INTEGRAL CALCULUS

[FOR B.A. & B.Sc. STUDENTS]

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PREFACE TO THE FIRST EDITION

This book is intended to serve as a companion volume to my "Differential Calculus", and is designed to meet the requirements of the B.A. and B.Sc. students of our universities.

The book opens with a chapter on the definition of indefinite integrals based on the anti-derivative notion and proceeds to give the geometrical interpretation of definite integrals. A short and simple account of improper integrals has also been given in this chapter. The second chapter has been devoted to explaining the two methods of integration, *viz.* "Integration by substitution" and "Integration by parts". A systematic account of the integration of Algebraic rational function, Trigonometric functions and Algebraic irrational functions, has then been given in Chapters III, IV and V respectively. This division of the book into chapters according to the class of functions to be integrated, and, not according to the methods of integration, will greatly help in enabling the student to obtain mastery over the technique of integration. Then follows the application of integration to Quadrature, Rectification and Volumes and Surfaces of revolution. Next comes the exhibition of a definite integral as the limit of a sum. This exhibition has based on purely geometrical, rather than on purely analytical considerations. I have deliberately refrained from giving the analytical proof as the rigorous analytical proof is beyond the comprehension of the students at this stage and the proof usually given is unsatisfactory and thus misleading. No separate chapter has been devoted to reduction formulae and all such cases have been considered at their appropriate places. The last two chapters have been devoted to a short Elementary course on Differential Equations.

The book contains a large number of examples to illustrate the various types.

I am greatly indebted to Prof. Sita Ram Gupta, M.A., P.E.S of the Government College, Lahore, who kindly went through a part of the manuscript and made some valuable suggestions. I am also deeply grateful to my friend and colleague Prof. Om Prakash M.A., of the D.A.V. College Lahore and Prof. Ramji Dass Syal M.A., of the Dayanand Technical Institute, Lahore who have helped me in the preparation of this book.

I shall be thankful to those who suggest improvements or point some errors which might have escaped my notice.

Lahore,
Dec., 1942

SHANTI NARAYAN

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Definitions

TABLE OF STANDARD RESULTS

1·1. Integral of a function. Integrand. If the differential coefficient of a function $f(x)$ is $F(x)$, *i.e.*, if

$$\frac{df(x)}{dx} = F(x),$$

we say that $f(x)$ is *an Integral* or a *Primitive* of $F(x)$ and, in symbols, write

$$\int F(x)dx = f(x).$$

For example,

$$\frac{d \sin x}{dx} = \cos x \Rightarrow \int \cos x \, dx = \sin x.$$

As another example, we see that

$$\frac{d \log x}{dx} = \frac{1}{x} \Rightarrow \int \frac{1}{x} dx = \log x$$

The letter x in dx, denotes that the integration is to be performed with respect to the variable x.

The process of determining an integral of a function is called *Integration* and the function to be integrated is called *Integrand.*

Since integration and differentiation are inverse processes, we have

$$\frac{d}{dx}\left[\int f(x) \, dx\right] = f(x).$$

Ex. *Show that*

(*i*) $\int x \, dx = x^2/2$. (*ii*) $\int \sin x \, dx = -\cos x + 2$.

(*iii*) $\int at^3 \, dt = at^4/4$. (*iv*) $\int 1/y \, dy = \log y - 3$.

1·2. The study of Integral Calculus consists in developing techniques for the determination of integral of a given function. This subject finds extensive applications to Geometry, Natural and Social sciences. In this book, we shall be concerned with applications in relation to the determination of Plane areas, Lengths of arcs and Volumes and Surfaces of solids of revolution, Centre of gravity and Moment of Inertia.

Historically, the subject arose in connection with the determination of areas of plane regions and was based on the notion of the limit of a type of a sum when the number of terms in the sum tends to infinity and each term tends to zero In fact, the name Integral Calculus has its origin in this process of summation and the words '*To integrate*' literally means '*To find the sum of*'. It is only afterwards that it was seen that the subject of Integration can also be viewed from the point of view of the Inverse of differentiation. We shall not develop the subject in the historical order and, as done above, start by defining Integration as the Inverse of differentiation. In Chapter 9, we shall consider the summation aspect of Integration also.

1·3. Indefinite valuedness of Integration. General Integral. Arbitrary Constant. If $f(x)$ is an integral of $F(x)$, then $f(x)+c$ is also an integral of $F(x)$; c being a constant whatsoever, for

$$\frac{d\,f(x)}{dx} = F(x) \Rightarrow \frac{d\,[\,f(x)+c]}{dx} = F(x).$$

Again, let $f(x)$, $\varphi(x)$ be two integrals of $F(x)$ so that we have

$$f'(x) = \varphi'(x) = F(x).$$

*As the differential coefficients of the functions $f(x)$ and $\varphi(x)$ are equal, the functions differ by some constant, *i.e.*,

$$f(x) - \varphi(x) = c \Leftrightarrow f(x) = \varphi(x)+c,$$

where c is a constant.

From these considerations, we deduce that the *Integral of a function is not unique* and that if $f(x)$ be any one integral of $F(x)$, then

(*i*) $f(x)+c$ is also its integral ; c being any constant whatsoever ;

(*ii*) every integral of $F(x)$ can be obtained from, $f(x)+c$, by giving some suitable value to c.

Thus if $f(x)$ be *any* one integral of $F(x)$, then $f(x)+c$ is its *General integral.*

From this it follows that *any two integrals of the same function differ by a constant.*

*Chapter IV of Author's *Differential Calculus.*

The constant, c, is called *Constant of integration.* The constant of integration will generally be *omitted* and the symbol $\int F(x)\,dx$ will denote any one integral of $F(x)$. But it must be remembered that the symbol $\int F(x)\,dx$ is *really infinite valued.*

It may happen that, by different methods of integration, we obtain different integrals of the same function, but it will always be seen that they differ from each other merely by a constant.

Ex. *Show that*

(*i*) x^4+c is the general integral of $4x^3$,

(*ii*) $\sin^{-1} x+c$ is the general integral of $1/\sqrt{(1-x^2)}$,

(*iii*) $\sec x+c$ is the general integral of $\sec x \tan x$,

where, c, denotes, an arbitrary constant.

1·4. Table of Elementary Integrals. We now give a table of elementary integrals based on the corresponding table of the differential coefficients of elementary functions.

$$^*\int x^n\,dx = \frac{x^{n+1}}{n+1},\ (n \neq -1) \qquad \because \frac{1}{n+1}\frac{d(x^{n+1})}{dx} = x^n.$$

$$\int \frac{1}{x}\,dx = \log x, \qquad \because \frac{d(\log x)}{dx} = \frac{1}{x}.$$

$$\int e^x\,dx = e^x, \qquad \because \frac{d(e^x)}{dx} = e^x.$$

$$\int a^x\,dx = \frac{a^x}{\log a}, \qquad \because \frac{d(a^x/\log a)}{dx} = a^x.$$

$$\int \sin x\,dx = -\cos x, \qquad \because \frac{d(-\cos x)}{dx} = \sin x.$$

$$\int \cos x\,dx = \sin x, \qquad \because \frac{d(\sin x)}{dx} = \cos x.$$

$$\int \sec^2 x\,dx = \tan x, \qquad \because \frac{d(\tan x)}{dx} = \sec^2 x.$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x \qquad \because \frac{d(-\cot x)}{dx} = \operatorname{cosec}^2 x.$$

$$\int \sec x \tan x\,dx = \sec x, \qquad \because \frac{d(\sec x)}{dx} = \sec x \tan x.$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x, \qquad \because \frac{d(-\operatorname{cosec} x)}{dx} = \cot x \operatorname{cosec} x.$$

$$\int \frac{1}{\sqrt{(1-x^2)}}\,dx = \sin^{-1} x \text{ or } -\cos^{-1} x,$$

$$\because \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{(1-x^2)}} = \frac{d(-\cos^{-1} x)}{dx}$$

---

*When $n = -1$, we have $x^n = x^{-1} = 1/x$ whose integral is $\log x$.

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x \quad \text{or} \quad -\cot^{-1}x, \quad \because \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$
$$= \frac{d(-\cot^{-1}x)}{dx}$$

$$\int \frac{1}{x\sqrt{(x^2-1)}}\,dx = \sec^{-1} x \text{ or } -\operatorname{cosec}^{-1}x, \quad \because \frac{d(\sec^{-1}x)}{dx} = \frac{1}{x\sqrt{(x^2-1)}}$$
$$= \frac{d(-\operatorname{cosec}^{-1}x)}{dx}$$

$$\int \cosh x\,dx = \sinh x, \quad \because \frac{d(\sinh x)}{dx} = \cosh x$$

$$\int \sinh x\,dx = \cosh x, \quad \because \frac{d(\cosh x)}{dx} = \sinh x.$$

It is important to notice that when $n \neq -1$, *the integral of x^n is obtained on increasing the index n by 1 and dividing by the increased index, $n+1$.* Thus, for example,

$$\int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}x^{3/2},$$

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = -\frac{1}{x}.$$

Note 1. The result

$$\int \frac{dx}{x} = \log x,$$

requires some explanation. We know that $\log x$ is defined for positive values of x only and, therefore, in the statement $d(\log x)/dx = 1/x$ or, in its equivalent statement $\int(1/x)dx = \log x$, it is implied that x is positive.

Now, when x is negative, $-x$ is positive, and therefore $\log(-x)$ has a meaning. In this case, we have

$$\frac{d[\log(-x)]}{dx} = \frac{1}{-x} \times -1 = \frac{1}{x}.$$

Therefore

$$\int \frac{1}{x}\,dx = \log(-x), \text{ if } x \text{ is negative.}$$

Thus the integral of $1/x$ is $\log x$ or $\log(-x)$ according as x is positive or negative. Both these results are included in the single statement

$$\int \frac{1}{x}\,dx = \log|x|,$$

where $|x|$ denotes the absolute value of x.

Note 2. The inverse trigonometrical functions in the above table are single-valued functions as defined in Chapter II of the author's *Differential Calculus*. Thus

$$\sin^{-1} x,\ \tan^{-1} x,\quad \cot^{-1} x,\quad \operatorname{cosec}^{-1} x$$

are the angles, lying between $-\pi/2$ and $\pi/2$, whose sine, tangent, cotangent and cosecant are x ; also

$$\cos^{-1} x,\ \sec^{-1} x,$$

are the angles, lying between 0 and π whose cosine and secant are x.

Note 3. From the above table, we see that both $\sin^{-1}x$ and $-\cos^{-1} x$ are integrals of $1/\sqrt{(1-x^2)}$. From this we cannot deduce the equality of $\sin^{-1} x$ and $-\cos^{-1} x$. The only legitimate conclusion is that they differ by some constant.

In fact, from elementary trigonometry, we know that

$$\sin^{-1} x - (-\cos^{-1} x) = \sin^{-1} x + \cos^{-1} x = \tfrac{1}{2}\pi.$$

Ex. Write down the integrals of

(*i*) x^3, (*ii*) $\sqrt{x}$, (*iii*) $\sqrt[3]{x^2}$, (*iv*) $\sqrt{x^{-3}}$,

(*v*) 2^x, (*vi*) $(\frac{1}{2})^x$, (*vii*) a^{2x}, (*viii*) e^{3x}.

Ans. (*i*) $\frac{1}{4}x^4$, (*ii*) $\frac{2}{3}x^{\frac{3}{2}}$, (*iii*) $\frac{3}{5}x^{\frac{5}{3}}$ (*iv*) $-2x^{-\frac{1}{2}}$

(*v*) $\dfrac{2^x}{\log 2}$, (*vi*) $\dfrac{-(\frac{1}{2})^x}{\log 2}$, (*vii*) $\dfrac{a^{2x}}{2\log a}$, (*viii*) $\dfrac{e^{3x}}{3}$.

1·5. Two simple theorems.

1·51. First Theorem.

$$\int a\, f(x)\, dx = a \int f(x)\, dx. \qquad \ldots(A)$$

i.e. the integral of the product of a constant and a function is equal to the product of the constant and the integral of the function.

The proof will follow from the corresponding theorem of Differential Calculus which states that the derivative of the product of a constant and a function is equal to the product of a constant and the derivative of the function.

Differentiating the right-hand side of (A), we obtain

$$\frac{d}{dx}\left[\, a \int f(x)\, dx \,\right] = a\, \frac{d}{dx}\int f(x)\, dx = af(x),$$

$$\Rightarrow \qquad \int af(x)\, dx = a \int f(x)\, dx.$$

Second theorem.

$$\int [f(x) \pm F(x)]\, dx = \int f(x)\, dx \pm \int F(x)\, dx \qquad \ldots(B)$$

i.e., The integral of the sum or difference of two functions is equal to the sum or difference of their integrals.

The proof will follow from the corresponding theorem of Differential Calculus which states that the derivative of the sum or difference of two functions is equal to the sum or difference of their derivatives.

Differentiating the right-hand side of (B), we obtain

$$\frac{d}{dx}\left[\int f(x)\,dx \pm \int F(x)\,dx\right] = \frac{d}{dx}\int f(x)\,dx \pm \frac{d}{dx}\int F(x)\,dx$$
$$= f(x) \pm F(x)$$
$$\Rightarrow \qquad \int [f(x) \pm F(x)]\,dx = \int f(x)\,dx \pm \int F(x)\,dx.$$

The theorem can easily be generalised to the case of the algebraic sum of a *finite* number of functions so that we have

$$\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots\ldots \pm f_n(x)]dx$$
$$= \int f_1(x)\,dx \pm \int f_2(x)dx \pm \ldots\ldots \pm \int f_n(x)\,dx.$$

Note. The two theorems prove useful when the integrand can be decomposed into the sum of a number of functions whose integrals are known. In fact this *decomposition of an integrand into the sum of a number of functions with known integrals* constitutes an important technique of integration as will be seen later on.

Examples

1. $\displaystyle\int 3x^3\,dx = 3\int x^3\,dx = \frac{3x^4}{4}$

2. $\int (a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n)\,dx$

$$= \int a_0 dx + \int a_1 x\,dx + \int a_2 x^2\,dx + \ldots\ldots\ldots + \int a_n x^n\,dx$$
$$= a_0 \int 1 dx + a_1 \int x\,dx + a_2 \int x^2\,dx + \ldots\ldots + a_n \int x^n\,dx$$
$$= a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_n \frac{x^{n+1}}{n+1}$$

3. $$\int \left(\cos x + \frac{2}{x} - e^x\right) dx = \int \cos x\,dx + 2\int \frac{1}{x}\,dx - \int e^x\,dx$$
$$= \sin x + 2 \log x - e^x$$

4. $$\int \frac{3 - 5x^2 + 7x^4 - 9x^6}{x^6}\,dx = \int \left(\frac{3}{x^6} - \frac{5}{x^4} + \frac{7}{x^2} - 9\right) dx$$
$$= \int \frac{3}{x^6}\,dx - \int \frac{5}{x^4}\,dx + \int \frac{7}{x^2}\,dx - \int 9\,dx$$
$$= -\frac{3}{5x^5} + \frac{5}{3x^3} - \frac{7}{x} - 9x$$
$$= \frac{-9 + 25x^2 - 105x^4 - 135x^6}{15x^5}$$

5. $$\int \frac{x^2}{1+x^2}\,dx = \int \frac{(x^2+1) - 1}{x^2+1}\,dx$$
$$= \int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \int 1\,dx - \int \frac{1}{x^2+1}\,dx = x - \tan^{-1} x.$$

6. $$\int \frac{x^4}{x^2+1}\,dx = \int \frac{x^4-1+1}{x^2+1}\,dx = \int\left(x^2-1+\frac{1}{x^2+1}\right)dx$$
$$= \frac{x^3}{3} - x + \tan^{-1} x.$$

7. $$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\,dx = \int \frac{\sin^3 x}{\sin^2 x \cos^2 x}\,dx + \int \frac{\cos^3 x}{\sin^2 x \cos^2 x}\,dx$$
$$= \int \tan x \sec x\,dx + \int \cot x \operatorname{cosec} x\,dx$$
$$= \sec x - \operatorname{cosec} x.$$

8. $$\int \sqrt{(1-\sin 2x)}\,dx = \int \sqrt{(\sin^2 x + \cos^2 x - 2\sin x \cos x)}\,dx$$
$$= \int \sqrt{(\sin x - \cos x)^2}\,dx$$
$$\int (\sin x - \cos x)\,dx$$
$$= -\cos x - \sin x = -(\cos x + \sin x)$$

9. $$\int \sqrt{(1-\cos 2x)}\,dx = \int \sqrt{(2\sin^2 x)}\,dx$$
$$= \int \sqrt{2}\sin x\,dx$$
$$= \sqrt{2}\int \sin x\,dx = -\sqrt{2}\cos x.$$

Exercises

Integrate the following functions :—

1. $x^{\frac{1}{3}} + x^{\frac{1}{2}} + x^{\frac{3}{5}}$.
2. $(x^2+2x+3)/x^4$.
3. $\dfrac{(1+2x)^3}{x^4}$,
4. $\dfrac{2x^4+3}{x^2+1}$.
5. $\dfrac{x^2-1}{x^2+1}$.
6. $\dfrac{(\sqrt{x}+\sqrt[3]{x^2})^2}{x}$.
7. $\dfrac{x^4+x^2+1}{2(x^2+1)}$.
8. $\dfrac{x^6-1}{x^2+1}$.
9. $5\cos x - 3\sin x - \dfrac{2}{\cos^2 x}$.
10. $\dfrac{5\cos^3 x + 7\sin^3 x}{2\sin^2 x \cos^2 x}$.
11. $\dfrac{\cos 2x}{\cos^2 x \sin^2 x}$.
12. $\sec^2 x \operatorname{cosec}^2 x$.
13. $\dfrac{3\cos x - 4}{\sin^2 x}$.
14. $\dfrac{1+2\sin x}{\cos^2 x}$.
15. $\tan^2 x$.
16. $\cot^2 x$.
17. $(\tan x + \cot x)^2$.
18. $(1-\cos 2x)/(1+\cos 2x)$.
19. $\sqrt{(1+\sin 2x)}$.
20. $\sqrt{(1+\cos 2x)}$.

Answers

1. $\frac{3}{4}x^{4/3}+\frac{2}{3}x^{3/2}+\frac{5}{8}x^{8/5}$.	2. $(x^2+x+1)/x^3$
3. $-(1+9x-36x^2)/3x^3+8\log x$.	4. $\frac{1}{3}(x^3-2x)+5\tan^{-1}x$
5. $x-2\tan^{-1}x$.	6. $x+\frac{3}{4}x^{4/3}+\frac{12}{7}x^{7/6}$
7. $\frac{1}{6}(x^3+3\tan^{-1}x)$.	8. $\frac{1}{5}x^5-\frac{1}{3}x^2+x-2\tan^{-1}x$
9. $5\sin x-3\cos x-2\tan x$	10. $\frac{7}{3}\sec x-\frac{5}{3}\operatorname{cosec} x$
11. $-\sec x\operatorname{cosec} x$.	12. $-2\cot 2x$.
13. $-4\cot x+3\operatorname{cosec} x$.	14. $\tan x+2\sec x$
15. $\tan x-x$.	16. $-\cot x-x$.
17. $-2\cot 2x$.	18. $\tan x-x$
19. $-\cos x+\sin x$	20. $\sqrt{2}\cos x$

1·6. Definite Integral. In geometrical and other applications of Integral Calculus, it becomes necessary to find the difference in the values of an integral of a function $f(x)$ for two assigned values of the independent variable x, say, a, b. This difference is called the *Definite integral of* $f(x)$ over the interval $[a, b]$ and is denoted by

$$\int_a^b f(x)\,dx.$$

Thus

$$\int_a^b f(x)\,dx = F(b)-F(a),$$

where $F(x)$ is an integral of $f(x)$.

The difference $[F(b)-F(a)]$ is sometimes denoted as

$$\Big|F(x)\Big|_a^b$$

Thus if $F(x)$ is an integral of $f(x)$, we write

$$\int_a^b f(x)dx = \Big|F(x)\Big|_a^b = F(b)-F(a).$$

The number, a, is called the *Lower limit* and the number, b, the *Upper limit* of integration.

It should be seen that the *value of a definite integral is unique and is independent of the particular integral which we may employ to calculate it.* Considering $F(x)+c$ instead of $F(x)$, we obtain

$$\int_a^b f(x)\,dx = \Big|F(x)+c\Big|_a^b = \Big\{F(b)+c\Big\}-\Big\{F(a)+c\Big\}$$
$$= F(b)-F(a),$$

so that the arbitrary constant, c, disappears in the process and we get the same value as on considering $F(x)$.

Examples

1. $\displaystyle\int_1^2 x\,dx = \left|\frac{x^2}{2}\right|_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}.$

2. $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx = \left|\tan^{-1}x\right|_0^1 = \tan^{-1}1 - \tan^{-1}0$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

3. $\displaystyle\int_0^{\pi/2} \sin x\,dx = \left|-\cos x\right|_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 1.$

Exercises

Evaluate the following definite integrals :

1. $\displaystyle\int_0^2 x^3\,dx.$
2. $\displaystyle\int_0^2 (2x+3x^2)\,dx.$
3. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1-x^2)}}\,dx.$
4. $\displaystyle\int_1^2 \frac{1}{x}\,dx.$
5. $\displaystyle\int_a^b e^x\,dx.$
6. $\displaystyle\int_0^1 x^4(1+x)^3\,dx.$
7. $\displaystyle\int_0^{\frac{1}{2}\pi} \cos x\,dx.$
8. $\displaystyle\int_0^{\frac{1}{4}\pi} \sec^2 x\,dx.$
9. $\displaystyle\int_0^2 \frac{x^4+1}{x^2+1}\,dx.$
10. $\displaystyle\int_0^{\frac{1}{3}\pi} \frac{2+3\sin x}{\cos^2 x}\,dx.$
11. $\displaystyle\int_{-1}^{+1} \cosh x\,dx$
12. $\displaystyle\int_0^1 \sqrt{(1+\cosh 2x)}\,dx.$

Answers

1. 4.	2. 12.	3. $\pi/6$.
4. log 2.	5. $e^b - e^a$	6. $\dfrac{351}{280}$.
7. 1.	8. 1.	9. $2/3 + 2\tan^{-1}2$.
10. $2\sqrt{3}+3$.	11. $(e^2-1)/e$.	12. $2\sinh 1$.

1·7. Two important properties of definite integrals.

(*i*) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

(*ii*) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where c, is a point inside or outside the interval $[a, b]$.

The proofs are quite simple.

Let $F(x)$ be an integral of $f(x)$, so that we have

$$\int_b^a f(x)dx = F(b)-F(a).$$

(*i*) We have

$$\int_b^a f(x)dx = F(x)\Big|_b^a = F(a)-F(b)$$
$$= -[F(b)-F(a)]$$
$$= -\int_a^b f(x)\,dx.$$

Hence the first result.

(*ii*) Now,

$$\begin{cases} \displaystyle\int_a^c f(x)\,dx = \Big|F(x)\Big|_a^c = F(c)-F(a). \\ \displaystyle\int_c^b f(x)\,dx = \Big|F(x)\Big|_c^b = F(b)-F(c). \end{cases}$$

$$\Rightarrow \quad \int_a^c f(x)\,dx+\int_c^b f(x)\,dx$$
$$= [F(c)-F(a)]+[F(b)-F(c)]$$
$$= F(b)-F(a) = \int_a^b f(x)\,dx.$$

Hence the second result.

1·8. Geometrical interpretation of a definite integral.

To show that the definite integral

$$\int_a^b f(x)\,dx$$

denotes the area bounded by the curve $y = f(x)$, *the axis of* x, *and the two ordinates* $x = a$ *and* $x = b$.

Let $y = f(x)$ be the equation of a curve referred to two rectangular axes. Let, A, denote the area bounded by the curve, the axis of x, a fixed ordinate AG, $(OA = a)$, and a variable ordinate MP.

Let $OM = x$ so that

$$MP = y = f(x).$$

The area A, depends on the position of the ordinate MP whose abscissa is x, and is, therefore, a function of x.

We take a point $Q(x+\Delta x, y+\Delta y)$ on the curve which lies so near P that, as a point moves along the curve from P to Q, its ordinate either *constantly* increases (Fig. 1) or constantly decreases (Fig. 2).

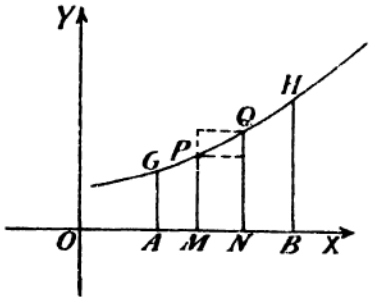

Fig. 1.

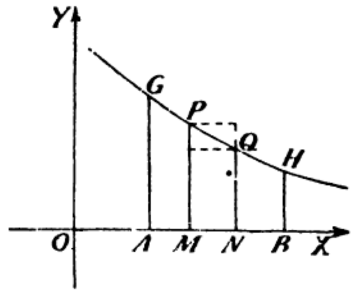

Fig. 2.

We have

$$ON = x + \Delta x, \quad NQ = y + \Delta y. \quad MN = \Delta x$$

The increment ΔA in A, consequent to the change Δx in x, is the area of the region $MNQPM$.

The area ΔA of the figure $MNQPM$ lies between the areas $(y + \Delta y)\,\Delta x$ and $y\,\Delta x$ of the two rectangles QM, PN.

For figure 1, we have

$$(y + \Delta y)\,\Delta x > \Delta A > y\,\Delta x$$

$$\Rightarrow \qquad (y + \Delta y) > \frac{\Delta A}{\Delta x} > y. \qquad \ldots(i)$$

Let $Q \to P$ so that $\Delta x \to 0$. Then from (i), we obtain

$$\frac{dA}{dx} = y = f(x).$$

For figure 2, we have

$$y\,\Delta x > \Delta A > (y + \Delta y)\,\Delta x$$

$$\Rightarrow \qquad y > \frac{\Delta A}{\Delta x} > (y + \Delta y),$$

so that, for this case also, we obtain in the limit

$$\frac{dA}{dx} = y = f(x).$$

Let BH be the ordinate $x = b$. We have

$$\int_a^b f(x)\,dx = \int_a^b \frac{dA}{dx}\,dx$$

$$= \Big| A \Big|_a^b$$

$$= \text{the value of } A \text{ for } x \text{ equal to } b.$$
$$- \text{the value of } A \text{ for } x \text{ equal to } a.$$
$$= \text{Area of the region } GABHGA - 0$$
$$= \text{Area of the region } GABHGA,$$

which is the area bounded by the curve $y = f(x)$, x-axis and the two ordinates $x = a$, $x = b$.

Note. The definition of the area-function, A, as given above, is not complete. To adequately define A so as to cover all possible cases, we agree to define by A the algebraic sum of the areas of all the portions enclosed by the curve, the axis of x and the two ordinates ; each portion being equipped with a proper sign $+$ or $-$, according to the following convention :—

(*i*) the areas of the portions to the right of the fixed ordinate GA lying above x-axis and also the areas of the portions to the left of GA lying below x-axis are positive ;

(*ii*) the areas of the portions to the left of GA lying above x-axis and the areas of the portions to the right of GA lying below x-axis are negative.

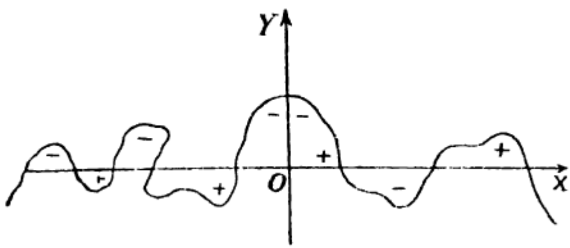

Fig. 3.

It is easy to show that with these conventions as to the meaning of the areas A, the result of § 1·8 holds whatever be the portion of the variable ordinate MP relative to GA.

Examples

1. *Find by integration, the area of the triangle the equations of whose sides are* $y = x$, $y = 0$ *and* $x = 2$; *also verify your result by elementary geometry.*

The area, in question, is enclosed by the curve $y = x$, the axis of x and the two ordinates $x = 0$ and $x = 2$. Therefore, the required area

$$= \int_0^2 y\,dx = \int_0^2 x\,dx$$

$$= \left| \frac{x^2}{2} \right|_0^2 = 2.$$

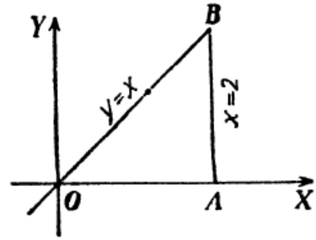

Fig. 4.

Also, by elementary geometry, the area of the triangle

$$= \tfrac{1}{2}\, OA.AB = \tfrac{1}{2}.2.2 = 2.$$

Hence the verification.

2. *Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 4x$.*

The equations to the parabola and the straight line are $y^2 = 4x$ and $y = 4x$.

Their points of intersection are $(0, 0)$, $(\frac{1}{4}, 1)$.

Therefore the required area

$$= \int_0^{\frac{1}{4}} (\sqrt{4x} - 4x)\,dx$$

$$= \left| 2 \cdot \frac{x^{3/2}}{\frac{3}{2}} - 4 \cdot \frac{x^2}{2} \right|_0^{\frac{1}{4}}$$

$$= \tfrac{4}{3} (\tfrac{1}{4})^{3/2} - 2 (\tfrac{1}{4})^2$$

$$= \frac{4}{3} \cdot \frac{1}{8} - 2 \cdot \frac{1}{16} = \frac{1}{24}$$

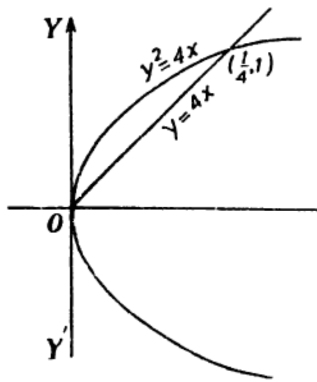

Fig. 5.

Exercises

1. Find the area between the x-axis and the curve $y = \sin x$ from $x = 0$ to $x = \pi$.

2. Find the area enclosed by the curve $y = \sec^2 x$; x-axis, y-axis and the ordinate $x = \frac{1}{4}\pi$.

3. Find the area enclosed by the curve $y = e^x$, x-axis and the two ordinates $x = -1$, $x = 1$.

4. Trace the curves $y = \sin x$, $y = \cos x$ as x varies from 0 to $\frac{1}{2}\pi$ and find the area of the region enclosed by them and the axis of x.

5. Show that the area of the region enclosed by the hyperbola $xy = 1$, x-axis and the two ordinates $x = 1$, $x = 2$ is $\log 2$.

6. Find the area of the region bounded by the parabola $y^2 = 4ax$ and its latus rectum.

Answers

1. 2, **2.** 1. **3.** $(e^2 - 1)/e$ **4.** $2-\sqrt{2}$. **6.** $\frac{8}{3}a^2$.

1·9. Improper definite integrals. In the definition of the definite integral

$$\int_a^b f(x)\,dx$$

and its geometrical interpretation, it is understood that

(*i*) both the limits a, b are finite, and

(*ii*) $f(x)$ is continuous in $[a, b]$.

We now generalise the definition so as to include the case of definite integrals for which either

(*i*) a or b or both are infinite or

(*ii*) for which $f(x)$ becomes infinite at some point of the interval $[a, b]$ *i.e.*, there exists point c of $[a, b]$ such that $f(x)$ tends to ∞ as x tends to c.

Integrals of these types are called *Generalised*, *Improper* or *Infinite* integrals.

1·91. Improper definite integrals of the first type. *Suppose that the upper limit b is ∞.*

To obtain the value of the integral

$$\int_a^\infty f(x)\,dx,$$

we first evaluate the definite integral

$$\int_a^t f(x)\,dx,$$

where t is a number $> a$. We then examine the limit of the definite integral whose value, of course, depends upon 't' as $t \to \infty$.

This limit, if it exists finitely, is defined to be the value of the symbol

$$\int_a^\infty f(x)\,dx.$$

In case the limit does not exist finitely, we cannot assign any meaning to

$$\int_a^\infty f(x)\,dx$$

and we, then, say this infinite integral does not exist.

Illustrations

1. To evaluate

$$\int_0^\infty \frac{1}{1+x^2}\,dx,$$

we first calculate the integral

$$\int_0^t \frac{1}{1+x^2}\,dx.$$

We have

$$\int_0^t \frac{1}{1+x^2}\,dx = \Big| \tan^{-1} x \Big|_0^t = \tan^{-1} t - \tan^{-1} 0 = \tan^{-1} t$$

$$\Rightarrow \qquad \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$$

$$\Rightarrow \qquad \int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

2. To evaluate

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx,$$

we first calculate the integral

$$\int_1^t \frac{dx}{\sqrt{x}}$$

We have

$$\int_1^t \frac{dx}{\sqrt{x}} = \Big| 2\sqrt{x} \Big|_1^t = 2\sqrt{t}-2$$

which $\to +\infty$ as $t \to \infty$.

Thus, we see that the infinite integral $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ does not exist.

1·92. *Now suppose that the lower limit is* $-\infty$. To obtain

$$\int_{-\infty}^b f(x)\,dx,$$

we first evaluate

$$\int_t^b f(x)\,dx.$$

The limit of this integral, as $t \to -\infty$, if it exists finitely, is to be the value of the integral

$$\int_{-\infty}^b f(x)\,dx.$$

1·93. To examine and evaluate, if possible, the infinite integral

$$\int_{-\infty}^\infty f(x)\,dx,$$

we examine the two integrals

$$\int_{-\infty}^{a} f(x)\,dx \text{ and } \int_{a}^{\infty} f(x)\,dx,$$

and then, if these two latter integrals exist finitely, we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx.$$

Ex. 1. *Evaluate the following improper integrals*

$$(i) \int_{1}^{\infty} \frac{1}{x^2}\,dx,\ (ii) \int_{4}^{\infty} \frac{dx}{\sqrt[2]{x^3}},\ (iii) \int_{-\infty}^{-1} \frac{dx}{x^4},\ (iv) \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$$

Ans. (*i*) 1, (*ii*) 1, (*iii*) $\frac{1}{3}$, (*iv*) π.

Ex. 2. *Show that the improper integrals*

$$\int_{1}^{\infty} \frac{1}{\sqrt{x}}\,dx \text{ and } \int_{1}^{\infty} \frac{1}{x}\,dx$$

do not exist.

Ex. 3. *Find the area bounded by the curve $y = 1/x^2$, the axis of x and the ordinate $x = 1$.*

In this case the region whose area is required extends to infinity so that, in the ordinary sense, we cannot speak of its area. A meaning may, however, be assigned to it by means of passage to the limit as follows.

Let $OA = a$.

We take a *variable* ordinate MP where $OM = t$ and consider the area of the finite region $GAMPG$. If this area tends to a finite limit as the ordinate MP recedes to infinity, then this limit is said to be the area of the infinite region under consideration.

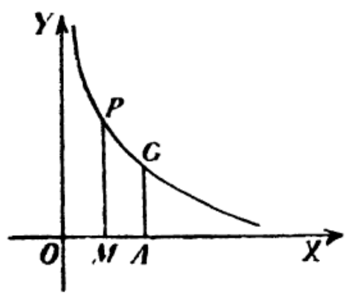

Fig. 6

Thus we have

$$\text{Area } GAMPG = \int_{1}^{t} y\,dx = \int_{1}^{t} \frac{1}{x^2}\,dx$$

$$= -\left|\frac{1}{x}\right|_{1}^{t} = -\frac{1}{t} + 1$$

which $\rightarrow 1$ as $t \rightarrow \infty$.

Thus the infinite region has a finite area which is equal to 1. When we say that the area of the infinite region is 1, we mean that the area of the region $GAMPG$ can be made as near, 1, as we like by taking MP sufficiently far off.

Ex. 4. *Examine the area bounded by the curve $y = 1/\sqrt{x}$, x-axis and the ordinate $x = 1$.*

Let $OM = t$, $OA = 1$

Area $GAMPG$

$$= \int_1^t \frac{1}{\sqrt{x}}\, dx = \Big| 2\sqrt{x} \Big|_1^t$$

$$= 2(\sqrt{t} - 1),$$

which $\to \infty$ as $t \to \infty$ so that the areas of the infinite region in this case is not finite.

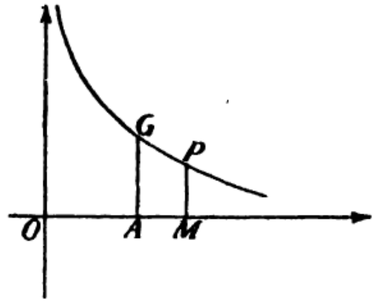

Fig. 7.

This means that the area of the region $GAMPG$ can be made as large as we like by taking MP sufficiently far off.

Note. The fact that the area of the infinite region in Ex. 3 is finite, whereas that of the infinite region in Ex. 4 is infinite, is intuitively explained by the fact that in the former case the curve approaches the X-axis much more rapidly than in the latter.

Ex. 5. *Examine the area lying in the second quadrant and bounded by the curve $y = e^x$ and the two co-ordinate axes.*

1·94. Improper definite Integrals of the second type. *Let $f(x)$ tend to infinity as x tends to ∞ and at no other point.* Let h be a positive number. We evaluate the proper integral

$$\int_{a+h}^{b} f(x)\, dx$$

whose value is a function of h. If this function of h tends to a finite limit as h tends to 0, then this finite limit is defined to be the value of the improper integral

$$\int_a^b f(x)\, dx.$$

In case the limit does not exist finitely, then

$$\int_a^b f(x)\, dx$$

has no meaning.

Similarly, if $f(x)$ tends to infinity as $x \to b$ and for no other point, we examine the limit of the proper integral

$$\int_a^{b-h} f(x)\, dx,$$

as h tends to 0. This limit, if it exists finitely, is defined to be the value of

$$\int_a^b f(x)\,dx.$$

If $f(x)$ tends to infinity at some point $x = c$ within the interval $[a, b]$, then we examine the two improper integrals

$$\int_a^c f(x)\,dx \text{ and } \int_c^b f(x)\,dx$$

and if they both exist finitely, we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Ex. 1. *Examine the improper integral*

$$\int_0^1 \frac{1}{\sqrt{(1-x^2)}}\,dx.$$

The integrand $1/\sqrt{(1-x^2)}$ tends to infinity as x tends to 1. To examine the value of the integral, we evaluate

$$\int_0^{1-h} \frac{1}{\sqrt{(1-x^2)}}\,dx.$$

We have

$$\int_0^{1-h} \frac{1}{\sqrt{(1-x^2)}}\,dx = \Big| \sin^{-1} x \Big|_0^{1-h}$$
$$= \sin^{-1}(1-h) - \sin^{-1} 0$$
$$= \sin^{-1}(1-h).$$

$$\Rightarrow \lim_{h \to 0} \int_0^{1-h} \frac{1}{\sqrt{(1-x^2)}}\,dx = \lim_{h \to 0} \sin^{-1}(1-h) = \sin^{-1} 1 = \frac{\pi}{2}$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{(1-x^2)}}\,dx = \frac{\pi}{2}.$$

Ex. 2. *Examine the area bounded by the curve, $y = 1/\sqrt{x}$, Y-axis and the ordinate $x = 1$.*

Here the region extends to infinity and we proceed as in Ex. 3 of § 1·93.

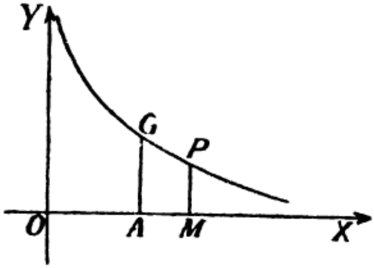

Fig. 8.

Let $OA = 1$.

We take an ordinate MP whose abscissa is h and examine the limit of the area of the region $MPGAM$ as the ordinate MP tends towards Y-axis, *i.e.*, as $h \to 0$. The area of the region

$$MPGAM = \int_h^1 y\,dx$$

$$= \int_h^1 \frac{1}{\sqrt{x}}\, dx$$

$$= \Big|\, 2\sqrt{x}\, \Big|_h^1 = 2-2\sqrt{h},$$

which $\to$ the finite limit 2, as $h \to 0$.

Thus in this case we say that the area of the infinite region under consideration is 2.

Ex. 3. *Examine the improper integrals*

$$(i) \int_0^1 \frac{1}{\sqrt[3]{x}}\, dx, \qquad (ii) \int_0^1 \frac{1}{x^2}\, dx.$$

Ex. 4. *Show that the area of the region bounded by the curve $y = 1/x^4$, Y-axis and the ordinate $x = 2$ is not finite.*

Answers

(*i*) $\frac{3}{2}$ (*ii*) Does not exist.

2

Methods of Integration

Integration by Substitution and Integration by Parts.

2·1. Methods of Integration. The following are the four principal methods of Integration :

I. *Decomposition of the given integrand as a sum of integrands with known integrals.*

II. *Integration by substitution.*

III. *Integration by parts.*

IV. *Integration by successive reduction.*

The first method of integration which depends upon the two theorems proved in §1.5, p. 5, has already been illustrated in the preceding chapter. It will be seen in chapter 3 that this method of integration is very largely employed for the integration of algebraic rational functions.

The other methods will be taken up in this chapter. It will be seen that the method of *Integrating by parts* is essentially a method by successive reduction for, with its help, we are enabled to express the integral of a product of two functions in terms of another whose evaluation may be simpler. The method of integration by successive reduction is thus also only a development of the method of integration by parts.

In the present chapter, we shall be laying emphasis mainly on the different *methods* of integration and in the following three chapters we shall consider the various *Classes of functions* and indicate the method of integrating functions belonging to any given class.

Note. The process of integration is largely of a tentative nature and is not so systematic as that of differentiation. In general, experience is the best guide for suggesting the quickest and the simplest method for integrating a given function.

2·2. Integration by substitution. This method consists in expressing the integral $\int f(x)\, dx$, where x is the independent variable, in terms of another integral where some other variable, say t, is the independent variable; x and t being connected by some suitable relation $x = \phi(t)$.

