Chapter 15

Trigonometry

15.1 Circular Measures

Definition

Trigonometry is the aspect of mathematics that deals with the relationship between the sides and angles of triangles.

An angle is described as a measure of rotation or the point where two lines meet.

Types of Angles

There are five types of angles namely acute angle, right angle, obtuse angle, straight angle and reflex angle.

(1) Acute angle: These are are angles between 0° and 90° (i.e $0^\circ < \theta < 90^\circ$)
(2) Right angle: This is an angle that has an exact value of 90°
(3) Obtuse angle: These are angles between 90° and 180° (i.e $90^\circ < \theta < 180^\circ$)
(4) Straight angle: This is an angle that has exact value of 180°
(5) Reflex angle: These are angles between 180° and 360° (i.e $180^\circ < \theta < 360^\circ$)

Measurement of Angle

If given a fixed position OX and a line OP rotates from OX to some other position OP anticlockwisely than the movement POX is called angle POX and the angle is positive since the sense of rotation is anti-clockwise. (see figure 15.1a)

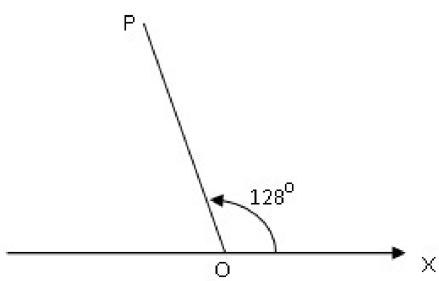


Figure 15.1a : $<POX = P\hat{O}X = 128^0$

If the movement is clockwise the angle POX is negative . (see figure 1b)

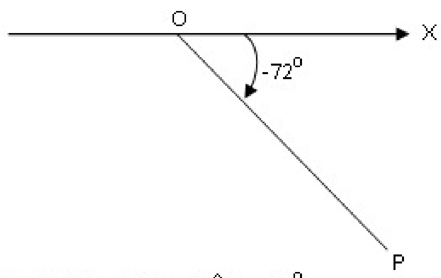


Figure 15.1b : <POX = PÔX = -72°

Generally angles are measured in degree or radians (where 360°=one revolution). The relationship between degree and radian. Let PQ be an arc of a circle with center O and radius r, If length of arc PQ is equal to the length of the radius than $\angle POQ$ is one radian. But if arc PQ is of length K then $\angle POQ$
$\angle POQ = \frac{K}{r}$ radians

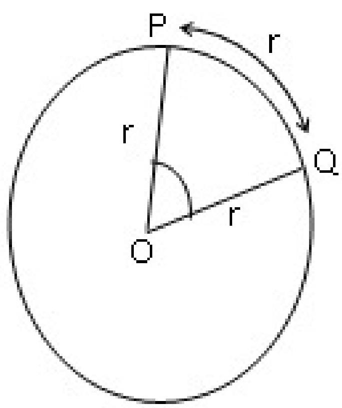


Figure 15.2a: <POQ = $\frac{r}{r}$ = 1

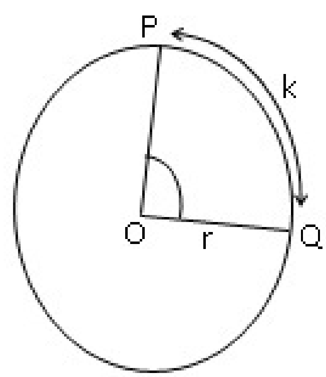


Figure 15.2b : <POQ = $\frac{k}{r}$ radians

From the statement above that 360° = 0ne revolution, the circumference of one revolution is $2\pi r$ Thus from figure15.2b, one complete revolution will be equivalent to $\frac{2\pi r}{r} = 2\pi$ radians

Therefore, 2π radians $= 360^\circ$
π radian $= 180^\circ$

$$1 radian = \frac{180^\circ}{\pi} = 57.2958 \tag{15.1}$$

Also

$$1^\circ = \frac{\pi}{180} \text{ radians} \tag{15.2}$$

Examples 15.1
(a) Express the following angles in radians
(i) 235° (ii) $-128^\circ 10'$
(b) Express the following angles in degrees and minutes correct, correct to the nearest minute
(i) 5.2 radians (ii) $\frac{\pi}{7}$ radian.
(c) Which of the following pair of angles is greater ?
129° or 2.16 radians.
Solutions
To express 235° radians from (5.2)
$1^\circ = \frac{\pi}{180}$ radians $235^\circ = \frac{235^\circ \times \pi}{180} = 4.1021$ radians
To express $-128^\circ 10'$ in radians convert $10'$ to degree 60'$= 1^\circ$
1' $= (\frac{1}{60})^\circ$
10'$= (\frac{1\times 10}{60})^\circ = (\frac{1}{6})^\circ$ $-128^\circ = -128\frac{1}{6}^\circ = -128.166^\circ$ $-128^\circ 10' = \frac{-128.166\times\pi}{180} radians$
To express 5.2 radians in degree, from equation (5.1) 1 radian $= \frac{180}{\pi}^\circ$ 5.2 radians $= \frac{180\times 5.2}{\pi}^\circ$ $=297.9381^\circ$ $=297^\circ 56'$

To express $\frac{\pi}{7}$ radians in degrees

$$\begin{aligned} 1 \text{ radian } &= \left(\frac{180}{\pi}\right)^\circ \\ \frac{\pi}{7} \text{ radian } &= \left(\frac{180^\circ}{\pi} \times \frac{\pi}{7}\right)^\circ = 25.7143^\circ \\ &= 25^\circ 43' \end{aligned}$$

3. To find the greater angle between

$$129^\circ \text{ or } 2.16 \text{ radians,}$$

Convert both the same unit (i.e degree or radian)
convert 2.16 radians to degrees

$$\begin{aligned} 1 \text{ radian } &= \left(\frac{180}{\pi}\right)^\circ \\ 2.16 \text{ radian } &= \left(\frac{180 \times 2.16}{\pi}\right)^\circ \\ &= 123.7589^\circ = 123^\circ \cdot 44' \end{aligned}$$

Since 129° is greater than $123^\circ 46'$ $\therefore$ 1290° is greater
or convert 129° to radians

$$\begin{aligned} 1^\circ &= \frac{180}{\pi} \text{ radians} \\ 129^\circ &= \left(\frac{180}{\pi} \times 129\right) \text{ radians} \\ &= 2.25 \text{radians} \end{aligned}$$

Since 2.25 radians is greater than 2.16 radians hence 129° is the greater angle.

Exercises 5a

1. Express the following angles

   (i) -2.25 radians in degrees
   (ii) $\frac{\pi}{3}$ radians in degrees
   (iii) $32^\circ 14'$ in radians
   (iv) $-228^\circ 23'$ in radians

2. Which is greater of the following pairs?
   (i) 19° or $\frac{2}{3}$ radian (ii) 138° or 3.16 radians
3. For which of the following angles will the positions of OA be same? $< A0B$
   (i) -120° (ii) 135° (iii) 600° (iv) 240° (v) -225° (vi) 30°

5.1 Trigonometric Ratios and Identities

5.1.1 The trigonometric ratios for an acute angle

In the study of an acute angle θ, a right angled triangle ABC is considered (see figure 3)

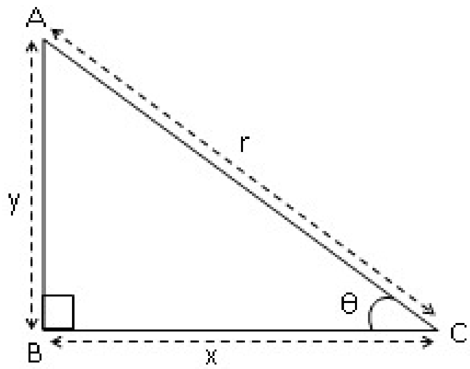


Figure 15.3

where $\overline{AC}$= r, which is the hypotenuse(i.e the line opposite the right angle), $\overline{AB}$=y, which is the side opposite to the angle θ and $\overline{BC}$=x, which is the side adjacent to the angle θ.

For an acute angle θ, the trigonometric ratios are defined as follows:

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r};\ \cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r}\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opposite}{adjacent} = \frac{y}{x}; \tag{5.3}$$

$$cosec\theta = \frac{1}{\sin\theta} = \frac{r}{y};\ \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x};\ \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y} \tag{5.4}$$

Also , from Figure 3, by Pythagora's theorem

(1) $$x^2 + y^2 = r^2$$

Dividing equation () through by r^2, we have

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

From the relationship in equation() above we have

(2) $$\cos^2\theta + \sin^2\theta = 1$$

Equations () may be written in forms of

(3) $$sin^2\theta = 1 - \cos^2\theta \text{ or } \cos^2\theta = 1 - \sin^2\theta$$

Furthermore, by dividing equation () by $\cos^2\theta$, we obtain

(4) $$1 + \tan^2\theta = \sec^2\theta$$

Also by dividing equation () by $\sin^2\theta$, we obtain

$$\cot^2\theta + 1 = \; cosec^2\theta \tag{5}$$

Knowing the relationships () to () made it easy to calculate all the trigonometric ratios if one is known and are generally true for any acute angle. It also allow us to rewrite trigonometric expressions in alternative ways and simpler forms. **Examples** 1. Given that $\tan\theta = \frac{3}{2}$ and $0^\circ \leq \theta \leq 90^\circ$. obtain the values of the other trigonometric ratios of the anlgle θ.

2. Show that $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$
3. $\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta} = 2\sec\theta$
4. If $a\cos^2\theta + b\sin^2\theta = c$, Show that $\tan^2\theta = \frac{c-a}{b-c}$

Solution

Solutions

1. Given that $\tan\theta = \frac{3}{2}$ and $0^\circ \leq \theta \leq 90^\circ$ from the relation ()

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \left(\frac{3}{2}\right)^2 = \sec^2\theta$$
$$1 + \frac{9}{4} = \sec^2\theta$$
$$\sqrt{\frac{4+9}{4}} = \sec^2\theta$$
$$1 + \frac{\sqrt{13}}{2} = \sec^2\theta$$

from the relation ()

$$\sec\theta = \frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{1}{\sec\theta}$$
$$\cos\theta = \frac{1}{\frac{\sqrt{13}}{2}} = 1 \times \frac{2}{\sqrt{13}} = \frac{2}{\sqrt{13}}$$

From the relation ()

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{2}{\sqrt{13}}\right)^2 = \frac{13-4}{13} = \frac{9}{13}$$
$$\sin\theta = \sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}}$$

From the relation ()

$$cosec\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{3}{\sqrt{13}}} = \frac{\sqrt{13}}{3}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$$\sin\theta = \frac{3}{\sqrt{13}},\ \cos\theta = \frac{2}{\sqrt{13}},\ \sec\theta = \frac{1}{2}\sqrt{13},\ cosec\theta = \frac{1}{3}\sqrt{13},\ \cot\theta = \frac{2}{3}$$

2. To show that

$$(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$$

From the L.H.S

$$= [(\sin\theta + \cos\theta)(\sin\theta + \cos\theta)] + [(\sin\theta - \cos\theta)(\sin\theta - \cos\theta)]$$

$$= [\sin^2\theta + \sin\theta\cos\theta + \sin\theta\cos\theta + \cos^2\theta] + [\sin^2\theta - \sin\theta\cos\theta - \sin\theta\cos\theta + \cos^2\theta]$$

$$= [\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta] + [\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta]$$

since $\sin^2\theta + \cos^2\theta = 1$

$$= [1 + 2\sin\theta\cos\theta] + [1 - 2\sin\theta\cos\theta]$$

$$= \ 1 + 2\sin\theta\cos\theta + 1 - 2\sin\theta\cos\theta$$

$$= \ 2$$

3. $\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta} = 2\sec\theta$
From the L.H.S

$$\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta} = \frac{\cos^2\theta + (1+\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)}$$

$$= \frac{\cos^2\theta + 1 + 2\sin\theta + \sin^2\theta}{\cos\theta(1+\sin\theta)}$$

$$= \frac{\cos^2\theta + \sin^2\theta + 1 + 2\sin\theta}{\cos\theta + \sin\theta\cos\theta}$$

But $\cos^2\theta + \sin^2\theta = 1$

$$= \frac{1 + 1 + 2\sin\theta}{\cos\theta(1+\sin\theta)}$$

$$= \frac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)}$$

But $\sec\theta = \dfrac{1}{\cos\theta}$

$$= 2\sec\theta$$

4. Given $a\cos^2\theta + b\sin^2\theta = c$
To show that

$$\tan^2\theta = \frac{c-a}{b-c}$$

From the R.H.S substitute c to have

$$\begin{aligned}
\frac{c-a}{b-c} &= \frac{(a\cos^2\theta + b\sin^2\theta) - a}{b - (a\cos^2\theta + b\sin^2\theta)} \\
&= \frac{-a(1-\cos^2\theta) + b\sin^2\theta}{b(1-\sin^2\theta) - a\cos^2\theta} \\
&= \frac{-a(\sin^2\theta + b\sin^2\theta}{b(\cos^2\theta) - a\cos^2\theta} \\
&= \frac{(b-a)\sin^2\theta}{(b-a)\cos^2\theta} \\
&= \tan^2\theta
\end{aligned}$$

Exercise 5b
1. Given that $\cos\theta = \frac{12}{13}$ and $0^\circ \leq \theta \leq 90^\circ$, evaluate $\sin\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $cosec\theta$.

