

Introduction to Aerial Robotics

Lecture 8

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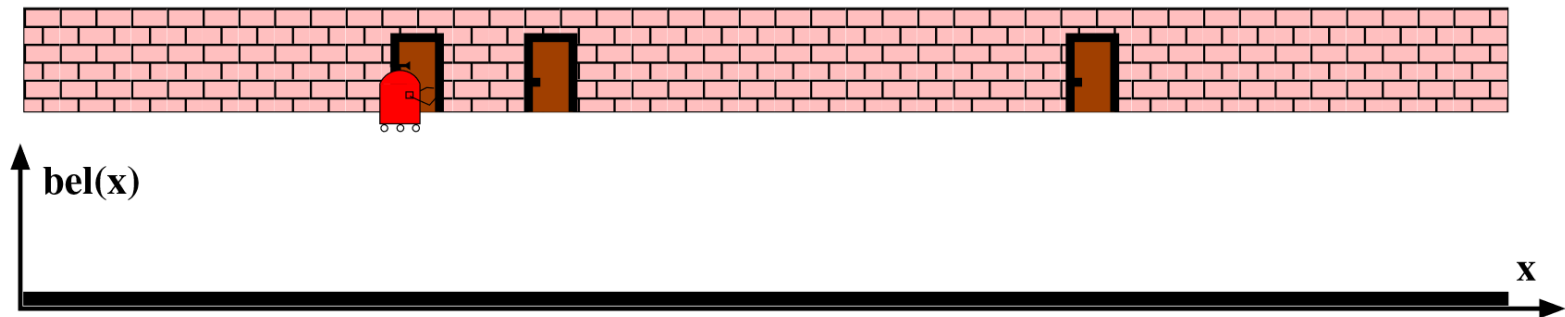
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Outline

- Bayesian Filtering
 - Introduction to Probability
 - Bayes' Filter
- Kalman Filtering
 - Gaussian Random Variables
 - The Kalman Filter
 - Continuous Time Systems

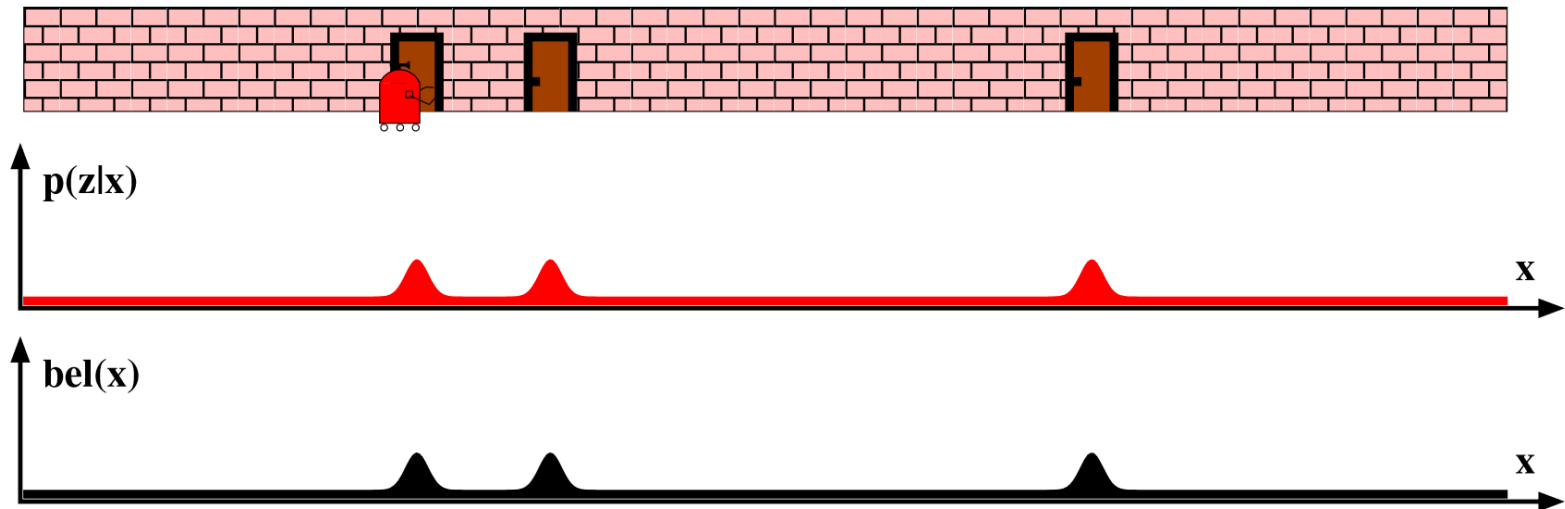
Bayesian Filtering

Example Problem



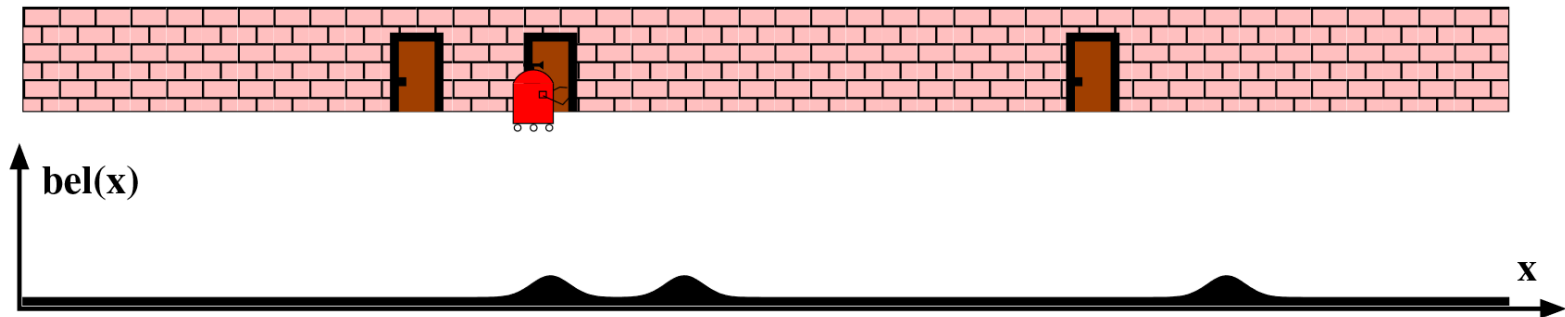
[Thrun, Burgard, and Fox. *Probabilistic Robotics*. 2005]

Example Problem



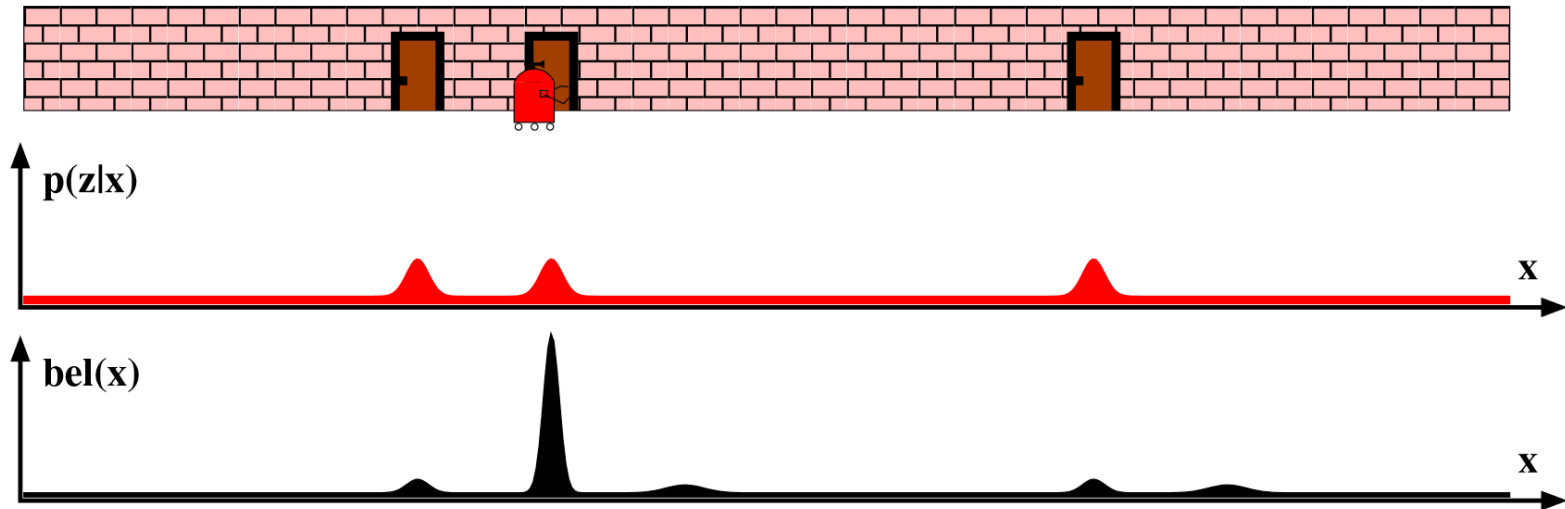
[Thrun, Burgard, and Fox. *Probabilistic Robotics*. 2005]