It leads to the result

$$\int \mathbf{f(x)\, dx} = \int \mathbf{f[\phi(t)]\, \phi'(t)\, dt},$$

which is proved as follows :—

Let $v = \int f(x)\, dx \Rightarrow \dfrac{dv}{dx} = f(x).$

We have

$$\frac{dv}{d} = \frac{dv}{dx} \cdot \frac{dx}{dt} = f(x) . \frac{dx}{dt} .$$

$$\Rightarrow \quad v = \int f(x) \frac{dx}{dt}\, dt = \int f[\phi(t)]\, \phi'(t)\, dt \text{ for } x = \phi(t).$$

Thus we have shown that

the integral of a function $f(x)$ with respect to x is equal to the integral of, $f(x)\, dx/dt$ with respect to t.

Here x is to be replaced by $\phi(t)$.

This method proves useful only when a relation $x = \phi(t)$ can be so selected that the new integrand $f(x)(dx/dt)$ is of a form whose integral is known.

An Important Note. It may be noted that in the result

$$\int f(x)\, dx = \int f[\phi(t)]\, \phi'(t) dt$$

dx has been replaced by $\phi'(t)dt$ and this equality can be obtained from $dx/dt = \phi'(t)$, *by supposing* that dx and dt are separate quantities. This supposition greatly simplifies the presentation of the process of integration by substitution. [Refer Ex. 3, on next page].

The logical justification for this supposition is not required here, for it has only been *formally* introduced for the sake of convenience.

Examples

1. *Integrate* $e^x \sin e^x$.

We put $e^x = t \Rightarrow e^x \dfrac{dx}{dt} = 1 \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{e^x}.$

$$\therefore \quad \int e^x \sin e^x\, dx = \int \left(e^x \sin e^x \right) \frac{dx}{dt}\, dt$$

$$= \int \left(e^x \sin e^x \right) \frac{1}{e^x}\, dt$$

$$= \int \sin t \, dt = -\cos t$$
$$= -\cos e^x.$$

2. *Integrate* $\cos^3 x \sin x$.

We put $\cos x = t \quad \Rightarrow -\sin x \dfrac{dx}{dt} = 1 \Rightarrow \dfrac{dx}{dt} = \dfrac{-1}{\sin x}$.

$$\therefore \int \cos^3 x \sin x \, dx = \int \cos^3 x \sin x . \frac{dx}{dt} dt$$
$$= \int \cos^3 x \sin x . \frac{-1}{\sin x} dt$$
$$= \int -t^3 \, dt = -\frac{t^4}{4}$$
$$= -\frac{1}{4} \cos^4 x.$$

3. *Evaluate* $\displaystyle\int \frac{x^5}{1+x^{12}} dx$.

We put $x^6 = t \quad \Rightarrow 6x^5 \dfrac{dx}{dt} = 1 \Rightarrow 6x^5 \, dx = dt.$

$$\therefore \int \frac{x^5}{1+x^{12}} dx = \int \frac{dt}{6(1+t^2)} = \frac{1}{6} \tan^{-1} t$$
$$= \frac{1}{6} \tan^{-1} x^6 .$$

Exercises

Find the integrals of the following functions :—

1. (*i*) $e^x \cos e^x$, (*ii*) $2xe^{x^2}$, (*iii*) $x^3 e^{x^4}$,

(*iv*) $e^{\tan t} \sec^2 t$, (*v*) $e^{\log x} / x$, (*vi*) $\dfrac{e^{\tan^{-1} x}}{1+x^2}$,

(*vii*) $\dfrac{e^{\sin^{-1} x}}{\sqrt{(1-x^2)}}$, (*viii*) $\dfrac{e^{\sqrt{x}}}{3\sqrt{x}}$, (*ix*) $\dfrac{e^x(1+x)}{\cos^2(xe^x)}$.

2. (*i*) $\sin^2 x \cos x$, (*ii*) $\sqrt[3]{\sin x} \cos x$, (*iii*) $\sin x \cos x$,

(*iv*) $3 \sin x \sec^4 x$.

3. (*i*) $\dfrac{\cos x}{1+\sin^2 x}$, (*ii*) $\dfrac{\tan^2 x \sec^2 x}{1+\tan^6 x}$,

(*iii*) $\dfrac{2}{x[1+(\log x)^2]}$, (*iv*) $\dfrac{3e^{2x}}{1+e^{4x}}$, (*v*) $\dfrac{2x}{1+x^4}$,

(*vi*) $\dfrac{x^5}{1+x^{12}}$, (*vii*) $\dfrac{2x^3}{1+x^8}$, (*viii*) $\dfrac{1}{e^x+e^{-x}}$.

4. (*i*) $\dfrac{\cos x}{(1+\sin x)^2}$, (*ii*) $\dfrac{\sec^2 x}{(1+\tan x)^3}$, (*iii*) $\dfrac{(1-\log x)^2}{x}$.

5. (*i*) $4x^3 \operatorname{cosec}^2 (x^4)$, (*ii*) $x^4 \sec^2 (x^5)$, (*iii*) $x^3 \sin x^4$,
   (*iv*) $\cos \sqrt{x}/\sqrt{x}$, (*v*) $e^x \sec^2 (e^x)$.

6. (*i*) $\dfrac{\cos (\log x)}{x}$, (*ii*) $\dfrac{\sec^2 (\log x)}{x}$,
   (*iii*) $\dfrac{\sin (2+3 \log x)}{x}$, (*iv*) $e^x \tan (e^x) \sec (e^x)$.

7. (*i*) $\dfrac{2x}{\sqrt{(1-x^4)}}$, (*ii*) $\dfrac{x^2}{\sqrt{(1-x^6)}}$,
   (*iii*) $\dfrac{2}{\sqrt{[2-(2x+3)^2]}}$, (*iv*) $\dfrac{\sec^2 x}{\sqrt{(1-\tan^2 x)}}$.

8. (*i*) $\dfrac{x^2 \tan^{-1} x^3}{1+x^6}$, (*ii*) $\dfrac{2x \sin^{-1} x^2}{\sqrt{(1-x^4)}}$,
   (*iii*) $\dfrac{\tan \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}$.

Answers

1. (*i*) $\sin e^x$ (*ii*) e^{x^2} (*iii*) $\frac{1}{3} e^{x^3}$
   (*iv*) $e^{\tan t}$ (*v*) x (*vi*) $e^{\tan^{-1} x}$
   (*vii*) $e^{\sin^{-1} x}$ (*viii*) $\frac{2}{3} e^{\sqrt{x}}$ (*ix*) $\tan (xe^x)$

2. (*i*) $\frac{1}{3} \sin^3 x$ (*ii*) $\frac{3}{4} \sin^{4/3} x$ (*iii*) $-\frac{1}{4} \cos 2x$
   (*iv*) $\sec^3 x$

3. (*i*) $\tan^{-1} (\sin x)$ (*ii*) $\frac{1}{3} \tan^{-1} (\tan^3 x)$
   (*iii*) $2 \tan^{-1} (\log x)$ (*iv*) $\frac{3}{5} \tan^{-1} (e^{2x})$
   (*v*) $\tan^{-1} x^2$ (*vi*) $\frac{1}{5} \tan^{-1} x^5$ (*vii*) $\frac{1}{4} \tan^{-1} x^4$
   (*viii*) $\tan^{-1}(e^x)$

4. (*i*) $-1/(1+\sin x)$ (*ii*) $-\frac{1}{2}(1+\tan x)^2$
   (*iii*) $-\frac{1}{3}(1-\log x)^3$

5. (*i*) $-\cot x^4$ (*ii*) $\frac{1}{5} \tan x^5$ (*iii*) $-\frac{1}{4} \cos x^4$
   (*iv*) $2 \sin \sqrt{x}$ (*v*) $\tan e^x$

6. (*i*) $\sin (\log x)$ (*ii*) $\tan (\log x)$ (*iii*) $-\frac{1}{3} \cos (2+3 \log x)$
   (*iv*) $\sec e^x$

7. (*i*) $\sin^{-1} x^2$ (*ii*) $\frac{1}{3} \sin^{-1} x^3$ (*iii*) $\sin^{-1} [(2x+3)/\sqrt{2}]$
   (*iv*) $\sin^{-1} (\tan x)$

8. (*i*) $\frac{1}{6} (\tan^{-1} x^3)^2$ (*ii*) $\frac{1}{2} (\sin^{-1} x^2)^2$ (*iii*) $(\tan \sqrt{x})^2$

2.3. Three important forms of integrals

2·31. $\displaystyle\int \frac{\mathbf{f'(x)}}{\mathbf{f(x)}}\,\mathbf{dx} = \mathbf{log\ f(x)}.$

We put $f(x) = t \Rightarrow f'(x)\,dx = dt$.

$$\therefore \qquad \int \frac{f'(x)}{f(x)}\,dx = \int \frac{dt}{t} = \log t = \log f(x).$$

Thus we see, that

the integral of a fraction whose numerator is the derivative of its denominator is equal to the logarithm of the denominator.

Illustrations

(1) $\displaystyle\int \frac{3x^2}{1+x^3}\,dx = \log(1+x^3)$, for the numerator $3x^2$ is the derivative of the denominator $1+x^3$.

(2) $\displaystyle\int \frac{e^x}{1+e^x}\,dx = \log(1+e^x)$.

2·311. Integrals of tan x, cot x, sec x, cosec x.

The result given above enables us to obtain the integrals of $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ as shown below :—

I. $\displaystyle\int \tan x\ dx = \int \frac{\sec x \tan x}{\sec x}\,dx = \log \sec x.$

$$\int \mathbf{tan\ x\ dx = log\ sec\ x}.$$

II. $\displaystyle\int \cot x\ dx = \int \frac{\cos x}{\sin x}\,dx = \log \sin x.$

$$\int \mathbf{cot\ x\ dx = log\ sin\ x}.$$

III. $\displaystyle\int \sec x\ dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx$

$$= \log(\sec x+\tan x),$$

for the numerator, $\sec^2 x+\sec x \tan x$, is the derivative of the denominator $\sec x+\tan x$.

To put the result in another form, we write

$$\sec x+\tan x = \frac{1+\sin x}{\cos x}$$

and employ the results

$$\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)},\quad \cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)},$$

so that we obtain

$$\sec x+\tan x = \frac{1+\tan(x/2)}{1-\tan(x/2)} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

Thus $\qquad \displaystyle\int \mathbf{sec\ x\ dx = log\ tan}\left(\frac{\pi}{4}+\frac{\mathbf{x}}{\mathbf{2}}\right).$

IV. $\displaystyle\int \operatorname{cosec} x\ dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}\,dx$

$$= \log\,(\operatorname{cosec} x - \cot x).$$

As in III above, we have

$$\operatorname{cosec} x - \cot x = \frac{1-\cos x}{\sin x} = \frac{1-\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}{\dfrac{2\tan^2(x/2)}{1+\tan^2(x/2)}} = \tan\frac{x}{2}$$

Thus $$\int \operatorname{cosec} x \, dx = \log \tan \frac{x}{2}.$$

Exercises

1. Integrate the following :

(*i*) $\dfrac{x^2}{1+x^3}$, (*ii*) $\dfrac{2x+1}{x^2+x+1}$,

(*iii*) $\dfrac{1}{\sqrt{(1-x^2)}\,\sin^{-1} x}$, (*iv*) $\dfrac{1}{(1+x^2)\tan^{-1} x}$, (*v*) $\tanh x$,

(*vi*) $\coth x$, (*vii*) $\dfrac{\cos x}{a+b\sin x}$, (*viii*) $\dfrac{\cot x}{\log \sin x}$,

(*ix*) $\dfrac{\sec^2 x}{1+\tan x}$, (*x*) $\dfrac{\operatorname{cosec}^2 x}{1+\cot x}$, (*xi*) $\dfrac{1}{x\log x}$,

(*xii*) $\dfrac{x^{e-1}+e^{x-1}}{x^e+e^x}$, (*xiii*) $\dfrac{ax^{n-1}}{bx^n+c}$, (*xiv*) $\dfrac{1}{\sqrt{x}+x}$,

(*xv*) $\dfrac{1}{x\log x\,[\log(\log x)]}$, (*xvi*) $\dfrac{\sin 2x}{a\cos^2 x+b\sin^2 x}$,

(*xvii*) $\dfrac{1-\tan x}{1+\tan x}$, (*xviii*) $\dfrac{\cos 2x}{(\sin x+\cos x)^2}$,

(*xix*) $\dfrac{\sec x \operatorname{cosec} x}{\log(\tan x)}$

2. Integrate the following functions by changing the independent variable :

(*i*) $e^x \tan(e^x)$, (*ii*) $x^2 \tan x^3$, (*iii*) $\cos x \cot(\sin x)$,

(*iv*) $\dfrac{\cot(\log x)}{x}$, (*v*) $\dfrac{\tan(\sin^{-1} x)}{\sqrt{(1-x^2)}}$, (*vi*) $\dfrac{\cot\sqrt{x}}{\sqrt{x}}$.

(*vii*) $\cot(2x+3)$, (*viii*) $\tan(3x+4)$, (*ix*) $e^x \cot(e^x)$.

3. Integrate

(*i*) $\sec(\tan x)\sec^2 x$. (*ii*) $x^{n-1}\operatorname{cosec} x^n$,

(*iii*) $\sec(ax+b)$, (*iv*) $\operatorname{cosec}(ax+b)$,

(*v*) $\operatorname{cosec}(\operatorname{cosec} x)\cot x \operatorname{cosec} x$, (*vi*) $\dfrac{\operatorname{cosec}(\tan^{-1} x)}{1+x^2}$.

4. Evaluate

$$\int \frac{dx}{\sin(x-a)\sin(x-b)}.$$

Answers

1. (*i*) $\frac{1}{3}\log(1+x^3)$ (*ii*) $\log(x^2+x+1)$
(*iii*) $\log\sin^{-1}x$ (*iv*) $\log\tan^{-1}x$
(*v*) $\log\cosh x$ (*vi*) $\log\sinh x$
(*vii*) $(1/b)\log(a+b\sin x)$ (*viii*) $\log\log(\sin x)$
(*ix*) $\log(1+\tan x)$ (*x*) $-\log(1+\cot x)$
(*xi*) $\log\log x$ (*xii*) $(1/e)\log(x^e+e^x)$
(*xiii*) $(a/nb)\log(bx^n+c)$ (*xiv*) $2\log(1+\sqrt{x})$
(*xv*) $\log[\log(\log x)]$ (*xvi*) $[1/(b-a)]\log(a\cos^2 x+b\sin^2 x)$
(*xvii*) $\log(\sin x+\cos x)$ (*xviii*) $\log(\sin x+\cos x)$
(*xix*) $\log[\log(\tan x)]$

2. (*i*) $\log(\sec e^x)$ (*ii*) $\frac{1}{3}\log(\sec x^3)$ (*iii*) $\log[\sin(\sin x)]$
(*iv*) $\log[\sin(\log x)]$ (*v*) $\log\sec(\sin^{-1}x)$ (*vi*) $2\log(\sin\sqrt{x})$
(*vii*) $\frac{1}{2}\log[\sin(2x+3)]$ (*viii*) $\frac{1}{3}\log[\sec(3x+4)]$ (*ix*) $\frac{1}{2}\log(\sin e^x)$

3. (*i*) $\log\left[\tan\left(\frac{\pi}{4}+\frac{1}{2}\tan x\right)\right]$ (*ii*) $\left(\frac{1}{n}\right)\log\left[\tan\left(\frac{1}{2}x^n\right)\right]$
(*iii*) $(1/a)\log\tan\left(\frac{1}{4}\pi+\frac{1}{2}(ax+b)\right)$ (*iv*) $(1/a)\log\left(\frac{ax+b}{2}\right)$
(*v*) $-\log\tan(\frac{1}{2}\operatorname{cosec}x)$ (*vi*) $\log\tan(\frac{1}{2}\tan^{-1}x)$

4. $\frac{1}{\sin(a-b)}\log\frac{\sin(x-a)}{\sin(x-b)}$

2·32. $\displaystyle\int[f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$, **when** $n \neq -1$.

We put $f(x) = t \Rightarrow f'(x)\,dx = dt$.

$$\therefore \quad \int[f(x)]^n f'(x)\,dx = \int t^n\,dt = \frac{t^{n+1}}{n+1},\ \text{for } n\neq -1$$
$$= \frac{[f(x)]^{n+1}}{n+1}.$$

It should be noticed that the integrand consists of the product of a power of a function $f(x)$ and the derivative $f'(x)$ of $f(x)$. The integral then is obtained on increasing the index by unity and dividing by the increased index.

The case of $n = -1$ corresponds to that of the preceding subsection 2·31.

Illustration. Evaluate

$$\int \sin^4 x\cos x\,dx$$

Taking $f(x) = \sin x$, we see that the given integral is of the form

$$\int[f(x)]^4 f'(x)\,dx$$

so that

$$\int\sin^4 x\cos x\,dx = \frac{\sin^{4+1}x}{4+1} = \frac{1}{5}\sin^5 x$$

Exercises

1. Integrate the following :—

(i) $\sqrt{\sin x}\cos x$, (ii) $\tan^4 x \sec^2 x$,

(iii) $\operatorname{cosec}^2 x\sqrt{\cot x}$, (iv) $\dfrac{(\tan^{-1}x)^3}{1+x^2}$,

(v) $\dfrac{1}{(\tan^{-1}x)^2(1+x^2)}$, (vi) $\dfrac{\sin^{-1}x}{\sqrt{(1-x^2)}}$,

(vii) $\dfrac{1}{\sqrt{(\sin^{-1}x)}\sqrt{(1-x^2)}}$, (viii) $\dfrac{x}{\sqrt{(1-x^2)}}$,

(ix) $\dfrac{x}{\sqrt[3]{(x^2+1)}}$, (x) $x\sqrt{(x^2+1)}$,

(xi) $e^x\sqrt{(1+e^x)}$, (xii) $\sin^3 x\cos x$,

(xiii) $\dfrac{2x+3}{\sqrt{(x^2+3x-4)}}$, (xiv) $\dfrac{(x+1)(x+\log x)^2}{2x}$,

(xv) $\dfrac{1}{x(1+\log x)^3}$, (xvi) $\dfrac{\log x}{x}$,

(xvii) $\sqrt{(2+\sec^2 x)}\sec^2 x\tan x$,

(xviii) $(e^x+e^{-x})(e^x-e^{-x})$, (xix) $\sec x\log(\sec x+\tan x)$.

Answers

1. (i) $\frac{2}{3}(\sin x)^{3/2}$ (ii) $\frac{1}{5}\tan^5 x$ (iii) $-\frac{2}{3}\cot^{2/3}x$.
(iv) $\frac{1}{4}(\tan^{-1}x)^4$ (v) $-1/\tan^{-1}x$ (vi) $\frac{1}{2}(\sin^{-1}x)^2$
(vii) $2\sqrt{(\sin^{-1}x)}$ (viii) $-\sqrt{(1-x^2)}$ (ix) $\frac{3}{4}(1+x^2)^{2/3}$
(x) $\frac{1}{3}(1+x^2)^{3/2}$ (xi) $\frac{2}{3}(1+e^x)^{3/2}$ (xii) $\frac{1}{4}\sin^4 x$
(xiii) $2\sqrt{(x^2+3x-4)}$ (xiv) $\frac{1}{6}(x+\log x)^3$ (xv) $-\frac{1}{2}(1+\log x)^2$
(xvi) $\frac{1}{2}(\log x)^2$ (xvii) $\frac{1}{3}(2+\sec^2 x)^{3/2}$ (xviii) $\frac{1}{2}(e^x-e^{-x})^2$
(xix) $\frac{1}{2}[\log(\sec x+\tan x)]^2$

2·33. $\displaystyle\int \mathbf{f'(ax+b)\,dx = \frac{f(ax+b)}{a}}$.

We put

$$ax+b=t \Rightarrow a\,dx=dt \Rightarrow dx \Rightarrow dt/a.$$

$$\therefore \qquad \int f(ax+b)\,dx=\int f'(t)\cdot\frac{dt}{a}$$

$$=\frac{1}{a}\int f'(t)\,dt$$

$$=\frac{1}{a}\cdot f(t)=\frac{1}{a}f(ax+b)$$

Thus the integral of a function of $(ax+b)$ is of the same form as the integral of the same function of x divided by a, which is the coefficient of x.

Examples

1. (*i*) The integral of $\cos x$ is $\sin x \Rightarrow$ the integral of $\cos(ax+b)$ is $\dfrac{1}{a}\sin(ax+b)$.

(*ii*) The integral of x^3 is $\dfrac{x^4}{4} \Rightarrow$ the integral of $(2x+3)^3$ is

$$\frac{(2x+3)^4}{4\cdot 2} = \frac{(2x+3)^4}{8}.$$

2. *Integrate* (*i*) $\sin^2 x$, (*ii*) $\sin^3 x$, (*iii*) $\cos x \cos 2x$.

(*i*) We know that

$$\cos 2x = 1-2\sin^2 x$$

$$\therefore \quad \sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$

$$\therefore \quad \int \sin^2 x\, dx = \tfrac{1}{2}\int (1-\cos 2x)\, dx$$

$$= \tfrac{1}{2}\left[\int 1\, dx - \int \cos 2x\, dx\right]$$

$$= \frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)$$

$$= \tfrac{1}{2}(x-\sin x\cos x)$$

(*ii*) We know that

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\therefore \quad \sin^3 x = \tfrac{1}{4}(3\sin x - \sin 3x)$$

$$\int \sin^3 x\, dx = \tfrac{1}{4}\left[\int 3\sin x\, dx - \int \sin 3x\, dx\right]$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]$$

$$= 1/12\,[\cos 3x - 9\cos x]$$

(*iii*) We know that

$$\cos x\cos 2x = \tfrac{1}{2}[\cos 3x + \cos x]$$

$$\therefore \quad \int \cos x\cos 2x\, dx = \tfrac{1}{2}\left[\int \cos 3x\, dx + \int \cos x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{\sin 3x}{3} + \sin x\right]$$

$$= \frac{\sin 3x + 3\sin x}{6}.$$

3. *Integrate* $\dfrac{1}{a\sin x + b\cos x}$.

We find two numbers r and α such that

$$a = r\cos\alpha,\ b = r\sin\alpha.$$

From these, squaring and adding, we obtain

$$r = \sqrt{(a^2+b^2)},$$

and on dividing one by the other, we obtain

$$\tan \alpha = b/a \quad \text{or} \quad \alpha = \tan^{-1}(b/a)$$

$$\therefore \; a \sin x + b \cos x = r(\sin x \cos \alpha + \cos x \sin \alpha)$$
$$= r \sin (x+\alpha),$$

$$\therefore \int \frac{dx}{a \sin x + b \cos x} = \frac{1}{r} \int \frac{dx}{\sin (x+\alpha)}$$
$$= \frac{1}{r} \int \operatorname{cosec} (x+\alpha) dx$$
$$= \frac{1}{r} \log \tan \frac{x+\alpha}{2},$$

where $r = \sqrt{(a^2+b^2)}$, $\alpha = \tan^{-1}(b/a)$.

Exercises

1. Integrate :

(*i*) $\cos^2 x$, (*ii*) $\cos^3 x$, (*iii*) $\sin 4x \cos 2x$,
(*iv*) $\sin 5x \sin 3x$, (*v*) $\cos x \cos 2x \cos 3x$.

2. Integrate :

(*i*) $\sin^4 x$, (*ii*) $\cos^4 x$.

3. Integrate :

(*i*) $1/(ax+b)$, (*ii*) $\sec^2 (2x+3)$, (*iii*) $\cot (4x+5)$,
(*iv*) $\cos \frac{x}{2}$, (*v*) $\sec^2 \frac{x}{2} \operatorname{cosec}^2 \frac{x}{2}$, (*vi*) $\frac{1}{1+\cos x}$
(*vii*) $\sqrt{(1-\cos x)}$, (*viii*) $\sqrt{(1+\sin x)}$, (*ix*) $\sqrt{(1+\cos x)}$,
(*x*) $\sqrt{(1-\sin x)}$, (*xi*) $\sec (ax+b)$, (*xii*) $\operatorname{cosec} (ax+b)$.
(*xiii*) $\frac{1}{3 \sin x + 4 \cos x}$, (*xiv*) $\frac{1}{5 \cos x - 12 \sin x}$

Answers

1. (*i*) $\frac{1}{2}(x + \sin x \cos x)$ (*ii*) $1/12\,(9 \sin x + \sin 3x)$
(*iii*) $(-1/12)(\cos 6x + 3 \cos 2x)$ (*iv*) $(1/16)(4 \sin 2x - \sin 8x)$
(*v*) $(1/48)(12x + 6 \sin 2x + 3 \sin 4x + 2 \sin 6x)$

2. (*i*) $(1/32)(12x - 8 \sin 2x + \sin 4x)$ (*ii*) $(1/32)(12x + 8 \sin 2x + \sin 4x)$

3. (*i*) $(\frac{1}{a}) \log (ax+b)$. (*ii*) $\frac{1}{2} \tan (2x+3)$.
(*iii*) $\frac{1}{4} \log \sin (4x+5)$ (*iv*) $2 \sin \frac{1}{2} x$
(*v*) $-4 \cot x$ (*vi*) $\tan \frac{1}{2} x$
(*vii*) $-2\sqrt{2} \cos x/2$ (*viii*) $2 (\sin \frac{1}{2} x - \cos \frac{1}{2} x)$
(*ix*) $2\sqrt{2} \sin \frac{1}{2} x$ (*x*) $2 (\sin \frac{1}{2} x + \cos \frac{1}{2} x)$
(*xi*) $(1/a) \log \tan [\frac{1}{2}(ax+b) + \frac{1}{4}\pi]$ (*xii*) $(1/a) \log \tan \frac{ax+b}{2}$
(*xiii*) $(1/5) \log \tan [\frac{1}{2}(x + \tan^{-1} 4/3)]$
(*xiv*) $\frac{1}{13} \log \tan [\frac{1}{2}(x + \tan^{-1} 12/5) + \frac{1}{4}\pi]$

$$= \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} - \theta\right]$$
$$= \frac{a^2}{2}\left[\sinh\theta\cosh\theta - \theta\right]$$
$$= \frac{a^2}{2}\left[\sqrt{(\cosh^2\theta - 1)}\cosh\theta - \theta\right]$$
$$= \frac{a^2}{2}\left[\sqrt{\left(\frac{x^2}{a^2} - 1\right)}\frac{x}{a} - \cosh^{-1}\frac{x}{a}\right]$$
$$= \frac{x\sqrt{(x^2-a^2)}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$
$$= \frac{x\sqrt{(x^2-a^2)}}{2} - \frac{a^2}{2}\log\frac{x+\sqrt{(x^2-a^2)}}{a}.$$

We have thus obtained the following *six* results :—

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a}.$$
$$\int \sqrt{(a^2-x^2)}\,dx = \frac{x\sqrt{(a^2-x^2)}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$
$$\int \frac{1}{\sqrt{(a^2+x^2)}}\,dx = \sinh^{-1}\frac{x}{a}.$$
$$\int \sqrt{(a^2+x^2)}\,dx = \frac{x\sqrt{(a^2+x^2)}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}.$$
$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \cosh^{-1}\frac{x}{a}.$$
$$\int \sqrt{(x^2-a^2)}\,dx = \frac{x\sqrt{(x^2-a^2)}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}.$$

The *minus* sign in the last case should be specially noted.

Examples

1. *Integrate*

(*i*) $\dfrac{1}{\sqrt{[a^2-(bx+c)^2]}}$, (*ii*) $\dfrac{1}{\sqrt{[a^2+(bx+c)^2]}}$,

(*iii*) $\dfrac{1}{\sqrt{[(bx+c)^2-a^2]}}$.

We have, (Refer § 2·23),

$$\int \frac{dx}{\sqrt{[a^2-(bx+c)^2]}} = \frac{1}{b}\sin^{-1}\frac{bx+c}{a}.$$
$$\int \frac{dx}{\sqrt{[a^2+(bx+c)^2]}} = \frac{1}{b}\sinh^{-1}\frac{bx+c}{a}.$$
$$\int \frac{dx}{\sqrt{[(bx+c)^2-a^2]}} = \frac{1}{b}\cosh^{-1}\frac{bx+c}{a}.$$

2. *Integrate*

(*i*) $\dfrac{1}{\sqrt{(x^2+2x+2)}}$, (*ii*) $\dfrac{1}{\sqrt{(x^2+4x+2)}}$

(*iii*) $\dfrac{1}{\sqrt{(-2x^2+3x+4)}}$

(*i*) We have

$$(x^2+2x+2) = [(x+1)^2+1^2]$$

$$\Rightarrow \int \frac{1}{\sqrt{(x^2+2x+2)}}dx = \int \frac{dx}{\sqrt{[(x+1)^2+1^2]}}$$

$$= \sinh^{-1}(x+1)$$

(*ii*) $$(x^2+4x+2) = (x+2)^2-(\sqrt{2})^2$$

$$\Rightarrow \int \frac{1}{\sqrt{(x^2+4x+2)}}dx = \int \frac{dx}{\sqrt{[(x+2)^2-(\sqrt{2})^2]}}$$

$$= \cosh^{-1}\frac{x+2}{\sqrt{2}}$$

(*iii*) We have

$$-2x^2+3x+4 = -2[x^2-\tfrac{3}{2}x-2]$$

$$= -2\left[\left(x-\frac{3}{4}\right)^2-\left(\frac{\sqrt{41}}{4}\right)^2\right]$$

$$= 2\left[\left(\frac{\sqrt{41}}{4}\right)^2-\left(x-\frac{3}{4}\right)^2\right],$$

$$\int \frac{1}{\sqrt{(-2x^2+3x+4)}}dx = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left[\left(\frac{\sqrt{41}}{4}\right)^2-\left(x-\frac{3}{4}\right)^2\right]}}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{x-\frac{3}{4}}{\frac{\sqrt{41}}{4}} = \frac{1}{\sqrt{2}}\sin^{-1}\frac{4x-3}{\sqrt{41}}.$$

Exercises

1. Integrate :

(*i*) $\dfrac{1}{\sqrt{(2x^2+3x+4)}}$, (*ii*) $\dfrac{1}{\sqrt{(3x^2+4x+1)}}$, (*iii*) $\dfrac{2}{\sqrt{(x^2+x+1)}}$

2. Integrate :

(*i*) $\sqrt{(1-x^2)}$. (*ii*) $\sqrt{(1+x^2)}$, (*iii*) $\sqrt{(x^2-1)}$.

3. Integrate :

(*i*) $\sqrt{(2x^2+3x+4)}$, (*ii*) $\sqrt{(3x^2+4x+1)}$,

(*iii*) $\sqrt{(-2x^2+3x+4)}$.

4. Integrate :

(*i*) $\dfrac{x^2}{\sqrt{(x^2+1)}}$, (*ii*) $\dfrac{2x^2}{\sqrt{(x^2-1)}}$, (*iii*) $\dfrac{2x+3}{\sqrt{(x^2+1)}}$

(*iv*) $(x+2)\sqrt{(x^2+1)}$, (*v*) $(2x+4)\sqrt{(2x^2+3x+1)}$.

(*vi*) $\dfrac{2x+5}{\sqrt{(x^2-2x+2)}}$.

5. Integrate :

(*i*) $\dfrac{x^2}{\sqrt{(x^6+2x^3+2)}}$, (*ii*) $\dfrac{x^3}{\sqrt{(x^8+1)}}$,

(*iii*) $\dfrac{\cos x}{\sqrt{(2\sin^2 x+3\sin x+4)}}$.

Answers

1. (*i*) $(1/\sqrt{2})\sinh^{-1}\dfrac{4x+3}{\sqrt{23}}$ (*ii*) $(1/\sqrt{3})\cosh^{-1}(3x+2)$
   (*iii*) $\sinh^{-1}[(2x+1)/\sqrt{3}]$
2. (*i*) $\frac{1}{2}x\sqrt{(1-x^2)}+\frac{1}{2}\sin^{-1}x$ (*ii*) $\frac{1}{2}x\sqrt{(x^2+1)}+\frac{1}{2}\sinh^{-1}x$
   (*iii*) $\frac{1}{2}x\sqrt{(x^2-1)}-\frac{1}{2}\cosh^{-1}x$
3. (*i*) $\frac{1}{8}(4x+3)\sqrt{(2x^2+3x+4)}+\frac{23}{32}\sqrt{2}\sinh^{-1}[(4x+3)/\sqrt{23}]$
   (*ii*) $\frac{1}{6}(3x+2)\sqrt{(3x^2+4x+1)}-1/18\sqrt{3}\cosh^{-1}(3x+2)$
   (*iii*) $\frac{1}{8}(4x-3)\sqrt{(-2x^2+3x+4)}+41/32\sqrt{2}\sin^{-1}[(4x-3)/\sqrt{41}]$
4. (*i*) $\frac{1}{2}x\sqrt{(x^2+1)}-\frac{1}{2}\sinh^{-1}x$ (*ii*) $x\sqrt{(x^2-1)}+\cosh^{-1}x$
   (*iii*) $2\sqrt{(x^2+1)}+3\sinh^{-1}x$ (*iv*) $\frac{1}{3}\sqrt{(x^2+1)}(x^2+3x+1)+\sinh^{-1}x$
   (*v*) $\frac{1}{48}\sqrt{(2x^2+3x+1)}(32x^2+108x+61)-\dfrac{5\sqrt{2}}{64}\cosh^{-1}(4x+3)$
   (*vi*) $2\sqrt{(x^2-2x+2)}+7\sinh^{-1}(x-1)$
5. (*i*) $\frac{1}{3}\sinh^{-1}(x^3+1)$ (*ii*) $\frac{1}{4}\sinh^{-1}x^4$
   (*iii*) $(1/\sqrt{2})\sinh^{-1}\dfrac{4\sin x+3}{\sqrt{23}}$

2·5. Integration by parts

If u, v be two functions of x, we have

$$\frac{d(uv)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$$

Integrating both sides, we get

$$uv=\int u\frac{dv}{dx}\,dx+\int v\frac{du}{dx}\,dx$$

$$\Rightarrow \int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx. \qquad \ldots(i)$$

Let

$$u=f(x) \text{ and } \frac{dv}{dx}=\phi(x).$$

Therefore

$$\frac{du}{dx}=f'(x) \text{ and } v=\int \phi(x)\,dx.$$

The statement (1) may now be rewritten as

$$\int f(x)\,\phi(x)\,dx = f(x)\int \phi(x)\,dx - \int \left[\int \phi(x)\,dx\right].f'(x)\,dx.$$

In words, this formula states that

The integral of the product of two functions

= first function × integral of second

– integral of (diff. coefficient of first × integral of second)

Examples

1. *Evaluate* $\int xe^x\,dx$.

Let

$$x = f(x),\ e^x = \phi(x).$$

$$\therefore \qquad \int xe^x\,dx = x\,.\,e^x - \int e^x\,.\,1\,dx = xe^x - e^x$$

Note. Suppose we take

$$f(x) = e^x,\ \phi(x) = x$$

$$\therefore \qquad \int x\,e^x\,dx = \tfrac{1}{2}x^2 e^x - \tfrac{1}{2}\int x^2 e^x\,dx,$$

so that the given integral $\int x\,e^x dx$ is reduced to a comparatively more complicated integral $\int x^2\,e^x\,dx$; the index of x having increased.

Thus a proper choice of the order of factors is sometimes necessary.

2. *Evaluate* $\int x^2 \cos x\,dx$.

We take $f(x) = x^2$, $\phi(x) = \cos x$.

$$\therefore \quad \int x^2 \cos x\,dx = x^2\,.\,\sin x - \int 2x \sin x\,dx.$$

[To evaluate $\int x \sin x\,dx$, we have again to apply the rule of integration by parts.]

$$\begin{aligned} &= x^2 \sin x - 2\left[x(-\cos x) - \int (-\cos x)\,dx\right] \\ &= x^2 \sin x + 2x \cos x - 2 \sin x \\ &= (x^2 - 2) \sin x + 2x \cos x. \end{aligned}$$

We thus see that the *rule of integration by parts may have to be repeated several times.*

3. *Evaluate* $\int \cos^{-1} x\,dx$

Here *we take unity as one factor.* We have

$$\begin{aligned} \int \cos^{-1} x\,dx &= \int \cos^{-1} x\,.\,1\,dx \\ &= \cos^{-1} x\,.\,x - \int \frac{-1}{\sqrt{(1-x^2)}}\,x\,dx \\ &= x\cos^{-1} x - \frac{1}{2}\int \left(1 - x^2\right)^{-\frac{1}{2}}\left(-2x\right)dx \\ &= x\cos^{-1} x - \tfrac{1}{2}\,.\,\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \end{aligned}$$

$$= x\cos^{-1}x-\sqrt{(1-x^2)}$$

$$\therefore \quad \int \cos^{-1}x\,dx = x\cos^{-1}x-\sqrt{(1-x^2)}$$

Exercises

Integrate the following functions :

1. (*i*) x^2e^x, (*ii*) x^3e^x, (*iii*) $x\sinh x$.
2. (*i*) $\log x$, (*ii*) $x\log x$, (*iii*) $x^2\log x$, (*iv*) $x^n\log x$.
3. (*i*) $x(\log x)^2$, (*ii*) $(x\log x)^3$ (*iii*) $\sqrt{x}\,(\log x)^2$.
4. (*i*) $\tan^{-1}x$, (*ii*) $x\tan^{-1}x$, (*iii*) $\cot^{-1}x$.
   (*iv*) $x\cot^{-1}x$, (*v*) $x^3\tan^{-1}x$.
5. (*i*) $\sin^{-1}x$, (*ii*) $x\sin^{-1}x$, (*iii*) $\sec^{-1}x$, (*iv*) $x\sec^{-1}x$.
6. (*i*) $x\cos x$, (*ii*) $x^2\sin x$, (*iii*) $x^3\cos x$, (*iv*) $x^2\sin^2x$,
   (*v*) $x^2\sin x\cos x$, (*vi*) $x\cos x\cos 2x$,
   (*vii*) $x\sin x\sec^3x$, (*viii*) $x\cos^3x\sin x$.
7. (*i*) $\dfrac{x}{\sin^2x}$, (*ii*) $\dfrac{x}{\cos^2x}$, (*iii*) $x\log(1+x)$.

Answers

1. (*i*) $(x^2-2x+2)e^x$. (*ii*) $(x^3-3x^2+6x-6)e^x$.
   (*ii*) $x\cosh x-\sinh x$.
2. (*i*) $x\log(x/e)$. (*ii*) $\frac{1}{4}x^2\log(x^2/e)$.
   (*iii*) $\frac{1}{9}x^3\log(x^3/e)$. (*iv*) $[x^{n+1}/(n+1)^2]\log(x^{n+1}/e)$.
3. (*i*) $\frac{1}{4}x^2[2(\log x)^2-2\log x+1]$.
   (*ii*) $\frac{1}{128}x^4[32(\log x)^3-24(\log x)^2+12\log x-3]$.
   (*iii*) $\frac{1}{27}[18(\log x)^2-24\log x+16]x^{3/2}$.
4. (*i*) $x\tan^{-1}x-\log\sqrt{(x^2+1)}$. (*ii*) $\frac{1}{2}(x^2+1)\tan^{-1}x-\frac{1}{2}x$.
   (*iii*) $x\cot^{-1}x+\log\sqrt{(x^2+1)}$, (*iv*) $\frac{1}{2}(x^2+1)\cot^{-1}x+\frac{1}{2}x$.
   (*v*) $\frac{1}{4}(x^4-1)\tan^{-1}x-1/12(x^3-3x)$.
5. (*i*) $x\sin^{-1}x+\sqrt{(1-x^2)}$. (*ii*) $\frac{1}{4}(2x^2-1)\sin^{-1}x+\frac{1}{4}x\sqrt{(1-x^2)}$.
   (*iii*) $x\sec^{-1}x-\cosh^{-1}x$. (*iv*) $\frac{1}{2}[x^2\sec^{-1}x-\sqrt{(x^2-1)}]$.
6. (*i*) $x\sin x+\cos x$. (*ii*) $-x^2\cos x+2(x\sin x+\cos x)$.
   (*iii*) $(x^3-6x)\sin x+3(x^2-2)\cos x$.
   (*iv*) $\frac{1}{6}x^3+\frac{1}{8}(1-2x^2)\sin 2x-\frac{1}{4}x\cos 2x$.
   (*v*) $\frac{1}{8}(1-2x^2)\cos 2x+\frac{1}{4}x\sin 2x$.
   (*vi*) $\frac{1}{18}(3x\sin 3x+\cos 3x+\frac{1}{9}x\sin x+\cos x)$.
   (*vii*) $\frac{1}{2}(x\sec^2x-\tan x)$.
   (*viii*) $-\frac{1}{4}x\cos^4x+\frac{1}{128}(12x+8\sin 2x+\sin 4x)$.
7. (*i*) $-x\cot x+\log\sin x$. (*ii*) $x\tan x-\log\sec x$.
   (*iii*) $\frac{1}{2}(x^2-1)\log(1+x)-\frac{1}{4}(x^2-2x)$.

2·51. To evaluate the integral

$$\int e^x\,[f(x)+f'(x)]\,dx.$$

Integrating by parts, we have

$$\int e^x f(x)dx = e^x f(x) - \int e^x f'(x)\,dx$$

$$\Rightarrow \qquad \int e^x[f(x)+f'(x)]\,dx = e^x f(x).$$

This form of integral is quite important.