2. Given that $\sin\theta = \frac{1}{\sqrt{3}}$ and $0^\circ \leq \theta \leq 90^\circ$, obtain the values of the other trigonometric ratios of the angle θ.

3. Simplify (i) $\sqrt{\left(1+\frac{x^2}{a^2}\right)}$ (ii) $\frac{1}{a^2+x^2}$, given that x = $a\tan\theta$.

4. Show that
(i) $\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\left(\frac{\sec\theta-1}{\sec\theta+1}\right) = 1$ (ii) $(cosec\theta - \cot\theta)(cosec\theta + \cot\theta) = 1$
(iii) $4 - 3\cos^2\theta = 3\sin^2\theta + 1$ (iv) $\frac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$

5. Given that c=$\cos\theta$, express the following in terms of c.
(i) $3\sin^2\theta - 2\cos\theta$ (ii) $\tan^2\theta + 2\cos\theta$.

6(a) Given that $p = x\cos\theta + y\sin\theta$ and $q = x\sin\theta - y\cos\theta$.
Show that $p^2 + q^2 = x^2 + y^2$

(b) Given that $p = r\sin\theta\cos\theta$, $q = r\sin\theta\sin\theta$ and $s = r\cos\theta$.
Show that $p^2 + q^2 + s^2 = r^2$

5.1.2 The Trigonometric Ratios of the Angles 30°, 45° and 60°
In order to obtain the trigonometric ratios of the angles above, we considered an equilateral triangle and an isosceles right angled triangle. Thus, the isosceles right angled triangle ABC in which $\overline{AB}$=$\overline{BC}$=1 unit is considered (see Figure 5.4)

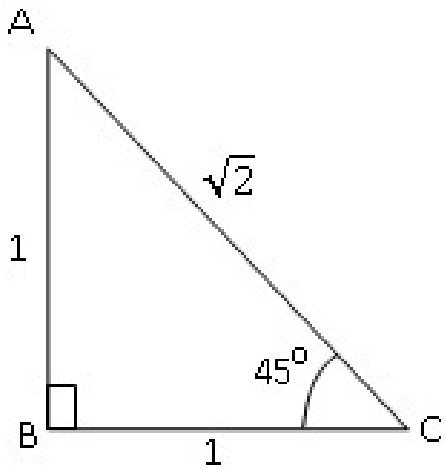


Figure 15.4

From the Figure 5.4

(6) $\quad \sin 45^\circ = \dfrac{|AB|}{|AC|} = \dfrac{1}{\sqrt{2}},\ \cos 45^\circ = \dfrac{|BC|}{|AC|} = \dfrac{1}{\sqrt{2}}$ and $\tan 45^\circ = \dfrac{|AB|}{|BC|} = 1$

Now, considering an equilateral triangle ABC of side i unit.(see Figure 5.5) $\overline{BN}$

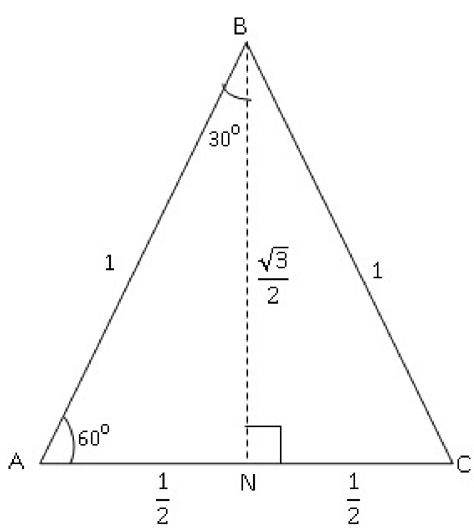


Figure 15.5

is the altitude.

Hence, considering right angled $\triangleleft$ ANB

$$\sin 30^\circ = \frac{|AN|}{|AB|} = \frac{\frac{1}{2}}{1} = \frac{1}{2};\ \cos 60^\circ = \frac{|AN|}{|AB|} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$$

$$\cos 30^\circ = \frac{|BN|}{|AB|} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2};\ \sin 60^\circ = \frac{|BN|}{|AB|} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{|AN|}{|BN|} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}};\ \cot 60^\circ = \frac{|AN|}{|BN|} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}}$$

$$\tan 60^\circ = \frac{|BN|}{|AN|} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3};\ \cot 30^\circ = \frac{|BN|}{|AN|} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\tan 30^\circ = \cot 60^\circ = \sqrt{3}$$

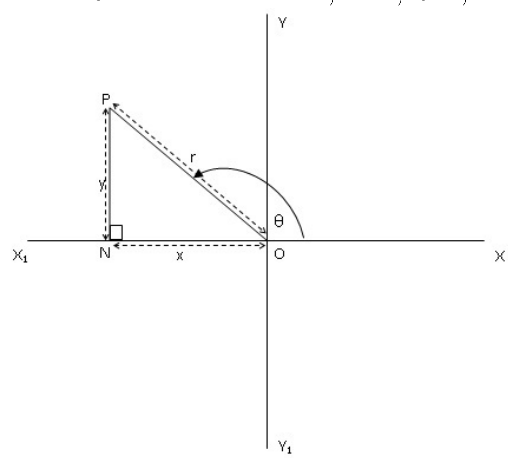
The summary of the obtained results are shown in table 1

Table 1: Summary of Trigonometric Ratios of angles $0°$, $30° < 45°, 60°$, and $90°$

Angles(θ)	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
$cosec\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

5.1.3 The trigonometric ratios for any angle

This section define the trigonometric ratios of angle of any size. Even, when applied to acute angles, it will yield the same results as in section 5.11. The angles will be measured from a fixed line $X'OX$ on the plane while line YOY' is a perpendicular line to line $X'OX$. This pair pf lines divided the plane into four quadrants, XOY, YOX', $X'OY'$ and $Y'OX$ which are $1st$, $2nd$, $3rd$, $4th$



quadrants respectively.(see Figure 5.6)

Figure 15.6

From the (Figure 5.6), let OP be any line in the plane through O and let X$\bar{O}$P= θ. This definition is with respect to the convection given in section 5.0

Let x and y be the cartesian co-ordinate of point P with reference to the X and Y axes with the usual sign convection. Now, let OP = r, be measured as positive for all position of P. Then the trigonometric ratios for X$\bar{O}P$ are defined as

(7) $$\sin\theta = \frac{y}{r} \ \cos\theta = \frac{x}{r} \ \tan\theta = \frac{y}{x}$$

and

(8) $$cosec\theta = \frac{1}{\sin\theta} \ \sec\theta = \frac{1}{\cos\theta} \ \cot\theta = \frac{1}{\tan\theta}$$

The application of these definition to an acute angle (i.e angles in the first quadrant the results are the same with those in section (5.1.1) see Figure 5.7

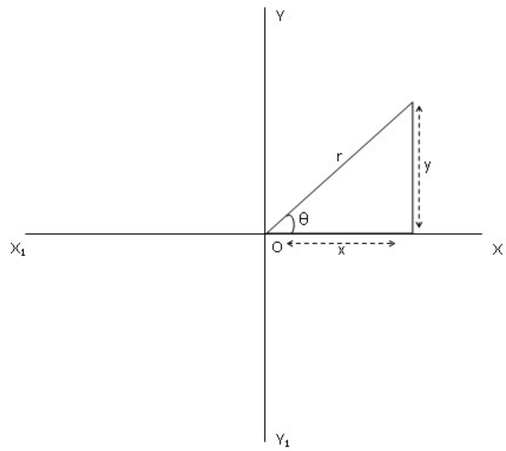


Figure 15.7

The signs of x and y are taken into consideration for any angle. If P is in the first quadrant, then x,y and r are all positive, which makes sine, cosine and tangent all positive. If P is in the second quadrant y is positive and x is negative, which makes sine positive but cosine and tangent are negative. If P is in the third quadrant, x and y are negative, which makes sine and cosine negative but tangent is positive. If P is in the fourth quadrant x is positive and y is negative, which makes cosine positive but sine and tangent are negative. These results is shown in the below stating which of three ratios (sine, cosine and tangent) are positive in each quadrant.

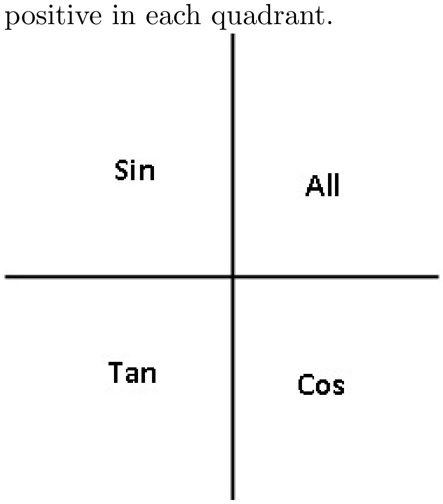
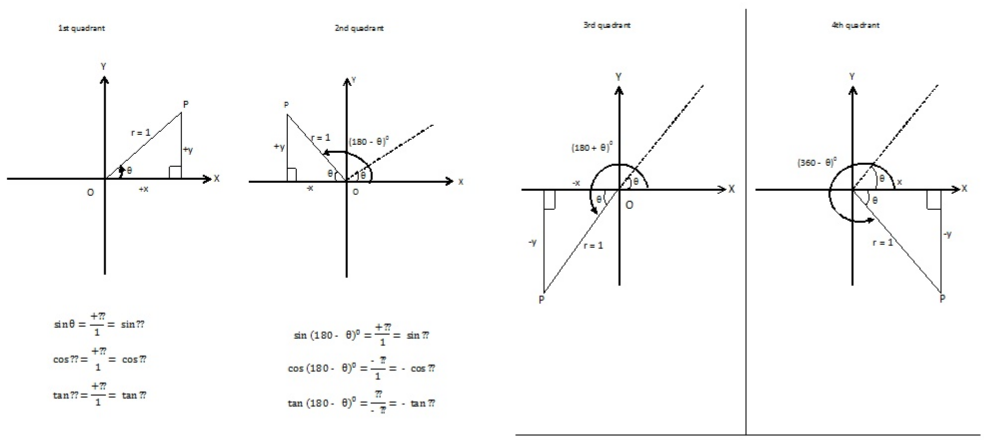


Figure 15.8

Using definitions (5.11) (5.12) the trigonometric ratios for any angle is defined. Trigonometric function table exist only for angle in the range of 0° to 90°. These tables are sufficient for the trigonometric ratios of an angle that maybe expressed in terms of the trigonometric ratio of an acute angle.



