Singular Value Decomposition (SVD) for Image Processing

AM 205

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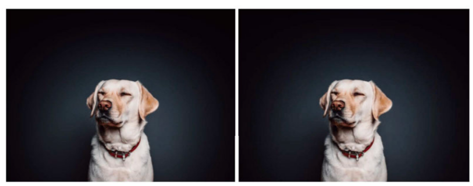
Motivation

Singular Value Decomposition (SVD) has been applied in a wide range of fields:

- Computer vision: image compression and denoising



Original (28KB) Lossy Compression (14KB, 50%)

- Computer vision: steganography

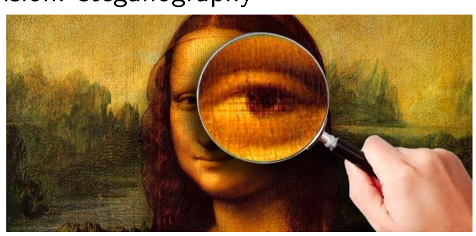
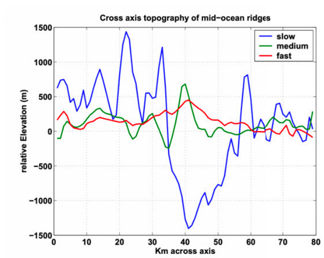


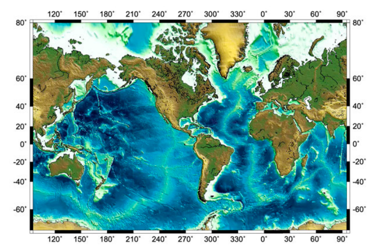
image credits: google image

Motivation

- Scientific computing: 3D reconstruction



(a) Topography data



(b) 3D reconstruction

image credits: http://www.columbia.edu/

Motivation

- Machine learning: feature extraction

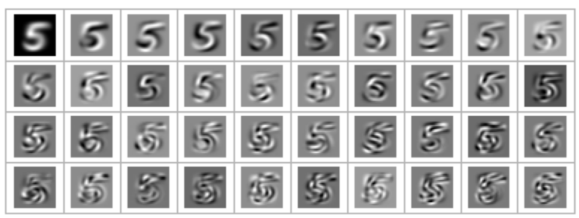


image credits: https://mathematicaforprediction.wordpress.com/

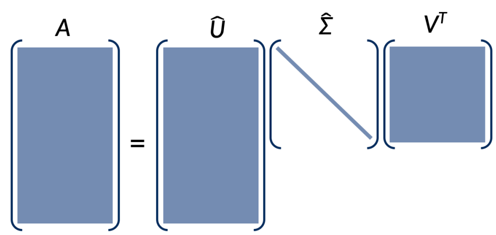
Singular Value Decomposition

The SVD of a matrix $A \in \mathbb{R}^{m \times n}$ is a factorization $A = \hat{U}\hat{\Sigma}V^T$ where

- $\hat{\Sigma} \in \mathbb{R}^{n \times n}$ is a diagonal matrix of singular values sorted in descending order, $\sigma_1 \geq \sigma_2 \geq ...\sigma_n$
- $\hat{U} \in \mathbb{R}^{m \times n}$ has orthonormal columns - left singular vectors
- $V \in \mathbb{R}^{n \times n}$ has orthonormal columns - right singular vectors

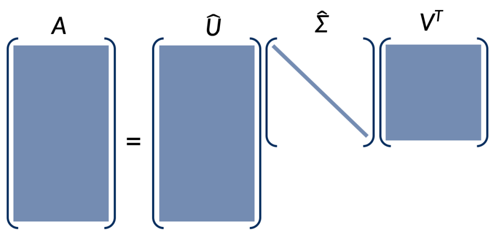
Singular Value Decomposition

In applications, we will often think of A as a tall, thin matrix, representing relatively few n samples in a high m-dimensional space, though the SVD is defined for *any* matrix. For $m > n$, the columns of $\hat{U}$ can be padded with $m - n$ arbitrary orthonormal vectors to obtain a full $m \times m$ matrix U, and $\hat{\Sigma}$ padded with rows of zeros to null the contribution of these columns.

