MATA33 Quiz 3 Name: Td2 Sols utorid: mark:

Q1 [8 marks]For each of the following matrices, find its inverse or explain (with justification) that why its inverse does not exist: $A = \begin{pmatrix} 3 & 6 & 9 \\ 6 & 12 & 18 \\ 9 & 18 & 27 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 1 \end{pmatrix}$

$$A = \begin{bmatrix} 3 & 6 & 9 \\ 6 & 12 & 18 \\ 9 & 18 & 27 \end{bmatrix} \overset{R_2: R_2 - 2R_1}{\sim} \begin{bmatrix} 3 & 6 & 9 \\ 0 & 0 & 0 \\ 9 & 18 & 27 \end{bmatrix}$$

$$\det(A) = \det\begin{pmatrix} 3 & 6 & 9 \\ 0 & 0 & 0 \\ 9 & 18 & 27 \end{pmatrix} = 0+0+0 = 0$$

$$B = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

$$B|I = \left[\begin{array}{ccc|ccc} 0 & 2 & 0 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1: \frac{1}{2}R_1$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ -2 & 0 & 2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3: R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ -2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$R_2: R_2 - 2R_3$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ -2 & 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$R_2: -\frac{1}{2}R_2$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Q2[6 marks] Suppose P and Q are 6 by 6 matrices with det(P) = 3, det(Q) = -2

Compute (a) $\det(P^{-2}\det(Q^{-1}))$ (b) $\det((2PQ)^{-1})$

(a) $\det(P^{-2}\det(Q^{-1}))$

$= \det(P^{-2}\frac{1}{\det(Q)})$

$= \det(P^{-2}(\frac{1}{-2}))$

$= (-\frac{1}{2})^6 \det(P^{-2})$

$= \frac{1}{64} \cdot \frac{1}{\det(P)^2}$

$= \frac{1}{64} \cdot \frac{1}{3^2} = \frac{1}{64} \cdot \frac{1}{9} = \frac{1}{576}$

$64 \times 9 = 576$

(b) $\det((2PQ)^{-1})$

$= \det(\frac{1}{2} Q^{-1}P^{-1})$

$= (\frac{1}{2})^6 \det(Q^{-1})\det(P^{-1})$

$= \frac{1}{64} \cdot \frac{1}{\det(Q)} \cdot \frac{1}{\det(P)}$

$= \frac{1}{64} \cdot \frac{1}{(-2)} \cdot \frac{1}{3}$

$= -\frac{1}{384}$

$64 \times 6 = 384$

Q3[2 marks] Given a system of linear equations in matrix form AX=b with det(A)≠0. State the solutions of this system when this system is solved by Cramer's rule. (Don't forget to define your variables…)

let $X = X_1, X_2 \ldots X_n$

$X_1 = \frac{\det(A_{X_1})}{\det(A)}$, $X_2 = \frac{\det(A_{X_2})}{\det(A)}$ — $X_n = \frac{\det(A_{X_n})}{\det(A)}$

$A_{X_1}, A_{X_2}, \ldots A_{X_n}$ is A with the n'th column replaced by b

Q4[6 marks] Determine the value(s) of the real number parameter t for which the system

$$Ax=b \begin{cases} 3tx - 2y = 4 \\ -2x + 5ty = 7 \end{cases}$$

has a unique solution. For these value(s) of t, use Cramer's rule to solve for x and y. (Use Cramer's rule ONLY. No method of solution other than Cramer's rule will earn any credit)

$$Ax=b \Leftrightarrow \begin{bmatrix} 3t & -2 \\ -2 & 5t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad D = \det(A) = \begin{vmatrix} 3t & -2 \\ -2 & 5t \end{vmatrix} = 15t^2 - (-2)(-2) = 15t^2 - 4$$

$$D_x = \begin{vmatrix} 4 & -2 \\ 7 & 5t \end{vmatrix} = 4(5t) - (-2)(7) = 20t + 14 \qquad D_y = \begin{vmatrix} 3t & 4 \\ -2 & 7 \end{vmatrix} = 3t(7) - (4)(-2) = 21t + 8$$

$$X = \frac{D_x}{D} = \frac{20t+14}{15t^2-4} \qquad Y = \frac{D_y}{D} = \frac{21t+8}{15t^2-4}$$

$t \in \mathbb{R}$, b/c numerator $\in \mathbb{R}$

$15t^2 - 4 \neq 0$ b/c denominator $\neq 0$ or else undefined

$\to t \neq \pm\sqrt{\frac{4}{15}}$

$\to 15t^2 - 4 \neq 0$

$15t^2 \neq 4$

$t^2 \neq \left(\frac{4}{15}\right)$

$\to t \in \mathbb{R}$ b/c numerator $\in \mathbb{R}$

$t \neq \pm\sqrt{\frac{4}{15}}$

Q5 [3 marks]Suppose B is a 3 by 3 matrix such that $B^4 - 3B^3 + 10B^2 - I = 0$. Is B invertible ? Explain

$B^4 - 3B^3 + 10B^2 - I = 0$

$B^4 - 3B^3 + 10B^2 = I$

$B\underbrace{(B^3 - 3B^2 + 10B)}_{} = I$

$\to B^3 - 3B^2 + 10B = B^{-1}$

Q6 [5 marks] suppose A is an n by n matrix with $n \geq 2$ and $A^2 = A$. Suppose c is a real number and c$\neq$1,

find the inverse of the matrix $I - cA$.

$\to A^2 = A$

$A^{-1}A^2 = A^{-1}A$

$A = I$

let $C = I - cA$

want to find C^{-1}

$C = I - c(I)$

$C = (1-c)I$

$C^{-1} = [(1-c)I]^{-1}$

$C^{-1} = \frac{1}{1-c} I$ $\qquad I^{-1} = I$ $\qquad (1-c)^{-1} = \frac{1}{1-c}$