Whenever we have to find a ratio for an angle $> 90^\circ$, just see which quadrant the angles lies in an and convert the angle to one the forms $(180^\circ - \theta)$, $(180^\circ + \theta)$ or $(360^\circ - \theta)$ where θ is an acute angle as shown in the table (5.1)above

Example

Determine the values of (i) $\sin 160^\circ$ (ii) $\cos 240^\circ$ (iii) $\tan 330^\circ$ **solution**

(i) Angle 160° is in the $2nd$ quadrant, so use $(180 - \theta)^\circ$ and $\sin 160^\circ$ will be positive (+)

$$\sin 160^\circ = \sin(180 - 20)^\circ = +\sin 20^\circ = +0.3420$$

(ii) Angle $\cos 240^\circ$ is in the $3rd$ quadrant, so use $(180 + \theta)^\circ$ and $\cos 240^\circ$ will be negative (-)

$$\cos 240^\circ = \cos(180 + 60)^\circ = -\cos 60^\circ = -0.500$$

(iii) Angle 330° is in the $4th$ quadrant, so use $(360 - \theta)^\circ$ and $\tan 330^\circ$ will be negative (-)

$$\tan 330^\circ = \tan(360 - 60)^\circ = -\tan 30^\circ = -1.0774$$

Example

Express the following in terms of positive trigonometric ratios of acute angles (i) $\cos -150°$ (ii) $\tan -210°$ (iii) $\cos -300°$ (iv) $\sin -500°$

Solution

(i) $\cos -150° = \cos 150° = -\cos(180-30)° = -\cos 30°$

(ii) $\tan -210° = -\tan 210° = -\tan(180+30)° = -\tan 30°$

(iii) $\cos -300° = \cos 300° = \cos(360-60)° = \cos 60°$

(iv) $\sin -500° = -\sin 500° = -\sin(360+40)°$

$= -\sin 140° = -\sin(180-40)° = -\sin 40°$

To find the value of θ given $\sin\theta$, $\cos\theta$ or $\tan\theta$, this is the inverse of problem in example 1. Note that there will be two values of θ between $0°$ and $360°$ and the sign will decides in which two quadrants the angles will lie.

Example

1. Find the values of θ in the following

(a) $\sin\theta = 0.1537$ (b) $\cos\theta$ =-0.2764 (c) $\tan\theta$= 1.271

solution

(a) Since $\sin\theta$ is positive, θ must lie in the $1st$ or $2nd$ quadrants. From the table, find where 0.1537 is, the corresponding acute angle is $8.8°$. Hence θ will be $8.8°$ (i.e $8°.50'$) or $(180-8.84)° = 171.16°$ (i.e $171°10'$)

(b) Since $\cos\theta$ is negative, θ must lie in the $2nd$ or $3rd$ quadrants. From the table, 0.2764 will be found under $73.95°$. Hence $\theta = (180-73.95)° = 106.05°$ (i.e $106°.3$') or $(180+73.95)° = 253.95°(253°.57')$.

(c) Since $\tan\theta$ is positive, θ must lie in the $1st$ or $3rd$ quadrants . From the table, 1.271 will be found under $51.8°$. Hence $\theta = 51.8°$ (i.e $51°48'$) or $(180+57.8)° = 231.8°$ (i.e $231°48'$)

Example

If $\tan\theta$=$\frac{3}{4}$ and θ is in the third quadrant, evaluate $\sin\theta$ and $\cos\theta$.

Solution

Given that

$$\tan\theta = \frac{3}{4} = \frac{Opp}{Adj}$$

$$\sin\theta = \frac{3}{5} \text{ and } \cos\theta = \frac{4}{5}$$

Since $\sin\theta$and $\cos\theta$ are in the third quadrants

$$\sin\theta = -\frac{3}{5} \text{ and } \cos\theta = -\frac{4}{5}$$

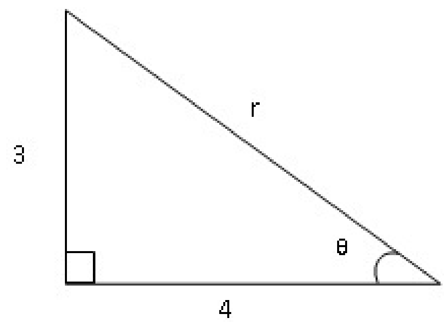


Figure 15.9

Complementary angles

Complementary angles are angles whose sum is 90°. see Figure 10, α and β are complementary angles

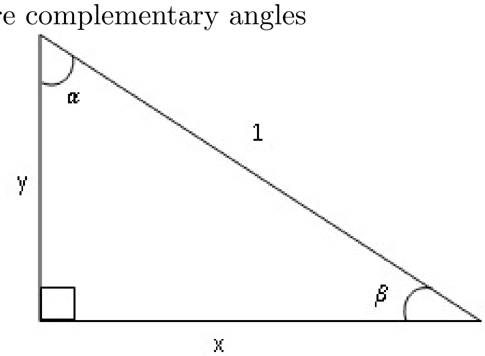


Figure 15.10

Then $\alpha = 90^\circ - \beta$ and $\beta = 90^\circ - \alpha$

From Figure 10

$$\sin\alpha = \frac{x}{1} = \cos\beta = \cos(90^\circ - \alpha)$$

Also

$$\cos\alpha = \frac{y}{1} = \sin\beta = \sin(90^\circ - \alpha)$$

$$\cos 72^\circ = \sin(90^\circ - 72^\circ) = \sin 18^\circ$$

Example

Given that $\frac{1-\tan^2 67\frac{1}{2}^\circ}{1+\tan^2 67\frac{1}{2}^\circ} = \cos 135^\circ$, find $\tan 67\frac{1}{2}^\circ$ in surd form.

Solution

$$\cos 135^\circ = \cos(180-45)^\circ = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$$

Hence

$$\frac{1-\tan^2 67\frac{1}{2}^\circ}{1+\tan^2 67\frac{1}{2}^\circ} = -\frac{1}{\sqrt{2}}$$

Let $\tan 67\frac{1}{2}^\circ = P$

$$\frac{1-P^2}{1+P^2} = -\frac{1}{\sqrt{2}}$$

$$(1-P^2)\sqrt{2} = -1(1+P^2)$$

$$\sqrt{2} - P^2\sqrt{2} = -1 - P^2$$

$$P^2 - P^2\sqrt{2} = -1 - \sqrt{2}$$

$$(1-\sqrt{2})P^2 = -1 - \sqrt{2}$$

$$P^2 = \frac{-1-\sqrt{2}}{1-\sqrt{2}}$$

$$P^2 = \frac{-(\sqrt{2}+1)}{-(\sqrt{2}-1)}$$

$$P^2 = \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = (\sqrt{2}+1)^2$$

$$P = \sqrt{2}+1$$

since $\tan 67\frac{1}{2}^\circ$ = P = $\sqrt{2}+1$

Exercise 5c

1. Determine the values of the following
(a) $\sin 150^\circ$ (b) $\cos -400^\circ$ (c) $\tan 520^\circ$ (d) $\sin -200^\circ$
2. Given $\sin\theta = \frac{1}{\sqrt{3}}$ and θ is obtuse find $\cos\theta$ and $\tan\theta$.
3. Find θ if (a) $\cos\theta = 0.2751$ and $\sin\theta$ is negative.
(b) $\tan\theta$ = -1.25 and $\cos\theta$ is negative.
(c) $\sin\theta = 0.2689$ and $\tan\theta$ is negative.
4. Express in terms of the trigonometric ratios of positive acute angles
(a) $\cos 190^\circ$ (b) $\tan -410^\circ$ (c) $\cos 300^\circ$ (d) $\sin -740^\circ$
5. Evaluate (a) $\sin(\frac{3\pi}{2})$ (b) $\cos(-\frac{9\pi}{4}$ (c) $\sin(-\frac{8pi}{3}$

6. Given that $\frac{4\tan 75^\circ}{1-\tan^2 75^\circ} = \frac{1}{\cos 150^\circ}$ find $\tan 75^\circ$
7. If $\frac{2\tan 15^\circ}{1+\tan^2 15^\circ} = \sin 30^\circ$, prove that $\tan 15^\circ = 2-\sqrt{3}$
8. Find the values of the following in simplified surd form
(a) $\frac{1+\cos 240^\circ}{1-\cos 240^\circ}$ (b) $\frac{\tan 300^\circ + \tan 210^\circ}{1-\tan 300^\circ \tan 210^\circ}$
(c) $\cos(-60^\circ)$ (d) $\tan^2 135^\circ$ (e) $\sin(-120^\circ)$
9. Given that $\sin\theta = \frac{3}{5}$ and θ is the third quadrant; calculate $\cos\theta$ and $\tan\theta$
10. If $\cos\theta = \frac{3}{2\sqrt{3}}$ and θ is obtuse find $\cos\theta$ and $\tan\theta$.

5.2 Graph of the trigonometric functions

The sine function

To draw the graph of the sine function $y = \sin\theta$. We make a table of values of $\sin\theta$ taking θ from 0° to 360° for every 30° and taking each value correct to 2 places of decimal.

Table 5.2

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$	0	0.50	0.87	1	0.87	0.50	0	-0.50	-0.87	-1	-0.87	-0.50	0

The curve is shown in Figure 11 with θ on the horizontal axis and $\sin\theta$ on the vertical axis.

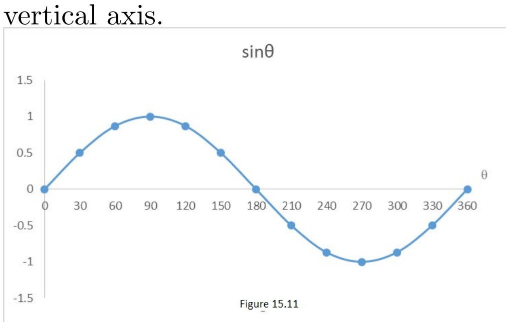


Figure 15.11

the curve in Figure 11 shows one circle of the sine curve. The curve repeats itself in successive cycles every 360°, both for values of θ less than 0° or greater than 360° (see Figure 12)

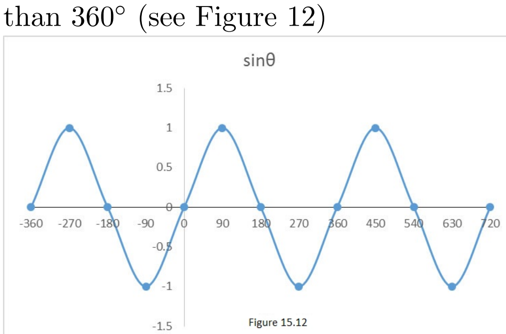
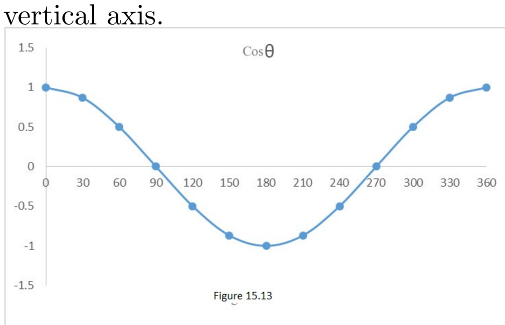


Figure 15.12

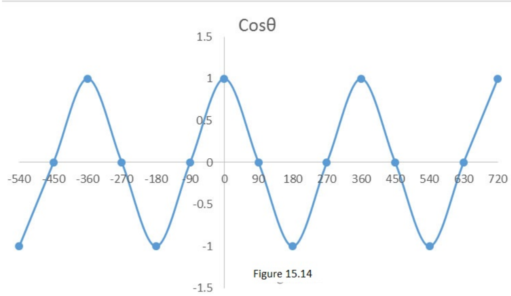
The Cosine Function
To draw the graph of the cosine function $y = \cos\theta$. We make a similar table of values for $\cos\theta$ taking θ from $0°$ to $360°$ for every $30°$ and taking each value correct to 2 places of decimal.
Table 5.3

θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$\cos\theta$	1	0.87	0.50	0	-0.50	-0.87	-1	-0.87	-0.50	0	0.50	0.87	1

The curve is shown in figure 13 with θ on the horizontal axis and $\cos\theta$ on the vertical axis.



