1) let $A = \begin{pmatrix} -1 & -3 \\ 5 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 \\ 7 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

a) find $2A + B^T - C$

$$2A + B^T - C = \begin{pmatrix} -2 & -6 \\ 10 & 8 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 14 & 12 \end{pmatrix}$$

b) evaluate $\det(2A - BA)$

let $D = 2A - BA$

$$= \begin{pmatrix} -2 & -6 \\ 10 & 8 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 5 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -6 \\ 10 & 8 \end{pmatrix} - \begin{pmatrix} -2+20 & -6+16 \\ -7+15 & -21+12 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -6 \\ 10 & 8 \end{pmatrix} - \begin{pmatrix} 18 & 10 \\ 8 & -9 \end{pmatrix}$$

$$= \begin{pmatrix} -20 & -16 \\ 2 & 17 \end{pmatrix}$$

$$\det(D) = (-20)(17) - (2)(-16)$$
$$= -340 + 32$$
$$= -308$$

c) find B^{-1}

$$B^{-1} = \frac{1}{\det(B)} \cdot \operatorname{adj}(B) \longleftarrow \operatorname{adj}(B) = [\operatorname{cofactor}(B_{ij})]^T$$

$$= \begin{bmatrix} 3 & -7 \\ -4 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & -4 \\ -7 & 2 \end{bmatrix}$$

$$= \frac{1}{6-28} \cdot \begin{bmatrix} 3 & -4 \\ -7 & 2 \end{bmatrix}$$

$$= \frac{1}{-22} \begin{bmatrix} 3 & -4 \\ -7 & 2 \end{bmatrix}$$

1d) Describe the matrices of the form $P = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ s.t. $PC = CP$ where x, y, z and w are real numbers. That is, describe what mathematical conditions must be true about x, y, z, w

Want: $PC = CP$

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$\begin{bmatrix} w & -w+x \\ y & -y+z \end{bmatrix} = \begin{bmatrix} w-y & x-z \\ y & z \end{bmatrix}$$

① $w = w - y \Rightarrow y = 0$

② $y = y \Rightarrow y = y$

③ $-w + x = x - z \Rightarrow w = z$

④ $-y + z = z \Rightarrow y = 0$

$\therefore y = 0,\ w = z,\ x \in \mathbb{R},\ z \in \mathbb{R}$

2a) let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

$1-2(-2)$

$3-2(-3)$
$= 3+6$

2a) use row reduction to find A^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$R_2: R_2 - 2R_1$, $R_3: R_3 - R_1$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right)$$

$R_1: R_1 - 2R_2$, $R_3: R_3 + 2R_2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$R_3: -R_3$

$R_1: R_1 - 9R_3$, $R_2: R_2 + 3R_3$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \quad \therefore A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

$-5 - 9(5) = -46$

$-2 - 9(-2) = -2 + 18$

$0 - 1.$

2b) Find all solutions to the equation $\det(xA) + 25x = 0$, where x is a variable

$$\det(xA) = (x)^3 \det(A)$$
$$= (x)^3\left[(-1)^4(1)\begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} + (-1)^6(8)\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}\right]$$
$$= (x)^3\left[(6-15) + (8)(5-4)\right]$$
$$= (x)^3\left[(-9) + 8\right]$$
$$= -x^3$$

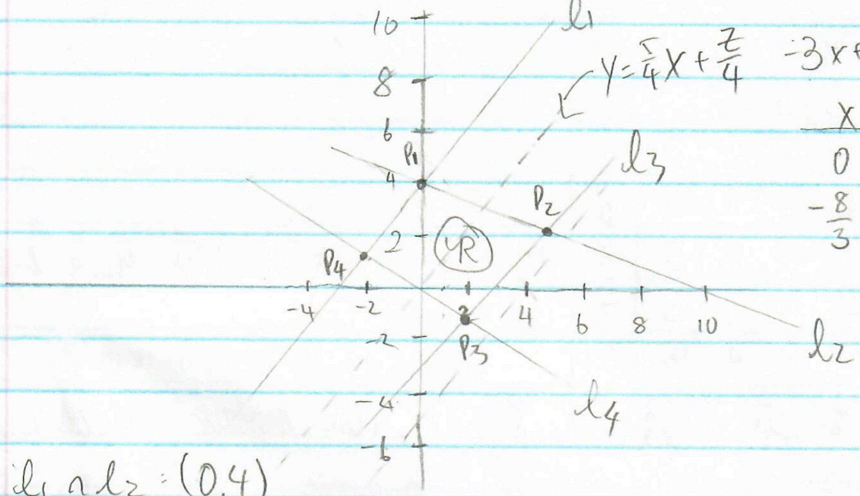
$$\det(xA) + 25x = 0$$
$$-x^3 + 25x = 0$$
$$-x(x^2 - 25) = 0$$
$$-x(x-5)(x+5)$$

