Probability

Statistical Inference

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Notation

- The **sample space**, \Omega, is the collection of possible outcomes of an experiment
  - Example: die roll \Omega = \{1,2,3,4,5,6\}
- An **event**, say E, is a subset of \Omega
  - Example: die roll is even E = \{2,4,6\}
- An **elementary** or **simple** event is a particular result of an experiment
  - Example: die roll is a four, \omega = 4
- \emptyset is called the **null event** or the **empty set**

Interpretation of set operations

Normal set operations have particular interpretations in this setting

1. $\omega \in E$ implies that E occurs when ω occurs
2. $\omega \not\in E$ implies that E does not occur when ω occurs
3. $E \subset F$ implies that the occurrence of E implies the occurrence of F
4. $E \cap F$ implies the event that both E and F occur
5. $E \cup F$ implies the event that at least one of E or F occur
6. $E \cap F=\emptyset$ means that E and F are **mutually exclusive**, or cannot both occur
7. E^c or $\bar E$ is the event that E does not occur

Probability

A **probability measure**, P, is a function from the collection of possible events so that the following hold

1. For an event $E\subset \Omega$, $0 \leq P(E) \leq 1$
2. $P(\Omega) = 1$
3. If E_1 and E_2 are mutually exclusive events $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

Part 3 of the definition implies **finite additivity**

$P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$ where the $\{A_i\}$ are mutually exclusive. (Note a more general version of additivity is used in advanced classes.)

Example consequences

- $P(\emptyset) = 0$
- $P(E) = 1 - P(E^c)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- if $A \subset B$ then $P(A) \leq P(B)$
- $P\left(A \cup B\right) = 1 - P(A^c \cap B^c)$
- $P(A \cap B^c) = P(A) - P(A \cap B)$
- $P(\cup_{i=1}^n E_i) \leq \sum_{i=1}^n P(E_i)$
- $P(\cup_{i=1}^n E_i) \geq \max_i P(E_i)$

Example

The National Sleep Foundation (www.sleepfoundation.org) reports that around 3% of the American population has sleep apnea. They also report that around 10% of the North American and European population has restless leg syndrome. Does this imply that 13% of people will have at least one sleep problems of these sorts?

Example continued

Answer: No, the events are not mutually exclusive. To elaborate let:

\begin{eqnarray*} A_1 & = & \{\mbox{Person has sleep apnea}\} \\ A_2 & = & \{\mbox{Person has RLS}\} \end{eqnarray*}

Then

\begin{eqnarray*} P(A_1 \cup A_2) & = & P(A_1) + P(A_2) - P(A_1 \cap A_2) \\ & = & 0.13 - \mbox{Probability of having both} \end{eqnarray*} Likely, some fraction of the population has both.

Random variables

- A **random variable** is a numerical outcome of an experiment.
- The random variables that we study will come in two varieties, **discrete** or **continuous**.
- Discrete random variable are random variables that take on only a countable number of possibilities.
  - P(X = k)
- Continuous random variable can take any value on the real line or some subset of the real line.
  - P(X \in A)

Examples of variables that can be thought of as random variables

- The (0-1) outcome of the flip of a coin
- The outcome from the roll of a die
- The BMI of a subject four years after a baseline measurement
- The hypertension status of a subject randomly drawn from a population

PMF

A probability mass function evaluated at a value corresponds to the probability that a random variable takes that value. To be a valid pmf a function, p, must satisfy

1. $p(x) \geq 0$ for all x
2. $\sum_{x} p(x) = 1$

The sum is taken over all of the possible values for x.

Example

Let X be the result of a coin flip where $X=0$ represents tails and $X = 1$ represents heads. $p(x) = (1/2)^{x} (1/2)^{1-x} ~~\mbox{ for }~~x = 0,1$ Suppose that we do not know whether or not the coin is fair; Let θ be the probability of a head expressed as a proportion (between 0 and 1). $p(x) = \theta^{x} (1 - \theta)^{1-x} ~~\mbox{ for }~~x = 0,1$

PDF

A probability density function (pdf), is a function associated with a continuous random variable

Areas under pdfs correspond to probabilities for that random variable

To be a valid pdf, a function f must satisfy

1. $f(x) \geq 0$ for all x
2. The area under $f(x)$ is one.

