0) Relationship between extremas, boundness & emptyness

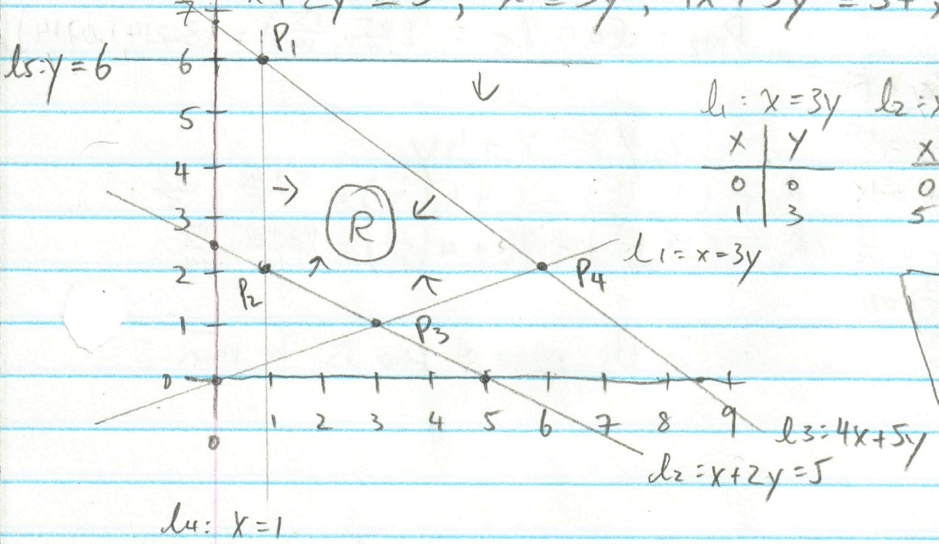
Theorem of Linear Programming

- if bounded and nonempty then must have max and min
- if unbounded and nonempty then may have max or min, **but not both**

- bounded: the region is contained
- non-empty: at least one point exists

1a) Optimize $Z = x + 4y$ subject to the constraints:

$x + 2y \geq 5$, $x \leq 3y$, $4x + 5y \leq 34$, $x \geq 1$, $y \leq 6$



$l_1: x=3y$

x	y
0	0
1	3

$l_2: x+2y=5$

x	y
0	$\frac{5}{2}$
5	0

$l_3: 4x+5y=34$

x	y
0	$\frac{34}{5}$ (6.8)
(8.5) $\frac{34}{4}$	0

$l_4: x=1$ $l_5: y=6$

$\boxed{\text{by Thm of linear programming, the area R is non-empty \& bounded} \therefore \text{we must have max and min}}$

Graphical method: $4 = x+4y$
$8 = x+4y$

z=8

z=4

as x & y ↑ objective func ↑

$\therefore$ max is P_1 & min is P_3

Extrema method:

$P_1: (1,6)$

$P_2: l_2 \cap l_4: (1,2)$

$P_3: l_2 \cap l_1: (3,1)$

$P_4: l_3 \cap l_1: (6,2)$

$Z(x,y) = x+4y$

$Z(1,6) = 1+4(6) = 25 \Rightarrow$ max

$Z(1,2) = 1+4(2) = 9$

$Z(3,1) = 3+4(1) = 7 \Rightarrow$ min

$Z(6,2) = 6+4(2) = 14$

$$\begin{aligned} &x+2y=5 \\ \ominus\ &x = 1 \\ \hline &2y = 4 \\ &y=2 \\ &x+2(2)=5 \\ &x=1 \end{aligned}$$