Example Problem



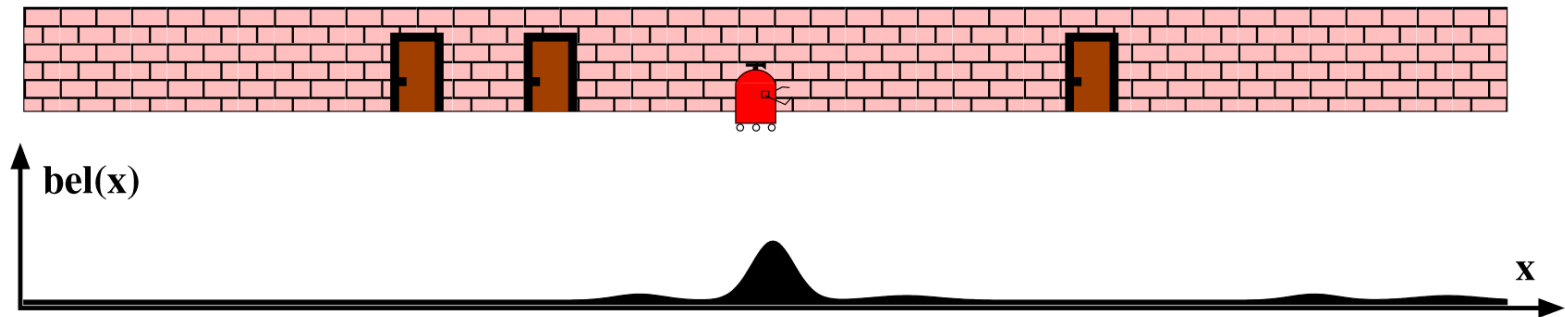
[Thrun, Burgard, and Fox. *Probabilistic Robotics*. 2005]

Example Problem



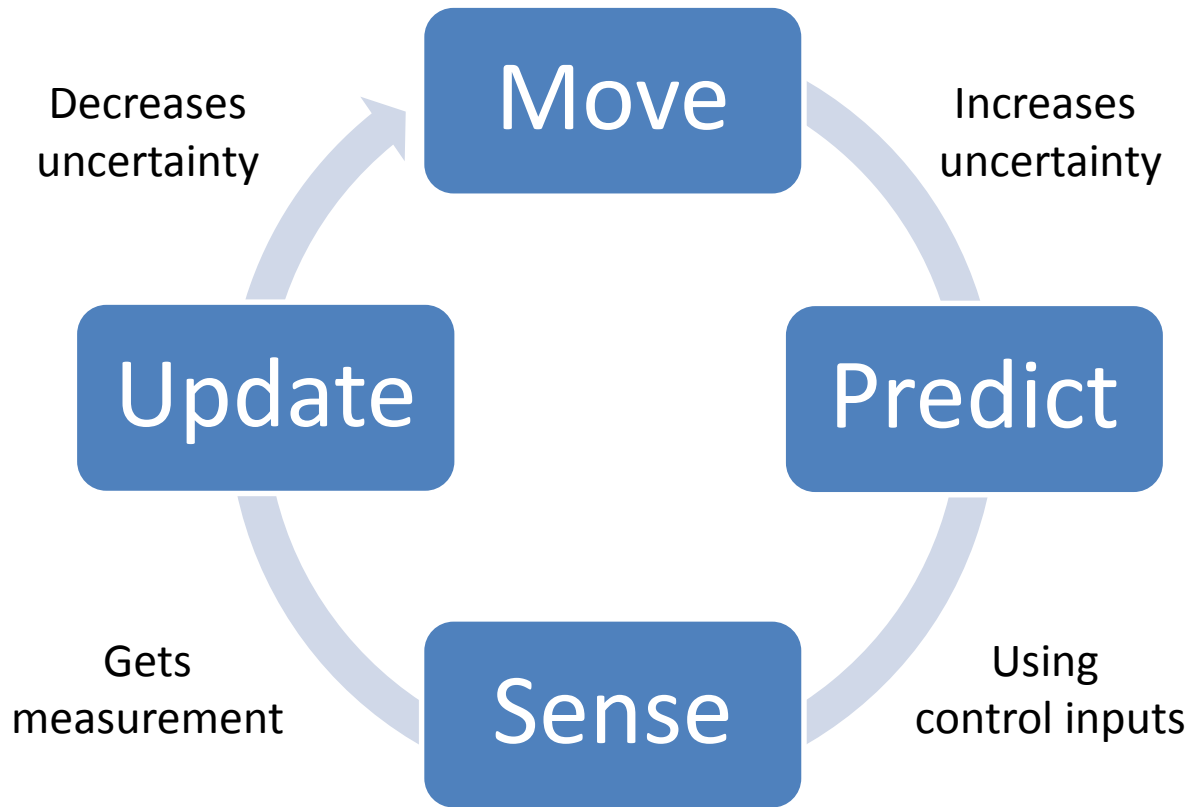
[Thrun, Burgard, and Fox. *Probabilistic Robotics*. 2005]

Example Problem



[Thrun, Burgard, and Fox. *Probabilistic Robotics*. 2005]

Problem Overview



Questions

- What are sources of uncertainty?
- How do we mathematically represent uncertainty in the system?
- How do we use collected evidence to update our belief?
- What can we observe?
- What can we not observe?

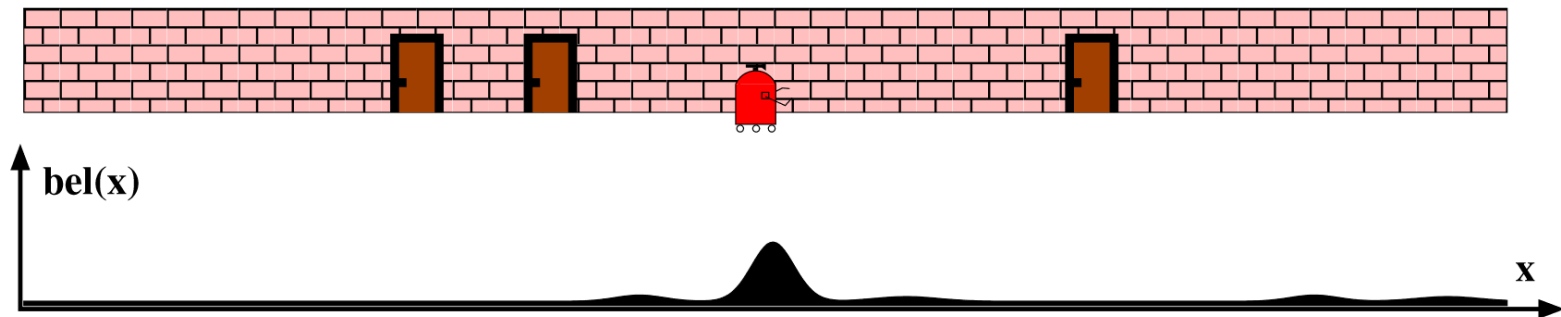
Introduction to Probability

Random Variables

- **Definition:** Variable whose value is subject to change due to randomness or chance
- **Properties:**
 - Can be continuous (e.g., position in 3D) or discrete (e.g., roll of a die)
 - Observed values of random variables are called *realizations*
- **Example:**
 - Pose of a robot, $p(X = x)$, or value of a rolled die, $p(D = d)$

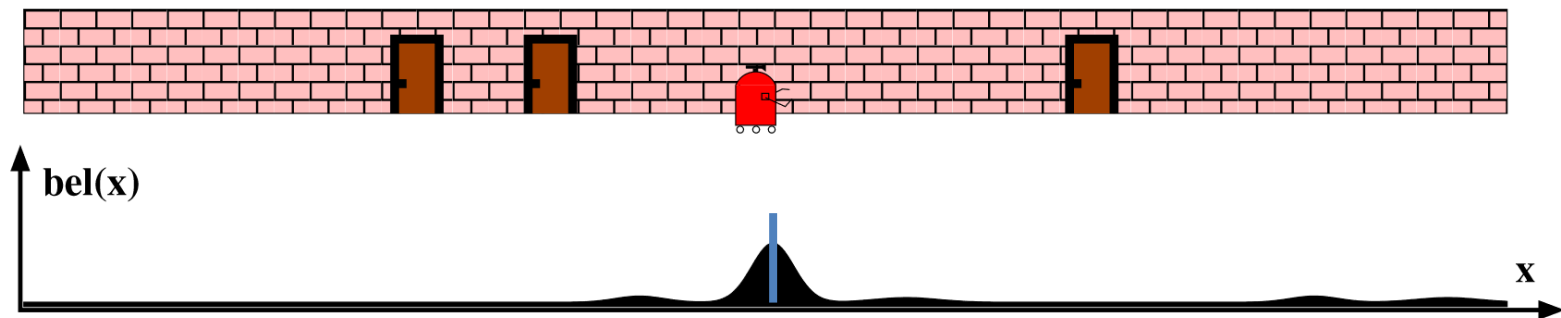
Probability Density Function

- **Definition:** Function describing the likelihood that a random variable X will take on a particular value x
- **Properties:**
 - Total probability is 1, $\int p(X = x)dx = 1$,
 $\sum_x p(X = x) = 1$
 - Non-negative, $p(X = x) \geq 0$



