MATH 217 - LINEAR ALGEBRA
HOMEWORK 1, DUE Friday, September 16, 2016

Hand in Part A and Part B as two *separate* assignments. Include the following information in the top left corner of every assignment:

- your full name,
- instructor's last name and section number,
- homework number,
- whether they are Part A problems or Part B problems.

A few words about solution writing:

- **Unless you are explicitly told otherwise for a particular problem, you are always expected to show your work and to give justification for your answers.** In particular, when asked if a statement is true or false, you will need to explain *why* it is true or false to receive full credit.
- Write down your solutions in full, as if you were writing them for another student in the class to read and understand.
- Don't be sloppy, since your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 ...").
- If you compute something by observation, say so and make sure that your imaginary fellow student who is reading your proof can also clearly see what you are claiming.
- Justify each step in writing and leave nothing to the imagination.

Part A (15 points)

Solve the following problems from the book:
Section 1.1: 18, 36, 44
Section 1.2: 10, 36, 48

Part B (25 points)

The following part covers material that is in the "Joy of Sets" and "Mathematical Hygiene" handouts, available on the Canvas site. Some of the material in these handouts, but not all of it, is recapped here.

1. Logical Connectives.

Every mathematical statement is either true or false. Starting from given mathematical statements, we can use logical operations to form new mathematical statements which are again either true or false. Let P and Q be two statements. Here are four basic logical constructions:

- The statement "P *and* Q" is true exactly when both P and Q are true statements.
- The statement "P *or* Q" is true exactly when at least one (possibly both!) of P or Q is true.
- The statement "*if* P *then* Q" is true exactly when Q is true or P is false. The shorthand notation for *"if P then Q"* is $P \Rightarrow Q$, read "P implies Q."

- The statement "*P if and only if Q*" is true exactly when both $P \implies Q$ and $Q \implies P$ are true statements. The shorthand notation for "*P if and only if Q*" is $P \iff Q$.

Problem 1. Decide whether the following statements are true or false. Justify your answers.

(a) 7 is prime or 16 is a perfect square.
(b) If all squares are rectangles and all triangles are squares, then 5 is divisible by 3.
(c) $\frac{d}{dx}(x^2) = 2x$ if and only if $\sin(\pi/6) = \sqrt{3}/2$.
(d) If every complex number has a complex square root, then every real number has a real square root.
(e) 10 is even and 10^{10} is odd if and only if the set of prime numbers is finite.[1]

2. Quantifiers.

Starting from a statement which involves a variable, we can form a new statement by quantifying the given variable.

- The quantifier "for all" indicates that something is true about every element in a given set and is abbreviated $\forall$. It is often appropriate to read "for all" as "for every" or "for each". For example, the truth value of the statement $x^2 > 0$ depends on the value of x. So the quantified statement "$\forall x \in \mathbb{R},\ x^2 > 0$" is false, since it fails for $x = 0$.
- The quantifier "there exists" indicates that something is true for at least one element in a given set and is abbreviated $\exists$. It is often read as "for some", where "some" is not necessarily plural. For example, the truth value of the statement $x^2 = 0$ depends on the value of x. So the quantified statement "$\exists x \in \mathbb{R}$ such that $x^2 = 0$" is true, since it holds for $x = 0$.
- The abbreviation "s.t." stands for "such that". (Yes, mathematicians can be lazy!)

Problem 2.

(a) Let $P(x)$ be a statement whose truth value depends on x. An *example* is a value of x that makes $P(x)$ true, and a *counterexample* is a value of x that makes $P(x)$ false. Fill in the blank spaces with "is true", "is false", or "nothing" as appropriate:

	"$\forall x, P(x)$"	"$\exists x$ s.t. $P(x)$"
An example proves		
A counterexample proves		

In (b) – (g), determine whether the given statement is true or false, and briefly justify your answer.

(b) Some triangles have three vertices.
(c) Every integer is even or every integer is odd.
(d) Every integer is even or odd.
(e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x + y = 0$.
(f) For all $x \in \mathbb{R}$ there is $y \in \mathbb{R}$ such that $x + y = 0$.
(g) For every positive real number a, there exists a unique[2] real number x such that $x^2 = a$.

[1]By convention, the "if ... then" and "if and only if" connectives bind more strongly than do the connectives "and" and "or." This means that you should read the statement in Problem 1(e) as "(10 is even and 10^{10} is odd) if and only if (the set of prime numbers is finite)." *Negation*, which is signified by the word "not," binds even more strongly than "and" and "or" do.

[2]The statement "there exists unique $x \in X$ such that $P(x)$" means that there is one and only one element in the set X having property P. For those who like fancy symbolisms, this is sometimes abbreviated "$\exists! x \in X$ s.t. $P(x)$."

3. Negation.

The *negation* of a statement P, denoted "*not* P," is a statement that is true whenever P is false and false whenever P is true. There may be many different ways to formulate the negation of P, but all of them will be logically equivalent. Note that the negation of an if-then statement, $P \Longrightarrow Q$, is P and (not) Q, as $P \Longrightarrow Q$ is false if and only if P is true and Q is false.

Problem 3. Formulate the negation of each of the statements below in a meaningful way (some of the statements have been recycled from Problems 1 and 2). Note: just writing "It is not the case that ..." before each statement will not receive credit, as that does not help the reader understand the meaning of the negation.

