A trip to Asymptopia

Statistical Inference

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Asymptotics

- Asymptotics is the term for the behavior of statistics as the sample size (or some other relevant quantity) limits to infinity (or some other relevant number)
- (Asymptopia is my name for the land of asymptotics, where everything works out well and there's no messes. The land of infinite data is nice that way.)
- Asymptotics are incredibly useful for simple statistical inference and approximations
- (Not covered in this class) Asymptotics often lead to nice understanding of procedures
- Asymptotics generally give no assurances about finite sample performance
  - The kinds of asymptotics that do are orders of magnitude more difficult to work with

Asymptotics form the basis for frequency interpretation of probabilities (the long run proportion of

Numerical limits

- Imagine a sequence
  - a_1 = .9,
  - a_2 = .99,
  - a_3 = .999, ...
- Clearly this sequence converges to 1
- Definition of a limit: For any fixed distance we can find a point in the sequence so that the sequence is closer to the limit than that distance from that point on

Limits of random variables

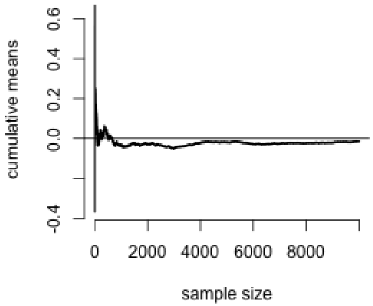
- The problem is harder for random variables
- Consider $\bar X_n$ the sample average of the first n of a collection of iid observations
    - Example $\bar X_n$ could be the average of the result of n coin flips (i.e. the sample proportion of heads)
- We say that $\bar X_n$ converges in probability to a limit if for any fixed distance the probability of $\bar X_n$ being closer (further away) than that distance from the limit converges to one (zero)

The Law of Large Numbers

- Establishing that a random sequence converges to a limit is hard
- Fortunately, we have a theorem that does all the work for us, called the **Law of Large Numbers**
- The law of large numbers states that if $X_1,\ldots X_n$ are iid from a population with mean μ and variance σ^2 then $\bar X_n$ converges in probability to μ
- (There are many variations on the LLN; we are using a particularly lazy version, my favorite kind of version)

Law of large numbers in action

```r
n <- 10000
means <- cumsum(rnorm(n))/(1:n)
plot(1:n, means, type = "l", lwd = 2, frame = FALSE, ylab = "cumulative means", xlab = "sample size")
abline(h = 0)
```



Discussion

- An estimator is **consistent** if it converges to what you want to estimate
  - Consistency is neither necessary nor sufficient for one estimator to be better than another
  - Typically, good estimators are consistent; it's not too much to ask that if we go to the trouble of collecting an infinite amount of data that we get the right answer
- The LLN basically states that the sample mean is consistent
- The sample variance and the sample standard deviation are consistent as well
- Recall also that the sample mean and the sample variance are unbiased as well
- (The sample standard deviation is biased, by the way)

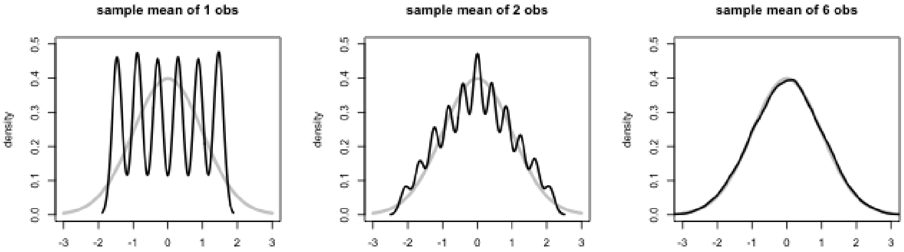
The Central Limit Theorem

- The **Central Limit Theorem** (CLT) is one of the most important theorems in statistics
- For our purposes, the CLT states that the distribution of averages of iid variables, properly normalized, becomes that of a standard normal as the sample size increases
- The CLT applies in an endless variety of settings
- Let $X_1,\ldots,X_n$ be a collection of iid random variables with mean μ and variance σ^2
- Let $\bar X_n$ be their sample average
- Then $\frac{\bar X_n - \mu}{\sigma / \sqrt{n}}$ has a distribution like that of a standard normal for large n.
- Remember the form $\frac{\bar X_n - \mu}{\sigma / \sqrt{n}} = \frac{\mbox{Estimate} - \mbox{Mean of estimate}}{\mbox{Std. Err. of estimate}}$.
- Usually, replacing the standard error by its estimated value doesn't change the CLT

Example

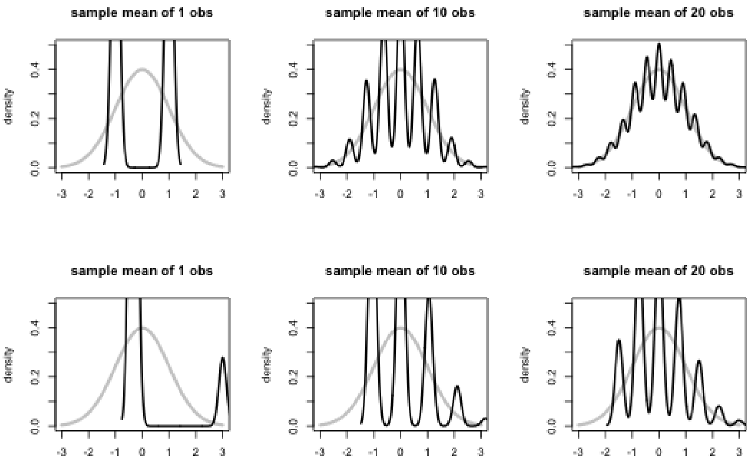
- Simulate a standard normal random variable by rolling n (six sided)
- Let X_i be the outcome for die i
- Then note that $\mu = E[X_i] = 3.5$
- $Var(X_i) = 2.92$
- SE $\sqrt{2.92 / n} = 1.71 / \sqrt{n}$
- Standardized mean $\frac{\bar X_n - 3.5}{1.71/\sqrt{n}}$

