$z + x^2 = 1$

① Sketching Surfaces

$z = x^2 + y^2$

- find all the two dimensional coordinate graphs
  ie. xz axis, yz axis, xy axis

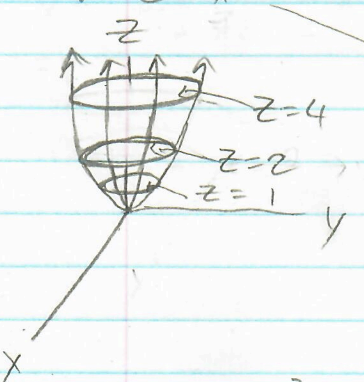
let $y=0$ | let $x=0$ | let z be a level curve

A: $z = x^2$ | B: $z = y^2$ | $x^2 + y^2 = z$

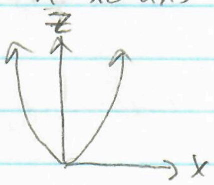
$z = 16 = x^2 + y^2$

$z = 4 = x^2 + y^2$

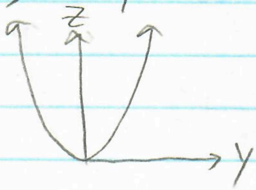
$z = 1 = x^2 + y^2$



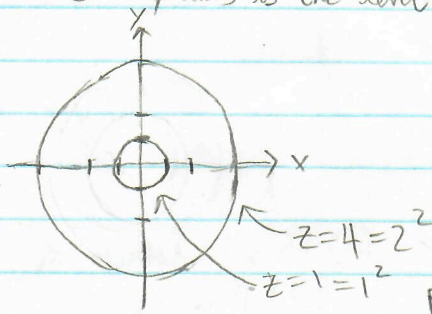
A: xz axis



B: yz axis

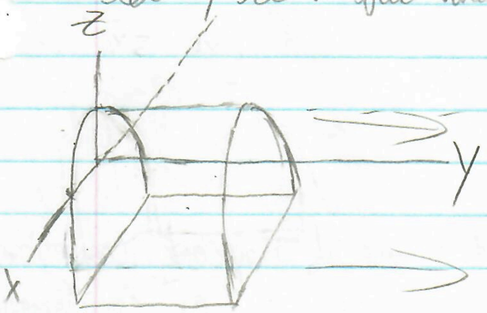


C: xy axis is the level curve

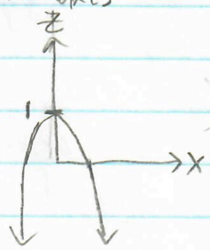


$z = 1 - x^2$

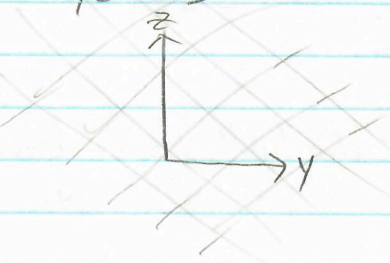
let y be a free variable, such that $y \in \mathbb{R}$, ∴ our shape is centered at y



xz-axis



yz-axis



② level curves

- the values of z from a bird's eye point of view

$f(x,y) = \sqrt{x^2 - 4y + 8}$, find the func $y = h(x)$ that gives the level curve of f passing through $(2,1)$

$$f(2,1) = \sqrt{(2)^2 - 4(1) + 8} = \sqrt{4 - 4 + 8} = \sqrt{8}$$

$$\Rightarrow \sqrt{8} = \sqrt{x^2 - 4y + 8}$$

$$8 = x^2 - 4y + 8$$

$$0 = x^2 - 4y$$

$$4y = x^2$$

$$y = \frac{x^2}{4}$$

③ Domain of a multivariable calc

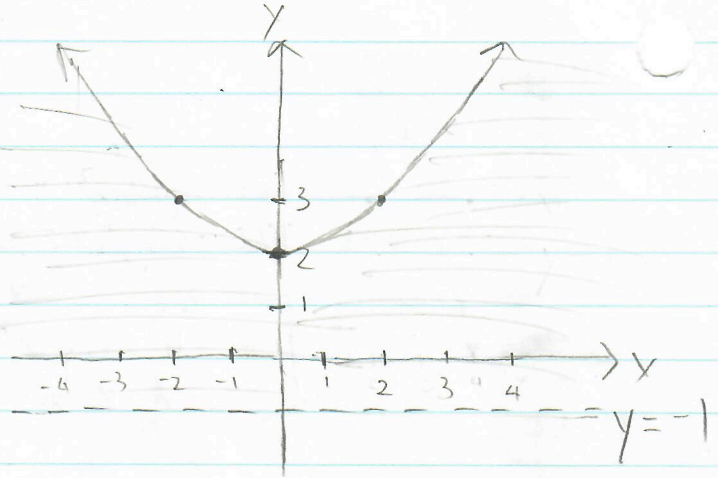
$f(x,y) = \sqrt{x^2-4y+8} + \ln(y+1)$

$x^2-4y+8 \geq 0$ $\quad$ $y > -1$

$-4y \geq -x^2-8$

$4y \leq x^2+8$

$y \leq \frac{x^2}{4}+2$



④ Partial Derivatives

$Z = f(x,y) = 2x^2 - 320x + 60\sqrt{y} + 4xy + 123$

- take the derivative with respect to either x, or y
- if we choose x, then assume y is a constant