(a) 5 is even and 7 is not odd.
(b) $\frac{d}{dx}(x^2) = 2x$ if and only if $\sin(\pi/6) = \sqrt{3}/2$.
(c) If every complex number has a complex square root, then every real number has a real square root.
(d) Every integer is even or every integer is odd.
(e) For all $x \in \mathbb{R}$ there is $y \in \mathbb{R}$ such that $x + y = 0$.

4. Converse and Contrapositive.

There are two additional logical statements that can be formed from a given "if-then" statement:

- The *converse* of the statement $P \Longrightarrow Q$ is the statement $Q \Longrightarrow P$. The converse may be true or false, independent of the truth value of the original "if-then" statement. To see this, compare the truth tables for both statements:

P	Q	$P \Longrightarrow Q$	$Q \Longrightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

The last two columns do not coincide.

- The *contrapositive* of the statement $P \Longrightarrow Q$ is the statement not $Q \Longrightarrow$ not P. The original "if-then" statement and its contrapositive have the *same* truth value. To see this, compare the truth tables for both statements:

P	Q	$P \Longrightarrow Q$	not Q	not P	not $Q \Longrightarrow$ not P
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The columns corresponding to $P \Longrightarrow Q$ and not $Q \Longrightarrow$ not P coincide.

Problem 4. Write both the converse and the contrapositive of the following "if-then" statements.

(a) If n is an odd number, then $3n + 1$ is an even number.
(b) If last month was August or next month is October, then this month is September.
(c) If the square matrix A is orthogonal, then both the transpose of A and the inverse of A are also orthogonal.

5. Sets.

A *set* is a container with no distinguishing feature other than its contents. The objects contained in a set are called the *elements* of the set. We write $a \in S$ to signify that the object a is an element of the set S. The number of elements in a set S is called the *cardinality* of the set, and it is denoted by $|S|$.

Since a set has no distinguishing feature other than its contents, there is a unique set containing no elements which is called the *empty set* and is denoted $\emptyset$. Some other very common sets are the set $\mathbb{R}$ of all real numbers, the set $\mathbb{Q}$ of all rational numbers, the set $\mathbb{Z}$ of all integers, and the set $\mathbb{C}$ of all complex numbers.

There are two important ways to specify a set.

- *Enumeration.* One can list the contents of the set, in which case the set is denoted by enclosing the list in curly braces. For example, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- *Comprehension.* One can describe the contents of the set by a property of its elements. If $P(a)$ is a property of the object a, then the set of all objects a such that $P(a)$ is true is denoted by $\{a \mid P(a)\}$, or equivalently $\{a : P(a)\}$. For example,

$$\mathbb{Q} \;=\; \{x \in \mathbb{R} \mid x = \tfrac{a}{b} \text{ for some } a, b \in \mathbb{Z} \text{ with } b \neq 0\}.$$

Let X and S be sets. We say that S is a *subset* of X if $a \in S \implies a \in X$ holds for all objects a. We write $S \subseteq X$ to signify that S is a subset of X. This means that S is a set each of whose elements also belongs to X. The subset of X consisting of all elements a of X such that property $P(a)$ holds true is denoted $\{a \in X \mid P(a)\}$.

Problem 5.

(a) Give common English descriptions of the following sets:
  (i) $\{n \in \mathbb{N} \mid \text{for all } m \in \mathbb{N}, \text{ if } n \text{ is a multiple of } m \text{ then } m = 1 \text{ or } m = n\}$.
  (ii) $\{z \in \mathbb{C} \mid z = x^2 \text{ for some } x \in \mathbb{R}\}$.

(b) Use set comprehension notation to give a description of each of the following sets:
  (i) The set of all integers that are powers of prime numbers.
  (ii) The plane consisting of all points in $\mathbb{R}^3$ whose coordinate entries sum to 1.

(c) Let $S = \{a, b, c\}$. One subset of S is the set $\{a, b\}$, so we may write $\{a, b\} \subseteq \{a, b, c\}$. List *all eight* of the subsets of S.

6. Set Operations.

Starting from given sets, we can use set operations to form new sets.

- Given sets X and Y, the *intersection* of X and Y is defined as
$$X \cap Y = \{a \mid a \in X \text{ and } a \in Y\}.$$
- Given sets X and Y, the *union* of X and Y is defined as
$$X \cup Y = \{a \mid a \in X \text{ or } a \in Y\}.$$
- Given sets X and Y, the *difference* of X and Y, denoted $X \setminus Y$ or $X - Y$, is the set
$$\{x \in X \mid x \notin Y\}.$$
- Given a set Y inside some larger set X, the *complement* of Y with respect to X, denoted Y^C, is $X \setminus Y$. (The larger set X, sometimes referred to as the *universe*, is often suppressed in the notation).

Problem 6. For each rational number q, let $q\mathbb{Z} = \{qm \mid m \in \mathbb{Z}\}$, so that we have $q\mathbb{Z} \subseteq \mathbb{Q}$.

(a) Use enumeration to describe the sets $\frac{1}{2}\mathbb{Z}$, $\frac{1}{3}\mathbb{Z}$, $\frac{1}{2}\mathbb{Z} \cap \frac{1}{3}\mathbb{Z}$, $\frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z}$, $\frac{1}{2}\mathbb{Z} \setminus \frac{1}{3}\mathbb{Z}$, and $(3\mathbb{Z})^C$ (where the complement is taken inside $\mathbb{Z}$).

(b) What is the smallest natural number n such that every set from part (a) is contained in $\frac{1}{n}\mathbb{Z}$?