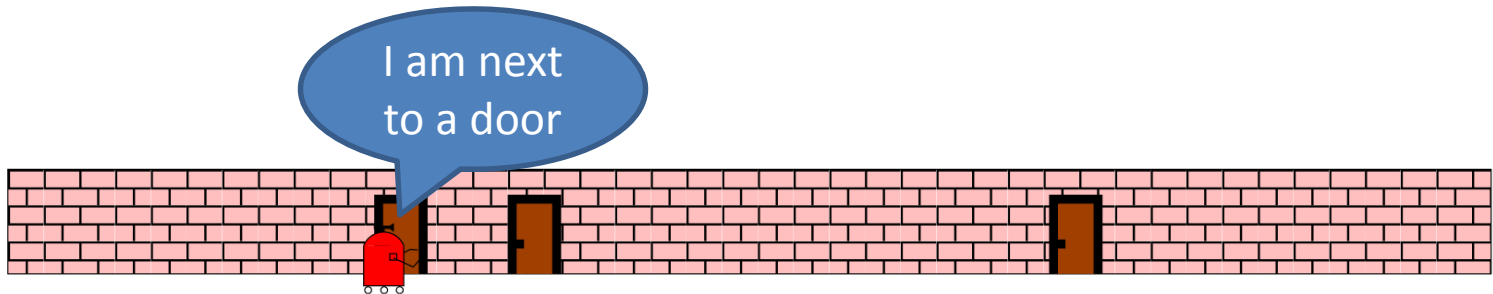
Expected Value

- **Definition:** Probability-weighted average value
 - $E[X] = \int p(X = x) x dx$
- **Intuition:** “Center of mass” of the probability distribution



Joint Probability Distribution

- **Definition:** The probability density function of a set of two or more random variables
- Also called a ***multivariate distribution***
- **Example:**
 - $p(X = x, Z = z)$ = a robot having a pose x and receiving a measurement z



Covariance

- **Definition:** A measure of how two random variables change together
 - $\sigma(X, Y) = E[(X - E[X])(Y - E[Y])]$
- The ***variance*** is a special case where the two random variables are identical
 - $\sigma^2(X) = \sigma(X, X)$
- **Intuition:** The “moment of inertia” of the probability distribution

Covariance Matrix

- For a multivariate distribution over $\mathbf{X} = [X_1, X_2, \dots, X_n]^T$ we define the **covariance matrix** to be
- $\mathbf{\Sigma} = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T]$

$$= \begin{bmatrix} \sigma^2(X_1) & \sigma(X_1, X_2) & \cdots & \sigma(X_1, X_n) \\ \sigma(X_2, X_1) & \sigma^2(X_2) & \cdots & \sigma(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(X_n, X_1) & \sigma(X_n, X_2) & \cdots & \sigma^2(X_n) \end{bmatrix}$$

- The covariance matrix is symmetric and positive semi-definite

Independence

- **Definition:** Two random variables are independent if the outcome of one has *no effect* on the outcome of the other
- $p(x, z) = p(x) p(z)$
- **Example:**
 - If X, Z are the outcomes of two dice rolls
- **Properties:**
 - Independent random variables are ***uncorrelated***,
 $\sigma(X, Z) = 0$
 - Uncorrelated random variables are ***not*** necessarily independent

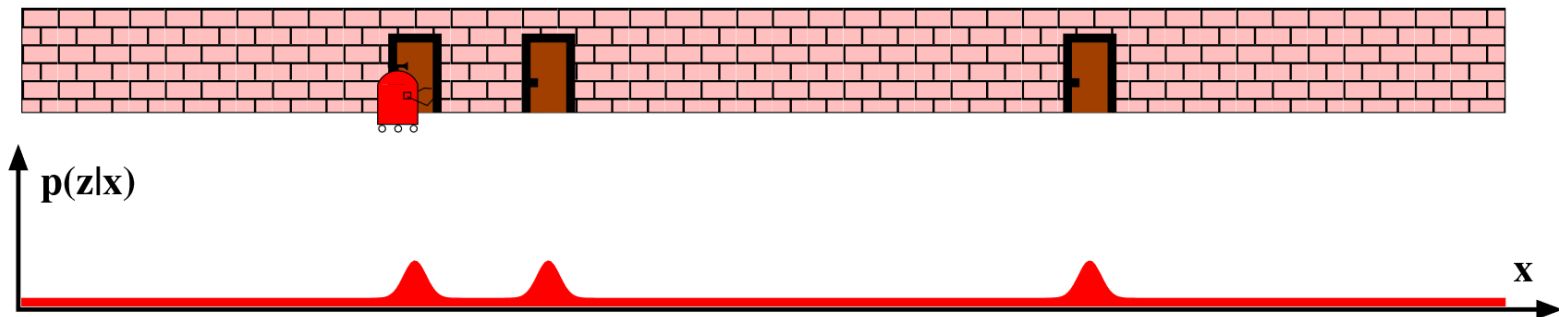
Conditional Probability

- **Definition:** Probability of an event z occurring conditioned on another event x occurring

- $$p(z \mid x) = \frac{p(x, z)}{p(x)} \quad \Leftrightarrow \quad p(x, z) = p(z \mid x) p(x)$$

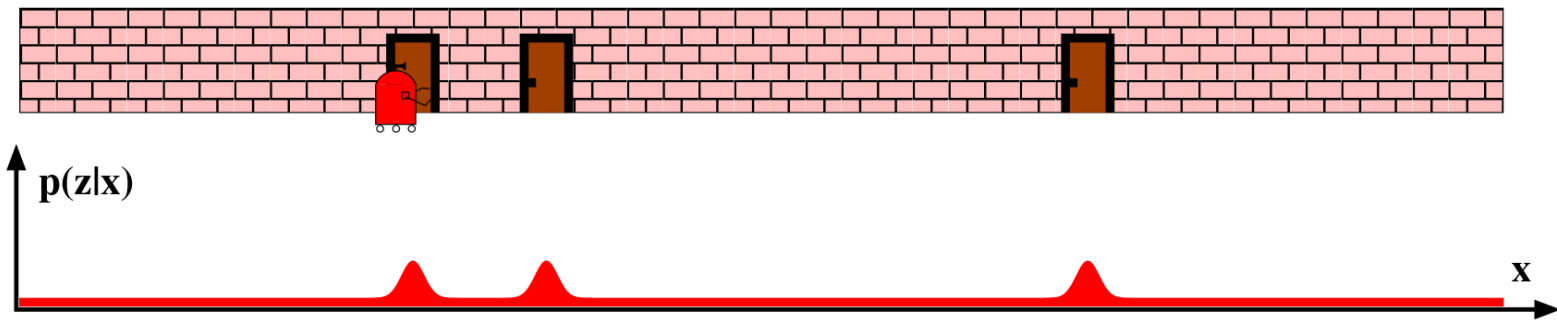
- **Example:**

– $z = \{\text{there is a door next to the robot}\}$



Conditional Independence

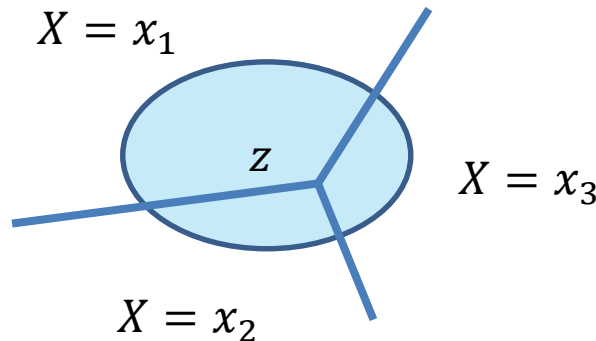
- **Definition:** Two random variables are *conditionally independent* if the outcome of one has *no effect* on the outcome of the other when conditioned on the outcome of a third random variable
- $p(z_1, z_2 | x) = p(z_1 | x) p(z_2 | x)$
- **Example:**
 - Let Z_1, Z_2 be two measurements taken from the same place
 - Z_1, Z_2 are conditionally independent given X



- **Question:** Are Z_1, Z_2 are independent?