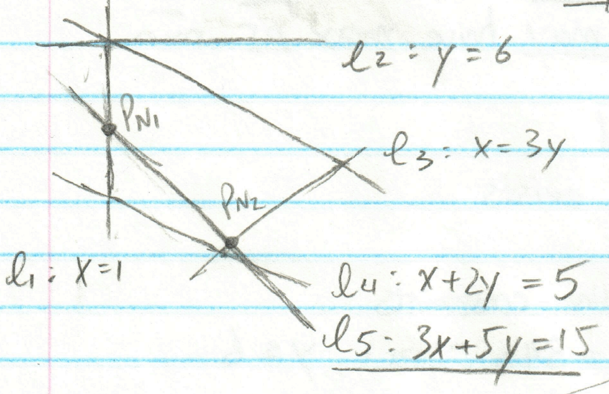
$$\begin{aligned} &x+2y=5 \\ \ominus\ &x-3y=0 \\ \hline &0+5y=5 \\ &y=1 \\ &x+2(1)=5 \\ &x=3 \end{aligned}$$

$$\begin{aligned} &4x+5y=34 \\ ① - 4⑤\ &x-3y=0 \\ \hline &0+17y=34 \\ &y=2 \\ &x-3(2)=0 \\ &x=6 \end{aligned}$$

1b) Adding additional constraint

$3x + 5y = 15$

notice the constraint is an <u>equality</u>, not <u>inequality</u>



since this new constraint strictly bounds the area to the line, the new corner points occur when l_5 intersects with another line

$P_{N1}: l_1 \cap l_5 = \left(1, \frac{12}{5}\right)$

$P_{N2}: l_4 \cap l_5 = \left\{\frac{45}{14}, \frac{15}{14}\right\} = (3.214, 1.0714)$

$$\begin{aligned} x &= 1 \\ 3x + 5y &= 15 \\ \hline 3① - ② \quad 0 - 5y &= -12 \\ y &= \frac{12}{5} \\ 3x + 5\left(\frac{12}{5}\right) &= 15 \\ 3x + 12 &= 15 \\ x &= 1 \end{aligned}$$

$$\begin{aligned} x - 3y &= 0 \\ 3x + 5y &= 15 \\ \hline 3① - ② \quad 0 - 14y &= -15 \\ y &= \frac{15}{14} \\ x - 3\left(\frac{15}{14}\right) &= 0 \\ x &= \frac{45}{14} \end{aligned}$$

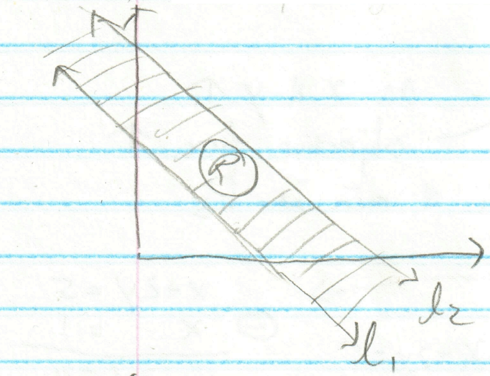
$Z(x, y) = x + 4y$

$P_{N1}: Z\left(1, \frac{12}{5}\right) = 1 + 4\left(\frac{12}{5}\right) = \frac{5 + 48}{5} = \frac{53}{5}$

$P_{N2}: Z\left(\frac{45}{14}, \frac{15}{14}\right) = \frac{45}{14} + 4\left(\frac{15}{14}\right) = \frac{45 + 60}{14} = \frac{105}{14}$

$\therefore P_{N1}$ is the max & P_{N2} is the min

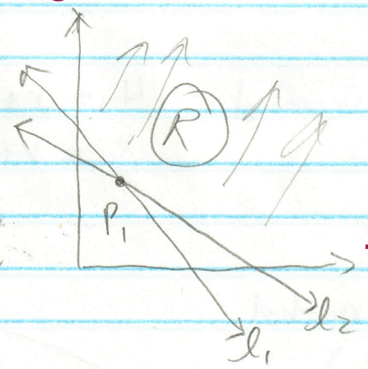
2) **Cases of unbounded linear programming**