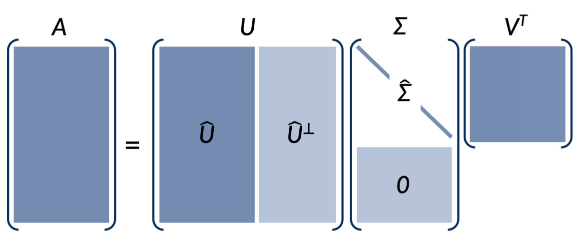


Singular Value Decomposition

Reduced SVD of A:



Full SVD of A:



Singular Value Decomposition

In Python:

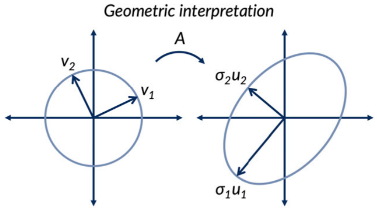
```python
import numpy as np
A = np.random.rand(20, 5)
U, s, Vt = np.linalg.svd(A) # full SVD
U, s, Vt = np.linalg.svd(A, full_matrices=False) # reduced SVD
```

In MATLAB:

```matlab
A = randn(20,5);
[U,S,V] = svd(A); % full SVD
[U,S,V] = svd(A,'econ'); % reduced SVD
```

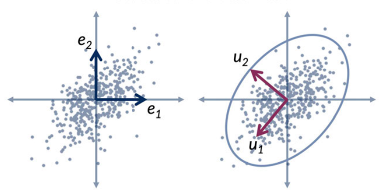
SVD viewed under different lenses

Geometric interpretation



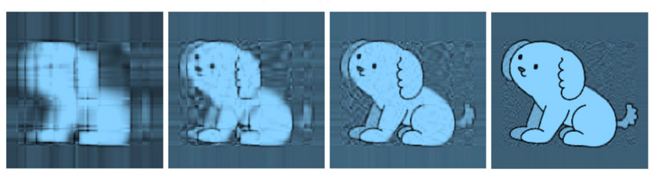
U performs a rotation
Σ performs a stretching

Statistical characterization



u_i are directions of highest variance
σ_i^2/N give the data variance

Optimal approximation



Compress an image: $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$

Low-Rank Approximation

The SVD provides a natural hierarchy of approximations we can make to A, expressed as a sum of rank-one matrices. If

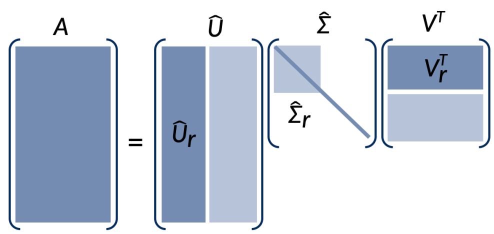
$$A = \sum_{j=1}^{n} \sigma_j u_j v_j^T,$$

where each $u_j v_j^T$ is a rank-one matrix whose columns are all scalar multiples of each other, then a rank-r approximation A_r of A is

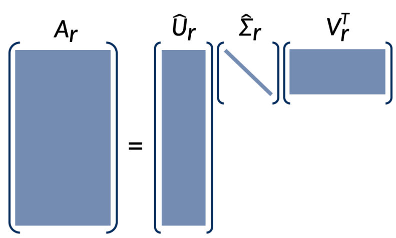
$$A_r = \sum_{j=1}^{r} \sigma_j u_j v_j^T.$$

Low-Rank Approximation

Reduced SVD of A:



Rank-r approximation of A:



Low-Rank Approximation

Note that by the orthogonality of the columns of u,

$$u_k^T A = u_k^T \left(\sum_{j=1}^{n} \sigma_j u_j v_j^T \right) = \sigma_k v_k^T,$$

so for a particular data point a_i that is the i^{th} column of A,

$$u_k^T a_i = \sigma_k v_{ki}^T \rightarrow (u_k^T a_i) u_k = \sigma_k u_k v_{ki}^T$$

is the projection of a_i onto the k^{th} left singular vector u_k. So another way to think about the low-rank approximation is that it is a sum of projections onto a limited number of left singular vectors.

Image Compression

The low-rank approximation gives us a useful algorithm for compressing data and images.