$x = 0,\ x = 5,\ x = -5$

3a) Find the max & min values (and the pts where these occur) of the linear objective function $Z = -5x + 4y$ for the feasible region $\mathcal{R}$ that is defined by the following constraints

$l_1: -3x + 2y \le 8$ $l_2: 2x + 5y \le 20$ $l_3: x - y \le 3$ $l_4: x + 2y \ge 0$

to earn full pts, your solution must include a neat labeled diagram of $\mathcal{R}$



$-3x + 2y = 8$

x	y
0	4
$-\frac{8}{3}$	0

$2x + 5y = 20$

x	y
0	4
10	0

$x - y = 3$

x	y
3	0
0	-3

$Z = -5x + 4y$

x	y
4	-5
-4	5

$x + 2y = 0$

x	y
0	0
2	-1
-2	1

$4y = 5x + Z$

$y = \frac{5}{4}x + \frac{Z}{4}$

$P_1: l_1 \cap l_2: (0,4)$

$P_2: l_3 \cap l_2: \quad 2x + 5y = 20 \quad (5,2)$

$\ominus \; 2x - 2y = 6$

$7y = 14$

$y = 2$

$2x + 10 = 20$

$x = 5$

$P_3: l_3 \cap l_4: x - y = 3 \quad (4,-1)$

$\ominus \; x + 2y = 0$

$-3y = 3$

$y = -1$

$x - (-1) = 3$

$x = 2$

$P_4: l_1 \cap l_4: -3x + 2y = 8 \quad (-2,-1)$

$\ominus \; x + 2y = 0$

$-4x = 8$

$x = -2$

$-2 + 2y = 0$

$y = 1$

b/c the feasible region is not empty, bounded and standard, a max & min must exist

$Z(0,4) = 16$

$Z(5,2) = -25 + 8 = -17$

$Z(2,-1) = -10 - 4 = -14$

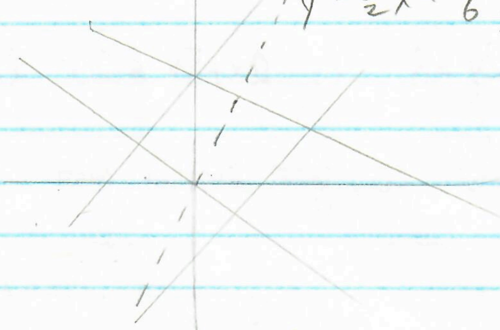
$Z(-2,1) = 10 + 4 = 14$

$\therefore$ max is $Z = 16$ @ pt $(0,4)$

min is $Z = -17$ @ pt $(5,2)$

3b) let ℓ represent the set of points in feasible region R in part (a) for which the new objective function $W = -9x + 6y$ has constant value equal to 0. Clearly show & label ℓ on diagram above

$y = \frac{3}{2}x + \frac{w}{6}, \; w = 0$



$W = -9x + 6y$

$\hookrightarrow 6y = 9x + W$

$y = \frac{9}{6}x + \frac{W}{6}$

let $W = 0$

$\hookrightarrow y = \frac{3}{2}x$

x	y
0	0
2	3

4) let $A = \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$ be the augmented matrix of a system

$18 - 4(3) = 18 - 12 = 6$

of four linear equ. in six variables $x_1, x_2 \ldots x_6$. Use matrix reduction to solve the linear system

