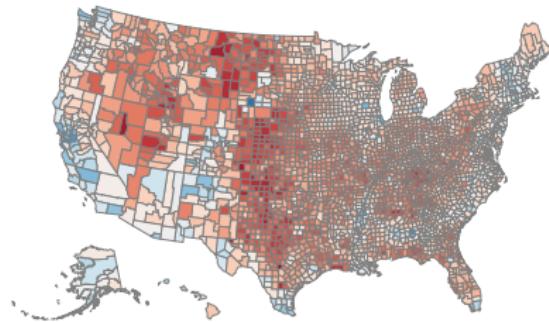


MODELING AND INFERRING ATTRIBUTED GRAPHS

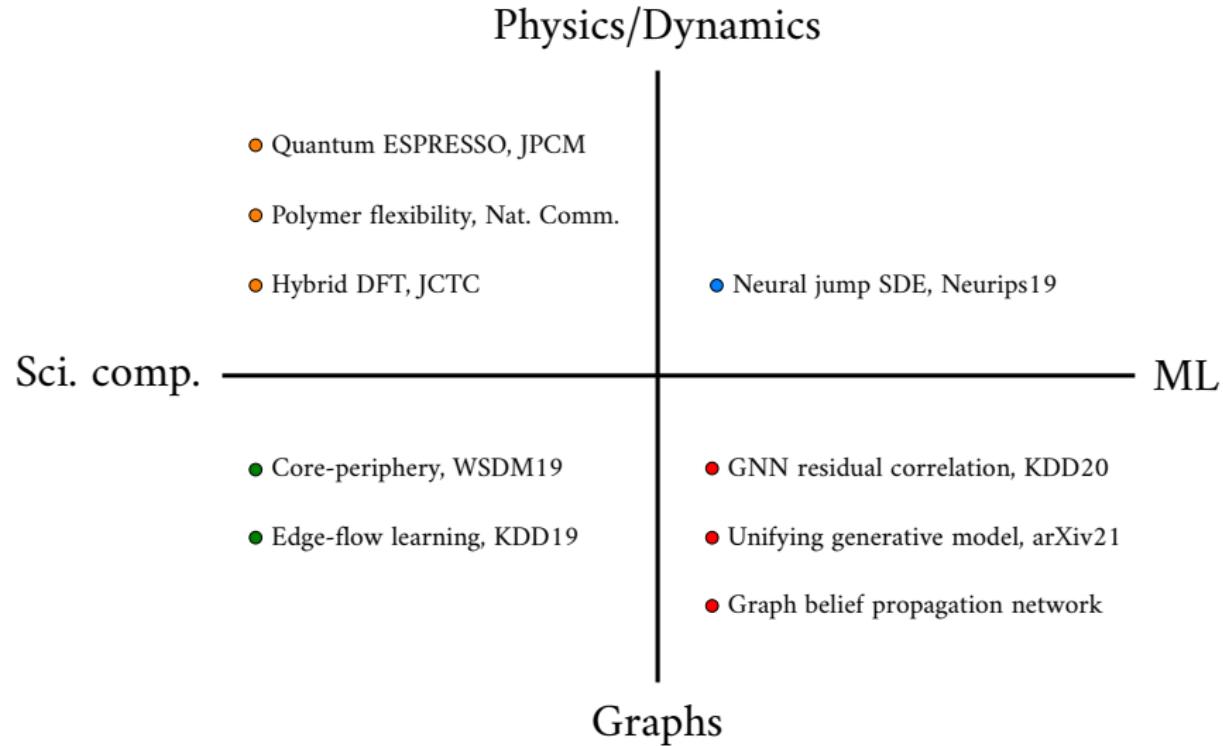


Junteng Jia

Department of Computer Science, Cornell University, Ithaca, NY 14853

Ph.D. THESIS DEFENSE – APRIL 12, 2021

RESEARCH OVERVIEW



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Physics/Dynamics

Sci. comp.

ML

Graphs



- Quantum ESPRESSO, JPCM
- Polymer flexibility, Nat. Comm.
- Hybrid DFT, JCTC
- Neural jump SDE, Neurips19
- Core-periphery, WSDM19
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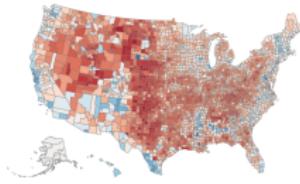
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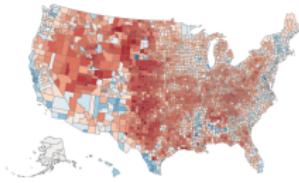
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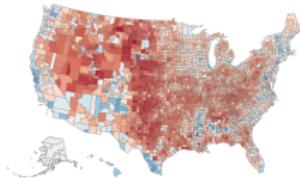
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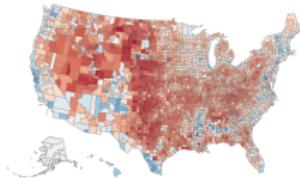
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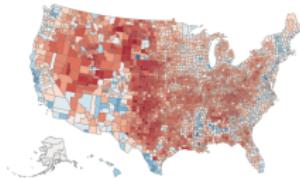
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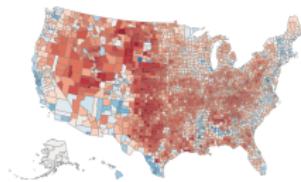
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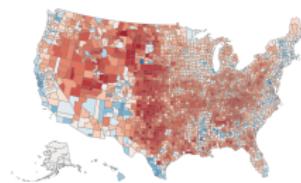
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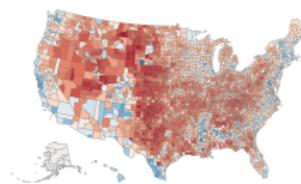
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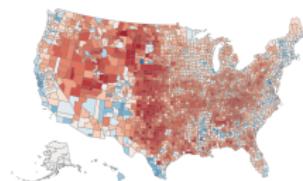
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- however, research often driven by improving on benchmark datasets
 - classification in citation/coauthor network [Lu-Getoor 03; Kipf-Welling 17; Shchur+ 18]

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- First, we develop theory and algorithms on **regression** problems
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- Later, we extend to the **classification** setting
Labels are discrete-valued like cat/dog, male/female ...

NODE LABEL PREDICTION IN ATTRIBUTED GRAPHS

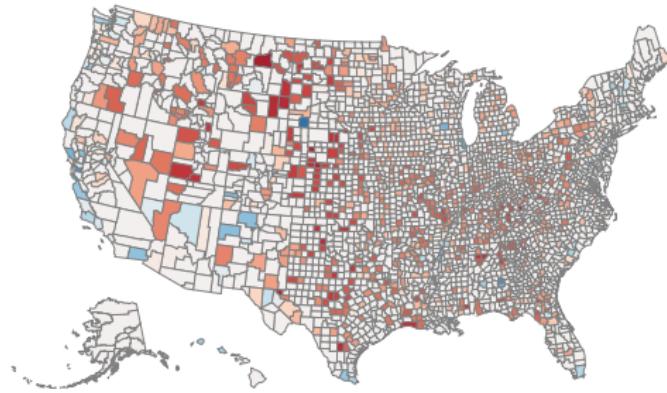
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- Each county as a node; bordering counties are connected

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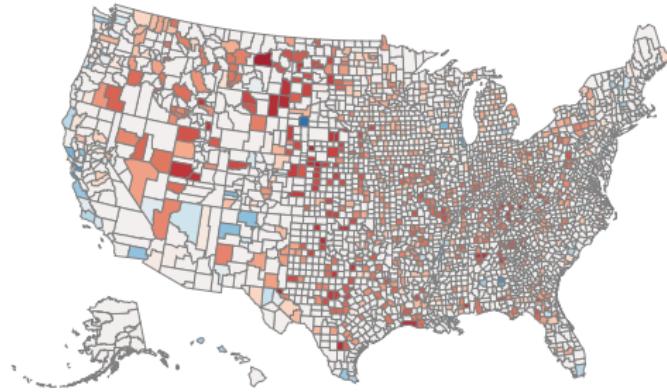
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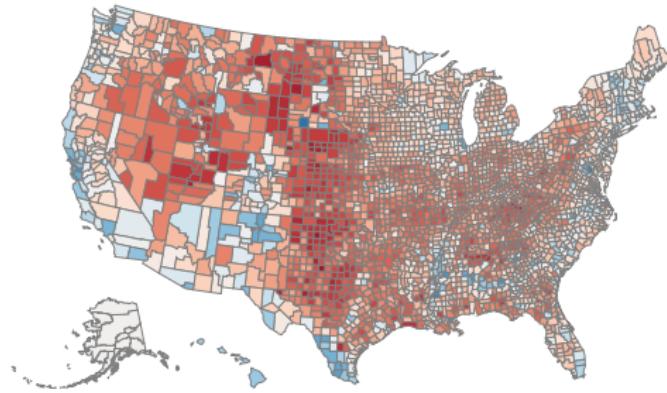
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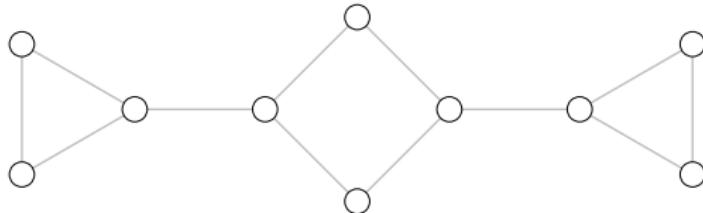
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 - estimate political leaning for other counties

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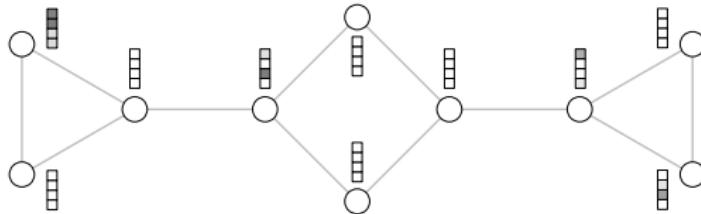
Problem statement



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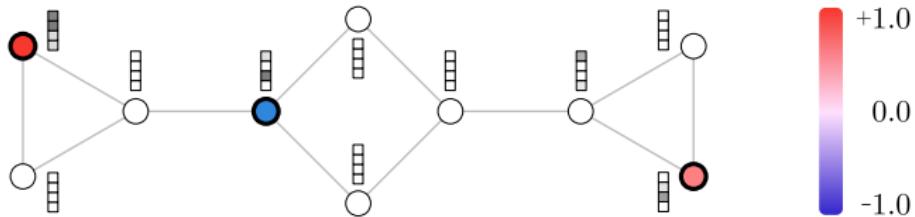
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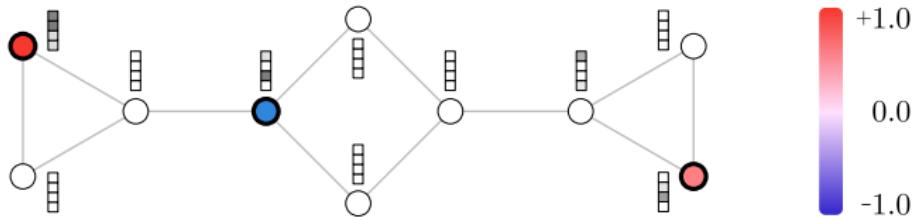
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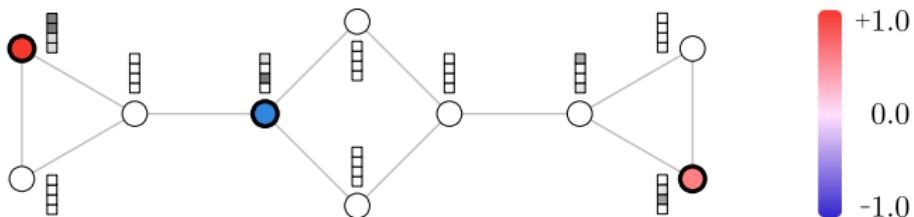
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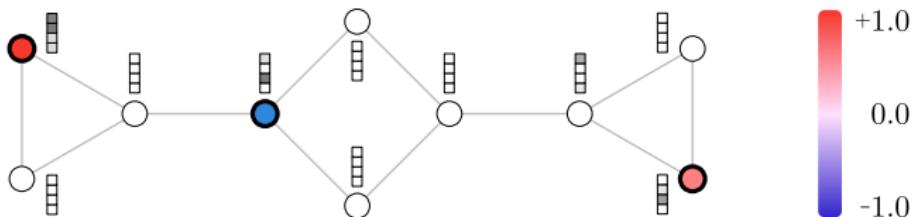
THERE ARE TWO BROAD CLASSES OF METHODS

Label propagation is a transductive method [early 2000s]



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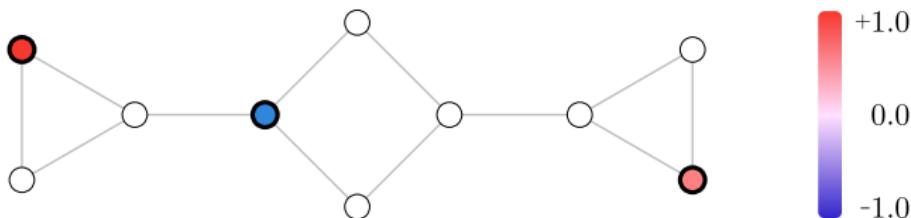
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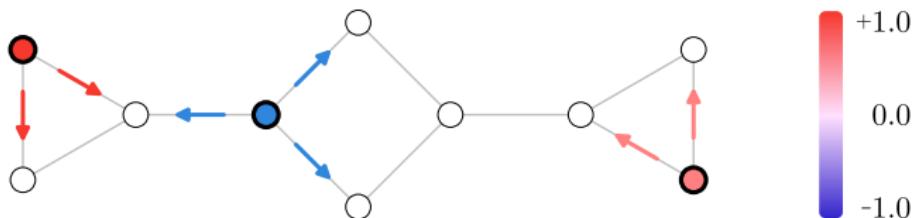
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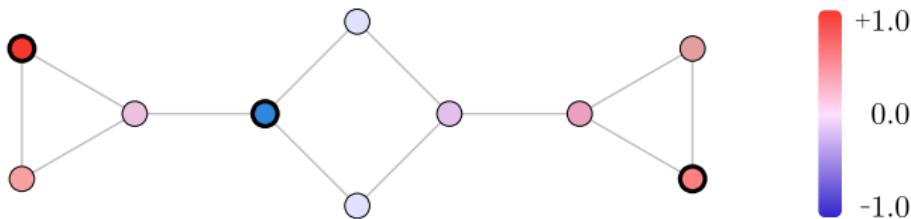


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[Zhu+ 03; Zhou+ 04; Wang-Zhang 06]

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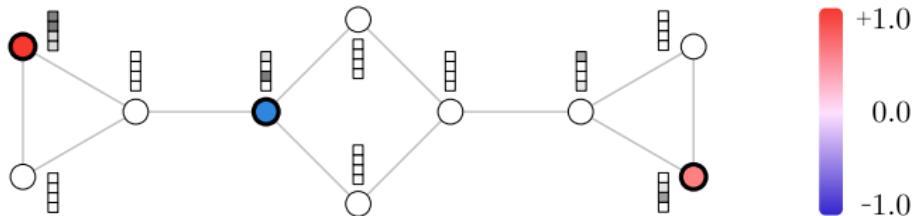
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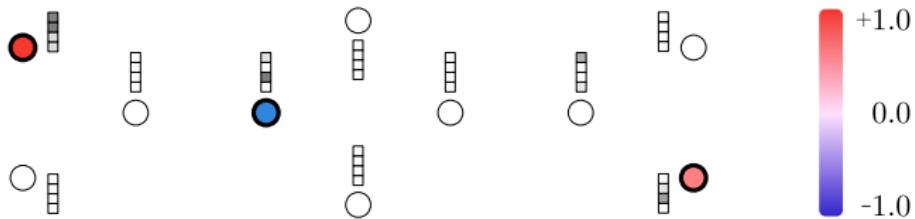
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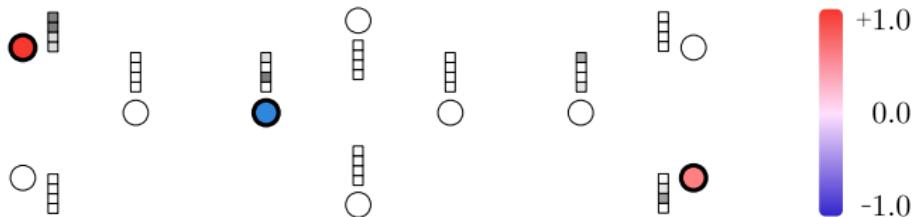
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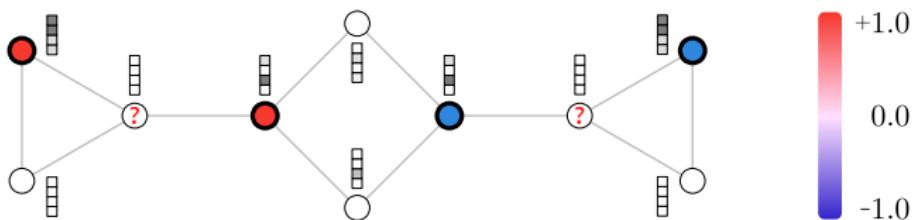
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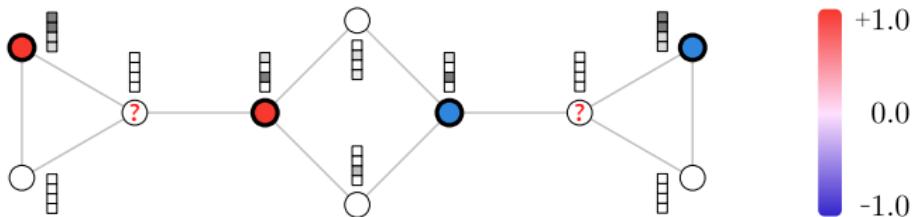
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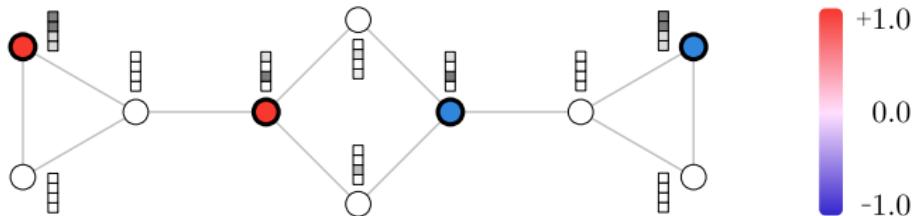
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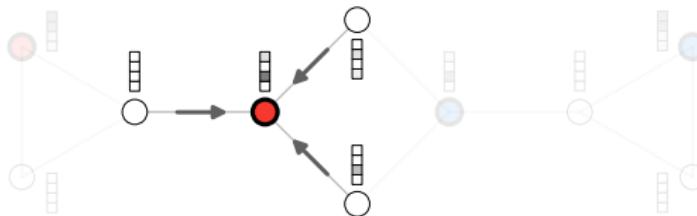
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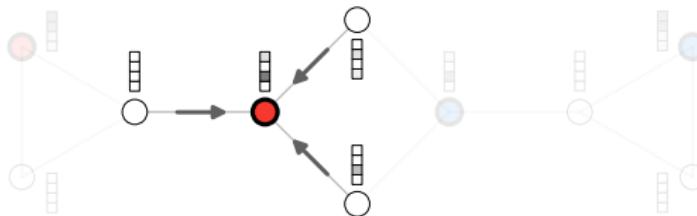