Marginal Distribution

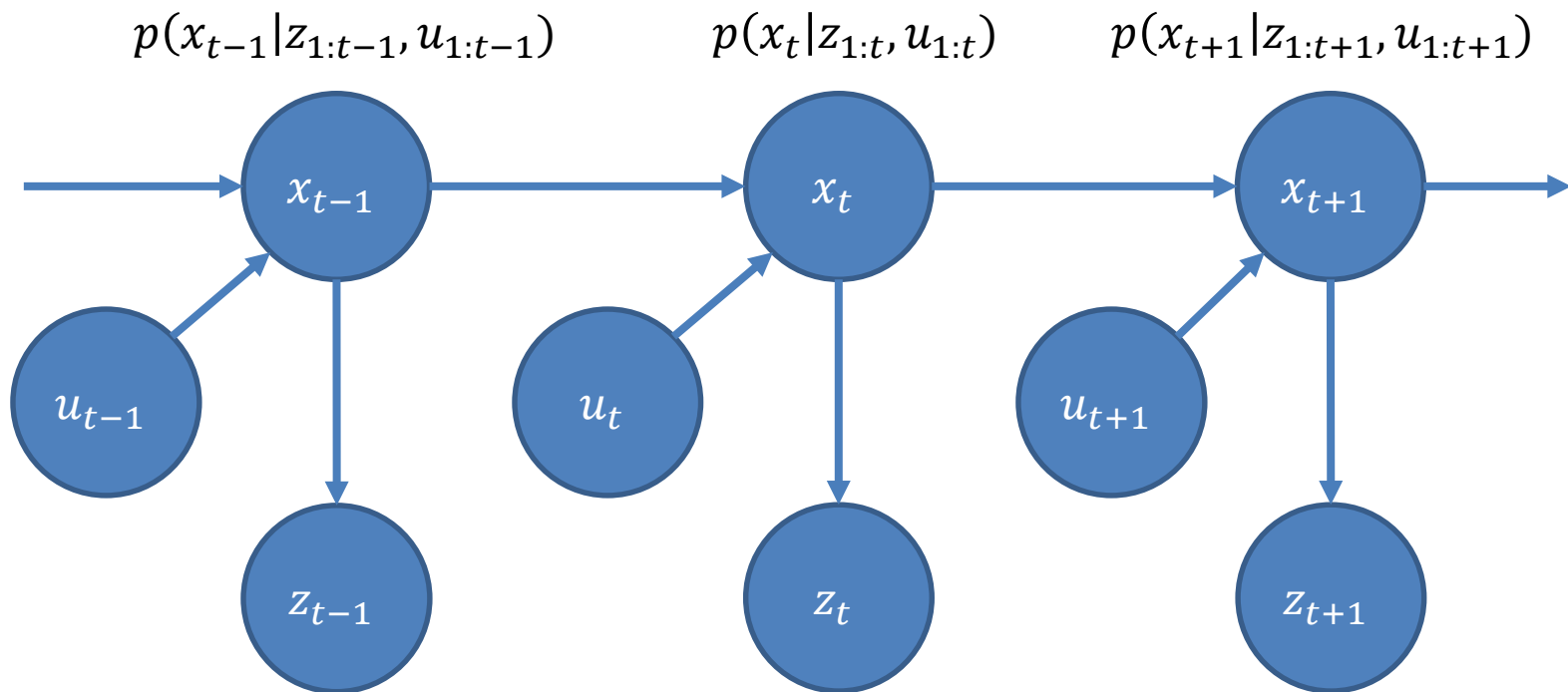
- **Definition:** The probability distribution of the subset of a collection of random variables
- $p(z) = \int p(x, z) dx$
- Also known as the ***Law of Total Probability***



$$p(z) = \sum_{i=1}^3 p(z | X = x_i) p(X = x_i)$$

Bayes' Filter

Bayesian Filter



Bayes' Theorem

- $$p(x | z) = \frac{p(z | x) p(x)}{p(z)} = \frac{p(z | x) p(x)}{\int p(z | x') p(x') dx'}$$
- **Intuition:** Describes how the belief about a random variable X should change to account for the collected evidence z
- **Derivation:**
 - $p(x, z) = p(z | x) p(x) = p(x | z) p(z)$

Markov Property

- **Definition:** The future state of the system is conditionally independent of the past states given the current state
 - $p(x_{t+1} | x_{0:t}) = p(x_{t+1} | x_t)$
 - $p(z_t | x_t, z_{1:t-1}, u_{1:t}) = p(z_t | x_t)$
 - $p(x_t | x_{t-1}, z_{1:t-1}, u_{1:t}) = p(x_t | x_{t-1}, u_t)$
- **Question:**
 - Which of the following satisfy the Markov assumption?
 - A first order system with $x = [\text{position}]$
 - A second order system with $x = [\text{position}]$
 - A second order system with $x = [\text{position}, \text{velocity}]$

Bayes' Filter Derivation

- **Goal:** Want to update the probability distribution of the robot pose using the realizations of the control input and measurement
- $$p(x_t \mid z_{1:t}, u_{1:t}) = \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})}$$
- **Note:** The measurement is *conditionally independent* of the past measurements and control inputs given the current state of the robot
- $$p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$
- **Note:** The denominator can be found as a *marginal distribution* of the numerator
- $$p(z_t \mid z_{1:t-1}, u_{1:t}) = \int p(x_t, z_t \mid z_{1:t-1}, u_{1:t}) dx_t$$

Process Model

- Also known as the ***transition model*** or ***motion model***
 - $p(x_t \mid z_{1:t-1}, u_{1:t})$
 - Note:** Can find the current pose via marginalization
 - $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int p(x_t, x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1}$
 - $= \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1}$
 - Note:** The future state is *conditionally independent* of the past measurements and control inputs given the current state and input
 - $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int p(x_t \mid x_{t-1}, u_t) p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}) dx_{t-1}$
- Prediction

Process model

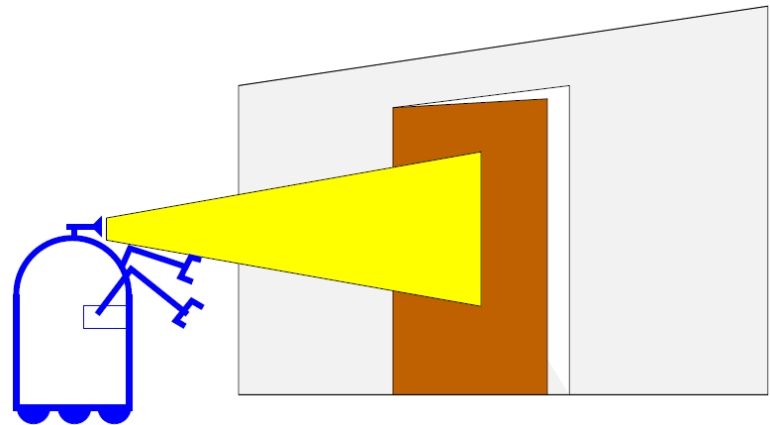
Prior

Bayes' Filter

- **Prior:** $p(x_0)$
- **Process model:** $f(x_t | x_{t-1}, u_t)$
- **Measurement model:** $g(z_t | x_t)$
- **Prediction step:**
- $p(x_t | z_{1:t-1}, u_{1:t}) = \int f(x_t | x_{t-1}, u_t) p(x_{t-1} | z_{1:t-1}, u_{1:t-1}) dx_{t-1}$
- **Update step:**
- $$p(x_t | z_{1:t}, u_{1:t}) = \frac{g(z_t | x_t) p(x_t | z_{1:t-1}, u_{1:t})}{\int g(z_t | x'_t) p(x'_t | z_{1:t-1}, u_{1:t}) dx'_t}$$

Measurements

- Robots collect noisy information using sensors
 - Exteroceptive
 - Laser scanner
 - 3D depth sensor
 - Magnetometer
 - *Camera*
 - Proprioceptive
 - Motor encoder
 - *Gyroscope*
 - *Accelerometer*



[Thrun, Burgard, and Fox. *Probabilistic Robotics*. 2005]