The curve in Figure 13 shows one circle of the cosine curve. The curve repeats itself in successive cycles every $360°$, both for values of θ less than $0°$ or greater than $360°$ (See Figure 14)



The Tan Function
A similar table of values for $\tan\theta$ taking θ from $0°$ to $360°$ for every $30°$ is prepared to draw the graph of $y = \tan\theta$.
Table 5.4

θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$\tan\theta$	0	0.58	1.73	∞	-1.73	-0.58	0	-0.58	-1.73	∞	-1.73	-0.58	0

The curve is shown in Figure 15 with θ on the horizontal axis and $\tan\theta$ on the vertical axis.

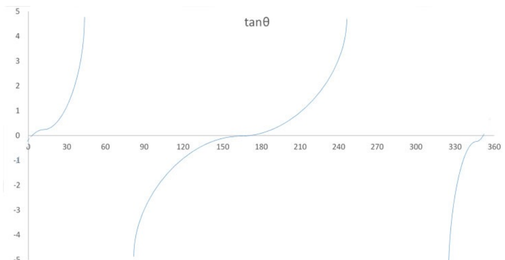


Figure 15.15

Composite trigonometrical functions

The addition or subtraction of two trigonometric function such as $\cos 2\theta + 2\sin\theta$ are referred to as composite functio. This can be solved graphically from $0°$ to $360°$

Example

Draw the graph of $y = \cos 2\theta + 2\sin\theta$ for $0° \leq \theta \leq 360°$. On the same axes, draw the graph of $y = \cos 2\theta$ and $y = 2\sin\theta$. Using your graphs,

(a) solve the equations $\cos 2\theta + 2\sin\theta = 0$ and $\cos 2\theta + 2\sin\theta = 1.5$

(b) Find the range(s) of values of θ for which

(i) $\cos 2\theta + 2\theta \leq -1$

(ii) $\cos 2\theta \geq 2\sin\theta$

Solution

Given $y = \cos 2\theta + 2\sin\theta$

We prepare a table for values of θ from $0°$ to $360°$

Table 5.5

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos 2\theta$	1.00	0.50	-0.50	-1.00	-0.50	0.50	1.00	0.50	-0.50	-1.00	-0.50	0.50	1.00
$2\sin\theta$	0	1.00	1.73	2.00	1.73	1.00	0	-1.00	-1.73	-2.00	-1.73	-1.00	0
y	1.00	1.50	1.23	1.00	1.23	1.50	1.00	-0.50	-2.23	-3.00	-2.23	-0.50	1.00

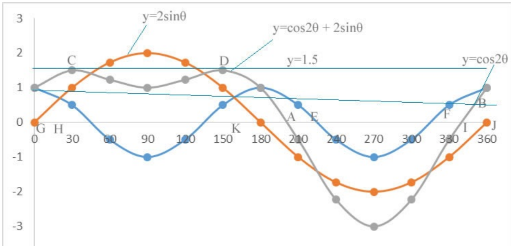


Figure 15.16

(a) (i) $\cos 2\theta + 2\sin\theta = 0$where the curve cuts the θ-axis i.e at A($\theta = 205°$) and BA($\theta = 342°$)

(ii) $\cos\theta + 2\sin\theta = 1.5$

Draw a line y = 1.5 parallel to the θ-axis. This torch the curve y = $\cos 2\theta + 2\sin\theta$ at $\theta = 30°$ and $150°$ (point C and D respectively on the graph) which are the solution of the equation.

(b) (i) Draw a line at y=1, parallel to the θ-axis. This cuts the curve y = $\cos\theta + 2\sin\theta$ at $\theta = 220°$ and $325°$ (point E and F respectively on the graph) so the range is $218° \leq \theta \leq 322°$

(ii) If $\cos 2\theta \geq 2\sin\theta$, the $\cos 2\theta$ curve will be above the $2\sin\theta$ curve, So we look for intervals in which this occurs. These points are G to H ($0° \leq \theta \leq 21°$) and points K to J ($158° \leq \theta \leq 360°$)

Exercise 5

1. Given that y = 1 - $2\sin\theta$ and y = $\cos\theta$, draw the graph on the same axis for $0° \leq \theta \leq 360°$, taking values of θ at 30° intervals. Using your graphs

(a) solve the equation $1 - 2\sin\theta = \cos\theta$

(b) Find the range of values of θ for which $2\sin\theta \geq \cos\theta$ and $1-2\sin\theta \leq \cos\theta$

2. Plot the graph of y= $3\sin x + 2\cos x$ for $0° \leq x \leq 360°$ using your graphs

(a) Solve the equations (i) $3\sin x + 2\cos x$ (ii) $3\sin x + 2\cos x = 1.5$

(b) find the range of values of x for which $3\sin x + 2\cos x$

(c) find the maximum and minimum values of the function $3\sin x + 2\cos x$ stating the values of x where there occur.

3. Prepare a table of values at 30° intervals, for the function y=$2\sin\theta - \cos\theta$, for $0° \leq \theta \leq 360°$. Draw the graph of this function and hence

(a) Solve the equations

(i) $2\sin\theta = \cos\theta$ (ii) $2\sin\theta - \cos\theta = -0.5$

(ii) $2\sin\theta - \cos\theta = 0$

(b) find the range of values of θ for which (i) $2\sin\theta \geq \cos\theta$ and $2\sin\theta < \cos\theta$

(c) Find the maximum and minimum values of $2\sin\theta - \cos\theta$ and the values of θ where they occur.

5.3 The addition Formulae

The formulae for the trigonometric ratios of the sum and difference of two angles in terms of the trigonometric ratios of those angle will be treated in this section.

If A and B are two angles, then expressions for $\sin(A+B)$, $\cos(A+B)$, $\sin(A-B)$, $\cos(A-B)$, etc in terms of sines and cosines of A and B are obtained by deriving formulae for compound angles such A + B and A-B .

Let consider the Figure 5.17 , which shows two angles, where ¡SOQ =A , ¡TOS = B and $\overline{OT}$=1 unit long while $\overline{OS}$ is perpendicular to $\overline{ST}$. Other lines are drawn as shown and ¡STR =A

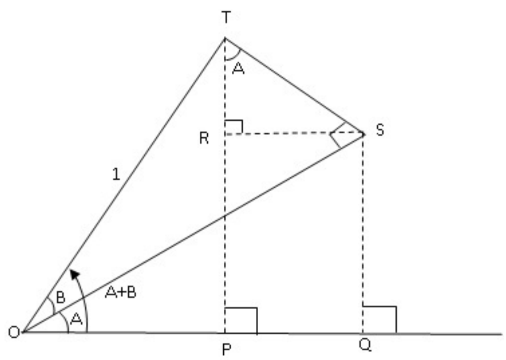


Figure 15.17

Hence, Sum of two angles, (A+B)

Also, Difference of two angles, (A-B)

Now substitute -B for B in these formulae

$$\begin{aligned} \text{Then } \sin(A-B) &= \sin A\cos(-B) + \cos A\sin(-B) \\ &= \sin A\cos B - \cos A\sin B \\ \text{and } \cos(A-B) &= \cos A\cos(-B) - \sin A\sin(-B) \\ &= \cos A\cos B + \sin A\sin B \end{aligned}$$

These four formulae are very important and should be memorized

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

From these results, we obtain by division the corresponding formulae for the $\tan(A+B)$ and $\tan(A-B)$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$
$$\equiv \frac{\frac{\sin A cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}}$$
$$\tan(A+b) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B}$$

dividing each term by $\cos A\cos B$, we have

$$\tan(A-B) = \frac{\frac{\sin A\cos B}{\cos A\cos B} - \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} + \frac{\sin A\sin B}{\cos A\cos B}}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} \ldots$$

Example

1. Find in surd form the value of (a) $\sin 75°$ (b) $\cos 105°$ (c) $\sin 15°$

Solution

$$\text{(a)}\ \sin 75° = \sin(45° + 30°) = \sin 45°\cos 30° + \cos 45°\sin 30°$$
$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$\text{(b)}\ \cos 105° = \cos(60° + 45°) = \cos 60°\cos 45° - \sin 60°\sin 45°$$
$$= \left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right) = \frac{1-\sqrt{3}}{2\sqrt{2}}$$
$$\text{(c)}\ \sin 15° = \sin(45° - 30°) = \sin 45°\cos 30° - \cos 45°\sin 30°$$
$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

2. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{6}{13}$, find the values of $\sin(A+B)$ and $\cos(A+B)$ when (a) A and B are both acute (b) A is acute and B is acute.

If $\sin A\frac{3}{5}$, then $\cos A = \sqrt{1-\sin^2 A} = \pm\frac{4}{5}$ (+ if acute , - if obtuse).

If $\cos B = \frac{5}{13}$, then $\sin B = \sqrt{1-\cos^{B}} = \pm\frac{12}{13}$ (B is acute).

$$
\begin{aligned}
(a)\ \sin(A+B) &= \sin A\cos B + \cos A\sin B\\
&= \left(\frac{3}{5}\times\frac{5}{13}\right)+\left(\frac{4}{5}\times\frac{12}{13}\right)=\frac{63}{65}\\
\cos(A+B) &= \cos A\cos B - \sin A\sin B\\
&= \left(\frac{4}{5}\times\frac{5}{13}\right)-\left(\frac{3}{5}\times\frac{12}{13}\right)=-\frac{16}{65}\\
(b)\ \sin(A+B) &= \sin A\cos B + \cos A\sin B\\
&= \left(\frac{4}{5}\times\frac{5}{13}\right)+\left(-\frac{4}{5}\times\frac{12}{13}\right)=-\frac{28}{65}\\
\cos(A+B) &= \cos A\cos B - \sin A\sin B\\
&= \left(-\frac{4}{5}\times\frac{5}{13}\right)-\left(\frac{3}{5}\times\frac{12}{13}\right)=-\frac{56}{65}
\end{aligned}
$$

3. Solve the equation $\sin 40^\circ \cos x + \cos 40^\circ \sin x = \frac{1}{2}$ for $0^\circ \leq x \leq 360^\circ$

Solution

Given $\sin 40^\circ \cos x + \cos 40^\circ \sin x = \frac{1}{2}$

The equation can be written as

$$\sin(40^\circ + x) = \frac{1}{2}$$

$$\text{Then } 40^\circ + x = \arcsin\left(\frac{1}{2}\right) = 30^\circ \text{ or } 150^\circ$$

$$x = -10^\circ 0r 110^\circ$$

$$\text{i.e } x = 350^\circ \text{ or } 110^\circ$$

4. Show that

$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$$

From L.H.S

$$\begin{aligned}
\sin(x+y)\sin(x-y) &= (\sin x\cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)\\
&= \sin^2 x\cos^2 y - \cos^2 x \sin^2 y\\
&= sin^2 x(1-\sin^2 y) - (1-\sin^2 x)\sin^2 y\\
&= \sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x \sin^2 y\\
&= \sin^2 x - \sin^2 y\\
&= R.H.S
\end{aligned}$$

5. Express $\tan 45°$ in surd form

$$\begin{aligned}
\tan 15° = \tan(45° - 30°) &= \frac{\tan 45° - \tan 30°}{1+\tan 45° \tan 30°}\\
&= \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}\\
&= 2-\sqrt{3}
\end{aligned}$$

Exercise

1. Find the values of the following in surd form
(a) $\cos 15°$ (b) $\sin 105°$ (c) $\tan 105°$ (d) $\cos 75°$
2. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, evaluate without using tables (i) $\sin(A+B)$ (ii) $\cos(A-B)$ (iii) $\tan(A+B)$, if A and B are acute. Is (A+B) an acute angle?
3. If $\tan(x+y) = \frac{4}{3}$ and $\tan x = \frac{1}{2}$. evaluate tan y.
4. Simplify
(i) $\sin 40° \cos 30° - \cos 40° \sin 30°$
(ii) $\cos 50° \cos 60° - \sin 50° \sin 60°$
(iii) $\cos 40° \cos 30° + \cos 40° \sin 30°$
(iv) $\cos 150° \cos 160° + \sin 150° \sin 160°$
(v) $\frac{\tan 30° + \tan 40°}{1-\tan 30° \tan 40°}$
(vi) $\frac{\tan 60° - \tan 30°}{1+\tan 60° \tan 30°}$
5. Show that (i) $\tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$
(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$
(iii) $\tan\left(x+\frac{1}{4}\pi\right) = \frac{\cos x + \sin x}{\cos x - \sin x}$
6. Express in terms of the sines and cosines of A, B, and C. (i) sin (A+B+C) (ii) cos (A+B+C)
7. Show that

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

5.4 Multiple and Sub-multiple Angles Formulae

The formulae obtained previously can be used to derive important formulae for multiple and sub-multiple angles.