Examples

1. *Evaluate the following integrals :*

(*i*) $\displaystyle\int \frac{xe^x}{(x+1)^2}\,dx,$ (*ii*) $\displaystyle\int e^x\,\frac{1-\sin x}{1-\cos x}\,dx.$

We have

(*i*) $$\int \frac{xe^x}{(x+1)^2}\,dx = \int \frac{x+1-1}{(x+1)^2}\,e^x\,dx,$$

$$= \int \left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right] e^x\,dx$$

Taking $f(x) = \dfrac{1}{x+1}$ so that $f'(x) = -\dfrac{1}{(x+1)^2}$

we see that the function to be integrated is of the form

$$e^x[f(x)+f'(x)]$$

and accordingly, the integral is

$$= e^x f(x) = \frac{e^x}{x+1}.$$

We may also, however, proceed independently as follows :

Integrating $\displaystyle\int \frac{1}{x+1}e^x\,dx$ by parts, we get

$$\int \frac{1}{x+1}\cdot e^x\,dx = \frac{1}{x+1}\cdot e^x - \int -\frac{1}{(x+1)^2}\,e^x\,dx.$$

$$\Rightarrow \qquad \int \left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right] e^x\,dx = \frac{1}{x+1}\,e^x.$$

(*ii*) Now, $$e^x\left(\frac{1-\sin x}{1-\cos x}\right) = e^x\left\{\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right\}$$

$$= e^x\left\{\tfrac{1}{2}\,\text{cosec}^2\,\frac{x}{2} - \cot\frac{x}{2}\right\}$$

so that $f(x) = \cot\dfrac{x}{2}$. We see that the integrand is of the form

$$e^x\,[f(x)+f'(x)].$$

Integrating $e^x \cot \frac{x}{2}$ by parts we obtain

$$\int e^x \cot \frac{x}{2}\, dx = e^x \cot \frac{x}{2} - \int -\tfrac{1}{2} \operatorname{cosec}^2 x\, dx$$

$$\therefore \quad \int e^x\left(\cot \frac{x}{2} - \tfrac{1}{2} \operatorname{cosec}^2 \frac{x}{2}\right) dx = e^x \cot \frac{x}{2}$$

$$\therefore \quad \int e^x\left(\tfrac{1}{2} \operatorname{cosec}^2 \frac{x}{2} - \cot \frac{x}{2}\right) dx = -e^x \cot \frac{x}{2}$$

Exercises

1. Integrate the following functions :

(*i*) $\dfrac{1+x}{(2+x)^2} e^x$, (*ii*) $\dfrac{1+\sin x}{1+\cos x} e^x$,

(*iii*) $\dfrac{1+x \log x}{x} e^x$, (*iv*) $\dfrac{2+\sin 2x}{1+\cos 2x} e^x$,

(*v*) $\dfrac{2-\sin 2x}{1-\cos 2x} e^x$, (*vi*) $\dfrac{1+\sin 2x}{1+\cos 2x} e^{2x}$,

(*vii*) $\dfrac{\sqrt{(1-\sin x)}}{1+\cos} e^{-\frac{1}{2}x}$, (*viii*) $\dfrac{(x^2+1)}{(x+1)^2} e^x$,

(*ix*) $e^x (\tan x - \log \cos x)$.

Answers

1. (*i*) $e^x/(2+x)$. (*ii*) $e^x \tan \frac{x}{2}$. (*iii*) $e^x \log x$.

(*iv*) $e^x \tan x$. (*v*) $-e^x \cot x$. (*vi*) $\frac{1}{2}e^{2x} \tan x$.

(*vii*) $e^{-\frac{1}{2}x} \sec \frac{1}{2} x$. (*viii*) $(x-1)\, e^x/(x+1)$. (*ix*) $e^x \log \sec x$.

2·52. Integrals of

(*i*) $e^{ax} \cos (bx+c)$ (*ii*) $e^{ax} \sin (bx+c)$.

Applying the rule of integration by parts, we obtain

$\int e^{ax} \cos (bx+c)\, dx$

$$= \frac{e^{ax}}{a} \cos (bx+c) - \int -\frac{e^{ax}}{a} b \sin (bx+c)\, dx$$

$$= \frac{e^{ax}}{a} \cos (bx+c) + \frac{b}{a} \int e^{ax} \sin (bx+c)\, dx. \qquad \ldots(i)$$

Similarly we have

$\int e^{ax} \sin (bx+c)\, dx$

$$= \frac{e^{ax}}{a} \sin (bx+c) - \int \frac{e^{ax}}{a} . b \cos (bx+c)\, dx$$

$$= \frac{e^{ax}}{a} \sin (bx+c) - \frac{b}{a} \int e^{ax} \cos (bx+c)\, dx. \qquad \ldots(ii)$$

If the value of $\int e^{ax} \cos(bx+c)dx$ be required, we substitute the R.H.S. of (*ii*) for the last term of (*i*) and if the value of $\int e^{ax} \sin(bx+c)\,dx$ be required, we substitute the R.H.S. of (*i*) for the last term of (*ii*). In the former case we get

$$\int e^{ax} \cos(bx+c)\,dx$$

$$= \frac{e^{ax}}{a}\cos(bx+c)+\frac{b}{a^2}e^{ax}\sin(bx+c)-\frac{b^2}{a^2}\int e^{ax}\cos(bx+c)dx$$

$$\Rightarrow \quad \left(1+\frac{b^2}{a^2}\right)\int e^{ax}\cos(bx+c)\,dx$$

$$= e^{ax}\frac{a\cos(bx+c)+b\sin(bx+c)}{a^2},$$

$$\Rightarrow \int e^{ax}\cos(bx+c)\,dx = e^{ax}\frac{a\cos(bx+c)+b\sin(bx+c)}{a^2+b^2}.$$

Similarly we have

$$\int e^{ax}\sin(bx+c)\,dx = e^{ax}\frac{a\sin(bx+c)-b\cos(bx+c)}{a^2+b^2}.$$

To put the results in another form, we determine two numbers r and α such that

$$a = r\cos\alpha \text{ and } b = r\sin\alpha.$$

These give

$$r = \sqrt{(a^2+b^2)},\ \alpha = \tan^{-1}(b/a).$$

$$\therefore \quad \int e^{ax}\cos(bx+c)\,dx = e^{ax}\frac{r\cos(bx+c-\alpha)}{a^2+b^2}$$

$$= e^{ax}\frac{\cos\left(bx+c-\tan^{-1}\dfrac{b}{a}\right)}{\sqrt{(a^2+b^2)}}.$$

Similarly

$$\int e^{ax}\sin(bx+c)\,dx = e^{ax}\frac{\sin\left(bx+c-\tan^{-1}\dfrac{b}{a}\right)}{\sqrt{(a^2+b^2)}}.$$

Examples

1. *Evaluate*

(*i*) $\int e^{3x}\sin 4x\,dx$, (*ii*) $\int e^{4x}\cos 2x\cos 4x\,dx$,
(*iii*) $\int xe^{3x}\cos x\,dx$.

(*i*) From the formula proved above in § 2·52, we get

$$\int e^{3x}\sin 4x\,dx = \frac{e^{3x}}{\sqrt{(3^2+4^2)}}\sin\left(4x-\tan^{-1}\frac{4}{3}\right)$$

$$= \frac{e^{3x}}{5}\sin\left(4x-\tan^{-1}\frac{4}{3}\right).$$

(*ii*) Now, $\cos 2x \cos 4x = \frac{1}{2}[2 \cos 2x \cos 4x]$

$$= \tfrac{1}{2}(\cos 6x + \cos 2x)$$

$\therefore \quad \int e^{4x} \cos 2x \cos 4x \, dx$

$$= \tfrac{1}{2} \int e^{4x} \cos 6x \, dx + \tfrac{1}{2} \int e^{4x} \cos 2x \, dx$$

$$= \frac{1}{2} \cdot \frac{e^{4x}}{\sqrt{(4^2+6^2)}} \cos\left(6x - \tan^{-1}\frac{6}{4}\right)$$

$$+ \frac{1}{2} \cdot \frac{e^{4x}}{\sqrt{(4^2+2^2)}} \cos\left(2x - \tan^{-1}\frac{2}{4}\right)$$

$$= \frac{e^{4x}}{2}\left[\frac{1}{\sqrt{52}} \cos\left(6x - \tan^{-1}\frac{3}{2}\right) + \frac{1}{\sqrt{20}} \cos\left(2x - \tan^{-1}\frac{1}{2}\right)\right].$$

(*iii*) To evaluate $\int x e^{2x} \cos x \, dx$, we apply the rule of integration by parts. Taking x and $e^{2x} \cos x$ as two factors, we have

$$\int x e^{2x} \cos x \, dx = x \cdot \frac{e^{2x}}{\sqrt{5}} \cos\left(x - \tan^{-1}\frac{1}{2}\right)$$

$$- \int 1 \cdot \frac{e^{2x}}{\sqrt{5}} \cos\left(x - \tan^{-1}\frac{1}{2}\right) dx.$$

Again,

$$\int e^{2x} \cos\left(x - \tan^{-1}\tfrac{1}{2}\right) dx$$

$$= \frac{e^{2x}}{\sqrt{5}} \cos\left(x - 2\tan^{-1}\frac{1}{2}\right)$$

$$\therefore \quad \int x e^{2x} \cos x \, dx = e^{2x}\left[\frac{x}{\sqrt{5}} \cos\left(x - \tan^{-1}\frac{1}{2}\right)\right.$$

$$\left. - \frac{1}{5} \cos\left(x - 2\tan^{-1}\frac{1}{2}\right.\right.$$

Exercises

1. Evaluate the following integrals :

(*i*) $\int e^{4x} \cos 5x \, dx$, (*ii*) $\int e^{x} \cos^2 x \, dx$.

(*iii*) $\int e^{2x} \cos^2 x \, dx$, (*iv*) $\int e^{x} \sin x \sin 2x \sin 3x \, dx$,

(*v*) $\int x^2 e^{x} \sin x \, dx$, (*vi*) $\int \sinh 2x \sin 2x \, dx$.

2. Integrate

(*i*) $\cos(\log x)$, (*ii*) $x \sin(2 \log x)$, (*iii*) $x^3 \sin(a \log x)$.

Answers

1. (*i*) $\sqrt{\frac{1}{41}}\, e^{4x} \cos\left(5x - \tan^{-1}\frac{5}{4}\right)$.

(*ii*) $\frac{1}{2}e^{x} + \frac{1}{2}\sqrt{\frac{1}{5}}\, e^{x} \cos\left(2x - \tan^{-1} 2\right)$.

(*iii*) $\frac{1}{4} e^{2x}\left[\frac{3}{\sqrt{5}} \cos\left(x - \tan^{-1}\frac{1}{2}\right)\right.$

$$\left. + \frac{1}{\sqrt{13}} \cos\left(3x - \tan^{-1}\frac{3}{2}\right)\right]$$

(*iv*) $\frac{1}{4}e^{x}[\sqrt{\frac{1}{5}}\sin(2x-\tan^{-1}2)-\sqrt{\frac{1}{37}}\sin(6x-\tan^{-1}6)$
$+\sqrt{\frac{1}{17}}\sin(4x-\tan^{-1}4)]$.

(*v*) $\sqrt{\frac{1}{2}}e^{x}[x^2\sin(x-\frac{1}{4}\pi)+\sin(x-\frac{3}{4}\pi)]+xe^{x}\cos x$.

(*vi*) $\frac{1}{4}\cosh 2x\sin 2x-\sinh 2x\cos 2x)$.

2. (*i*) $\sqrt{\frac{1}{2}}\,.\,x\cos[\log x-\frac{1}{4}\pi]$. (*ii*) $\sqrt{\frac{1}{5}}\,x^2\sin(2\log x-\frac{1}{4}\pi)$.

(*iii*) $\dfrac{x^4}{\sqrt{(16+a^2)}}\sin\left[a\log x-\tan^{-1}\dfrac{a}{4}\right]$.

2·6. Sometimes both the methods of integration have to be applied in one and the same question. We will illustrate the procedure by two examples.

Examples

1. *Evaluate*

$$\int\frac{x\sin^{-1}x}{\sqrt{(1-x^2)}}\,dx.$$

We put

$$x=\sin\theta\Rightarrow dx=\cos\theta\,d\theta$$

$$\therefore\quad\int\frac{x\sin^{-1}x}{\sqrt{(1-x^2)}}\,dx=\int\frac{\sin\theta.\theta}{\cos\theta}\cos\theta\,d\theta=\int\theta\sin\theta\,d\theta.$$

To evaluate $\int\theta\sin\theta\,d\theta$, we apply the rule of integration by parts and obtain

$$\int\theta\sin\theta\,d\theta=(-\theta\cos\theta)-\int-\cos\theta.1\,d\theta.$$
$$=-\theta\cos\theta+\int\cos\theta\,d\theta=-\theta\cos\theta+\sin\theta.$$

$$\therefore\quad\int\frac{x\sin^{-1}x}{\sqrt{(1-x^2)}}\,dx=-\theta\cos\theta+\sin\theta$$
$$=-\sqrt{(1-x^2)}\sin^{-1}x+x.$$

2. *Integrate* $\tan^{-1}\sqrt{\left(\dfrac{1-x}{1+x}\right)}$.

We put

$$x=\cos\theta\Rightarrow dx=-\sin\theta\,d\theta.$$

Also

$$\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}=\tan^{-1}\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}$$
$$=\tan^{-1}\sqrt{\left[\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}\right]}$$
$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}.$$

$$\therefore \quad \int \tan^{-1} \sqrt{\left(\frac{1-x}{1+x}\right)}\, dx = \int \frac{\theta}{2}\, (-\sin\theta)\, d\theta$$
$$= -\tfrac{1}{2} \int \theta \sin\theta \, d\theta$$
$$= -\tfrac{1}{2}\, [-\theta \cos\theta + \sin\theta]$$
$$= -\tfrac{1}{2}\, [-x \cos^{-1} x + \sqrt{(1-x^2)}].$$

Exercises

Integrate the following :

1. $x^5 \tan^{-1} x^3$.
2. $x \tan^{-1} x/(1+x^2)^{3/2}$.
3. $x^5 \cos x^2$.
4. $\tan^{-1} [2x/(1-x^2)]$.
5. $\dfrac{e^{m \tan^{-1} x}}{(1+x^2)^{3/2}}$.
6. $\dfrac{e^{m \tan^{-1} x}}{(1+x^2)^2}$.
7. $\dfrac{x e^{m \sin^{-1} x}}{\sqrt{(1-x^2)}}$.
8. $\dfrac{x^2 e^{m \sin^{-1} x}}{\sqrt{(1-x^2)}}$.

Answers

1. $\frac{1}{6}[(x^6+1) \tan^{-1} x^3 - x^3]$.
2. $(x-\tan^{-1}x)/\sqrt{(1+x^2)}$.
3. $\frac{1}{2}(x^4-2) \sin x^2 + x^2 \cos x^2$.
4. $2x \tan^{-1} x - \log (1+x^2)$.
5. $\dfrac{e^{m\theta}}{\sqrt{(m^2+1)}} \cos (\theta - \cot^{-1} m)$, where $\theta = \tan^{-1} x$.
6. $\dfrac{e^{m\theta}}{2}\left[\dfrac{1}{m} + \dfrac{1}{\sqrt{(m^2+1)}} \cos\left(2\theta - \tan^{-1} \dfrac{2}{m}\right)\right]$, where $\theta = \tan^{-1} x$.
7. $\dfrac{e^{m\theta}}{\sqrt{(m^2+1)}} \sin (\theta - \cot^{-1} m)$, where $\theta = \sin^{-1} x$.
8. $\dfrac{e^{m\theta}}{2}\left[\dfrac{1}{m} - \dfrac{1}{\sqrt{(m^2+4)}} \cos\left(2\theta - \tan^{-1} \dfrac{2}{m}\right)\right]$, where $\theta = \sin^{-1} x$

2·7. Definite Integrals. The application of the methods of integrating by substitution and of integrating by parts for evaluating definite integrals shall be illustrated by means of examples.

The validity of the procedure will be seen to be quite apparent.

When the variable x in a definite integral

$$\int_a^b f(x)\, dx$$

is changed, we usually change the limits also ; the new limits being the values of the new variable which correspond to the values a and b of x.

Examples

1. *Evaluate* $\int_0^{\frac{1}{2}\pi} \frac{\cos x}{1+\sin^2 x}\,dx.$

We put

$$\sin x = t \Rightarrow \cos x dx = dt \qquad \ldots(1)$$

From (1), we see that

$$x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = 1 .$$

Thus 0 and 1 are the limits for the new variable t. We have

$$\int_0^{\frac{1}{2}\pi} \frac{\cos x\,dx}{1+\sin^2 x} = \int_0^1 \frac{dt}{1+t^2} = \left|\tan^{-1} t\right|_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} .$$

2. *Evaluate* $\int_0^1 x^2 e^{2x}\,dx.$

Integrating by parts, we have

$$\int x^2 e^{2x}\,dx = \frac{x^2 e^{x}}{2} - \int \frac{2x}{2} . e^{2x}\,dx = \frac{x^2 e^{2x}}{2} - \int xe^{2x}\,dx.$$

$$\Rightarrow \quad \int_0^1 x^2 e^{2x}\,dx = \left|\frac{x^2 e^{2x}}{2}\right|_0^1 - \int_0^1 x\,e^{2x}\,dx$$

$$= \frac{e^2}{2} - \int_0^1 x\,e^{2x}\,dx.$$

Again,

$$\int x\,e^{2x}\,dx = \frac{x\,e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx$$

$$= \frac{x\,e^{2x}}{2} - \frac{e^{2x}}{4} .$$

$$\Rightarrow \quad \int_0^1 x\,e^{2x}\,dx = \left|\frac{x\ e^{2x}}{2}\right|_0^1 - \left|\frac{e^{2x}}{4}\right|_0^1$$

$$= \frac{e^2}{4} + \frac{1}{4} .$$

$$\therefore \quad \int_0^1 x^2 e^{2x}\,dx = \frac{e^2}{2} - \left(\frac{e^2}{4} + \frac{1}{4}\right) = \frac{e^2-1}{4} .$$

3. *Prove that*

$$\int_0^{\sqrt{\frac{1}{2}}} \frac{\sin^{-1} x\,dx}{(1-x^2)^{\frac{3}{2}}} = \frac{\pi}{4} - \frac{1}{4}\log 2.$$

We put

$$x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta. \qquad \ldots(1)$$

From (1), we see that

$\theta = 0$ when $x = 0$ and $\theta = \frac{1}{4}\pi$ when $x = \sqrt{\frac{1}{2}}$.

Thus the given integral

$$= \int_0^{\frac{1}{4}\pi} \theta \sec^2 \theta \, d\theta.$$

Integrating by parts, we obtain

$$\int \theta \sec^2 \theta \, d\theta = \theta \tan \theta - \int 1 . \tan \theta \, d\theta$$
$$= \theta \tan \theta - \log \sec \theta$$

$$\therefore \int_0^{\frac{1}{4}\pi} \theta \sec^2 \theta \, d\theta = \Big| \theta \tan \theta - \log \sec \theta \Big|_0^{\frac{1}{4}\pi}$$

$$= \frac{\pi}{4} . 1 - \log \sqrt{2} = \frac{\pi}{4} - \frac{1}{2} \log 2.$$

Exercises

1. Evaluate the following integrals :

(i) $\int_0^1 \frac{2x}{1+x^2} dx$, (ii) $\int_0^{\frac{1}{4}\pi} \cos^2 t \, dt$, (iii) $\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2} dx$,

(iv) $\int_0^{\frac{1}{2}\pi} \sin^3 t \cos t \, dt$, (v) $\int_0^{\frac{1}{2}\pi} \sin^3 t \, dt$, (vi) $\int_0^1 \frac{\sqrt{(\tan^{-1} x)}}{1+x^2} dx$,

(vii) $\int_0^{\pi} x \cos x \, dx$, (viii) $\int_0^{\pi} x \sin^{-1} x \, dx$,

(ix) $\int_0^1 x \tan^{-1} x \, dx^2$ (x) $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{dx}{\sqrt{(x-x^2)}}$.

2. Prove that

$$\int_0^1 x(\tan^{-1} x)^2 \, dx = \frac{\pi}{4}\left(\frac{\pi}{4} - 1\right) + \frac{1}{2} \log 2.$$

Answers

1. (i) $\log 2$. (ii) $\pi/4$. (iii) $\pi^3/192$. (iv) $1/4$.
(v) $2/3$. (vi) $\pi^{3/2}/12$. (vii) -2. (viii) $\pi/8$.
(ix) $(\pi-2)/4$. (x) $\pi/6$.

2·71. Improper definite Integrals. *While calculating improper definite integrals, we often need the following limits :

$$(i) \lim_{x \to \infty} \frac{x^n}{e^x} = 0,$$

$$(ii) \lim_{x \to \infty} \frac{\log x}{x^n} = 0, \quad (iii) \lim_{x \to 0} (x^n \log x) = 0,$$

n being a positive number.

*These results have been proved in Chapter XIII of the Author's *Differential Calculus.*

Examples

1. *Evaluate*

$$\int_0^1 \log x\, dx$$

We know that $\log x \to -\infty$ as $x \to 0$ so that here we are concerned with an improper integral

We firstly evaluate

$$\int_h^1 \log x\, dx$$

We have

$$\begin{aligned}\int_h^1 \log x\, dx &= \int_h^1 1.\log x\, dx \\ &= \Big| x\log x \Big|_h^1 - \int_h^1 1\,.\,dx \\ &= \Big| x\log x \Big|_h^1 - \Big| x \Big|_h^1 \\ &= -h\log h-(1-h), \text{ for } \log 1=0.\end{aligned}$$

As $h \to 0$, $h \log h \to 0$ and, therefore, the right hand side $\to -1$.

Thus $\int_0^1 \log x\, dx$ exists finitely and is equal to -1.

2. *Show that, a being positive,*

$$\int_0^\infty e^{-ax}\cos bx\; dx = \frac{a}{a^2+b^2}\int_0^\infty e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2}.$$

We write

$$I = \int_0^\infty e^{-ax}\cos bx\, dx,\; J = \int_0^\infty e^{-ax}\sin bx\, dx.$$

Let x be a positive number. Integrating by parts, we have

$$\int_0^x e^{-ax}\cos bx\, dx = \Big|\frac{e^{-ax}\sin bx}{b}\Big|_0^x + \frac{a}{b}\int_0^x e^{-ax}\sin bx\; dx \quad \text{...(1)}$$

Now

$$\Big| e^{-ax}\frac{\sin bx}{b}\Big|_0^x = \frac{\sin bx}{be^{ax}} - 0 = \frac{\sin bx}{be^{ax}},$$

which tends to 0 as $x \to \infty$, for $1/e^{ax} \to 0$ and $\sin bx$ remains numerically less than 1.

Thus letting $x \to \infty$, we obtain, from (1)

$$I = \frac{a}{b}J. \quad \text{...(2)}$$

Again, we have

$$\int_0^x e^{-ax}\sin bx\, dx = \Big| -e^{-ax}\frac{\cos bx}{b}\Big|_0^x - \frac{a}{b}\int_0^x e^{-ax}\cos bx\, dx \quad \text{...(3)}$$

Now

$$\Big| -e^{-ax}\frac{\cos bx}{b}\Big|_0^x = -\frac{\cos bx}{be^{ax}}+\frac{1}{b},$$

which tends to $1/b$ as $x \to \infty$.

Therefore, letting $x \to \infty$, we obtain, from (3)

$$J=\frac{1}{b}-\frac{a}{b}I. \qquad \ldots(4)$$

Solving (2) and (4) for I and J, we obtain

$$I=\frac{a}{a^2+b^2},\ J=\frac{b}{a^2+b^2}.$$

Exercises

Evaluate the following improper integrals :

(i) $\int_0^1 \frac{\sin^{-1} x}{\sqrt{(1-x^2)}}\,dx$, (ii) $\int_0^\infty x^2e^{-x}\,dx$, (iii) $\int_0^\infty x^3e^{-x}\,dx$

(iv) $\int_0^1 x\log x\,dx$, (v) $\int_0^\infty \frac{\log x}{x^2}\,dx$,

(vi) $\int_1^\infty \frac{\log x}{x^3}\,dx$, (vii) $\int_0^a \frac{dx}{\sqrt{(ax-x^2)}}$,

(viii) $\int_0^a \sqrt{\left(\frac{a-x}{x}\right)}\,dx$, (ix) $\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{2x}}\,dx$,

(x) $\int_0^\infty e^{-ax}\sin nx\,dx$, $(a>0)$,

(xi) $\int_0^\infty e^{-2x}\sin x\sin 3x\,dx$, (xii) $\int_{-\infty}^0 e^x\sin x\,dx$.

Answers

(i) $\pi^2/8$.	(ii) 2.	(iii) 6.	(iv) $-\frac{1}{4}$.
(v) 1.	(vi) 1/4.	(vii) π.	(viii) $a\pi/2$.
(ix) $\pi/2$.	(x) $n/(a^2+n^2)$.	(xi) 3/40.	(xii) $-1/2$.

2·8. Reduction Formulae. In the general sense of the term, any formula which expresses an integral in terms of another which is simpler, is a reduction formula for the first integral. The common practice is, however, to confine the use of the term to cases in which the integral is a member of a class of functions and the formula expresses the integral of the general member of class in terms of one or two integrals of the same class. The successive application of the reduction formula enables us to express the integral of the general member of the class of functions in terms of that of simplest member of the class.

The reduction formulae are *generally obtained by applying the rule of integration by parts.*

A large number of reduction formulae will be obtained in the following chapters. Here we will consider some illustrative cases only.

Examples

1. *Establish a reduction formula for $\int x^n e^{ax}\, dx$ and apply it to evaluate $\int x^3 e^{ax}\, dx$.*

Integrating by parts, we have

$$\int x^n e^{ax}\, dx = x^n \cdot \frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx,$$

which is the required reduction formula.

Putting $n = 3, 2, 1$ successively in it, we obtain

$$\int x^3 e^{ax}\, dx = \frac{x^3 e^{ax}}{a} - \frac{3}{a}\int x^2 e^{ax}\, dx \qquad \ldots(i)$$

$$\int x^2 e^{ax}\, dx = \frac{x^2 e^{ax}}{a} - \frac{2}{a}\int x\, e^{ax}\, dx \qquad \ldots(ii)$$

$$\int x^2 e^{ax}\, dx = \frac{x\, e^{ax}}{a} - \frac{1}{a}\int e^{ax}\, dx$$

$$= \frac{x\, e^{ax}}{a} - \frac{e^{ax}}{a^2}. \qquad \ldots(iii)$$

From (*ii*) and (*iii*), we get

$$\int x^2 e^{ax}\, dx = \frac{x^2 e^{ax}}{a} - \frac{2x\, e^{ax}}{a^2} + \frac{2e^{ax}}{a^3}. \qquad \ldots(iv)$$

Again from (*i*) and (*iv*), we obtain

$$\int x^3 e^{ax}\, dx = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6x\, e^{ax}}{a^3} - \frac{6}{a^4} e^{ax}$$

$$= \frac{e^{ax}}{a^4}(a^3x^3 - 3a^2x^2 + 6ax - 6).$$

2. *Obtain a reduction formula for $\int x^m \sin nx\, dx$.*

We have, integrating by parts,

$$\int x^m \sin nx\, dx = -\frac{x^m \cos nx}{n} + \frac{m}{n}\int x^{m-1} \cos nx\, dx \ \ldots(i)$$

Again,

$$\int x^{m-1} \cos nx\, dx = \frac{x^{m-1} \sin nx}{n} - \frac{m-1}{n}\int x^{m-2} \sin nx\, dx \qquad \ldots(ii)$$

Substituting this value in (*i*), we get

$$\int x^m \sin nx\, dx = -\frac{x^m \cos nx}{n} + \frac{mx^{m-1} \sin nx}{n^2}$$

$$- \frac{m(m-1)}{n^2}\int x^{m-2} \sin nx\, dx,$$

which is the required reduction formula.

3. *Obtain a reduction formula for* $\int x^n e^{-x}\, dx$ *and hence show that the improper integral*

$$\int_0^\infty x^n e^{-x}\, dx = n\,!,$$

where n is any positive integer.

Integrating by parts, we have

$$\int_0^t x^n e^{-x}\, dx = \Big| -x^n e^{-x} \Big|_0^t + n \int_0^t x^{n-1} e^{-x}\, dx$$

$$= -t^n e^{-t} + n \int_0^t x^{n-1} e^{-x}\, dx$$

Now $\lim\limits_{t \to \infty} (-t^n e^{-t}) = \lim\limits_{t \to \infty} \left(\dfrac{t^n}{e^t} \right) = 0$

$$\therefore \quad \int_0^\infty x^n e^{-x}\, dx = \int_0^\infty x^{n-1} e^{-x}\, dx$$

Denoting $\int_0^\infty x^n e^{-x}\, dx$ by I_n, we write

$$I_n = nI_{n-1}.$$

Changing n to $n-1$, $n-2$,......2, 1, we get

$$I_{n-1} = (n-1)\, I_{n-2},$$
$$I_{n-2} = (n-2)\, I_{n-3},$$
$$I_{n-3} = (n-3)\, I_{n-4},$$
$$\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots$$
$$I_1 = 1.I_0 = I_0.$$

From these we obtain

$$I_n = n\,!\,.\, I_0.$$

Now,

$$I_0 = \int_0^\infty e^{-x}\, dx = \lim_{t \to \infty} \left(\int_0^t e^{-x}\, dx \right)$$

$$= \lim_{t \to \infty} \left. -e^{-t} + 1 \right| = 1$$

$$\therefore \quad I_n = n\,!.$$

4. *Obtain a reduction formula for* $\int x^m (\log x)^n\, dx$ *and apply it to evaluate* $\int_0^1 x^4 (\log x)^3\, dx$.

Integrating by parts, we have

$$\int x^m (\log x)^n \, dx$$
$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^{m+1}(\log x)^{n-1}.\frac{1}{x}\,dx$$
$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1}\,dx,$$

as the required reduction formula.

Putting $m = 4$, we get

$$\int x^4 (\log x)^n \, dx = \frac{x^5}{5}(\log x)^n - \frac{n}{5}\int x^4 (\log x)^{n-1}\,dx.$$

Putting $n = 3, 2, 1$ successively, we get

$$\int x^4 (\log x)^3 \, dx = \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int x^4 (\log x)^2\,dx \qquad ...(i)$$

$$\int x^4 (\log x)^2 \, dx = \frac{x^5}{5}(\log x)^2 - \frac{2}{5}\int x^4 \log x\,dx \qquad ...(ii)$$

$$\int x^4 \log x \, dx = \frac{x^5}{5}\log x - \frac{1}{5}\int x^4\,dx$$
$$= \frac{x^5}{5}\log x - \frac{x^5}{25}.$$

From these, we obtain

$$\int x^4 (\log x)^3 \, dx = \frac{x^5}{5}(\log x)^3 - \frac{3x^5}{25}(\log x)^2 + \frac{6x^5}{125}\log x - \frac{6}{625}x^5.$$

$$\Rightarrow \int_0^1 x^4 (\log x)^3 \, dx = \left| \frac{x^5}{5}(\log x)^3 - \frac{3x^5}{25}(\log x)^2 + \frac{6x^5}{125}\log x - \frac{6}{625}x^5 \right|_0^1$$

When $x \to 0$, $x^5(\log x)^3$, $x^5(\log x)^2$, $x^5 \log x$ all $\to 0$.

$$\therefore \quad \int_0^1 x^4 (\log x)^3 \, dx = -\frac{6}{625}.$$

Exercises

1. Obtain a reduction formula for $\int x^m \cos nx dx$ and apply it to evaluate $\int x^3 \cos 3x \, dx$.

2. If

$$u_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, dx \text{ and } n > 1,$$

prove that

$$u_n + n(n-1)u_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}.$$

3. Show that

$$\int x^m(1-x)^{n-1}\,dx = \frac{x^{m+1}(1-x)^{n-1}}{m+n} + \frac{n-1}{m+1}\int x^m(1-x)^{n-2}\,dx.$$

Hence deduce that

$$\int_0^1 x^m(1-x)^{n-1}\,dx = \frac{(n-1)\,!\;m\,!}{m+n},$$

m, n, being positive integers.

4. If m and n are positive integers, prove that

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= \frac{1.\,2.\,3.\,\ldots\,(m-1)}{n(n+1)(n+2)\ldots\,(n+m-1)}.$$

Answers

1. $\int x^m \cos nx\,dx = x^{m-1}\,\dfrac{nx\sin nx + m\cos nx}{n^2} - \dfrac{m(m-1)}{n^2}\int x^{m-2}\cos nx\,dx.$

$\int x^3 \cos 3x\,dx = \frac{1}{27}(9x^2-2)\cos 3x + \frac{1}{9}(3x^3-2x)\sin 3x.$

EXERCISES ON CHAPTER II

1. Prove that if m and n are unequal integers

$$\int_0^\pi \sin mx \sin nx\,dx = 0,\quad \int_0^\pi \sin^2 mx\,dx = \frac{\pi}{2}.$$

2. Evaluate $\int e^x (x-2)(2x+3)\,dx$.
3. Evaluate $\int x^3 \log(1+x^2)\,dx$.
4. Prove that if n is an integer > 1,

$$\int_0^\infty \frac{dx}{[x+\sqrt{(1+x^2)}]^n} = \frac{n}{n^2-1}.$$

(Put $x = \sinh\theta$)

5. Evaluate $\displaystyle\int \frac{\cos^{-1} x}{x^3}\,dx$.

6. Evaluate $\displaystyle\int \frac{\sin(\log x)}{x^3}\,dx$.

7. Show that

(*i*) $\displaystyle\int \frac{x}{1+\cos x}\,dx = x\tan\tfrac{1}{2}x + 2\log\cos\tfrac{1}{2}x.$

(*ii*) $\displaystyle\int \frac{x}{1-\cos x}\,dx = -x\cot\tfrac{1}{2}x + 2\log\sin\tfrac{1}{2}x.$

(iii) $\int \frac{x}{1+\sin x} dx = -x \tan \frac{1}{2}(\frac{1}{2}\pi - x) + 2 \log \cos \frac{1}{2}(\frac{1}{2}\pi - x)$.

(iv) $\int \frac{x}{1-\sin x} dx = x \cot \frac{1}{2}(\frac{1}{2}\pi - x) + 2 \log \sin \frac{1}{2}(\frac{1}{2}\pi - x)$.

8. Show that $\int_a^b \frac{\log x}{x} dx = \frac{1}{2} \log\left(\frac{b}{a}\right) . \log (ab)$.

9. Evaluate $\int_\alpha^\beta \left[\log \log x + \frac{1}{(\log x)^2}\right] dx$.

10. Evaluate the following integrals :—

(i) $\int \frac{\cos 2x}{\cos x} dx$, (ii) $\int \frac{x - \sin x}{1 - \cos x} dx$, (iii) $\int \sec^3 x \, dx$.

11. Evaluate

(i) $\int x^2 \tan^{-1} x \, dx$, (ii) $\int x^8 \tan^{-1} x^3 \, dx$,

(iii) $\int \frac{(1-x)e^x \, dx}{x^2}$, (iv) $\int (\sin x + \cos x) e^x \, dx$,

(v) $\int_0^1 \frac{x^3 \sin^{-1} x}{\sqrt{(1-x^2)}} dx$, (vi) $\int_0^\infty \frac{x (\tan^{-1} x)^2}{(1+x^2)^{3/2}} dx$,

(vii) $\int_0^\pi x e^{2x} \sin x \, dx$, (viii) $\int_0^1 x^2 e^x \sin \pi x \, dx$.

12. Integrate

(i) $\frac{1}{\sqrt{(x+a)} + \sqrt{(x+b)}}$, (ii) $\frac{\sqrt{(a^2 - x^2)}}{x}$,

(ii) $\log [x + \sqrt{(x^2 + a^2)}]$, (iv) $x \log [x + \sqrt{(x^2 + a^2)}]$.

13. Integrate

(i) $\sqrt{\left(\frac{x}{a+x}\right)}$, (ii) $\frac{\sin x}{\sin (x-a)}$,

(iii) $\sqrt{\left(\frac{x+a}{x}\right)}$, (iv) $\tan^{-1} \frac{3x - x^3}{1 - 3x^2}$.

Answers

2. $(2x^2 - 5x - 1)e^x$. **3.** $\frac{1}{4}(x^4 - 1) \log (1 + x^2) - \frac{1}{8}(x^4 - 2x^2)$.

5. $[x\sqrt{(1-x^2)} - \cos^{-1} x]/2x^2$.

6. $-[2 \sin (\log x) + \cos (\log x)]/5x^2$.

9. $\beta \log (\log \beta) - \alpha \log (\log \alpha) - [\beta (\log \beta)^{-1} - \alpha (\log \alpha)^{-1}]$.

10. (i) $2 \sin x - \log \tan (\frac{1}{2}x + \frac{1}{4}\pi)$. (ii) $-x \cot (\frac{1}{2}x)$.

(iii) $\frac{1}{2} \tan x \sec x + \frac{1}{2} \log \tan (\frac{1}{2}x + \frac{1}{4}\pi)$.

11. (*i*) $\frac{1}{3}x^3 \tan^{-1} x+\frac{1}{6}[\log(x^2+1)-x^2]$.

(*ii*) $\frac{1}{9}x^9 \tan^{-1} x^3+\frac{1}{18}[\log(x^6+1)-x^6]$.

(*iii*) $-e^x/x$. (*iv*) $e^x \sin x$. (*v*) $\frac{7}{5}$. (*vi*) $(\pi-2)$.

(*vii*) $\frac{1}{25}(5\pi-4e^{2\pi}-\frac{4}{25})$. (*viii*) $\pi[e(\pi^4-4\pi^2+3)-2\pi^2+6]/(1+\pi^2)^3$

12. (*i*) $\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]$.

(*ii*) $\sqrt{(a^2-x^2)}+a \log \frac{a-\sqrt{(a^2-x^2)}}{x}$.

(*iii*) $x \log[x+\sqrt{(a^2+x^2)}]-\sqrt{(a^2+x^2)}$.

(*iv*) $\frac{1}{2}x^2 \log[x+\sqrt{(a^2+x^2)}]-\frac{1}{4}[x\sqrt{(a^2+x^2)}-a^2 \sinh^{-1}(x/a)]$.

13. (*i*) $(a+x)\tan^{-1}\sqrt{(x/a)}-\sqrt{(ax)}$.

(*ii*) $\sin a \log \sin(x-a)+(x-a)\cos a$.

(*iii*) $a \tan\theta \sec\theta+a \log \tan(\frac{1}{2}\theta+\frac{1}{4}\pi)$, where $x=a\tan^2\theta$.

(*iv*) $3x \tan^{-1} x-\frac{3}{2}\log(1+x^2)$.

3

Integration of Algebraic Rational Functions

3·1. In this chapter we shall be concerned with the integration of rational functions $f(x)/\varphi(x)$, where $f(x)$ and $\varphi(x)$ are polynomials, say

$$f(x) = a_0x^m+a_1x^{m-1}+\ldots\ldots+a_{m-1}x+a_m,$$
$$\varphi(x) = b_0x^n+b_1x^{n-1}+\ldots\ldots+b_{n-1}x+b_n.$$

We can assume that the degree m of the numerator $f(x)$ is smaller than the degree n of the denominator. For, otherwise, we may divide the numerator by the denominator, till we get a remainder whose degree is smaller than that of the denominator. Thus we have

$$\frac{f(x)}{\varphi(x)} = Q\,(x) + \frac{R(x)}{\varphi(x)}\,,$$

where the polynomial $Q(x)$ is the quotient and the polynomial $R(x)$ is the remainder obtained on dividing $f(x)$ by $\varphi(x)$, the degree of $R(x)$ being smaller than that of $\varphi(x)$.

The part $Q(x)$ which is a polynomial can be at once integrated term by term.

We will now see how the integration of $R(x)/\varphi(x)$ can be effected, the degree of $R(x)$ being smaller than that of $\varphi(x)$.

From Algebra, we know that every polynomial $\varphi(x)$ can be resolved into real factors of the first and second degree so that we may write

$$\varphi(x) = A\,(a_1x+b_1)^{p_1}\,(a_2x+b_2)^{p_2}\ldots\ldots$$
$$\times(A_1x^2+2B_1x+C_1)^{q_1}\,(A_2x^2+2B_2x+C_2)^{q_2}$$

Here $p_1, p_2, p_3,\ldots\ldots$; $q_1, q_2, q_3,\ldots\ldots$are positive integers denoting the number of times each factor is repeated.

Also, some or all of $p_1, p_2, p_3, \ldots\ldots$ and of $q_1, q_2, \ldots\ldots$ may be unity. Thus, we have four types of factors of the denominator $\varphi(x)$:—

(*i*) *Linear non-repeated.* (*ii*) *Linear repeated.*

(*iii*) *Quadratic non-repeated.* (*iv*) *Quadratic repeated.*

From the theory of partial fractions, we know that $R(x)/\varphi(x)$ can be decomposed into the sum of a number of fractions such that

(*i*) to a linear non-repeated factor, $(ax+b)$, corresponds a partial fraction of the form

$$\frac{L}{ax+b};$$

(*ii*) to a linear repeated factor, $(ax+b)^p$, corresponds a sum of p, fractions of the form

$$\frac{L_1}{ax+b}+\frac{L_2}{(ax+b)^2}+\ldots\ldots+\frac{L_p}{(ax+b)^p};$$

(*iii*) to a quadratic non-repeated factor, $Ax^2+2Bx+C$, which cannot be further resolved into real linear factors, corresponds a fraction of the form

$$\frac{Lx+M}{Ax^2+2Bx+C};$$

(*iv*) to a quadratic repeated factor $(Ax^2+2Bx+C)^q$, corresponds a sum of q fractions of the form

$$\frac{L_1x+M_1}{Ax^2+2Bx+C}+\frac{L_2x+M_2}{(Ax^2+2Bx+C)^2}+\ldots+\frac{L_qx+M_q}{(Ax^2+2Bx+C)^q}.$$

The method of determining the constants L and M etc. is explained in books on elementary Algebra which give an account of partial fractions.

Thus to be able to complete the integration of an algebraic rational function, we have to learn to integrate fractions of the types

(*i*) $\dfrac{L}{ax+b}$, (*ii*) $\dfrac{L}{(ax+b)^r}$,

(*iii*) $\dfrac{Lx+M}{Ax^2+2Bx+C}$, (*iv*) $\dfrac{Lx+M}{(Ax^2+2Bx+C)^r}$.

We consider these types one by one :

(*i*) $\displaystyle\int\frac{L}{ax+b}dx=\frac{L}{a}\log(ax+b).$

(*ii*) $\displaystyle\int\frac{L}{(ax+b)^r}dx=\frac{L}{a(1-r)}\cdot\frac{1}{(ax+b)^{r-1}},$ for $r\neq 1$.