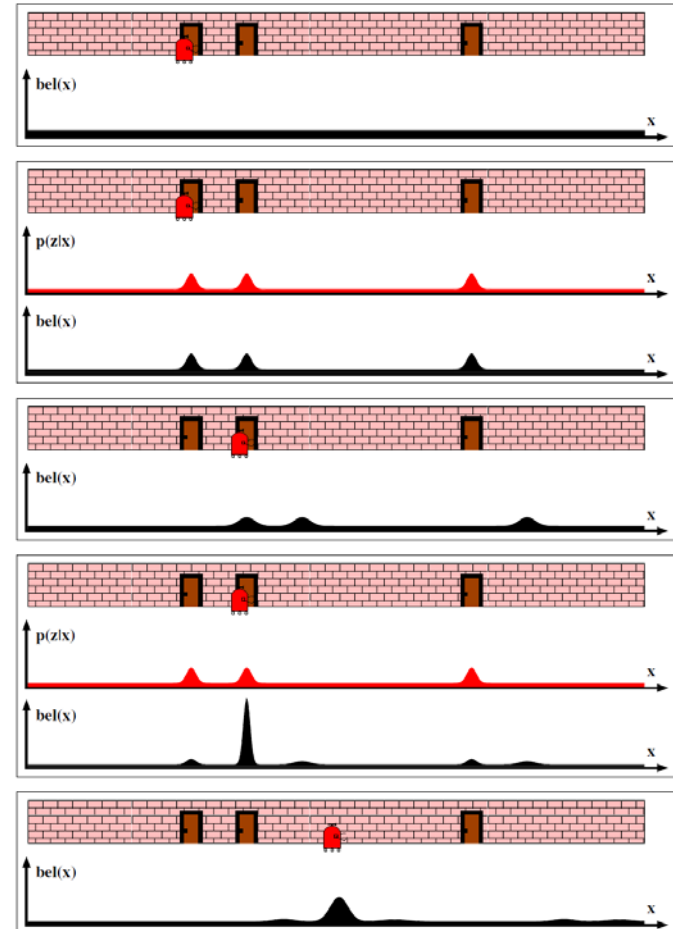
Applications

- Can use Bayesian filtering in many other domains
 - Map building
 - Simultaneous localization and mapping (SLAM)
 - Feature tracking
 - Pose estimation
 - Target tracking

Kalman Filter

Motivation

- Real systems have uncertainty
 - Initial conditions
 - Aerodynamics
 - Friction
 - Disturbances
 - Wind gust
 - Wheel slip
- Errors will compound over time if not corrected



Bayes' Filter

- **Prior:** $p(x_0)$ ← State
- **Process model:** $f(x_t | x_{t-1}, u_t)$ ← Control input
- **Measurement model:** $g(z_t | x_t)$ ← Measurement
- **Prediction step:**
- $p(x_t | z_{1:t-1}, u_{1:t}) = \int f(x_t | x_{t-1}, u_t) p(x_{t-1} | z_{1:t-1}, u_{1:t-1}) dx_{t-1}$
- **Update step:**
- $$p(x_t | z_{1:t}, u_{1:t}) = \frac{g(z_t | x_t) p(x_t | z_{1:t-1}, u_{1:t})}{\int g(z_t | x'_t) p(x'_t | z_{1:t-1}, u_{1:t}) dx'_t}$$

Markov Property

- **Definition:** The future state of the system is conditionally independent of the past states given the current state
 - $p(x_{t+1} | x_{0:t}) = p(x_{t+1} | x_t)$
 - $p(z_t | x_t, z_{1:t-1}, u_{1:t}) = p(z_t | x_t)$
 - $p(x_t | x_{t-1}, z_{1:t-1}, u_{1:t}) = p(x_t | x_{t-1}, u_t)$
- **Question:**
 - Which of the following satisfy the Markov assumption?
 - A first order system with $x = [\text{position}]$ Yes!
 - A second order system with $x = [\text{position}]$ No!
 - A second order system with $x = [\text{position}, \text{velocity}]$ Yes!

Assumptions

- The prior state of the robot is represented by a Gaussian distribution
 - $p(x_0) \sim N(\mu_0, \Sigma_0)$
- The process model $f(x_t \mid x_{t-1}, u_t)$ is linear with additive Gaussian white noise
 - $x_t = A_t x_{t-1} + B_t u_t + n_t$
 - $n_t \sim N(0, Q_t)$
 - $x_t, n_t \in \mathbf{R}^n, u_t \in \mathbf{R}^m, A_t, Q_t \in \mathbf{R}^{n \times n}$, and $B_t \in \mathbf{R}^{n \times m}$
- The measurement model $g(z_t \mid x_t)$ is linear with additive Gaussian white noise
 - $z_t = C_t x_t + v_t$
 - $v_t \sim N(0, R_t)$
 - $z_t, v_t \in \mathbf{R}^p, C_t \in \mathbf{R}^{p \times n}$, and $R_t \in \mathbf{R}^{p \times p}$

Gaussian Random Variables

Multivariate Normal (Gaussian) Distribution

- Let X be a vector of n random variables
- A multivariate normal distribution takes the form

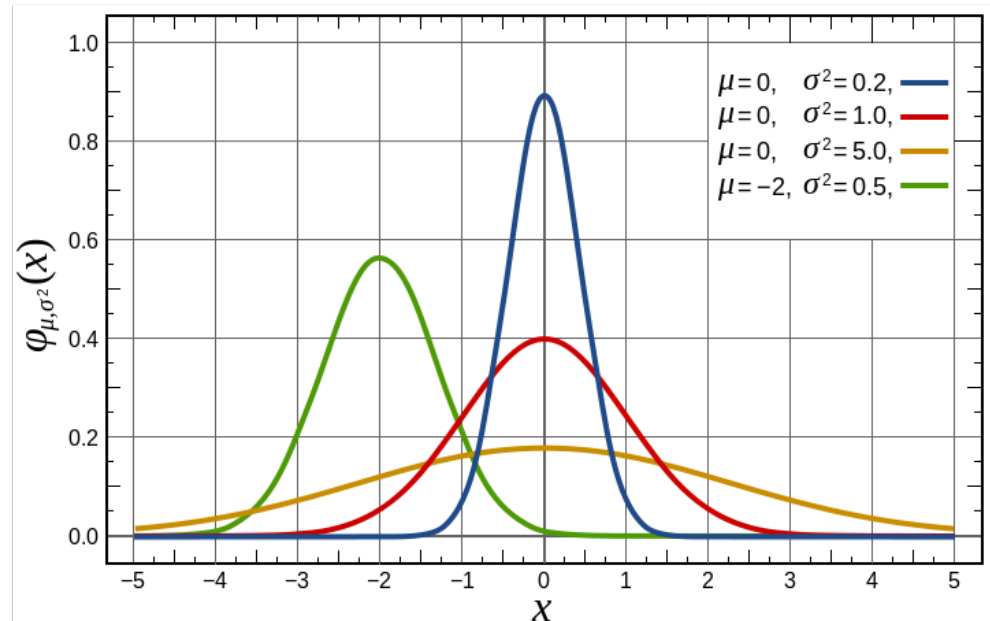
- $$f_X(x) = \frac{1}{(2\pi)^{n/2} \sqrt{\det(\Sigma)}} e^{\frac{-(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}}$$

- where $\mu \in \mathbf{R}^n$ and $\Sigma \in \mathbf{R}^{n \times n}$

Mean

Covariance

- Fully parameterized by μ, Σ



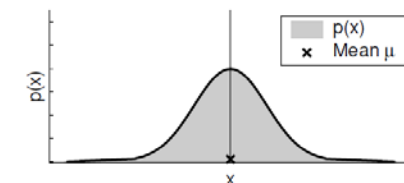
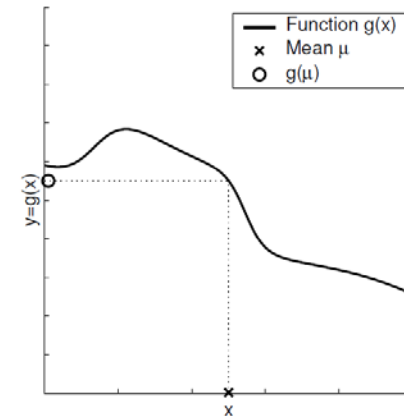
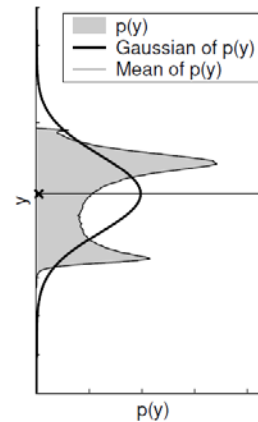
[http://en.wikipedia.org/wiki/Normal_distribution]

Affine Transformations

- Affine transformation of Gaussian distributions are Gaussian
- If $X \sim N(\mu_X, \Sigma_X)$ and $Y = AX + b$ then $Y \sim N(\mu_Y, \Sigma_Y)$ where
- $\mu_Y = A \mu_X + b$ and $\Sigma_Y = A \Sigma_X A^T$

