Likelihood

Statistical Inference

Brian Caffo, Roger Peng, Jeff Leek
Johns Hopkins Bloomberg School of Public Health

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Likelihood

- A common and fruitful approach to statistics is to assume that the data arises from a family of distributions indexed by a parameter that represents a useful summary of the distribution
- The **likelihood** of a collection of data is the joint density evaluated as a function of the parameters with the data fixed
- Likelihood analysis of data uses the likelihood to perform inference regarding the unknown parameter

Likelihood

Given a statistical probability mass function or density, say $f(x, \theta)$, where θ is an unknown parameter, the **likelihood** is f viewed as a function of θ for a fixed, observed value of x.

Interpretations of likelihoods

The likelihood has the following properties:

1. Ratios of likelihood values measure the relative evidence of one value of the unknown parameter to another.
2. Given a statistical model and observed data, all of the relevant information contained in the data regarding the unknown parameter is contained in the likelihood.
3. If $\{X_i\}$ are independent random variables, then their likelihoods multiply. That is, the likelihood of the parameters given all of the X_i is simply the product of the individual likelihoods.

Example

- Suppose that we flip a coin with success probability θ
- Recall that the mass function for x $f(x,\theta) = \theta^x(1 - \theta)^{1 - x} ~~~\mbox{for}~~~ \theta \in [0,1]$. where x is either 0 (Tails) or 1 (Heads)
- Suppose that the result is a head
- The likelihood is ${\cal L}(\theta, 1) = \theta^1 (1 - \theta)^{1 - 1} = \theta ~~~\mbox{for} ~~~ \theta \in [0,1]$.
- Therefore, ${\cal L}(.5, 1) / {\cal L}(.25, 1) = 2$,
- There is twice as much evidence supporting the hypothesis that $\theta = .5$ to the hypothesis that $\theta = .25$

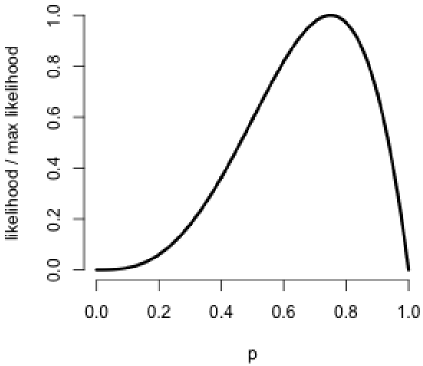
Example continued

- Suppose now that we flip our coin from the previous example 4 times and get the sequence 1, 0, 1, 1
- The likelihood is: $$\begin{eqnarray*} {\cal L}(\theta, 1,0,1,1) & = & \theta^1 (1 - \theta)^{1 - 1} \theta^0 (1 - \theta)^{1 - 0} \\ & \times & \theta^1 (1 - \theta)^{1 - 1} \theta^1 (1 - \theta)^{1 - 1}\\ & = & \theta^3(1 - \theta)^1 \end{eqnarray*}$$
- This likelihood only depends on the total number of heads and the total number of tails; we might write ${\cal L}(\theta, 1, 3)$ for shorthand
- Now consider ${\cal L}(.5, 1, 3) / {\cal L}(.25, 1, 3) = 5.33$
- There is over five times as much evidence supporting the hypothesis that $\theta = .5$ over that $\theta = .25$

Plotting likelihoods

- Generally, we want to consider all the values of \theta between 0 and 1
- A **likelihood plot** displays \theta by {\cal L}(\theta,x)
- Because the likelihood measures *relative evidence*, dividing the curve by its maximum value (or any other value for that matter) does not change its interpretation

```r
pvals <- seq(0, 1, length = 1000)
plot(pvals, dbinom(3, 4, pvals)/dbinom(3, 4, 3/4), type = "l", frame = FALSE, lwd = 3, xlab = "p", ylab
```



Maximum likelihood

- The value of \theta where the curve reaches its maximum has a special meaning
- It is the value of \theta that is most well supported by the data
- This point is called the **maximum likelihood estimate** (or MLE) of \theta MLE = \mathrm{argmax}_\theta {\cal L}(\theta, x).
- Another interpretation of the MLE is that it is the value of \theta that would make the data that we observed most probable

Some results

- $X_1, \ldots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ the MLE of μ is $\bar X$ and the ML of σ^2 is the biased sample variance estimate.
- If $X_1,\ldots, X_n \stackrel{iid}{\sim} Bernoulli(p)$ then the MLE of p is $\bar X$ (the sample proportion of 1s).
- If $X_i \stackrel{iid}{\sim} Binomial(n_i, p)$ then the MLE of p is $\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n n_i}$ (the sample proportion of 1s).
- If $X \stackrel{iid}{\sim} Poisson(\lambda t)$ then the MLE of λ is X/t.
- If $X_i \stackrel{iid}{\sim} Poisson(\lambda t_i)$ then the MLE of λ is $\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n t_i}$

Example

- You saw 5 failure events per 94 days of monitoring a nuclear pump.
- Assuming Poisson, plot the likelihood

```r
lambda <- seq(0, 0.2, length = 1000)
likelihood <- dpois(5, 94 * lambda)/dpois(5, 5)
plot(lambda, likelihood, frame = FALSE, lwd = 3, type = "l", xlab = expression(lambda))
lines(rep(5/94, 2), 0:1, col = "red", lwd = 3)
lines(range(lambda[likelihood > 1/16]), rep(1/16, 2), lwd = 2)
lines(range(lambda[likelihood > 1/8]), rep(1/8, 2), lwd = 2)
```