3.1-1

$$\max(f(n), g(n)) = \Theta(f(n) + g(n))$$

if $f(n) = \max(f(n), g(n))$ or $g(n)$ is min
$\Rightarrow g(n)\ O(f(n))$ $\Rightarrow \Omega(g(n))$

otherwise $f(n) = \min(f(n), g(n))$ or $g(n) = \max(f(n), g(n))$
$\Rightarrow \Omega(f(n))$ $\Rightarrow O(g(n))$.

$\Rightarrow \Theta(f(n) + g(n))$

3.1-2

$$(n+a)^b = \Theta(n^b)$$

$$\lim_{n\to\infty} \frac{(n+a)^b}{n^b} = 1.$$

$(n+a)^b \le c n^b$

$\Rightarrow \left(\frac{n+a}{n}\right)^b \le c$

$\Rightarrow c \ge \left(1 + \frac{a}{n}\right)^b.$

or $c \ge 1$ because for any arbitrarily large no. $\frac{a}{n} < 1$

$\Rightarrow 1 \le 1 + \frac{a}{n} \ll$

$\Rightarrow$ for $c = 2$ $(n+a)^b = O(n^b)$.
& $c = \frac{1}{2}$ $(n+a)^b = \Omega(n^b)$

$\Theta(n^b)$.

Let $f(n)$ be $\max(f, g)$.
~~max~~

$\max(f, g) = f \le 1(f+g)$ Θ.

$\ge \frac{1}{2}(f+g)$

3.1-4

$2^{n+1} = O(2^n)$?

$2^{n+1} = 2^n \cdot 2 \leq c \cdot 2^n$

$c \geq 2 \Rightarrow$ for $\underline{c=3}$ $(2^{n+1}) = O(2^n)$.

$2^{2n} = 2^n \cdot 2^n$

For $2^{2n} \leq c2^n$

$\Rightarrow 2^n \cdot 2^n \leq c2^n$

$\Rightarrow c \geq 2^n$ which would not be possible for an arbitrarily large $\underline{n}$

3.1-5

3 - 1

$$p(n) = \sum_{i=0}^{d} a_i n^i$$

(a) if $k \geq d$.

prove that $p(n) = O(n^k)$

$$\Rightarrow \sum_{i=0}^{d} a_i n^i \leq c n^k$$

OR $\sum_{i=0}^{d} a_i n^{i-k} \leq c$ [Dividing both sides by n^k]

$$\Rightarrow c \geq \sum_{i=0}^{d} a_i n^{i-k}$$

since $k \geq d \Rightarrow i-k \leq 0$

$$\Rightarrow c \geq a_d$$

$\therefore$ For $c = a_d + 1$, $p(n) = O(n^k)$ $\forall$ $n \geq n_0$

(b) if $k \leq d$

prove that $p(n) = \Omega(n^k)$.

$$\sum_{i=0}^{d} a_i n^i \geq c n^k$$

$\Rightarrow \sum_{i=0}^{d} a_i n^{i-k} \geq c$ [Dividing both sides by n^k]

$$\Rightarrow c \leq \sum_{i=0}^{d} a_i n^{i-k}$$

since $k \leq d \Rightarrow i-k \geq 0$

$$\Rightarrow c \leq a_0$$

for $c = a_0 - 1$ $p(n) = \Omega(n^k)$ for all n_0.

(c). $k = d$

prove that $p(n) = \Theta(n^k)$.

$$c_1 n^k \leq \sum_{i=0}^{d} a_i n^i \leq c_2 n^k$$

$$\Rightarrow \quad c_1 \leq \sum_{i=0}^{d} a_i n^{i-k} \leq c_2$$ [Dividing all sides by n^k]

Since $k = d \Rightarrow i-k \leq 0$

$$\Rightarrow \quad c_1 \leq \frac{a_0}{n^k} + \frac{a_1}{n^{k-1}} + \dots + a_d \leq c_2$$

OR $c_1 \leq a_d \leq c_2$ {for arbitrarily large n}.