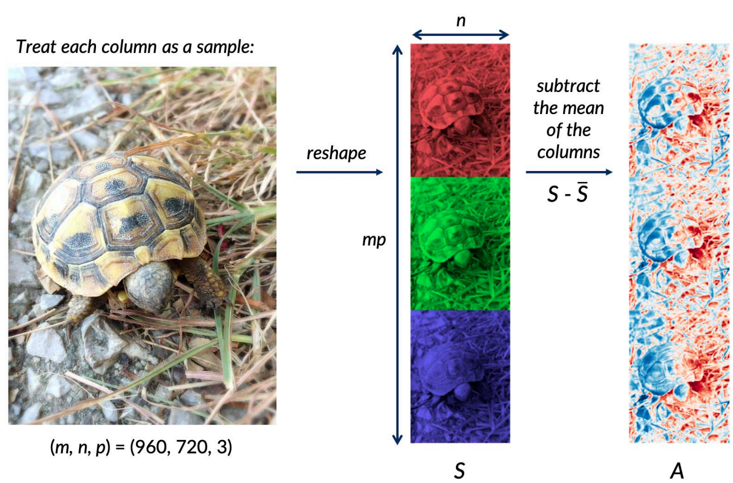


Image Compression

We factor the centered $A = S - \bar{S}$, where $\bar{S} = \frac{1}{n}\sum_{j=1}^{n} S_j$.
Then, $S_r = \bar{S} + \sum_{j=1}^{r} \sigma_j u_j v_j^T$.

r = 10 *r = 50* *r = 100*

Image Compression

The following are two possible metrics we can use to quantify the fraction of our image reconstructed by a rank-r approximation, as well as the fraction of storage space required.

- Explained variance ratio:

$$p(r) = \frac{\sum_{j=1}^{r} \sigma_j^2}{\sum_{j=1}^{n} \sigma_j^2}$$

- Compression ratio:

$$c(r) = \frac{r(\overbrace{1 + 3m + n}^{\sigma, u, v^T}) + \overbrace{3m}^{\bar{S}}}{\underbrace{3mn}_{S}}$$

Image Denoising

Retaining a low-rank approximation of an image can also be a technique for denoising.

Consider the set of $N = 22$ stamps below, which all have similar features, but are obscured by different black mail markings.

Image Denoising

In this example, we consider each $m \times n \times p$ color image as a single sample of length mnp, where $p = 3$, and assemble a matrix $S \in \mathbb{R}^{3mn \times N}$.

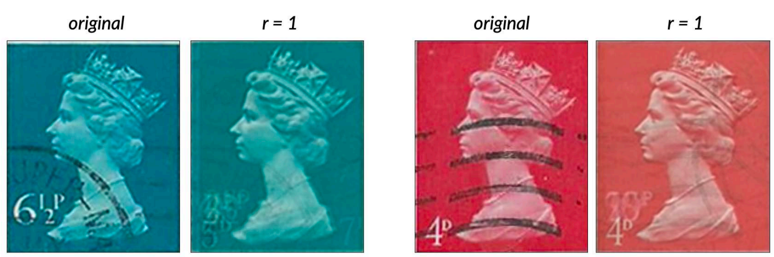
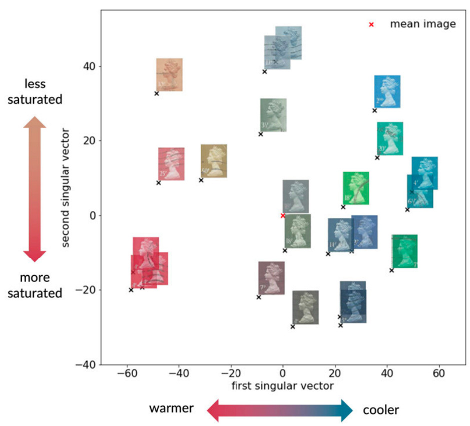


Image Denoising

The singular vectors can be used to construct a lower-dimensional space that captures the most significant features of the data. This forms the basis of PCA and can be used to uncover features such as clustering in an unsupervised way.