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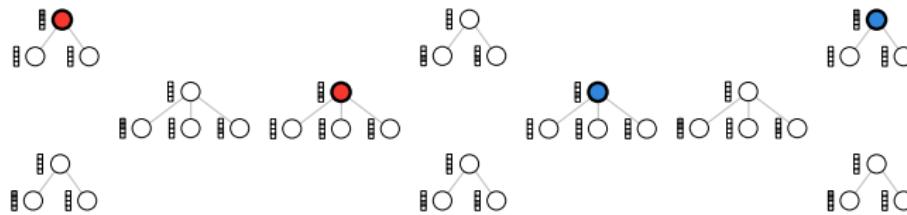
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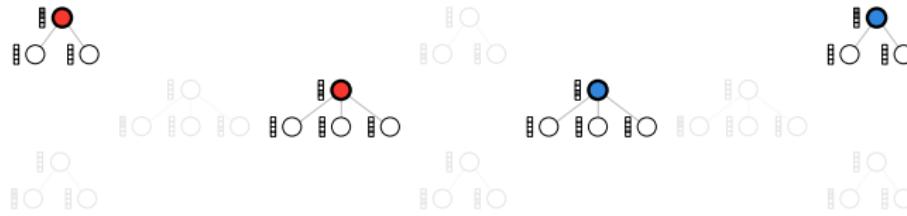
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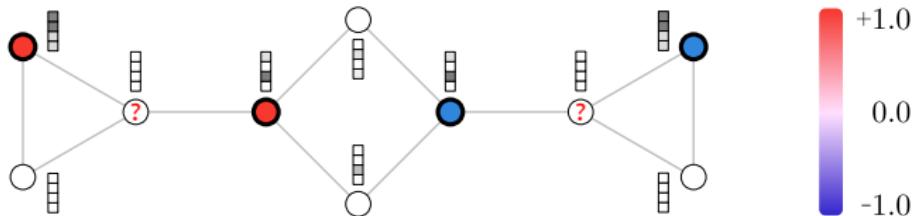
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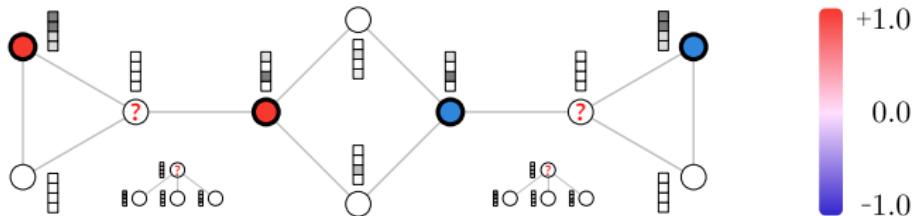
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[Kipf-Welling 16; Hamilton+ 17; Zhou+ 18; Velickovic+ 18; Xu+ 19; ~10,000 papers in 5 years]

COMBINING GNN WITH LABEL PROPAGATION

	ego features	neighboring features	neighboring labels
Label Propagation			●
OLS, MLP	●		
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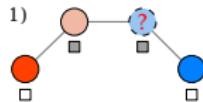
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conceptually similar to boosting; the two “weak predictors” uses different inputs

COMBING GNN WITH LABEL PROPAGATION

Ground Truth Labels $\{y_u\}_{u \in V}$



- decompose outcomes as **learnable from features** and **correlated residuals**

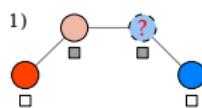
$$y_u = f(\{\mathbf{x}_v\}_{v \in N_K(u)}) + r_u$$

conceptually similar to boosting; the two “weak predictors” uses different inputs

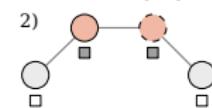
- sequentially deploy **GNN** and **label propagation**

COMBINING GNN WITH LABEL PROPAGATION

Ground Truth Labels $\{y_u\}_{u \in V}$



Predictions $\{\hat{y}_u\}_{u \in V}$



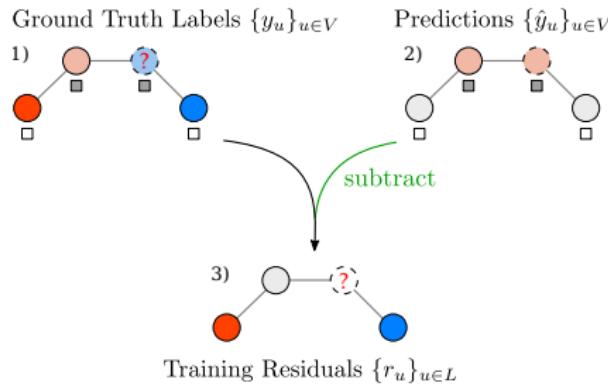
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 - train a GNN inductively;

COMBINING GNN WITH LABEL PROPAGATION



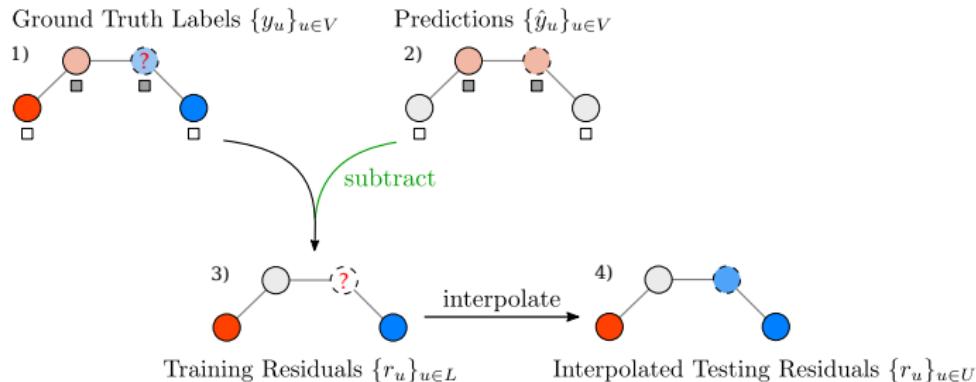
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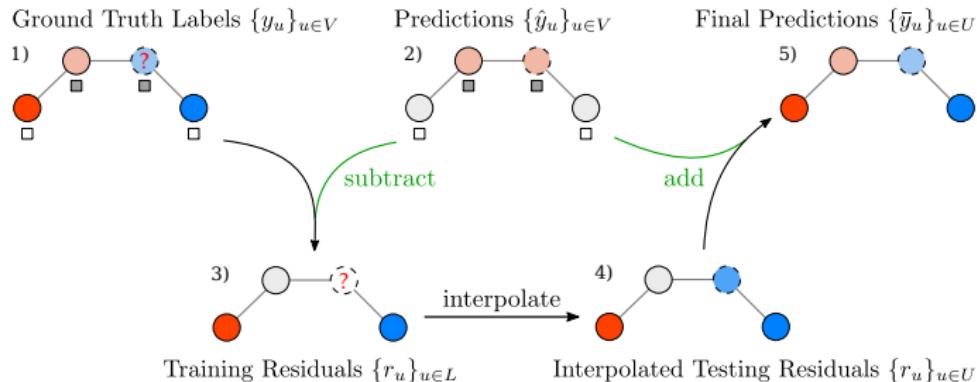
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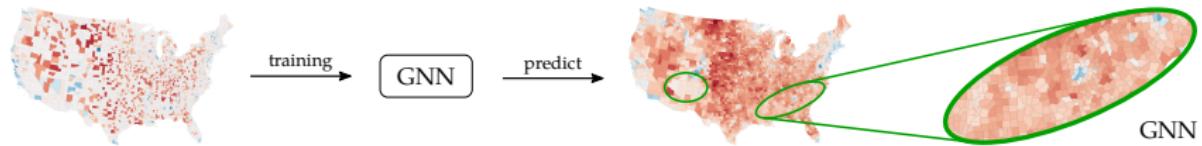
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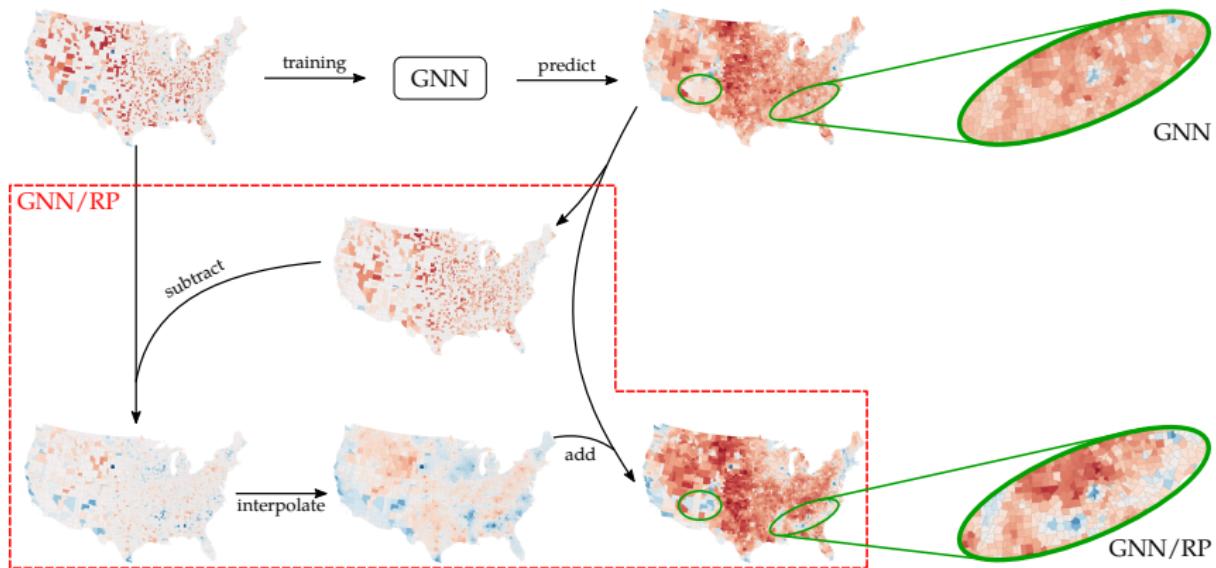
conceptually similar to boosting; the two “weak predictors” uses different inputs

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 - final prediction = inductive prediction + estimated residual

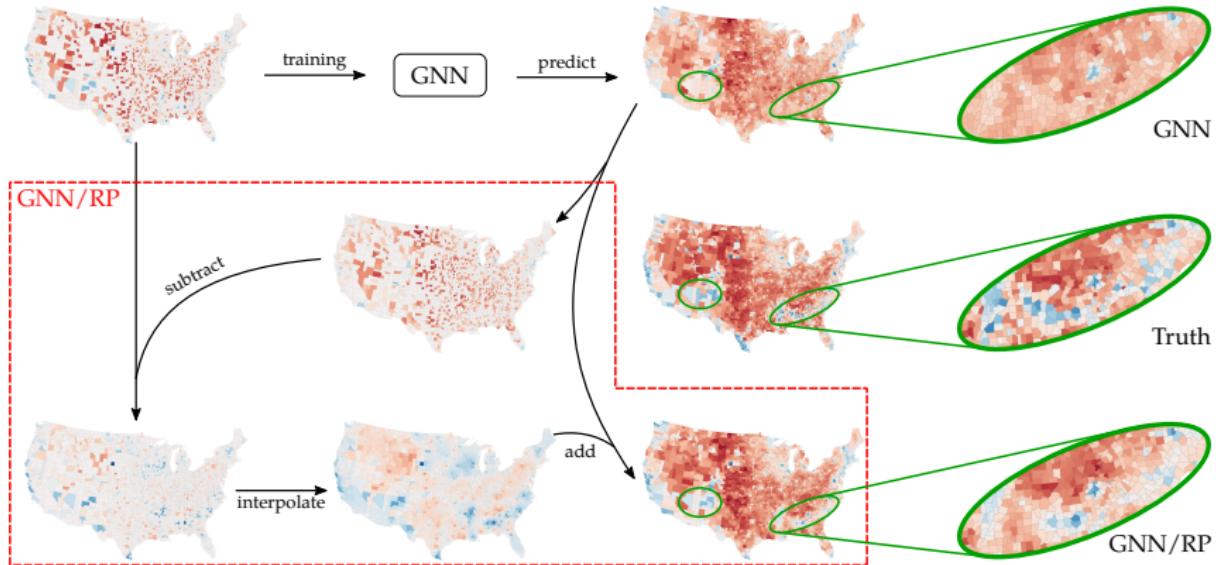
GNN/RP WORKS SUPER WELL IN PRACTICE



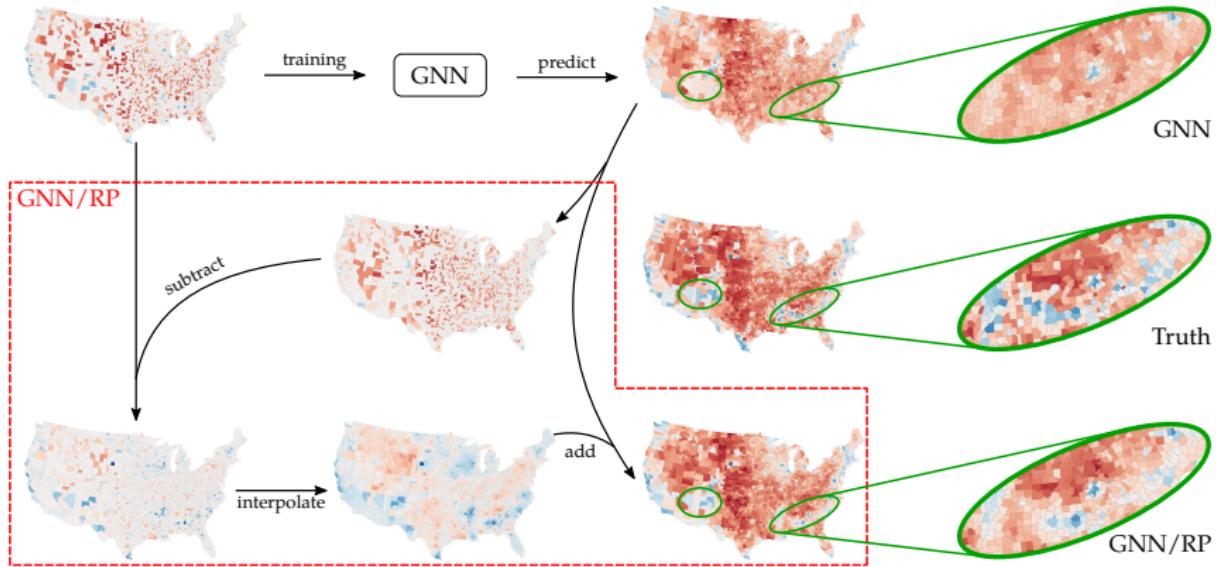
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- coefficient of determination (R^2) increases from 0.51 to 0.69 on test set

RP ASSUMES LINEARLY CORRELATED RESIDUALS

	ego features	neighboring features	neighboring labels
Label Propagation			●
OLS, MLP	●		
GNN	●	●	
GNN/RP	●	●	●