If B in the previous formulae are replaced by A. Then

$$\begin{aligned} \sin 2A = \sin(A + A) &= \sin A \cos A + \cos A \sin A \\ &= 2 \sin A \cos A \end{aligned} \quad (*)$$

$$\begin{aligned} \cos 2A &= \cos A \cos A - \sin A \sin A \\ &= \cos^2 A - \sin^2 A \end{aligned} \quad (**)$$

Since $\cos^2 A = 1 - \sin^2 A$ and $\sin^2 A = 1 - \cos^2 A$, then the above result can be written in either of the form

$$\begin{aligned} \cos 2A &= \cos^2 A - (1 - \cos^2 A) \\ &= \cos^2 A - 1 + \cos^2 A \\ \cos 2A &= 2\cos^2 A - 1 \end{aligned} \quad (***)$$

or

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 1 - \sin^2 A - \sin^2 A \\ \cos 2A &= 1 - 2\sin^2 A \end{aligned} \quad (***)$$

Also

$$\begin{aligned} \tan 2A = \tan(A + A) &= \frac{\tan A + \tan A}{1 + \tan A \tan A} \\ &= \frac{2\tan A}{1 + \tan^2 A} \end{aligned} \quad (*)$$

The results above can be used to obtain expressions for $\sin 3A$, $\cos 3A$, $\tan 3A$ e.t.c

$$\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$$

$$\text{since } \sin 2A = 2\sin A \cos A \text{ and } \cos 2A = \cos^2 A - \sin^2 A$$

$$\begin{aligned} &= (2\sin A \cos A)\cos A + (\cos^2 A - \sin^2 A)\sin A \\ &= 2\sin A \cos^2 A + \sin A \cos^2 A - \sin^3 A \\ &= 3\sin A \cos^2 A - \sin^3 A \end{aligned}$$

$$\text{But } \cos^2 A = 1 - \sin^2 A$$

$$= 3\sin A(1-\sin^2 A) - \sin^3 A$$
$$= 3\sin A - 3\sin^3 A - \sin^3 A \qquad = 3\sin A - 4\sin^3 A$$

Also

$$\cos 3A = \cos(2A + A) = \cos 2A\cos A + \sin 2A\sin A$$
$$\text{But } \cos 2A = 2\cos^2 A - 1 \text{ and } \sin 2A = 2\sin\cos A$$

$$\cos 3A = (2\cos^2 A - 1)\cos A - (2\sin A\cos A)\sin A$$
$$= 2\cos^3 A - \cos A - 2\sin^2 A\cos A$$
$$\text{But } \sin^2 A = 1 - \cos^2 A$$

$$= 2\cos^3 A - \cos A - 2(1-\cos^2 A)\cos A$$
$$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$$
$$\cos 3A = 4\cos^3 A - 3\cos A$$

Likewise

$$\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 + \tan 2A\tan A}$$
$$= \frac{2\tan A}{1-\tan^2 A}$$

But $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

$$\tan 3A = \frac{\frac{2\tan A}{1+\tan^2 A} + \tan A}{1 - \left(\frac{2\tan A}{1+\tan^2 A}\right)\tan A}$$
$$= \frac{\frac{2\tan A + 2\tan A(1+\tan^2 A)}{1-\tan^2 A}}{\frac{(1-\tan^2 A)-(2\tan^2 A)}{1-\tan^2 A}} \quad = \frac{2\tan A + \tan A - \tan^3 A}{1-\tan^2 A}\cdot\frac{1-\tan^2 A}{1-3\tan^2 A}$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Replacing A by $\frac{1}{2}\theta$ in (*), (**), (***), (****) and (*****), we obtain

$$\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$$

$$\begin{aligned}\cos\theta &= \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta\\ &= 2\cos^2\frac{1}{2}\theta - 1\\ &= 1 - 2\sin^2\frac{1}{2}\theta\end{aligned}$$

$$\tan\theta = \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}$$

The above result make it possible for us to express $\sin\theta$, $\cos\theta$ ans $\tan\theta$ in terms of $\tan\frac{1}{2}\theta$.
Let $\tan\frac{1}{2}\theta = t$, from above we have

$$\tan\theta = \frac{2t}{1-t^2}$$

$$\begin{aligned}\sin\theta &= 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\\ &= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta+\sin^2\frac{1}{2}\theta} \quad \left(\text{since }\ \cos^2\frac{1}{2}\theta+\sin^2\frac{1}{2}\theta = 1\right)\end{aligned}$$

Dividing the numerator by and denominator by $\cos^2\frac{1}{2}\theta$

$$\sin\theta = \frac{2\frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}}{1+\frac{\sin^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta}} = \frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$$

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\begin{aligned}\cos\theta &= \frac{\cos^2\frac{1}{2}\theta-\sin^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta+\sin^2\frac{1}{2}\theta} = \frac{1-\frac{\cos^2\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta}}{1+\frac{\cos^2\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta}} = \frac{1-t^2}{1+t^2}\\ &= \frac{1-t^2}{1+t^2}\end{aligned}$$

Examples

(1) If $\cos 2A = \frac{3}{5}$, find $\tan A$ where A is acute.
Solution

$$\cos 2A = 2\cos^2 A - 1 \text{ and } \cos 2A = 1 - 2\sin^2 A$$
$$\frac{3}{5} = 2\cos^2 A - 1 \text{ and } \frac{3}{5} = 1 - 2\sin^2 A$$
$$\cos 2A = \frac{1}{2}\left(\frac{3}{5} + 1\right) \text{ and } \sin^2 A = \frac{1}{2}\left(1 - \frac{3}{5}\right)$$
$$\cos^2 A = \frac{4}{5} \text{ and } \sin^2 A = \frac{1}{5}$$
$$\cos A = \sqrt{\frac{4}{5}} \text{ and } \sin A = \sqrt{\frac{1}{5}}$$
$$\therefore \tan A = \frac{\sin A}{\cos A} = \frac{\sqrt{\frac{1}{5}}}{\sqrt{\frac{4}{5}}} = \frac{1}{2}$$

(2) (a.) Show that

$\cos 2\theta = \cos\theta$
$2\cos^2\theta - 1 = \cos\theta$
$2\cos^2\theta - \cos\theta - 1 = 0$
$\cos^2\theta + \cos^2\theta - \cos\theta - 1 = 0$
$(\cos^2\theta - 1) + \cos^2\theta - \cos\theta = 0$
$(\cos\theta - 1)(\cos^2\theta + 1) + \cos\theta(\cos^2\theta - 1)$
$(\cos\theta - 1)[\cos\theta + 1 + \cos\theta] = 0$
$(\cos\theta - 1) = 0 or 2\cos\theta + 1 = 0$
$\cos\theta = 1$ or $\cos\theta = -\frac{1}{2}$
$\theta = 0^0, 360$ or $\theta = 120^0, 240^0$
$\theta = 0^0, 120^0, 240^0$ and 360^0

3 If $\tan\theta = \frac{24}{7}$ and θ is acute, calculate $\tan\frac{\theta}{2}$ **Solution**

$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$ Let $\tan\frac{\theta}{2} = t$ then $\tan\theta = \frac{2t}{1-t} = \frac{24}{7}$ $24 - 24t^2$ $24t^2 + 14 + -24 = 0$ $12t^2 + 7t - 12 = 0$ $(4t-3)(3t+4) = 0$ $t = \frac{3}{4}$ or $t = \frac{-4}{3}$

But since θ is acute,so is $\frac{\theta}{2}$, so that $\tan\frac{\theta}{2}$ is positive. Therefore, $\tan\frac{\theta}{2} = \frac{3}{4}$

Exercises

1. If $\cos A = \frac{4}{5}$, find without tables $\sin 2A, \cos\frac{A}{2}$ and $\tan\frac{A}{2}$
2. If θ is an acute angle and $\cos 2\theta = \frac{119}{169}$, find the values of $\sin\theta$ and $\cos\theta$ without using tables.
3. Without using tables, find the values of (a) $2\sin 15^0\cos 15^0$ (b) $1 - 2\sin^2 15$ (c) $\frac{2\tan\frac{3\pi}{8}}{1-\tan^2\frac{3\pi}{8}}$ (d) $\sin^2 22\frac{1}{2}^0 - \cos^2 22\frac{1}{2}^0$

4. Prove that (a) $\frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$, (b) $\tan 2\theta - \tan\theta = \tan\theta \sec 2\theta$ (c) $\frac{1+\cos x+\cos 2x}{\sin x+\sin 2x} = \cot x$ (d)$\cos^4\theta - \sin^4\theta = \cos 2\theta$
5. Given that $2\cos 36^0 = 1{+}2\cos 72^0$, show that $4\cos^2 36^0 - 2\cos 36^0 - 1 = 0$ and hence deduce that $\cos 36^0 = \frac{1+\sqrt{5}}{4}$
6. If $\cos A = \frac{24}{25}$ and $\sin B = \frac{4}{5}$, where A is acute and B is obtuse, find without using tables the values of (a) $\sin 2A$ (b) $\cos 2B$ (c) $\sin(A-B)$ (d) $\tan 2B$
7. $\tan^2(45^0+\theta) = \frac{1+sin2\theta}{1-\sin 2\theta}$
8. If $\tan^2\theta = 1 + 2\tan^2\alpha$, Show that $\cos 2\theta + \sin^2\alpha = 0$

5.4 Factor Formulae

In this section, the sums anf differences of sines and cosines are expressed as products of sines and cosines From
$\sin(A+B) = \sin A\cos B + \cos A \sin B$ $\sin(A-B) = \sin A\cos B - \cos A\sin B$ On addition, we have
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$ and on subtraction, we have
$2\cos A\sin B = \sin(A+B) - \sin(A-B)$ Similarly from
$\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\cos(A-B) = \cos A\cos B + \sin A\sin B$
on addition and subtraction,we have $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$. With these formular product of series and cosines are expressed as a sum or a difference.
Let $A+B = C$ and $A-B = C$ So that $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$, Therefore
$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
$\cos C + \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

5.5 Solution of Triangle

Triangles are plane shape with three sides and three angles. There are five types of triangle namely Scalene triangle, Right angled triangle, Obtuse angled triangle, Isosceles triangle and Equilateral triangle.
Scalene triangle is a triangle in which non of its three sides and three angles are equal.

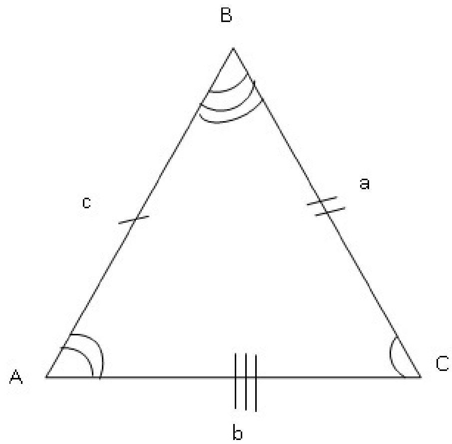


Figure 15.18

Right angled triangle is a triangle in which one of its angles is 90°.