Hence, there exists a c_1 and c_2 such that $p(n) = \Theta(n^k)$.

(d). $k > d$

$p(n) = o(n^k)$.

$$\sum_{i=0}^{d} a_i n^i < c_1 n^k.$$

$$\Rightarrow \quad \sum_{i=0}^{d} a_i n^{i-k} < c_1$$

since $k > d \Rightarrow i-k < 0$

$$\Rightarrow \quad c_1 > \frac{a_d}{n^{k-d}} + \dots + \frac{a_0}{n^k}$$

$\Rightarrow$ for any value of $c_1 > a_d$

$p(n) = o(n^k)$.

(e). $k < d$

$$p(n) = \omega(n^k).$$

$$\sum_{i=0}^{d} a_i n^i \quad > c n^k$$

$$\Rightarrow \quad \sum_{i=0}^{d} a_i n^{i-k} > c$$

Since $k < d \Rightarrow$ ~~$i-k >$~~ $-k < i-k < d-k$.

$$\Rightarrow \quad c < a_d n^{d-k} + \cdots + \frac{a_0}{n^k}$$

$\Rightarrow$ For any value $c < a_d$.

$$p(n) = \omega(n^k)$$

3-2

(a) $\log^k n$ v/s n^ϵ

$$\lim_{n \to \infty} \frac{\log^k n}{n^\epsilon} = 0$$

$\Rightarrow \log^k n = o(n^\epsilon)$.

(b) n^k v/s c^n

$$\lim_{n \to \infty} \frac{n^k}{c^n} = 0 \quad \text{for } \underline{c > 1}$$

$\Rightarrow n^k = o(c^n)$.

(c) $\sqrt{n}$ v/s $n^{\sin n}$.

$n^{\sin n} \Rightarrow \sin n$ oscillates between 0 & 2 for $n \to \infty$.

$$\Rightarrow \lim_{n \to \infty} \frac{\sqrt{n}}{n^{\sin n}} = \lim_{n \to \infty} n^{1/2 - \sin n}$$

$\sim$ oscillates b/w $\underline{0 \text{ \& } 2}$

or $n^{\sin n} = o(\sqrt{n})$.

or $\sqrt{n} = \omega(n^{\sin n})$

(d) 2^n v/s $2^{n/2}$

$$\lim_{n \to \infty} \frac{2^n}{\sqrt{2^n}} = \lim_{n \to \infty} \sqrt{2^n} = \infty$$

$\Rightarrow 2^n = \omega(2^{n/2})$

3-3

(a) Order correctly:

Exponential:

$2^{2^{n+1}}$, 2^n, $2^{\sqrt{2\log n}}$, $4^{\log n}$,

$\log(\log^* n)$, $2^{\log^* n}$, $(\sqrt{2})^{\log n} = \sqrt{n}$, n^2, $n!$,

$(\log n)!$, $\left(\frac{3}{2}\right)^n$, n^3, $\log^2 n$, $\log(n!)$, 2^{2^n}, $n^{1/\log n} =$

$\ln \ln n$, $\log^* n$, $n.2^n$, $n^{\log\log n}$, $\ln n$, $\cancel{1}$,

$2^{\log n}$, $(\log n)^{\log n}$, e^n, $4^{\log n}$, $(n+1)!$, $\sqrt{\log n}$

$\log^*(\log n)$, $2^{\sqrt{2\log n}}$, n, 2^n, $n\log n$, $2^{2^{n+1}}$

Order increasing.

① $\left.\begin{array}{l} 1 \\ n^{1/\log n} = 2^{\log n/\log n} = 2. \end{array}\right\}$ Same. $O(1)$.

$\log\log^* n$, $\ln n$, $\log^* n$, $\ln\ln n$, $\log^* \log n$