① Matrix Invertibility by Row Reduction

$A|I$ eg. find the inverse of $A = \begin{bmatrix} -1 & -1 \\ 4 & 5 \end{bmatrix}$ $\therefore A^{-1} = \begin{bmatrix} -5 & 1 \\ 4 & 1 \end{bmatrix}$

$A^{-1}A | A^{-1}I$

b/c $A^{-1}A = I$ $\quad I | A^{-1}$

$$\left[\begin{array}{cc|cc} -1 & -1 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to -R_1} \sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 : R_2 - 4R_1} \sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & 1 & 4 & 1 \end{array}\right] \xrightarrow{R_1 : R_1 - R_2} \sim \left[\begin{array}{cc|cc} 1 & 0 & -5 & 1 \\ 0 & 1 & 4 & 1 \end{array}\right]$$

② Algebraic Manipulation of Inverses

eg. given only A^{-1} & b in $Ax = b$, find A

- could solve by finding inverse of A^{-1} (which is A) like question above, or do this

$Ax = b$

$A^{-1}Ax = A^{-1}b$

$Ix = A^{-1}b$ — we know the values of A^{-1} & b, just matrix multiply to solve

$x = A^{-1}b$

③ The Determinant (2x2 only)

shortcut for 2x2 matrix: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, ie. $\det\begin{pmatrix} -1 & 1 \\ 4 & 5 \end{pmatrix} = (-1)(5) - (1)(4) = -9$

- however, does not work for 3x3, 4x4 ... nxn matrices
- In general, use this formula for nxn matrix

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (-1)^{i+j} a_{11} \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ \cancel{a_{21}} & a_{22} \end{vmatrix} + (-1)^{i+j} a_{12} \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & \cancel{a_{22}} \end{vmatrix}$$

$$\det\begin{pmatrix} -1 & 1 \\ 4 & 5 \end{pmatrix} = (-1)^2(-1) \begin{vmatrix} \cancel{-1} & \cancel{1} \\ \cancel{4} & 5 \end{vmatrix} + (-1)^3(4) \begin{vmatrix} \cancel{-1} & \cancel{1} \\ 4 & \cancel{5} \end{vmatrix}$$

$$= (-1)^2(-1)(5) + (-1)^3(1)(4)$$

$$= -5 - 4 = -9$$

- note, we can pick any row, column, or diagonal to do the operation on

④ a 3x3 matrix

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (-1)^{i+j}(a_{11}) \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix} + \ldots$$

$$= (-1)^{i+j}(a_{11}) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \ldots$$

$$= (-1)(a_{11})[a_{22}a_{33} - a_{23}a_{32}] + \ldots$$

trix

⑤ Solving Determinants Efficiently

$$\det\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} = (-1)^{i+j}(0)\begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{vmatrix} + (-1)^{i+j}(0)\begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{vmatrix} + (-1)^{i+j}(0)\begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

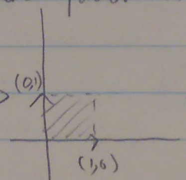
$$= 0 + 0 + 0 = 0$$

⑥ determinant, what does this mean? (graphically)

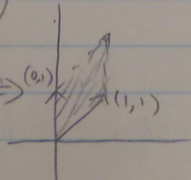
- In general, what is a determinant?

↳ measures the factor in which a region increases or decreases

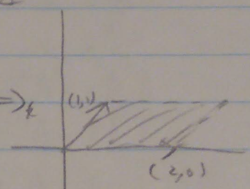
eg. $\begin{array}{cc} x & y \end{array}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Longleftrightarrow$



eg. $\begin{array}{cc} x & y \end{array}$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Longleftrightarrow$



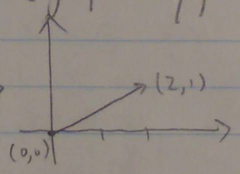
eg. $\begin{array}{cc} x & y \end{array}$ $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \Longleftrightarrow$



$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (1)(1) - (0)(0) = 1$ $\quad \det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (1)(1) - 0 = 1$ $\quad \det\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = (2)(1) - 0 = 2$

⑦ $\det(A) = 0$, what does this mean? (graphically)

eg. $\begin{array}{cc} x & y \end{array}$ $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ $\det\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = (0) - (0) = 0 \Longleftrightarrow$



Problems
- area is a line, so area = 0
- a line in 2-D (lower dimension)

Implications of $\det(A) = 0$
- not invertible, ie. if $\det(A) = 0$ then A^{-1} DNE

⑧ Properties of determinants

$\det(2A) = 2^{n}\det(A)$ → size of matrix $(n \times n)$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

(8) $A = [a_{ij}]$, is an $n \times n$ matrix, where $n \geq 2$ and A is not a zero matrix
assume there are n non-zero real numbers $X_1, X_2 \ldots X_n$ such that $Ax = 0$
Prove A is not invertible

A is not invertible <=> $\det(A) = 0$

now, we try to prove $\det(A) = 0$
Start off with $Ax = 0$, as given

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}X_1 + a_{12}X_2 \\ a_{21}X_1 + a_{22}X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if we convert this to a system of equations

$a_{11}X_1 + a_{12}X_2 = 0$
$a_{21}X_1 + a_{22}X_2 = 0$

given $X_1, X_2 \ldots X_n$ is non-zero
as an example, $X_1 = 7$, & $X_2 = 2$

$7a_{11} + 2a_{12} = 0$
$7a_{21} + 2a_{22} = 0$

① $7a_{11} = -2a_{12}$
② $7a_{21} = -2a_{22}$

we can see ① & ② are the same equation

$\therefore$ $a_{11} = a_{21}$ & $a_{12} = a_{22}$

$$\therefore A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{bmatrix}$$

option 1: row reduce

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{bmatrix} \xrightarrow{R_2 : R_2 - R_1} \sim \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} = a_{11}(0) - a_{12}(0) = 0$$

option 2: calculate det. right away

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{pmatrix} = a_{11}a_{12} - a_{11}a_{12} = 0$$

$\therefore$ b/c $\det(A) = 0$
THEN A is not invertible