- Example:**

- $x_t = A_t x_{t-1} + B_t u_t + n_t$



Affine Transformations

- **Fact:**

- Expectation is a linear operator of x

- $E[X] = \int p(x) x dx$

$$\mu_Y = E[Y]$$

$$= E[AX + b]$$

$$= A E[X] + b$$

$$= A \mu_X + b$$

$$\Sigma_Y = E[(Y - \mu_Y)(Y - \mu_Y)^T]$$

$$= E[(AX + b - A\mu_X - b)(AX + b - A\mu_X - b)^T]$$

$$= E[(A(X - \mu_X))(A(X - \mu_X))^T]$$

$$= A E[(X - \mu_X)(X - \mu_X)^T] A^T$$

$$= A \Sigma_X A^T$$

Independence

- Let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ where X_1, X_2 are uncorrelated, i.e., the covariance is of the form $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ where $\Sigma_{12} = \Sigma_{21} = 0$
- Then X_1, X_2 are independent and $f_X(X) = f_{X_1}(X_1) f_{X_2}(X_2)$
- **Note:** The converse is always true, i.e., if two random variables are independent then they are uncorrelated
- **Example:** We assume that the noise is independent of the state of the system

Sum of Independent Gaussians

- Let X, Y be independent multivariate Gaussian random variables with mean μ_X, μ_Y and covariance Σ_X, Σ_Y
- The sum $Z = X + Y$ is also Gaussian with mean $\mu_Z = \mu_X + \mu_Y$ and covariance $\Sigma_Z = \Sigma_X + \Sigma_Y$
- **Example:**
 - $x_t = x_{t-1} + n_t$
 - $z_t = x_t + v_t$

Jointly Normal Random Vectors

- Let X be a multivariate Gaussian random variable and let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$
- Then X_1, X_2 are both (multivariate) Gaussian random variables and are jointly normally distributed
- **Note:** If X_1, X_2 are both (multivariate) Gaussian random variables then it does *not* necessarily imply that $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is also Gaussian

Conditional Distributions

- Let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a multivariate Gaussian with mean $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and covariance $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$
- Then the conditional density $f_{X_1|X_2}(x_1|X_2 = x_2)$ is a multivariate normal distribution with
 - mean $\mu_{X_1|X_2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$
 - covariance $\Sigma_{X_1|X_2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$
- **Note:** $\Sigma_{X_1|X_2}$ is the Schur complement of Σ_{22}

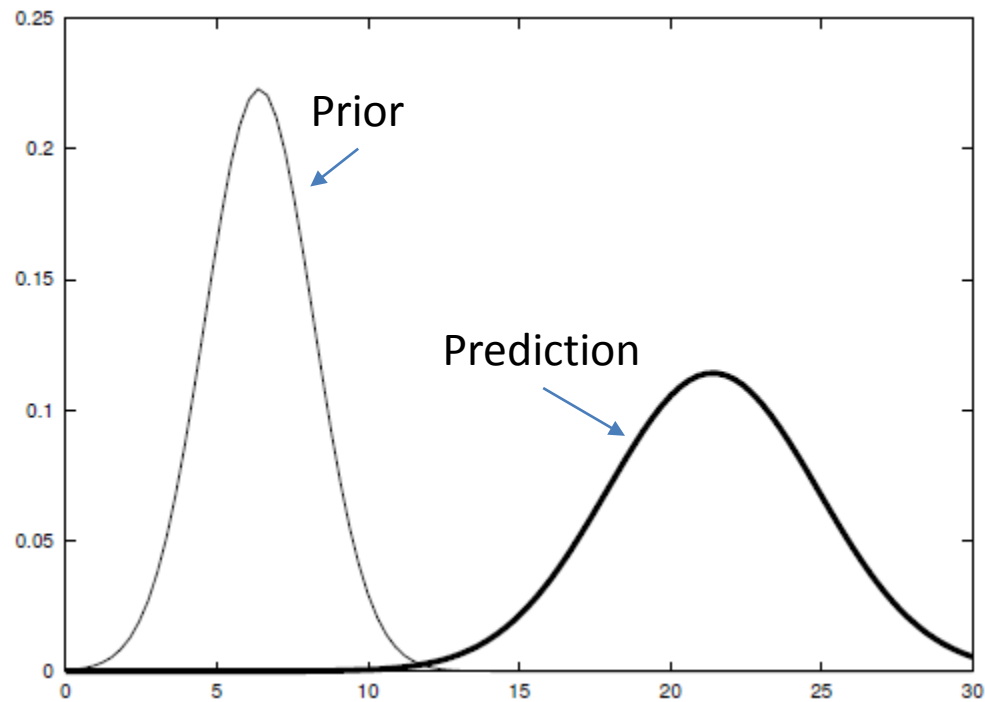
The Kalman Filter

System Model

- The prior state of the robot is represented by a Gaussian distribution
 - $p(x_0) \sim N(\mu_0, \Sigma_0)$
- The process model $f(x_t | x_{t-1}, u_t)$ is linear with additive Gaussian white noise
 - $x_t = A_t x_{t-1} + B_t u_t + n_t$
 - $n_t \sim N(0, Q_t)$
- The measurement model $g(z_t | x_t)$ is linear with additive Gaussian white noise
 - $z_t = C_t x_t + v_t$
 - $v_t \sim N(0, R_t)$

Kalman Filter – Prediction

- Bayes: $p(x_t | z_{1:t-1}, u_{1:t}) = \int f(x_t | x_{t-1}, u_t) p(x_{t-1} | z_{1:t-1}, u_{1:t-1}) dx_{t-1}$



[Thrun, Burgard, and Fox. *Probabilistic Robotics*. 2005]

Kalman Filter – Prediction

- Bayes: $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int f(x_t \mid x_{t-1}, u_t) p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}) dx_{t-1}$
- $x_t = A_t x_{t-1} + B_t u_t + n_t$
- $n_t \sim N(0, Q_t)$
- Prior: $p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}) \sim N(\mu_{t-1}, \Sigma_{t-1})$
- Prediction:
 - $\bar{\mu}_t = A \mu_{t-1} + B u_t$
 - $\bar{\Sigma}_t = A \Sigma_{t-1} A^T + Q$

Kalman Filter – Update

- Bayes: $p(x_t | z_{1:t}, u_{1:t}) = \frac{g(z_t | x_t) p(x_t | z_{1:t-1}, u_{1:t})}{\int g(z_t | x'_t) p(x'_t | z_{1:t-1}, u_{1:t}) dx'_t}$
- The observation model is $z_t = C_t \bar{x}_t + v_t$, $v_t \sim N(0, R_t)$
- The best update without a measurement is to set $x_t = \bar{x}_t$
- $\begin{bmatrix} x_t \\ z_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ v_t \end{bmatrix}$
- **Question:** Is this a jointly normal distribution?
- $\mu = \begin{bmatrix} \bar{\mu}_t \\ C\bar{\mu}_t \end{bmatrix}$
- $\Sigma = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} \bar{\Sigma}_t & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & C^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} \bar{\Sigma}_t & \bar{\Sigma}_t C^T \\ C\bar{\Sigma}_t & C\bar{\Sigma}_t C^T + R \end{bmatrix}$

Kalman Filter – Update

- The distribution of x_t conditioned on z_t is thus normal with

- $\mu_{x_t|z_t} = \bar{\mu}_t + \bar{\Sigma}_t C^T (C \bar{\Sigma}_t C^T + R)^{-1} (z_t - C \bar{\mu}_t)$

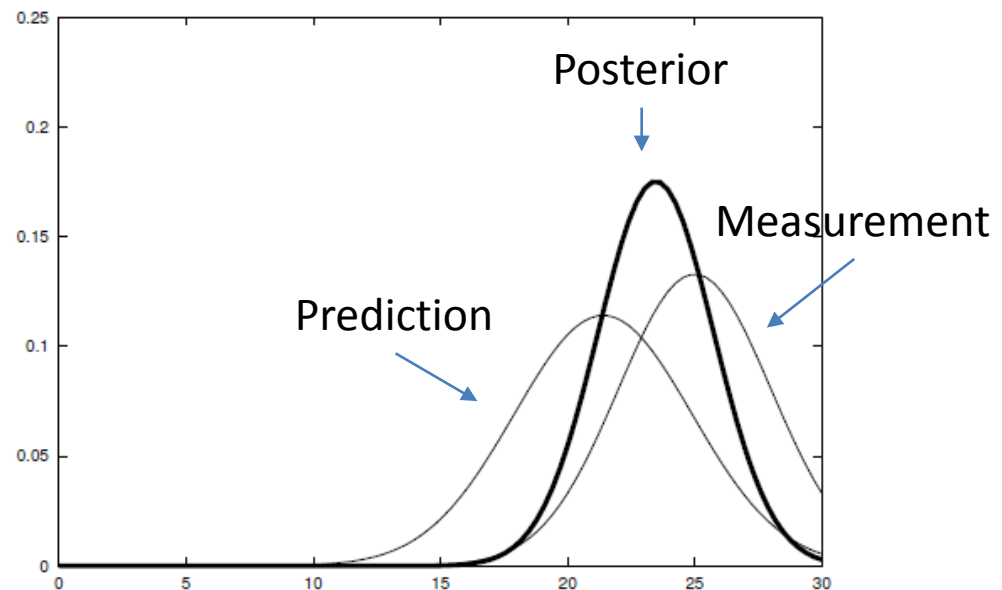
- $\Sigma_{x_t|z_t} = \bar{\Sigma}_t - \bar{\Sigma}_t C^T (C \bar{\Sigma}_t C^T + R)^{-1} C \bar{\Sigma}_t$