Principal Component Analysis

The left singular vectors $u_1, u_2, \ldots, u_n$ form a rotated, orthonormal basis for the m-dimensional space occupied by the columns of A, $a_1, a_2, \ldots, a_n$ (data points).

This new basis is oriented such that u_1 points in the direction along which the data has the largest variance, u_2 points along the direction of next-largest variance orthogonal to u_1, and so on. How?

Consider the covariance matrix $C = \frac{1}{n}AA^T \in \mathbb{R}^{m \times m}$, where

- the diagonals $c_{ii} = \frac{1}{n}\sum_{j=1}^{n} a_{ij}^2$ give the variance of the data along the i^{th} axis
- the off-diagonals $c_{ik} = \frac{1}{n}\sum_{j=1}^{n} a_{ij}a_{jk}$ give the covariance along the i^{th} and k^{th} axes.

Principal Component Analysis

If A $= \hat{U}\hat{\Sigma}V^T$, then

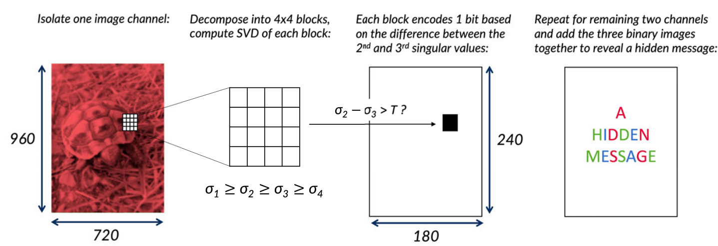
$$C = \frac{1}{n}(\hat{U}\hat{\Sigma}V^T)(\hat{U}\hat{\Sigma}V^T)^T = \frac{1}{n}\hat{U}\hat{\Sigma}(V^TV)\hat{\Sigma}^T\hat{U}^T$$
$$= \frac{1}{n}\hat{U}\hat{\Sigma}\hat{\Sigma}^T\hat{U}^T = \hat{U}\left(\frac{\hat{\Sigma}^2}{n}\right)\hat{U}^T$$

Since $Cu_j = \frac{1}{n}\sigma_j^2 u_j$ for any u_j, the columns of $\hat{U}$ are the eigenvectors and the diagonals of Σ^2/n the eigenvalues of the covariance matrix.

The eigenvalues once again represent the variance of the data, now along the rotated axes $u_1, \ldots u_n$. These axes are called principal components in PCA, but they are the same as left singular vectors!

Steganography

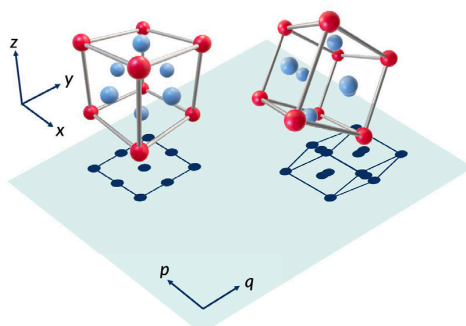
Steganography is the art of concealing hidden messages within non-secret data. The first exercise will feature decoding a hidden message encoded in the singular values of an image.



Key idea: Since the information contained in later singular vectors, which have correspondingly smaller singular values, is less important, this part of the decomposition can be manipulated to encode hidden messages in plain sight!

3D Reconstruction

SVD can also be used to perform 3D reconstruction from a sequence of 2D projections[1].



Here we will consider a rotating object characterized by N control points on its surface.

[1] Reference: Muller, N. *et al.* (2004). Singular value decomposition, eigenfaces, and 3D reconstructions. SIAM review, 46(3), 518-545.