by thm of LP, it is unbounded and non-empty

Z = 6x + 3y

- has no max or min

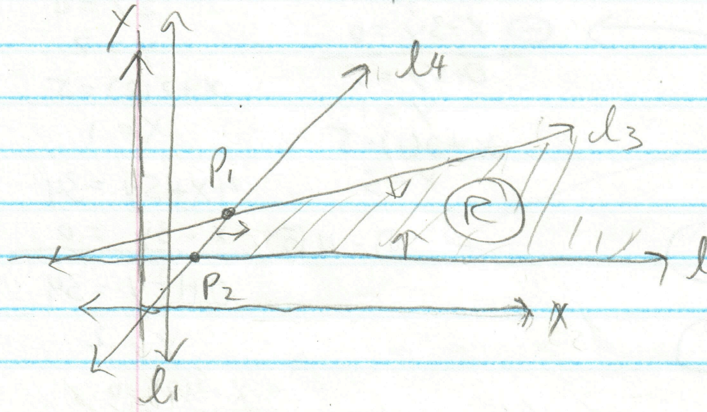


by thm of LP it is unbounded, and non-empty

Z = 4x + 4y

- has a min but no max

based on objective function



by thm of LP, it is <u>unbounded</u> and non-empty

- notice here we have 2 corner points however, by LP, we can only have a min OR a max, not both **(unbounded)**
- depending on the objective function **Z = +/- ax +/- by** either P_1 or P_2 is the min, or max

eg. if P1 is the min, then P2 is nothing

different objective functions can demonstrate drastically different general behaviour

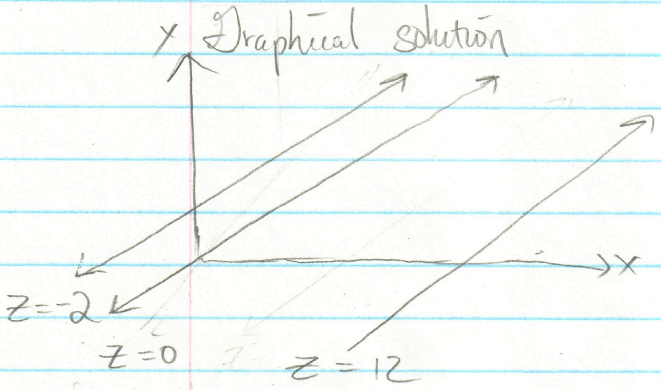
$Z = 4x + 4y$	$Z = -4x - 4y$	**the two functions to the left have**
$4 = 4x + 4y$	$4 = -4x - 4y$	**complete opposite**
$12 = 4x + 4y$	$12 = -4x - 4y$	**increasing / decreasing behaviour**

2) Continued.

- based on last example, how do we know if min or max?
  - ↳ must know the objective function's <u>general</u> behaviour

eg. if $Z = 3x - 4y$

Graphical solution



- from the general behaviour, we observe
  - as $Z \to -\infty$, bounded by $P_1 (l_3 \cap l_4)$ eventually
  - as $Z \to \infty$, not bounded by any corner pts

Algebraic Solution (not really)

$$\lim_{\substack{x \to \infty \\ y=0}} Z(x,0) = \lim_{x \to \infty} 3x - 4(0) = \lim_{x \to \infty} 3x = \infty \text{ (no line bounds as } x \to \infty \text{ on the x-axis)}$$

$$\lim_{\substack{y \to \infty \\ x=0}} Z(0,y) = \lim_{y \to \infty} 3(0) - 4y = \lim_{y \to \infty} 4y \leq y \text{ value of } l_3$$

3) Small mining company operates 2 mines with following conditions

- saddle mine costs \$100K/day, yields 5kg of gold & 30kg of silver
- horseshoe mine costs \$150K/day, yields 7.5kg of gold & 10kg of silver
- must produce <u>at least</u> 65kg of gold & 180kg of silver

	gold	silver	cost
Saddle mine (x)	5	30	100K
horse shoe mine (y)	7.5	10	150K
min	65	180	N/A
max	N/A	N/A	N/A

Objective func / cost func:

$Z = 100K\,x + 150K\,y$

constraints: $5x + 7.5y \geq 65$

$30x + 10y \geq 180$

non-neg constraints: $x \geq 0$

$y \geq 0$