(*iii*) *To integrate*

$$(Lx+M)/(Ax^2+2Bx+C),$$

we determine two constants λ, μ such that

$$Lx+M \equiv \lambda\,(2Ax+2B)+\mu, \qquad ...(1)$$

the co-factor, $2Ax+2B$, of λ being the derivative of the denominator $Ax^2+2Bx+C$. From (1), we have

$$L = 2A\lambda,\ M = 2B\lambda+\mu.$$

$$\Rightarrow \qquad \lambda = \frac{L}{2A}, \qquad \mu = \frac{AM-BL}{A}.$$

$$\int \frac{Lx+M}{Ax^2+2Bx+C}dx$$

$$= \lambda \int \frac{2Ax+2B}{Ax^2+2Bx+C}\,dx + \mu \int \frac{dx}{Ax^2+2Bx+C}$$

$$= \lambda \log (Ax^2+2Bx+C) + \mu \int \frac{dx}{Ax^2+2Bx+C}$$

Again,

$$\int \frac{dx}{Ax^2+2Bx+C} = \frac{1}{A} \int \frac{dx}{x^2+\frac{2B}{A}x+\frac{C}{A}}$$

$$= \frac{1}{A} \int \frac{dx}{\left(x+\frac{B}{A}\right)^2 + \left(\frac{\sqrt{(AC-B^2)}}{A}\right)^2}.$$

We assume that $AC-B^2$ is positive, for otherwise $Ax^2+2Bx+C=0$ would have real roots and $Ax^2+2Bx+C$ would be capable of being expressed as the product of real linear factors.

$$\therefore \quad \int \frac{dx}{Ax^2+2Bx+C} = \frac{1}{\sqrt{(AC-B^2)}} \tan^{-1} \frac{Ax+B}{\sqrt{(AC-B^2)}}$$

Thus we have completed the integration of

$$(Lx+M)/(Ax^2+2Bx+C).$$

Note. The integral of $1/(x^2+a^2)$ is $(1/a) \tan^{-1} (x/a)$.

(*iv*) The method for integrating fractions of the type

$$\frac{Lx+M}{(Ax^2+2Bx+C)^r}$$

will be given in § 3·5 on page 62.

Exercises

1. *Evaluate*

(*i*) $\int \frac{dx}{3x-4}$, (*ii*) $\int \frac{dx}{5-2x}$, (*iii*) $\int \frac{2dx}{3-2x}$.

2. *Evaluate*

(*i*) $\int \frac{dx}{(2x-3)^3}$, (*ii*) $\int \frac{dx}{(5+3x)^2}$, (*iii*) $\int \frac{dx}{(3-2x)^4}$.

3. *Evaluate*

(*i*) $\int \frac{dx}{x^2+3x+4}$, (*ii*) $\int \frac{(x+2)dx}{x^2+2x+3}$, (*iii*) $\int \frac{(2x-3)dx}{3x^2+4x+5}$

4. Show that

$$\int_0^1 \frac{(4x^2+3)\,dx}{8x^2+4x+5} = \tfrac{1}{2} - \tfrac{1}{8}\log \tfrac{17}{5} + \tfrac{1}{6}\tan^{-1} \tfrac{6}{7}.$$

Answers

1. (*i*) $\frac{1}{3}\log(3x-4)$. (*ii*) $-\frac{1}{2}\log(5-2x)$. (*iii*) $-\log(3-2x)$.

2. (*i*) $-\frac{1}{4}\cdot\frac{1}{(2x-3)^2}$. (*ii*) $-\frac{1}{3}\cdot\frac{1}{5+3x}$. (*iii*) $\frac{1}{6}\cdot\frac{1}{(3-2x)^3}$.

3 (*i*) $\frac{2}{\sqrt{7}}\tan^{-1}\frac{2x+3}{\sqrt{7}}$.

(*ii*) $\frac{1}{2}\log(x^2+2x+3)+\sqrt{\tfrac{1}{2}}\tan^{-1}[(x+1)/\sqrt{2}]$.

(*iv*) $\frac{1}{3}\log(3x^2+4x+5) - \frac{13}{3\sqrt{11}}\tan^{-1}\left(\frac{3x+2}{\sqrt{11}}\right)$.

3·2. Case of non-repeated linear factors only in the denominator

Example

Evaluate $\int \frac{x^2+5x+41}{(x+3)(x-1)(2x-1)}\,dx$.

We have

$$\frac{x^2+5x+41}{(x+3)(x-1)(2x-1)} = \frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{2x-1}.$$

$$\Rightarrow\quad x^2+5x+41 = A(x-1)(2x-1)+B(x+3)(2x-1)$$
$$+C(x+3)(x-1).$$

Putting $x = -3, 1, \frac{1}{2}$ respectively, we get

$$A = \tfrac{5}{4},\ B = \tfrac{47}{4},\ C = -25.$$

$$\therefore\quad \int \frac{x^2+5x+41}{(x+3)(x-1)(2x-1)}\,dx$$
$$= \frac{5}{4}\int\frac{dx}{x+3}+\frac{47}{4}\int\frac{dx}{x-1}-25\int\frac{dx}{2x-1}$$
$$= \tfrac{5}{4}\log(x+3)+\tfrac{47}{4}\log(x-1)-\tfrac{25}{2}\log(2x-1).$$

Exercises

1. *Evaluate*

(*i*) $\int \frac{dx}{x^2-1}$, (*ii*) $\int \frac{x^2+1}{x^2-1}\,dx$,

(iii) $\displaystyle\int \frac{dx}{(x+1)(x+2)(x+3)}$, (iv) $\displaystyle\int \frac{(2x-3)\,dx}{(x^2-1)(2x+3)}$,

(v) $\displaystyle\int \frac{(x-1)(x-2)(x-3)}{(x+1)(x+2)(x+3)}\,dx$. (vi) $\displaystyle\int \frac{x^2\,dx}{(x+1)(x-2)(x+3)}$,

(vii) $\displaystyle\int \frac{x^2\,dx}{(x+1)(x+2)(x+3)}$, (viii) $\displaystyle\int \frac{x^3+3}{x^3-3x}\,dx$.

(ix) $\displaystyle\int \frac{(x^2+1)\,dx}{(x^2-1)(x^2-4)}$, (x) $\displaystyle\int \frac{x^5\,dx}{x^3-2x^2-5x+6}$,

(xi) $\displaystyle\int_2^3 \frac{(x^2+1)}{(2x+1)(x^2-1)}\,dx$, (xii) $\displaystyle\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1}$,

$(0 \leqslant \alpha < \pi)$.

Answers

1. (i) $\frac{1}{2}\log[(x-1)/(x+1)]$. (ii) $x+\log[(x-1)/(x+1)]$.

(iii) $\frac{1}{2}\log[(x+1)(x+3)/(x+2)^2]$.

(iv) $\frac{5}{2}\log(x+1)-\frac{1}{10}\log(x-1)-\frac{12}{5}\log(2x+3)$.

(v) $x+12\log[(x+2)^3/(x+1)(x+3)^5]$.

(vi) $\frac{9}{10}\log(x+3)+\frac{4}{15}\log(x-2)-\frac{1}{6}\log(1+x)$.

(vii) $\frac{1}{2}\log(x+1)-4\log(x+2)+\frac{9}{2}\log(x+3)$.

(viii) $x-\log x+\frac{1}{2}(1+\sqrt{3})\log(x-\sqrt{3})+\frac{1}{2}(1-\sqrt{3})\log(x+\sqrt{3})$.

(ix) $\frac{5}{12}\log[(x-2)/(x+2)]-\frac{1}{3}\log[(x-1)/(x+1)]$.

(x) $\frac{1}{3}x^3+x^2+9x-\frac{1}{6}\log(x-1)-\frac{32}{15}\log(x+2)+\frac{243}{10}\log(x-3)$.

(xi) $-\log 3-\frac{5}{6}\log\frac{7}{5}+\frac{7}{3}\log 2$.

(xii) $\alpha/2\sin\alpha$, if $\alpha\neq 0$; $\frac{1}{2}$ if $\alpha=0$.

3·3. Case of non-repeated linear or repeated linear factors only in the denominator.

Example

Evaluate

$$\int \frac{1}{x^3(x-1)^2(x+1)}\,dx.$$

We suppose

$$\frac{1}{x^3(x-1)^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}+\frac{E}{(x-1)^2}+\frac{F}{x+1}$$

$$\therefore\quad 1 \equiv Ax^2(x-1)^2(x+1)+Bx(x-1)^2(x+1)+C(x-1)^2(x+1)$$
$$+Dx^3(x-1)(x+1)+Ex^3(x+1)+Fx^3(x-1)^2 \qquad \ldots(1)$$

Putting $x=0,\,-1$, we get

$$C=1,\ E=\tfrac{1}{2},\ F=-\tfrac{1}{4}.$$

To obtain A, B, D we equate the coefficients of x^5, x^4, x^3 on both sides of (*i*) and get

$$0 = A+D+F,$$
$$0 = -A+B+E+2F,$$
$$0 = -A-B+C-D+E+F$$

Solving these equations we obtain

$$A = 2, \quad B = 1, \quad D = -\tfrac{7}{4},$$

$$\int \frac{1}{x^3(x-1)^2(x+1)}\,dx = 2\log x - \frac{1}{x} - \frac{1}{2x^2} - \frac{7}{4}\log(x-1)$$
$$- \frac{1}{2(x-1)} - \frac{1}{4}\log(x+1)$$

Exercises

1. Integrate the following functions :

(*i*) $\dfrac{x^2+x+1}{(x+1)^2(x+2)}$, (*ii*) $\dfrac{x+1}{(x-1)^2(x+2)^2}$,

(*iii*) $\dfrac{x^2+1}{(x+2)^2(x-1)}$, (*iv*) $\dfrac{x^3+2}{(x-1)(x-2)^3}$,

(*v*) $\dfrac{x^3-4x^2+5x-2}{x^3+4x^2+5x+2}$, (*vi*) $\dfrac{(1+x)^3}{(1-x)^3}$,

(*vii*) $\dfrac{57x^3-25x^2+9x-1}{(x-1)^2(2x-1)(5x-1)}$.

2. Evaluate the following definite integrals :

(*i*) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{(1-x^2)^2}$, (*ii*) $\displaystyle\int_0^{\infty} \frac{dx}{(1+x)^2(2+x)}$

Answers

1. (*i*) $-\dfrac{1}{x+1} + \log\dfrac{(x+2)^3}{(x+1)^2}$.

(*iii*) $\dfrac{1}{27}\log\dfrac{x+2}{x-1} - \dfrac{2}{9}\cdot\dfrac{1}{x-1} + \dfrac{1}{9}\cdot\dfrac{1}{x+2}$.

(*ii*) $\dfrac{2}{27}\log\dfrac{x-1}{x+2} - \dfrac{7}{9}\cdot\dfrac{1}{x+2} + \dfrac{5}{6}\cdot\dfrac{1}{(x+2)^2}$.

(*iv*) $4\log(x-2) - 3\log(x-1) - \dfrac{2}{x-2} - \dfrac{5}{(x-2)^2}$.

(*v*) $x+12(x+1)^{-1}+28\log(x+1)-36\log(x+2)$.

(*vi*) $-x-12(1-x)^{-1}+4(1-x)^{-2}-6\log(1-x)$.

(*vii*) $-10(x-1)^{-1}+35/6\log(2x-1)-2/15\log(5x-1)$.

2. (*i*) $1/12\log(27e^4)$. (*ii*) $\frac{1}{2}\log(4/e)$.

3.4. Case of linear or quadratic non-repeated factors only in the denominator.

Example

Evaluate

$$\int \frac{dx}{(x-1)^2(x-2)(x^2+4)}.$$

We suppose

$$\frac{1}{(x-1)^2(x-2)(x^2+4)} \equiv \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-2)}+\frac{Dx+E}{x^2+4}$$

$$\Rightarrow 1 \equiv A(x-1)\,(x-2)\,(x^2+4)+B(x-2)\,(x^2+4)$$
$$+C(x-1)^2(x^2+4)+(Dx+E)\,(x-1)^2\,(x-2)$$

Putting $x = 1, 2$, we get

$$B = -\frac{1}{5} \text{ and } C = \frac{1}{8}$$

To find A, D, E, we equate the coefficients of x^4, x^3, x^2, so that we get

$$A+C+D = 0$$
$$-3A+B-2C-2D+E = 0$$
$$6A-2B+5C+4D-2E = 0$$

Solving these equations, we get

$A = -3/25,\ D = -1/200,\ E = 7/100$

$$\therefore \int \frac{1}{(x-1)^2\,(x-2)\,(x^2+4)}\,dx = -\frac{3}{25}\int\frac{dx}{x-1}-\frac{1}{5}\int\frac{dx}{(x-1)^2}$$
$$+\frac{1}{8}\int\frac{dx}{x-2}-\frac{1}{400}\int\frac{2x}{x^2+4}\,dx+\frac{7}{100}\int\frac{dx}{x^2+4}$$
$$= -\frac{3}{25}\log(x-1)+\frac{1}{5(x-1)}+\frac{1}{8}\log(x-2)-\frac{1}{400}\log(x^2+4)$$
$$+\frac{7}{200}\tan^{-1}x/2$$

Exercises

1. Evaluate

(*i*) $\displaystyle\int\frac{x^3}{x^2+1}dx,$ (*ii*) $\displaystyle\int\frac{x^4}{(x^2-1)(x^2+4)}\,dx,$

(*iii*) $\int \frac{x}{1+x^3}\,dx,$ (*iv*) $\int \frac{x^5}{(x^2+1)(x-4)}\,dx,$

(*v*) $\int \frac{x^2+1}{x^3+1}\,dx,$ (*vi*) $\int \frac{x^2+6x-25}{(x^2+4x)^2-25}\,dx,$

(*vii*) $\int \frac{dx}{(x+1)^2(x^2+1)},$ (*viii*) $\int \frac{dx}{1-x^6},$

(*ix*) $\int \frac{(x+a)dx}{x^2(x-a)(x^2+a^2)},$ (*x*) $\int \frac{x\,dx}{(x+1)^3(x^2+1)},$

(*xi*) $\int \frac{dx}{(x-1)^2(x^2+4)},$ (*xii*) $\int \frac{x\,dx}{x^3+x^2+x+1},$

(*xiii*) $\int \frac{x^3\,dx}{x^4+3x^2+2}.$

2. Evaluate the following definite integrals :

(*i*) $\int_1^\infty \frac{dx}{(x+1)^2(x^2+1)},$ (*ii*) $\int_0^2 \frac{4-x}{x(x^2-2x+2)}\,dx.$

Answers

1. (*i*) $\frac{1}{2}x^2-\frac{1}{2}\log(x^2+1).$

(*ii*) $x+\frac{1}{10}\log[(x-1)/(x+1)]-\frac{8}{5}\tan^{-1}(x/2).$

(*iii*) $\frac{1}{6}\log(x^2-x+1)-\frac{1}{3}\log(x+1)+\sqrt{\frac{1}{3}}\tan^{-1}[(2x-1)/\sqrt{3}].$

(*iv*) $\frac{1}{3}x^3+2x^2+15x+\frac{1024}{17}\log(x-4)+\frac{2}{17}\log(x^2+1)+\frac{1}{17}\tan^{-1}x.$

(*v*) $\frac{2}{3}\log(x+1)+\frac{1}{6}\log(x^2-x+1)+\sqrt{\frac{1}{3}}\tan^{-1}[(2x-1)/\sqrt{3}].$

(*vi*) $\frac{1}{2}\log(x+5)-\frac{3}{10}\log(x-1)-\frac{1}{10}\log(x^2+4x+5)+\frac{17}{5}\tan^{-1}(x+2).$

(*vii*) $\frac{1}{2}\log(x+1)-\frac{1}{4}\log(x^2+1)-\frac{1}{2}(x+1)^{2}.$

(*viii*) $\frac{1}{6}\log\frac{x+1}{1-x}+\frac{1}{12}\log\frac{x^2+x+1}{x^2-x+1}$

$$+\frac{1}{2\sqrt{3}}\left(\tan^{-1}\frac{2x+1}{\sqrt{3}}+\tan^{-1}\frac{2x-1}{\sqrt{3}}\right).$$

(*ix*) $\frac{1}{a^3}\log\frac{(x-a)\sqrt{(x^2+a^2)}}{x^2}+\frac{1}{a^2x}.$

(*x*) $\frac{1}{4}\log(x+1)-\frac{1}{8}\log(x^2+1)+\frac{1}{4}\tan^{-1}x+\frac{1}{4}(x+1)^{-2}$

(*xi*) $-\frac{2}{25}\log(x-1)-\frac{1}{5(x-1)}+\frac{1}{25}\log(x^2+4)-\frac{3}{50}\tan^{-1}\frac{x}{2}.$

(*xii*) $\frac{1}{2}\log\frac{\sqrt{(x^2+1)}}{x+1}+\frac{1}{2}\tan^{-1}x.$

(*xiii*) $\log(x^2+2)-\frac{1}{2}\log(x^2+1).$

2. (*i*) $\frac{1}{4}.$ (*ii*) $\frac{1}{4}\pi+\log 2.$

3·41. Case of the integrand consisting of even powers of x only.

Examples

Integrate

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}.$$

We note that the fraction is a function of x^2. On this account its decomposition into partial fractions is more conveniently effected by putting $x^2=y$. We have

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=\frac{(y+1)(y+2)}{(y+3)(y+4)}$$

$$=1+\frac{(y+1)(y+2)}{(y+3)(y+4)}-1$$

$$=1+\frac{(y+1)(y+2)-(y+3)(y+4)}{(y+3)(y+4)}$$

Let

$$\frac{(y+1)(y+2)-(y+3)(y+4)}{(y+3)(y+4)}=\frac{A}{y+3}+\frac{B}{y+4}.$$

$\Rightarrow \quad (y+1)(y+2)-(y+3)(y+4)=A(y+4)+B(y+3).$

Putting $y=-3$ and $y=-4$ on both the sides, we get

$$A=2, B=-6.$$

$$\therefore \quad \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=1+\frac{2}{y+3}-\frac{6}{y+4}$$

$$=1+\frac{2}{x^2+3}-\frac{6}{x^2+4}$$

$$\Rightarrow \quad \int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx=x+\frac{x}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}.$$

Exercises

1. Evaluate the following integrals :

(*i*) $\displaystyle\int\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx,$ (*ii*) $\displaystyle\int\frac{x^2}{x^4+x^2-2}dx.$

2. *Evaluate*

$$\int\frac{\cos^2\theta}{\cos^2\theta+4\sin^2\theta}d\theta$$

Answers

1. (*i*) $\dfrac{1}{(a^2-b^2)}\left[a\tan^{-1}\dfrac{x}{a}-b\tan^{-1}\dfrac{x}{b}\right]$.

   (*ii*) $1/6 \log[(x-1)/(x+1)]+(\sqrt{2}/3)\tan^{-1}(x/\sqrt{2})$.

2. $-\frac{1}{3}\tan^{-1}(\tan\theta)-2/3\tan^{-1}(2\tan\theta)$.

3·5. Integration of

$$\frac{Lx+M}{(Ax^2+2Bx+C)^n},$$

where n is a positive integer different from 1.

To evaluate

$$\int\frac{(Lx+M)\,dx}{(Ax^2+2Bx+C)^n},$$

we determine two constants λ and μ, such that

$$Lx+M=\lambda(2Ax+2B)+\mu, \qquad \text{[Refer § 3·1]}$$

where the co-factor $2Ax+2Bx$ of λ is the derivative of $Ax^2+2Bx+C$.

We thus have

$$\int\frac{Lx+M}{(Ax^2+2Bx+C)^n}dx$$

$$=\lambda\int\frac{2Ax+2B}{(Ax^2+2Bx+C)^n}\,dx+\mu\int\frac{dx}{(Ax^2+2Bx+C)^n}$$

$$=\frac{\lambda}{(1-n)}\cdot\frac{1}{(Ax^2+2Bx+C)^{n-1}}+\mu\,\frac{dx}{(Ax^2+2Bx+C)^n}.$$

To evaluate the integral

$$\int\frac{dx}{(Ax^2+2Bx+C)^n}$$

we obtain a reduction formula, we write

$$Ax^2+2Bx+C=A\left[\left(x+\frac{B}{A}\right)^2+\left(\frac{\sqrt{(AC-B^2)}}{A}\right)^2\right]$$

$$=A\left[\left(x+\frac{B}{A}\right)^2+k^2\right],$$

where, for the sake of simplicity, we have written

$$\sqrt{(AC-B^2)}/A=k.$$

Also, putting $x+B/A=y$, we get

$$\int\frac{dx}{(Ax^2+2Bx+C)^n}=\frac{1}{A^n}\int\frac{dy}{(y^2+k^2)^n}.$$

The integral

$$\int \frac{dy}{(y^2+k^2)^n}$$

is evaluated with the help of a *reduction formula* which is obtained in the following section.

3·51. Reduction Formula for

$$\int \frac{\mathbf{1}}{(\mathbf{y}^2+\mathbf{k}^2)^n}\mathbf{dy}.$$

To obtain the required reduction formula, we write

$$\int \frac{dy}{(x^2+k^2)^n} = \int 1.\ \frac{1}{(y^2+k^2)^n}\,dy,$$

and apply the rule of integration by parts. Thus, we have

$$\int \frac{dy}{(y^2+k^2)^n} = \frac{y}{(y^2+k^2)^n} - \int y.\ (-n)(y^2+k^2)^{-n-1}.\ 2y\,dy$$

$$= \frac{y}{(y^2+k^2)^n} + 2n\int \frac{y}{(y^2+k^2)^{n+1}}\,dy$$

$$= \frac{y}{(y^2+k^2)^n} + 2n\int \frac{y^2+k^2-k^2}{(y^2+k^2)^{n+1}}\,dy$$

$$= \frac{y}{(y^2+k^2)^n} + 2n\int \frac{dy}{(y^2+k^2)^n} - 2nk^2\int \frac{dy}{(y^2+k^2)^{n+1}}.$$

$$\therefore\ 2nk^2\int \frac{dy}{(y^2+k^2)^{n+1}} = \frac{y}{(y^2+k^2)^n} + (2n-1)\int \frac{dy}{(y^2+k^2)^n}.$$

$$\Rightarrow \qquad \int \frac{dy}{(y^2+k^2)^{n+1}} = \frac{y}{2nk^2\,(y^2+k^2)^n} + \frac{2n-1}{2nk^2}\int \frac{dy}{(y^2+k^2)^n},$$

Changing n to $n-1$, we get

$$\int \frac{dy}{(y^2+k^2)^n} = \frac{y}{2(n-1)k^2(y^2+k^2)^{n-1}} + \frac{2n-3}{2(n-1)k^2}\int \frac{dy}{(y^2+k^2)^{n-1}}.$$

which is the required reduction formula. With its help, we can integrate the given integral by successive reduction.

Examples

Evaluate

$$\int \frac{dx}{(x^2+1)^4}.$$

Changing k to 1 and y to x in the reduction formula above, we get

$$\frac{dx}{(x^2+1)^n} = \frac{x}{2(n-1)(x^2+1)^{n-1}}+\frac{2n-3}{2(n-1)}\int\frac{dx}{(x^2+1)^{n-1}}.$$

Changing n to 4, 3, 2, successively, we get

$$\int\frac{dx}{(x^2+1)^4} = \frac{x}{6(x^2+1)^3}+\frac{5}{6}\int\frac{dx}{(x^2+1)^3},$$

$$\int\frac{dx}{(x^2+1)^3} = \frac{x}{4(x^2+1)^2}+\frac{3}{4}\int\frac{dx}{(x^2+1)^2}.$$

$$\int\frac{dx}{(x^2+1)^2} = \frac{x}{2(x^2+1)}+\frac{1}{2}\int\frac{dx}{x^2+1}.$$

$$= \frac{x}{2(x^2+1)}+\frac{1}{2}\tan^{-1}x.$$

From this system of equalities, we get

$$\int\frac{dx}{(x^2+1)^4} = \frac{x}{6(x^2+1)^3}+\frac{5x}{24(x^2+1)^2}+\frac{5x}{16(x^2+1)}+\frac{5}{16}\tan^{-1}x.$$

Note. The integral

$$\int\frac{dy}{(y^2+1)^2}$$

can also be evaluated by the substitution $y = \tan\theta$, so that we make no use of the reduction formula. We have

$$\int\frac{dy}{(y^2+1)^2} = \int\frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int\cos^2\theta\, d\theta$$

$$= \tfrac{1}{2}\int(1+\text{cas } 2\theta)\, d\theta$$

$$= \tfrac{1}{2}(\theta+\sin\theta\cos\theta),$$

$$= \tfrac{1}{2}\left[\tan^{-1}y+\frac{y}{y^2+1}\right]$$

which agrees with the result obtained above with the help of the reduction formula.

Exercises

1. Evaluate the following integrals :

(i) $\displaystyle\int\frac{2x-3}{(x^2+x+1)^2}dx,$ (ii) $\displaystyle\int\frac{x+2}{(x+1)(x^2+1)^2}dx,$

(iii) $\displaystyle\int\frac{x^3-x^2-1}{(x^2+1)^2(x^2-1)}dx,$ (iv) $\displaystyle\int\frac{x^2+1}{(x^2-1)(x^2+2)^2}dx$

(v) $\displaystyle\int \frac{x(2x^2-x+5)}{(x^2+2x+2)^2}\,dx$

(vi) $\displaystyle\int \frac{x^2-21}{(x^2-2x+6)^3}\,dx$, (vii) $\displaystyle\int \frac{x^2-2}{(x^2+2)^3}\,dx$.

(viii) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)^2(x^2+2)^2}\,dx$.

2. Show that

$$\int_0^\infty \frac{dx}{(x^2+1)^n} = \frac{2n-3}{2n-2}\int_0^\infty \frac{dx}{(x^2+1)^{n-1}},$$

and deduce that

$$\int_0^\infty \frac{dx}{(1+x^2)^3} = \frac{\pi}{2}\cdot\frac{1.3.5.7}{2.4.6.8}.$$

Answers

1. (*i*) $-\dfrac{1}{x^2+x+1}-\dfrac{16}{3\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}-\dfrac{4}{3}\cdot\dfrac{2x+1}{x^2+x+1}$.

(*ii*) $\dfrac{1+3x}{4(1+x^2)}+\tan^{-1}x+\dfrac{1}{8}\log\dfrac{(x+1)^2}{x^2+1}$

(*iii*) $\dfrac{1}{4}\cdot\dfrac{1}{x^2+1}+\dfrac{1}{2}\tan^{-1}x+\dfrac{1}{8}\log\dfrac{(x+1)^3}{(x^2+1)(x-1)}$.

(*iv*) $\dfrac{1}{12}\cdot\dfrac{x}{2+x^2}-\dfrac{5\sqrt{2}}{72}\tan^{-1}\dfrac{x}{\sqrt{2}}+\dfrac{1}{9}\log\dfrac{x-1}{x+1}$.

(*v*) $\log(x^2+2x+2)-\dfrac{15}{2}\tan^{-1}(x+1)-\dfrac{11}{2(x^2+2x+2)}-\dfrac{x+1}{2(x^2+2x+2)}$

(*vi*) $-\dfrac{11x-11}{40(x^2-2x+6)}-\dfrac{5(x-1)+2}{4(x^2-2x+6)^2}-\dfrac{11\sqrt{5}}{200}\tan^{-1}\dfrac{x-1}{\sqrt{5}}$.

(*vii*) $-\dfrac{x}{8(x^2+2)}-\dfrac{x}{2(x^2+2)^2}-\dfrac{\sqrt{2}}{16}\tan^{-1}\dfrac{x}{\sqrt{2}}$.

(*viii*) $(10-7\sqrt{2})\pi/4$.

3·6. Integration of algebraic rational functions by substitution. The process of integrating an algebraic rational function can sometimes be shortened by the use of a suitable substitution.

Examples

1. *Evaluate*

$$\int \frac{dx}{x(x^4+1)^3}.$$

The denominator consists of a repeated quadratic factor.

Putting $x^2+1 = y$, we get

$$\int \frac{dx}{x(x^2+1)^3} = \int \frac{dy}{2y^3(y-1)}.$$

In this new integrand we have linear factors in the denominator

Let

$$\frac{1}{y^3(y-1)} = \frac{A}{y} + \frac{B}{y^2} + \frac{C}{y^3} + \frac{D}{y-1}.$$

$$\Rightarrow \qquad 1 = Ay^2(y-1)+By(y-1)+C(y-1)+Dy^3.$$

Putting $y = 0$ and 1, we get

$$C = -1,\ D = 1.$$

Equating the coefficients of y^3 and y^2, we get

$$0 = A+D$$
$$0 = -A+B,$$
$$\Rightarrow \qquad A = -1,\ B = -1.$$

$$\therefore \int \frac{1}{y^3(y-1)}\, dy = -\log y + \frac{1}{y} + \frac{1}{2y^2} + \log(y-1),$$

where $\qquad y = x^2+1.$

The process may now be easily completed.

Note. It is easy to see that the process of splitting up into partial fractions the integrand which is obtained after substitution is shorter than the one for so splitting up the given integrand.

2. *Evaluate* $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}$

Putting $x = \tan\theta$, we get

$$\int \frac{dx}{(x^2+1)^2} = \int \cos^2\theta\, d\theta$$
$$= \tfrac{1}{2} \int (1+\cos 2\theta)\, d\theta$$
$$= \tfrac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) = \tfrac{1}{2}\left(\theta+\sin\theta\cos\theta\right).$$

Also as θ varies from 0 to $\pi/2$, x varies from 0 to ∞. Therefore the limits for θ are 0 and $\pi/2$.

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \tfrac{1}{2}\Big|\theta+\sin\theta\cos\theta\Big|_0^{\frac{1}{2}\pi} = \frac{\pi}{4}.$$

3. *Evaluate* $\displaystyle\int \frac{1+x^2}{1+x^4}\,dx.$

We write

$$\frac{1+x^2}{1+x^4}=\frac{1/x^2+1}{1/x^2+x^2}=\frac{1+1/x^2}{(x-1/x)^2+2}$$

Put $\quad x-\dfrac{1}{x}=y$ so that $\left(1+\dfrac{1}{x^2}\right)dx=dy$

$$\int\frac{1+x^2}{1+x^4}\,dx=\int\frac{dy}{y^2+2}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}}$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-1/x}{\sqrt{2}}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{\sqrt{2}x}.$$

4. *Evaluate* $\displaystyle\int\frac{x^2-1}{1+x^2}\,dx.$

Now

$$\int\frac{x^2-1}{1+x^4}\,dx=\int\frac{1-1/x^2}{1/x^2+x^2}\,dx=\int\frac{(1-1/x^2)}{(x+1/x)^2-2}\,dx.$$

Put $\quad x+\dfrac{1}{x}=y$ so that $\left(1-\dfrac{1}{x^2}\right)dx=dy$

$$\therefore\ \text{integral}=\int\frac{dy}{y^2-2}=\frac{1}{2\sqrt{2}}\int\left(\frac{1}{y-\sqrt{2}}-\frac{1}{y+\sqrt{2}}\right)dy$$

$$=\frac{1}{2\sqrt{2}}\log\frac{y-\sqrt{2}}{y+\sqrt{2}}=\frac{1}{2\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}.$$

Exercises

1. Evaluate the integrals of the following functions :

(*i*) $\dfrac{x}{x^4+x^2+1}$; (*ii*) $\dfrac{2x}{(x^2+1)(x^2+3)}$,

(*iii*) $\dfrac{x^3}{(x^2+1)^4}$, (*iv*) $\dfrac{1}{x(x^2+1)(x^2+2)^2}$,

(*v*) $\dfrac{1}{x(x^5+1)}$, (*vi*) $\dfrac{1}{x(x^4-1)}$.

(*vii*) $\dfrac{1}{x(1+x^n)}$.

2. Evaluate

(*i*) $\displaystyle\int\frac{dx}{x^4+1}$, (*ii*) $\displaystyle\int_0^\infty\frac{x^2}{1+x^4}\,dx$,

(*iii*) $\displaystyle\int\frac{x^2+1}{x^4+x^2+1}\,dx$, (*iv*) $\displaystyle\int\frac{dx}{x^4+x^2+1}$, (*v*) $\displaystyle\int \frac{x^2+3x+1}{x^4-x^2+1}\,dx$, (*vi*) $\displaystyle\int_0^{\frac{1}{4}\pi} \sqrt{(\tan x)}\,dx$,

(*vii*) $\int \sqrt{(\cot x)}\,dx$.

3. Evaluate

$$\int \frac{x^3}{(x^2+1)^3}\,dx,$$

by the substitutions (*a*) $x = \tan\theta$, (*b*) $x^2+1 = u$, and show that the results you obtain by the two methods are in accordance.

Answers

1. (*i*) $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x^2+1}{\sqrt{3}}$ (*ii*) $\dfrac{1}{2}\log\dfrac{x^2+1}{x^2+3}$.

(*iii*) $-\dfrac{3x^2+1}{12(x^2+1)^3}$. (*iv*) $\dfrac{1}{8}\log\dfrac{x^2(x^2+2)^3}{(x^2+1)^4}-\dfrac{1}{4(x^2+2)}$.

(*v*) $\dfrac{1}{5}\log\dfrac{x^5}{x^5+1}$. (*vi*) $\dfrac{1}{4}\log\dfrac{x^4}{x^4-1}-\dfrac{1}{4(x^4-1)}$.

(*vii*) $\dfrac{1}{n}\log\dfrac{x^n}{1+x^n}$

2. (*i*) $\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{x^2-1}{\sqrt{2}x}-\dfrac{1}{4\sqrt{2}}\log\dfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}$.

(*ii*) $\dfrac{\pi}{2\sqrt{2}}$. (*iii*) $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{x^2-1}{\sqrt{3}x}$.

(*iv*) $\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{x^2-1}{\sqrt{3}x}-\dfrac{1}{4}\log\dfrac{x^2-x+1}{x^2+x+1}$.

(*v*) $\tan^{-1}\dfrac{x^2-1}{x}+\sqrt{3}\tan^{-1}\dfrac{2x^2-1}{\sqrt{3}}$.

(*vi*) $\frac{1}{2}\sqrt{2}\log(\sqrt{2}-1)+\frac{1}{4}\sqrt{2}\pi$.

(*vii*) $\dfrac{1}{2\sqrt{2}}\log\dfrac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}-\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{t^2-1}{\sqrt{2}t}$ where $t=\sqrt{(\cot x)}$.

3·7. Integration of algebraic rational functions of e^x. An integral of a rational function of e^x is transformed into an integral of a rational function of t by means of the substitution $e^x = t$ as is illustrated below.

Examples

1. *Evaluate*

$$\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}\,dx.$$

We put $e^x = t$, so that $e^x\,dx = dt$.

$$\therefore \quad \int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}\,dx$$

$$= \int \frac{4t+6/t}{9t-4/t}\cdot\frac{dt}{t}$$

$$\int \frac{4t^2+6}{t(9t^2-4)}\,dt$$

$$= -\frac{3}{2}\int\frac{dt}{t}+\frac{35}{12}\int\frac{dt}{3t-2}+\frac{35}{12}\int\cdot\frac{dt}{3t+2}$$

$$= -\frac{3}{2}\log t+\frac{35}{36}\log(3t-2)+\frac{35}{36}\log(3t+2)$$

$$= -\frac{3}{2}\log t+\frac{35}{36}\log(9t^2-4)$$

$$= -\frac{3}{2}x+\frac{35}{36}\log(9e^{2x}-4).$$

2. *Evaluate*

$$\int_0^\infty \operatorname{sech} x\,dx.$$

Now,

$$\int \operatorname{sech} x\,dx = \int\frac{2}{e^x+e^{-x}}\,dx = \int\frac{2e^x}{1+e^{2x}}\,dx.$$

Putting $e^x = y$, we see that

$$\int \operatorname{sech} x\,dx = \int\frac{2dy}{1+y^2} = 2\tan^{-1} y = 2\tan^{-1} e^x$$

$$\therefore \quad \int_0^\infty \operatorname{sech} x\,dx = \Big|\, 2\tan^{-1}(e^x)\,\Big|_0^\infty$$

$$= 2\left[\frac{\pi}{2}-\frac{\pi}{4}\right] = \frac{\pi}{2},$$

for $\quad \lim\limits_{x\to\infty}(\tan^{-1} e^x) = \dfrac{\pi}{2}.$

Exercises

1. Evaluate

(*i*) $\displaystyle\int\frac{dx}{e^x+e^{2x}}$, (*ii*) $\displaystyle\int\frac{dx}{(e^x-1)^2}$.

(*iii*) $\displaystyle\int \frac{dx}{(1+e^{x})(1+e^{-x})}$, (*iv*) $\displaystyle\int_0^\infty \frac{e^{x}}{(e^{x}-1)^2}\,dx$,

(*v*) $\displaystyle\int_1^\infty \operatorname{cosech} x\,dx$, (*vi*) $\displaystyle\int \frac{dx}{1+\cosh x}$.

Answers

1. (*i*) $\log(1+e^{-x})-e^{-x}$. (*ii*) $(1-e^{x})^{-1}-\log(1-e^{-x})$.

(*iii*) $-\dfrac{1}{1+e^{x}}$. (*iv*) $2\log\dfrac{e-1}{e}+\dfrac{2e-1}{e^2-e}$.

(*v*) $\log[(e+1)/(e-1)]$. (*vi*) $\tanh\frac{1}{2}x$.

EXERCISES ON CHAPTER III

Integrate the following functions :

1. (*i*) $\dfrac{2x^3+7x^2+4x+2}{2x+3}$, (*ii*) $\dfrac{3x^3-4x^2+5x-6}{x^2+x-6}$,

(*iii*) $\dfrac{x^2-3x+3}{x^3-4x^2-7x+10}$, (*iv*) $\dfrac{17x^2-x-26}{(x^2-1)(x^2-4)}$.

2. (*i*) $\dfrac{3x^2-5x+4}{x^3-2x^2+3x+6}$, (*ii*) $\dfrac{1}{(2+x^2)(1-x^2)}$,

(*iii*) $\dfrac{x^2-3x}{x^4-1}$, (*iv*) $\dfrac{2x^2-x+3}{x^3+1}$.

3. (*i*) $\dfrac{x^3+1}{x^4-3x^3+3x^2-x}$, (*ii*) $\dfrac{(x^2+1)(x^2+2)(x^2+3)}{(x^2+4)(x^2+5)(x^2+6)}$,

(*iii*) $\dfrac{1}{x^4+1}$. (*iv*) $\dfrac{3x^2+x-2}{(x+1)^3(x^2+1)}$.

4. (*i*) $\dfrac{x^5}{x^3-1}$, (*ii*) $\dfrac{1}{x(x^2+a^2)^2}$,

(*iii*) $\dfrac{1}{x^{13}(x^6-1)}$, (*iv*) $\dfrac{1}{(x-1)^3(x^3+1)}$.

5. (*i*) $\dfrac{1}{x(x-1)^2(x^2+1)}$, (*ii*) $\dfrac{x^2+2}{(x^2+1)^2}$,

(*iii*) $\dfrac{2x+1}{(x^2+1)^3}$, (*iv*) $\dfrac{2x}{(1+x)(1+x^2)^2}$.

(*v*) $\dfrac{1}{(x^2-a^2)(x^2+a^2)^2}$, (*vi*) $\dfrac{3x+4}{(2x^2-x+2)^2}$.

6. Evaluate $\displaystyle\int_0^1 \frac{24t^3}{(1+t^2)^4}\,dt$.

7. Show that $\displaystyle\int_1^\infty \frac{(x^2+3)\,dx}{x^6(x^2+1)} = \frac{1}{30}(58-15\pi)$.

8. Show that

$$\int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2(a+b)(b+c)(c+a)}$$

9. Show that

$$\int_0^1 \frac{1-x^2}{1+x^2+x^4}\,dx = \frac{1}{2}\log 3.$$

10. Integrate :

(*i*) $\dfrac{\log(x^2+a^2)}{x^2}$, (*ii*) $x\log(x^2+1)$,

(*iii*) $e^x \log(e^{2x}+5e^x+6)$. (*iv*) $\dfrac{\tan^{-1} x}{(1+x)^2}$.

Answers

1. (*i*) $\frac{1}{3}(x^3+3x^2-3x)+\frac{5}{8}\log(2x+3)$.

(*ii*) $\frac{1}{2}(3x^2-14x)+\frac{186}{5}\log(x+3)+\frac{12}{5}\log(x-2)$.

(*iii*) $\frac{4}{21}\log(x+2)+\frac{25}{28}\log(x-5)-\frac{1}{12}\log(x-1)$.

(*iv*) $\frac{1}{6}\log[(x-1)^5(x-2)^{10}/(x+1)^4(x+2)^{11}]$.

2. (*i*) $\frac{6}{5}\log(x+1)+\frac{9}{10}\log(x^2-3x+6)-(1/\sqrt{15})\tan^{-1}[(2x-3)/\sqrt{15}]$.

(*ii*) $(1/3\sqrt{2})\tan^{-1}(x/\sqrt{2})+\frac{1}{6}[\log(1+x)-\log(1-x)]$.

(*iii*) $\frac{1}{2}\tan^{-1}x+\frac{3}{4}\log(x^2+1)-\frac{1}{2}\log(x-1)-\log(x+1)$.

(*iv*) $2\log(x+1)+(2/\sqrt{3})\tan^{-1}[(2x-1)/\sqrt{3}]$.

3. (*i*) $\log(x-1)^2-\log x-x/(x-1)^2$.

(*ii*) $x-\dfrac{3}{2}\tan^{-1}\dfrac{x}{2}+\dfrac{24}{\sqrt{5}}\tan^{-1}\dfrac{x}{\sqrt{5}}\ \ \dfrac{30}{\sqrt{6}}\tan^{-1}\dfrac{x}{\sqrt{6}}$.

(*iii*) $\dfrac{1}{3}\tan^{-1}x-\dfrac{1}{4\sqrt{3}}\log\dfrac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}+\dfrac{1}{6}\tan^{-1}\dfrac{x^2-1}{x}$.