$R_2: R_2 - 2R_1$, $R_4: R_4 - 2R_1$; $R_2 \to -R_2$; $R_3: R_3 - 5R_2$, $R_4: R_4 - 4R_2$; $R_3 \leftrightarrow R_4$, $R_3: \frac{1}{6}R_3$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$0 - 2(-2)$

$R_2: R_2 - 3R_3$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -3x_2 + 2x_3 - 2x_5$

$x_2 \in \mathbb{R}$ $\qquad x_2 = a \in \mathbb{R}$

$x_3 = -2x_4$ $\qquad$ let $x_4 = b \in \mathbb{R}$

$x_4 \in \mathbb{R}$ $\qquad x_5 = c \in \mathbb{R}$

$x_5 \in \mathbb{R}$

$x_6 = \frac{1}{3}$

$$\longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} + a\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$x_3 = -2b$

$2x_3 = -4b$

$Z = 16x + 12y$

$12y = -16x + Z$

$y = -\frac{16}{12}x + \frac{Z}{12}$

5) Products X, Y are purchased from a single supplier then sold in two cities A, B

- The same # of product X is sold in cities A, B, and same for Y
- The supplier charges \$8/unit of X and \$6/unit of Y

constraints: (i) • In city A, each unit of X sells for \$30, each unit of Y sells for \$10

- the total sales ~~\$~~ must be at least \$600

(ii) • In city B, each unit of X sells for \$10, each unit of Y sells for \$20

- the total sales must be at least \$500

(iii) • the sum total number of products X and Y sold in city A must be at least 40

Calculate how many units of X & Y should be sold in each city so that the total wholesale purchase cost to cover both cities is minimized

let X = # of products sold in cities A & B

Y = # of products sold in cities A & B

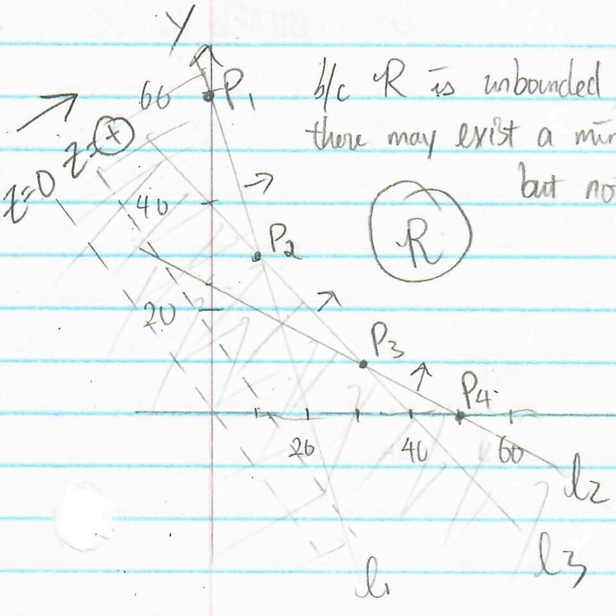
	A	B	min
X	\$30	\$10	
Y	\$10	\$20	
min	\$600	\$500	

$$\text{min}: Z = \underset{A}{(8x + 6y)} + \underset{B}{(8x + 6y)} = 16x + 12y$$

l_1 constraints: $30x + 10y \geq 600 \Rightarrow 3x + y \geq 60$

l_2 $\quad 10x + 20y \geq 500 \Rightarrow x + 2y \geq 50$

l_3 $\quad x + y \geq 40$, # of products X, Y sold in B can be anything, need to only satisfy # of products x, y sold in A is at least 40



b/c R is unbounded, non-empty, there may exist a min or max but not both

$l_1: 3x + y = 60$

x	y
20	0
0	60

$l_2: x + 2y = 50$

x	y
50	0
0	25

$l_3: x + y = 40$

x	y
40	0
0	40

$P_1: (0, 60) \quad Z(0,60) = 12(60) = 720$

$P_2: (10, 30) \quad Z(10, 30)$

$P_3: (30, 10) \quad = 16(10) + 12(30)$

$P_4: (50, 0) \quad = 160 + 360 = 520$

$l_3: x + y = 40$

$\ominus\ l_1: 3x + y = 60$

$-2x = -20$

$x = 10$

$y = 30$

$Z(30, 10)$

$= 30(10) + 10(30)$

$= 300 + 300 = 600$

$Z(50, 0) = 16(50) = 800$

$l_2: x + 2y = 50$

$\ominus\ l_3: x + y = 40$

$y = 10$

$x = 30$

∴ min exists $Z = 520$ @ pt $(10, 30)$

& isoprofit line ↓ as x ↑ $(y = -\frac{16}{12}x + \frac{Z}{12})$

6) let $A = \begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix}$. find all values of the real number t for which there exists a unique solution to the matrix equation $(A+tI)Y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$
use cramer's rule to find the unique sol. for these values of t

