⓪ Matrices
- they are weird

eg. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- somehow, the product of two non-zero matrices is 0!
- but they are quite useful as data holders to extract certain types of information

① Applications of matrix data storage
- a company has 4 locations L_1, L_2, L_3, L_4 & sells products $(X_1, X_2 \dots X_8)$ at location $(L_1, L_2 \dots L_4)$
- let $P = [P_{ij}]$ be the profit matrix defined by P_{ij} when one unit of the product X_j is sold at location L_i

a) the company's profit from selling 7 units of product X_2 at location L_4 is 84k, value of $P_{4,2}$?

row $1 \dots i$ = location

column $1 \dots j$ = product

$P_{4,2}$ = profit of selling one unit of product 2 at location 4

given 7 units, so just divide given profit by 7

$P_{4,2} = \frac{84k}{7} = 12k$

b) State matrix Q such that $A = QP$ where $A = [A_{1,j}]$ & $A_{1,j}$ is company's average profit from selling one unit of product X_j at all 4 locations

$1\times8 = (1\times4)(4\times8)$

$A = QP$ $\quad A = [a_{11}, a_{12} \dots a_{18}] = [q_{11}, q_{12} \dots q_{14}]\begin{bmatrix} P_{11} & P_{12} & \dots & P_{18} \\ P_{21} & & & \\ \vdots & & & \\ P_{41} & & & P_{88} \end{bmatrix}$

defined by A_{ij}
∴ 1 row, j columns

eg. $a_{11} = q_{11} \cdot P_{11} + q_{12} \cdot P_{21} + q_{13} \cdot P_{31} + q_{14} q_{41}$
$= q_{11}(P_{11} + P_{21} + P_{31} + P_{41})$

average profit of product 1 at all 4 locations

⇓

Avg. Profit of 1 = $\frac{P_{11} + P_{21} + P_{31} + P_{41}}{\#\text{ of locations}}$

4 locations

∴ $q_{11} = \frac{1}{4}$, by generalization, $q_{12} \dots q_{14}$ is also $\frac{1}{4}$

∴ $Q = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}] = \frac{1}{4}[1, 1, 1, 1]$

c) State matrices E & C so that the single entry in matrix product EPC is the total profit from selling exactly j units of product X_j at all 4 locations

$$\underset{1\times1}{B} = [b_{11}] = \underset{1\times4}{E}\ \underset{4\times8}{P}\ \underset{8\times1}{C} = [e_{11}, e_{12} \dots e_{14}] \begin{bmatrix} P_{11} & P_{12} & \dots & P_{18} \\ P_{21} & \ddots & & \\ \vdots & & & \\ P_{41} & & & P_{88} \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \\ \vdots \\ C_{81} \end{bmatrix}$$

$E \cdot P : ep_{11} = e_{11} P_{11} + e_{12} P_{21} + \dots \ e_{14} P_{41}$

profit of 1 unit of product 1 at all 4 locations — count of location number 1 — profit of 1 unit of product 1 at location number 1

its quite obvious there should only be one of each location

$\therefore E = [1, 1, 1, 1]$

$(EP) \cdot C = ep_{11} C_{11} + ep_{12} C_{21} + \dots \ ep_{18} C_{81}$

profit of 1 unit of product 1 at all 4 locations — units of product 1

based on the given, we want j units of product j

$$\therefore C = \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \\ C_{41} \\ C_{51} \\ C_{61} \\ C_{71} \\ C_{81} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

② Matrix Multiplication

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}$$

$$AA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix}$$

eg. a matrix whose square is an identity matrix

$a^2+bc=1$, $ab+bd=0$, $ac+dc=0$, $bc+d^2=1$

from pattern recongition, let $c=1$, $b=1$, $a=0$, $d=0$

verify: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0\cdot 0+1\cdot 1 & 0\cdot 1+1\cdot 0 \\ 1\cdot 0+0\cdot 1 & 1\cdot 1+0\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

eg. $B = \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 17 \\ -3 & 16 \end{bmatrix}$, find $AB = C$

