Inverse Kinematics

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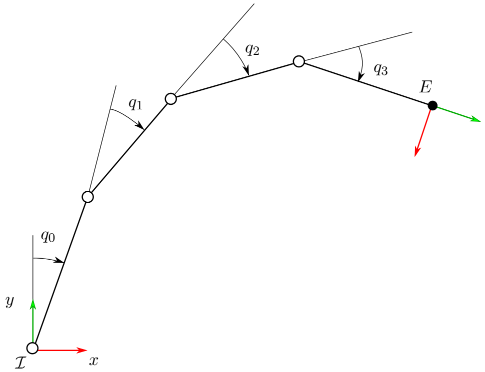
- Introduction
- Forward and Inverse Kinematics
- Forward Differential Kinematics
- Pseudo Inverse Matrices
- Inverse Differential Kinematics
- Numerical Approach To Solve Inverse Kinematics Problems

What is *kinematics*?

- Greek: $\kappa\iota\nu\eta\mu\alpha$, *kinema*, movement
- Movement of a body can be described by positions, velocities, accelerations
- Kinematics does not describe what causes the movement (i.e. no forces and torques)



Generalized Coordinates

$$q = (q_1, q_2, \ldots, q_n)^T$$

Endeffector Position and Orientation (Cartesian Coordinates)

$$X_E = (x, y, z, \theta_x, \theta_y, \theta_z)^T$$

Forward Kinematics $X_E = f_F(q)$

- Straightforward
- Use to obtain endeffector position and orientation given all joint angles q

Inverse Kinematics $q = f_I(X_E)$

- Hard or even impossible to obtain the function f_I
- If closed-form expression can be found, very fast
- Multiple solutions may be obtained

Forward Differential Kinematics $\dot{X}_E = J_{e0}(q)\dot{q}$

- *Linear* map: $J_{e0}(q) : \mathbb{R}^n \to \mathbb{R}^6$
- Linearization of robot configuration around a given joint configuration

Geometric Endeffector Jacobian $J_{e0}(q) \in \mathbb{R}^{6\times n}$

- Structure is only a function of the robot geometry
- In general $J_{e0}^{-1}(q)$ is not defined (i.e. $n \neq 6$), hence we need a *Pseudo Inverse*

Inverse Differential Kinematics $\dot{q} = J_{e0}^{+}(q)\dot{X}_E$

- *Linear* map: $J_{e0}^{+}(q) : \mathbb{R}^6 \to \mathbb{R}^n$
- Note: $J_{e0}^{+}(q)$ is called the *Pseudo Inverse* of $J_{e0}(q)$

Derive Moore-Penrose inverse

Consider the system

$$Ax = b$$

where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ with $m \neq n$

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1. Left multiply by A^T: $\quad A^T A x = A^T b$

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2. Note that $A^T A$ is a symmetric positive semidefinite matrix. It is invertible iff A has full rank. Hence, left-multiply by $(A^T A)^{-1}$

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3. $x = (A^T A)^{-1} A^T b$

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2. Note that $A^T A$ is a symmetric positive semidefinite matrix. It is invertible iff A has full rank. Hence, left-multiply by $(A^T A)^{-1}$
3. $x = (A^T A)^{-1} A^T b$
4. Define $A^+ := (A^T A)^{-1} A^T$

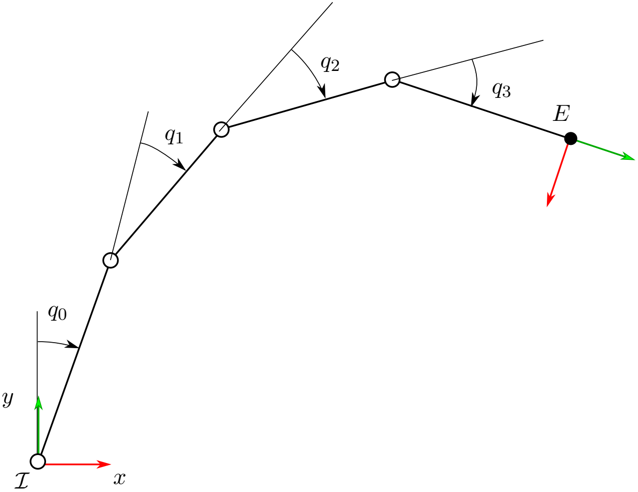
Damped Least-Squares Solution (DLS)

- Moore-Penrose inverse can become singular (i.e. $A^T A$ is not invertible)
- Add small positive values to diagonal entries of $A^T A$, then matrix will *always* be invertible
- Define $A^+ := (A^T A + \lambda \mathbb{I})^{-1} A^T$
- Note that the introduced error becomes larger as λ becomes larger.