(*iv*) $\dfrac{1}{2}\log\dfrac{x^2+1}{(x+1)^2}+\dfrac{3}{2}\tan^{-1}x+\dfrac{5}{2(x+1)}$.

4. (*i*) $\frac{1}{3}x^3+\frac{1}{3}\log(x^3-1)$.

(*ii*) $\dfrac{1}{2a^2}\cdot\dfrac{1}{x^2+a^2}-\dfrac{1}{2a^4}\log\dfrac{x^2+a^2}{x^2}$.

(*iii*) $\frac{1}{6}\{\log[(x^6-1)/x^6]+x^{-6}+\frac{1}{2}x^{-12}\}$.

(*iv*) $\frac{8}{9}\log(x-1)-\frac{1}{24}\log(x+1)-\frac{1}{6}\log(x^2-x+1)$

$+(\sqrt{3}/3)\tan^{-1}[(2x-1)/\sqrt{3}]+\frac{1}{4}(3x-4)/(x-1)^2$.

5. (*i*) $\log[x/(x-1)]+\frac{1}{2}\tan^{-1}x-\frac{1}{2}(x-1)^{-1}$.

(*ii*) $\frac{3}{2}\tan^{-1}x+\frac{1}{2}x/(1+x^2)$.

(*iii*) $\dfrac{x-2}{4(x^2+1)^2}+\dfrac{3x}{8(x^2+1)}+\dfrac{3}{8}\tan^{-1}x$.

(iv) $\frac{1}{2}\,\frac{x-1}{x^2+1}+\frac{1}{4}\log\frac{x^2+1}{(x+1)^2}$.

(v) $\frac{1}{8a^5}\log\frac{x-a}{x+a}-\frac{1}{2a^5}\tan^{-1}\frac{x}{a}-\frac{1}{4a^4}\cdot\frac{x}{x^2+a^2}$.

(vi) $\frac{19x-16}{15(2x^2-x+2)}+\frac{38\sqrt{15}}{225}\tan^{-1}\frac{4x-1}{\sqrt{15}}$.

6. 1.

10. (i) $\frac{2}{a}\tan^{-1}\frac{x}{a}-\frac{1}{x}\log(x^2+a^2)$.

(ii) $\frac{1}{2}x^2\log(x^3+1)-\frac{1}{2}\log(x+1)+\frac{1}{4}\log(x^2-x+1)$
$+\sqrt{(3/2)}\tan^{-1}[(2x-1)/\sqrt{3}]-\frac{3}{4}x^3$.

(iii) $e^x\log(e^{2x}+5e^x+6)+2\log(e^x+2)+3\log(e^x+3)-2e^x$

(iv) $\frac{(x-1)\tan^{-1}x}{2(x+1)}+\log\sqrt{\left[\frac{1+x}{\sqrt{(1+x^2)}}\right]}$.

4

Integration of Trigonometric Functions

4·1. Integration of $\sin^n x$ where n is a positive integer

When n is a positive integer, odd or even, the function $\sin^n x$ may be integrated with the help of a reduction formula. If, however, n is an odd positive integer, the function can be more easily integrated by means of the substitution $\cos x = t$ so that the reduction formula may only be employed when n is an even positive integer.

Let n be an odd positive integer, say $2k+1$, where k is a positive integer.

We put $\cos x = t$ so that $-\sin x\, dx = dt$.

$$\therefore \int \sin^n x\, dx = \int \sin^{2k+1} x\, dx$$
$$= \int \sin^{2k} x \sin x\, dx$$
$$= -\int (1-\cos^2 x)^k (-\sin x)\, dx = -\int (1-t^2)^k\, dt.$$

Since k is a positive integer, the integrand $(1-t^2)^k$ can, by the Binomial theorem, be expanded as a sum of $(k+1)$ terms each of which may be easily integrated.

Reduction Formula for $\int \sin^n x\, dx$.

We write

$$\int \sin^n x\, dx = \int \sin x . \sin^{n-1} x\, dx.$$

Integrating by parts, we get

$$\int \sin^n x\, dx = -\cos x \sin^{-1} x - \int (-\cos x)(n-1) \sin^{n-2} x \cos x\, dx$$
$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\, (1-\sin^2 x)\, dx$$
$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx.$$

Transposing $(n-1)\int \sin^n x\, dx$, we get

$$n\int \sin^n x\, dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx,$$

$$\Rightarrow \int \sin^n x\, dx = \frac{-\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx \qquad \ldots(A)$$

which is the required reduction formula.

This formula which connects the integrals

$$\int \sin^n x\, dx \text{ and } \int \sin^{n-2} x\, dx.$$

enables us to successively integrate any positive integral power of $\sin x$, odd or even. But it need be applied only when, n, is even.

Examples

Integrate

(*i*) $\sin^3 x$ (*ii*) $\sin^4 x$

(*i*) The index, 3, being odd, we proceed by substitution.

We put $\cos x = t$, so that

$$\begin{aligned}\int \sin^3 x\, dx &= -\int (1-t^2)\, dt\\ &= -\left(t-\frac{t^3}{3}\right)\\ &= -\left(\cos x - \frac{\cos^3 x}{3}\right)\\ &= -\cos x + \tfrac{1}{3}\cos^3 x\end{aligned}$$

(*ii*) The index, 4, being even, we employ the reduction formula. To integrate $\sin^4 x$, we first obtain the reduction formula (A). Putting $n = 4, 2$ successively in the reduction formula, we obtain

$$\int \sin^4 x\, dx = -\frac{\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x\, dx \qquad \ldots(i)$$

$$\begin{aligned}\int \sin^2 x\, dx &= -\frac{\cos x \sin x}{2} + \tfrac{1}{2}\int \sin^0 x\, dx\\ &= -\frac{\cos x \sin x}{2} + \tfrac{1}{2} x \qquad \ldots(ii)\end{aligned}$$

From (*i*) and (*ii*), we obtain

$$\int \sin^4 x = -\frac{\cos x \sin^3 x}{4} - \frac{3}{8}\cos x \sin x + \frac{3}{8} x.$$

Exercises

Evaluate

1. (*i*) $\int \sin^2 x\, dx$. (*ii*) $\int \sin^5 x\, dx$. (*iii*) $\int \sin^6 x\, dx$.

2. (*i*) $\int_0^{\frac{1}{2}\pi} \sin^7 x\, dx$. (*ii*) $\int_0^{\frac{1}{2}\pi} \sin^8 x\, dx$. (*iii*) $\int_0^{\frac{1}{2}\pi} \sin^9 x\, dx$.

Answers

1. (*i*) $\frac{1}{2}(x-\sin x \cos x)$. (*ii*) $-[\cos x-\frac{2}{3}\cos^3 x+\frac{1}{5}\cos^5 x]$.

(*iii*) $\frac{1}{6}\cos x \sin^5 x-\frac{5}{24}\cos x \sin^3 x-\frac{5}{16}\cos x \sin x+\frac{5}{16}x$.

2. (*i*) 16/35. (*ii*) $35\pi/256$. (*iii*) 128/315.

4·11. Evaluation of the definite integral

$$\int_0^{\frac{1}{2}\pi} \sin^n x\, dx,$$

where, n, is a positive integer.

Firstly, we obtain the reduction formula (*A*), *viz.*,

$$\int \sin^n x\, dx = -\frac{\cos x \,.\, \sin^{n-1} x}{n}+\frac{n-1}{n}\int \sin^{n-2} x\, dx.$$

This gives

$$\int_0^{\frac{1}{2}\pi} \sin^n x\, dx = \left|-\frac{\cos x \,.\, \sin^{n-1} x}{n}\right|_0^{\frac{1}{2}\pi}+\frac{n-1}{n}\int_0^{\frac{1}{2}\pi} \sin^{n-2} x\, dx$$

$$= 0+\frac{n-1}{n}\int_0^{\frac{1}{2}\pi} \sin^{n-2} x\, dx.$$

Writing I_n for $\int_0^{\frac{1}{2}\pi} \sin^n x\, dx$, we obtain the connection formula

$$I_n = \frac{n-1}{n} I_{n-2}.$$

With the help of this formula, we will successively connect I_{n-2} with I_{n-4}, I_{n-4} with I_{n-6} etc., etc. and finally, I_3 with I_1 or I_2 with I_0, according as n is odd or even. Thus we have

$$I_n = \frac{n-1}{n} I_{n-2},$$

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4},$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6},$$

$$\begin{cases} I_3 = \frac{2}{3} I_1, \text{ if } n \text{ is odd} \\ I_2 = \frac{1}{2} I_0, \text{ if } n \text{ is even.} \end{cases}$$

From these, we get

$$I_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3}\, I_1, \text{ when } n \text{ is odd,} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{1}{2}\, I_0, \text{ when } n \text{ is even} \end{cases}$$

Now,

$$I_1 = \int_0^{\frac{1}{2}\pi} \sin x \; dx = \left| -\cos x \right|_0^{\frac{1}{2}\pi} = 1,$$

$$I_0 = \int_0^{\frac{1}{2}\pi} \sin^0 x \, dx = \int_0^{\frac{1}{2}\pi} 1 \, . \, dx = \frac{\pi}{2}.$$

$$\therefore \; I_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3}, \; \textit{when } n \textit{ is odd.} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, \; \textit{when } n \textit{ is even.} \end{cases}$$

Example

Evaluate :

(*i*) $\int_0^{\frac{1}{2}\pi} \sin^7 x \, dx.$ (*ii*) $\int_0^{\frac{1}{2}\pi} \sin^8 x \, dx.$

We have

$$\int_0^{\frac{1}{2}\pi} \sin^7 x \; dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{16}{35},$$

$$\int_0^{\frac{1}{2}\pi} \sin^8 x \; dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35\pi}{256}.$$

Exercises

1. Evaluate :

(*i*) $\int_0^{\frac{1}{2}\pi} \sin^5 x \, dx,$ (*ii*) $\int_0^{\frac{1}{2}\pi} \sin^6 x \, dx,$ (*iii*) $\int_0^{\frac{1}{2}\pi} \sin^{10} x \, dx.$

2. Evaluate :

(*i*) $\int_0^{\frac{1}{4}\pi} \sin^4 2x \, dx,$ (*ii*) $\int_0^{\frac{1}{6}\pi} \sin^6 3x \, dx,$ (*iii*) $\int_0^{\pi} \sin^5 (\frac{1}{2}x) \, dx.$

3. Evaluate :

(*i*) $\int_0^{\pi} \frac{\sin^4 \theta \, d\theta}{(1+\cos \theta)^2},$ (*ii*) $\int_0^{\pi} \sin^2 \theta \frac{\sqrt{(1-\cos \theta)}}{1+\cos \theta} \, d\theta.$

Answers

1. (*i*) 8/15.	(*ii*) $5\pi/32$.	(*iii*) $63\pi/512$.
2. (*i*) $3\pi/32$.	(*ii*) $5\pi/96$.	(*iii*) 16/25.
3. (*i*) $3\pi/2$.	(*ii*) $8\sqrt{2}/3$.	

4·2. Integration of $\cos^n x$ where n is a positive integer. When n is a positive integer, the function $\cos^n x$ may be integrated with the help of a reduction formula. If, however, n is a *odd* positive integer, the function can be more easily integrated by means of the substitution $\sin x = t$.

Let n be an odd positive integer, say $2k+1$, where k is a positive integer.

We put $\sin x = t \Rightarrow \cos x\, dx = dt$.

We have

$$\begin{aligned}\int \cos^n x\, dx &= \int \cos^{2k+1} x\ dx\\ &= \int \cos^{2k} x \,.\, \cos x\, dx\\ &= \int (1-\sin^2 x)^k \,.\, \cos x\, dx = \int (1-t^2)^k\, dt,\end{aligned}$$

which may be easily evaluated by expanding $(1-t^2)^k$ by the Binomial theorem.

Reduction formula for $\int \cos^n x\, dx$.

We write

$$\int \cos^n x\, dx = \int \cos x \,.\, \cos^{n-1} x\, dx,$$

so that, on integrating by parts, we obtain

$$\begin{aligned}\int \cos^n x\, dx &= \sin x \,.\, \cos^{n-1} x - \int \sin x.\, (n-1) \cos^{n-2} x.\, (-\sin x) dx\\ &= \sin x \,.\, \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, (1-\cos^2 x)\, dx\\ &= \sin x \,.\, \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, dx - (n-1) \int \cos^n x\, dx\\ &\Rightarrow\ n \int \cos^n x\, dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, dx,\end{aligned}$$

$$\Rightarrow \int \cos^n x\, dx = \frac{\sin x \,.\, \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx \quad ...(B)$$

is the required reduction formula.

4·21. Evaluation of the definite integral

$$\int_0^{\frac{1}{2}\pi} \cos^n x\, dx$$

As in §4·11, we can show that

$$\int_0^{\frac{1}{2}\pi} \cos^n x\, dx = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{2}{3}, & \textit{when } n \textit{ is odd} \\[2ex] \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2} & \textit{when } n \textit{ is even.} \end{cases}$$

On comparing the values of

$$\int_0^{\frac{1}{2}\pi} \cos^n x\, dx \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin^n x\, dx,$$

we see that

$$\int_0^{\frac{1}{2}\pi} \cos^n x\, dx = \int_0^{\frac{1}{2}\pi} \sin^n x\, dx,$$

for every value of n.

This result is also capable of a direct proof on putting

$$x = \tfrac{1}{2}\pi - y.$$

Exercises

Write down the values of

(*i*) $\int_0^{\frac{1}{2}\pi} \cos^7 x\, dx$, (*ii*) $\int_0^{\frac{1}{2}\pi} \cos^8 x\, dx$, (*iii*) $\int_0^{\frac{1}{4}\pi} \cos^6 2t\, dt$.

Answers

(*i*) 16/35. (*ii*) $35\pi/256$, (*iii*) $5\pi/64$.

Cor. *The values of the definite integrals*

$$\int_0^{\frac{1}{2}\pi} \sin^n x\, dx \text{ and } \int_0^{\frac{1}{2}\pi} \cos^n x\, dx$$

enable us at once to obtain the values of the following definite integrals,

(*i*) $\displaystyle\int_0^1 \frac{x^n}{\sqrt{(1-x^2)}}\, dx$, (*ii*) $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^n}\, dx$, (*iii*) $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^{n+\frac{1}{2}}}$,

where n is a positive integer.

(*i*) We put $x = \sin\theta$ and note that $\theta = 0$ when $x = 0$ and $\theta = \pi/2$ when $x = 1$, so that limits of the new variable θ are 0 and $\pi/2$. Making this substitution, we get

$$\int_0^1 \frac{x^n}{\sqrt{(1-x^2)}}\, dx = \int_0^{\frac{1}{2}\pi} \frac{\sin^n\theta\cos\theta d\theta}{\cos\theta} = \int_0^{\frac{1}{2}\pi} \sin^n\theta\, d\theta.$$

(*ii*) We put $x = \tan\theta$ and note that

$\theta = 0$ when $x = 0$ and $\theta \to \pi/2$ when $x \to \infty$

so that the limits of the new variable θ are 0 and $\pi/2$. Making this substitution, we get

$$\int_0^\infty \frac{dx}{(1+x^2)^n} = \int_0^{\frac{1}{2}\pi} \frac{\sec^2\theta\, d\theta}{\sec^{2n}\theta} = \int_0^{\frac{1}{2}\pi} \cos^{2n-2}\theta\, d\theta$$

(*iii*) Putting $x = \tan\theta$, we see that

$$\int_0^{\infty} \frac{dx}{(1+x^2)^{n+\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi} \frac{\sec^2\theta\, d\theta}{\sec^{2n+1}\theta} = \int_0^{\frac{1}{2}\pi} \cos^{2n-1}\theta\, d\theta$$

Exercises

Evaluate the following definite integrals :

1. $\displaystyle\int_0^1 \frac{x^5}{\sqrt{(1-x^2)}}\, dx.$

2. $\displaystyle\int_0^1 \frac{x^6}{\sqrt{(1-x^2)}}\, dx.$

3. $\displaystyle\int_0^{\infty} \frac{1}{(1+x^2)^5}\, dx.$

4. $\displaystyle\int_0^{\infty} \frac{1}{(1+x^2)^{\frac{7}{2}}}\, dx.$

5. $\displaystyle\int_0^1 \frac{x^2(2-x^2)}{\sqrt{(1-x^2)}}\, dx.$

6. $\displaystyle\int_0^1 x^2(1-x^2)^{\frac{3}{2}}\, dx.$

7. $\displaystyle\int_0^a x^2(a^2-x^2)^{\frac{9}{2}}\, dx.$

8. $\displaystyle\int_0^a x^4\sqrt{(a^2-x^2)}\, dx,$

9. $\displaystyle\int_0^a \frac{x^4}{\sqrt{(a^2-x^2)}}\, dx.$

10. $\displaystyle\int_0^1 \frac{x^7}{\sqrt{(1-x^4)}}\, dx.$

11. $\displaystyle\int_0^1 x^5 \sqrt{\left(\frac{1+x^2}{1-x^2}\right)} dx.$

12. $\displaystyle\int_0^{\infty} \frac{x^2}{\sqrt{(1+x^6)^7}}\, dx.$

13. $\displaystyle\int_0^1 x^5 \sin^{-1} x\, dx.$

14. $\displaystyle\int_0^1 x^6 \sin^{-1} x\, dx.$

Answers

1. $\dfrac{8}{15}$.
2. $\dfrac{5\pi}{32}$.
3. $\dfrac{35\pi}{256}$.
4. $\dfrac{8}{15}$.
5. $\dfrac{5\pi}{16}$.
6. $\dfrac{\pi}{32}$.
7. $\dfrac{21\pi}{2048} a^{12}$.
8. $\dfrac{\pi}{32} a^6$.
9. $\dfrac{3\pi}{16} a^4$.
10. $\dfrac{1}{3}$.
11. $\dfrac{3\pi+8}{24}$.
12. $\dfrac{8}{45}$.
13. $\dfrac{11\pi}{192}$.
14. $\dfrac{\pi}{14} - \dfrac{16}{245}$.

4·3. Integration of $\sin^p x \cos^q x$ where p, q are positive integers. This can be integrated by the substitution $\cos x = t$ or $\sin x = t$ in case p is odd or q is odd. In case, however, p, q are *both* even, the integration is accomplished with the help of a reduction formula.

Let p be odd, say $2k+1$. Putting $\cos x = t$, we get

$$\int \sin^{2k+1} x \cos^q x\, dx = -\int (1-t^2)^k t^q\, dt,$$

which may be evaluated on expanding $(1-t^2)^k$ by the Binomial theorem.

It may similarly be shown that the function can be integrated by the substitution $\sin x = t$, if q is odd.

In case p, q are both odd, we may proceed either way.

Reduction Formula for

$$\int \sin^p x \cos^q x \, dx$$

We write

$$\int \sin^p x \cos^q x \, dx = \int \sin^{p-1} x (\sin x \cos^q x) \, dx$$

and apply the rule of integration by parts to obtain

$$\int \sin^p x \cos^q x \, dx$$

$$= -\frac{\cos^{q+1} x}{q+1} \sin^{p-1} x + \int \frac{\cos^{q+1} x}{q+1} \cdot (p-1) \sin^{p-2} x \cos x \, dx$$

$$= -\frac{\cos^{q+1} x \sin^{p-1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^q x (1-\sin^2 x) \, dx.$$

$$= -\frac{\cos^{q+1} x \sin^{p-1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^q x \, dx$$

$$- \frac{p-1}{q+1} \int \sin^p x \cos^q x \, dx$$

Transposing

$$\frac{p-1}{q+1} \int \sin^p x \cos^q x \, dx,$$

to the left hand side and dividing by

$$1+(p-1)/(q+1), \text{ i.e., } (p+q)/(q+1) \text{ ;}$$

we obtain

$$\int \sin^p x \cos^q x \, dx = -\frac{\cos^{q+1} x \sin^{p-1} x}{p+q}$$

$$+ \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x \, dx, \qquad \ldots(C)$$

which is the required reduction formula.

It may be similarly shown that

$$\int \sin^p x \cos^q x \, dx = \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^p x \cos^{q-2} x \, dx.$$

Note. The substitution $\cos x = t$ proves effective when p is an odd positive integer even if q is not an integer.

Exercises

1. Integrate

   (*i*) $\sin^3 x \cos^4 x$, (*ii*) $\sin^4 x \cos^3 x$, (*iii*) $\tan^3 x \sec^3 x$.

2. Evaluate $\int_0^{\frac{1}{2}\pi} \sin^{3/2} x \cos^3 x \, dx$.

3. Show that $\int_0^{\frac{1}{2}\pi} \sqrt{\sin \theta} \cos^5 \theta \, d\theta = \dfrac{64}{231}$.

4. Show that

   (*i*) $\displaystyle\int \frac{\sin^5 x}{\cos^4 x} dx = \frac{1}{3 \cos^3 x} - \frac{2}{\cos x} - \cos x.$

   (*ii*) $\displaystyle\int \frac{\cos^5 x}{\sin x} dx = \frac{\sin^4 x}{4} - \sin^2 x + \log \sin x.$

5. Show that

$$\int \frac{\sin^6 x}{\cos x} dx = -\frac{\sin^5 x}{5} - \frac{\sin^3 x}{3} - \sin x + \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

6. Evaluate the following integrals :—

   (*i*) $\int \sin^2 x \cos^2 x \, dx$, (*ii*) $\int \sin^2 x \cos^4 x \, dx$

   (*iii*) $\int \sin^4 x \cos^4 x \, dx$, (*iv*) $\int \sin^2 x \cos^6 x \, dx$.

Answers

1. (*i*) $\frac{1}{7} \cos^7 x - \frac{1}{5} \sin^5 x$. (*ii*) $\frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x$. (*iii*) $\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x$.

2. $\frac{8}{77}$.

6. (*i*) $-\frac{1}{4} \sin x \cos^3 x + \frac{1}{8} (x + \sin x \cos x)$.

   (*ii*) $\frac{1}{6} \cos^3 x \sin^3 x + \frac{1}{8} \sin^3 x \cos x + \frac{1}{16} (x - \sin x \cos x)$

   (*iii*) $-\frac{1}{8} \cos^5 x \sin^3 x - \frac{1}{16} \cos^5 x \sin x + \frac{1}{64} \sin x \cos^3 x$
   $+ \frac{3}{128} (x + \sin x \cos x)$.

   (*iv*) $\frac{1}{8} \sin^3 x \cos^5 x + \frac{5}{48} \sin^3 x \cos^3 x + \frac{5}{64} \sin^3 x \cos x$
   $+ \frac{5}{128} (x - \sin x \cos x)$.

4·31. Evaluation of the definite integral

$$\int_0^{\frac{1}{2}\pi} \sin^p x \cos^q x \, dx,$$

where p, q are positive integers.

We write

$$I_{p,\,q} = \int_0^{\frac{1}{2}\pi} \sin^p x \cos^q x \, dx.$$

From the reduction formula (C) § 4·3, we obtain

$$\int_0^{\frac{1}{2}\pi} \sin^p x \cos^q x \, dx = -\left|\frac{\cos^{q+1} x \sin^{p-1} x}{p+q}\right|_0^{\frac{1}{2}\pi} + \frac{p-1}{p+q}\int_0^{\frac{1}{2}\pi} \sin^{p-2} x \cos^q x \, dx$$

$$= \frac{p-1}{p+q}\int_0^{\frac{1}{2}\pi} \sin^{p-2} x \cos^q x \, dx.$$

In terms of our notation, we have

$$I_{p,q} = \frac{p-1}{p+q} \cdot I_{p-2,q}$$

From this we get, changing p to $p-2$, $p-4$, and so on,

$$I_{p-2,q} = \frac{p-3}{p+q-2} \cdot I_{p-4,q},$$

$$I_{p-4,q} = \frac{p-5}{p+q-4} \cdot I_{p-6,q},$$

....

....

Finally,
$$\begin{cases} I_{3,q} = \dfrac{2}{3+q} \cdot I_{1,q}, \text{ when } p \text{ is odd,} \\ I_{2,q} = \dfrac{1}{2+q} \cdot I_{0,q}, \text{ when } p \text{ is even.} \end{cases}$$

Thus

$$I_{p,q} = \begin{cases} \dfrac{p-1}{p+q} \cdot \dfrac{p-3}{p+q-2} \cdots\cdots \dfrac{2}{3+q} \cdot I_{1,q}, \text{ when } p \text{ is odd ;} \\ \dfrac{p-1}{p+q} \cdot \dfrac{p-3}{p+q-2} \cdots\cdots \dfrac{1}{2+q} \cdot I_{0,q}, \text{ when } p \text{ is even ;} \end{cases}$$

Now

$$I_{1,q} = \int_0^{\frac{1}{2}\pi} \sin x \cos^q x \, dx = -\left|\frac{\cos^{q+1} x}{q+1}\right|_0^{\frac{1}{2}\pi} = \frac{1}{q+1}$$

$$I_{0,q} = \int_0^{\frac{1}{2}\pi} \sin^0 x \cos^q x \, dx = \int_0^{\frac{1}{2}\pi} \cos^q x \, dx,$$

which has been evaluated in § 4·21.

Thus *when p is odd, we have*

$$\mathbf{I}_{p,q} = \frac{\mathbf{p-1}}{\mathbf{p+q}} \cdot \frac{\mathbf{p-3}}{\mathbf{p+q-2}} \cdots\cdots \frac{\mathbf{2}}{\mathbf{3+q}} \cdot \frac{\mathbf{1}}{\mathbf{q+1}},$$

where q may be odd or even.

When p is even and q is odd.

$$\mathbf{I}_{p,q} = \frac{\mathbf{p-1}}{\mathbf{p+q}} \cdot \frac{\mathbf{p-3}}{\mathbf{p+q-2}} \cdot \frac{\mathbf{p-5}}{\mathbf{p+q-4}} \cdots\cdots \frac{\mathbf{1}}{\mathbf{2+q}} \times \frac{\mathbf{q-1}}{\mathbf{q}} \cdot \frac{\mathbf{q-3}}{\mathbf{q-2}} \cdots\cdots \frac{\mathbf{2}}{\mathbf{3}}$$

When p is even and q is even

$$\mathrm{I}_{p,q} = \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdot \frac{p-5}{p+q-4} \cdots\cdots \frac{1}{2+q}$$

$$\times \frac{q-1}{q} \cdot \frac{p-3}{q-2} \cdots\cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

We have a simple rule for writing down the value of the integral in that

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{(p-1)(p-3)\ldots(q-1)(q-3)\ldots}{(p+q)(p+q-2)\ldots\ldots}$$

ultimately followed by the factor $\frac{1}{2}\pi$ only when p and q are both even ; the three sets of factors, starting with $(p-1)$, $(q-1)$, $(p+q)$, and diminishing by 2 at a time, descend to either 1 or 2 according as the first factor of the set is odd or even.

Examples

Write down the values of

(*i*) $\int_0^{\frac{1}{2}\pi} \sin^5 x \cos^6 x \, dx$, (*ii*) $\int_0^{\frac{1}{2}\pi} \sin^6 x \cos^8 x \, dx$.

We have

(*i*) $\int_0^{\frac{1}{2}\pi} \sin^5 x \cos^6 x \, dx = \frac{4.2.5.3.1}{11.9.7.5.3.1} = \frac{8}{693}$,

(*ii*) $\int_0^{\frac{1}{2}\pi} \sin^6 x \cos^8 x \, dx = \frac{5.3.1.7.5.3.1}{14.12.10.8.6.4.2} \cdot \frac{\pi}{2} = \frac{5\pi}{4096}$

Exercises

Evaluate

(*i*) $\int_0^{\frac{1}{2}\pi} \sin^3 \theta \cos^4 \theta \, d\theta$, (*ii*) $\int_0^{\frac{1}{2}\pi} \cos^5 x \sin^4 x \, dx$

(*iii*) $\int_0^{\frac{1}{2}\pi} \cos^4 x \sin 3x \, dx$, (*iv*) $\int_0^{\frac{1}{2}\pi} \cos^3 2x \sin^4 4x \, dx$.

Answers

(*i*) 2/35 (*ii*) 8/315. (*iii*) 13/35. (*iv*) 128/1155.

Cor. *The value of the definite integral*

$$\int_0^{\frac{1}{2}\pi} \sin^p x \cos^q x \, dx$$

enables us to write down the values of the following integrals :

$$(i)\ \int_0^\infty \frac{x^n}{(1+x^2)^m}\,dx, \qquad (ii)\ \int_0^\infty \frac{x^n\,dx}{(1+x^2)^{(m+\frac{1}{2})}},$$

$$(iii)\ \int_0^{2a} x^m \sqrt{(2ax-x^2)}\,dx,$$

where n, m are positive integers.

(*i*) Putting $x = \tan\theta$, we see that

$$\int_0^\infty \frac{x^n}{(1+x^2)^m}\,dx = \int_0^{\frac{1}{2}\pi} \frac{\sin^n\theta\cos^{2m}\theta}{\cos^n\theta}.\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{1}{2}\pi} \sin^n\theta\cos^{2m-(n+2)}\theta\,d\theta,$$

whose value is known.

(*ii*) Putting $x = \tan\theta$, we see that

$$\int_0^\infty \frac{x^n}{(1+x^2)^{(m+\frac{1}{2})}}\,dx = \int_0^{\frac{1}{2}\pi} \sin^n\theta\cos^{2m-(n+1)}\theta\,d\theta,$$

whose value is known.

(*iii*) We write

$$\int_0^{2a} x^m\sqrt{(2ax-x^2)}\,dx = \int_0^{2a} x^m\sqrt{[a^2-(a-x)^2]}\,dx.$$

We put $\qquad a-x = a\cos\theta$

so that $\qquad dx = a\sin\theta\,d\theta.$

Also $\qquad x = a(1-\cos\theta) = 2a\sin^2(\theta/2).$

Now, $\quad \theta = 0$ and π when $x = 0$ and $2a$ respectively.

$\therefore$ the given integral

$$= \int_0^\pi 2^m a^m \sin^{2m}\frac{\theta}{2}.\,a\sin\theta\,.\,a\sin\theta\,d\theta$$

$$= (2a)^{m+2}\int_0^\pi \sin^{2m+2}\frac{\theta}{2}\cos^2\frac{\theta}{2}\,d\theta.$$

We now put $(\theta/2) = \phi$ and see that the integral

$$(2a)^{m+2}.\,2\int_0^{\frac{1}{2}\pi} \sin^{2m+2}\phi\cos^2\phi\,d\phi$$

whose value could now be put down.

Exercises

Evaluate the following definite integrals :

1. $\displaystyle\int_0^\infty \frac{x^2}{(1+x^2)^4}\,dx.$ $\qquad$ 2. $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^3}\,dx.$

3. $\displaystyle\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}}\,dx.$ $\qquad$ 4. $\displaystyle\int_0^\infty \frac{x^3}{(1+x^2)^{9/2}}\,dx.$

5. $\displaystyle\int_0^\infty \frac{t^4 dt}{(1+t^2)^4}$

6. $\displaystyle\int_0^{2a} x^2\sqrt{(2ax-x^2)}\, dx.$

7. $\displaystyle\int_0^2 x^3\sqrt{(2x-x^2)}\, dx.$

8. $\displaystyle\int_0^2 x^{5/2}\sqrt{(2-x)}\, dx.$

9. $\displaystyle\int_0^{2a} x^{9/2}\,(2a-x)^{-\frac{1}{2}}\, dx.$

10. $\displaystyle\int_0^\infty \frac{x^3 dx}{(4+x^2)^2}.$

11. $\displaystyle\int_0^1 x^{3/2}\,(1-x)^{3/2}\, dx.$

Answers

1. $\dfrac{\pi}{32}$. 2. $\dfrac{1}{4}$. 3. $\dfrac{2}{15}$. 4. $\dfrac{2}{35}$. 5. $\dfrac{\pi}{32}$.

6. $\dfrac{5\pi}{8}a^4$. 7. $\dfrac{7\pi}{8}$. 8. $\dfrac{5\pi}{8}$. 9. $\dfrac{63\pi}{8}a^5$. 10. $\dfrac{1}{3}$.

11. $\dfrac{3\pi}{128}$.

4·4. Integration of $\tan^n x$ and $\cot^n x$ where n is a positive integer. Evaluation of the integral of a positive integral power of $\tan x$ and $\cot x$ can be effected by means of the reduction formula which we now proceed to obtain.

4·41. Reduction formula for $\int \tan^n x\, dx$.

We have

$$\begin{aligned}\int \tan^n x\, dx &= \int \tan^{n-2} x \tan^2 x\, dx\\ &= \int \tan^{n-2} x\,(\sec^2 x-1)\, dx\\ &= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx\\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx, \qquad \ldots(D)\end{aligned}$$

which is the required reduction formula.

4·42. Reduction formula for $\int \cot^n x\, dx$.

We have

$$\begin{aligned}\int \cot^n x\, dx &= \int \cot^{n-2} x\,.\,\cot^2 x\, dx\\ &= \int \cot^{n-2} x\,.\,(\operatorname{cosec}^2 x-1)dx\\ &= \int \cot^{n-2} x \operatorname{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx\\ &= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx, \qquad \ldots(E)\end{aligned}$$

which is the required reduction formula.

Examples

Evaluate :

1. $\int \tan^6 x\, dx$.

1. Putting $n = 6, 4, 2$ successively in (D), we get

$$\int \tan^6 x\, dx = \frac{\tan^5 x}{5} - \int \tan^4 x\, dx. \qquad \ldots(i)$$

$$\int \tan^4 x\, dx = \frac{\tan^3 x}{3} - \int \tan^2 x\, dx. \qquad \ldots(ii)$$

$$\int \tan^2 x\, dx = \frac{\tan x}{1} - \int \tan^0 x\, dx = \tan x - x. \qquad \ldots(iii)$$

From (*ii*) and (*iii*), we obtain

$$\int \tan^4 x\, dx = \frac{\tan^3 x}{3} - \tan x + x \qquad \ldots(iv)$$

From (*i*) and (*iv*), we obtain

$$\int \tan^6 x\, dx = -\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x.$$

Exercises

Evaluate

(*i*) $\int \tan^3 x\, dx$, (*ii*) $\int \cot^5 x\, dx$, (*iii*) $\int \cot^6 x\, dx$,

(*iv*) $\int_0^{\frac{1}{4}\pi} \tan^4 x\, dx$, (*v*) $\int_{\pi/4}^{\pi/2} \cot^4 x\, dx$.

Answers

(*i*) $\frac{1}{2}\tan^2 x - \log \sec x$. (*ii*) $-\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log \sin x$.

(*iii*) $-\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x$.

(*iv*) $(3\pi - 8)/12$. (*v*) $(3\pi - 8)/12$.

4·51. Integration of $\sec^n x$ where n is a positive integer.

When n is a positive integer, the function $\sec^n x$ may be integrated with the help of a reduction formula. But if n is an even positive integer, the function can also be integrated by means of the substitution $\tan x = t$ and this method is simpler than that of reduction formula.

Let n be an even positive integer, say, $2k$ where k is a positive integer.

Put $\tan x = t$ so that $\sec^2 x\, dx = dt$.

We see that

$$\begin{aligned}\int \sec^n x\, dx &= \int \sec^{2k} x\, dx\\ &= \int \sec^{2k-2} x \sec^2 x\, dx\\ &= \int (1+t^2)^{k-1}\, dt,\end{aligned}$$

which may be evaluated by expanding $(1+t^2)^{k-1}$ by means of the Binomial theorem.

Reduction formula for $\int \sec^n x\, dx$.

We have

$\int \sec^n x\, dx = \int \sec^{n-2} x \,.\, \sec^2 x\, dx.$

Integrating by parts, we obtain

$$\begin{aligned}\int \sec^n x\, dx &= \tan x \,.\, \sec^{n-2} x - \int \tan x \,.\, (n-2) \sec^{n-2} x \,.\, \tan x\, dx\\ &= \tan x \,.\, \sec^{n-2} x - (n-2) \int \sec^{n-2} x\, (\sec^2 x - 1)\, dx\\ &= \tan x \,.\, \sec^{n-2} x - (n-2)\left[\int \sec^n x\, dx - \int \sec^{n-2} x\, dx\right]\end{aligned}$$

Transposing $(n-2) \int \sec^n x\, dx$ to the left and dividing by $1+(n-2)$ *i.e.*, $n-1$, we get

$$\int \sec^n x\, dx = \frac{\tan x \,.\, \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx \qquad \ldots(F)$$

which is the formula connecting the integrals of $\sec^n x$ and $\sec^{n-2} x$.

4·52. Integration of $\operatorname{cosec}^n x$ where n is a positive integer. We can easily show that when n is even, $\operatorname{cosec}^n x$ can be integrated by the substitution $\cot x = t$, and that when n is odd we require a reduction formula which can also be obtained as in § 4·51.

Reduction formula for

$$\int \operatorname{cosec}^n x\, dx$$

is

$$\int \operatorname{cosec}^n x\, dx = -\frac{\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\, dx \qquad \ldots(G)$$

Exercises

1. Evaluate :

(*i*) $\int \sec^3 x\, dx$, (*ii*) $\int \sec^4 x\, dx$,

(*iii*) $\int \sec^5 x\, dx$, (*iv*) $\int \sec^6 x\, dx$.

2. Evaluate :

(*i*) $\int \operatorname{cosec}^3 x\, dx$, (*ii*) $\int \operatorname{cosec}^4 x\, dx$,

(*iii*) $\int \operatorname{cosec}^5 x\, dx$.

Answers

1. (*i*) $\frac{1}{2}\tan x \sec x + \frac{1}{2}\log(\sec x + \tan x)$.
   (*ii*) $\frac{1}{3}\tan x \sec^2 x + \frac{2}{3}\tan x$.
   (*iii*) $1/4 \tan x \sec^3 x + 3/8 \tan x \sec x + 3/8 \log(\sec x + \tan x)$.
   (*iv*) $1/5 \tan x \sec^4 x + 4/15 \tan x \sec^2 x + 8/15 \tan x$.
2. (*i*) $-\frac{1}{2}\cot x \operatorname{cosec} x + \frac{1}{2}\log\tan(\frac{1}{2}x)$.
   (*ii*) $-\frac{1}{3}\cot x \operatorname{cosec}^2 x - 2/3 \cot x$.
   (*iii*) $-\frac{1}{4}\cot x \operatorname{cosec}^3 x - 3/8 \cot x \operatorname{cosec} x + 3/8 \log\tan(\frac{1}{2}x)$.

Note. As the negative integral powers of $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$ are the positive integral powers of $\operatorname{cosec} x$, $\sec x$, $\cot x$, $\tan x$, $\cos x$, $\sin x$, respectively, we see that, with the help of § 4·1—4·5, any positive or negative integral power of $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ can be integrated.

The integrals of the fractional powers of these functions such as

$$\sin^{1/2} x, \quad \tan^{1/5} x$$

cannot, in general, be integrated in terms of the *elementary functions*. There is, however, a special case where $\sin^p x \cos^q x$ can be integrated. This is considered in the next article.

4·6. The integration of $\sin^p x \cos^q x$, when $p+q$ is a negative even integer.

Let $p+q = -2n$, where n is a positive integer. In this case we make the substitution.

$$\tan x = t \Rightarrow \sec^2 x\, dx = dt \Rightarrow dx = dt/(1+t^2)$$

Also $$\sin x = \frac{t}{\sqrt{(1+t^2)}}, \cos x = \frac{1}{\sqrt{(1+t^2)}}.$$

$$\int \sin^p x \cos^q x\, dx = \int \frac{t^p}{(1+t^2)^{p/2}} \cdot \frac{1}{(1+t^2)^{q/2}} \cdot \frac{dt}{1+t^2}$$
$$= \int t^p \cdot \frac{1}{(1+t^2)^{(p+q+2)/2}}\, dt$$
$$= \int t^p (1+t^2)^{n-1}\, dt$$

which may be evaluated on expanding $(1+t^2)^{n-1}$ by the Binomial theorem.

Example

Evaluate $\int \sqrt{\cot x} \, . \sec^4 x\, dx.$

We have

$$\sqrt{\cot x} \, . \sec^4 x = \sin^{-\frac{1}{2}} x \cos^{-\frac{7}{2}} x,$$

so that $p+q = -4$, is a negative even integer.

Putting $\tan x = t \Rightarrow dx = dt/(1+t^2)$, we obtain

$$\int \sqrt{\cot x}\,.\sec^4 dx = \int \frac{1}{\sqrt{t}}\cdot(1+t^2)^2 . \frac{dt}{1+t^2}$$

$$= \int \frac{1+t^2}{\sqrt{t}}\,dt$$

$$= 2\sqrt{t}+\tfrac{2}{5}t^{\frac{5}{2}}$$

$$= 2\sqrt{\tan x}+\tfrac{2}{5}\sqrt{\tan^5 x}.$$

Exercises

1. Show that

$$\int \frac{dx}{\sin^3 x \cos x} = -\frac{1}{2\tan^2 x}+\log\tan x.$$

$$\int \frac{dx}{\sin^2 x \cos^4 x} = \frac{\sin x\,(1+2\cos^2 x)}{3\cos^3 x} - 2\cot 2x.$$

2. Evaluate :

(*i*) $\displaystyle\int \frac{dx}{\sin^{3/2} x \cos^{5/2} x}$, (*ii*) $\displaystyle\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx$,

(*iii*) $\displaystyle\int \frac{(1-\cos x)^{2/5}}{(1+\cos x)^{8/5}}\,dx$, (*iv*) $\displaystyle\int \sec^{4/7} x \operatorname{cosec}^{10/7} x\,dx$.

3. Evaluate :

(*i*) $\displaystyle\int \frac{(1-\cos x)^{2/7}}{(1+\cos x)^{6/7}}\,dx$, (*ii*) $\displaystyle\int \frac{1}{\sin\theta\cos^3\theta}\,d\theta$,

(*iii*) $\displaystyle\int \sec^{8/9} x \operatorname{cosec}^{10/9} x\,dx$, (*iv*) $\displaystyle\int \sec^{2/3} x \operatorname{cosec}^{4/3} x\,dx$.