③ Diagonal Matrix

$A = \begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -55 \\ -33 & 4 \end{bmatrix}$, given $D^2 - A^2 = B$ & D is diagonal find D

$D^2 - A^2 = B$

$D^2 = B + A^2 = \begin{bmatrix} 6 & -55 \\ -33 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 6 & -55 \\ -33 & 4 \end{bmatrix} + \begin{bmatrix} 19 & 55 \\ 33 & 96 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix}$

$D^2 = \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}}$ $\iff$ $D^2 = \begin{bmatrix} a^2 & 0 \\ 0 & d^2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

- this relationship only holds for diagonal matrices
  - as a reference, the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I = I^2 = I^{\infty} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - in linear algebra, the identity matrix acts like "1" on the real number system

③ More diagonals

let $a = [a_{ij}]$ be an $n \times n$ matrix, $n \geq 2$, let $P = A^T A$

Prove: if diagonal entries in P are all equal to 0, then $A = 0$

$$P = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$P = A^T A = \begin{bmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & \cdots & & \\ a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} & \ddots & & \\ \vdots & & & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix}$$

$0 = a_{11}^2 + a_{12}^2 + a_{13}^2 \implies$ generalized. $0 = \sum_{j=1}^{n} a_{1j} \quad 0 = \sum_{j=1}^{n} a_{2j} \quad 0 = \sum_{j=1}^{n} a_{3j}$

$\Downarrow$

only solution is, $a_{11} = a_{12} = a_{13} = 0$ $\qquad 0 + 0 + 0 = \sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \sum_{j=1}^{n} a_{3j}$

$\Downarrow$

$0 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$ ■

④ Important Transpose property and why

$(AB)^T = B^T A^T$

a weak proof by matrix size

$A = m \times n$

$B = n \times w$

$$\begin{array}{cc} (AB)^T & B^T A^T \\ \Downarrow & \Downarrow \\ (m \times n \;\; n \times w)^T & (n \times w)^T (m \times n)^T \\ \Downarrow & \Downarrow \\ (m \times w)^T & (w \times n)(n \times m) \\ \Downarrow & \Downarrow \\ (w \times m) & (w \times m) \end{array}$$

(5) Upper Triangular Matrix

let $U = [U_{ij}]$ be an 3×3 upper triangular matrix such that $U_{ij} = -(i-j+3)^2$. What is largest element in U?

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix}$$

- for U_{ij} where $i < j$, a) max of $j = 3$, max of $i = 2$

$$U_{ij} = -(2-3+3)^2 = -(2)^2 = -4$$

b) max of $j = 3$, min of $i = 1$

$$U_{ij} = -(1-3+3)^2 = -(1)^2 = -1$$

- for U_{ij} where $i = j$, $U_{ii} = -(i-i+3)^2 = -(3)^2 = -9$
- for U_{ji} where $i > j$ $U_{ij} = 0$

(6) Symmetric Matrix

- $A^T = A$
- let $n \geq 3$, what is the max number of distinct entries?
- $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 5 \\ 3 & 5 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} O & O & O \\ X & O & O \\ X & X & O \end{bmatrix}$

↳ in other words, the number of non-zero elements in an $n \times n$ upper / lower triangular matrix

1 2 3 ... sum $= 1+2+3 \ldots n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(7) Augmented Matrix Reduction

- a student has money in 3 bank accounts that pay 5%, 7%, 8% interest
- there is 3 times as much in the 8% account vs. the 5% account
- total investment is \$1600 & interest is \$115
- what is invested in each account?

let X_1, X_2, X_3 equal to amount in 5%, 7%, 8% account

$X_1 + X_2 + X_3 = 1600$ (1)

$3X_1 = X_3$

$3X_1 \quad - X_3 = 0$ (2)

$0.05X_1 + 0.07X_2 + 0.08X_3 = 115$

$5X_1 + 7X_2 + 8X_3 = 11500$ (3)