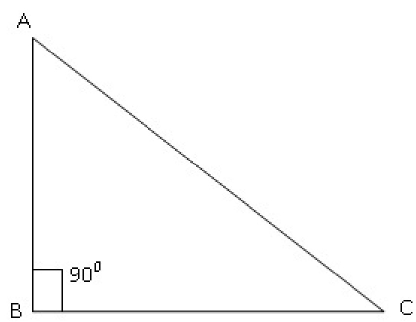


Figure 15.19

where $\overline{AC}$ is the hypoteneous.

Obtuse angled triangle is a triangle in which one of its angles is greater than 90° but less than 180° (i.e $90° \leq 0 \leq 180°$).

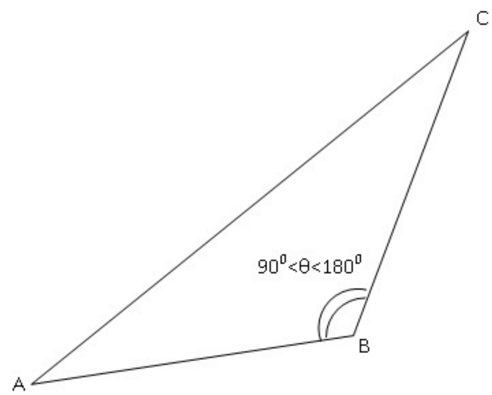


Figure 15.20

Isosceles triangle is a triangle in which two of its sides are equal and the angle formed by the base line to these two sides are equal (see figure 5.21 a, b and c)'

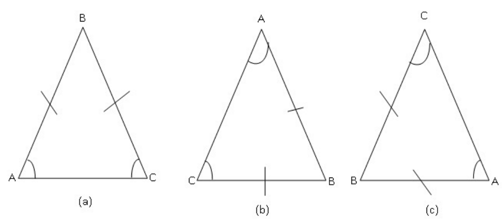


Figure 15.21

Equilateral triangle is a triangle in which all the three sides and the three angles are equal. (i.e The angles are of value 60° each).

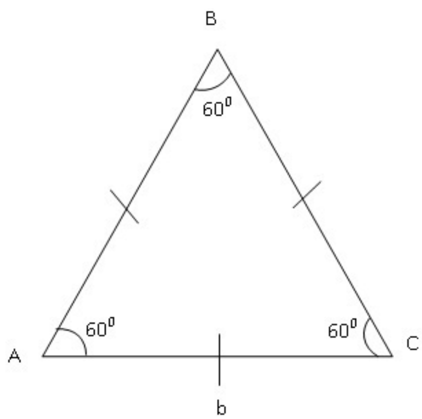


Figure 15.22

Note

i The sum of the angles in a triangle is $180°$ or Π.

ii The greatest side is opposite the greatest angle and the smallest side is opposite the smallest angle.

iii For a right-angled triangle, Pythagoras' theorem and simple trigonometrical ratios are sufficient to solve any puzzle on it.

For non-right angled triangles two important relations are required, those are Sine rule and Cosine rule.

The Sine Rule

Given a triangle ABC ($\triangle$ ABC), it is standard to label the side opposite to angle A as a, side opposite to angle B s b and that opposite to angle C as c.

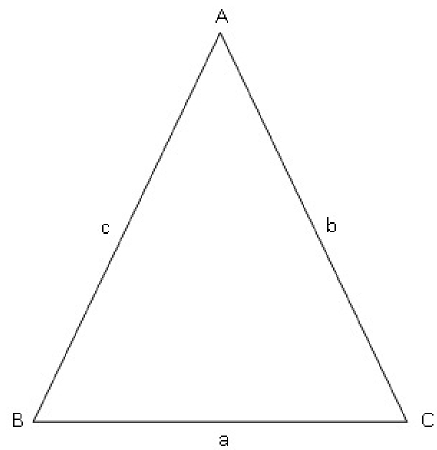


Figure 15.23

The Sine rule states that in a triangle each sides is proportional to the sine of the angle opposite it.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Cosine Rule
Given a triangle ABC ($\triangle$ ABC), where two sides are known with an angle included between the known sides, then the third sides which is opposite the known angle is calculated by the use of Cosine rule which state that

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Examples

1 Given $\triangle ABC$ as shown in figure 5.25, calculate the size of angles B and A and the lenght of a.

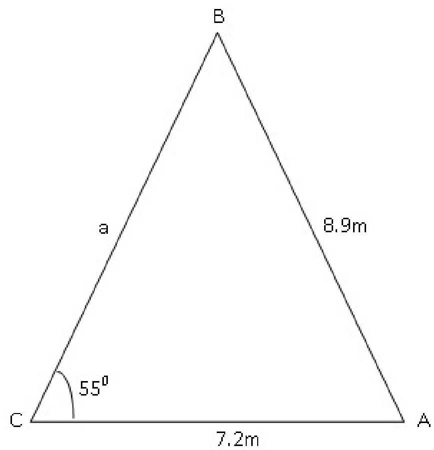


Figure 15.24

Solution: By Sine rule which states

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

from figure 5.25
$a = ?$, $b = 7.2m$, $c = 8.9m$, $A = ?$, $B =?$ and $C = 55°$
$\therefore$

$$\frac{a}{\sin A} = \frac{7.2m}{\sin B} = \frac{8.9m}{\sin C}$$

Hence

$$\frac{7.2m}{\sin B} = \frac{8.9m}{\sin 55°}$$

$$\sin B = \frac{7.2m \sin 55°}{8.9m}$$

$$\sin B = 0.6627$$

$$B = \sin^(- 1)(0.6627)$$

which gives $B = 41.51°$ or $138.49°$ but the value of B will be $41.51°$ because sum of angles in a triangle is $180°$
Then $A = 180° - (41.51° + 55°) = 83.49$
Thus

$$\frac{a}{\sin 83.49°} = \frac{8.9m}{\sin 55°}$$

$$\sin a = \frac{8.9m \sin 83.49°}{\sin 55°}$$

$$a = 10.79m$$

item [2] Find p in △ PQR in figure 5.26

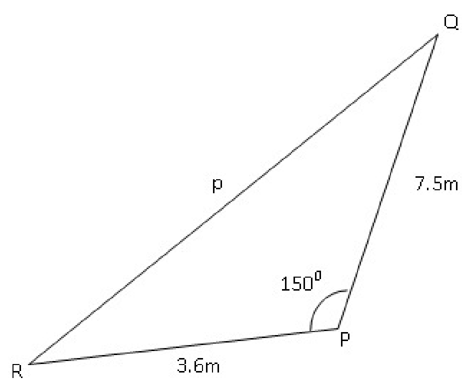


Figure 15.25

Solution:- In this figure Cosine rule will be used since we are given two sides and an included angle.
Thus

$$p^2 = q^2 + r^2 - 2qr \cos P$$

where
$p = ?$, $q = 3.6m$, $r = 7.5m$ and $P = 150°$

$$p^2 = (3.6m)^2 + (7.5m)^2 - 2(3.6m)(7.5m) \cos 150°$$

$$= 12.96m^2 + 56.25m^2 - 54m^2 \times (-\cos 30°)$$

$$= 69.21m^2 + (54m^2 \times \frac{\sqrt{3}}{2}) = 69.21m^2 + 46.77m^2$$

$$= 115.98m^2$$

$$P = \sqrt{115.98m^2} = 10.77m$$

3 Find the lenght of the smallest side in figure 5.27 and solve the $\triangle$ in figure 5.27b completely.

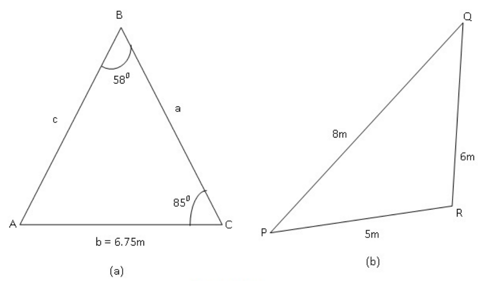


Figure 15.26

Solution

(a) Considering figure 5.27 (a), the smallest side is the side that is opposite the smallest angle. $< A = 180° - (85° + 58°) = 37°$, therefore the smallest side will be side a. Using Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$a = ?$, $b = 6.75m$, $< A = 37°$ and $< B = 58°$

$$a = \frac{b \sin A}{\sin B}$$

$$a = \frac{6.75m \sin 37°}{\sin 58°}$$

$$a = 4.79$$

(b) To solve figure 5.27b completely we have to find the three angles, using Cosine rule

$$p^2 = q^2 + r^2 - 2qr \cos P$$

where $p = 6m$, $q = 5m$ and $r = 8m$ and $P = ?$

$$80m^2 \cos Pp^2 = 25m^2 + 64m^2 - 36m^2$$

$$\cos P = \frac{25m^2 + 64m^2 - 36m^2}{80m^2}$$

$$\cos P = 0.6625$$

$$P = 48.51^\circ$$

We then use Sine or Cosine rule to find next angle Q or R.

$$\frac{q}{\sin Q} = \frac{p}{\sin P}, q = 5m, p = 6m, P = 48.51^\circ$$

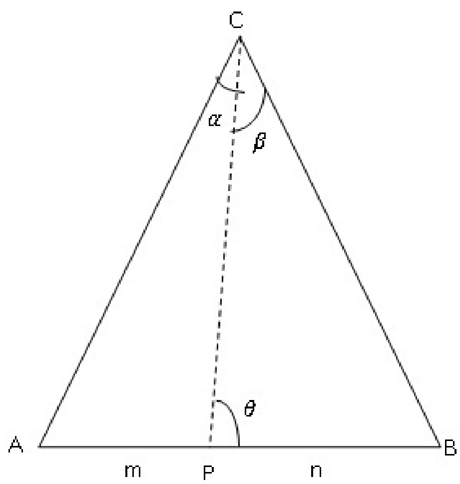


Figure 15.27

From (i) and (ii)

(9) $$\frac{m}{n} = \frac{AP}{PB} = \frac{AP}{CP} * \frac{CP}{PB} = \frac{\sin\alpha}{\sin A} * \frac{\sin B}{\sin\beta}$$

But

(10)
$\theta = A+\alpha\,(Exterior\ <\ of\ a\ \triangle)\ and\ \theta = 180^\circ-(B+\beta), \alpha = A-\theta\ and\ \beta = 180^\circ-(B+\theta)$

substituting eqn(iv) into (iii) to eliminate α and β

(11) $$\frac{m}{n} = \frac{\sin(A-\theta)}{\sin A} = \frac{\sin B}{\sin[180^\circ-(B+\theta)]}$$

(12) $$\frac{m}{n} = \frac{\sin(A-\theta)}{\sin A} = \frac{\sin B}{(B+\theta)}$$

$$\frac{m}{n} = \frac{\sin B(\sin A\cos\theta - \cos A\sin\theta)}{\sin A(\sin B\cos\theta + \cos B\sin\theta)} \tag{13}$$

Dividing the numerator and denominator by $\sin A\sin B\sin\theta$

$$\frac{m}{n} = \frac{\frac{\sin A\sin B\cos\theta}{\sin A\sin B\sin\theta} - \frac{\cos A\sin B\sin\theta}{\sin A\sin B\sin\theta}}{\frac{\sin A\sin B\sin\theta}{\sin A\sin B\sin\theta} + \frac{\sin A\cos B\sin\theta}{\sin A\sin B\sin\theta}} \tag{14}$$

$$\frac{m}{n} = \frac{\cot\theta - \cot A}{\cot\theta + \cot B} \tag{15}$$

$$m\cot\theta + m\cot B = n\cot\theta - n\cot A \tag{16}$$

$$n\cot A + m\cot B = n\cot\theta - m\cot\theta \tag{17}$$

$$n\cot A + m\cot B = (n-m)\cot\theta \tag{18}$$

5. Given a triangle whose sides are in the ratio 4 : 5 :, prove without use of table that one angle is twice another angle.
   Solution Given the sides in ratio 4 : 5 :, their lengths are 4k, 5k and 6k (i.e. a, b, c) where k is a constant.
   Using Cosine rule

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{19}$$

Hence

$$\cos A = \frac{(5k)^2 + (6k)^2 - (4k)^2}{2(5k)(6k)} = \frac{25k^2 + 36k^2 - 16k^2}{60k^2} = \frac{3}{4} \tag{20}$$

$$\cos B = \frac{(6k)^2 + (4k)^2 - (5k)^2}{2(6k)(4k)} = \frac{36k^2 + 16k^2 - 25k^2}{48k^2} = \frac{9}{16} \tag{21}$$

$$\cos C = \frac{(4k)^2 + (5k)^2 - (6k)^2}{2(5k)(6k)} = \frac{16k^2 + 25k^2 - 36k^2}{40k^2} = \frac{1}{8} \tag{22}$$

The smallest angle will be angle A since it opposite the smallest side 4k

$$\cos 2A = 2\cos^2 A - 1 = 2(\frac{3}{4})^2 - 1 = 2(\frac{9}{16}) - 1 = \frac{1}{8}\cos 2A = \cos C \tag{23}$$

$$2A = 2n\pi \pm C$$

But, since A and C are angles of triangle $2A = C$ is the only solution which proves the fact.
Exercise 5.

