BDA session 1 approximation derivations

October 2022

1 Approximation of $(1+a)^b \approx e^{ab}$

First we will look at the following limit:

$$y = \lim_{x \to \inf} (1 + \frac{1}{x})^x \tag{1}$$

$$\ln y = \lim_{x \to \inf} \ln(1 + \frac{1}{x})^x \tag{2}$$

$$\ln y = \lim_{x \to \inf} x \ln(1 + \frac{1}{x}) \tag{3}$$

$$\ln y = \lim_{x \to \inf} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} \tag{4}$$

If we substitute x directly we get $\frac{0}{0}$ (indeterminate form), therefore, we can apply L'Hopital's rule.

$$\ln y = \lim_{x \to \inf} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} = \lim_{x \to \inf} \frac{\frac{\partial}{\partial x} \ln(1 + \frac{1}{x})}{\frac{\partial}{\partial x}(\frac{1}{x})} = \lim_{x \to \inf} \frac{\left(\frac{1}{1+(\frac{1}{x})}\right)(-x^{-2})}{(-x^{-2})} \tag{5}$$

$$\ln y = \lim_{x \to \inf} \left(\frac{1}{1 + \frac{1}{x}}\right) \tag{6}$$

Substitute x for ∞:

$$\ln y = (\frac{1}{1+0}) \tag{7}$$

$$\ln y = 1 \tag{8}$$

$$y = e^1 = e \tag{9}$$

Therefore, we have that:

$$\lim_{x \to \inf} (1 + \frac{1}{x})^x = e$$

Now we go back to our original equation:

$$(1+a)^b = (1+a)^{b\frac{a}{a}} = \left((1+a)^{\frac{1}{a}}\right)^{ba} \tag{10}$$

We can substitute $a = \frac{1}{x}$:

$$\left((1+a)^{\frac{1}{a}}\right)^{ba} = \left((1 + \frac{1}{x})^x\right)^{ba} \tag{11}$$

And using the previously found limit we can say that:

$$\left((1 + \frac{1}{x})^x\right)^{ba} \approx e^{ba} \tag{12}$$

The approximation is made under the assumption that $a \to \infty$ and subsequently that $x \to 0$, therefore, it is only valid if x is indeed close to 0.

2 Derivation of Taylor expansion for e^x

The general formulation of a Taylor polynomial of order k, generated by f(x) at $x = a$ is given by:

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... + \frac{f^k(a)}{k!}(x-a)^k$$

To get the approximation we set $f(x) = e^x$ and $a = 0$. We also take advantage of the fact that $\frac{\partial}{\partial x}e^x = e^x$:

$$e^x = e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + ... + \frac{e^a}{k!}(x-a)^k$$

We substitute $a = 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$