let $C = A + tI$, $C = \begin{pmatrix} -1+t & -3 \\ 2 & 4+t \end{pmatrix}$

Want: $CY = b$, $\begin{pmatrix} -1+t & -3 \\ 2 & 4+t \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ with unique solution

let D be $\det(C)$, let D_x be $\det(C_x)$, let D_y be $\det(C_y)$

$$\det(C) = \det\begin{pmatrix} -1+t & -3 \\ 2 & 4+t \end{pmatrix} = (-1+t)(4+t) - (2)(-3) = -4 - t + 4t + t^2 + 6 = t^2 + 3t + 2$$

$$\det(C_x) = \det\begin{pmatrix} 1 & -3 \\ 2 & 4+t \end{pmatrix} = 4+t-6 = t-2$$

$$\det(C_y) = \det\begin{pmatrix} -1+t & 1 \\ 2 & 2 \end{pmatrix} = -1+t-2 = t-3$$

$$x = \frac{\det(C_x)}{\det(C)} = \frac{t-2}{t^2+3t+2} \qquad y = \frac{\det(C_y)}{\det(C)} = \frac{t-3}{t^2+3t+2}$$

$\hookrightarrow t^2+3t+2 > 0$ or else no solution

$(t+2)(t+1) > 0$

$t \neq -2 \quad t \neq -1$

$\hookrightarrow t^2+3t+2 > 0$

$\hookrightarrow$ same constraints as x

$\therefore$ only unique sol when $t \neq -2, t \neq -1$

7) let $n \geq 3$, let $A = (a_{ij})$ be the $n \times n$ matrix s.t $a_{ij} = 1$ for all $i,j = 1, 2 \ldots n$

Prove $(I-A)^{-1} = I - \left(\frac{1}{n-1}\right)A$

$$
\begin{aligned}
(I-A)^{-1}(I-A) &= \left[I - \left(\tfrac{1}{n-1}\right)A\right](I-A) \\
&= I^2 - A - \left(\tfrac{1}{n-1}\right)A + \left(\tfrac{1}{n-1}\right)A^2 \\
&= I - A - \left(\tfrac{1}{n-1}\right)A + \left(\tfrac{1}{n-1}\right)nA \\
&= I + A\left(-1 - \left(\tfrac{1}{n-1}\right) + \left(\tfrac{n}{n-1}\right)\right) \\
&= I + A\left(\tfrac{-(n-1) - 1 + n}{n-1}\right) \\
&= I + A\left(\tfrac{-n+1-1+n}{n-1}\right) \\
&= I + A\left(\tfrac{0}{n-1}\right) \\
&= I
\end{aligned}
$$

let $C = I - A$,

THEN $C^{-1}C = I$, as proven

$\therefore$ since we satisfy the property,

it is shown that $(I-A)^{-1} = I - \left(\frac{1}{n-1}\right)A$

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & \\ \vdots & & & \\ 1 & & & 1 \end{bmatrix}$$

$$AA = \begin{bmatrix} 1 & 1 & & 1 \\ 1 & \ddots & & \\ \vdots & & & \\ 1 & & & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & \\ \vdots & & & \\ 1 & & & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+1+1+\cdots+1 & \cdots & 1+1+1\cdots+1 \\ 1+1+1+\cdots+1 & \ddots & \\ \vdots & & \\ 1+1+1\cdots+1 & & 1+1\cdots+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} n & n & \cdots & n \\ n & \ddots & & \\ \vdots & & & \\ n & & & n \end{bmatrix}$$

$$A^2 = n\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & \\ \vdots & & & \\ 1 & & & 1 \end{bmatrix}$$

$$A^2 = nA$$