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- for regression, the typically training objective is the sum-of-square loss

$$\min_{\theta} \sum_{u \in L} (y_u - \hat{y}_u)^2, \quad \hat{y}_u = f(\mathbf{x}_u, \{\mathbf{x}_v : v \in N_K(u)\}; \theta)$$

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STATISTICAL MODEL FOR NODE ATTRIBUTES

**Is there a common statistical framework
that formalizes all three correlations?**

STATISTICAL MODEL FOR NODE ATTRIBUTES

- Problem input:
 - graph topology $G(V, E)$
 - features on all vertices, $\mathbf{X} = [\mathbf{x}_u]_{u \in V}$
 - labels on a subset of vertices $L \subset V$, training labels $\mathbf{y}_L = [y_u]_{u \in L}$
- Problem output:
 - labels on the rest of vertices $U \equiv V \setminus L$, testing labels $\mathbf{y}_U = [y_u]_{u \in U}$

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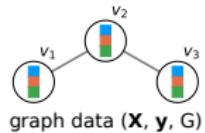
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 - different observations gives different algorithms

STATISTICAL MODEL FOR NODE ATTRIBUTES

Data Type

Corresponding Gaussian MRF

Learning Algorithm



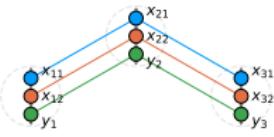
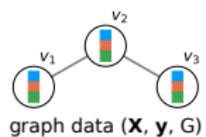
graph data (\mathbf{X} , \mathbf{y} , G)

STATISTICAL MODEL FOR NODE ATTRIBUTES

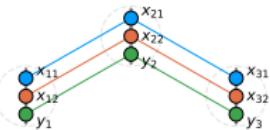
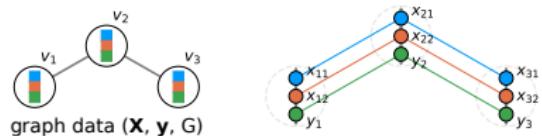
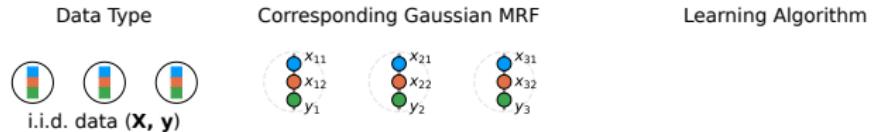
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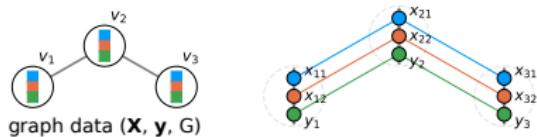


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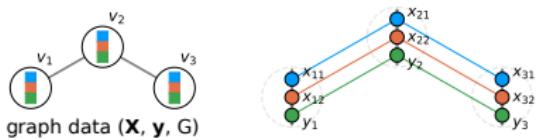
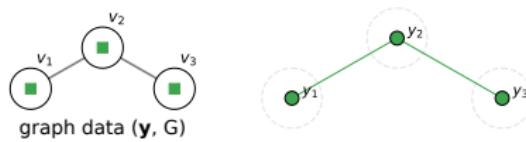
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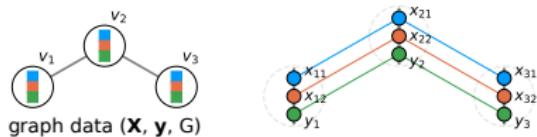
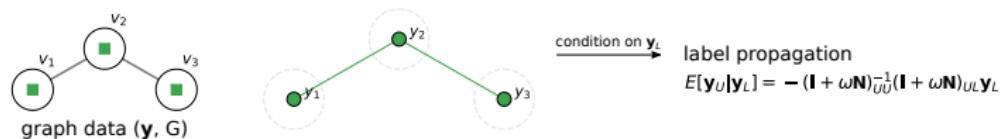
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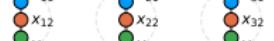
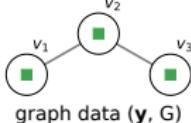
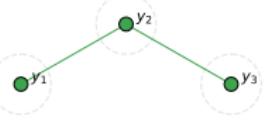
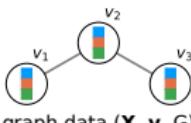
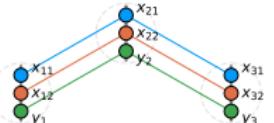


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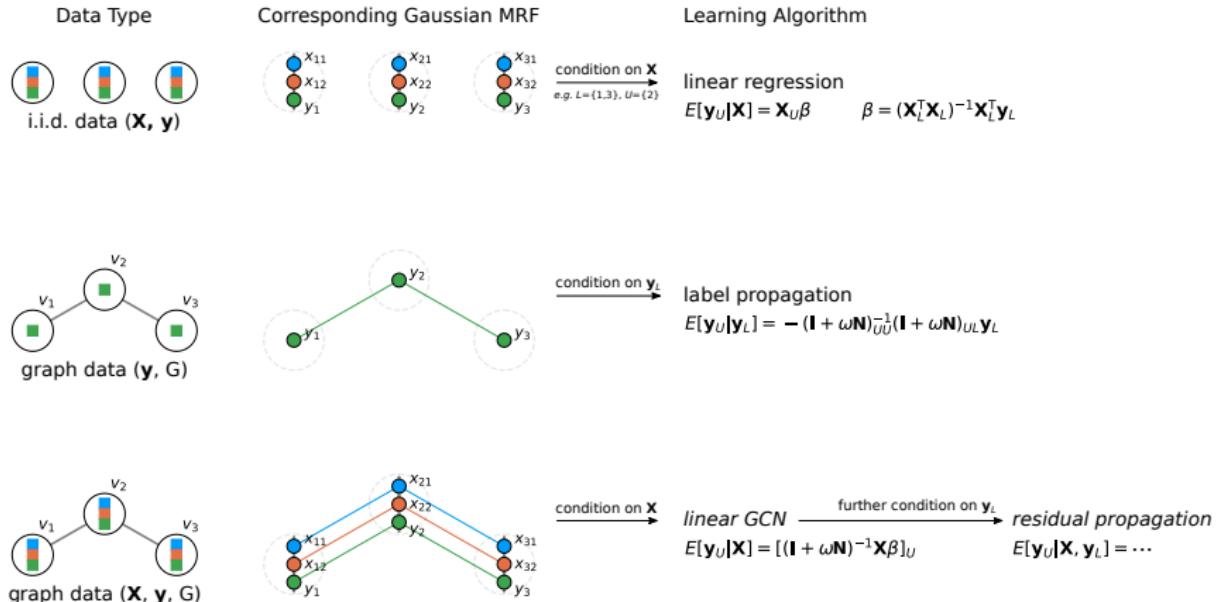
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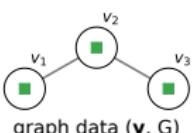
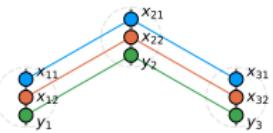
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 graph data (\mathbf{y}, G)		$\xrightarrow{\text{condition on } \mathbf{y}_L}$ label propagation $E[\mathbf{y}_U \mathbf{y}_L] = -(\mathbf{I} + \omega \mathbf{N})_{UU}^{-1} (\mathbf{I} + \omega \mathbf{N})_{UL} \mathbf{y}_L$
 graph data ($\mathbf{X}, \mathbf{y}, G$)		$\xrightarrow{\text{condition on } \mathbf{X}}$ linear GCN $E[\mathbf{y}_U \mathbf{X}] = [(\mathbf{I} + \omega \mathbf{N})^{-1} \mathbf{X} \boldsymbol{\beta}]_U$

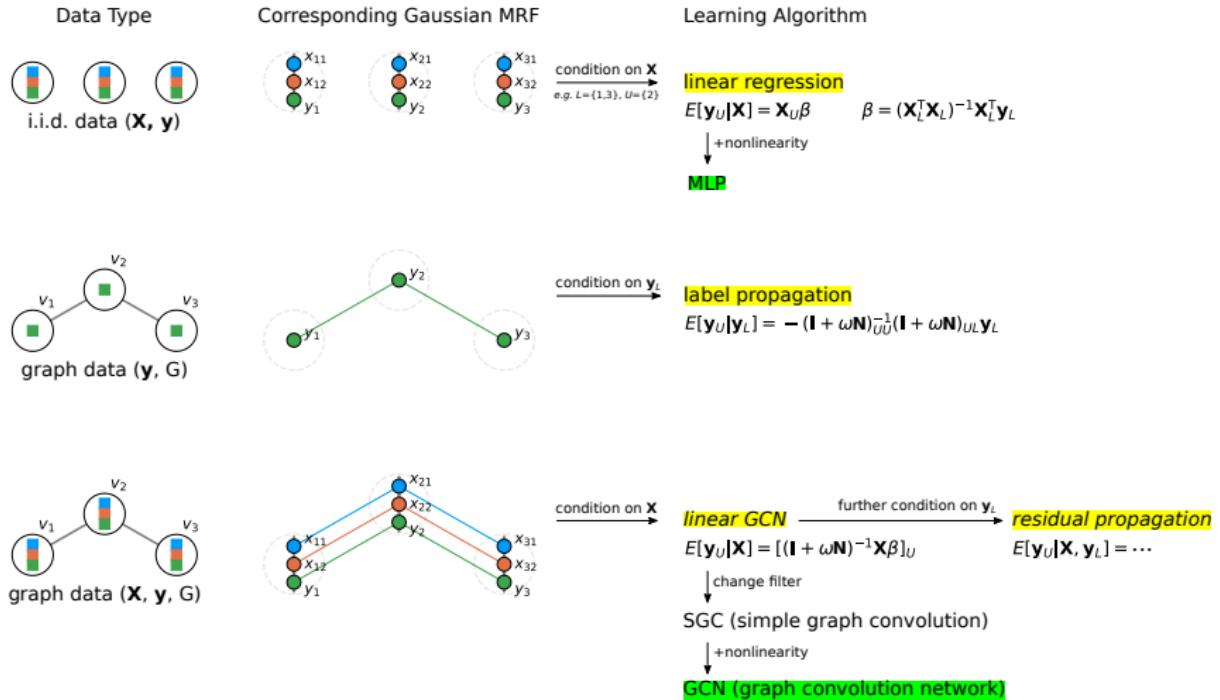
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graph data $(\mathbf{X}, \mathbf{y}, G)$		$\xrightarrow{\text{condition on } \mathbf{X}}$ linear GCN $\xrightarrow{\text{further condition on } \mathbf{y}_L}$ residual propagation $E[\mathbf{y}_U \mathbf{X}] = [(\mathbf{I} + \omega \mathbf{N})^{-1} \mathbf{X} \boldsymbol{\beta}]_U$ $E[\mathbf{y}_U \mathbf{X}, \mathbf{y}_L] = \dots$

STATISTICAL MODEL FOR NODE ATTRIBUTES



STATISTICAL MODEL FOR NODE ATTRIBUTES

The joint distribution of all node attributes is a Gaussian MRF

- Real-valued attributes $\mathbf{a}_u = [\mathbf{x}_u; y_u] \in \mathbb{R}^{p+1}$ on each node u .
- $A_i = i$ th attribute over all nodes.
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This is a well-known result for standard linear models

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β is learned directly with OLS in a **discriminative** manner on the training nodes

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In practice, ω or α can be tuned with cross-validation on the training set.

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Computes expectation of y in the full model conditioned only on features

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- The joint distribution of all attributes is given by:

$$\text{vec}(\mathbf{A}) \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Gamma}^{-1}), \quad \boldsymbol{\Gamma} = \mathbf{H} \otimes \mathbf{I}_n + \text{diag}(\mathbf{h}) \otimes \mathbf{N}$$

- Predict \mathbf{y} with its expectation conditioned on the **features**:

$$\begin{aligned}\bar{\mathbf{y}} &= E[\mathbf{y} | \mathbf{X} = \mathbf{X}] = (H_{p+1,p+1} \mathbf{I}_n + h_{p+1} \mathbf{N})^{-1} \left(-\mathbf{H}_{1:p,p+1}^\top \otimes \mathbf{I}_n \right) \text{vec}(\mathbf{X}) \\ &= \underbrace{(\mathbf{I}_n + \omega \mathbf{N})^{-1} \mathbf{X} \beta}_{\text{LP on features!}}\end{aligned}$$

$(\mathbf{I}_n + \omega \mathbf{N})^{-1} \mathbf{X}$ is the solution to label propagation for the features:

$$\mathbf{X}^{(k)} = \alpha \cdot \mathbf{S} \mathbf{X}^{(k-1)} + (1 - \alpha) \cdot \mathbf{X}$$

- Linear graph convolution (LGC)

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- Linear graph convolution (LGC)

- use label propagation to smooth/preprocess features
- just like linear regression, learn β with OLS on the training nodes

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$$\begin{array}{ll} \text{LGC} & (1 - \alpha)(\mathbf{I} + \alpha\mathbf{S} + \alpha^2\mathbf{S}^2 + \dots)\mathbf{X}\beta \quad \mathbf{S} = \mathbf{D}^{-1/2}\mathbf{W}\mathbf{D}^{-1/2} \\ [\text{Jia-Benson } 21] & \end{array}$$

$$\begin{array}{ll} \text{SGC} & \tilde{\mathbf{S}}^K \mathbf{X}\beta \quad \tilde{\mathbf{S}} = (\mathbf{D} + \mathbf{I})^{-1/2}(\mathbf{W} + \mathbf{I})(\mathbf{D} + \mathbf{I})^{-1/2} \\ [\text{Wu+ } 19] & \end{array}$$

$$\begin{array}{ll} \text{GCN} & \sigma(\tilde{\mathbf{S}} \dots \sigma(\tilde{\mathbf{S}} \mathbf{X} \Theta^{(1)}) \dots \Theta^{(K)})\beta \\ [\text{Kipf-Welling } 17] & \end{array}$$

- connection to the graph convolutional network:
 - LGC sums decaying contributions from features on increasingly distant neighbors

Neumann expansion, $(\mathbf{I} + \omega\mathbf{N})^{-1} = \left(1 - \frac{\omega}{1+\omega}\right) \left(\mathbf{I} - \frac{\omega}{1+\omega}\mathbf{S}\right)^{-1} = (1 - \alpha)(\mathbf{I} + \alpha\mathbf{S} + \alpha^2\mathbf{S}^2 + \dots)$

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- LGC computes $E[\mathbf{y}_U | \mathbf{X}]$, **why not conditioning on \mathbf{y}_L as well?**

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CASE 4. RESIDUAL PROPAGATION

- The conditional distribution of all labels conditioned on features:

$$\mathbf{y} | \mathbf{X} = \mathbf{X} \sim \mathcal{N}(\bar{\mathbf{y}}, \boldsymbol{\Gamma}_{PP}^{-1}), \quad \boldsymbol{\Gamma}_{PP} = H_{p+1,p+1} \mathbf{I}_n + h_{p+1} \mathbf{N}$$

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- Linear graph convolution with residual propagation (LGC/RP)

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- Linear graph convolution with residual propagation (LGC/RP)
 - train LGC to predict both training nodes and testing nodes
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- Residual propagation post-processing is compatible with SGC, GCN, ...