Simulation of mean of n dice



Coin CLT

- Let X_i be the 0 or 1 result of the i^{th} flip of a possibly unfair coin
  - The sample proportion, say $\hat p$, is the average of the coin flips
  - $E[X_i] = p$ and $Var(X_i) = p(1-p)$
  - Standard error of the mean is $\sqrt{p(1-p)/n}$
  - Then $\frac{\hat p - p}{\sqrt{p(1-p)/n}}$ will be approximately normally distributed


sample mean of 1 obs
sample mean of 10 obs
sample mean of 20 obs
sample mean of 1 obs
sample mean of 10 obs
sample mean of 20 obs
density
0.0
0.2
0.4
-3
-2
-1
0
1
2
3

CLT in practice

- In practice the CLT is mostly useful as an approximation P\left(\frac{\bar X_n - \mu}{\sigma / \sqrt{n}} \leq z \right) \approx \Phi(z).
- Recall 1.96 is a good approximation to the .975^{th} quantile of the standard normal
- Consider \begin{eqnarray*} .95 & \approx & P\left(-1.96 \leq \frac{\bar X_n - \mu}{\sigma / \sqrt{n}} \leq 1.96 \right)\\ \\ & = & P\left(\bar X_n +1.96 \sigma/\sqrt{n} \geq \mu \geq \bar X_n - 1.96\sigma/\sqrt{n} \right),\\ \end{eqnarray*}

Confidence intervals

- Therefore, according to the CLT, the probability that the random interval $\bar X_n \pm z_{1-\alpha/2}\sigma / \sqrt{n}$ contains μ is approximately 100$(1-\alpha)$%, where $z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution
- This is called a $100(1 - \alpha)$% **confidence interval** for μ
- We can replace the unknown σ with s

Give a confidence interval for the average height of sons

in Galton's data

```
library(UsingR)
data(father.son)
x <- father.son$sheight
(mean(x) + c(-1, 1) * qnorm(0.975) * sd(x)/sqrt(length(x)))/12
```

```
## [1] 5.710 5.738
```

Sample proportions

- In the event that each X_i is 0 or 1 with common success probability p then $\sigma^2 = p(1 - p)$
- The interval takes the form $\hat p \pm z_{1 - \alpha/2} \sqrt{\frac{p(1 - p)}{n}}$
- Replacing p by $\hat p$ in the standard error results in what is called a Wald confidence interval for p
- Also note that $p(1-p) \leq 1/4$ for $0 \leq p \leq 1$
- Let $\alpha = .05$ so that $z_{1 -\alpha/2} = 1.96 \approx 2$ then $2 \sqrt{\frac{p(1 - p)}{n}} \leq 2 \sqrt{\frac{1}{4n}} = \frac{1}{\sqrt{n}}$
- Therefore $\hat p \pm \frac{1}{\sqrt{n}}$ is a quick CI estimate for p

Example

- Your campaign advisor told you that in a random sample of 100 likely voters, 56 intent to vote for you.
  - Can you relax? Do you have this race in the bag?
  - Without access to a computer or calculator, how precise is this estimate?
- `1/sqrt(100)=.1` so a back of the envelope calculation gives an approximate 95% interval of `(0.46, 0.66)`
  - Not enough for you to relax, better go do more campaigning!
- Rough guidelines, 100 for 1 decimal place, 10,000 for 2, 1,000,000 for 3.

```
round(1/sqrt(10^(1:6)), 3)
```

```
## [1] 0.316 0.100 0.032 0.010 0.003 0.001
```

Poisson interval

- A nuclear pump failed 5 times out of 94.32 days, give a 95% confidence interval for the failure rate per day?
- $X \sim Poisson(\lambda t)$.
- Estimate $\hat \lambda = X/t$
- $Var(\hat \lambda) = \lambda / t$ $\frac{\hat \lambda - \lambda}{\sqrt{\hat \lambda / t}} = \frac{X - t \lambda}{\sqrt{X}} \rightarrow N(0,1)$
- This isn't the best interval.
    - There are better asymptotic intervals.
    - You can get an exact CI in this case.

R code

```
x <- 5
t <- 94.32
lambda <- x/t
round(lambda + c(-1, 1) * qnorm(0.975) * sqrt(lambda/t), 3)
```

```
## [1] 0.007 0.099
```

```
poisson.test(x, T = 94.32)$conf
```

```
## [1] 0.01721 0.12371
## attr(,"conf.level")
## [1] 0.95
```

In the regression class

```
exp(confint(glm(x ~ 1 + offset(log(t)), family = poisson(link = log))))
```

```
## Waiting for profiling to be done...
```

```
##   2.5 %  97.5 %
## 0.01901 0.11393
```