Answers

2. (*i*) $2/3\sqrt{\tan^3 x}-2\sqrt{\cot x}$. (*ii*) $2\sqrt{\tan x}$.
   (*iii*) $5/15\,(\tan\tfrac{1}{2}x)^{11/5}$. (*iv*) $-7/3\,(\cot x)^{3/7}$.
3. (*i*) $7/11\,(\tan\tfrac{1}{2}x)^{11/7}$. (*ii*) $\log\tan\theta+\tfrac{1}{2}\tan^2\theta$.
   (*iii*) $-9\,(\cot x)^{1/9}$ (*iv*) $-3\,(\cot x)^{1/3}$.

4·61. Evaluation of

$$\int \sin^p x \cos^q x\,dx,$$

where either p, q or both are negative integers.

We will now obtain a system of reduction formulae for the integral $\int \sin^p x \cos^q x\,dx$

I. As in § 4·3, we have

$\int \sin^p x \cos^q x \, dx$

$$= -\int \sin^{p-1} x . (\sin x \cos^q x) \, dx$$

$$= -\frac{\cos^{q+1} x}{q+1} \sin^{p-1} x + \int \frac{\cos^{q+1} x}{q+1} (p-1) \sin^{p-2} x \cos x \, dx$$

$$= -\frac{\cos^{q+1} x \sin^{p-1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^{q+2} x \, dx \qquad \text{...(A)}$$

This reduction formula proves useful when q is a negative integer and p is a positive jnteger.

II. If we write

$$\int \sin^p x \cos^q x \, dx = \int \cos^{q-1} x (\cos x \sin^p x) \, dx,$$

and, as above, apply the rule of integration by parts, we will obtain the reduction formula

$$\int \sin^p x \cos^q x \, dx$$

$$= \frac{\sin^{p+1} x \cos^{p-1} x}{1+p} + \frac{q-1}{1+p} \int \sin^{p+2} x \cos^{q-2} x \, dx \qquad \text{...(B)}$$

which proves useful when p is a negative and q a positive integer.

III. If in the formula (A) above, we change

$$\cos^{q+2} x \text{ to } \cos^q x \, (1-\sin^2 x),$$

then, as in §4·3, we prove that

$$\int \sin^p x \cos^q x \, dx$$

$$= -\frac{\cos^{q+1} x \sin^{p-1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x \, dx$$

Changing p to $p+2$ and making some necessary changes we get

$$\int \sin^p x \cos^q x \, dx$$

$$= \frac{\cos^{q+1} x \sin^{p+1} x}{p+1} + \frac{p-2+q}{p+1} \int \sin^{p+2} x \cos^q x \, dx$$

which proves useful when q is a negative integer.

IV. We may similarly obtain a formula connecting

$$\int \sin^p x \cos^q x \, dx \text{ with } \int \sin^p x \cos^{q+2} x \, dx$$

which proves useful when q is a negative integer.

Note. From the above it appears that the integral

$$\int \sin^p x \cos^q x \, dx$$

can be connected with any one of the *six* following integrals :—

$$\int \sin^{p-2} x \cos^q x \, dx \; ; \qquad \int \sin^p x \cos^{q-2} x \, dx \; ;$$
$$\int \sin^{p-2} x \cos^{q+2} x \, dx \; ; \qquad \int \sin^{p+2} x \cos^{q-2} x \, dx \; ;$$
$$\int \sin^{p+2} x \cos^q x \, dx \; ; \qquad \int \sin^p x \cos^{q+2} x \, dx.$$

These reduction formulae can also be obtained in another way as we now show.

Another Method. Instead of obtaining the reduction formulae by applying the rule of integration by parts, as is done above, we can also proceed differently as follows :—

(*i*) Put

$$P = \sin^{\lambda+1} x \cos^{\mu+1} x,$$

where λ and μ are the smaller of the indices of $\sin x$ and $\cos x$ respectively in the two integrands whose integrals are to be connected.

(*ii*) Find (dP/dx) and re-arrange its value as a linear combination of the two integrands whose integrals are to be connected.

(*iii*) Finally integrate to obtain the required reduction formula.

Illustration. Connect the integrals

$$\int \sin^p x \cos^q x \, dx \text{ and } \int \sin^{p-2} x \cos^q x \, dx$$

Here

$$\lambda = p-2, \mu = q,$$

so that we write

$$P = \sin^{p-1} x \cos^{q+1} x.$$

$$\Rightarrow \quad \frac{dP}{dx} = (p-1) \sin^{p-2} x \cos^{q+2} x - (q+1) \sin^p x \cos^q x.$$

On changing $\cos^{q+2} x$ to $\cos^q x \, (1-\sin^2 x)$, we get

$$\frac{dP}{dx} = (p-1) \sin^{p-2} x \cos^q x \, (1-\sin^2 x) - (q+1) \sin^p x \cos^q x.$$
$$= (p-1) \sin^{p-2} x \cos^q x - (p+q) \sin^p x \cos^q x$$

Integrating, we get

$$P = (p-1) \int \sin^{p-2} x \cos^q x \, dx - (p+q) \int \sin^p x \cos^q x \, dx.$$

$$\Rightarrow \quad \int \sin^p x \cos^q x \, dx$$
$$= -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x \, dx.$$

We may similarly obtain the remaining reduction formulae left out.

Exercises

Show that

(*i*) $\int \frac{\sin^3 x}{\cos^4 x}\, dx = \frac{\sin^2 x}{3 \cos^3 x} - \frac{2}{3} \sec x.$

(*ii*) $\int \frac{\sin^4 x}{\cos^6 x}\, dx = \frac{\sin^3 x}{3 \cos^4 x} - \frac{3 \sin x}{8 \cos^2 x}$

$+ \frac{3}{8} \log \tan \left(\frac{x}{2} + \frac{\pi}{4} \right).$

(*iii*) $\int \frac{\cos^5 x}{\sin^4 x}\, dx = -\frac{\cos^4 x}{3 \sin^3 x} + \frac{4 \cos^2 x}{3 \sin x} + \frac{8 \sin x}{3}.$

4·7. Integration of R **(cos x, sin x)** where $R(\cos x, \sin x)$ denotes an expression which is rational in the two functions $\sin x$ and $\cos x$ such as

$$\frac{1}{4+5 \cos x},\ \frac{1}{3+4 \cos x+5 \sin x},\ \frac{2 \sin x+3 \cos x}{4+5 \cos^2 x+6 \sin x}.$$

The transformation

$$t = \tan \left(\frac{x}{2} \right),$$

converts the integral of a rational function of $\cos x$ *and* $\sin x$ *into that of a rational function of* t which may then be evaluated by the methods as given in Chapter III. We have

$$dt = \frac{1}{2} \sec^2 \frac{x}{2}\, dx \Rightarrow dx = \frac{2t}{1+t^2},$$

Also
$$\sin x = \frac{2 \tan (x/2)}{1+\tan^2 (x/2)} = \frac{2dt}{1+t^2},$$

$$\cos x = \frac{1-\tan^2 (x/2)}{1+\tan^2 (x/2)} = \frac{1-t^2}{1+t^2}.$$

$$\therefore \int R(\cos x, \sin x) dx = \int R \left[\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right] \frac{2dt}{1+t^2}.$$

The integrand, now, is a rational function of t.

Examples

1. *Evaluate* $\int \frac{dx}{a+b \cos x}.$

Putting $t = \tan x/2$, we get

$$\int \frac{dx}{a+b \cos x} = \int \frac{2dt}{(1+t^2)\left[a+b\, \frac{1-t^2}{1+t^2} \right]}$$

$$=\int \frac{2dt}{(a+b)+(a-b)t^2}$$

$$=\frac{2}{a-b}\int \frac{dt}{t^2+(a+b)/(a-b)}, \text{ if } a \neq b.$$

Writing

$$\frac{a+b}{a-b}=\frac{(a+b)^2}{a^2-b^2},$$

we see that $(a+b)/(a-b)$ is positive or negative according as

$$a^2 > \text{ or } < b^2.$$

First Case. Let

$$a^2 > b^2 \text{ so that } (a+b)/(a-b)$$

is positive.

We have

$$\int \frac{dx}{a+b\cos x}=\frac{2}{a-b}\int \frac{dt}{t^2+\left[\sqrt{\left(\frac{a+b}{a-b}\right)}\right]^2}$$

$$=\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\tan^{-1}\left[t\sqrt{\left(\frac{a-b}{a+b}\right)}\right]$$

$$=\frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left[\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\frac{x}{2}\right]$$

Second Case. Let

$$a^2 < b^2 \text{ so that } (a+b)/(a-b)$$

is negative.

We have

$$\int \frac{dx}{a+b\cos x}=\frac{2}{a-b}\int \frac{dt}{t^2-(b+a)/(b-a)}$$

$$=\frac{2}{a-b}\int \frac{dt}{t^2-\left[\sqrt{\left(\frac{b+a}{b-a}\right)}\right]^2}=\frac{2}{a-b}\int \frac{dt}{t^2-\alpha^2},$$

where we have written α for $\sqrt{\{(b+a)/(b-a)\}}$. Now

$$\int \frac{dt}{t^2-\alpha^2}=\frac{1}{2\alpha}\int\left(\frac{1}{t-\alpha}-\frac{1}{t+\alpha}\right)dt=\frac{1}{2\alpha}\log\frac{t-\alpha}{t+\alpha}$$

$$\therefore \quad \int \frac{dx}{a+b\cos x}=\frac{2}{a-b}\cdot\frac{1}{2\alpha}\log\frac{t-\alpha}{t+\alpha}$$

where $$\alpha=\sqrt{[(b+a)/(b-a)]}$$

Third Case. Let

$$a^2=b^2 \text{ so that } b=a \quad \text{or} \quad b=-a.$$

If $b = a$, we have

$$\int \frac{dx}{a+b\cos x} = \frac{1}{a}\int \frac{dx}{1+\cos x}$$

$$= \frac{1}{2a}\int \sec^2 \frac{x}{2}\, dx = \frac{1}{a} \tan \frac{x}{2}.$$

If $b = -a$, we have

$$\int \frac{dx}{a+b\cos x} = \frac{1}{a}\int \frac{dx}{1-\cos x}$$

$$= \frac{1}{2a}\int \operatorname{cosec}^2 \frac{x}{2}\, dx = -\frac{1}{a} \cot \frac{x}{2}$$

2. *Evaluate* $\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{dx}{5+7\cos x+\sin x}$.

Putting $\tan (x/2) = t$, we get

$$\int \frac{dx}{5+7\cos x+\sin x} = \int \frac{2dt}{(1+t^2)\left[5+7\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}\right]}$$

$$= \int \frac{dt}{-t^2+t+6}$$

$$= \int \frac{dt}{(3-t)(t+2)}$$

$$= \frac{1}{5}\int \left(\frac{1}{3-t}+\frac{1}{2+t}\right) dt$$

$$= \frac{1}{5}\left[-\log (3-t)+\log (2+t)\right].$$

Now, $t = 1$ when $x = \pi/2$, and $t = -1$ when $x = -\pi/2$.

$$\therefore \quad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{dx}{5+7\cos x+\sin x} = \frac{1}{5}\left| \log \frac{(2+t)}{(3-t)} \right|_{-1}^{1}$$

$$= \tfrac{1}{5} (\log 3/2 - \log 1/4) = 1/5 \log 6.$$

Exercises

1. Evaluate the following :

(i) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{5+4\cos x}$, (ii) $\displaystyle\int_0^{\pi} \frac{dx}{2+\cos x}$,

(iii) $\displaystyle\int_0^{\pi} \frac{d\theta}{5+3\cos\theta}$.

2. Evaluate

(i) $\displaystyle\int \frac{dx}{a+b\sin x}$, (ii) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{4+5\sin x}$, (iii) $\displaystyle\int \frac{\operatorname{cosec} x\, dx}{2+\operatorname{cosec} x}$.

3. Show that

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{1+2\cos\theta} = \frac{1}{\sqrt{3}} \log(2+\sqrt{3}).$$

4. Show that

(*i*) $\displaystyle\int_0^{\pi} \frac{dx}{3+2\sin x+\cos x} = \frac{\pi}{4}$

(*ii*) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{1+2\sin x+\cos x} = \frac{1}{2}\log 3$

(*iii*) $\displaystyle\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{(a^2-b^2)}}$, if $a^2 > b^2$.

5. Prove that

$$\int_0^{\alpha} \frac{d\theta}{\cos\alpha+\cos\theta} = \operatorname{cosec}\alpha \log\sec\alpha.$$

6. Prove that

$$\int_0^{\pi} \frac{dx}{a^2-2a\cos x+1} = \frac{\pi}{1-a^2} \text{ or } \frac{\pi}{a^2-1},$$

according as $a <$ or > 1.

7. Show that $\displaystyle\int_0^{\pi} \frac{dx}{(2+\cos x)^2} = \frac{2\pi}{3\sqrt{3}}$.

Answers

1. (*i*) $2/3 \tan^{-1} \frac{1}{3}$. (*ii*) $\pi/\sqrt{3}$. (*iii*) $\frac{1}{4}\pi$.

2. (*i*) $\displaystyle\frac{2}{\sqrt{(a^2-b^2)}} \tan^{-1} \frac{a\tan(x/2)+b}{\sqrt{(a^2-b^2)}}$ if $a^2 > b^2$

$\displaystyle\frac{2}{\sqrt{(b^2-a^2)}} \log \frac{a\tan(x/2)+b-\sqrt{(b^2-a^2)}}{a\tan(x/2)+b+\sqrt{(b^2-a^2)}}$, if $b^2 > a^2$

$\displaystyle-\frac{1}{a}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$ if $b = a$ and $\displaystyle\frac{1}{a}\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)$

if $b = -a$.

(*ii*) $\frac{1}{3}\log 2$. (*iii*) $\displaystyle\frac{1}{\sqrt{3}} \log \frac{\sin\frac{1}{2}x+(2-\sqrt{3})\cos\frac{1}{2}x}{\sin\frac{1}{2}x+(2+\sqrt{3})\cos\frac{1}{2}x}$.

4·71. Other important transformations. Theoretically, the transformation $\tan(x/2) = t$, enables us to integrate every rational function of $\sin x$ and $\cos x$. But it is not always the most convenient transformation, as the degree of the denominator of the rational function thus obtained is generally high. Sometimes other transformations, like $\sin x = t$, $\cos x = t$, $\tan x = t$, prove more con venient. We now solve a few examples to illustrate the various possibilities of frequent occurrence.

Examples

1. *Evaluate*

$$\int_0^{\frac{1}{2}\pi} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}.$$

We put $\sin x = t$ so that $\cos x\, dx = dt$

Also $t = 0$ when $x = 0$ and $t = 1$ when $x = \pi/2$

$$\therefore \quad \text{the integral} = \int_0^1 \frac{dt}{(1+t)(2+t)}$$

$$= \int_0^1 \left(\frac{1}{t+1} - \frac{1}{2+t}\right) dt$$

$$= \left| \log \frac{1+t}{2+t} \right|_0^1 = \log \frac{2}{3} - \log \frac{1}{2} = \log \frac{4}{3}$$

2. *Evaluate*

$$\int \frac{dx}{\sin x\,(a+b\cos x)}.$$

We put $\cos x = t$ so that $-\sin x\, dx = dt$. Thus

$$\int \frac{dx}{\sin x(a+b\cos x)}$$

$$= -\int \frac{dt}{\sin^2 x\,(a+b\cos x)}$$

$$= -\int \frac{dt}{(1-t)(1+t)(a+bt)}$$

$$= -\int \left(\frac{1}{2(a+b)} \cdot \frac{1}{1-t} + \frac{1}{2(a-b)} \cdot \frac{1}{1+t} + \frac{b^2}{b^2-a^2} \cdot \frac{1}{a+bt} \right) dt$$

$$= \frac{1}{2(a+b)} \log(1-t) - \frac{1}{2(a-b)} \log(1+t) + \frac{b}{a^2-b^2} \log(a+bt)$$

$$= \frac{1}{2(a+b)} \log(1-\cos x) - \frac{1}{2(a-b)} \log(1+\cos x)$$

$$+ \frac{b}{a^2-b^2} \log(a+b\cos x)$$

$$= \frac{1}{a+b} \log \sin \frac{x}{2} - \frac{1}{(a-b)} \log \cos \frac{x}{2}$$

$$+ \frac{b}{a^2-b^2} \log(a+b\cos x)$$

where we have omitted the constant $-b \log 2/(a^2-b^2)$.

3. *Evaluate*

$$\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}.$$

We write

$$\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \int \frac{\sec^2 x \, dx}{b^2 + a^2 \tan^2 x}.$$

Putting $t = \tan x$, we see that the given integral

$$= \int \frac{dt}{b^2 + a^2 t^2} = \frac{1}{a^2} \int \frac{dt}{(t^2 + b^2/a^2)}$$

$$= \frac{1}{a^2} \frac{a}{b} \tan^{-1}\left(\frac{at}{b}\right) = \frac{1}{ab} \tan^{-1}\left(\frac{a \tan x}{b}\right)$$

4. *Evaluate*

$$\int \frac{\sin x \cos x}{a^2 \cos^2 x + b^2 \sin^2 x} \, dx.$$

Dividing the numerator and denominator by $\cos^2 x$, we have

$$\int \frac{\sin x \cos x}{a^2 \cos^2 x + b^2 \sin^2 x} \, dx = \int \frac{\tan x \, dx}{a^2 + b^2 \tan^2 x}.$$

Putting $\tan^2 x = t$, we see that the integral

$$= \int \frac{dt}{2(1+t)(a^2 + b^2 t)}$$

$$= \frac{1}{2} \int \left(\frac{1}{a^2 + b^2} \cdot \frac{1}{1+t} + \frac{b^2}{b^2 - a^2} + \frac{1}{a^2 + b^2 t} \right) dt$$

$$= \frac{1}{2} \left[\frac{1}{a^2 - b^2} \log(1+t) + \frac{b^2}{b^2 - a^2} \cdot \frac{1}{b^2} \log(a^2 + b^2 t) \right]$$

$$= \frac{1}{2(a^2 - b^2)} \log \frac{\sec^2 x}{a^2 + b^2 \tan^2 x}$$

$$= -\frac{1}{2(a_2 - b_2)} \log(a^2 \cos^2 x + b^2 \sin^2 x).$$

Note. The integral may also be evaluated by putting

$$\sin^2 x = t, \text{ or } \cos^2 x = t.$$

Exercises

Evaluate the following integrals :

1. (*i*) $\displaystyle\int \frac{\cos x}{1 + \sin^2 x} \, dx$, (*ii*) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin^2 \theta \cos \theta}{1 + e^2 \sin^2 \theta} \, d\theta$,

(*iii*) $\displaystyle\int \frac{dx}{\sin x + \sin 2x}$, (*iv*) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} \, dx$.

2. (*i*) $\displaystyle\int \frac{2 - \sin x}{\sin x(1 - \cos x)} \, dx$, (*ii*) $\displaystyle\int \frac{\sec x}{1 + \operatorname{cosec} x} \, dx$,

(iii) $\int \frac{dx}{1+\cos^2 x}$, (iv) $\int_0^{\frac{1}{2}\pi} \frac{dx}{2+\sin^2 x}$

(v) $\int \frac{d\theta}{5+4\cos 2\theta}$,

3. (i) $\int \frac{dx}{1-\cos^4 x}$, (ii) $\int \frac{dx}{(2\sin x+\cos x)^2}$,

(iii) $\int_0^{\frac{1}{2}\pi} \frac{dx}{1+4\sin^2 x}$, (iv) $\int_0^{\frac{1}{2}\pi} \frac{\sin x\cos x}{a^2\sin^2 x+b^2\cos^2 x}\, dx$,

(v) $\int_0^{\frac{1}{2}\pi} \frac{dx}{a^2-b^2\cos^2 x}$, $(a^2 > b^2)$,

(vi) $\int \frac{dx}{3\sin x+\sin^3 x}$.

4. Show that

$$\int_0^{\frac{1}{4}\pi} \frac{\sin\theta+\cos\theta}{9+16\sin 2\theta} = \frac{1}{20}\log 3.$$

Answers

1. (i) $\tan^{-1}(\sin x)$. (ii) $(e-\tan^{-1} e)/e^3$,
   (iii) $\frac{1}{3}\log[\sin x(1+\cos x)/(1+2\cos x)^2]$. (iv) $\log(9/8)$.
2. (i) $\log\tan\frac{1}{2}x+\cot\frac{1}{2}x-\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}x$.
   (ii) $\frac{1}{4}\log[(1+\sin x)/(1-\sin x)]+\frac{1}{2}(1+\sin x)$.
   (iii) $\sqrt{\frac{1}{2}}\tan^{-1}(\tan x/\sqrt{2})$. (iv) $\sqrt{6}\pi/12$, (v) $\frac{1}{3}\tan^{-1}(\frac{1}{3}\tan\theta)$.
3. (i) $-\frac{1}{2}\cot x+\frac{1}{2}\sqrt{\frac{1}{2}}(\tan^{-1}\sqrt{\frac{1}{2}}\tan x)$.
   (ii) $-\frac{\cos x/2}{(2\sin x+\cos x)}$ (iii) $\frac{\pi}{2\sqrt{5}}$.
   (iv) $\frac{1}{(a^2-b^2)}\log\frac{a}{b}$, if $a^2 \neq b^2$ and $\frac{1}{2a^2}$ if $a^2=b^2$.
   (v) $\pi/2a\sqrt{(a^2-b^2)}$.
   (vi) $\frac{1}{6}\log\frac{t-1}{t+1}+\frac{1}{12}\log\frac{2+t}{2-t}$ where $t=\cos x$.

4·81. To integrate

(a cos x + b sin x)/(c cos x + d sin x).

We note that the derivative of the denominator $c\cos x+d\sin x$ is $-c\sin x+d\cos x$ and proceed to determine two constants λ and μ such that

$$a\cos x+b\sin x \equiv \lambda(-c\sin x+d\cos x)+\mu(c\cos x+d\sin x).$$

Equating the coefficients of $\cos x$ and $\sin x$, we see that the constants λ, μ are given by the equations

$$a = d\lambda+\mu, \quad b = -c\lambda+d\mu$$

$$\Rightarrow \quad \lambda = \frac{ad-bc}{d^2+c^2}, \mu = \frac{ac-bd}{d^2+c^2}$$

$$\therefore \quad \int \frac{a\cos x+b\sin x}{c\cos x+d\sin x}\,dx$$

$$= \lambda \int \frac{-c\sin x+d\cos x}{c\cos x+d\sin x}\,dx+\mu\int dx$$

$$= \lambda \log(c\cos x+d\sin x)+\mu x.$$

4·82. To integrate $\dfrac{\mathbf{a\cos x+b\sin x+c}}{\mathbf{d\cos x+e\sin x+f}}$.

We determine three constants λ, μ, ν such that

$$a\cos x+b\sin x+c = \lambda(d\cos x+e\sin x+f) +\mu(-d\sin x+e\cos x)+\nu.$$

These are given by the equations

$$a = d\lambda+e\mu,\ b = e\lambda-d\mu,\ c = f\lambda+\nu.$$

With these values of λ, μ, ν, we have

$$\int \frac{a\cos x+b\sin x+c}{d\cos x+e\sin x+f}\,dx$$

$$= \int \lambda\,dx+\mu\int \frac{-d\sin x+e\cos x}{d\cos x+e\sin x+f}\,dx +\nu\int\frac{dx}{d\cos x+e\sin x+f}$$

$$= \lambda x+\mu\log(d\cos x+e\sin x+f) +\nu\int\frac{dx}{d\cos x+e\sin x+f}$$

where $$\int\frac{dx}{d\cos x+e\sin x+f}$$

is to be evaluated with the substitution $\tan(x/2) = t$.

Exercises

Evaluate the following :

(*i*) $\displaystyle\int_0^{\frac{1}{2}\pi}\frac{dx}{1+\cot x}$, (*ii*) $\displaystyle\int\frac{2\sin x+3\cos x}{3\sin x+4\cos x}\,dx,$

(*iii*) $\displaystyle\int\frac{dx}{a+b\tan x}$, (*iv*) $\displaystyle\int\frac{3\cos x+4\sin x}{4\cos x+5\sin x}\,dx,$

(*v*) $\displaystyle\int\frac{\cos x+2\sin x+3}{4\cos x+5\sin x+6}\,dx,$ (*vi*) $\displaystyle\int\frac{5\cos x+6}{2\cos x+\sin x+3}\,dx.$

Answers

(*i*) $\pi/4$. (*ii*) $\frac{18}{25}x+\frac{1}{25}\log(3\sin x+4\cos x)$.

(*iii*) $[ax+b\log(a\cos x+b\sin x)]/(a^2+b^2)$.

(*iv*) $[3x-\log(4\cos x+5\sin x)]/41$.

(*v*) $\dfrac{14}{41}x-\dfrac{3}{41}\log(4\cos x+5\sin x+6)$

$$+\frac{39}{41}\cdot\frac{1}{\sqrt{5}}\log\frac{2\sin\frac{1}{2}x+(5-\sqrt{5})\cos\frac{1}{2}x}{2\sin\frac{1}{2}x+(5+\sqrt{5})\cos\frac{1}{2}x}.$$

(*vi*) $2x+\log(2\cos x+\sin x+3)$.

4·9. Two properties of definite integrals. We shall now prove two properties of definite integrals which will be useful in evaluating certain types of definite integrals.

4·91. First property. *For any function*, $\varphi(x)$.

$$\int_0^a \varphi(x)\,dx=\int_0^{\frac{1}{2}a}\varphi(x)\,dx+\int_0^{\frac{1}{2}a}\varphi(a-x)\,dx$$

From § 1·6, we have

$$\int_0^a \varphi(x)\,dx=\int_0^{\frac{1}{2}a}\varphi(x)\,dx+\int_{\frac{1}{2}a}^a\varphi(x)\,dx$$

We put

$$a-x=y\Rightarrow dx=-dy$$

Also

$$x=\frac{a}{2}\Rightarrow y=\frac{a}{2}\ ;\ x=a\Rightarrow y=0.$$

$$\therefore\quad \int_{\frac{1}{2}a}^a\varphi(x)\,dx=-\int_{\frac{1}{2}a}^0\varphi(a-y)\,dy=\int_0^{\frac{1}{2}a}\varphi(a-x)\,dx$$

$$\therefore\quad \int_0^a\varphi(x)\,dx=\int_0^{\frac{1}{2}a}\varphi(x)\,dx+\int_0^{\frac{1}{2}a}\varphi(a-x)\,dx \quad \ldots \text{(A)}$$

Cor. 1. From (A) we deduce that

$$\varphi(a-x)=\varphi(x)\quad\Rightarrow\int_0^a\varphi(x)\,dx=2\int_0^{\frac{1}{2}a}\varphi(x)\,dx$$

$$\varphi(a-x)=-\varphi(x)\Rightarrow\int_0^a\varphi(x)\,dx=0.$$

Cor. 2. The results obtained above in cor. 1 enable us to evaluate the integrals

$$\int_0^\pi \sin^p x\cos^q x\,dx \text{ and } \int_0^{2\pi}\sin^p x\cos^q x\,dx$$

where p, q are positive integers.

(*i*) To evaluate $\int_0^{\pi} \sin^p x \cos^q x\, dx$, we write

$$\varphi(x) = \sin^p x \cos^q x$$

$$\therefore \quad \varphi(\pi - x) = \sin^p (\pi - x) \cos^q (\pi - x)$$

$$= \sin^p x(-\cos x)^q = (-1)^q \sin^p x \cos^q x$$

which is equal to $\varphi(x)$ or $-\varphi(x)$ according as q is even or odd.

Thus

$$q \text{ is odd } \Rightarrow \int_0^{\pi} \sin^p x \cos^q x\, dx = 0$$

$$q \text{ is even } \Rightarrow \int_0^{\pi} \sin^p x \cos^q x\, dx = 2\int_0^{\frac{1}{2}\pi} \sin^p x \cos^q x\, dx$$

(*ii*) Again, let

$$\varphi(x) = \sin^p x \cos^q x.$$

$$\therefore \varphi(2\pi - x) = \sin^p (2\pi - x) \cos^q (2\pi - x) = (-1)^p \sin^p x \cos^q x,$$

which is equal to $\varphi(x)$ or $-\varphi(x)$ according as p is even or odd.

Thus

$$p \text{ is odd } \Rightarrow \int_0^{2\pi} \sin^p x \cos^q x\, dx = 0$$

Also if p is even, we have

$$\int_0^{2\pi} \sin^p x \cos^q x\, dx = 2\int_0^{\pi} \sin^p x \cos^q x\, dx$$

$$= \begin{cases} 0, \text{ if } q \text{ is odd,} \\ 4\displaystyle\int_0^{\frac{1}{2}\pi} \sin^p x \cos^q x, \text{ if } q \text{ is even} \end{cases}$$

Thus

$$\int_0^{2\pi} \sin^p x \cos^q x\, dx = 0$$

if either p or q or both are odd, and $= 4\int_0^{\frac{1}{2}\pi} \sin^p x \cos^q x\, dx$, if they are both even

Exercises

1. Evaluate

(*i*) $\int_0^{\pi} \sin^4 x \cos^5 x\, dx$, (*ii*) $\int_0^{\pi} \sin^4 x \cos^6 x\, dx$,

(*iii*) $\int_0^{2\pi} \sin^4 x \cos^5 x\, dx$, (*iv*) $\int_0^{2\pi} \sin^4 x \cos^6 x\, dx$,

(*v*) $\int_0^{\pi} \sin^3 x\, dx$, (*vi*) $\int_0^{\pi} \sin^4 x\, dx$,

(*vii*) $\int_0^{2\pi} \cos^3 x\, dx$, (*viii*) $\int_0^{2\pi} \cos^4 x\, dx$.

2. Prove that

$$\int_0^{\pi} \theta \sin^2 \theta \cos \theta \, d\theta = -\frac{4}{9}.$$

3. Show that

$$\int_0^{\frac{1}{2}\pi} \cos^4 3\varphi \sin^2 6\varphi \, d\varphi = \frac{5}{96}\pi.$$

4. Show that

$$\int_0^{\pi} \sin^3 \theta \, (1+2\cos\theta)\,(1+\cos\theta)^2 \, d\theta = \frac{8}{3}.$$

5. Show that

$$\int_0^{\frac{1}{2}\pi} \sin 2x \log \tan x \, dx = 0.$$

Answers

1. (*i*) 0. (*ii*) $3\pi/256$ (*iii*) 0. (*iv*) $3\pi/128$.
(*v*) $4/3$. (*vi*) $3\pi/8$. (*vii*) 0. (*viii*) $3\pi/4$.

4·92. Second property. *For any function* $\varphi(x)$,

$$\int_0^a \varphi(x)\,dx = \int_0^a \varphi(a-x)\,dx.$$

We put

$$a-x=y \Rightarrow dx=-dy.$$

Also $$x=0 \Rightarrow y=a,\ x=a \Rightarrow y=0$$

$\therefore$ $$\int_0^a \varphi(a-x)\,dx = -\int_a^0 \varphi(y)\,dy$$

$$= \int_0^a \varphi(y)\,dy = \int_0^a \varphi(x)\,dx.$$

Examples

1. ***Evaluate***

$$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\sin x+\cos x}\,dx.$$

Let $$\varphi(x) = \frac{\sin x}{\sin x+\cos x}.$$

$\therefore$ $$\varphi(\tfrac{1}{2}\pi-x) = \frac{\sin(\frac{1}{2}\pi-x)}{\sin(\frac{1}{2}\pi-x)+\cos(\frac{1}{2}\pi-x)}$$

$$= \frac{\cos x}{\cos x + \sin x}.$$

As $\int_0^{\frac{1}{2}\pi} \varphi(x)\,dx = \int_0^{\frac{1}{2}\pi} \varphi(\tfrac{1}{2}\pi - x)\,dx,$

$$\therefore \quad \int_0^{\frac{1}{2}\pi} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{1}{2}\pi} \frac{\cos x}{\sin x + \cos x}\,dx = I, \text{ say,}$$

$$\therefore \quad 2I = \int_0^{\frac{1}{2}\pi} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int_0^{\frac{1}{2}\pi} 1.\,dx = \frac{\pi}{2}$$

$$\Rightarrow \qquad I = \frac{\pi}{4}.$$

2. *Show that*

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2 x}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}} \log (1 + \sqrt{2}).$$

Let $\qquad \varphi(x) = \dfrac{\sin^2 x}{\sin x + \cos x}$

$$\therefore \qquad \varphi(\tfrac{1}{2}\pi - x) = \frac{\cos^2 x}{\cos x + \sin x}.$$

$$\therefore \quad \int_0^{\frac{1}{2}\pi} \frac{\sin^2 x}{\sin x + \cos x}\,dx = \int_0^{\frac{1}{2}\pi} \frac{\cos^2 x}{\cos x + \sin x}\,dx = I, \text{ say.}$$

$$\therefore \qquad 2I = \int_0^{\frac{1}{2}\pi} \frac{1}{\cos x + \sin x}\,dx.$$

Putting $\tan (x/2) = t$, we see that

$$2I = \int_0^{\frac{1}{2}\pi} \frac{dx}{\cos x + \sin x} = 2\int_0^1 \frac{d}{1 + 2t - t^2}$$

$$= 2\int_0^1 \frac{dt}{2 - (t-1)^2}$$

$$= 2\,\frac{1}{2\sqrt{2}} \int_0^1 \left[\frac{1}{\sqrt{2} - (t-1)} + \frac{1}{\sqrt{2} + (t-1)}\right] dt$$

$$= \frac{1}{\sqrt{2}} \left| \log \frac{\sqrt{2} + (t-1)}{\sqrt{2} - (t-1)} \right|_0^1 = \frac{1}{\sqrt{2}}\left[0 - \log \frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$= -\frac{1}{\sqrt{2}} \log \frac{1}{(\sqrt{2}+1)^2} = \frac{2}{\sqrt{2}} \log (\sqrt{2} + 1).$$

$$\therefore\ I = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1).$$

3. *Show that*

$$\int_0^{\frac{1}{2}\pi} \log \sin x\,dx = \frac{\pi}{2} \log \frac{1}{2} = -\frac{\pi}{2} \log 2.$$

Let $\varphi(x) = \log \sin x$. $\quad \therefore \quad \varphi(\tfrac{1}{2}\pi - x) = \log \cos x$.

$$\therefore \quad \int_0^{\frac{1}{2}\pi} \log \sin x \, dx = \int_0^{\frac{1}{2}\pi} \log \cos x \, dx = I, \text{ say.}$$

$$\therefore \quad 2I = \int_0^{\frac{1}{2}\pi} (\log \sin x + \log \cos x) \, dx$$

$$= \int_0^{\frac{1}{2}\pi} \log \sin x \cos x \, dx$$

$$= \int_0^{\frac{1}{2}\pi} \log \frac{\sin 2x}{2} \, dx$$

$$= \int_0^{\frac{1}{2}\pi} \log \sin 2x \, dx - \int_0^{\frac{1}{2}\pi} \log 2 \, dx$$

$$= \int_0^{\frac{1}{2}\pi} \log \sin 2x \, dx - \frac{\pi}{2} \log 2. \qquad \ldots(i)$$

Putting $2x = y$, we get

$$\int_0^{\frac{1}{2}\pi} \log \sin 2x \, dx = \frac{1}{2} \int_0^{\pi} \log \sin y \, dy = \frac{1}{2} \int_0^{\pi} \log \sin x \, dx \ldots(ii)$$

Now, $\varphi(\pi - x) = \log \sin(\pi - x) = \log \sin x = \varphi(x)$.

$\therefore$ from cor. 1 to § 4·91, page 100, we have

$$\int \log \sin x \, dx = 2 \int_0^{\frac{1}{2}\pi} \log \sin x \, dx = 2.I. \qquad \ldots(iii)$$

From (*ii*) and (*iii*), we obtain

$$\int_0^{\frac{1}{2}\pi} \log \sin 2x \, dx = I. \qquad \ldots(iv)$$

From (*i*) and (*iv*), we finally obtain

$$2I = I - \frac{\pi}{2} \log 2$$

$$I = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log \frac{1}{2}.$$

Exercises

Evaluate

1. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx$.
2. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin x - \cos x}{1 + \sin x \cos x} \, dx$.
3. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin^2 x \, dx}{1 + \sin x \cos x}$.
4. $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx$.

5. $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x+\tan x}\,dx.$ **6.** $\displaystyle\int_0^{\frac{1}{4}\pi} \log(1+\tan\theta)\,d\theta$

7. $\displaystyle\int_0^{1} \frac{\log(1+x)}{1+x^2}\,dx.$ **8.** $\displaystyle\int_0^{\frac{1}{2}\pi} \log\tan x\,dx.$

9. $\displaystyle\int_0^{\pi} x\cos^4 x\,dx.$ **10.** $\displaystyle\int_0^{\pi} x\sin x\cos^4 x\,dx.$

11. $\displaystyle\int_0^{\pi} x\sin^3 x\,dx.$ **12.** $\displaystyle\int_0^{\pi} x\sin^2 x\cos^4 x\,dx.$

13. $\displaystyle\int^{\pi} x\log\sin x\,dx.$

14. $\displaystyle\int_0^{\frac{1}{2}\pi} \log(\tan x+\cot x)\,dx.$

15. $\displaystyle\int_0^{\infty} \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}.$ **16.** $\displaystyle\int_0^{\pi} \log(1+\cos\theta)\,d\theta$

17. $\displaystyle\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}.$ **18.** $\displaystyle\int_0^{\pi} \frac{x\,dx}{1+e\sin x}.$

19. $\displaystyle\int_0^{\pi} x\sin^6 x\cos^4 x\,dx.$

Answers

1. $\frac{\pi}{4}$.	**2.** 0.	**3.** $\frac{\;}{3\sqrt{3}}$
4. $\frac{\pi^2}{4}$.	**5.** $\frac{\pi(\pi-2)}{2}$	**6.** $\frac{\pi}{8}\log 2$.
7. $\frac{\pi}{8}\log 2$.	**8.** 0.	**9.** $\frac{3}{16}\pi^2$.
10. $\frac{\pi}{5}$.	**11.** $\frac{2\pi}{3}$.	**12.** $\frac{\pi^2}{32}$.
13. $\frac{1}{2}\pi^2\log\frac{1}{2}$.	**14.** $\pi\log 2$.	**15.** $\pi\log 2$.
16. $\pi\log\frac{1}{2}$.	**17.** $\frac{\pi^2}{2ab}$.	**18.** $\frac{\pi\cos^{-1}e}{\sqrt{(1-e^2)}}\left(e^2<1\right)$.
19. $\frac{3\pi^2}{512}$.		

4·10. Reduction formula for $\int \cos^m x \cos nx\, dx$.

Applying the rule of Integration by parts, we obtain

$$\int \cos^m x\cos nx\,dx = \cos^m x.\,\frac{\sin nx}{n}$$

$$+\frac{m}{n}\int \cos^{m-1} x\sin x\sin nx\,dx, \qquad \ldots(i)$$

Now,

$$\cos(n-1)x = \cos nx \cos x + \sin nx \sin x.$$

Replacing $\sin nx \sin x$ by $\cos(n-1)x - \cos nx \cos x$ in the integral on the right of (*i*), we get

$$\int \cos^m x \cos nx\, dx = \frac{\cos^m x \sin nx}{n}$$

$$+\frac{m}{n} \cos^{m-1} x \cos(n-1)x\, dx - \frac{m}{n}\int \cos^m x \cos nx\, dx$$

$$\Rightarrow \quad \left(1+\frac{m}{n}\right)\int \cos^m x \cos nx\, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x\, dx$$

$$\Rightarrow \quad \int \cos^m x \cos nx\, dx = \frac{\cos^m x \sin nx}{m+n}$$

$$+\frac{m}{m+n}\int \cos^{m-1} x \cos(n-1)x\, dx$$

which is the required reduction formula.

Note 1. We can similarly obtain a reduction formula for the integral $\int \cos^m x \sin nx\, dx$.

Note 2. If, instead of replacing $\sin x \sin nx$ by

$$\cos(n-1)x - \cos nx \cos x$$

in the integral on the right of (*i*), we had integrated this integral by parts, taking $\sin nx$ as the one and $\cos^{m-1} x \sin x$ as the second factor, we would have got the reduction formula

$$\int \cos^m x \cos nx\, dx = \frac{n \sin nx \cos x - m \cos nx \sin x}{n^2 - m^2} \cos^{m-1} x$$

$$-\frac{m(m-1)}{n^2-m^2}\int \cos^{m-2} x \cos nx\, dx$$

Exercises

1. Prove that $\quad I_{m,n} = \int \cos^m x \sin nx\, dx,$

$$\Rightarrow \quad (m+n)\, I_{m,n} = -\cos^m x \cos nx + m\, I_{m-1,\, n-1}.$$

Hence or otherwise evaluate

$$\int_0^{\frac{1}{2}\pi} \cos^5 x \sin 3x\, dx.$$

2. Prove that

$$\int_0^{\frac{1}{2}\pi} \cos^m x \cos nx\, dx = m\int_0^{\frac{1}{2}\pi} \cos^{m-1} x \cos(n-1)\, x\, dx,$$

and then deduce the value of the integral when $m > n$.

In particular, show that

$$\int_0^{\frac{1}{2}\pi} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}}.$$

3. If $f(p, q) = \int_0^{\frac{1}{2}\pi} \cos^p x \cos qx \, dx$, prove that

$$f(p, q) = \frac{p(p-1)}{p^2-q^2} f(p-2, q) = \frac{p}{p+q} f(p-1, q-1).$$

4. Show that

$$\int \sin^m x \cos nx \, dx = \frac{m \cos x \cos nx + n \sin x \sin mx}{n^2-m^2} \sin^{m-1} x$$
$$- \frac{m(m-1)}{n^2-m^2} \int \sin^{m-2} x \cos nx \, dx$$

5. If $u_n = \int_0^{\frac{1}{2}\pi} x^n \sin mx \, dx$, prove that

$$u_n = \frac{n \, \pi^{n-1}}{m^2 \, 2^{n-1}} - \frac{n(n-1)}{m^2} u_{n-2},$$

if m is of the form $4r+1$.