THE DERIVED ALGORITHMS PERFORM SUPER WELL

Dataset	Outcome	LP	LR	LGC (α)	SGC (K)	GCN (K)	LGC/RP	SGC/RP	GCN/RP
U.S.	income	0.40	0.63	0.66 (0.46)	0.51 (1.0)	0.53 (1.3)	0.69	0.55	0.55
	education	0.31	0.71	0.71 (0.00)	0.43 (1.0)	0.47 (1.0)	0.71	0.46	0.48
	unemployment	0.47	0.34	0.39 (0.59)	0.32 (1.3)	0.45 (2.5)	0.54	0.52	0.53
	election	0.52	0.42	0.49 (0.68)	0.43 (1.1)	0.52 (2.1)	0.64	0.61	0.61
CDC	airT	0.95	0.85	0.86 (0.78)	0.86 (2.6)	0.95 (3.0)	0.96	0.97	0.97
	landT	0.89	0.81	0.81 (0.09)	0.79 (1.0)	0.91 (2.4)	0.90	0.93	0.93
	precipitation	0.89	0.59	0.61 (0.93)	0.61 (2.3)	0.79 (3.0)	0.89	0.90	0.90
	sunlight	0.96	0.75	0.81 (0.97)	0.80 (3.0)	0.90 (3.0)	0.96	0.97	0.97
	pm2.5	0.96	0.21	0.27 (0.99)	0.23 (2.7)	0.78 (3.0)	0.96	0.96	0.97
London	income	0.46	0.85	0.85 (0.00)	0.64 (1.0)	0.63 (1.0)	0.85	0.65	0.64
	education	0.65	0.81	0.83 (0.40)	0.74 (1.6)	0.79 (1.4)	0.86	0.77	0.79
	age	0.65	0.73	0.73 (0.17)	0.66 (1.2)	0.70 (1.7)	0.75	0.72	0.72
	election	0.67	0.73	0.81 (0.74)	0.74 (2.0)	0.76 (2.1)	0.85	0.78	0.78
Twitch	days	0.08	0.58	0.59 (0.67)	0.22 (1.4)	0.26 (1.7)	0.60	0.23	0.26

30% training; hyperparameter α , K tuned with cross-validation on training nodes

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- LGC consistently outperforms SGC on all datasets (explain from graph filtering later)

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Dataset	Outcome	LP	LR	LGC (α)	SGC (K)	GCN (K)	LGC/RP	SGC/RP	GCN/RP
U.S.	income	0.40	0.63	0.66 (0.46)	0.51 (1.0)	0.53 (1.3)	0.69	0.55	0.55
	education	0.31	0.71	0.71 (0.00)	0.43 (1.0)	0.47 (1.0)	0.71	0.46	0.48
	unemployment	0.47	0.34	0.39 (0.59)	0.32 (1.3)	0.45 (2.5)	0.54	0.52	0.53
	election	0.52	0.42	0.49 (0.68)	0.43 (1.1)	0.52 (2.1)	0.64	0.61	0.61
CDC	airT	0.95	0.85	0.86 (0.78)	0.86 (2.6)	0.95 (3.0)	0.96	0.97	0.97
	landT	0.89	0.81	0.81 (0.09)	0.79 (1.0)	0.91 (2.4)	0.90	0.93	0.93
	precipitation	0.89	0.59	0.61 (0.93)	0.61 (2.3)	0.79 (3.0)	0.89	0.90	0.90
	sunlight	0.96	0.75	0.81 (0.97)	0.80 (3.0)	0.90 (3.0)	0.96	0.97	0.97
	pm2.5	0.96	0.21	0.27 (0.99)	0.23 (2.7)	0.78 (3.0)	0.96	0.96	0.97
London	income	0.46	0.85	0.85 (0.00)	0.64 (1.0)	0.63 (1.0)	0.85	0.65	0.64
	education	0.65	0.81	0.83 (0.40)	0.74 (1.6)	0.79 (1.4)	0.86	0.77	0.79
	age	0.65	0.73	0.73 (0.17)	0.66 (1.2)	0.70 (1.7)	0.75	0.72	0.72
	election	0.67	0.73	0.81 (0.74)	0.74 (2.0)	0.76 (2.1)	0.85	0.78	0.78
Twitch	days	0.08	0.58	0.59 (0.67)	0.22 (1.4)	0.26 (1.7)	0.60	0.23	0.26

- LGC consistently outperforms SGC on all datasets (explain from graph filtering later)
- Inductive/RP always outperforms LP and the inductive base predictor

30% training; hyperparameter α , K tuned with cross-validation on training nodes

OUR MODEL CONNECTS A LOT OF DOTS

	ego features	neighboring features	neighboring labels
Label Propagation			●
OLS, MLP	●		
GNN	●	●	
GNN/RP	●	●	●

- Our generative model,

OUR MODEL CONNECTS A LOT OF DOTS

	ego features	neighboring features	neighboring labels
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GNN	●	●	
GNN/RP	●	●	●

- Our generative model,
 - formalizes three types of correlations that are helpful for learning node labels

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 - unifies two separate lines of research

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 - leads to new algorithms LGC, RP that greatly outperforms baselines

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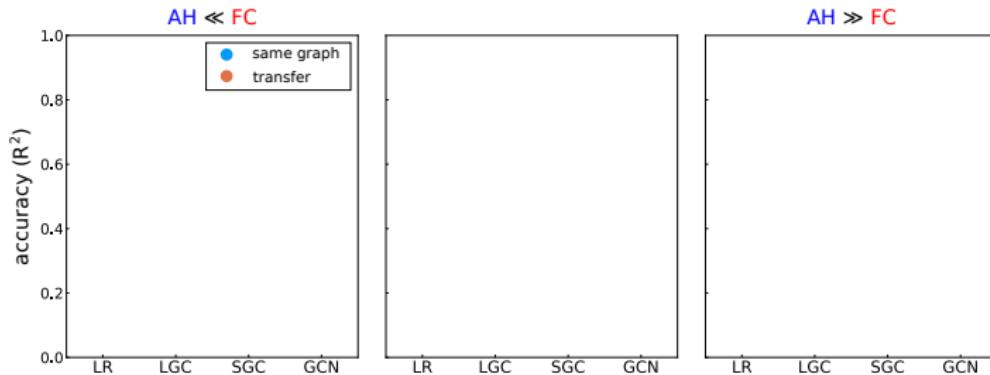
OUR MODEL CONNECTS A LOT OF DOTS

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 - we can use LGC to interpret empirical datasets

WE CAN SAMPLE DATA TO TEST ALGORITHMS

Transfer learning experiments on inductive methods

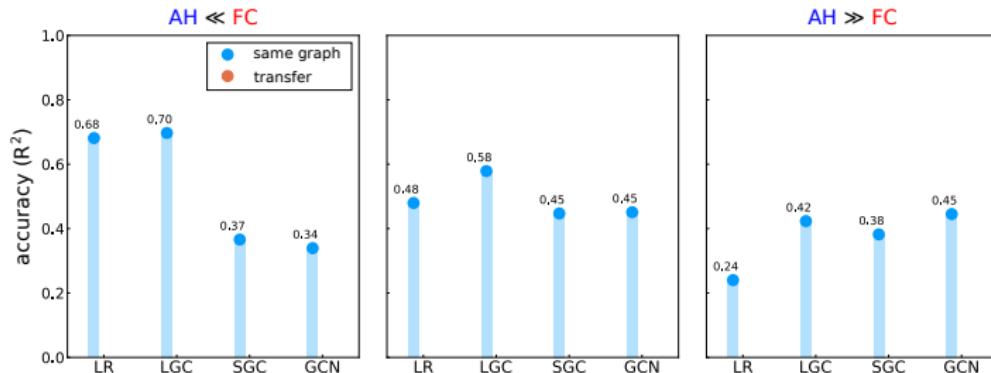


- same graph: sample one attributed graphs G_1 , train and test on different vertices

The real-world dataset we consider here is the election data from 2012/2016.

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Transfer learning experiments on inductive methods

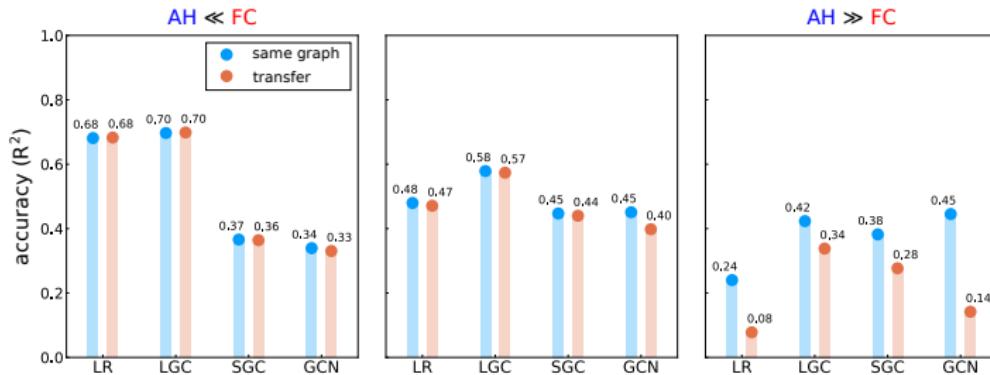


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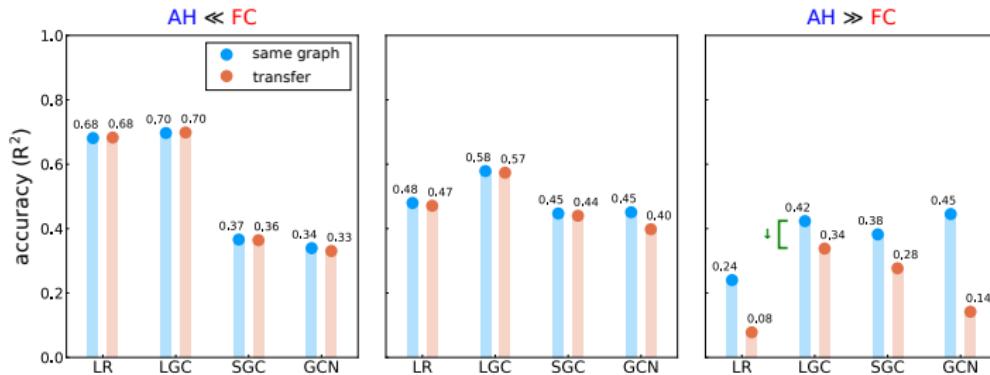


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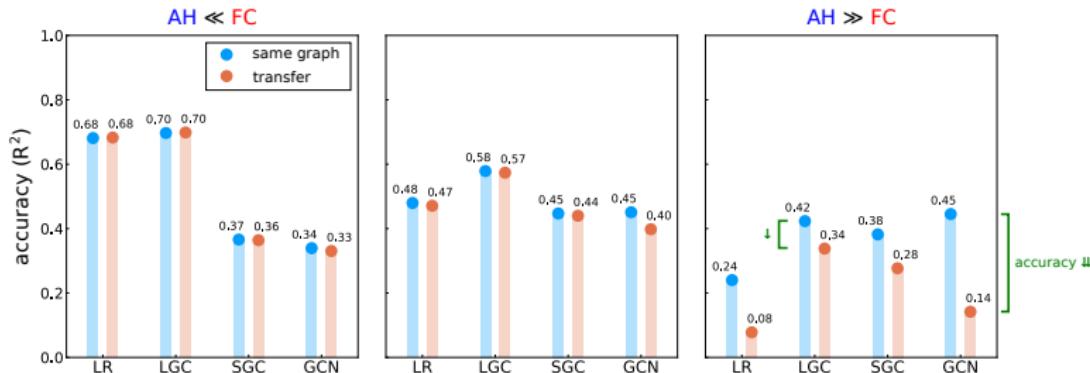


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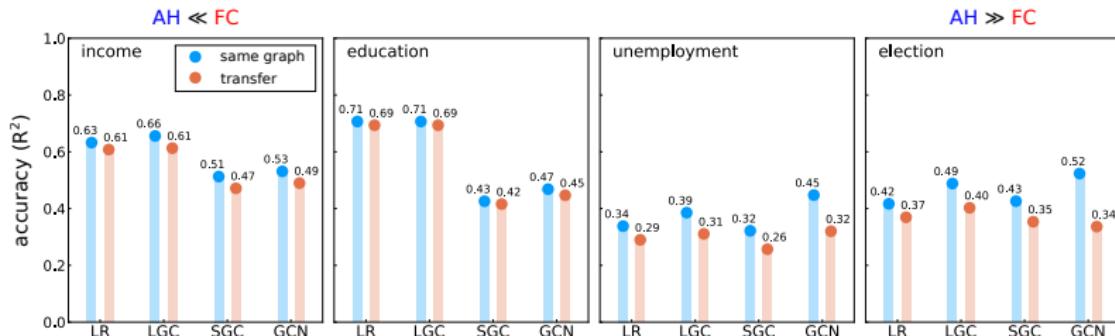


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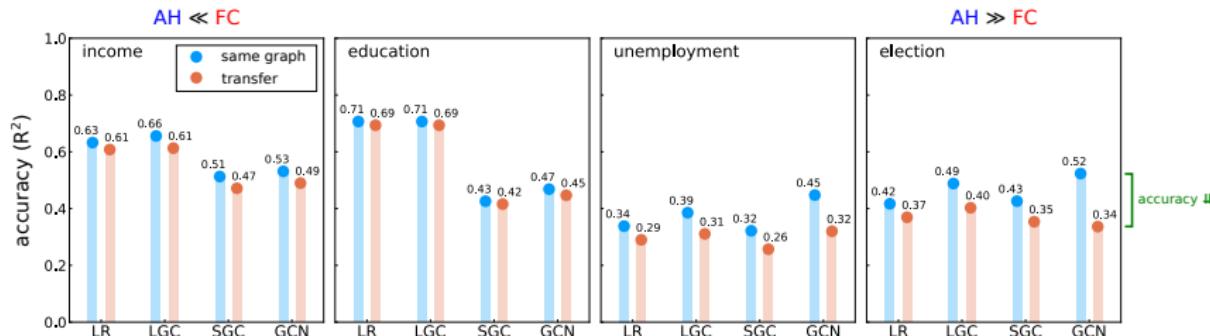


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GCN performs much worse than LGC due to memorization
on real-world datasets, we also observe similar patterns

The real-world dataset we consider here is the election data from 2012/2016.

OUR MODEL HELPS UNDERSTAND SMOOTHING



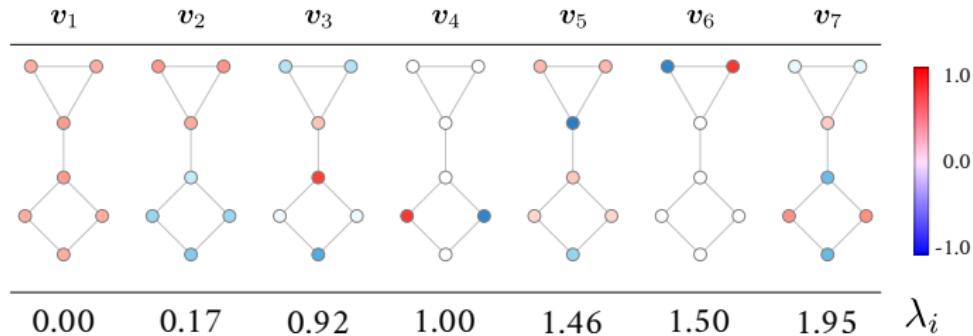
- Spectral graph theory measures smoothness with eigenvectors of $N = V\Lambda V^\top$.

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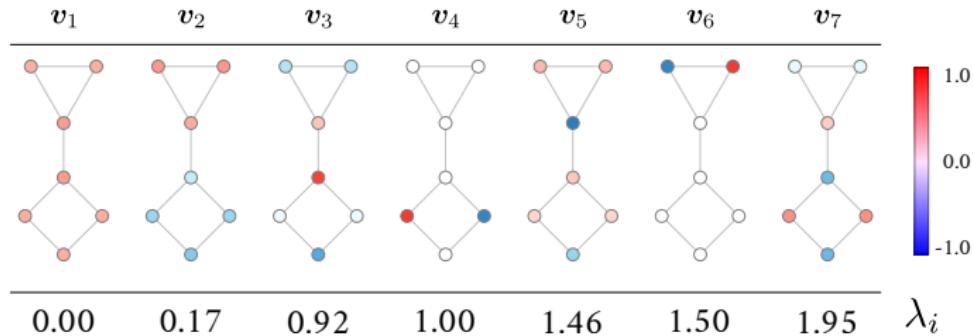
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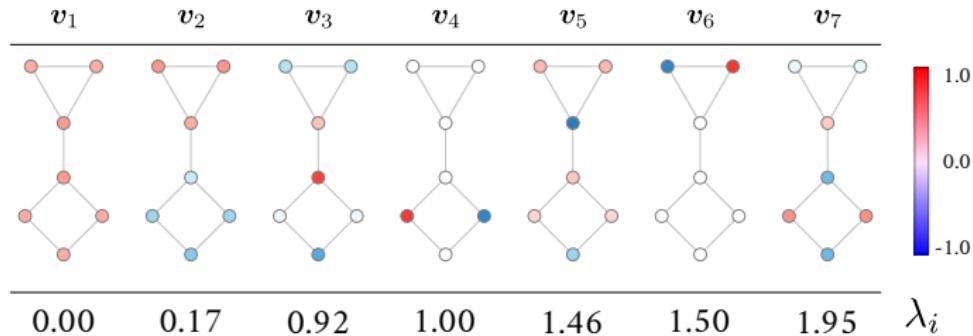
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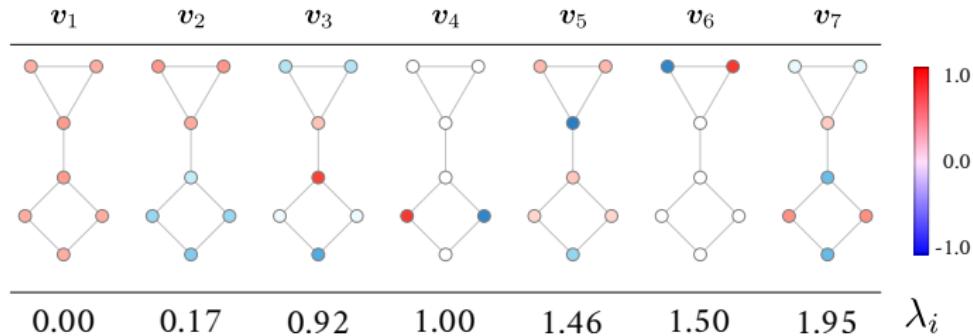
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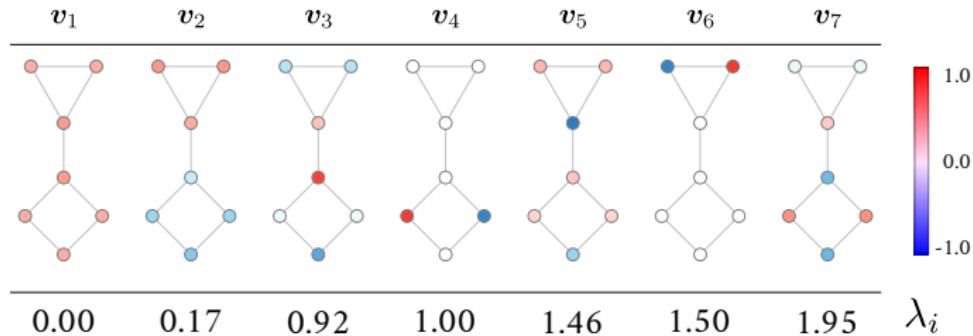
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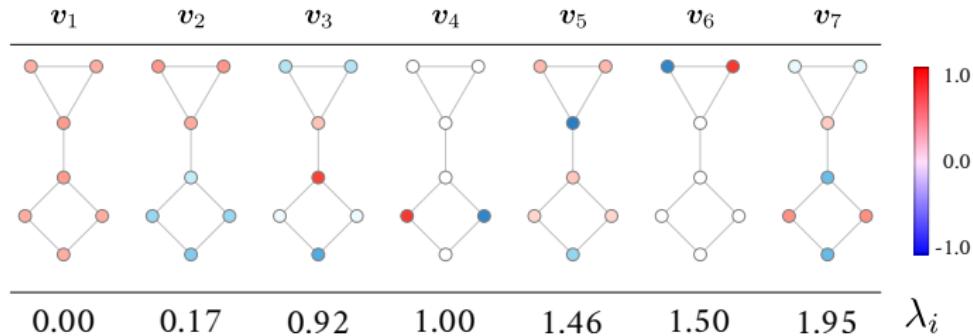
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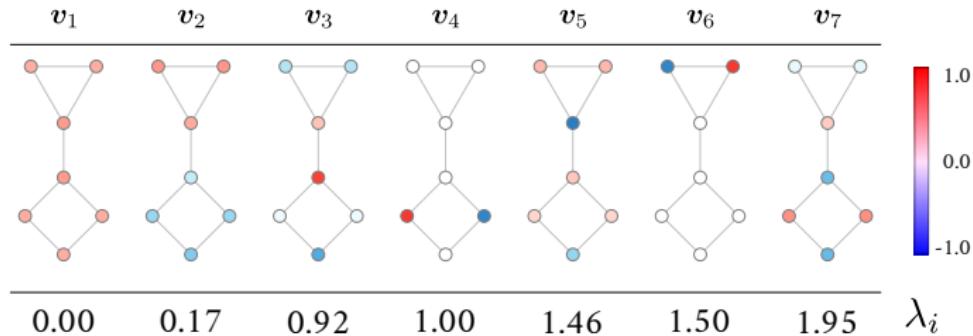