1 Solve the following triangles, given that
a $a = 5$, $b = 8$ and $c = 10$
b $a = 2.4$, $b = 4.5$ and $c = 72°$
c $A = 57°$, $B = 25°$ and $a = 3.6$
d $B = 52°$, $b = 7.1$ and $c = 5.5$
2 If the sides of a triangle are $3 + \sqrt{3}, 2\sqrt{3}$ and $3\sqrt{2}$, find without tables the sizes of the smallest angle.
3 In ΔABC, $a = 2$, $c = 3$ and $B = 95°$. Find the size of the smallest angle.
4 In a ΔABC the angle is $60°$. Show that $c^2 = a^2 - ab + b^2$. If a, b are teh roots of the equation $4x^2 - 10x + 3 = 0$, find the value of C and show that the length of the perpendicular from C to AB is $\frac{(3\sqrt{3})}{16}$.
5 The median AD of a triangle ABC makes angles β and γ respectively with AB, AC and $\angle ADB = \theta$. Show that $2\cot\theta = \cot\gamma - \cot\beta$. If $AD = 15m$, $\beta = 35°$, and $\gamma = 30°$, find B, C and a as accurately as the tables allows.
6 Given that point P divides the side AB of a ΔABC internally in the ratio $m : n$. If the $\angle ACP = \alpha$, $\angle BPC = \theta$, prove that $m\cot\alpha - n\cot\beta = (m + n)\cot\theta$.

Area of a triangle

In a ABC, let AD be the perpendicular from A to BC be the height of the ΔABC.

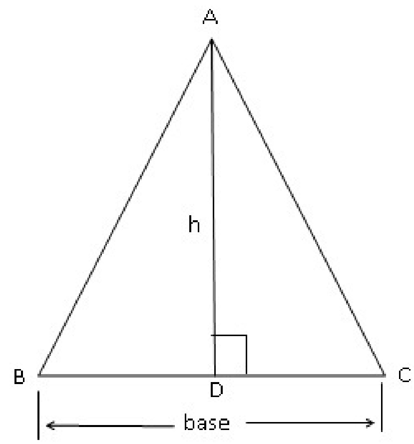


Figure 15.28

The standard formula for its area is

$Areaof\Delta ABC = \frac{1}{2}basexheight = \frac{1}{2}bh$ where $base = a////height = h$ An alternative formula is derived from figure 5.28

$$\sin B = \frac{h}{c}$$

$$h = c\sin B$$

$Areaof\Delta ABC = \frac{1}{2}ac\sin B$ Similarly by taking other sides as the "base" the area will be $Areaof\Delta ABC = \frac{1}{2}ab\sin C$ or $\frac{1}{2}bc\sin A$ Thus, the area of ΔABC is

$$Area = \frac{1}{2}basexheight$$

or $= \frac{1}{2}ac\sin B$ or $\frac{1}{2}ab\sin C$ or $\frac{1}{2}bc\sin A$ Also we have what it called the Hero's formula

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

where $S = \frac{a+b+c}{2}$

Examples

1 Given that sides of triangle are of length $a = 4.57m$, $b = 3.61m$ and $C = 4.72m$. find the area.

Solution

Given $a = 4.57m$, $b = 3.61m$ and $C = 4.72m$ To find the area, we use Hero's formula $Area = \sqrt{s(s-a)(s-b)(s-c)}$ where $S = \frac{a+b+c}{2} = \frac{4.57m+3.61m+4.72m}{2}$ $S = 6.44m$ $Area = \sqrt{6.44m(6.44m-4.57m)(6.44m-3.61m)(6.44m-4.72m)}$ $= \sqrt{6.44m(1.87m)(2.83m)(1.72m)} = \sqrt{58.62m^4}$ $Area = 7.66m^2$

2 In $\angle ABC$, $b = 6cm$, $c = 8cm$ and $A = 60°$. Find the area of the triangle. Let AD be the bisector of $\angle A$ where D lies on BC and $AD = k$. Find the area of triangles ABD and ACD in terms of K and hence Show that $AD = \frac{24\sqrt{3}}{7}$.

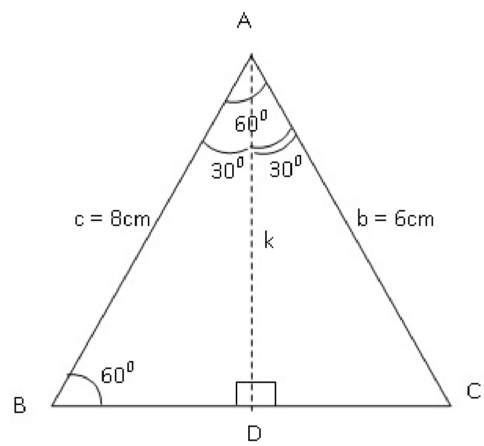


Figure 15.29

Solution

Given $\angle ABC$ $Areaof\angle ABC = \frac{1}{2}bc\sin A = \frac{1}{2}(8cm)(6cm)\sin 60° = 24cm^2\frac{\sqrt{3}}{2}$ $= 12\sqrt{3}cm^2$ $Areaof\angle ABD = \frac{1}{2}C\,|AD|\sin 30°$ $\frac{1}{2}(8cm)(kcm)(\frac{1}{2}) = 2kcm^2$

Area of $\angle ACD = \frac{1}{2}|AD|b\sin 30° = \frac{1}{2}(Kcm)(6cm)\frac{1}{2} = \frac{3}{2}kcm^2$

To show that $AD = \frac{24\sqrt{3}}{7}$ $AreaofC\angle ABD{+}Areaof\angle ACD = Areaof\angle ABC$ $2kcm^2 + \frac{3}{2}kcm^2 = 12\sqrt{3}cm^2$ $\frac{4k+3k}{2}cm^2 = 12\sqrt{3}cm^2$ $\frac{7k}{2} = 12\sqrt{3}$ $AD = k = \frac{24\sqrt{3}}{7}$

Exercises

1 In $\angle ABC$, $A = 47°$, $b = 4.8cm$ and $c = 6.9cm$. Find

a the area of the triangle

b a

c the length of the perpendicular from B to AC.

2 In $\angle ABC$, $a = 5cm$, $b = 4cm$ and the area of the triangle is $8.6cm^2$. Find angle C, given that it is obtuse.

3 Given that the sides of triangle are $15cm$, $9cm$ and $5cm$, find the sizes ot the angle and the area of the triangle.

4 ABC is a triangle with sides of length a, b, c, opposite A, B, C respectively. P is on the opposite side of BC to A and the triangle BCP is equilateral. Show that $AP^2 = a^2 + c^2 - ac\cos B + \sqrt{3}ac\sin B$.

5 Given that three sides of a triangle area $4, 6$ and $8cm$. Without usi9ng tables

a find the cosine of the largest angle

b show that the area of the triangle

c find the length of the shortest altitude of triangle.

Three dimensional problem

Three dimensional problems are usually solve by visualising the triangles involved in the solid shape or any other shape involved. All the right angles should be clearly marked, even through they may not look like right angles should be clearly marked, even though they may not look like right angles. It is necessary to recognize the angles between a line and a plane and that between two planes.

Definition

Planes: A flat surface like the top of a table is called a plane. It is determined by two intersecting straight lines each lies in the plane. (See Figure 3.30)

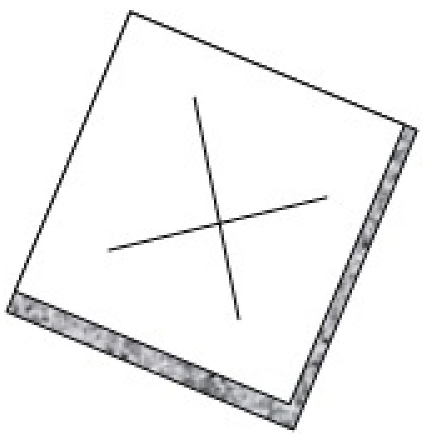


Figure 15.30

Normal to the Plane

If a pencil is standing on a plane where two lines intersecting at a point 0. the pencil will be perpendicular to both lines and every line in the plane, through 0. The pencil is then said to be a normal to the plane.

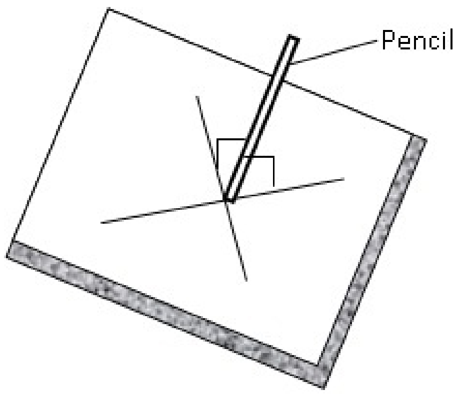


Figure 15.31

Angle between a line and a sphere
If the line AB meets the plane at B. AC is a normal to the plane BC is called the projection of AB on the plane. Then it is said that the angle between AB and the plane is the angle angle between AB and its projection. (See Figure 3.52)

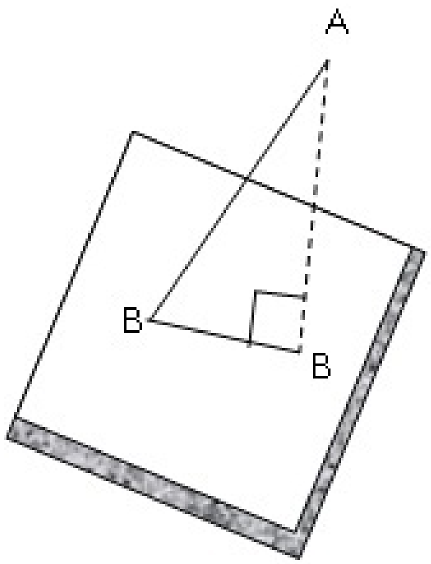


Figure 15.32

Angle between two planes
Let P_1 and P_2 be two planes intersecting on one common edge $\overline{XY}$. The angle between the plane P_1 and P_2 is the $\angle BAC$, where $\overline{AB}$ and $\overline{AC}$ are each perpendicular to the common edge $\overline{XY}$. See Figure 5.332

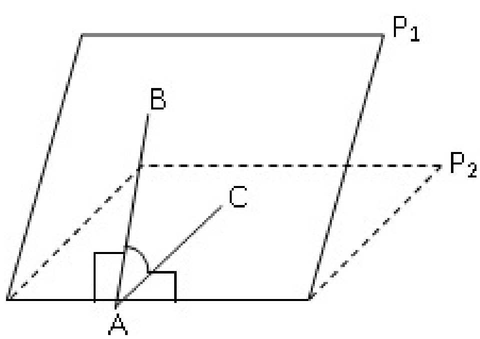


Figure 15.33

Angle of elevation, depression and bearings
From an horizontal point A on a plane, if one hole up at a point C, the act of turning the head up it makes an angle which is called angle of elevation of C from A is θ. (See Figure 5.34)