- Define the Kalman gain K_t

- $K_t = \bar{\Sigma}_t C^T (C \bar{\Sigma}_t C^T + R)^{-1}$

- $\mu_t = \bar{\mu}_t + K_t (z_t - C \bar{\mu}_t)$

- $\Sigma_t = \bar{\Sigma}_t - K_t C \bar{\Sigma}_t$



[Thrun, Burgard, and Fox. *Probabilistic Robotics*. 2005]

Kalman Gain

- $K_t = \bar{\Sigma}_t C^T (C \bar{\Sigma}_t C^T + R)^{-1}$
- **Intuition:** How much to trust the sensor vs. the prediction
- **Example:**
 - Perfect sensor $R = 0$
 - $K_t = \bar{\Sigma}_t C^T (C \bar{\Sigma}_t C^T + R)^{-1} = C^{-1}$
 - $\mu_t = \bar{\mu}_t + K_t(z_t - C \bar{\mu}_t) = C^{-1}z_t$
 - $\Sigma_t = \bar{\Sigma}_t - K_t C \bar{\Sigma}_t = 0$
 - Horrible sensor $R \rightarrow \infty$
 - $K_t = \bar{\Sigma}_t C^T (C \bar{\Sigma}_t C^T + R)^{-1} \rightarrow 0$
 - $\mu_t = \bar{\mu}_t + K_t(z_t - C \bar{\mu}_t) \rightarrow \bar{\mu}_t$
 - $\Sigma_t = \bar{\Sigma}_t - K_t C \bar{\Sigma}_t \rightarrow \bar{\Sigma}_t$

Kalman Filter

- Prior:
 - $p(x_0) \sim N(\mu_0, \Sigma_0)$
- Transition model:
 - $x_t = A_t x_{t-1} + B_t u_t + n_t$
 - $n_t \sim N(0, Q_t)$
- Measurement model:
 - $z_t = C_t x_t + v_t$
 - $v_t \sim N(0, R_t)$
- Prior:
 - μ_{t-1}, Σ_{t-1}
- Prediction:
 - $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
 - $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t$
- Update:
 - $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
 - $\Sigma_t = \bar{\Sigma}_t - K_t C_t \bar{\Sigma}_t$
 - $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$