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$$\text{LGC : } (I + \omega N)^{-1} f = \sum_i c_i v_i (1 + \omega \lambda_i)^{-1}$$

[Ortega+ 2018; Li+ 2018; Li+ 2019]

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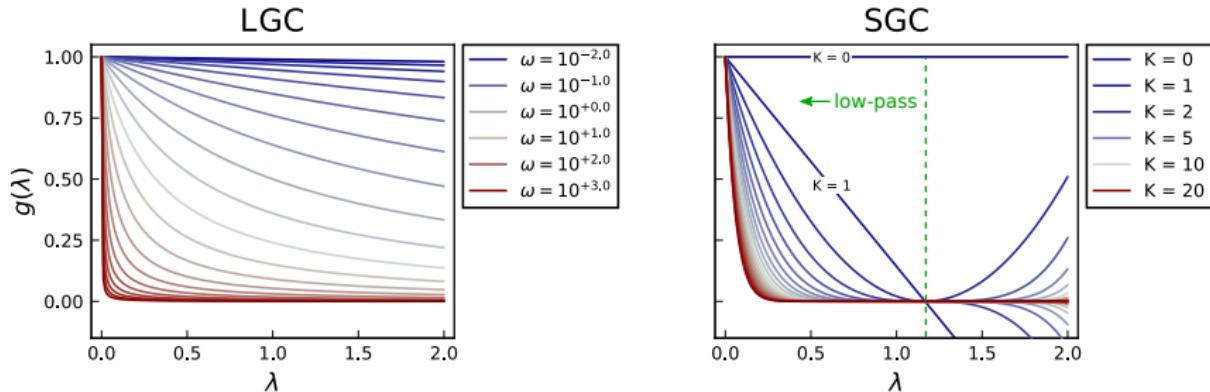
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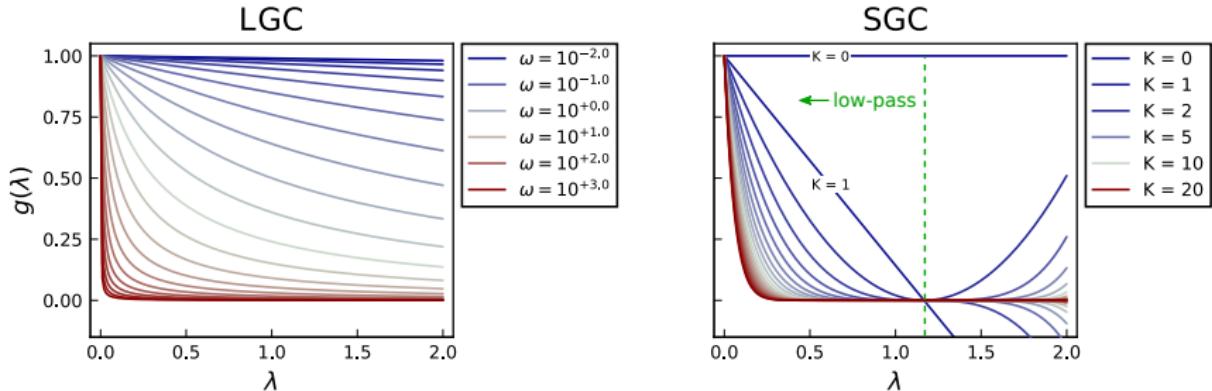
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OUR MODEL HELPS UNDERSTAND SMOOTHING

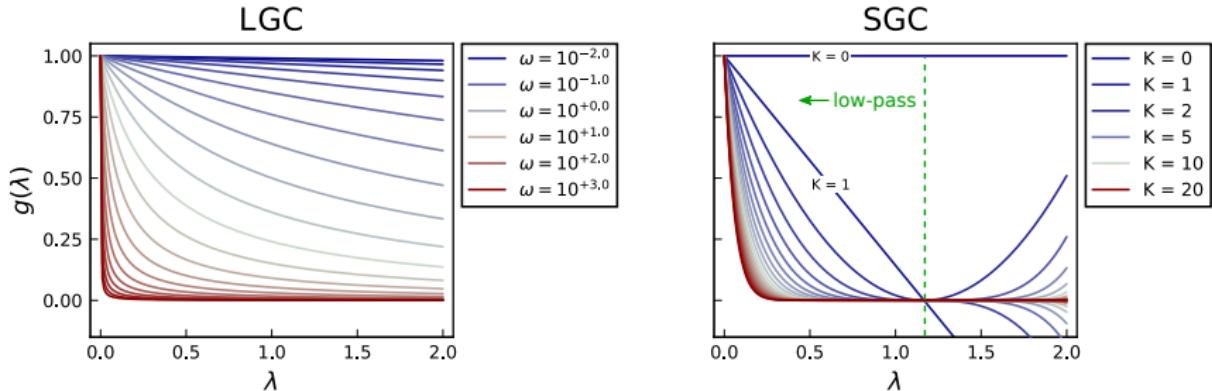


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- LGC is a low-pass filter on the entire eigenspectrum $[0, 2]$

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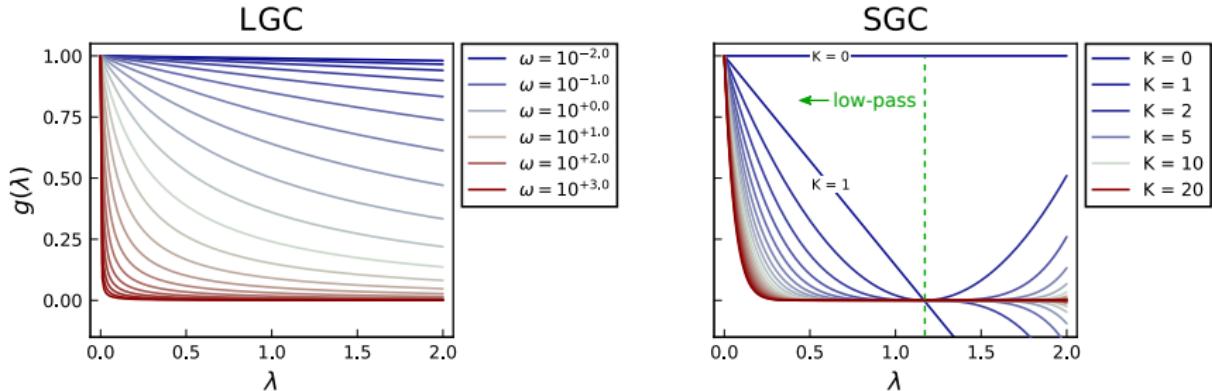


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- LGC is a low-pass filter on the entire eigenspectrum $[0, 2]$
- LGC is more flexible than SGC,

OUR MODEL HELPS UNDERSTAND SMOOTHING

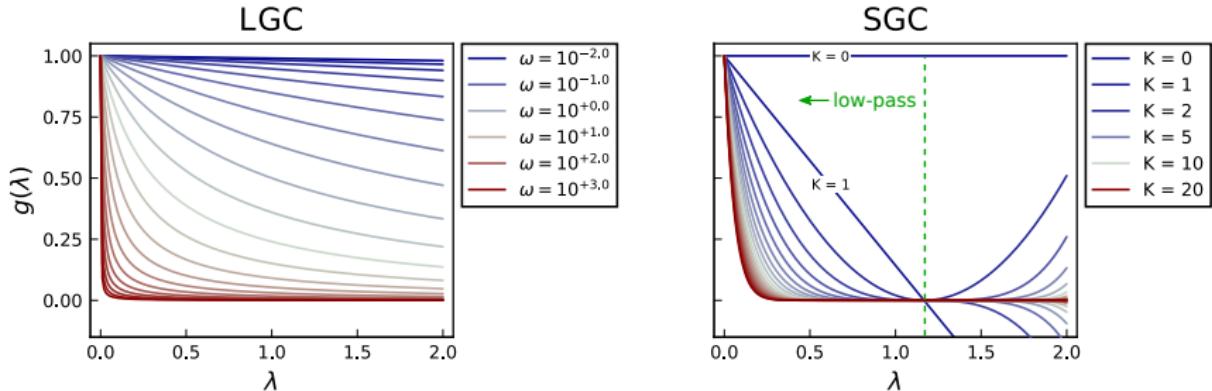


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better balance between preserving the signal ($\omega, K \downarrow$) and reducing the noise ($\omega, K \uparrow$).

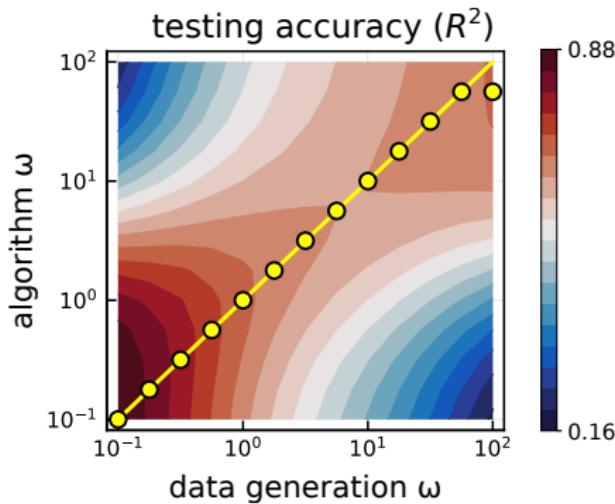
OUR MODEL HELPS UNDERSTAND SMOOTHING



$$(\mathbf{I} + \omega \mathbf{N})^{-1} \mathbf{f} = \sum_i c_i \mathbf{v}_i \ (1 + \omega \lambda_i)^{-1} \quad \tilde{\mathbf{S}}^K \mathbf{f} = \sum_i c_i \mathbf{v}_i \ (1 - d/(d+1)\lambda_i)^K$$

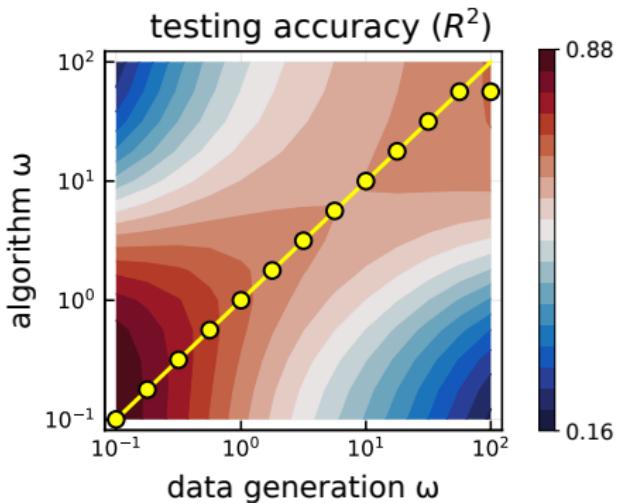
- LGC is a low-pass filter on the entire eigenspectrum $[0, 2]$
- LGC is more flexible than SGC, due to its continuous parameter ω
better balance between preserving the signal ($\omega, K \downarrow$) and reducing the noise ($\omega, K \uparrow$).

OUR MODEL HELPS UNDERSTAND SMOOTHING



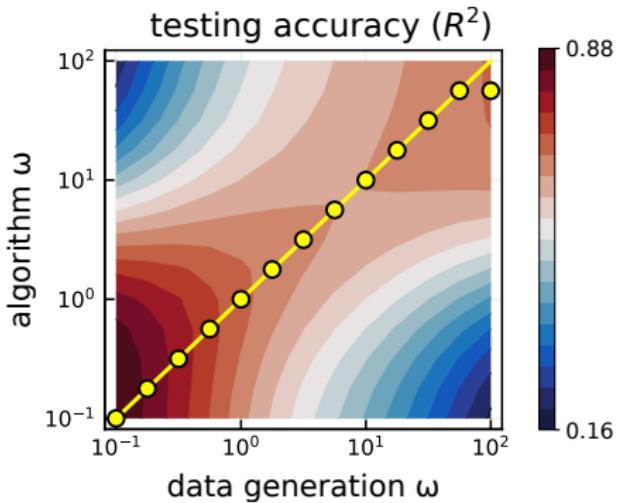
- Sample datasets from our generative model, with different $\omega = \textcolor{blue}{h}/\textcolor{red}{H}$

OUR MODEL HELPS UNDERSTAND SMOOTHING



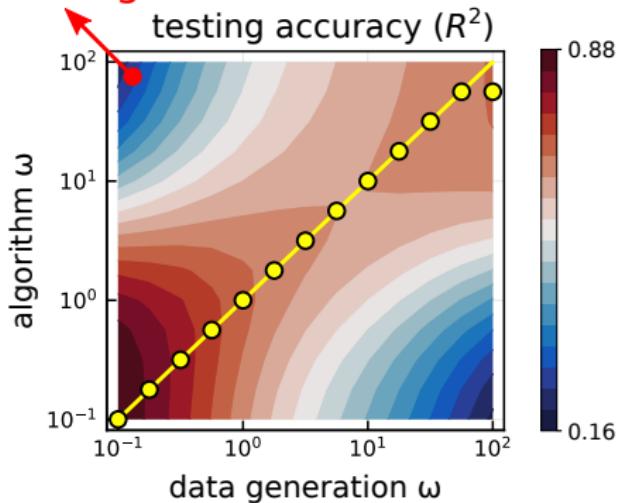
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For each dataset, run algo with different ω ; create a 2D performance plot

OUR MODEL HELPS UNDERSTAND SMOOTHING



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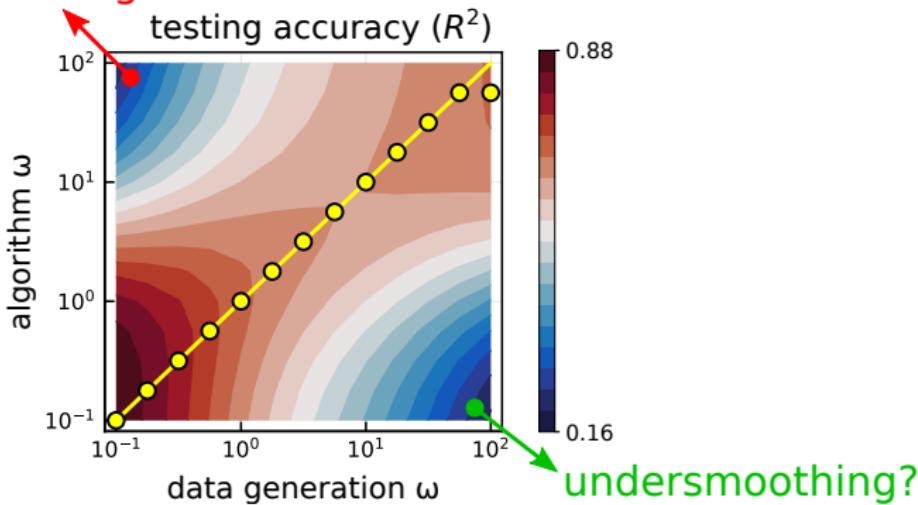
OUR MODEL HELPS UNDERSTAND SMOOTHING oversmoothing!