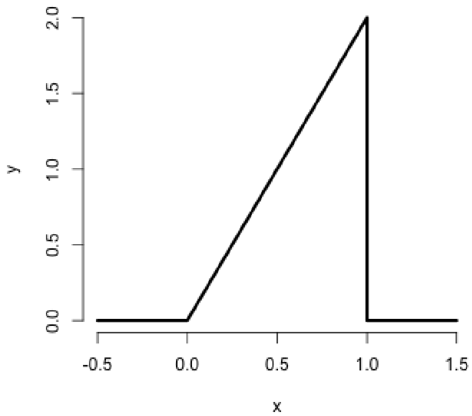
Example

Suppose that the proportion of help calls that get addressed in a random day by a help line is given by

$$f(x) = \left\{\begin{array}{ll} 2 x & \mbox{ for } 1 > x > 0 \\ 0 & \mbox{ otherwise} \end{array} \right.$$

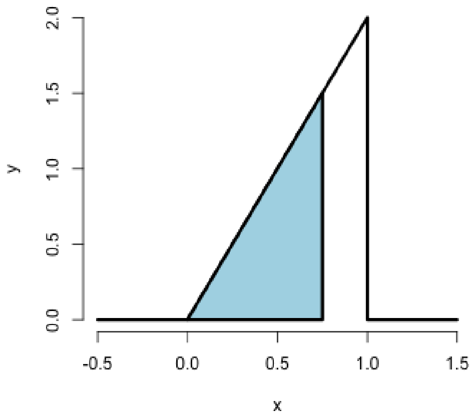
Is this a mathematically valid density?

```r
x <- c(-0.5, 0, 1, 1, 1.5)
y <- c(0, 0, 2, 0, 0)
plot(x, y, lwd = 3, frame = FALSE, type = "l")
```



Example continued

What is the probability that 75% or fewer of calls get addressed?



```r
1.5 * 0.75/2
```

```
## [1] 0.5625
```

```r
pbeta(0.75, 2, 1)
```

```
## [1] 0.5625
```

CDF and survival function

- The **cumulative distribution function** (CDF) of a random variable X is defined as the function $F(x) = P(X \leq x)$
- This definition applies regardless of whether X is discrete or continuous.
- The **survival function** of a random variable X is defined as $S(x) = P(X > x)$
- Notice that $S(x) = 1 - F(x)$
- For continuous random variables, the PDF is the derivative of the CDF

Example

What are the survival function and CDF from the density considered before?

For $1 \geq x \geq 0$ $F(x) = P(X \leq x) = \frac{1}{2} Base \times Height = \frac{1}{2} (x) \times (2 x) = x^2$

$S(x) = 1 - x^2$

```
pbeta(c(0.4, 0.5, 0.6), 2, 1)
```

```
## [1] 0.16 0.25 0.36
```

Quantiles

- The α^{th} **quantile** of a distribution with distribution function F is the point x_α so that $F(x_\alpha) = \alpha$
- A **percentile** is simply a quantile with α expressed as a percent
- The **median** is the 50^{th} percentile

Example

- We want to solve 0.5 = F(x) = x^2
- Resulting in the solution

```
sqrt(0.5)
```

```
## [1] 0.7071
```

- Therefore, about 0.7071 of calls being answered on a random day is the median.
- R can approximate quantiles for you for common distributions

```
qbeta(0.5, 2, 1)
```

```
## [1] 0.7071
```

Summary

- You might be wondering at this point "I've heard of a median before, it didn't require integration. Where's the data?"
- We're referring to are **population quantities**. Therefore, the median being discussed is the **population median**.
- A probability model connects the data to the population using assumptions.
- Therefore the median we're discussing is the **estimand**, the sample median will be the **estimator**