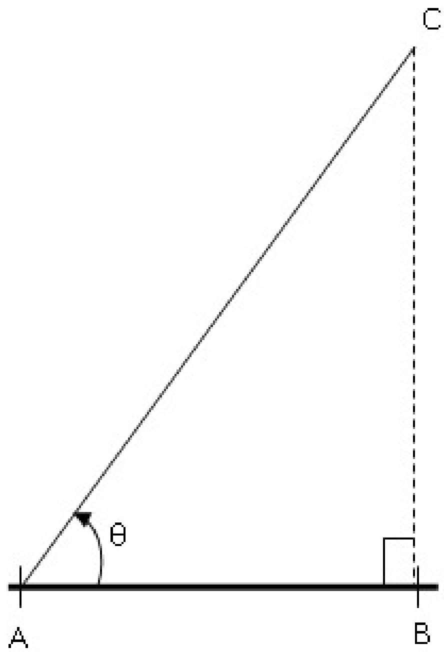


Figure 15.34

Also looking down from C to a point A, the head makes angle which is called angle of depression of A from C is θ. (See Figure 5.35)

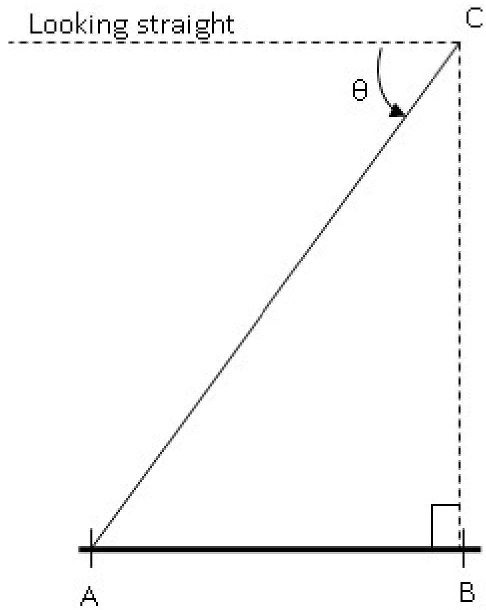


Figure 15.35

Examples

(1) The base of a rectangular box measures 30cm by 40cm and itis 50cm high. Find the length of the longest straight rod which could be put into the box and the angle it will make with the box.
Solution
Let the rectangular box be given

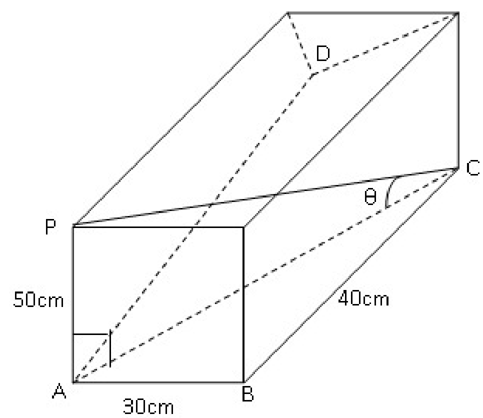


Figure 15.36

From the Figure 5.36 the longest straight rod is $\overline{PC}$.
To calculate $\overline{PC}$ we first find $\overline{AC}$ by Pythagoras theorem

$$|AB|^2 + |BC|^2 = |AC|^2$$
$$(30cm)^2 + (40cm)^2 = |AC|^2$$
$$900cm^2 + 1600cm^2 = |AC|^2$$
$$2500cm^2 = |AC|^2$$
$$|AC| = \sqrt{2500cm^2} = 50cm$$

Then

$$|PC|^2 = |PA|^2 + |AC|^2$$
$$|PC|^2 = (50cm)^2 + 2500cm^2$$
$$|PC|^2 = 2500cm^2 + 2500cm^2$$
$$|PC| = \sqrt{5000cm^2} = 70.71cm$$

The angle it makes with the base will be

$$\tan\theta \frac{|PA|}{|AC|} = \frac{50cm}{50cm} = 1$$
$$\theta = \tan^{-1}(1) = 45°$$

(2) ABC is an isosceles triangle in which $AB = AC = 35m$ and $\angle BAC = 45°$. A vertical radio mast stands at A and the angle of elevation of the top of this mast, from the midpoint of BC is 15°. Find the height of the mast.

Solution

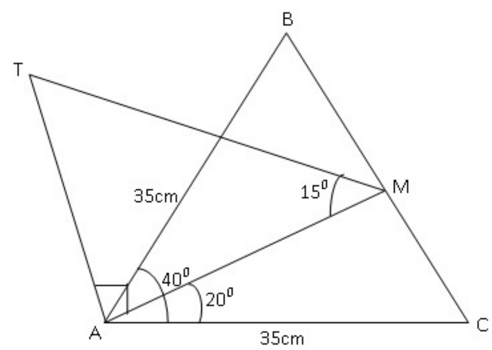


Figure 15.37

From Figure 5.37 to find the height of the mast $\overline{AT}$, first find $\overline{AM}$ which will be

$$\cos 20^\circ \frac{\overline{AM}}{\overline{AC}} = \frac{\overline{AM}}{35m}$$
$$\overline{AM} = 35m\cos 20^\circ$$
$$\overline{AM} = 32.89m$$

Also to calculate $\overline{AT}$;

$$\tan 15^\circ \frac{\overline{AT}}{\overline{AM}} = \frac{\overline{AT}}{32.89m}$$
$$\overline{AT} = 32.89m\tan 15^\circ$$
$$\overline{AT} = 8.81m$$

(3) Two point R and S are on a level ground 350m apart. The bearing of S from R is 147°. From R the bearing of a radio mast is 055° and from S, the bearing is 032°. The angle of elevation of the top of the mast from R is 20°. Find the height of the mast.

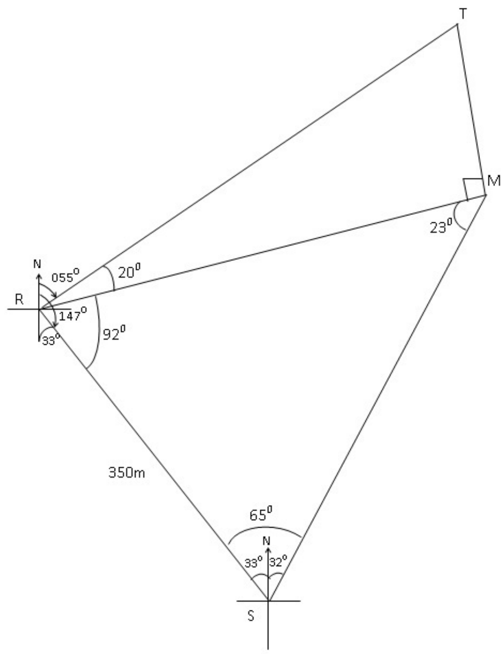


Figure 15.38

To find the lenght $\overline{MT}$ of the mast, first find $\overline{RM}$ by using Sine rule.

$$\frac{\overline{RM}}{\sin 65^\circ} = \frac{350m}{\sin 23^\circ}$$

$$\overline{RM} = \frac{350m \sin 65^\circ}{\sin 23^\circ}$$

$$\overline{RM} = 811.83m$$

Hence

$$\tan 20^\circ = \frac{\overline{MT}}{\overline{RM}}$$

$$811.83m \tan 20^\circ = \overline{MT}$$

$$\overline{MT} = 295.48m$$

Therefore, the height of the mast is 295.48m.
A vertical tower stands on a river bank. From a point on the other bank directly opposite and at a height h above the water level, the angle of elevation of the

top of the tower is α and the angle of depression of the reflection of the top of the tower β. (Assume the water is smooth and the reflection of any object in the water surface will appear to be as far below the surface as the object is above it) prove that the height of the top of the tower above the water is $h\sin(\alpha+\beta)\ cosec\ (\beta-\alpha)$ and the width of the river is $2h\cos\alpha\cos\beta\ cosec(\beta-\alpha)$.
Solution Let figure 5.37 show the river with the tower.

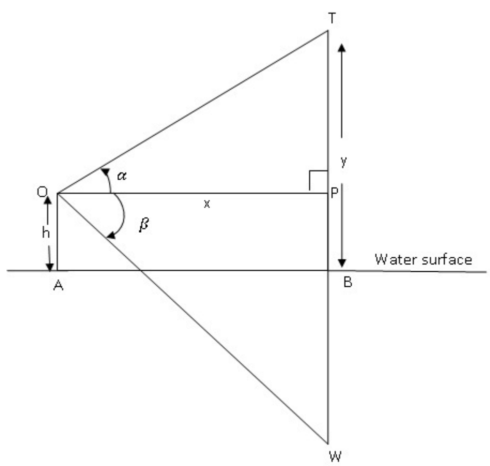


Figure 15.39

From figure 5.37, the tower is $\overline{TB}$, $\overline{OA}$ height of the observer, $\overline{OP}$ horizontal distance of observer from the tower. Let height of the tower $\overline{TB} = y$ and width of the river $\overline{AB} = x$.
i.e. $\overline{AB} = \overline{OP} = x$ and $\overline{TB} = BW = y$
In $\triangle OPT$, $\tan\alpha = \frac{\overline{TP}}{\overline{OP}}$

(i) $$\overline{TP} = \overline{OP}\tan\alpha$$

But $\overline{TP} = y - h$ and $\overline{OP} = x$
so eqn (i) becomes

(ii) $$y - h = x\tan\alpha$$

also
In $\triangle OPW$

$$\tan\beta = \frac{\overline{PW}}{\overline{OP}}$$

(iii) $$\overline{PW} = \overline{OP}\tan\beta$$

But $\overline{PW} = y + h$ and $\overline{OP} = x$

So eqn (iii) becomes

(iii) $$y + h = x\tan\beta$$

To eliminate y, we subtract (ii) from (iv)

$$2h = x(\tan\beta - \tan\alpha = x\left(\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha}\right) = x\left(\frac{\sin\beta\cos\alpha - \sin\alpha\cos\beta}{\cos\beta\cos\alpha} = x\left(\frac{\sin(\beta-\alpha)}{\cos\beta\cos\alpha}\right) 2h\cos\beta\cos\alpha$$

which proves the width of the river.
Also, to obtain the height of the tower y, multiply (ii) by $\tan\beta$ and (iv) by $\tan\alpha$ to obtain

(iv) $$y\tan\beta - h\tan\beta = x\tan\alpha\tan\beta$$

(iv) $$y\tan\beta - h\tan\beta = x\tan\alpha\tan\beta$$

(v) $$y\tan\alpha + h\tan\alpha = x\tan\alpha\tan\beta$$

Subtract eqn (vi) from eqn (v)

$$y\tan\beta - h\tan\beta - y\tan\alpha - h\tan\alpha = 0y(\tan\beta - \tan\alpha) - h(\tan\beta + \tan\alpha) = 0y(\tan\beta - \tan\alpha) = h(\tan\beta + \tan\alpha$$

Exercises

1. ABCD is a cube of side 8cm. E is the corner diagonally opposite A. Find (a) the length of $\overline{AE}$ (b) the angle $\overline{AE}$ makes with ABCD (c) $< BAE$.

2. From a point A, due South of a tower, the angle of elevation of the top is 45°. Another point B is due East of the tower and the bearing of B from A is 030°. Find the angle of elevation of the top of the tower from B.
3. ABC is an equilateral triangle on horizontal ground and each side is 65m long. At A there is a tree. The example of elevation of the top of the tree from C is 25°. Find (i) the height of the tree and (ii) the angle of elevation of the top of the tree from the midpoint of BC.
4. A column is hm high. A man is standing at a horizontal distance am from the base of the column, his eye level being at bm. He notices that a statue on top of the column subtends an angle θ at his eye. Find the height of the statue.
5. An observer O standing on top of a hill finds that the angles of depression to two points A and B on the since horizontal level are α and β respectively. If he is 300m vertically above AB and the angle AOB is γ, find the distance AB in terms of α, β and γ.
6. The base of a pyramid of vertex V is a square ABCD of side 2a. Each of the slant edges is of lenght $a\sqrt{3}$. Find (i) the angle between a slant face and the base (ii) the perpendicular distance of D from the edge VA and (iii) the angle between two adjacent slant faces.