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For each dataset, run algo with different ω ; create a 2D performance plot
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- Oversmoothing happens if data ω is small, but algorithm ω is large

OUR MODEL HELPS UNDERSTAND SMOOTHING

oversmoothing!



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WE CAN USE LGC TO INTERPRET EMPIRICAL DATA

year	sh050m	sh100m	sh500m	income	migration	birth	death	education	unemployment
2012	0.06	-0.42	0.24	0.22	0.16	-0.13	0.04	-0.90	-0.38
2016	-0.02	-0.38	0.22	0.70	0.21	-0.13	0.51	-1.53	-0.39

sh050m/sh100m/sh500m: shares of friends within 50/100/500 miles; features normalized to zero mean & unit variance

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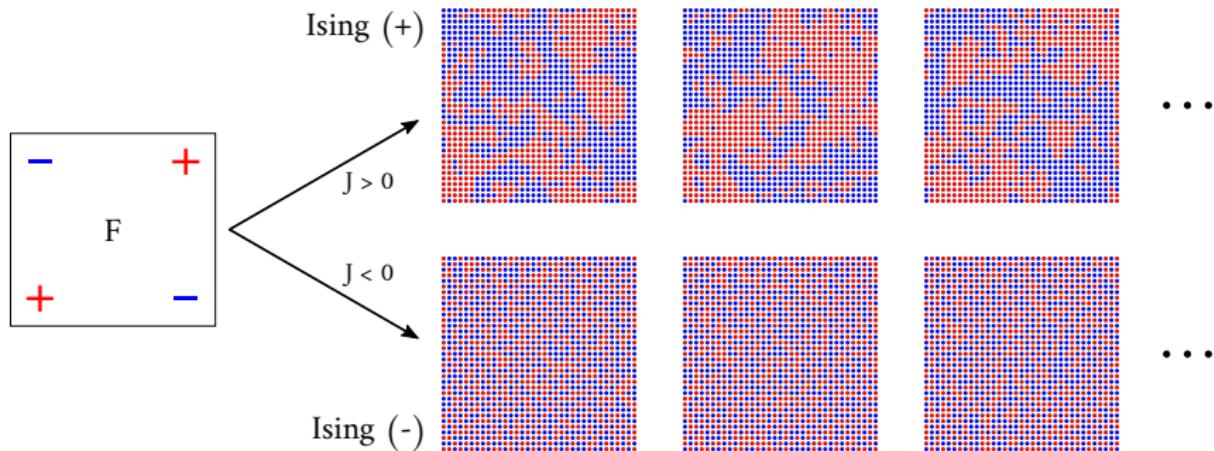
- predict election margin of victory (GOP positive, DEM negative)
- GOP leaning counties tend to have:
 - lower education levels, and higher income; this trend is stronger in 2016 than 2012
 - lower birth rate, and higher death rate → older population
 - lower unemployment rate, and higher migration rate → manufacturing hubs?
 - lower sh100m, and higher sh500m → rural areas?

sh050m/sh100m/sh500m: shares of friends within 50/100/500 miles; features normalized to zero mean & unit variance

WHERE DO WE GO FROM HERE?

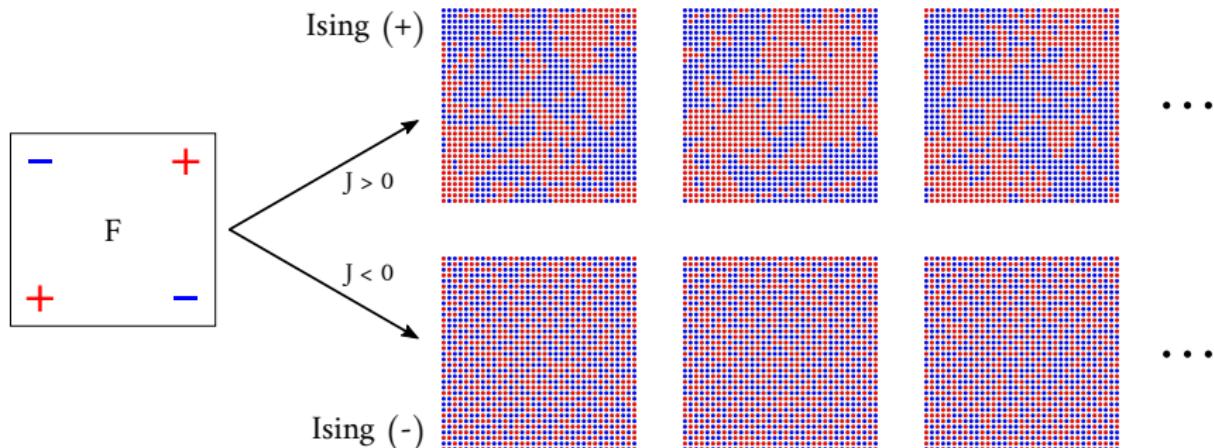
- What if the labels are not positively correlated?
(e.g. heterophily graphs)
- What if the labels are categorical variables?
(node classification setting)

WHAT IF LABELS AREN'T POSITIVELY CORRELATED?



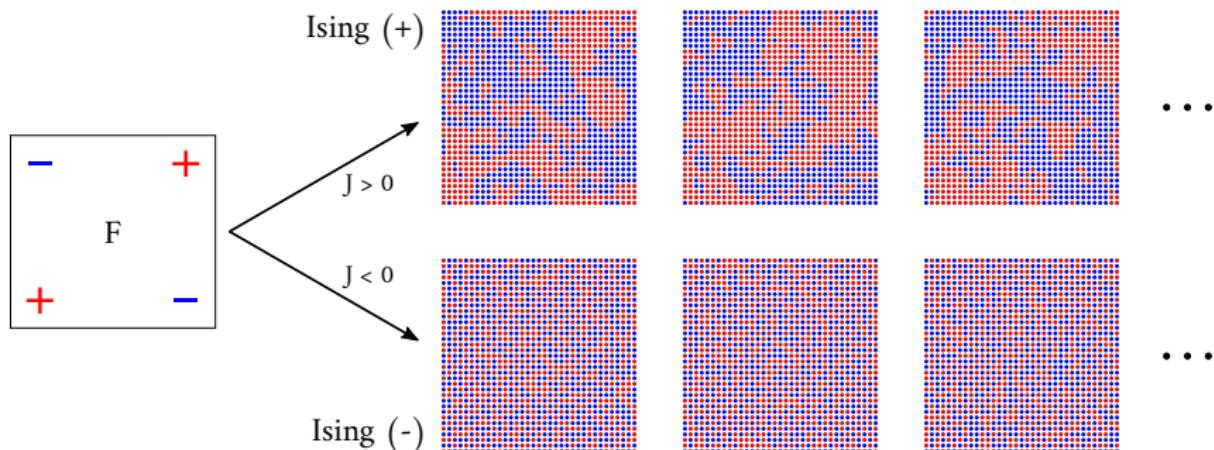
- Sometimes labels can be negatively correlated among neighbors.

WHAT IF LABELS AREN'T POSITIVELY CORRELATED?



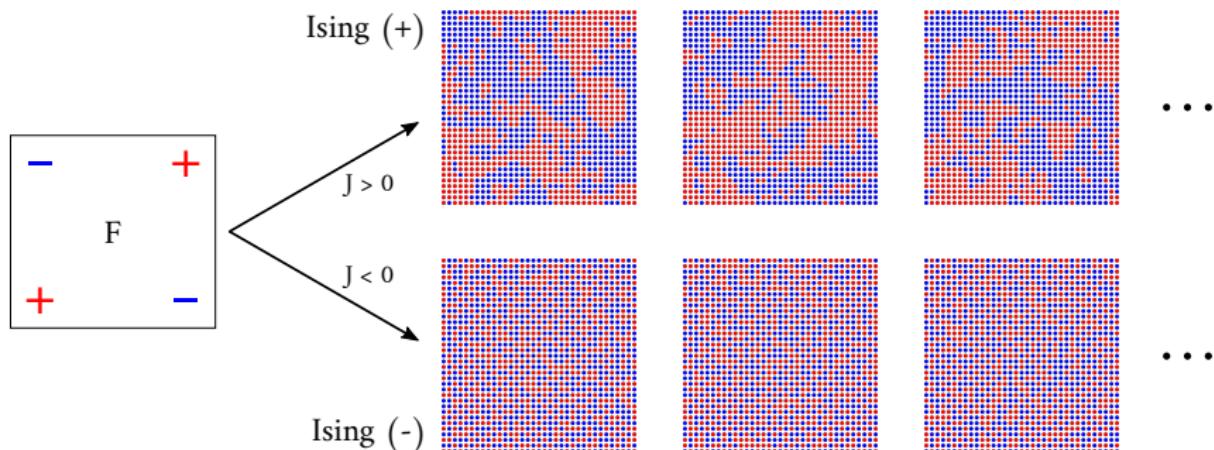
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 - Ising model with negative coupling J (antiferromagnetic materials)

WHAT IF LABELS AREN'T POSITIVELY CORRELATED?



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 - gender in sexual interaction network

WHAT IF LABELS AREN'T POSITIVELY CORRELATED?



- Sometimes labels can be negatively correlated among neighbors.
 - Ising model with negative coupling J (antiferromagnetic materials)
 - gender in sexual interaction network
- Residuals are negatively correlated among neighbors as well.

WE CAN LEARN THE CORRELATION DIRECTLY

- Model the residual with a multivariate Gaussian:

$$\mathbf{r} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Gamma}^{-1}), \quad \boldsymbol{\Gamma} = \beta(\mathbf{I} - \alpha \mathbf{S}), \quad \mathbf{S} = \mathbf{D}^{-1/2} \mathbf{W} \mathbf{D}^{-1/2}$$

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- $\beta > 0$ controls the magnitude of the residuals

WE CAN LEARN THE CORRELATION DIRECTLY

- Model the residual with a multivariate Gaussian:

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- $\beta > 0$ controls the magnitude of the residuals
- $\alpha = 0$, uncorrelated residuals; i.i.d. assumption and sum-of-square loss
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 - learn the optimal α, β with **maximum marginal likelihood**

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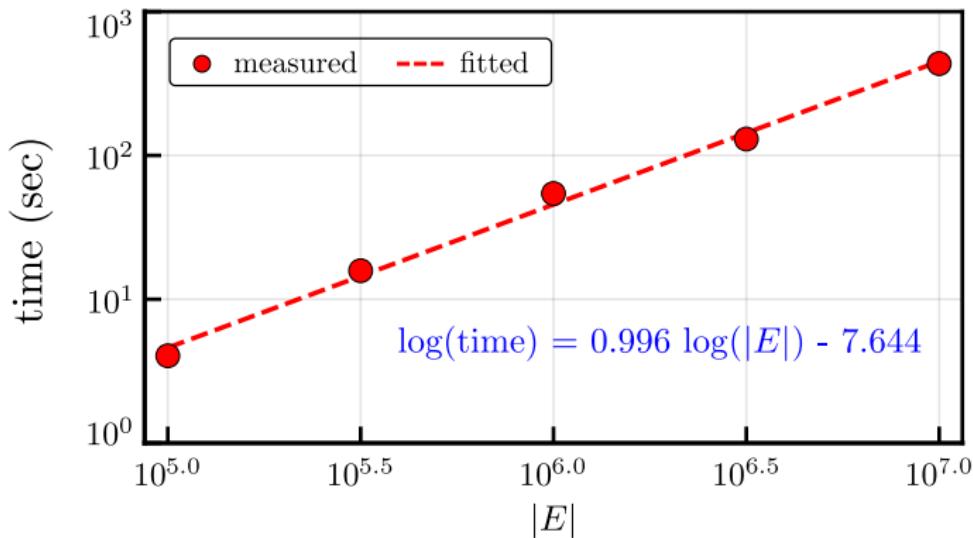
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- Estimate gradient with [stochastic trace estimator](#) & [CG solver](#)

$$\frac{\partial \log |\boldsymbol{\Gamma}|}{\partial \alpha} \approx \frac{1}{T} \sum_{t=1}^T (\boldsymbol{\Gamma}^{-1} \mathbf{z}_t)^\top \left(\frac{\partial \boldsymbol{\Gamma}}{\partial \alpha} \mathbf{z}_t \right)$$

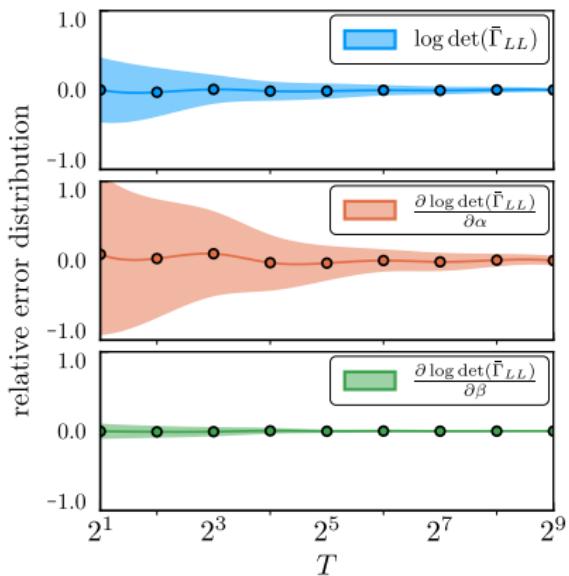
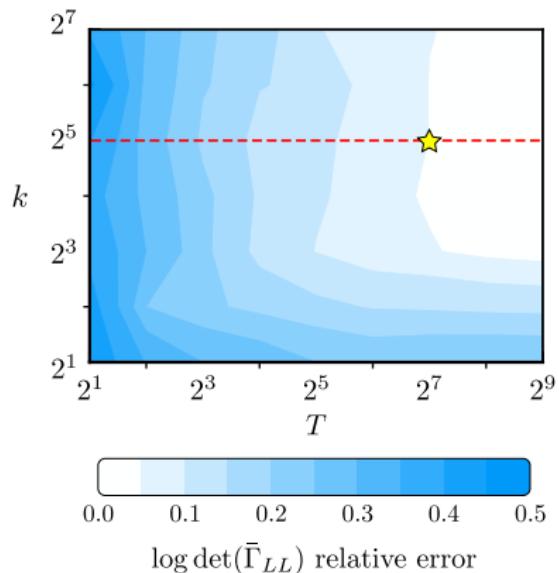
[Ubaru+ 2017; Gardner+ 2018]

STOCHASTIC ESTIMATION IS EFFICIENT



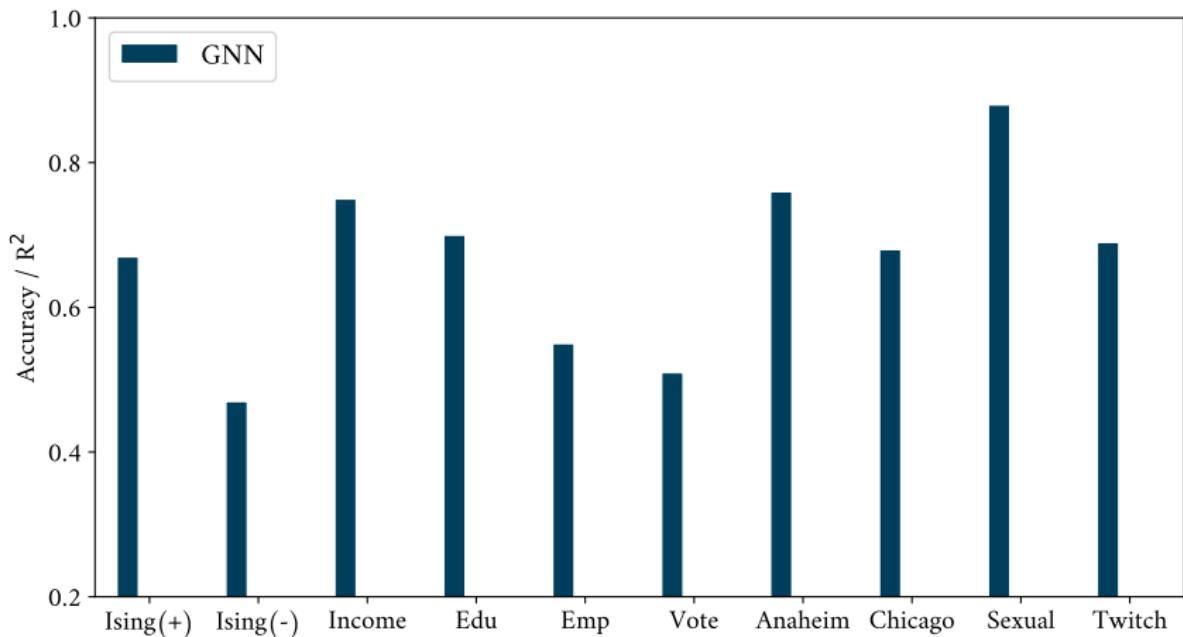
- Stochastic trace estimator cost $\mathcal{O}(|E|)$, linear in the # of edges