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 3 & 0 & -1 & 0 \\ 5 & 7 & 8 & 11500 \end{array}\right]$$

Solve by matrix Row Reduction

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 3 & 0 & -1 & 0 \\ 5 & 7 & 8 & 11500 \end{array}\right] \overset{\#1:\ R_2 : R_2 - 3R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 0 & -3 & -4 & -4800 \\ 5 & 7 & 8 & 11500 \end{array}\right] \overset{\#2: R_3 : R_3 - 5R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 0 & -3 & -4 & -4800 \\ 0 & 2 & 3 & 3500 \end{array}\right] \overset{\#3 R_3 : \frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 0 & -3 & -4 & -4800 \\ 0 & 1 & \frac{3}{2} & 1750 \end{array}\right]$$

$$\overset{\#4\ R_2 : R_2 + R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 0 & 0 & \frac{1}{2} & 450 \\ 0 & 1 & \frac{3}{2} & 1750 \end{array}\right] \overset{\#5\ R_3 : R_3 - 3R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 0 & 0 & \frac{1}{2} & 450 \\ 0 & 1 & 0 & 400 \end{array}\right] \overset{\#6\ R_2 : 2R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 0 & 0 & 1 & 900 \\ 0 & 1 & 0 & 400 \end{array}\right] \overset{\#7\ R_1 : R_1 - R_2,\ R_1 : R_1 - R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 900 \\ 0 & 1 & 0 & 400 \end{array}\right] \overset{\#8\ R_2 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 300 \\ 0 & 1 & 0 & 400 \\ 0 & 0 & 1 & 900 \end{array}\right]$$

$\therefore X_1 = 300,\ X_2 = 400,\ X_3 = 900$, can solve linear programming questions this way too

General Idea (Personal approach)

- try to reduce into upper triangular form (like #4)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ \boxed{0 \quad 0 \quad a_{33}} \end{bmatrix}$$

use this independent row to make a_{13}, a_{23} 0

same idea, apply independent row of a_{22} onto a_{12}

$$\sim \begin{bmatrix} a_{11} & a_{12} & 0 \\ \boxed{0 \quad a_{22} \quad 0} \\ 0 & 0 & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

- tip: avoid fractions whenever possible, like when setting up Q7 or row reducing at #3 → #4

⑧ Other types of results from Augmented Matrix ($Ax = b$)

(A) when there are more columns than rows

eg. $\left[\begin{array}{ccccc|c} 1 & 2 & -5 & 6 & 14 & 15 \\ 2 & 4 & -9 & 12 & 28 & 41 \\ -1 & -2 & 5 & -5 & -12 & -15 \end{array}\right]$

$\sim \left[\begin{array}{ccccc|c} \boxed{1} & 2 & \boxed{0} & \boxed{0} & 2 & 70 \\ 0 & 0 & 1 & 0 & 0 & 11 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$

let $r, s \in \mathbb{R}$

$X_1 = -2X_2 - 2X_5 + 70$ $\qquad$ $X_1 = -2r - 2s + 70$

X_2 = free variable (used in X_1) $\qquad$ $X_2 = r$

$X_3 = 11$ $\qquad \Rightarrow$ $X_3 = 11$

$X_4 = -2X_5$ $\qquad$ $X_4 = -2s$

X_5 = free variable (used in X_1, X_4) $\qquad$ $X_5 = s$

- Columns X_1, X_3, X_4 are independent columns
- columns X_2, X_5 are dependent columns
- we can plug in any value for X_2, X_5 (r, s) and the matrix still holds

$\therefore$ infinite solutions

(B) more rows than columns

eg. $\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$ $\quad X_1 = 7$, $\quad X_2 = 5$

$\therefore$ one solution

- more rows than columns does not mean anything (inconclusive)

(C) Inconsistent

eg $\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{array}\right]$ $\quad 0 \neq 3$

$\therefore$ no solution