3D Reconstruction

The object's state in 3D can be expressed as an object matrix:

$$O \in \mathbb{R}^{3\times N} = \begin{bmatrix} x^{(1)} & x^{(2)} & \cdots & x^{(N)} \\ y^{(1)} & y^{(2)} & \cdots & y^{(N)} \\ z^{(1)} & z^{(2)} & \cdots & z^{(N)} \end{bmatrix}$$

Its tracked motion is captured by the product of a time-varying rotation R_t and the orthographic projection P_z in the motion matrix:

$$M_t \in \mathbb{R}^{2\times 3} = P_z R_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} R_t$$

The coordinates of the control points in the 2D projected space are thereby captured by the measurement matrix:

$$A_t \in \mathbb{R}^{2\times N} = M_t O = \begin{bmatrix} q_t^{(1)} & q_t^{(2)} & \cdots & q_t^{(N)} \\ p_t^{(1)} & p_t^{(2)} & \cdots & p_t^{(N)} \end{bmatrix}$$

3D Reconstruction

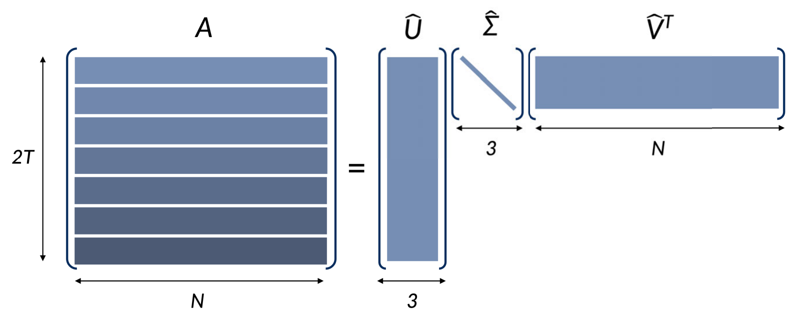
Measurements across T rotations can be stacked to form a combined measurement matrix:

$$A \in \mathbb{R}^{2T \times N} = MO = \begin{bmatrix} q_0^{(1)} & q_0^{(2)} & \cdots & q_0^{(N)} \\ p_0^{(1)} & p_0^{(2)} & \cdots & p_0^{(N)} \\ q_1^{(1)} & q_1^{(2)} & \cdots & q_1^{(N)} \\ p_1^{(1)} & p_1^{(2)} & \cdots & p_1^{(N)} \\ & & \vdots & \\ q_{T-1}^{(1)} & q_{T-1}^{(2)} & \cdots & q_{T-1}^{(N)} \\ p_{T-1}^{(1)} & p_{T-1}^{(2)} & \cdots & p_{T-1}^{(N)} \end{bmatrix}$$

where $M \in \mathbb{R}^{2T \times 3}$ is a stacked series of motion matrices.

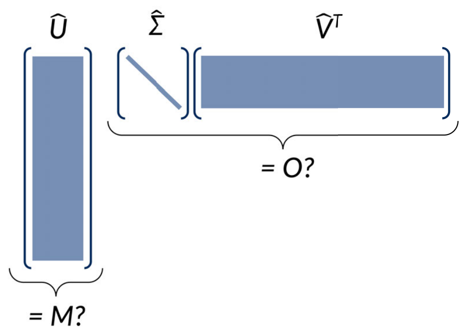
3D Reconstruction

- Our objective is to deduce the object matrix O given only A. Since $\text{rank}(O) = 3$, we expect for a general 3D rotation that $\text{rank}(A) = 3$.
- This means that we can represent A by a truncated SVD $A = \hat{U}\hat{\Sigma}\hat{V}^T$, where $\hat{U} \in \mathbb{R}^{2T\times 3}$, $\hat{\Sigma} \in \mathbb{R}^{3\times 3}$, and $\hat{V}^T \in \mathbb{R}^{3\times N}$.



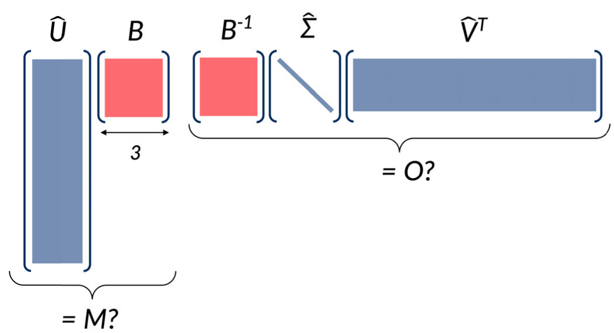
3D Reconstruction

- How can we obtain the factorization $A = MO$? Naively, we can propose $M = \hat{U}$, $O = \hat{\Sigma}\hat{V}^T$:



3D Reconstruction

- However, this is not a unique factorization; we could just as easily introduce $M = \hat{U}B$, $O = B^{-1}\hat{\Sigma}\hat{V}^T$, to obtain $MO = \hat{U}BB^{-1}\hat{\Sigma}\hat{V}^T = \hat{U}\hat{\Sigma}\hat{V}^T$ for some matrix $B \in \mathbb{R}^{3\times 3}$.