STOCHASTIC ESTIMATION IS ACCURATE

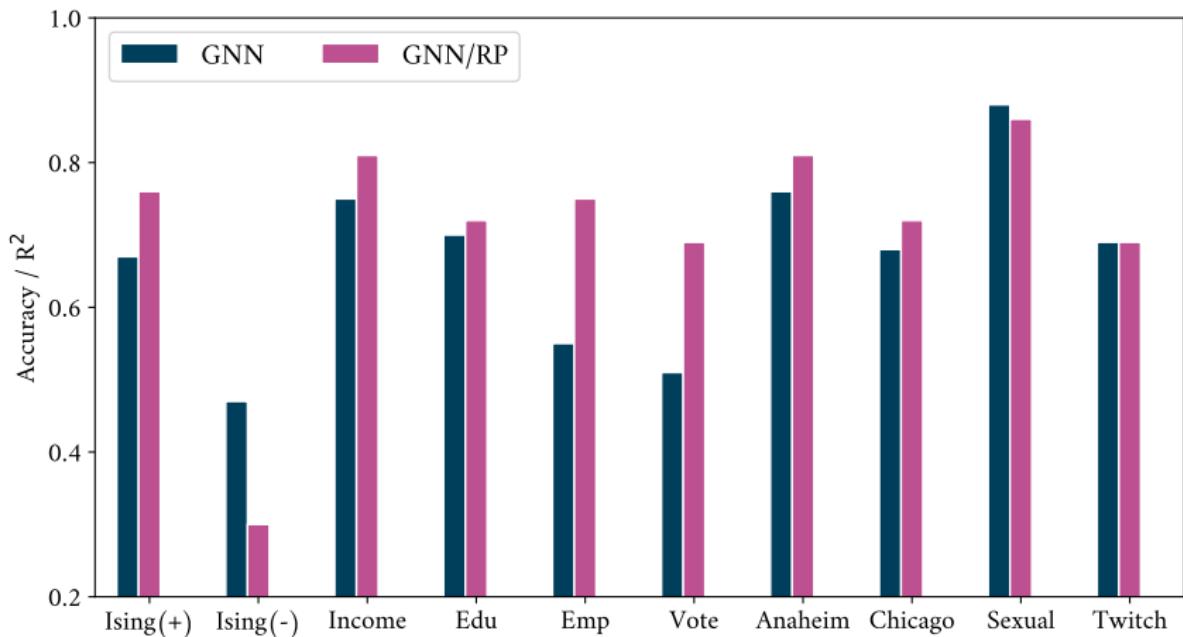


- Stochastic trace estimator gives accurate results
 - a small number of CG iterations, Gaussian trial vectors; $< 5\%$ error
 - unbiased estimation of gradient optimize with SGD

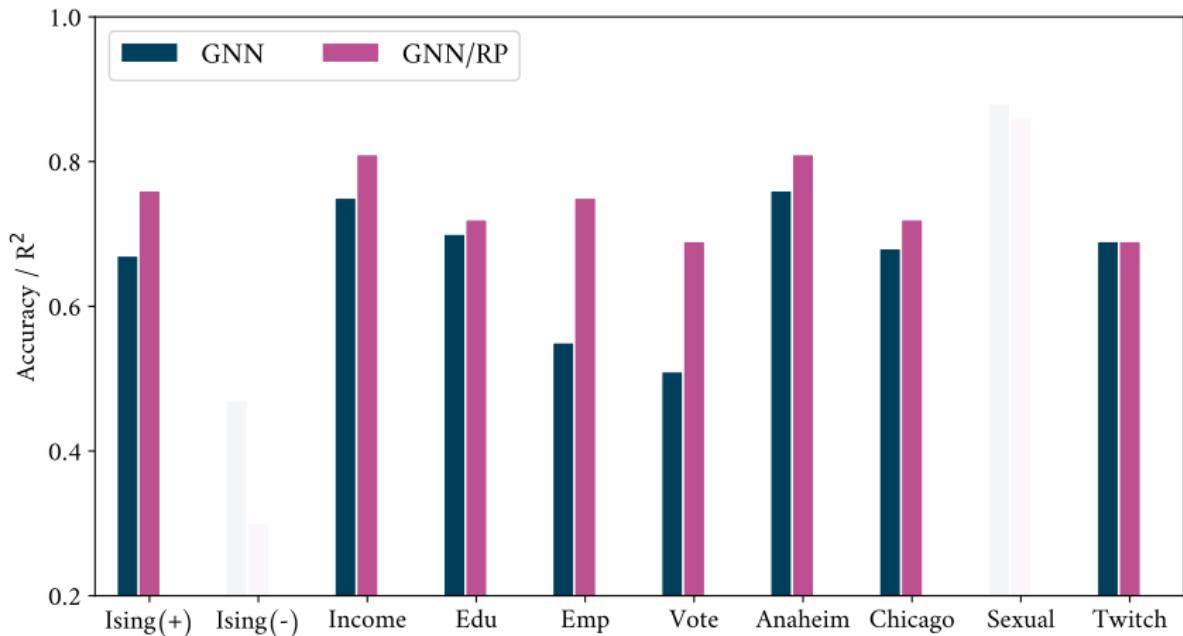
C-GNN PERFORMS WELL ON HETEROGRAPHY GRAPHS



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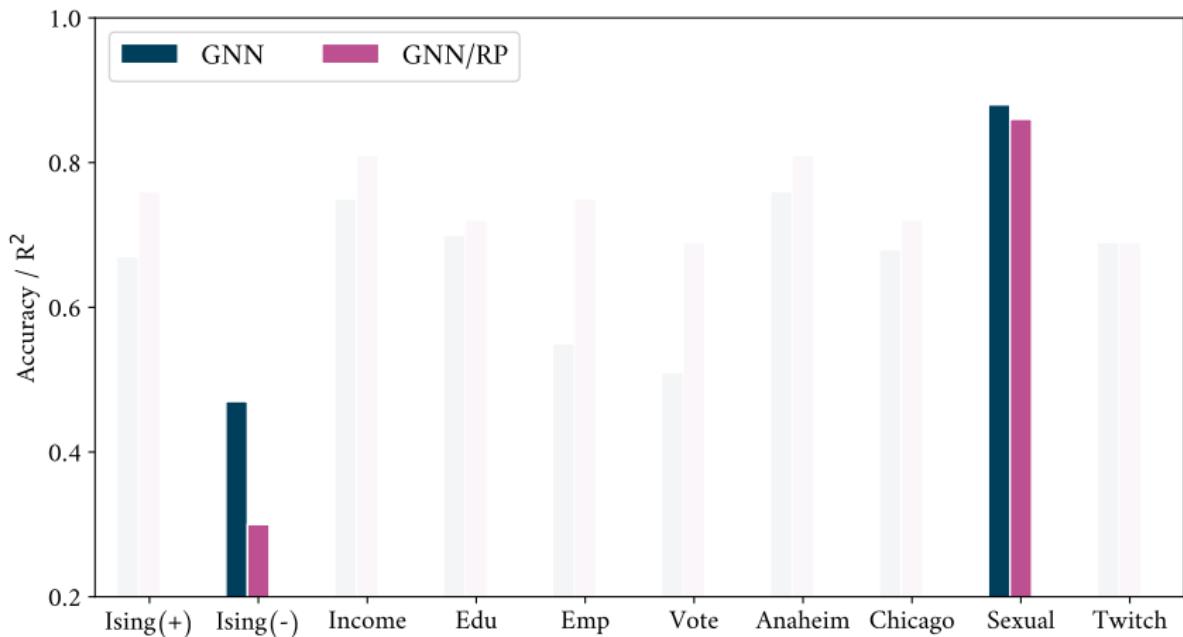


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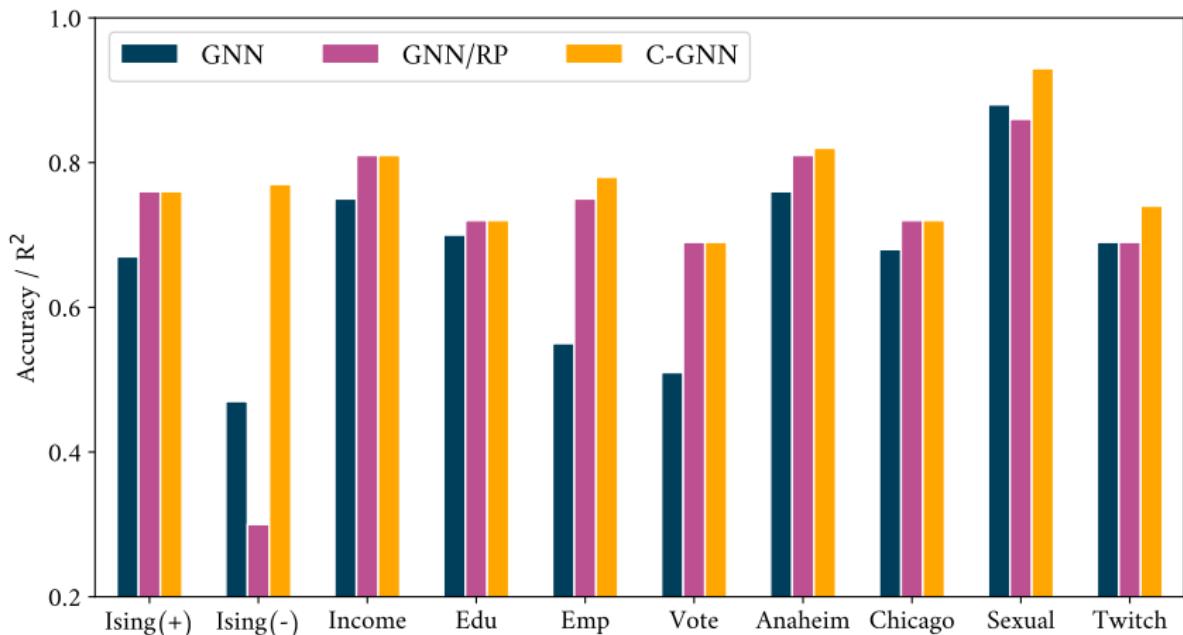
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- GNN/RP outperforms GNN on homophily graphs; underperforms if heterophily
- C-GNN performs the best on all datasets

WHERE DO WE GO FROM HERE?

- What if the labels are not positively correlated?
(e.g. heterophily graphs)
- What if the labels are categorical variables?
(node classification setting)

A STATISTICAL MODEL FOR ATTRIBUTED GRAPHS

- Consider a two-step process for generating node attributes:

$\mathbf{h} \in \mathbb{R}^c$, $\mathbf{H} \in \mathbb{R}^{c \times c}$ are model parameters with all positive entries

A STATISTICAL MODEL FOR ATTRIBUTED GRAPHS

- Consider a two-step process for generating node attributes:
 - Sample node labels jointly from a pairwise MRF

$$P(\mathbf{y}) = \frac{\varphi(\mathbf{y})}{\sum_{\mathbf{y}'} \varphi(\mathbf{y}')}, \quad \varphi(\mathbf{y}) = \prod_{u \in V} h(y_u) \prod_{(u,v) \in E} H(y_u, y_v)$$

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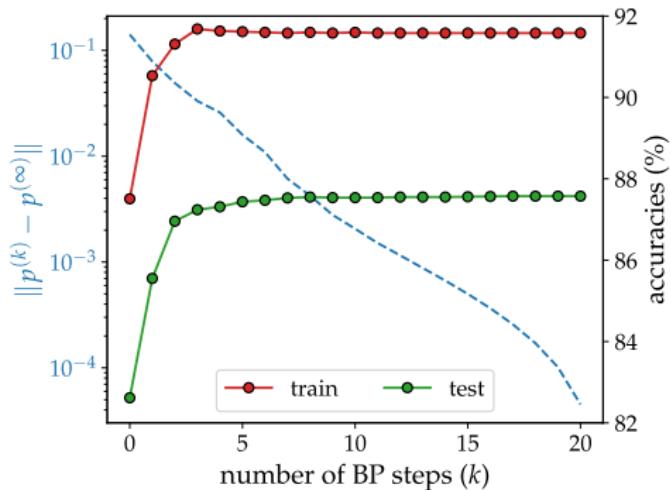
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- Or we can compute $P(y_u | \mathbf{X}, \mathbf{y}_L)$:
further conditioning on \mathbf{y}_L , resulting in a pairwise MRF on $G[U]$

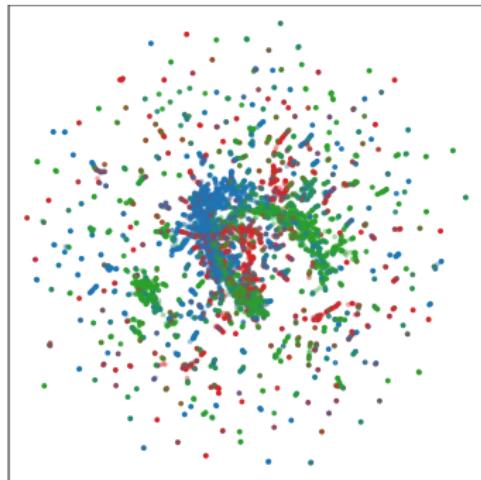
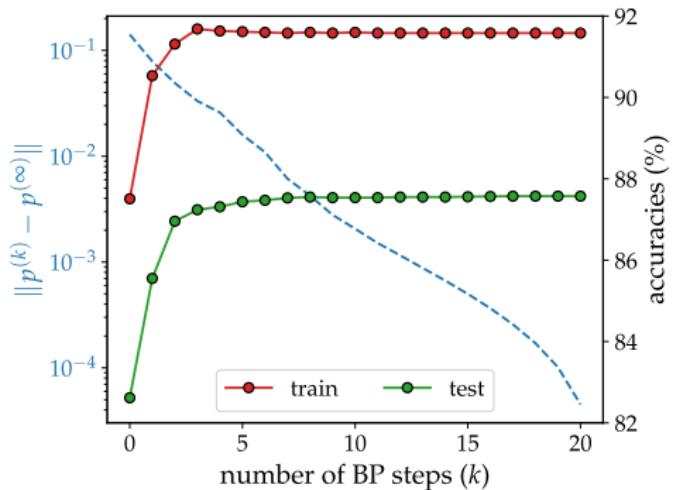
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GBPN CASE STUDY ON PUBMED DATASET



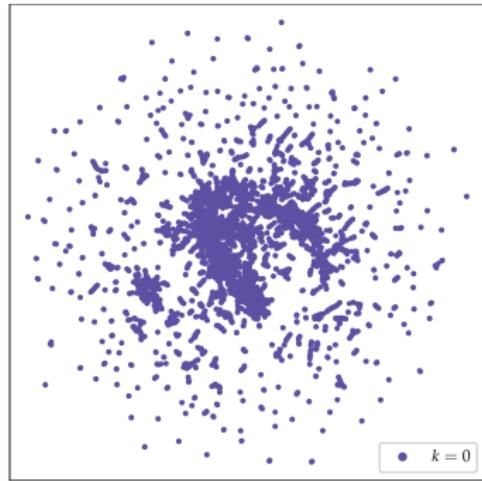
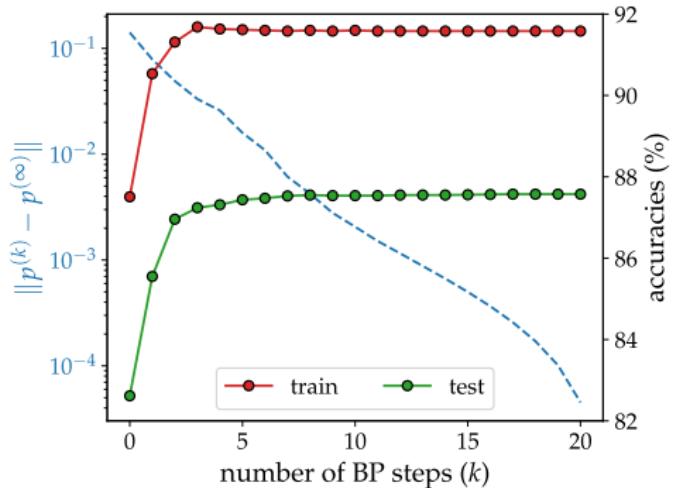
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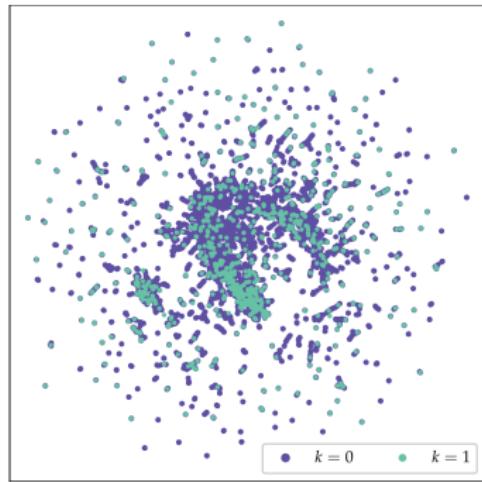
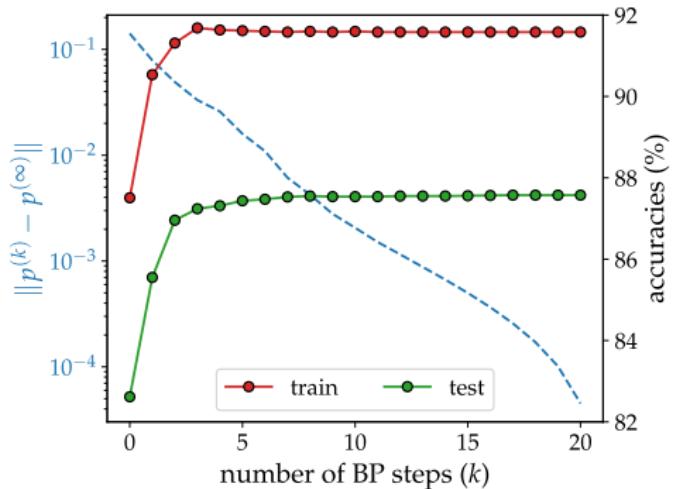
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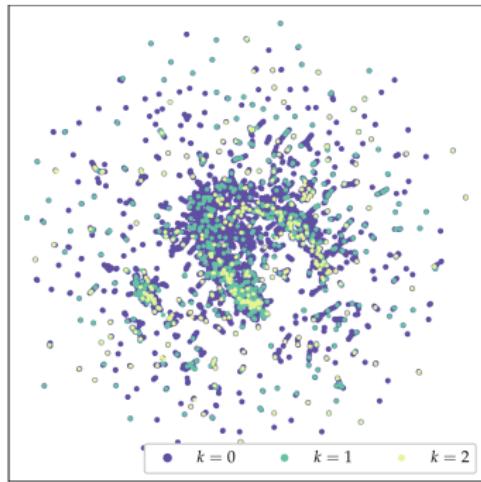
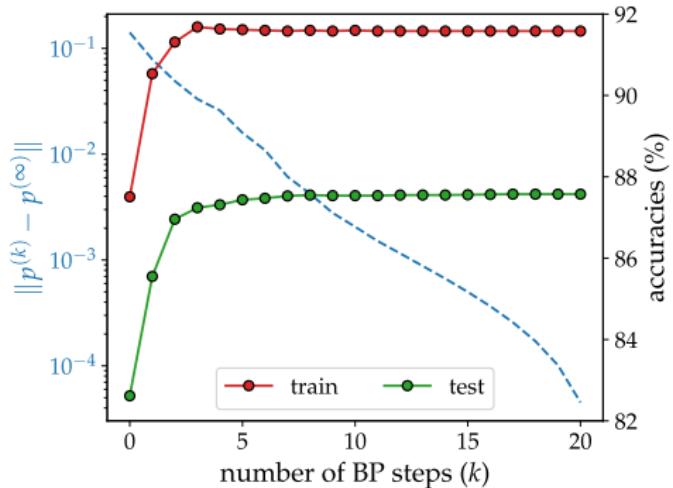
- Belief propagation converge quickly on real-world dataset
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- GBPN is interpretable (node correctly predicted by GBPN after k steps)