$\frac{\partial z}{\partial x} = 4x - 320 + 4y$ $\quad$ $\frac{\partial z}{\partial y} = 30y^{1/2} + 4x$

⑤ Multivariable Calc for competitive & complementary goods

- given joint price functions for product A & B determine if they are competitive or complementary

Demand of A: $q_A = -5P_A + 20P_B$

Demand of B: $q_B = -3P_A^2(P_B^{1/2})$

$\frac{\partial q_A}{\partial P_A} = -5$, $\frac{\partial q_A}{\partial P_B} = 20$, $\frac{\partial q_B}{\partial P_A} = -6P_AP_B^{1/2}$ $\quad$ $\frac{\partial q_B}{\partial P_B} = -\frac{3}{2}P_A^2P_B^{-1/2}$

$\frac{\partial q_A}{\partial P_A}, \frac{\partial q_B}{\partial P_B}$: these are usually negative (ie. $\frac{\partial q_A}{\partial P_A} = -5 < 0$), why? think about a normal economics demand curve for a single product

Competitive: $\frac{\partial q_A}{\partial P_B} > 0$ & $\frac{\partial q_B}{\partial P_A} > 0$, b/c if Price of B ↑ then Demand of A ↑ the same logic applies to demand of B with respect to price of A

Complementary: $\frac{\partial q_A}{\partial P_B} < 0$ & $\frac{\partial q_B}{\partial P_A} < 0$, by similar logic

note, for this question, $\frac{\partial q_B}{\partial P_A} < 0$ but $\frac{\partial q_A}{\partial P_B} > 0$ $\therefore$ not competitive nor complementary

⑥ Mixed Partials

$z = x^2y + y^2$

$z_x = 2xy$

$z_y = x^2 + 2y$

$z_{xx} = 2y$

$z_{xy} = 2x$

$z_{yx} = 2x$

$z_{yy} = 2$

$z_{xy} = z_{yx} \quad \Big| \quad \dfrac{\partial z}{\partial x \partial y} = \dfrac{\partial z}{\partial y \partial x}$

⑦ Chain Rule

$W = 2x^2(y+5)^{1/2} \qquad x = 2r^3 + 4s^2 \qquad y = (r+6)^{2/3}s$

use chain rule to evaluate partial derivative $\frac{\partial w}{\partial r}$ when $s=1$, $r=2$

$W = f(x,y)\ ,\quad x = g(r,s)\ ,\quad y = h(r,s)$

$\hookrightarrow W = f(g(r,s), h(r,s))$

$$\frac{\partial w}{\partial(r,s)} = \left[\frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial r} + \frac{\partial x}{\partial s}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial r} + \frac{\partial y}{\partial s}\right)\right]$$

$$= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

chain rule in 1-D

$[f(g(x))]' = f'(g(x)) \cdot g'(x)$

$= \dfrac{df}{dg} \cdot \dfrac{dg}{dx}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$= W_x X_r + W_y Y_r$

$= (240)(24) + \left(\frac{400}{3}\right)\left(\frac{1}{3}\right)$

$= 5760 + \frac{400}{9}$

$W_x = 4x(y+5)^{1/2} \qquad W_y = x^2(y+5)^{-1/2}$

$X_r = 6r^2 \qquad Y_r = \frac{2}{3}(r+6)^{-1/3}s$

$X_r|_{(r,s)=(2,1)} = 6(2)^2 = 24 \qquad Y_r|_{(r,s)=(2,1)} = \frac{2}{3}(2+6)^{-1/3}(1)$

$= \frac{2}{3}(8)^{-1/3}(1)$

$= \frac{2}{3}(2)^{-1}(1)$

$= \frac{1}{3}$

$X = 2(2)^3 + 4(1)^2 = 20$

$Y = (2+6)^{2/3}(1)$

$= 8^{2/3} = 2^2 = 4$

$12 \times 20 = 240$

$W_x|_{(x,y)=(20,4)} = 4(20)(9)^{1/2} = 4(20)(3) = 240$

$W_Y|_{(x,y)=(20,4)} = (20)^2(4+5)^{-1/2} = 400(9)^{-1/2} = 400\left(\frac{1}{3}\right) = \frac{400}{3}$

(8) Implicit Differentiation

$xz^2 + y^2 z = 14$ defines z implicitly as a function of independent vars x & y

find $\frac{\partial z}{\partial y}$ such that $\frac{\partial z}{\partial y} = -\frac{2yz}{2xz + y^2}$

$\frac{\partial z}{\partial y} \Rightarrow$ Method 1

$$2xzz_y + (2yz + y^2 z_y) = 0$$
$$2xzz_y + y^2 z_y = -2yz$$
$$z_y(2xz + y^2) = -2yz$$
$$z_y = -\frac{2yz}{2xz + y^2}$$

- when differentiating z with respect to y and the equation has both z & y ie. $y^2 z$, perform the operation twice, once on y, once on z

$$y^2 z \xrightarrow[\frac{\partial z}{\partial y}]{\text{implicit diff}} \left(2yz + y^2 \frac{\partial z}{\partial y}\right)$$

Method 2

$$\frac{\partial F}{\partial z} = 2xz + y^2$$
$$\frac{\partial F}{\partial y} = 2yz$$

$$\frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \frac{-2yz}{2xz + y^2}$$

- find $\frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial y}$

then $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$