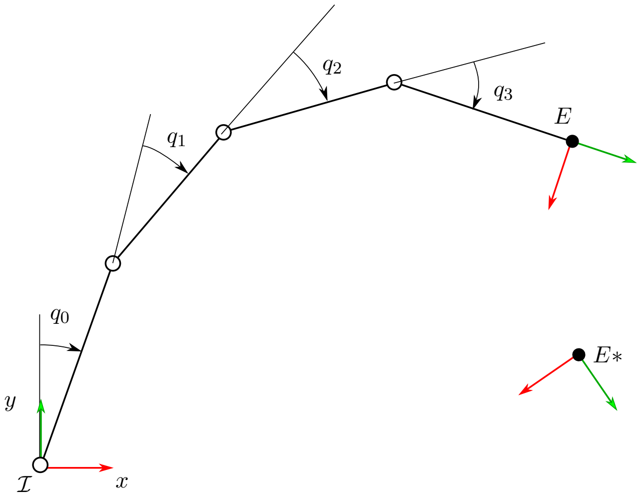
Iterative Inverse Kinematics

Given a desired endeffector position and orientation X_E^*, what are the corresponding joint angles q^*?

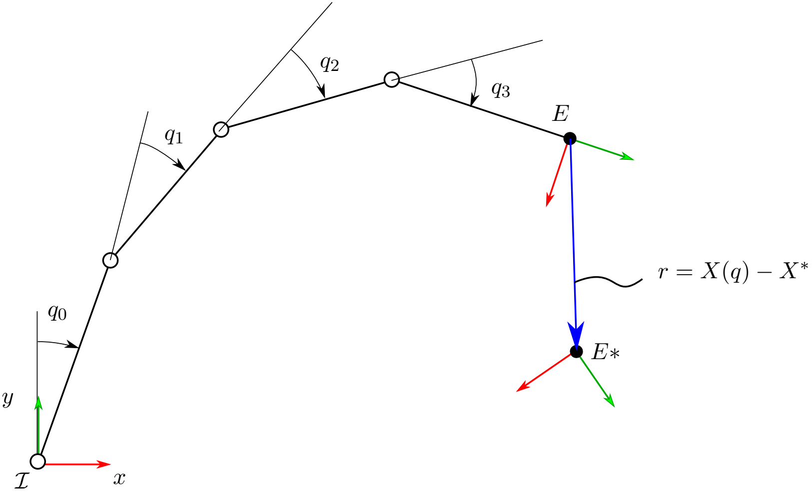
1. Initialize joints to some arbitrary, nonsingular position $q^{(0)}$
2. Compute the position and orientation of endeffector $X_E(q^{(0)})$ using simple Forward Kinematics
3. Compute pose difference $\Delta X_E = X_E^* - X_E(q^{(0)})$
4. Interpret ΔX_E as velocity vector pointing to the goal pose
5. Use Inverse Differential Kinematics to compute joint velocities that move the endeffector towards goal pose: $\dot{q}^{(0)} = J_{e0}^{+}(q^{(0)})\Delta X_E$
6. Apply a scaled version of $\dot{q}^{(0)}$ to the joints and update the robot pose, i.e. $q^{(1)} = q^{(0)} + k\dot{q}^{(0)}$ where $k \in \mathbb{R}^+$
7. Go to step 2, use updated joint configuration $q^{(1)}$