GBPN CASE STUDY ON PUBMED DATASET



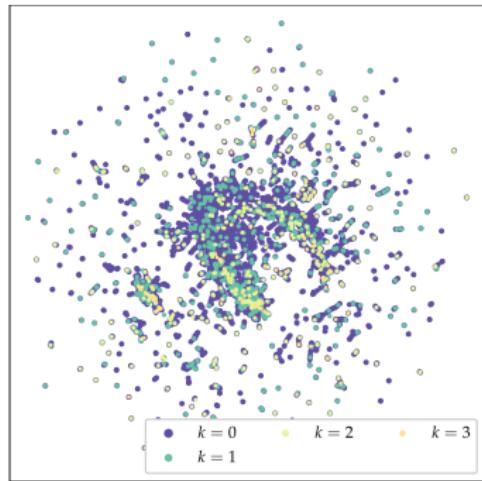
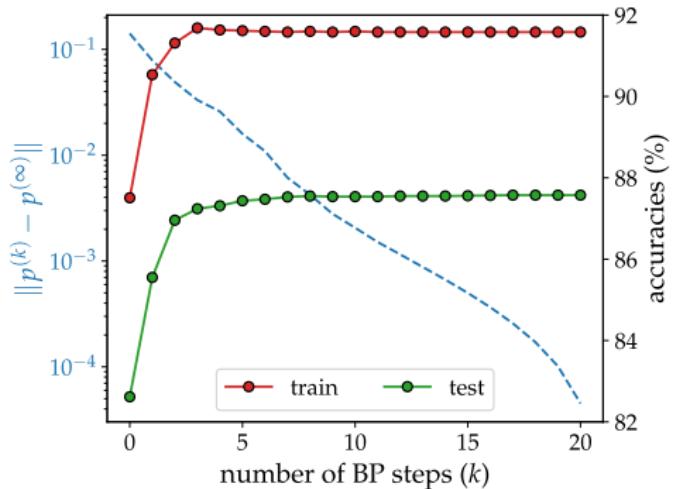
- Belief propagation converge quickly on real-world dataset
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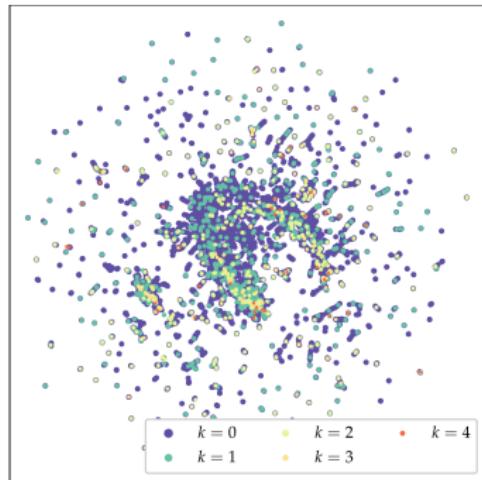
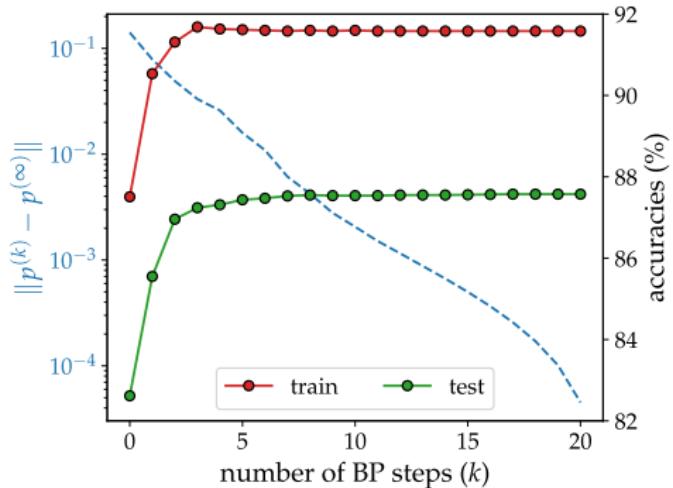
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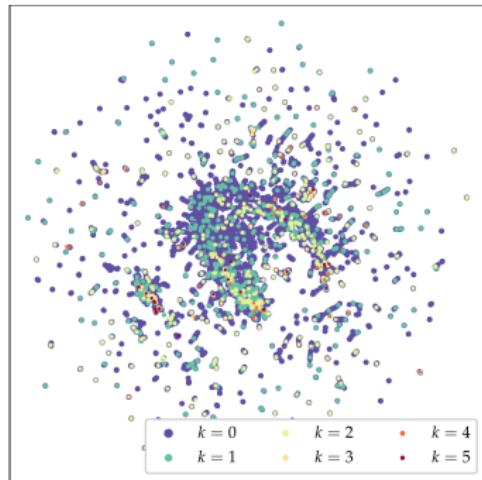
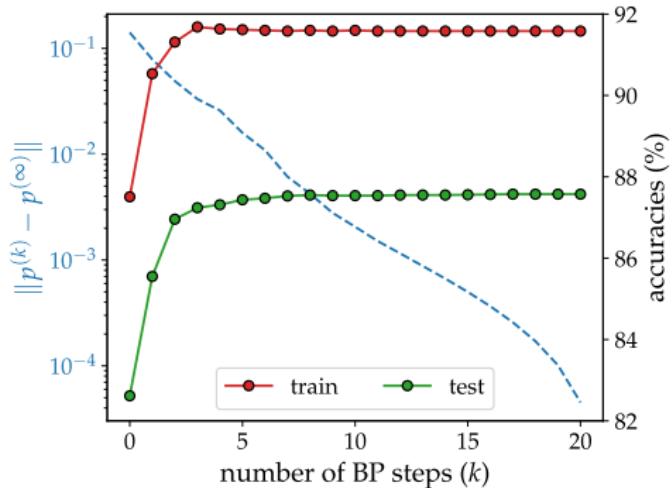
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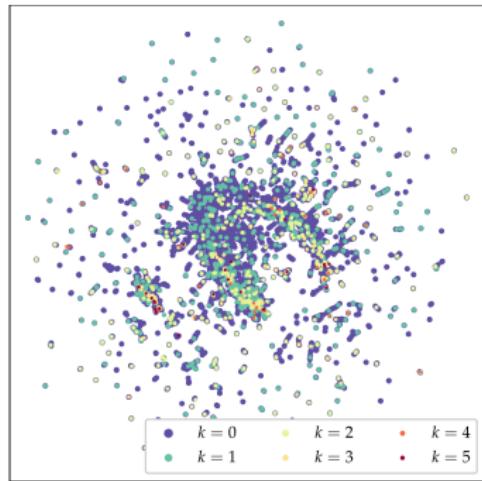
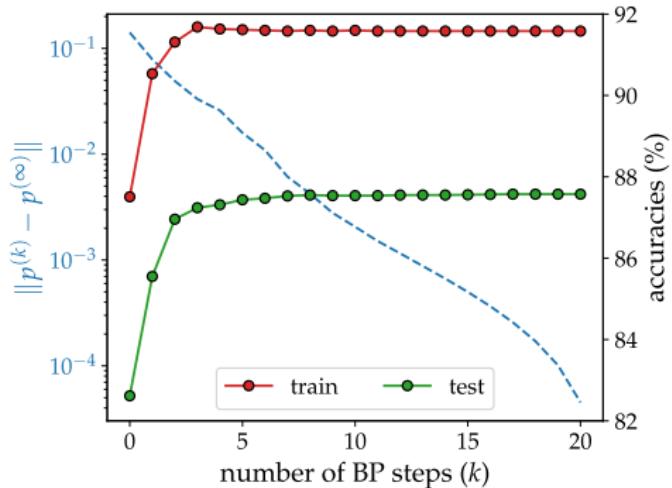
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first get easy examples right, then iteratively correct harder examples with neighbors

PERFORMANCE ON BENCHMARK DATASETS

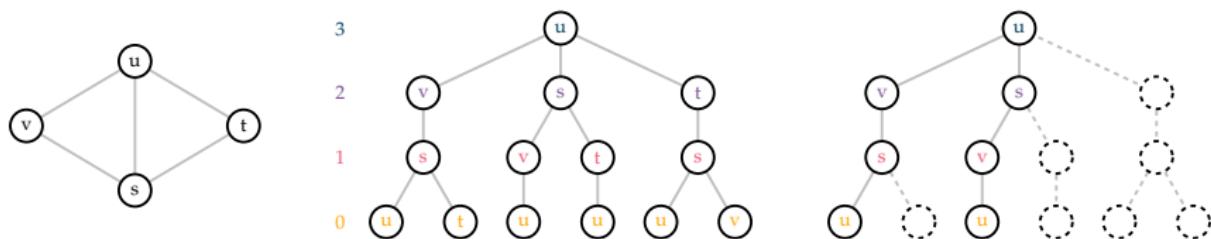
dataset	MLP	SGC	GCN	GraphSAGE	GAT	GBPN(I)	GBPN(T)
County	89.8 ± 0.6	88.2 ± 0.7	87.9 ± 0.7	90.9 ± 0.5	90.5 ± 0.4	90.1 ± 0.7	90.4 ± 0.8
Sexual	73.9 ± 1.4	76.1 ± 1.4	83.7 ± 1.1	93.0 ± 0.9	93.7 ± 0.9	97.0 ± 0.5	97.1 ± 0.5
Cora	72.4 ± 1.1	87.1 ± 0.7	87.1 ± 0.7	87.1 ± 0.8	86.7 ± 0.7	84.8 ± 0.8	84.8 ± 0.8
CiteSeer	70.5 ± 0.9	72.9 ± 1.1	73.4 ± 0.9	73.1 ± 0.8	72.5 ± 0.8	73.9 ± 0.7	73.7 ± 0.7
PubMed	86.6 ± 0.4	86.9 ± 0.3	87.0 ± 0.3	87.8 ± 0.3	88.0 ± 0.2	88.2 ± 0.3	88.2 ± 0.3
CS	94.1 ± 0.2	93.1 ± 0.2	93.1 ± 0.3	93.6 ± 0.3	94.3 ± 0.3	95.4 ± 0.1	95.4 ± 0.2
Physics	95.8 ± 0.2	96.1 ± 0.1	96.1 ± 0.1	96.2 ± 0.2	96.4 ± 0.1	96.8 ± 0.1	96.8 ± 0.1

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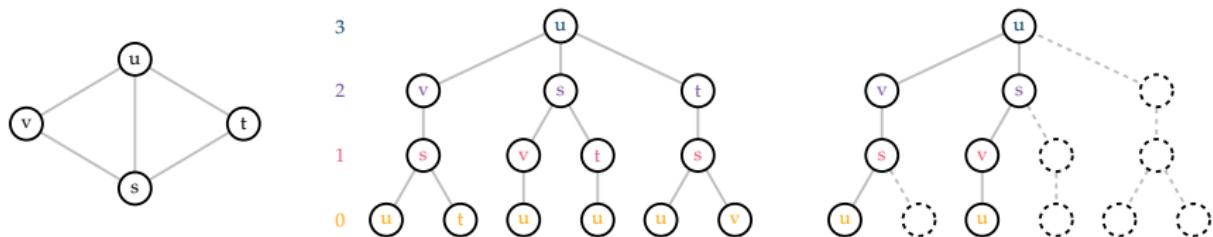
- GBPN is not only more interpretable, but also more accurate

PERFORMANCE ON LARGE-SCALE DATASETS



- We design a subsampling algorithm for mini-batch training

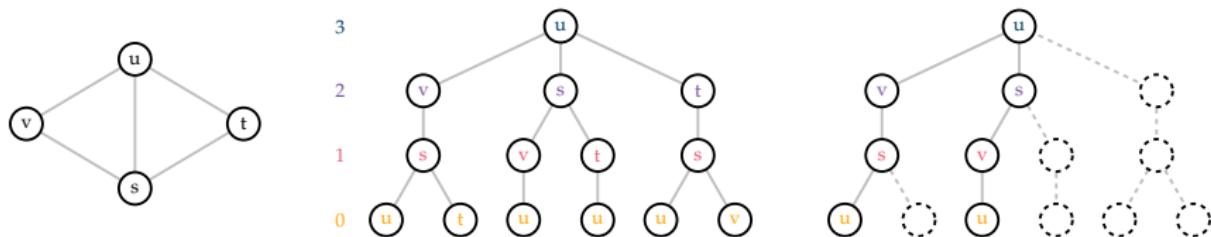
PERFORMANCE ON LARGE-SCALE DATASETS



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Chase	61.1 ± 0.3	77.5 ± 0.5	74.6 ± 0.8	83.0 ± 0.5	86.2 ± 0.5
arXiv	54.6 ± 0.3	70.4 ± 0.3	70.4 ± 0.2	69.4 ± 1.1	70.1 ± 0.1
Products	61.5 ± 0.2	78.1 ± 0.2	76.8 ± 0.0	81.4 ± 0.4	81.2 ± 0.1

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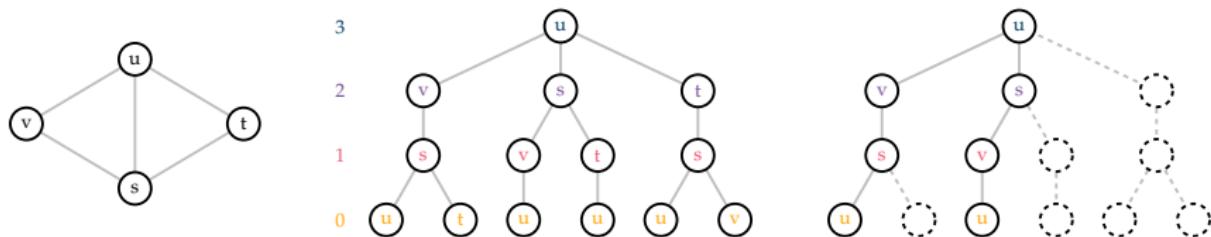
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- We design a subsampling algorithm for mini-batch training
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- GBPN(T) outperforms GBPN(I) by a noticeable margin on some dataset

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- One key aspect of semi-supervised node regression is to leverage feature correlation and attribute homophily.
- My thesis proposed a generative model that accounts for three different types of correlations in attributed graphs. It unifies existing algorithms and motivates new ones with the state-of-the-art performance.
- Our model successfully generalization to non-homophily graphs and node classification setting.

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- **Committee Members**

- Prof. Austin Benson
- Prof. David Bindel
- Prof. Jon Kleinberg



- **Collaborators**

- Prof. Michael Schaub
- Prof. Santiago Segarra



- **Supportors**

- Parents & Family
- Friends & Stacey



MY RESEARCH

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- Residual Correlation in Graph Neural Network Regression,
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