- We'd like to use this additional freedom with B to impose constraints in our factorization.

3D Reconstruction

- Recall that each pair of rows in M is given by

$$M_t = P_z R_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} R_t.$$

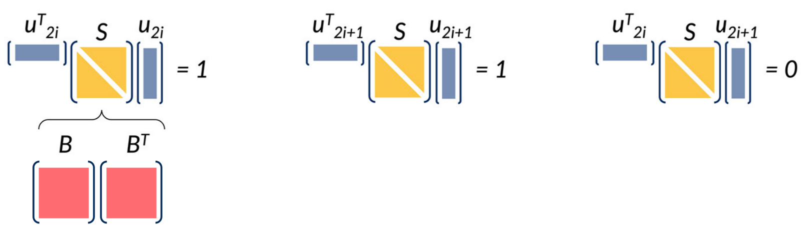
- The two rows of P_z pick out the first two rows of R_t, which is orthogonal as required for rotation matrices. Thus, we expect that the pairs of rows in $M = \hat{U}B$ to be orthogonal. If m_i and u_i denote the i^{th} row of M and $\hat{U}$, respectively,

$$\begin{aligned} m_{2i}^T m_{2i} &= (u_{2i}^T B)(B^T u_{2i}) = 1 \\ m_{2i+1}^T m_{2i+1} &= (u_{2i+1}^T B)(B^T u_{2i+1}) = 1 \\ m_{2i}^T m_{2i+1} &= (u_{2i}^T B)(B^T u_{2i+1}) = 0 \end{aligned}$$

for $i = 0, 1, ... T - 1$.

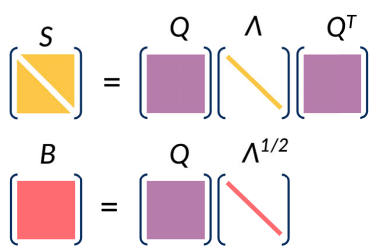
3D Reconstruction

- The 6 unknowns in the symmetric matrix $S = BB^T$ can be solved for by setting up a least squares problem from these orthogonality relations.



3D Reconstruction

- If $S = Q \Lambda Q^T$ is the eigendecomposition of S with orthogonal eigenvectors in Q and diagonal matrix of eigenvalues Λ, then B can be determined as $B = Q \Lambda^{1/2}$.



3D Reconstruction

- However, B is still not unique! Multiplication by an arbitrary rotation such as $B = Q\Lambda^{1/2}R$ still results in $BB^T = (Q\Lambda^{1/2}R)(R^T\Lambda^{1/2}Q^T) = Q\Lambda Q^T = S$ by the orthonormality of columns of a rotation matrix.
- It's acceptable to simply take $R = I$, but we acknowledge that our final solution for the object matrix O will be unique up to a rotation.
- Thus, our final factorization is

$$M = \hat{U}B = \hat{U}(Q\Lambda^{1/2}R)$$
$$O = B^{-1}\hat{\Sigma}\hat{V}^T = (R^T\Lambda^{-1/2}Q^T)\hat{\Sigma}\hat{V}^T$$

and the 3D object is recovered in O.

References

1. Brunton, Steven L., and J. Nathan Kutz. "Chapter 1: Singular Value Decomposition." *Data-driven science and engineering: Machine learning, dynamical systems, and control.* Cambridge University Press, 2019.
2. Chanu, Yambem Jina, Kh Manglem Singh, and Themrichon Tuithung. "A robust steganographic method based on singular value decomposition." *Int. J. Inf. Comput. Technol* 4.7 (2014): 717-726.
3. Muller, Neil, Lourenço Magaia, and Ben M. Herbst. "Singular value decomposition, eigenfaces, and 3D reconstructions." *SIAM review* 46.3 (2004): 518-545.