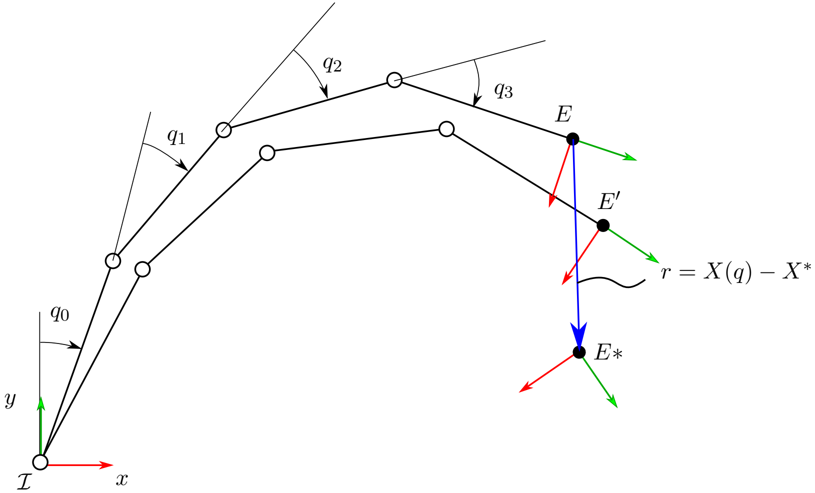


q_0
q_1
q_2
q_3
E
y
x
$\mathcal{I}$

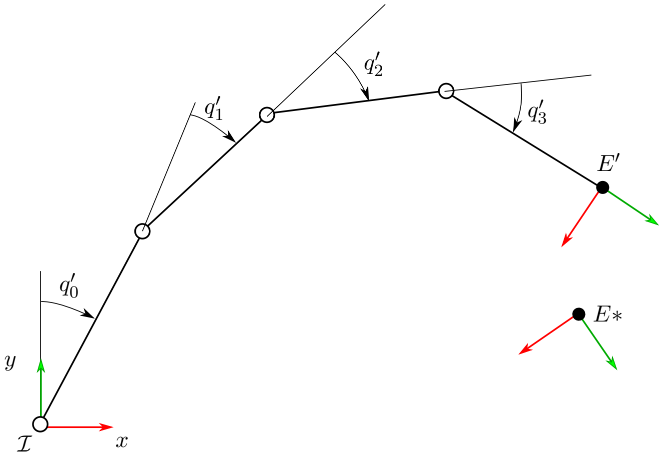


q_2
q_3
q_1
E
q_0
$E*$
y
$\mathcal{I}$
x

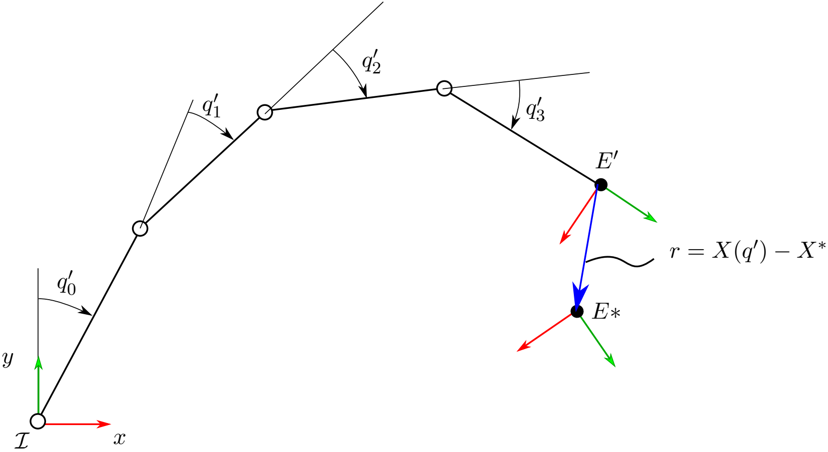

q_0
q_1
q_2
q_3
E
$E*$
$r = X(q) - X^*$
y
x
$\mathcal{I}$

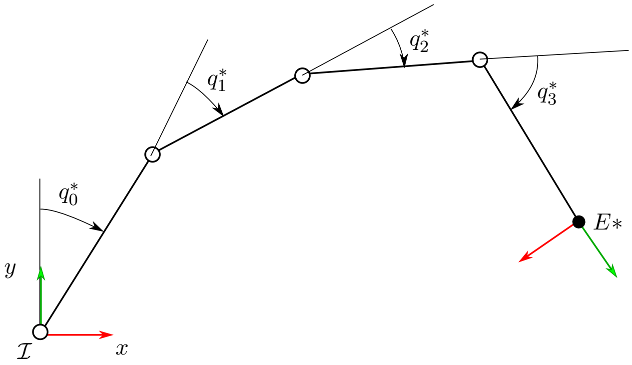


q_0
q_1
q_2
q_3
E
E'
$r = X(q) - X^*$
$E*$
y
x
$\mathcal{I}$



q_0'
q_1'
q_2'
q_3'
E'
$E*$
y
x
$\mathcal{I}$


q'_2
q'_1
q'_3
E'
$r = X(q') - X^*$
q'_0
$E*$
y
$\mathcal{I}$
x


q_0^*
q_1^*
q_2^*
q_3^*
$E*$
y
x
$\mathcal{I}$

Iterative Inverse Kinematics

What can go wrong?

- Initialize in singular position
- If $k\dot{q}^{(i)}$ is large, the linearization may no longer be a good approximation of the robots configuration
- No axis-limits are considered
- Moore-Penrose inverse may become singular $\rightarrow$ use DLS