6. $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin (2p+1) x \, dx$, prove that

$$(2p+1)^2 I_n + n(n-1) I_{n-2} = (-1)^p \cdot n \cdot (\pi/2)^{n-1},$$

n and p being positive integers.

Evaluate $\int_0^{\frac{1}{2}\pi} x^4 \sin 3x \, dx$.

7. Show how to evaluate the integral

$$\int x^m \cos nx \, dx.$$

8. Find a reduction formula for

$$\int e^{ax} \cos^n x \, dx,$$

n being a positive integer.

Answers

1. 1/3.

2. $-\dfrac{\pi}{2^{m+1}} \cdot \dfrac{\underline{|m}}{\underline{|\tfrac{1}{2}(m+n)} \cdot \underline{|\tfrac{1}{2}(m-n)}}$, if $(m-n)$ be even,

$2^{m-1} \, \underline{|m} \, \dfrac{\underline{|\tfrac{1}{2}(m+n-1)} \cdot \underline{|\tfrac{1}{2}(m-n-1)}}{\underline{|m+n} \cdot \underline{|m-n}}$, if $(m-n)$ be odd.

6. $(16+24\pi-9\pi^2)/162$

8. $$\int e^{ax}\cos^n x\,dx = \frac{a\cos x+n\sin x}{a^2+n^2}e^{ax}\cos^{n-1}x + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2}x\,dx.$$

EXERCISES ON CHAPTER IV

1. Show that

(i) $$\int_0^{\frac{1}{2}\pi}\frac{\sin^2 x}{a^2\sin^2 x+b^2\cos^2 x}dx=\frac{\pi}{2a(a+b)},$$

(ii) $$\int_0^{\infty}\frac{dx}{a^2\cosh^2 x+b^2\sinh^2 x}=\frac{1}{ab}\tan^{-1}\frac{b}{a}.$$

2. *Evaluate*

(i) $$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{a^2\cos^2\theta+b^2\sin^2\theta},\quad (ii)\ \int_0^{\frac{1}{2}\pi}\frac{d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^2}$$

(iii) $$\int_0^{\frac{1}{2}\pi}\cos^2 x\sin 4x\,dx,$$

(iv) $$\int\frac{\cos x}{\cos x+\sin x}dx.$$

3. *Evaluate*

(i) $$\int\frac{\cos 7\theta-\cos 8\theta}{\cos 2\theta-\cos 3\theta}d\theta,\quad (ii)\ \int_0^{\frac{1}{2}\pi}\frac{x+\cos x}{1+\cos 2x}dx.$$

4. Show that

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{1+\cos\alpha\sin 2\alpha}=\begin{cases}\alpha/\sin\alpha, & \text{if } 0<\alpha<\pi;\\ 1, & \text{if } \alpha=0.\end{cases}$$

5. Prove that

$$\int_0^{\frac{1}{2}\pi}\frac{dx}{1+\sin x\cos\lambda}=(\pi-\lambda)\operatorname{cosec}\lambda,\ (0<\lambda<\pi).$$

6. Show that

$$\int_0^{\frac{1}{2}\pi}\frac{\sin\theta\cos^2\theta\,d\theta}{\sqrt{(1+a^2\cos^2\theta)}}=\frac{\sqrt{(1+a^2)}}{2a^2}-\frac{\sinh^{-1}a}{2a^3}.$$

7. Show that, if $a^2>b^2$,

$$\int_0^{\pi}\frac{dx}{(a+b\cos x)^2}=\frac{\pi a}{(a^2-b^2)^{3/2}}.$$

8. *Evaluate*

(i) $$\int_0^{\frac{1}{2}\pi}\frac{1+2\cos x}{(2+\cos x)^2}dx,\quad (ii)\ \int_0^{\pi}\frac{a^2\sin^2 x+b^2\cos^2 x}{a^4\sin^2 x+b^4\cos^2 x}dx.$$

(*iii*) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{\cosh^2\alpha - \cos^2 x}$, (*iv*) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin 2x\, dx}{\sin^4 x + \cos^4 x}$.

9. Evaluate the integrals of the following functions :

(*i*) $\sqrt{(1+\sin x)}$, (*ii*) $\dfrac{\sin x}{\sqrt{(1+\sin x)}}$,

(*iii*) $\sqrt{(1+\sec x)}$, (*iv*) $\sqrt{(\sec x - 1)}$,

(*v*) $\dfrac{1}{\sin^4 x + \cos^4 x}$, (*vi*) $\sqrt{\tan x} + \sqrt{\cot x}$.

10. Integrate

(*i*) $\sec^2 x \log (1+\sin^2 x)$, (*ii*) $\dfrac{\sin x \log (\sin x + \cos x)}{\cos^2 x}$.

(*iii*) $\cos 2x \log (1+\tan x)$, (*iv*) $\dfrac{\cosh x + \sinh x \sin x}{1+\cos x}$.

11. Evaluate

$$\int_0^{\frac{1}{2}\pi} \sqrt[3]{\tan x}\, dx.$$

12 If $$I_n = \int_0^{\frac{1}{4}\pi} \tan^n \theta\, d\theta,$$

prove that when n is a positive integer,

$$n(I_{n-1} + I_{n+1}) = 1.$$

13. If $$U_n = \int_0^{\frac{1}{4}\pi} \tan^n x\, dx,$$

show that

$$U_n + U_{n-2} = \frac{1}{n-1},$$

and deduce the value of u_5.

14. If $u_n = \displaystyle\int_0^{\frac{1}{2}\pi} \theta \sin^n \theta\, d\theta$, and $n > 1$ prove that

$$u_n = \frac{n-1}{n} u_{n-2} + \frac{1}{n^2}.$$

Deduce that $u_5 = \dfrac{149}{225}$.

15. If $u_n = \int \cos n\theta \operatorname{cosec} \theta\, d\theta$, prove that

$$u_n - u_{n-2} = \frac{2\cos (n-1)\theta}{n-1}.$$

Hence, or otherwise, prove that

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 3\theta \sin 5\theta}{\sin \theta}\, d\theta = \frac{71}{105}.$$

16. Find the reduction formula connecting

$I_{m,n} = \int \cos^m x \sin^n x \, dx$ with $I_{m,\, n-2}$ and $I_{m-2,\, n}$.

Deduce

$$(m+n)(m+n-2)\, I_{m,\, n} = \{(n-1)\sin^2 x-(m-1)\cos^2 x\}\cos^{m-1} x \sin^{n-1} x + (m-1)(n-1)\, I_{m-2,\, n-2}.$$

17. If $u_n = \int \dfrac{dx}{\cosh^n x}$, prove that

$$(n-1)\, u_n = \frac{\tanh x}{\cosh^{n-2} x} + (n-2)\, u_{n-2}.$$

Evaluate the integral when $n = 5$.

18. Prove that if

$$u_n = \int_0^\pi \frac{1-\cos nx}{1-\cos x}\, dx,$$

where n is a positive integer or zero, then

$$u_{n+2}+u_n = 2u_{n+1}.$$

Prove that

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2 n\theta}{\sin^2 \theta}\, d\theta = \tfrac{1}{2}n\pi.$$

19. Show that if n is a positive integer, then

$$\int_0^{2\pi} \frac{\cos(n-1)x-\cos nx}{1-\cos x}\, dx = 2\pi$$

and deduce that

$$\int_0^{2\pi} \left(\frac{\sin \frac{1}{2}nx}{\sin \frac{1}{2}x}\right)^2 dx = 2n\pi.$$

20. Prove that

$$\int_0^\pi \frac{\sin n\theta}{\sin \theta}\, d\theta$$

is equal to 0 or π according as n is an even or odd positive integer.

By means of a reduction formula or otherwise, prove that

$$\int_0^\pi \frac{\sin^2 n\theta}{\sin \theta}\, d\theta = \pi,$$

where n is a positive integer.

21. Find a reduction formula for u_n, where

$$u_n = \int_0^\pi \frac{x \sin nx}{\sin x}\, dx,$$

by evaluating $u_n - u_{n-2}$, or by any other way. Show that if n is an odd positive integer,

$$\int_0^{\pi} \frac{x \sin nx}{\sin x}\, dx = \tfrac{1}{2}\pi^2,$$

and that if n is an even positive integer, the value of the same integral tends to $-\pi^2/2$, as n is increased indefinitely.

22. Evaluate

(*i*) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin 2\theta}$,

(*ii*) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta}\ (ab-h^2)>0$,

(*iii*) $\displaystyle\int_0^{2\pi} \frac{dx}{(1+\cos\alpha\cos x)^2}\ , 0<\alpha<\pi$,

(*iv*) $\displaystyle\int_0^{2\pi} \frac{dx}{(1+e\cos x)^2}$, where, $e<1$,

(*v*) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sec^2\alpha\, dx}{\tan x-\tan\alpha},\ \alpha>\frac{\pi}{4}$.

23. Evaluate the following definite integrals :

(*i*) $\displaystyle\int_0^{\frac{1}{2}\pi} x\cot x\, dx$, (*ii*) $\displaystyle\int_0^{\frac{1}{2}\pi} x^2 \operatorname{cosec}^2 x\, dx$,

(*iii*) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{x^2\cos x}{\sin^2 x}\, dx$.

24. Evaluate

(*i*) $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{x\, dx}{\cos x(\cos x+\sin x)}$,

(*ii*) $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{x^2(\sin 2x-\cos 2x)}{(1+\sin 2x)\cos^2 x}\, dx$.

25. Evaluate

(*i*) $\displaystyle\int_0^{\pi} \frac{x}{1+\sin x}\, dx$, (*ii*) $\displaystyle\int_0^{\pi} \frac{x^2\cos x}{(1+\sin x)^2}\, dx$.

26. Evaluate

(*i*) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{x\sin 2x}{\sin x+\cos x}\, dx$, (*ii*) $\displaystyle\int_0^{\pi} \frac{x^2\sin x\, dx}{(2x-\pi)(1+\cos^2 x)}$

(*iii*) $\displaystyle\int_0^{\pi} \frac{x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}\, dx$, (*iv*) $\displaystyle\int^{\pi} x\sin^6 x\cos^4 x\, dx$.

Answers

2. (*i*) $\pi/2ab$. (*ii*) $\pi(a^2+b^2)/4a^3b^3$.
   (*iii*) $1/3$. (*iv*) $\frac{1}{2}[x+\log(\sin x+\cos x)]$.

3. (*i*) $\theta+\frac{2}{5}\sin 5\theta$. (*ii*) $\frac{1}{2}[(\sqrt{2}-1)+\log(\sqrt{2}+1)]$.

8. (*i*) $1/2$. (*ii*) $\dfrac{2\pi}{a^2+b^2}$.

(*iii*) $\dfrac{\pi}{\sinh 2\alpha}$, if $\alpha \neq 0$. (*iv*) $\dfrac{\pi}{4}$.

9. (*i*) $-2\sqrt{(1-\sin x)}$.

(*ii*) $-2\sqrt{(1-\sin x)}-\sqrt{2}\log\tan\left(\frac{1}{4}\pi+\frac{1}{4}x\right)$.

(*iii*) $2\tan^{-1}\sqrt{(\sec x-1)}$. (*iv*) $-2\coth^{-1}[\sqrt{(1+\sec x)}]$.

(*v*) $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{\tan x-\cot x}{\sqrt{2}}\right)$.

(*vi*) $\sqrt{2}\tan^{-1}\left(\dfrac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)$.

10. (*i*) $\tan x\log(1+\sin^2 x)\quad 2x-\sqrt{2}\tan^{-1}(\cot x/\sqrt{2})$.

(*ii*) $\sec x\log(\sin x+\cos x)+\log\dfrac{\sin\frac{1}{2}x+\cos\frac{1}{2}x}{\cos\frac{1}{2}x-\sin\frac{1}{2}x}$

$$+\sqrt{2}\log\frac{\sin\frac{1}{2}x-(\sqrt{2}+1)\cos\frac{1}{2}x}{\sin\frac{1}{2}x+(\sqrt{2}-1)\cos\frac{1}{2}x}$$

(*iii*) $\frac{1}{2}\sin 2x\log(1+\tan x)-\frac{1}{2}x+\frac{1}{2}\log\sin\left(x+\frac{1}{4}\pi\right)$.

(*iv*) $\cosh x\tan\frac{1}{2}x$.

11. $\frac{1}{2}(\pi/\sqrt{3}-\log 2)$. **13.** $\frac{1}{4}\log(4/e)$.

17. $\dfrac{1}{4}\dfrac{\tanh x}{\cosh^2 x}+\dfrac{3}{8}\cdot\dfrac{\tanh x}{\cosh x}+\dfrac{3}{4}\tan^{-1}(e^x)$.

21. $u_n=\dfrac{2}{(n-1)^2}[(n-1)x\sin(n-1)x+\cos(n-1)x-1]+u_{n-2}$

22. (*i*) $\dfrac{2\pi}{\sqrt[3]{3}}$. (*ii*) $\dfrac{2\pi}{\sqrt{(ab-h^2)}}$.

(*iii*) $\dfrac{2\pi}{\sin^3\alpha}$. (*iv*) $\dfrac{2\pi}{(1-e^2)^{3/2}}$.

(*v*) $\log\dfrac{\sin\left(\alpha-\frac{1}{4}\pi\right)}{\sin\alpha}-\dfrac{\pi}{4}\tan\alpha$.

23. (*i*) $\frac{1}{8}\pi\log 2$. (*ii*) $\pi\log 2$.

(*iii*) $-\frac{1}{16}\pi^3+\frac{3}{8}\pi\log 2$.

24. (*i*) $\frac{1}{8}\pi\log 2$. (*ii*) $\frac{1}{16}\pi^2-\frac{1}{4}\pi\log 2$.

25. (*i*) π. (*ii*) $\pi(2-\pi)$.

26. (*i*) $\dfrac{\pi}{2}+\dfrac{\pi}{2\sqrt{2}}\log(\sqrt{2}-1)$ (*ii*) $\dfrac{\pi^2}{4}$.

(*iii*) $\dfrac{\pi^2}{32}$. (*iv*) $\dfrac{3\pi^2}{512}$.

5

Integration of Irrational Functions

5·1. Integration of rational function of x and a linear surd,

$$(\mathbf{ax+b})^{1/n},$$

where n is some positive integer.

The substitution $ax+b = t^n$ transforms the given integral into the integral of a rational function of t which may, then, be evaluated by the methods given in Chapter 3.

A rational function of x and of

$$\left(\frac{ax+b}{cx+d}\right)^{1/n},$$

may also be integrated by the transformation

$$\frac{ax+b}{cx+d} = t^n.$$

The following examples will make the procedure clear.

Examples

1. *Evaluate* $\displaystyle\int \frac{x}{\sqrt[3]{(a+bx)}}\, dx$

We put

$$a+bx = y^3 \Rightarrow b dx = 3y^2\, dy$$

$$\therefore \qquad \int \frac{x\, dx}{\sqrt[3]{(a+bx)}} = \int \frac{(y^3-a)}{b} \cdot \frac{1}{y} \cdot \frac{3y^2\, dy}{b}$$

$$= \frac{1}{b^2}\int (y^3-a)\, 3y\, dy$$

$$= \frac{3}{b^2}\int (y^4-ay)\, dy$$

$$= \frac{3}{b^2}\left[\frac{y^5}{5} - \frac{ay^2}{2}\right]$$

$$= \frac{3}{10\,b^2}\left[2y^5 - 5ay^2\right]$$

$$= \frac{3}{10\,b^2}\,y^2\left[2y^3 - 5a\right]$$

$$= \frac{3}{10\,b^2}\,(a+bx)^{2/3}\left[2a+2bx-5a\right]$$

$$= \frac{3}{10\,b^2}(a+bx)^{2/3}\,(2bx-3a)$$

2. *Evaluate* $\displaystyle\int \frac{1}{\sqrt{(x+1)} - \sqrt[4]{(x+1)}}\,dx$

We put

$$1+x = y^4 \Rightarrow dx = 4y^3\,dy$$

$$\therefore \quad \int \frac{dx}{\sqrt{(x+1)} - \sqrt[4]{(x+1)}} = \int \frac{4y^3\,dy}{y^2-y}$$

$$= \int \frac{4y^2\,dy}{y-1}$$

$$= 4\int\left[(y+1) + \frac{1}{y-1}\right]dy$$

$$= 4\left[\frac{y^2}{2} + y + \log(y-1)\right]$$

$$= 2y^2 + 4y + \log(y-1),$$

where $\qquad x+1 = y^4.$

3. *Evaluate* $\displaystyle\int\sqrt{\left(\frac{1-x}{1+x}\right)}\frac{dx}{x}.$

We put $\qquad \dfrac{1-x}{1+x} = y^2.$

$$\Rightarrow \quad x = \frac{1-y^2}{1+y^2} \Rightarrow dx = -\frac{4y\,dy}{(1+y^2)^2}$$

$$\therefore \quad \int\sqrt{\left(\frac{1-x}{1+x}\right)}\frac{dx}{x} = -4\int\frac{y^2}{(1-y^2)(1+y^2)}\,dy$$

$$= -2\int\left(\frac{1}{1-y^2} - \frac{1}{1+y^2}\right)dy$$

$$= -2\int\left[\frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right) - \frac{1}{1+y^2}\right]dy$$

$$= \log\frac{1-y}{1+y} + 2\tan^{-1} y$$

$$= \log \frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{\sqrt{(1+x)}+\sqrt{(1-x)}} + 2\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}.$$

Exercises

1. Find the integrals of the following functions :—

(*i*) $x\sqrt[3]{(a+bx)}$, (*ii*) $\dfrac{x^2}{\sqrt[3]{(x+2)}}$,

(*iii*) $\dfrac{x}{(x+2)\sqrt{(x+1)}}$, (*iv*) $\dfrac{\sqrt{x}}{x-1}$.

2. (*i*) $\dfrac{1}{x+\sqrt[3]{x}}$, (*ii*) $\dfrac{1+\sqrt{x}-\sqrt[3]{x^2}}{1+\sqrt[3]{x}}$,

(*iii*) $\dfrac{1}{\sqrt{(1+x)}+\sqrt[3]{(1+x)}}$, (*iv*) $\dfrac{\sqrt[4]{x}}{\sqrt{x}-1}$.

3. (*i*) $\dfrac{x^2}{\sqrt{(2x+3)}}$, (*ii*) $\dfrac{x^2}{(x+3)\sqrt{(3x+4)}}$,

(*iii*) $\dfrac{1}{(x^3+x^2)\sqrt{(1+x)}}$, (*iv*) $\dfrac{x+1}{(x+2)(x+3)^{3/2}}$,

(*v*) $\left(\dfrac{x+2}{2x+3}\right)^{1/2}\cdot\dfrac{1}{x}$, (*vi*) $\sqrt{\left(\dfrac{1+x^2}{x^2-x^4}\right)}$.

4. Evaluate

(*i*) $\displaystyle\int_0^\infty \frac{dx}{x\sqrt{(1+x)}}$, (*ii*) $\displaystyle\int_1^\infty \frac{dx}{x^2\sqrt{(1+x)}}$,

(*iii*) $\displaystyle\int_8^{15} \frac{dx}{(x-3)\sqrt{(x+1)}}$, (*iv*) $\displaystyle\int_3^8 \frac{2-3x}{x\sqrt{(1+x)}}\,dx$.

Answers

1. (*i*) $3(4b^2x^2+abx-3a^2)\sqrt[3]{(a+bx)}/28b^2$.

(*ii*) $\frac{3}{40}(x+2)^{2/3}(5x^2-12x+36)$

(*iii*) $2\sqrt{(x+1)}-4\tan^{-1}\sqrt{(x+1)}$.

(*iv*) $2\sqrt{x}+\log[(\sqrt{x}-1)/(\sqrt{x}+1)]$.

2. (*i*) $\frac{3}{2}\log(1+\sqrt[3]{x^2})$.

(*ii*) $6[\tan^{-1}y-y+\frac{1}{3}y^3-\frac{1}{5}y^5+\frac{1}{6}y^6+\frac{1}{7}y^7-\frac{1}{8}y^8]$, where $x=y^6$.

(*iii*) $2\sqrt{(1+x)}-3\sqrt[3]{(1+x)}+6\sqrt[6]{(1+x)}-6\log[1+\sqrt[6]{(1+x)}]$.

(*iv*) $2\log\dfrac{\sqrt[4]{x}-1}{\sqrt[4]{x}+1}+4\sqrt[4]{x}+\dfrac{4}{3}\sqrt[4]{x^3}$.

3. (i) $\frac{1}{5}(x^2-2x+6)\sqrt{(2x+3)}$.

(ii) $\frac{2}{27}(3x-35)\sqrt{(3x+4)}+\frac{18}{\sqrt5}\tan^{-1}\sqrt{\left(\frac{3x+4}{5}\right)}$.

(iii) $-\left(\frac{3x+1}{x}\right)\cdot\frac{1}{\sqrt{(1+x)}}+\frac{3}{2}\log\frac{\sqrt{(1+x)}+1}{\sqrt{(1+x)}-1}$.

(iv) $-\frac{4}{\sqrt{(x+3)}}+\log\frac{\sqrt{(x+3)}+1}{\sqrt{(x+3)}-1}$.

(v) $\frac{1}{\sqrt2}\log\frac{1+\sqrt2 y}{1-\sqrt2 y}-\sqrt{\frac{2}{3}}\log\frac{\sqrt3 y+\sqrt2}{\sqrt3 y-\sqrt2}$,

where $y=\sqrt{\left(\frac{x+2}{2x-3}\right)}$.

(vi) $\tan^{-1}\sqrt{\left(\frac{1+x^2}{1-x^2}\right)}+\frac{1}{2}\log\frac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}$.

4. (i) $2\log(\sqrt2+1)$.
(ii) $\sqrt2-\log(\sqrt2+1)$.
(iii) $(\log 5-\log 3)/2$
(iv) $2\log(3/2e^3)$.

5·2. To evaluate the integrals

(i) $\int\sqrt{(ax^2+bx+c)}\,dx$; (ii) $\int\frac{1}{\sqrt{(ax^2+bx+c)}}\,dx$.

If a be positive, we have

$$\sqrt{(ax^2+bx+c)}=\sqrt{\left[a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)\right]}$$
$$=\sqrt a\sqrt{\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]}.$$

(i) Let a and $4ac-b^2$ be both positive. We write

$$(4ac-b^2)/4a^2=k^2 \text{ and put } x+b/2a=y,$$

so that we have

$$\int\sqrt{(ax^2+bx+c)}\,dx=\sqrt a\int\sqrt{(y^2+k^2)}\,dy$$
$$=\sqrt a\left[\frac{y\sqrt{(y^2+k^2)}}{2}+\frac{k^2}{2}\sinh^{-1}\frac{y}{k}\right]$$

and
$$\int\frac{1}{\sqrt{(ax^2+bx+c)}}\,dx=\frac{1}{\sqrt a}\int\frac{dy}{\sqrt{(y^2+k^2)}}=\frac{1}{\sqrt a}\sinh^{-1}\frac{y}{k}.$$

(ii) Let a be positive and $4ac-b^2$ be negative. We write $(4ac-b^2)/4a^2=-t^2$ and put $x+b/2a=y$, so that we have, in this case,

$$\int\sqrt{(ax^2+bx+c)}\,dx=\sqrt a\int\sqrt{(y^2-t^2)}\,dy$$
$$=\sqrt a\left[\frac{y\sqrt{(y^2-t^2)}}{2}-\frac{t^2}{2}\cosh^{-1}\frac{y}{t}\right]$$

and $$\int \frac{1}{\sqrt{(ax^2+bx+c)}}\,dx = \frac{1}{\sqrt{a}}\int \frac{dy}{\sqrt{(y^2-t^2)}} = \frac{1}{\sqrt{a}}\cosh^{-1}\frac{y}{t}.$$

(*iii*) *Let a be negative.* We have

$$\sqrt{(ax^2+bx+c)} = \sqrt{(-a)}\sqrt{\left[-x^2-\frac{bx}{a}-\frac{c}{a}\right]}$$

$$= \sqrt{(-a)}\sqrt{\left[\frac{b^2-4ac}{4a^2}-\left(x+\frac{b}{2a}\right)^2\right]}$$

$$= \sqrt{(-a)}\sqrt{\left[\left(\frac{\sqrt{(b^2-4ac)}}{2a}\right)^2-\left(x+\frac{b}{2a}\right)^2\right]}$$

We have assumed that $b^2 > 4ac$, for if $b^2 < 4ac$, the expression under radical sign will always be negative.

As before, we write $(b^2-4ac)/4a^2 = t^2$ and $x+b/2a = y$, so that we have, in this case,

$$\int \sqrt{(ax^2+bx+c)}dx = \sqrt{(-a)}\int \sqrt{(t^2-y^2)}dy$$

$$= \sqrt{(-a)}\left[\frac{y\sqrt{(t^2-y^2)}}{2}+\frac{t^2}{2}\sin^{-1}\frac{y}{t}\right]$$

and $$\int \frac{1}{\sqrt{(ax^2+bx+c)}}\,dx = \frac{1}{\sqrt{(-a)}}\int \frac{dy}{\sqrt{(t^2-y^2)}} = \frac{1}{\sqrt{(-a)}}\sin^{-1}\frac{y}{t}.$$

Exercises

1. Integrate

(*i*) $\frac{1}{\sqrt{(2x^2+3x+4)}}$, (*ii*) $\sqrt{(2x^2+3x+4)}$,

(*iii*) $\frac{4}{\sqrt{(3x^2-4x+1)}}$, (*iv*) $\sqrt{(3x^2-4x+1)}$.

2. (*i*) $\frac{1}{\sqrt{(1+2x-3x^2)}}$, (*ii*) $\sqrt{(1+2x-3x^2)}$,

(*iii*) $\frac{1}{\sqrt{(5x-6-x^2)}}$.

Answers

1. (*i*) $\frac{1}{\sqrt{2}}\sinh^{-1}\frac{4x+3}{\sqrt{(23)}}$,

(*ii*) $\frac{4x+3}{8}\sqrt{(2x^2+3x+4)}+\frac{23\sqrt{2}}{32}\sinh^{-1}\frac{4x+3}{\sqrt{(23)}}$,

(*iii*) $\frac{4}{\sqrt{3}}\cosh^{-1}(3x-2)$.

(*iv*) $\frac{3x-2}{6}\sqrt{(3x^2-4x+1)}-\frac{\sqrt{3}}{18}\cosh^{-1}(3x-2)$.

2. (i) $\sqrt{\frac{1}{3}}\sin^{-1}[(3x-1)/2]$.

(ii) $\dfrac{3x-1}{6}\sqrt{(1+2x-3x^2)}+\dfrac{2\sqrt{3}}{9}\sin^{-1}\left(\dfrac{3x-1}{2}\right)$.

(iii) $\sin^{-1}(2x-5)$.

5·3. Integration of

(i) $\dfrac{Ax+B}{\sqrt{(ax^2+bx+c)}}$, (ii) $(Ax+B)\sqrt{(ax^2+bx+c)}$.

We determine two constants λ and μ such that

$$Ax+B=\lambda\,(2ax+b)+\mu,$$

where $2ax+b$ is the differential coefficient of ax^2+bx+c. The constants λ, μ are given by

$$A=2a\lambda,\quad B=b\lambda+\mu,$$

$$\Rightarrow \qquad \lambda=\frac{A}{2a},\ \mu=\frac{2aB-Ab}{2a}.$$

(i) We have

$$\int\frac{Ax+B}{\sqrt{(ax^2+bx+c)}}\,dx=\int\frac{\lambda(2ax+b)+\mu}{\sqrt{(ax^2+bx+c)}}\,dx$$

$$=\lambda\int(ax^2+bx+c)^{-\frac{1}{2}}\,(2ax+b)dx+\mu\int\frac{dx}{\sqrt{(ax^2+bx+c)}}$$

$$=2\lambda\,(ax^2+bx+c)^{\frac{1}{2}}+\mu\int\frac{dx}{\sqrt{(ax^2+bx+c)}}$$

and the integral on the right has been already considered in § 5.2.

(ii) Again, we have

$$\int(Ax+B)\sqrt{(ax^2+bx+c)}dx$$

$$=\lambda\int(2ax+b)\sqrt{(ax^2+bx+c)}dx+\mu\int\sqrt{(ax^2+bx+c)}\,dx$$

$$=\frac{2\lambda}{3}\,(ax^2+bx+c)^{3/2}+\mu\int\sqrt{(ax^2+bx+c)}\,dx,$$

and $\int\sqrt{(ax^2+bx+c)}\,dx$ has been already considered in § 5·2.

Exercises

1. Evaluate

(i) $\displaystyle\int x\sqrt{(1+x-x^2)}dx$, (ii) $\displaystyle\int\frac{x}{\sqrt{(8+x-x^2)}}\,dx$,

(iii) $\displaystyle\int\frac{2x+3}{\sqrt{(4x^2+5x+6)}}\,dx$, (iv) $\displaystyle\int\frac{x}{\sqrt{(3x^2+2x+1)}}\,dx$.

2. *Evaluate*

(*i*) $\int \frac{x^2-2x+3}{\sqrt{(x^2+1)}} dx$, (*ii*) $\int \frac{x^3-3}{\sqrt{(x^2+2)}} dx$,

(*iii*) $\int (2x-5)\sqrt{(2+3x-x^2)}dx$.

Answers

1. (*i*) $\frac{1}{24}(8x^2-2x-11)\sqrt{(1+x-x^2)}+\frac{5}{16}\sin^{-1}\frac{2x-1}{\sqrt{5}}$.

   (*ii*) $-\sqrt{(8+x-x^2)}+\frac{1}{2}\sin^{-1}\frac{2x-1}{\sqrt{33}}$.

   (*iii*) $\frac{\sqrt{(4x^2+5x+6)}}{2}+\frac{7}{8}\sinh^{-1}\frac{8x+5}{\sqrt{71}}$.

   (*iv*) $\frac{\sqrt{(3x^2+2x+1)}}{3}-\frac{1}{3\sqrt{3}}\sinh^{-1}\frac{3x+1}{\sqrt{2}}$.

2. (*i*) $\frac{1}{2}(x-4)\sqrt{(x^2+1)}+\frac{5}{2}\sinh^{-1}x$.

   (*ii*) $\frac{1}{3}(x^2-4)\sqrt{(x^2+2)}-3\sinh^{-1}(x/\sqrt{2})$.

   (*iii*) $\frac{1}{6}(4x^2-18x+1)\sqrt{(2+3x-x^2)}-\frac{17}{4}\sin^{-1}\frac{2x-3}{\sqrt{17}}$.

5·4. Reduction Formula for

$$\int \frac{x^n}{\sqrt{(ax^2+bx+c)}}\,dx,$$

where n is a positive integer.

Let $$I_n = \int \frac{x^n}{\sqrt{(ax^2+bx+c)}}\,dx.$$

We will obtain a reduction formula connecting I_n with I_{n-1} and I_{n-2}.

We note that, $2ax+b$ is the derivative of ax^2+bx+c.

We write

$$x^n = \frac{2ax+b-b}{2a}\cdot x^{n-1}$$

$$\therefore \quad I_n = \frac{1}{2a}\int\frac{(2ax+b)x^{n-1}}{\sqrt{(ax^2+bx+c)}}dx-\frac{b}{2a}\int\frac{x^{n-1}}{\sqrt{(ax^2+bx+c)}}dx. \quad ...(i)$$

Integrating the first integral on the right by parts, we obtain

$$\int \frac{2ax+b}{\sqrt{(ax^2+bx+c)}}x^{n-1}\,dx$$
$$= 2\sqrt{(ax^2+bx+c)}x^{n-1}-2\int (n-1)\,x^{n-2}\sqrt{(ax^2+bx+c)}dx$$
$$= 2\sqrt{(ax^2+bx+c)}x^{n-1}-2(n-1)\int\frac{x^{n-2}(ax^2+bx+c)}{\sqrt{(ax^2+bx+c)}}dx$$

$$= 2\sqrt{(ax^2+bx+c)}x^{n-1} - 2a(n-1)I_n - 2b(n-1)I_{n-1} - 2c(n-1)\,I_{n-2} \qquad \ldots(ii)$$

From (*i*) and (*ii*), we obtain

$$I_n = \frac{1}{a}\sqrt{(ax^2+bx+c)}x^{n-1} - (n-1)\,I_n - \frac{b(n-1)}{a}I_{n-1} - \frac{c(n-1)}{a}I_{n-2} - \frac{b}{2a}I_{n-1}.$$

Transposing $(n-1)\,I_n$ and dividing by n, we obtain

$$I_n = \frac{\sqrt{(ax^2+bx+c)}\,x^{n-1}}{an} - \frac{b(2n-1)}{2an}I_{n-1} - \frac{c(n-1)}{an}I_{n-2} \qquad \ldots(A)$$

which is the required reduction formula.

As I_1 and I_0 have already been determined, we can, by successive applications of this formula, determine

$$I_2, I_3, I_4, \text{ etc.}$$

Example

Evaluate

$$\int \frac{x^3}{\sqrt{(x^2-2x+2)}}\,dx.$$

Changing a to 1, b to -2, and c to 2 in (A), we get

$$\int \frac{x^n}{\sqrt{(x^2-2x+2)}}\,dx = \frac{\sqrt{(x^2-2x+2)}}{n}.\,x^{n-1} + \frac{2n-1}{n}\int\frac{x^{n-1}}{\sqrt{(x^2-2x+2)}}dx - \frac{2(n-1)}{n}\int\frac{x^{n-2}}{\sqrt{(x^2-2x+2)}}dx.$$

Putting $n = 3$ and $n = 2$, we get

$$\int \frac{x^3}{\sqrt{(x^2-2x+2)}}\,dx = \frac{\sqrt{(x^2-2x+2)}}{3}x^2 + \frac{5}{3}\int\frac{x^2}{\sqrt{(x^2-2x+2)}}dx - \frac{4}{3}\int\frac{x}{\sqrt{(x^2-2x+2)}}\,dx \qquad \ldots(i)$$

$$\int \frac{x^2}{\sqrt{(x^2-2x+2)}}\,dx = \frac{\sqrt{(x^2-2x+2)}}{2}x + \frac{3}{2}\int\frac{x}{\sqrt{(x^2-2x+2)}}\,dx - \int\frac{dx}{\sqrt{(x^2-2x+2)}} \qquad \ldots(ii)$$

Now,

$$\int\frac{x}{\sqrt{(x^2-2x+2)}}\,dx = \frac{1}{2}\int\left[\frac{2x-2}{\sqrt{(x^2-2x+2)}} + \frac{2}{\sqrt{[(x-1)^2+1)]}}\right]dx$$

$$= (x^2-2x+2)^{\frac{1}{2}} + \sinh^{-1}(x-1) \qquad \ldots(iii)$$

and $$\int\frac{dx}{\sqrt{(x^2-2x+2)}} = \sinh^{-1}(x-1). \qquad \ldots(iv)$$

From the system of equations (*i*), (*ii*), (*iii*) and (*iv*), we get

$$\int \frac{x^3}{\sqrt{(x^2-2x+2)}}\,dx = \frac{2x^2+5x+15}{6}\sqrt{(x^2-2x+2)}$$
$$+\frac{5}{6}\sinh^{-1}(x-1).$$

Note. The integration of $x^n/\sqrt{(ax^2+bx+c)}$ can sometimes be more conveniently effected in another manner which we shall now explain.

The method is also applicable to the more general integral

$$\int \frac{\varphi(x)}{\sqrt{(ax^2+bx+c)}}\,dx,$$

where $\varphi(x)$ is a polynomial in x.

The method depends upon the following result :—

If $\varphi(x)$ is a polynomial of degree n, then there exists a polynomial $f(x)$ of degree $(n-1)$ and a constant A, such that

$$\int \frac{\varphi(x)}{\sqrt{(ax^2+bx+c)}}\,dx = f(x)\sqrt{(ax^2+bx+c)}+A\int\frac{dx}{\sqrt{(ax^2+bx+c)}}.$$

Differentiating both sides and multiplying by

$$\sqrt{(ax^2+bx+c)},$$

we get

$$\varphi(x) = f'(x)(ax^2+bx+c)+\tfrac{1}{2}(2ax+b)f(x)+A.$$

Either side of this equation is a polynomial of degree n. Equating the coefficients of like powers of x, we will get $(n+1)$ *linear* equations which will uniquely determine the n coefficients occurring in the polynomial $f(x)$ and the constant A. Thus $f(x)$ and A have been completely determined.

We will now illustrate this method by an example.

Example

Evaluate

$$\int \frac{x^3+5x^2-3x+4}{\sqrt{(x^2+x+1)}}\,dx.$$

Here $\varphi(x) = x^3+5x^2-3x+4$ is a polynomial of third degree. We write

$$\int \frac{x^3+5x^2-3x+4}{\sqrt{(x^2+x+1)}}\,dx = (Ax^2+Bx+C)\sqrt{(x^2+x+1)}$$
$$+D\int\frac{1}{\sqrt{x^2+x+1)}}\,dx.$$

Differentiating and multiplying by $\sqrt{(x^2+x+1)}$, we get

$$x^3+5x^2-3x+4 = (2Ax+B)(x^2+x+1)$$
$$+\tfrac{1}{2}(Ax^2+Bx+C)(2x+1)+D.$$

Equating the coefficients of like powers of x, we get

$$\begin{aligned} 1 &= 3A, \\ 5 &= \tfrac{5}{2}A+2B, \\ -3 &= 2A+\tfrac{3}{2}B+C \\ 4 &= B+\tfrac{1}{2}C+D. \end{aligned}$$

From these, we obtain

$$A = 1/3\ ;\ B = 25/12\ ;\ C = -163/24\ ;\ D = 85/16.$$

Also, $$\int \frac{1}{\sqrt{(x^2+x+1)}}\,dx = \int \frac{dx}{\sqrt{[(x+\frac{1}{2})^2+(\sqrt{3}/2)^2]}}$$
$$= \sinh^{-1}\frac{2x+1}{\sqrt{3}}$$

$$\therefore \int \frac{x^3+5x^2-3x+4}{\sqrt{(x^2+x+1)}}\,dx = \left(\frac{x^2}{3}+\frac{25}{12}x-\frac{163}{24}\right)\sqrt{(x^2+x+1)}$$
$$+\frac{85}{16}\sinh^{-1}\frac{2x+1}{\sqrt{3}}.$$

Exercises

1. Find the integrals of the following functions :—

(i) $\dfrac{x^3+4x^2-6x+3}{\sqrt{(5+6x-x^2)}}$, (ii) $\dfrac{x^4}{\sqrt{(3+2x+x^2)}}$,

(iii) $\dfrac{x^3}{\sqrt{(ax^2+2bx+c)}}$.

2. Evaluate $\displaystyle\int \frac{x^3dx}{\sqrt{(x^2+2x+2)}}$ in two different ways.

Answers

1. (i) $\frac{1}{6}(2x^2+27x+227)\sqrt{(5+6x-x^2)}-139\sin^{-1}[(3-x)/\sqrt{(14)}]$.

(ii) $\frac{1}{24}(3x^3-7x^2+4x+30)\sqrt{(3+2x+x^2)}-\frac{7}{8}\sinh^{-1}[(x+1)/\sqrt{2}]$.

(iii) $\dfrac{1}{6a^3}(2a^2x^2-5abx+15b^2-4ac)\sqrt{(ax^2+2bx+c)}$
$$+\frac{3abc-5b^3}{2a^3}\int\frac{dx}{\sqrt{(ax^2+2bx+c)}}.$$

2. $\frac{1}{6}(2x^2-5x+7)\sqrt{(x^2+2x+2)}+\frac{1}{2}\sinh^{-1}(x+1)$.

5·5. To evaluate

$$\int \frac{\mathbf{dx}}{(\mathbf{Ax+B})\sqrt{(\mathbf{ax^2+bx+c})}}.$$

The substitution $Ax+B = 1/t$ will enable us to reduce it to an integral of the form considered in § 5·2.

We have $$dx = -\frac{1}{At^2}\,dt,$$

and $$x = \left(\frac{1}{t} - B\right)\frac{1}{A} = \frac{1-Bt}{At}.$$

Thus we have

$$\int \frac{dx}{(Ax+B)\sqrt{(ax^2+bx+c)}}$$

$$= -\int \frac{dt}{\sqrt{[(aB^2-bAB+cA^2)t^2+(bA-2aB)t+a]}},$$

which has been considered in § 5·2.

Example

Evaluate $$\int \frac{dx}{(x+1)\sqrt{(2x^2+3x+4)}}.$$

Putting $x+1 = 1/y$, we get

$$\int \frac{dx}{(x+1)\sqrt{(2x^2+3x+4)}} = -\int \frac{dy}{\sqrt{(3y^2-y+2)}}$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dy}{\sqrt{[(y-1/6)^2+(\sqrt{23}/6)^2]}}$$

$$= -\frac{1}{\sqrt{3}}\sinh^{-1}\frac{6y-1}{\sqrt{23}}$$

$$= -\frac{1}{\sqrt{3}}\sinh^{-1}\left[\frac{5-x}{\sqrt{23}\,(x+1)}\right].$$

Note. The integral

$$\int \frac{dx}{(Ax+B)^r\sqrt{(ax^2+bx+c)}}$$

where r is a positive integer, may also be evaluated by the substitution $Ax+B = 1/t$.

Exercises

Evaluate the following integrals :

1. (*i*) $\displaystyle\int \frac{dx}{(2-x)\sqrt{(1-2x+3x^2)}}$, (*ii*) $\displaystyle\int \frac{dx}{(3+2x)\sqrt{(x^2+x+1)}}$.

(iii) $\int \frac{dx}{(x+1)\sqrt{(x^2-1)}}$, (iv) $\int \frac{(x+1)dx}{(2+x)\sqrt{(2x^2-3x+1)}}$,

2. (i) $\int \frac{1+2x}{(1+3x)\sqrt{(x^2+2x+5)}}\, dx$, (ii) $\int \frac{\sqrt{x^2+1}}{x}\, dx$,

(iii) $\int \frac{x^2+2x-1}{(x+2)\sqrt{(2x^2+3x-4)}}\, dx$, (iv) $\int \frac{\sqrt{(1+x+x^2)}}{1+x}dx$,

3. (i) $\int \frac{dx}{(x-a)\sqrt{[(x-a)(b-x)]}}$, (ii) $\int \frac{(x^2+1)\, dx}{x\sqrt{(4x^2+1)}}$,

(iii) $\int \frac{x^3}{(x-1)\sqrt{(x^2-x+1)}}dx$, (iv) $\int \frac{dx}{(x+2)^2\sqrt{(3x+4x-5)}}$.

