol{I} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} l(F_2 - F_4) \\ l(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}$

A Planar Quadrotor

- $$\begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} F \\ M \end{bmatrix}$$





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Assignment

- Chapter 2.1-2.4 of "A Mathematical Introduction to Robotic Manipulation"
- Paper Reading: "The GRASP Multiple Micro-UAV Test Bed", Nathan Michael, Daniel Mellinger, Quentin Lindsey, and Vijay Kumar.

Next Lecture...

- Control Basics
- Quadrotor control

Logistics

- Project 1, phase 1 will be released on 14/9
  - Tentative due: 25/9
- Proposed change to lecture date:
  - 15/9 -> 19/9