Example Problem

$$x_t = x_{t-1} + u_t + n_t$$

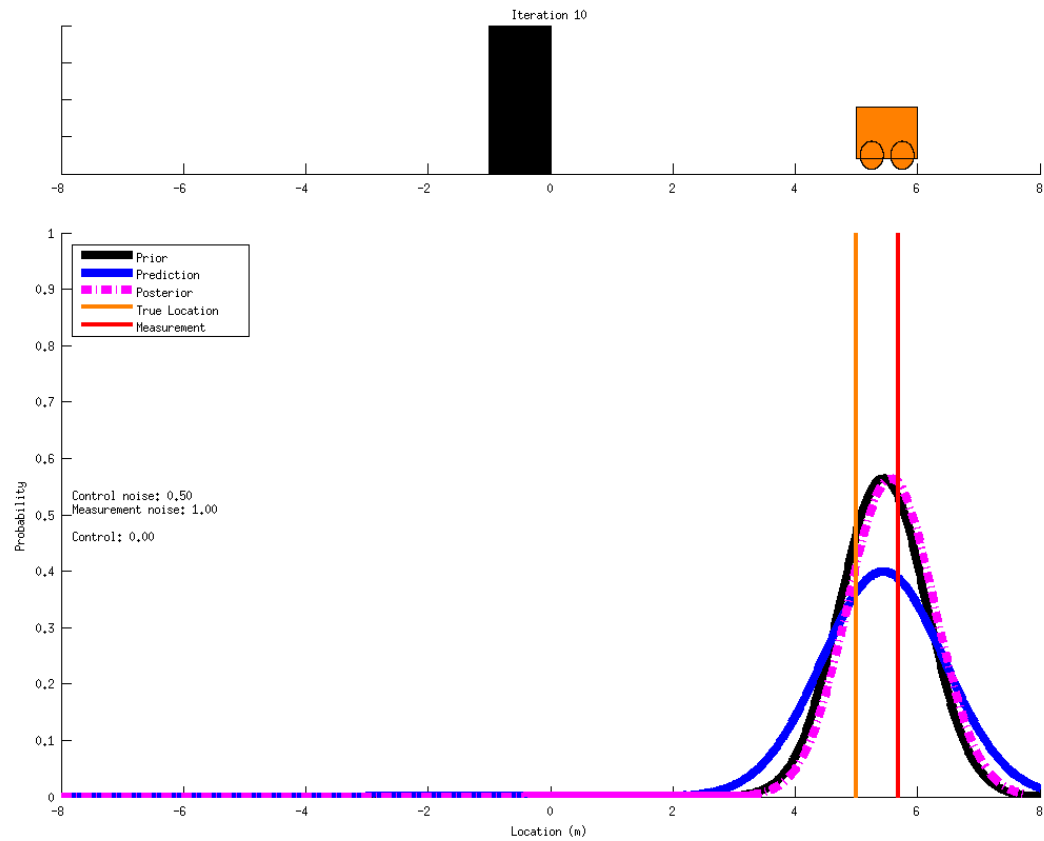
$$Q_t = 0.5$$

$$A_t = B_t = 1$$

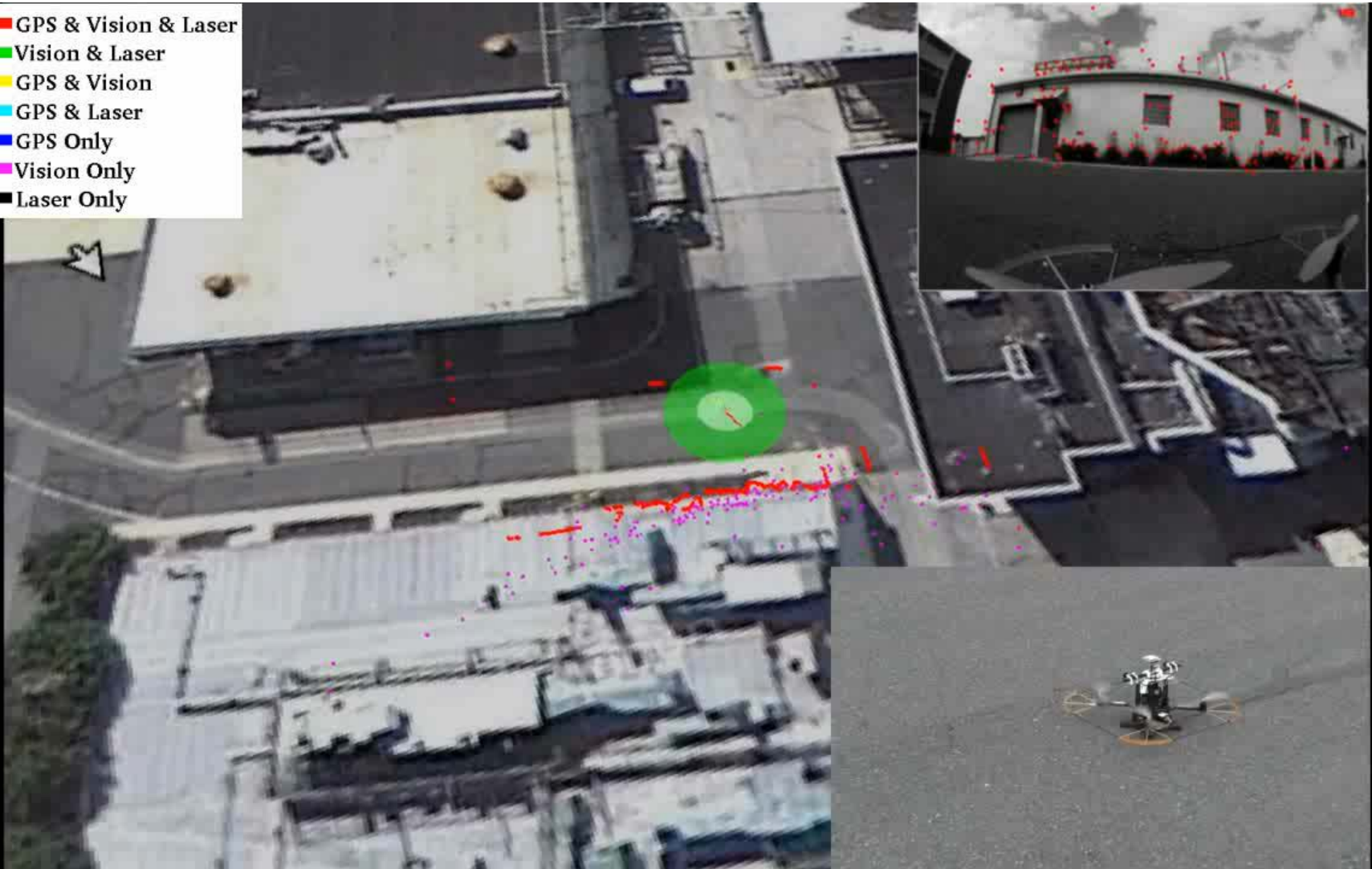
$$z_t = x_t + v_t$$

$$R_t = 1.0$$

$$C_t = 1$$



- GPS & Vision & Laser
- Vision & Laser
- GPS & Vision
- GPS & Laser
- GPS Only
- Vision Only
- Laser Only



Kalman Filter Facts

- If the distribution is not Gaussian, the Kalman filter is the minimum variance linear estimator
 - The noise must be uncorrelated with the initial state x_0
- The variance never increases due to receiving a measurement
- The variance update is independent of the measurement realization
- Prediction and update can happen in arbitrary order as long as they are sorted in temporally

Continuous Time Systems

Discrete vs. Continuous Time

Discrete Time

- Events occur at discrete points in time
- Time intervals often evenly spaced
- Example:
 - Kinematic cart
 - $x_t = x_{t-1} + u_t + n_t$

Continuous Time

- Events may occur infinitesimally close to each other in time
- Example:
 - Ballistic motion
 - $\ddot{x} = -g + u + n$

Continuous Time Systems

- There is a continuous time version of the Kalman Filter
 - Continuous dynamics
 - Continuous observations
- Often called the Kalman-Bucy Filter
- Much less commonly used
- *Not* covered in this course

Continuous Dynamics

- $\dot{x} = f(x, u, n) = A x + B u + U n$
- **Question:** How do we turn this into a discrete time system?
 - State-transition matrix
 - Numerical integration
- One-step Euler integration
 - $x_t = x_{t-1} + f(x_{t-1}, u_t, n_t) \delta t$
 - $x_t = (I + \delta t A) x_{t-1} + (\delta t B) u_t + (\delta t U) n_t$
 - $x_t = F x_{t-1} + G u_t + V n_t$
- Prediction:
 - $\bar{\mu}_t = F \mu_{t-1} + G u_t$
 - $\bar{\Sigma}_t = F \Sigma_{t-1} F^T + V Q V^T$

Example Problem

- Second order system $\mathbf{x} = [s, \dot{s}]^T$
- Input is a force $\ddot{s} = u$
- $\dot{\mathbf{x}} = f(\mathbf{x}, u, n) = A \mathbf{x} + B u + U n$
- $F = (I + \delta t A)$
- $G = \delta t B$
- $V = \delta t U$
- Prediction:
 - $\bar{\mu}_t = F \mu_{t-1} + G u_t$
 - $\bar{\Sigma}_t = F \Sigma_{t-1} F^T + V Q V^T$
- $\dot{\mathbf{x}} = \begin{bmatrix} \dot{s} \\ \ddot{s} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + n$
- $F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \delta t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \delta t \\ 0 & 1 \end{bmatrix}$
- $G = \delta t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \delta t \end{bmatrix}$
- $V = \delta t$

Recap

Bayes' Filter

- **Prior:** $p(x_0)$ ← State
- **Process model:** $f(x_t | x_{t-1}, u_t)$ ← Control input
- **Measurement model:** $g(z_t | x_t)$ ← Measurement
- **Prediction step:**
- $p(x_t | z_{1:t-1}, u_{1:t}) = \int f(x_t | x_{t-1}, u_t) p(x_{t-1} | z_{1:t-1}, u_{1:t-1}) dx_{t-1}$
- **Update step:**
- $$p(x_t | z_{1:t}, u_{1:t}) = \frac{g(z_t | x_t) p(x_t | z_{1:t-1}, u_{1:t})}{\int g(z_t | x'_t) p(x'_t | z_{1:t-1}, u_{1:t}) dx'_t}$$

Assumptions

- The prior state of the robot is represented by a Gaussian distribution
 - $p(x_0) \sim N(\mu_0, \Sigma_0)$
- The process model $f(x_t | x_{t-1}, u_t)$ is linear with additive Gaussian white noise
 - $x_t = A_t x_{t-1} + B_t u_t + n_t$
 - $n_t \sim N(0, Q_t)$
- The measurement model $g(z_t | x_t)$ is linear with additive Gaussian white noise
 - $z_t = C_t x_t + v_t$
 - $v_t \sim N(0, R_t)$

Kalman Filter

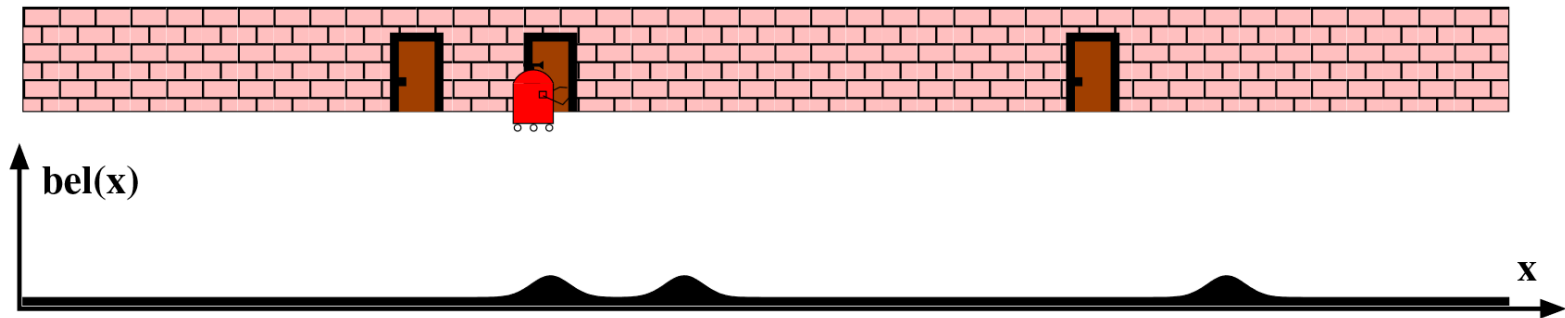
- Prior:
 - $p(x_0) \sim N(\mu_0, \Sigma_0)$
- Transition model:
 - $x_t = A_t x_{t-1} + B_t u_t + n_t$
 - $n_t \sim N(0, Q_t)$
- Measurement model:
 - $z_t = C_t x_t + v_t$
 - $v_t \sim N(0, R_t)$
- Prior:
 - μ_{t-1}, Σ_{t-1}
- Prediction:
 - $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
 - $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t$
- Update:
 - $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
 - $\Sigma_t = \bar{\Sigma}_t - K_t C_t \bar{\Sigma}_t$
 - $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$

Continuous Dynamics

- Can convert continuous time systems
- $\dot{x} = f(x, u, n) = A x + B u + U n$
- Into discrete time systems using one-step Euler integration
- $x_t = F x_{t-1} + G u_t + V n_t$
- $F = (I + \delta t A), G = \delta t B, V = \delta t U$
- This will introduce some error, but the observations can help correct it
- Prediction:
 - $\bar{\mu}_t = F \mu_{t-1} + G u_t$
 - $\bar{\Sigma}_t = F \Sigma_{t-1} F^T + V Q V^T$

Kalman Filter Discussion

- **Advantages:**
 - Simple
 - Purely matrix operations
 - Computationally efficient, even for high dimensional systems
- **Disadvantages:**
 - Assumes everything is linear and Gaussian
 - Unimodal distribution
 - Cannot handle multiple hypotheses



Reading

- “Probabilistic Robotics”, Sebastian Thrun, Wolfram Burgard, and Dieter Fox, Chapter 2, Chapter 3

Logistics

- (Delayed) Lab session this week
 - (Sorry ☹️) Previous lab session was canceled due to broken network switch
 - This week: Straight line & multi-waypoint trajectory tracking
- Project 2, Phase 1 will be released tomorrow
 - Vision-based position estimation
 - Due 13 Nov.