Answers

1. (i) $\frac{1}{3}\sinh^{-1}\left[\frac{5x-1}{\sqrt{2}(2-x)}\right]$.

(ii) $-\frac{1}{\sqrt{7}}\sinh^{-1}\left[\frac{1-4x}{\sqrt{3}(3+2x)}\right]$. (iii) $\sqrt{\left(\frac{x-1}{x+1}\right)}$.

(iv) $\frac{1}{\sqrt{2}}\cosh^{-1}(4x-3)+\frac{1}{\sqrt{15}}\cosh^{-1}\left(\frac{8-11x}{2+x}\right)$.

2. (i) $\frac{2}{3}\sinh^{-1}\frac{x+1}{2}-\frac{1}{3\sqrt{40}}\sinh^{-1}\left(\frac{7+x}{1+3x}\right)$.

(ii) $\sqrt{(x^2+1)}-\text{cosech}^{-1}x$.

(iii) $\frac{1}{2}\sqrt{(2x^2+3x-4)}-\frac{3}{4\sqrt{2}}\cosh^{-1}\frac{4x+3}{\sqrt{41}}+\frac{1}{\sqrt{2}}\sin^{-1}\frac{5x+14}{\sqrt{41}(x+2)}$.

(iv) $\sqrt{(x^2+x+1)}-\frac{1}{2}\sinh^{-1}\frac{2x+1}{\sqrt{3}}-\sinh^{-1}\left\{\frac{1-x}{\sqrt{3}(1+x)}\right\}$.

3. (i) $\frac{2}{a-b}\sqrt{\left(\frac{b-x}{x-a}\right)}$. (ii) $\frac{1}{4}\cdot\sqrt{(4x^2+1)}-\sinh^{-1}\left(\frac{1}{2x}\right)$.

(iii) $\frac{1}{4}(2x+7)\sqrt{(x^2-x+1)}+\frac{11}{8}\sinh^{-1}\frac{2x-1}{\sqrt{3}}-\sinh^{-1}\left[\frac{x+1}{\sqrt{3}(x-1)}\right]$

(iv) $\frac{\sqrt{(3x^2+4x-5)}}{x+2}+4\sin^{-1}\frac{4x+9}{\sqrt{19}(x+2)}$.

5·6. To evaluate $\int \frac{\mathbf{dx}}{\mathbf{(ax^2+bx+c)\sqrt{(Ax^2+Bx+C)}}}$.

The proper substitution for this case is

$$\frac{Ax^2+Bx+C}{ax^2+bx+c}=y^2.$$

An example will illustrate the process.

Example

Evaluate

$$\int \frac{dx}{(1+x^2)\sqrt{(1-x^2)}}$$

We put $$\frac{1-x^2}{1+x^2} = y^2 \Rightarrow x^2 = \frac{1-y^2}{1+y^2}$$

$$\Rightarrow \qquad 2x\,dx = -\frac{4y\,dy}{(1+y^2)^2}$$

Also $$1+x^2 = \frac{2}{1+y^2} \text{ and } 1-x^2 = \frac{2y^2}{1+y^2}.$$

$$\therefore \int \frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = -\frac{1}{\sqrt{2}}\int \frac{dy}{\sqrt{(1-y^2)}}$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1} y$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1} \sqrt{\left(\frac{1-x^2}{1+x^2}\right)}.$$

Exercises

Evaluate the following integrals :

1. (*i*) $\int \frac{dx}{(x^2-1)\sqrt{(x^2+1)}}$, (*ii*) $\int \frac{dx}{(2x^2+3)\sqrt{(3x^2-4)}}$.

   (*iii*) $\int \frac{dx}{x^2\sqrt{(x^2+1)}}$, (*iv*) $\int \frac{(x+1)\,dx}{(x^2+4)\sqrt{(x^2+9)}}$.

2. (*i*) $\int \frac{(x^2+2x+3)\,dx}{(x^2+1)\sqrt{(x^2-2)}}$, (*ii*) $\int \frac{(x^3+x^2+x+1)dx}{(x^2+2)\sqrt{(x^2-3)}}$,

   (*iii*) $\int_0^\infty \frac{dx}{(x^2+a^2)\sqrt{(x^2+b^2)}}$, $(a^2<b^2)$ (*iv*) $\int_0^{1/\sqrt{3}} \frac{dx}{(1+x^2)\sqrt{(1-x^2)}}$.

Answers

1. (*i*) $-\frac{1}{\sqrt{2}} \sinh^{-1} \sqrt{\left(\frac{x^2+1}{x^2-1}\right)}$.

   (*ii*) $\frac{1}{\sqrt{51}} \sinh^{-1} \sqrt{\left(\frac{9x^2-12}{8x^2+12}\right)}$. (*iii*) $-\frac{\sqrt{(1+x^2)}}{x}$.

   (*iv*) $\frac{\sqrt{5}}{10} \log \frac{\sqrt{(x^2+9)}-\sqrt{5}}{\sqrt{(x^2+9)}+\sqrt{5}} + \frac{\sqrt{5}}{10} \tan^{-1} \frac{\sqrt{5}x}{2\sqrt{(x^2+9)}}$.

2. (*i*) $\cosh^{-1} \frac{x}{\sqrt{2}} + \frac{2}{\sqrt{3}} \tan^{-1} \sqrt{\left(\frac{x^2-2}{3}\right)} + \frac{2}{\sqrt{3}} \sinh^{-1} \sqrt{\left(\frac{x^2-2}{2x^2+2}\right)}$.

(ii) $\sqrt{(x^2-3)}-\dfrac{1}{\sqrt{5}}\tan^{-1}\sqrt{\left(\dfrac{x^2-3}{5}\right)}+\cosh^{-1}\dfrac{x}{\sqrt{3}}$

$$-\frac{1}{\sqrt{10}}\sinh^{-1}\sqrt{\left(\frac{2x^2-5}{3x^2+6}\right)}.$$

(iii) $\dfrac{1}{a\sqrt{(b^2-a^2)}}\cos^{-1}\left(\dfrac{a}{b}\right)$. (iv) $\dfrac{\pi}{4\sqrt{2}}$.

5·7. Integration of irrational algebraic functions by trigonometric transformations. In some cases, a suitable trigonometric transformation greatly simplifies the process of integration of irrational algebraic functions. Examples of this type have already appeared in §2·4, p. 30.

For, integrating functions which involve

$$\sqrt{(a^2-x^2)},\ \sqrt{(a^2+x^2)} \text{ or } \sqrt{(x^2-a^2)}$$

we apply the transformations

$$x=a\sin\theta,\ x=a\tan\theta,\ x=a\sec\theta$$

respectively.

The transformations $x=a\sinh\theta$, $x=a\cosh\theta$ also prove useful for functions involving $\sqrt{(a^2+x^2)}$ and $\sqrt{(x^2-a^2)}$ respectively.

Examples

1. *Evaluate* $\displaystyle\int\frac{1}{x^2\sqrt{(1-x^2)}}dx.$

Putting $x=\sin\theta$, we see that the given integral

$$=\int\operatorname{cosec}^2\theta\,d\theta=-\cot\theta=-\frac{\sqrt{(1-x^2)}}{x}.$$

2. *Evaluate* $\displaystyle\int_0^1 x\sqrt{\left(\frac{1-x^2}{1+x^2}\right)}dx.$

We put $x^2=\cos\theta \Rightarrow 2x\,dx=-\sin\theta\,d\theta.$

Now, $\theta=\pi/2$ when $x=0$ and $\theta=0$ when $x=1$. Therefore the given integral

$$=-\int_{\frac{1}{2}\pi}^{0}\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}\frac{\sin\theta}{2}\,d\theta$$

$$=-\int_0^{\frac{1}{2}\pi}\frac{\sin(\theta/2)}{\cos(\theta/2)}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$$

$$=-\int_0^{\frac{1}{2}\pi}\sin^2\frac{\theta}{2}\,d\theta=\frac{1}{2}\int_0^{\frac{1}{2}\pi}(1-\cos\theta)\,d\theta=\frac{\pi-2}{4}$$

3. ***Evaluate*** $\displaystyle\int \frac{dx}{(1+x^2)^{3/2}}$.

Putting $x = \tan\theta$, we see that the given integral

$$= \int \cos\theta \, d\theta = \sin\theta = \frac{x}{\sqrt{(1+x^2)}}.$$

Exercises

1. Evaluate

(i) $\displaystyle\int \frac{1}{x^2\sqrt{(1+x^2)}}\,dx$, (ii) $\displaystyle\int \frac{dx}{(4+x^2)^{3/2}}$.

(iii) $\displaystyle\int \frac{1}{(2ax+x^2)^{3/2}}\,dx$, (iv) $\displaystyle\int \frac{dx}{(x^2+2x+10)^{3/2}}$,

2. (i) $\displaystyle\int x\sqrt{\left(\frac{1-x}{1+x}\right)}dx$, (ii) $\displaystyle\int \frac{\sqrt{(x^2-a^2)}}{x}\,dx$,

(iii) $\displaystyle\int_\alpha^\beta \frac{dx}{\sqrt{[(x-\alpha)(\beta-x)]}}$, $(\beta > \alpha)$,

(iv) $\displaystyle\int \frac{dx}{(x^2-a^2)^{3/2}}$, (v) $\displaystyle\int \frac{(1+x\cos\alpha)\,dx}{(1+2x\cos\alpha+x^2)^{3/2}}$.

3. Show that

$$\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{(1-x^2)}} = \frac{1}{2}\log(2+\sqrt{3}).$$

Answers

1. (i) $-\dfrac{\sqrt{(1+x^2)}}{x}$. (ii) $\dfrac{1}{4}\,\dfrac{x}{\sqrt{(4+x^2)}}$.

(iii) $-\dfrac{1}{a^2}\cdot\dfrac{a+x}{\sqrt{(2ax+x^2)}}$. (iv) $\dfrac{1}{9}\cdot\dfrac{x+1}{\sqrt{(x^2+2x+10)}}$.

2. (i) $\frac{1}{2}\cos^{-1}x+\frac{1}{2}(x-2)\sqrt{(1-x^2)}$.

(ii) $a\sqrt{(x^2-a^2)} - a\cos^{-1}(a/x)$. (iii) π.

(iv) $-\dfrac{x}{a^2\sqrt{(x^2-a^2)}}$. (v) $\dfrac{x}{\sqrt{(x^2+2x\cos\alpha+1)}}$

Example

If m be a positive integer, find a reduction formula for

$$\int x^m\sqrt{(2ax-x^2)}\,dx.$$

Hence obtain the value of

$$\int_0^{2a} x^3\sqrt{(2ax-x^2)}\,dx.$$

We have $\quad x^m(2ax-x^2)^{\frac{1}{2}} = x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}$.

We shall connect

$$\int x^{m+\frac{1}{2}}\,(2a-x)^{\frac{1}{2}}\,dx \text{ with } \int x^{m+\frac{1}{2}-1}\,(2a-x)\,dx$$

$$\text{i.e., } \int x^{m-\frac{1}{2}}\,(2a-x)^{\frac{1}{2}}\,dx.$$

We take

$$P = x^{m-\frac{1}{2}+1}(2a-x)^{\frac{1}{2}+1} = x^{m+\frac{1}{2}}(2a-x)^{\frac{3}{2}}$$

$$\Rightarrow \frac{dP}{dx} = (m+\tfrac{1}{2})x^{m-\frac{1}{2}}\,(2a-x)^{\frac{3}{2}} - \tfrac{3}{2}\,x^{m+\frac{1}{2}}\,(2a-x)^{\frac{1}{2}}$$

$$= (m+\tfrac{1}{2})x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}\,(2a-x) - \tfrac{3}{2}\,x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}$$

$$= (2m+1)ax^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}} - (m+\tfrac{1}{2})x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}$$

$$- \tfrac{3}{2}x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}$$

$$= (2m+1)ax^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}} - (m+2)x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}.$$

Integrating, we get

$$P = (2m+1)a\int x^{m-\frac{1}{2}}\,(2a-x)^{\frac{1}{2}}\,dx - (m+2)$$

$$\int x^{m+\frac{1}{2}}\,(2a-x)^{\frac{1}{2}}\,dx.$$

This gives on transposition, etc.

$$\int x^m\sqrt{(2ax-x^2)}\,dx = -\frac{x^{m+\frac{1}{2}}(2a-x)^{\frac{3}{2}}}{m+2}$$

$$+\frac{(2m+1)a}{m+2}\int x^{m-1}\sqrt{(2ax-x^2)}dx.$$

Taking limits from 0 to $2a$, we get

$$I_m = \int_0^{2a} x^m\sqrt{(2ax-x^2)}\,dx$$

$$= \frac{(2m+1)\,a}{m+2}\int_0^{2a} x^{m-1}\sqrt{(2ax-x^2)}\,dx$$

$$= \frac{(2m+1)\,a}{m+2}\,I_{m-1}.$$

$$I_3 = \frac{7a}{5}\,I_2,$$

$$I_2 = \frac{5a}{4}\,I_1.$$

$$I_1 = \frac{3a}{3}\,I_0.$$

Also
$$I_0 = \int_0^{2a} \sqrt{(2ax-x^2)}\, dx$$
$$= \int_0^{2a} \sqrt{[a^2-(x-a)^2]}\, dx$$
$$= \left| \frac{(x-a)\sqrt{[a^2-(x-a)^2]}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} \right|_0^{2a}$$
$$= \pi a^2/2$$
$$\therefore \quad I_3 = \tfrac{7}{8}\pi a^5.$$

Exercises

1. If n be a positive integer, find a reduction formula for
$$\int (a^2+x^2)^{n/2} dx,$$
and apply it to evaluate
$$\int (a^2+x^2)^{5/2}\, dx.$$

2. If $u_n = \int x^n \sqrt{(a^2-x^2)}\, dx$, prove that
$$u_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{n+2} + \frac{n-1}{n+2}\, a^2 u_{n-2}.$$
Evaluate $\displaystyle\int_0^a x^4 \sqrt{(a^2-x^2)}\, dx.$

3. Obtain suitable formulae of reduction for the integrals of
$$\frac{x^n}{\sqrt{(2ax-x^2)}} \quad \text{and} \quad \frac{1}{x^n\sqrt{(x^2-1)}}$$
and integrate each of them when $n = 3$.

4. If $I_n = \int x^n \sqrt{(a-x)}\, dx$, prove that
$$(2n+3)\, I_n = 2anI_{n-1} - 2x^n\, (a-x)^{3/2}$$
Evaluate $\displaystyle\int_0^a x^2 \sqrt{(ax-x^2)}\, dx.$

Answers

1. $\displaystyle\int (a^2+x^2)^{n/2} dx = \frac{x(a^2+x^2)^{n/2}}{n+1} + \frac{na^2}{n+1}\int (a^2+x^2)^{\frac{n-1}{2}} dx.$

$\int (a^2+x^2)^{5/2} dx = \frac{1}{48}\, x(8x^4+26a^2x^2+33a^4)\sqrt{(a^2+x^2)} + \frac{5}{16}a^6 \sinh^{-1}(x/a).$

2. $\pi a^6/32.$

3. $\displaystyle nI_n = -\frac{x^n(2a-x)}{\sqrt{(2ax-x^2)}} + (2n-1)\, aI_{n-1};$

$\displaystyle I_3 = \frac{2x^4+ax^3+5a^2x^2-30a^3x}{6\sqrt{(2ax-x^2)}} + \frac{5a^3}{2}\sin^{-1}\frac{x-a}{a}.$

$(n-1)\, I_n = \sqrt{(x^2-1)}/x^{n-1} + (n-2)\, I_{n-2};$

$I_3 = \sqrt{(x^2-1)}/2x^2 + \frac{1}{2}\sec^{-1} x;$

4. $5\pi a^4/128.$

EXERCISES ON CHAPTER 5

1. Evaluate

(i) $\int \frac{x\,dx}{\sqrt{(x+1)}}$, (ii) $\int \frac{2x^2+3}{\sqrt{(3-2x-x^2)}}\,dx$,

(iii) $\int \frac{dx}{(x-1)\sqrt{(x^2+2x+3)}}$, (iv) $\int \frac{dx}{(1+x)^2\sqrt{x}}$,

(v) $\int \sqrt{[(x-3)(4-x)]}\,dx$, (vi) $\int \frac{\sqrt{(1+x+x^2)}}{x+1}\,dx$.

2. Evaluate

(i) $\int x^3\sqrt{(a+bx^2)}\,dx$, (ii) $\int \frac{dx}{x\sqrt{[x(x-a)]}}$,

(iii) $\int \frac{2a+x}{a+x}\sqrt{\left(\frac{a-x}{a+x}\right)}\,dx$, (iv) $\int x^3\sqrt{\left(\frac{1+x^2}{1-x^2}\right)}\,dx$,

(v) $\int_1^\infty \frac{dx}{(x-\cos\alpha)\sqrt{(x^2-1)}}$, when $\pi > \alpha > 0$.

3. Show that

$$\int_1^2 \frac{dx}{(x+1)\sqrt{(x^2-1)}} = \frac{1}{\sqrt{3}}.$$

4. Evaluate

(i) $\int \frac{x\,dx}{(a-x)^2\sqrt{(2ax-x^2)}}$, (ii) $\int \frac{1-\sqrt{(1-x^2)}}{x\sqrt{(1-x^2)}}\,dx$.

5. Prove that

$$\int_a^\infty \frac{dx}{x^4\sqrt{(a^2+x^2)}} = \frac{2-\sqrt{2}}{3a^4}, \text{ when } a > 0.$$

6. With the help of the substitution $x=1/\sqrt{(t^2+1)}$, or otherwise, prove that

$$\int_0^\infty \frac{dx}{(9+25x^2)\sqrt{(1+x^2)}} = \frac{1}{12}\tan^{-1}\frac{4}{3}.$$

7. Show that

$$\int \frac{dx}{(x^2+6x+5)^{3/2}} = -\frac{x+3}{4\sqrt{(x^2+6x+5)}}.$$

8. Prove that the value of $\int_0^4 x^3\sqrt{(4x-x^2)}\,dx$ is nearly 88.

9. Prove that

(i) $\int_0^a \frac{dx}{x+\sqrt{(a^2-x^2)}} = \frac{1}{4}\pi.$

(ii) $\displaystyle\int_0^a \frac{a\,dx}{[x+\sqrt{(a^2-x^2)}]^2}=\frac{1}{\sqrt{2}}\log(1+\sqrt{2})$.

10. Evaluate

$$\int[x+\sqrt{(1+x^2)}]^n dx.$$

11. Evaluate

(i) $\displaystyle\int\sqrt{\left(\frac{e^x+a}{e^x-a}\right)}dx$, (ii) $\displaystyle\int\sqrt{\left(\frac{x+a}{x+b}\right)}\frac{dx}{x+c}$.

12. Integrate

(i) $\displaystyle\frac{1-x^2}{(1+x^2)\sqrt{(1+x^4)}}$ (ii) $\displaystyle\frac{x^4-1}{x^2\sqrt{(x^4+x^2+1)}}$

Answers

1. (i) $\frac{2}{3}(x-2)\sqrt{(1+x)}$. (ii) $9\sin^{-1}[(x+1)/2]+(3-x)\sqrt{(3-2x-x^2)}$.
(iii) $-\sqrt{\frac{1}{3}}\sinh^{-1}[(x+2)\sqrt{2}/(x-1)]$. (iv) $\tan^{-1}\sqrt{x}+\sqrt{x}/(1+x)$.
(v) $\frac{1}{4}(2x-7)\sqrt{(x-3)(4-x)}+\frac{1}{8}\sin^{-1}(2x-7)$.
(vi) $\sqrt{(x^2+x+1)}-\sinh^{-1}[(1-x)/\sqrt{3}(1+x)]-\frac{1}{2}\sinh^{-1}[(2x+1)/\sqrt{3})]$.

2. (i) $(3bx^2-2a)(a+bx^2)^{3/2}/15b^2$ (ii) $2\sqrt{(x-a)}/a\sqrt{x}$.
(iii) $\sqrt{(a^2-x^2)}-2a\sqrt{[(a-x)/(a+x)]}$.
(iv) $\frac{1}{2}\tan^{-1}y-y(1+3y^2)/2(1+y^2)^2$, where $\sqrt{(1+x^2)}=y\sqrt{(1-x^2)}$.
(v) $(\pi-\alpha)/\sin\alpha$.

4 (i) $\displaystyle\frac{1}{a}\left[\frac{\sqrt{(2ax-x^2)}}{a-x}-\log\frac{\sqrt{(2a-x)}+\sqrt{x}}{\sqrt{(2a-x)}-\sqrt{x}}\right]$.
(ii) $\sqrt{[\log x+\operatorname{sech}^{-1}x]}$.

10. $\displaystyle\frac{[x+\sqrt{(1+x^2)}]^{n+1}}{2(n+1)}+\frac{[x+\sqrt{(1+x^2)}]^{n-1}}{2(n-1)}$.

11. (i) $\cosh^{-1}(e^x/a)+\sec^{-1}(e^x/a)$.
(ii) $\displaystyle\cosh^{-1}\frac{2x+a+b}{a-b}$
$\displaystyle\pm\sqrt{\left(\frac{c-a}{c-b}\right)}\cosh^{-1}\frac{2(c-b)(c-a)+(a+b-2c)(x+c)}{(a-b)(c+x)}$.

12. (i) $\displaystyle\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}x}{1+x^2}\right)$. (ii) $\displaystyle\frac{\sqrt{(x^4+x^2+1)}}{x}$.

Miscellaneous Exercises

1. Evaluate :

(i) $\displaystyle\int_1^2\frac{dx}{x(1+x^4)}$. (ii) $\displaystyle\int\frac{x^2\,dx}{x^4-x^2-12}$,

(iii) $\displaystyle\int\frac{x^5\,dx}{(1+x^2)^3}$ (iv) $\displaystyle\int\frac{x^5\,dx}{(a^2-x^2)^2}$,

(v) $\displaystyle\int\frac{x^2}{(1-x^2)^3}dx$, (vi) $\displaystyle\int\frac{(x^3+x^2+2)dx}{x(x-1)^2(x^2+1)}$.

2. Integrate the following :

(*i*) $e^{ax} \sin^3 x$, (*ii*) $\dfrac{1}{1-\sin x \cos x}$,

(*iii*) $\dfrac{\cos 2x}{3+\cos x+\sin x}$, (*iv*) $\dfrac{1}{\cos 2\alpha-\cos x}$.

3. Evaluate

$$\int_0^{\frac{1}{2}\pi} \frac{dx}{2 \cos x+\sin x+a},$$

for the two cases when $a = 1$ and $a = 3$.

4. (*i*) Show that if $0 < \alpha < \pi$,

$$\int_0^\infty \frac{dx}{x^2+2x \cos \alpha+1} = \frac{\alpha}{\sin \alpha},$$

(*ii*) Evaluate

$$\int_0^\infty \frac{(x^2+1)\, dx}{x^4+2x^2 \cos \alpha+1} \text{ and } \int_0^\infty \frac{dx}{x^4+2x^2 \cos \alpha+1},$$

for the same range of α.

5. Show that

$$\int_0^1 \frac{dx}{\sqrt{x}+\sqrt{(2-x)}} = \sqrt{2}-\log(\sqrt{2}+1).$$

6. Find the value of

$$\int_0^4 x^2(4x-x^2)^{1/2}\, dx.$$

7. Prove that

$$\int \frac{x^2\, dx}{(x \sin x+\cos x)^2} = \frac{\sin x-x \cos x}{\cos x+x \sin x}.$$

[Write $\quad \dfrac{x^2\, dx}{(x \sin x+\cos x)^2} = \dfrac{x \cos x}{(x \sin x+\cos x)^2} \cdot x \sec x,$

and integrate by parts, remembering that $x \cos x$, is the derivative of $x \sin x+\cos x$].

8. Prove that $\displaystyle\int \frac{\sin x-\cos x}{\sqrt{\sin 2x}}\, dx = -\cosh^{-1}(\cos x+\sin x)$.

9. Evaluate $\int (\tan x+\sec x)^4 \sec^2 x\, dx$.

10. Show that $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{a\, d\theta}{a^2+\sin^2 \theta} = \frac{\pi}{2\sqrt{(a^2+1)}}$.

11. If $a > b > 0$, show that

$$\int_0^\pi \frac{dx}{a^2+b^2-2ab \cos x} = \frac{\pi}{a^2-b^2}.$$

What is the value of integral if $b > a > 0$?

12. Show that, if $c > a > 0$,

$$\int_0^a \frac{\sqrt{(a^2-x^2)}}{(c^2-x^2)}\,dx = \frac{\pi[c-\sqrt{(c^2-a^2)}]}{2c}.$$

13. Prove that

$$\int_0^1 \frac{dx}{(1+x)(2+x)\sqrt{[x(1-x)]}} = \pi\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}}\right).$$

[Put $x = \sin^2 t$].

14. Prove that, if $a > b > 0$,

$$\int_0^\pi \frac{\sin^2 x\,dx}{a^2-2ab\cos x+b^2} = \frac{\pi}{2a^2}.$$

What is the value of the integral if $b > a > 0$?

15. Evaluate :

(*i*) $\displaystyle\int \frac{x^4\,dx}{(1+x^2)^2}$, (*ii*) $\displaystyle\int \frac{x\,dx}{(x^3-1)^2}$,

(*iii*) $\displaystyle\int \frac{dx}{(x^3-a^3)^2}$, (*iv*) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2(x^2+b^2)}$.

16. Evaluate $\displaystyle\int_{-1}^{+1} (1-x^2)^m(1-x)^n\,dx$.

17. Evaluate $\displaystyle\int \frac{\sin x}{\sin 4x}\,dx$.

18. Evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin 4\theta}{\sin\theta}\,d\theta$.

19. By means of the substitution

$$\sqrt{(1+x^4)} = (1+x^2)\cos\theta,$$

or otherwise, prove that

$$\int_0^1 \frac{1-x^2}{1+x^2}\,\frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}.$$

20. Prove that $\displaystyle\int_0^{\frac{1}{2}\pi} \log\sin x\,dx = \frac{\pi}{2}\log\frac{1}{2}$ and deduce that

$$\int_0^1 \frac{x^2\log x}{\sqrt{(1-x^2)}}\,dx = \frac{\pi}{4}\left(\frac{1}{2}+\log\frac{1}{2}\right).$$

21. Evaluate $\displaystyle\int_0^{\frac{1}{4}\pi} (\pi x-4x^2)\log(1+\tan x)\,dx$.

22. Evaluate $\displaystyle\int_0^{\pi} \theta\sin\theta\cosh(\cos\theta)\,d\theta$.

23. Evaluate

$$\int \frac{\sin 2\theta}{a-b\tan\theta}\,d\theta.$$

24. Show that

$$\int_0^{\pi} \theta^2 \log \sin \theta \, d\theta = \frac{3\pi}{2} \int_0^{\pi} \theta^2 \log (\sqrt{2} \sin \theta) \, d\theta.$$

(Use § 4·92, p. 102).

25. Show that

$$\int_0^1 x^{\alpha-1} (\log x)^n \, dx = \frac{(-1)^n \, n!}{\alpha^{n+1}},$$

where n is a positive integer and α is a positive number.

26. Evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} (\sqrt{\sin x} + \sqrt{\cos x})^{-4} \, dx.$

27. If $y^2 = ax^2 + 2bx + c,$

and $\displaystyle u_n = \int \frac{x^n}{y} dx,$

prove that

$$(n+1)au_{n+1} + (2n+1)bu_n + ncu_{n-1} = x^n y,$$

and deduce that

$$au_1 = y - bu_0 \, ; \, 2a^2u_2 = y(ax - 3b) - (ac - 3b^2) \, u_0.$$

28. Show that

$$\int_0^1 \cot^{-1} (1 - x + x^2) \, dx = \frac{\pi}{2} - \log 2.$$

29. Show that

(i) $\displaystyle\int_0^{\frac{1}{2}\pi} \sin \theta \log \sin \theta \, d\theta = \log \frac{2}{e},$

(ii) $\displaystyle\int_0^{\pi} \cos 2x \log \sin x \, dx = -\frac{1}{2} \pi.$

30. By means of the substitution

$$v^2 = x + 1 + 1/x,$$

evaluate

$$\int \frac{x-1}{x+1} \frac{dx}{\sqrt{[x(x^2+x+1)]}}.$$

31. Evaluate $\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin 2\theta}{\sqrt{(1 + \sin \theta \sin 2\beta)}} \, d\theta,$

when $0 \leqslant \beta \leqslant \frac{1}{4}\pi.$

32. If $u_n = \int (\sin x + \cos x)^n \, dx$, then

$$nu_n = (\sin x + \cos x)^{n-1} (\sin x - \cos x) + 2(n-1)u_{n-2}.$$

33. If $I_{(m, n)} = \int_0^{\frac{1}{2}\pi} \sin^m x \cos nx \, dx,$

and $\quad J_{(m, n)} = \int_0^{\frac{1}{2}\pi} \sin^m x \sin nx \, dx, \; m > 1,$

prove that

$$(m+n) \, I_{(m, n)} = \sin\left(\tfrac{1}{2}n\pi\right) - m \, J_{(m-1, n-1)}$$

and express $I_{(m, n)}$ in terms of $I_{(m-2, n-2)}$.

34. Show how, by the differentiation of

$$\sin x/(a+b\cos x)^{n-1},$$

to obtain a formula of reduction for

$$\int \frac{dx}{(a+b\cos x)^n}.$$

35. If m and n are positive integers, and

$$I_{(m, n)} = \int_0^1 x^m (1-x)^n \, dx,$$

then

$$(m+n+1) \, . \, I_{(m, n)} = n \, I_{(m, n-1)}.$$

Deduce that

$$I_{(m, n)} = \frac{m \, ! \, n \, !}{(m+n+1) \, !}.$$

36. If m and n are positive integers, then, with the help of the substitution

$x = a+(b-a)\, y$, prove that

$$\int_a^b (x-a)^m (b-x)^n \, dx = \frac{m \, ! \, n \, ! \, . \, (b-a)^{m+n+1}}{(m+n+1) \, !}.$$

37. If $I_{(m, n)} = \int \frac{x^m \, dx}{(x^2+1)^n}$, prove that

$$(2n-2) \, I_{(m, n)} = -x^{m-1} (x^2+1)^{-n+1} + (m-1) \, I_{(m-2, n-1)}.$$

Answers

1. (*i*) $\frac{1}{4} \log \frac{32}{17}$. $\qquad$ (*ii*) $\frac{1}{7}\left[\log \frac{x-2}{x+2} + \sqrt{3} \tan^{-1} \frac{x}{\sqrt{3}}\right]$

(*iii*) $\frac{1}{2} \log (1+x^2) + (1+x^2)^{-1} - \frac{1}{4} (1+x^2)^{-2}$.

(*iv*) $a^2 \log (a^2-x^2) + x^2(2a^2-x^2)/2(a^2-x^2)$.

(*v*) $\frac{1}{16}\left[\frac{4x}{(1-x^2)^2} - \frac{2x}{1-x^2} + \log \frac{1-x}{1+x}\right]$.

(*vi*) $2 \log x - \frac{3}{2} \log (x-1) - \frac{1}{4} \log (1+x^2) - 2(x-1)^{-1} + \frac{1}{2} \tan^{-1} x$

2. (*i*) $\frac{e^{mx}}{4}\left[\frac{3}{\sqrt{(m^2+1)}} \sin (x - \cot^{-1} m) - \frac{1}{\sqrt{(m^2+9)}} \sin\left(3x - \tan^{-1} \frac{3}{m}\right)\right]$.

(*ii*) $\frac{2}{\sqrt{3}} \tan^{-1} \left[\frac{2 \tan x - 1}{\sqrt{3}}\right]$.

(*iii*) $(\sin x+\cos x)-3\log(3+\sin x+\cos x)$.

(*iv*) $\dfrac{1}{\sin 2\alpha}\log\dfrac{\sin(\frac{1}{2}x-\alpha)}{\sin(\frac{1}{2}x+\alpha)}$.

3. $\frac{1}{2}\log 3$; $\tan^{-1}\frac{1}{2}$. 4. (*ii*) $\frac{1}{2}\pi\sec(\frac{1}{2}\alpha)$, $\frac{1}{4}\pi\sec(\frac{1}{2}\alpha)$.

6. 10π. 9. $\frac{1}{10}(\tan x+\sec x)^5+\frac{1}{6}(\tan x+\sec x)^3$.

11. $\pi/(b^2-a^2)$. 14. $\pi/2b^3$.

15. (*i*) $x+\frac{1}{2}x(1+x^2)^{-1}-\frac{3}{2}\tan^{-1}x$.

(*ii*) $\dfrac{1}{18}\log\dfrac{x^2+x+1}{(x-1)^2}-\dfrac{1}{9}\cdot\dfrac{1}{x-1}-\dfrac{1}{9}\cdot\dfrac{2x+1}{x^2+x+1}-\dfrac{1}{3\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$.

(*iii*) $\dfrac{1}{9a^5}\left[\log\dfrac{a^2+ax+x^2}{(x-a)^2}-\dfrac{x}{x-a}+\dfrac{2ax+x^2}{a^2+ax+x^2}+\dfrac{6}{\sqrt{3}}\tan^{-1}\dfrac{2x+a}{\sqrt{3}a}\right]$.

(*iv*) $\pi(2a+b)/4a^3b(a+b)^2$.

16. $2^{2m+n+1}(m+n)!\,.\,(m!)/(2m+n+1)!$.

17. $\frac{1}{4}\log\tan(\frac{1}{4}\pi-\frac{1}{2}x)+(1/4\sqrt{2})\log\{(1+\sqrt{2}\sin x)/(1-\sqrt{2}\sin x)\}$.

18. $4/3$. 21. $\pi^3\log 2/192$. 22. $\frac{1}{2}\pi(e-e^{-1})$.

23. $-\dfrac{2ab^2}{(a^2-b^2)^2}\log(a\cos\theta-b\sin\theta)+\dfrac{b(a^2-b^2)}{(a^2+b^2)^2}\,\theta-\dfrac{(a\cos\theta+b\sin\theta)\cos\theta}{a^2+b^2}$.

26. $1/3$.

30. $-2\operatorname{cosec}^{-1}(x^{1/2}+x^{-1/2})$.

31. $-\frac{4}{3}\tan\beta\sec\beta$, if $0<\beta\leqslant\frac{1}{4}\pi$.
$-\frac{4}{3}\cot\beta\operatorname{cosec}\beta$, if $\frac{1}{4}\pi\leqslant\beta<\frac{1}{2}\pi$;
0, if $\beta=0$ or $\frac{1}{2}\pi$.

33. $I_{(m,n)}=\dfrac{1}{m+n}\sin\dfrac{n\pi}{2}+\dfrac{m}{(m+n)(m+n-2)}\cos\dfrac{(n-1)\pi}{2}-\dfrac{m(m-1)}{(m+n)(m+n-2)}I_{(m-2,\,n-2)}$.

34. If $u_n=\int dx/(a+b\cos x)^n$, then
$$(n-1)(a^2-b^2)\,u_n=-b\sin x/(a+b\cos x)^{n-1}+(2n-3)\,au_{n-1}-(n-2)u_{n-2}.$$

Miscellaneous Exercises II

1. Show that

(*i*) $\displaystyle\int_0^{\pi}x\,\varphi(\sin x)\,dx=\frac{\pi}{2}\int_0^{\pi}\varphi(\sin x)\,dx$,

(*ii*) $\displaystyle\int_a^{\pi-a}xf(\sin x)\,dx=\frac{\pi}{2}\int_a^{\pi-a}f(\sin x)\,dx$

2. Show that

(*i*) $\displaystyle\int_{-a}^{a}f(x)^2\,dx=2\int_0^{a}f(x)^2\,dx$,

(ii) $\int_{-a}^{a} xf(x)^2\,dx = 0.$

3. If
$$f(x+mp) = f(x)$$
for all integral values of m, prove that
$$\int_0^{np} f(x)\,dx = n\int_0^{\pi} f(x)\,dx,$$
where n is a positive or a negative integer.

4. Show that
$$\int_0^{n\pi} f(\cos^2 x)\,dx = n\int_0^{\pi} f(\cos^2 x)\,dx,$$
n being a positive integer.

5. Show that

(i) $\int_0^{b-c} f(x+c)\,dx = \int_c^b f(x)\,dx,$

(ii) $\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx.$

6. $f(x) = \frac{1}{2}a_0 + \sum_{m=1}^{n}(a_m \cos mx + b_m \sin mx)$, and $m \leqslant n$, then

$\pi a_0 = \int_0^{2\pi} f(x)\,dx\,;\ \pi a_m = \int_0^{2\pi} f(x)\cos mx\,dx,$

$\pi b_m = \int_0^{2\pi} f(x)\sin mx\,dx.$

7. If m, n are positive integers and $m > n$, prove that
$$\int_0^{\pi} \cos mx\,(\cos x)^n\,dx = 0.$$

8. Show that
$$\int_0^{\pi} \frac{\sin (n+\frac{1}{2})\,x}{\sin \frac{1}{2}x}\,dx = \pi\,;\ n \text{ being a positive integer.}$$
$[\frac{1}{2}\sin (n+\frac{1}{2})\,x \operatorname{cosec} \frac{1}{2}x = \frac{1}{2} + \cos x + \ldots + \cos nx].$

9. If p, q are positive integers,
$$\int_0^{\pi} \cos px \sin qx\,dx = \begin{cases} 2q/(q^2-p^2), & \text{if } (q-p) \text{ is odd,} \\ 0 & \text{if } (q-p) \text{ is even.}\end{cases}$$

10. Prove that if
$$u_n = \int_0^1 x^n \tan^{-1} x\,dx$$
when
$$(n+1)u_n + (n-1)u_{n-2} = -1/n.$$
Evaluate u_{4m} and prove that $4mu_{4m} \to \frac{1}{4}\log 2$ as $m \to \infty$.

11. If

$$I_n = \int_{-1}^{+1} (1-x^2)^n \cos mx \, dx,$$

prove that when n is not less than or equal to 2,

$$m^2 I_n = 2n(2n-1) I_{n-1} - 4n(n-1) I_{n-2}.$$

12. Prove that

$$\int f^n(x)\, g(x)\, dx = f^{n-1}(x)\, g(x) - f^{n-2}(x)\, g'(x) + \ldots$$
$$+ (-1)^n \int f(x)\, g^n(x)\, dx.$$

13. If

$$I_n = \int_0^{\pi} (\pi x - x^2)^n \cos mx \, dx,$$

prove that, if $n > 1$ and is an integer, then

$$m^2 I_n - 2n(2n-1) I_{n-1} + n(n-1)\, \pi^2 I_{n-2} = 0.$$

Prove that

$$\int_0^{\pi} (\pi x - x^2)^3 \cos 2x \, dx = \tfrac{3}{4}\pi(\pi^2 - 15).$$

14. Show that

$$\int \sin n\theta \sec \theta \, d\theta = -2 \cos(n-1)\theta/(n-1) - \int \sin(n-2)\,\theta \sec \theta \, d\theta.$$

15. Prove that

(*i*) $\displaystyle\int_{-a}^{+a} f(x) f(-x)\, dx = 2\int_0^{\infty} f(x) f(-x)\, dx,$

(*ii*) $\displaystyle\int_0^{\infty} \frac{dx}{(1+4x^2)\sqrt{(1+3x^2)}} = \frac{\pi}{6},$

(*iii*) $\displaystyle\int_0^{\infty} \frac{dx}{(1+x)\sqrt{(1+x+x^2)}} = \log 3.$

16. Find the reduction formula for $I_{m,\,n}$ where

$$I_{m,\,n} = \int_0^{\frac{1}{2}\pi} \cos^m x \sin nx \, dx.$$

Deduce that

$$I_{m,\,n} = \frac{1}{2^{m+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \ldots + \frac{2^m}{m}\right].$$

17. Evaluate the following :

(*i*) $\displaystyle\int_0^1 \frac{\sin^{-1} x}{x}\, dx,$ (*ii*) $\displaystyle\int \frac{1+\cos x}{\sin x \cos x}\, dx,$

(*iii*) $\displaystyle\int \frac{x}{1+\sin x}\, dx,$ (*iv*) $\displaystyle\int \frac{x+\sin x}{1+\cos x}\, dx,$

(*v*) $\displaystyle\int \left(\frac{\tan x^{-1}}{x}\right)^2 dx,$ (*vi*) $\displaystyle\int \frac{dx}{x\sqrt{(x^n-1)}},$

(*vii*) $\int \frac{dx}{(2\sin x+\cos x)^2}$ (*viii*) $\int \log[\sqrt{(x-a)}+\sqrt{(x-b)}]dx$,

(*ix*) $\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\, d\theta}{\sin^2\theta+4\cos^2\theta}$ (*x*) $\int_0^{\frac{1}{2}\pi} \cos^3 x \sin 5x\, dx$.

18. Prove that

$$\int_0^1 \frac{\sqrt{(1-x^2)}}{1-x^2\sin^2\alpha}\, dx = \frac{\pi}{4\cos^2\frac{1}{2}\alpha}.$$

19. Show that

(*i*) $\int \operatorname{sech}^3 x\, dx = \frac{1}{2}\sec^{-1}\cosh x+\frac{1}{2}\operatorname{sech} x \tanh x$.

(*ii*) $\int \frac{1}{(e^x-1)^2}\, dx = x-\log(e^x-1)-\frac{1}{e^x-1}$

20. Prove that

$$\int_0^{\frac{1}{2}\pi} \cos^{n-2} x \sin nx\, dx = \frac{1}{n-1}, \ (n > 1 \text{ and integral}).$$

Answers

17. (*i*) $\frac{1}{2}\pi\log 2$. (*ii*) $\log(\tan x \tan\frac{1}{2}x)$.

(*iii*) $x(\tan\frac{1}{2}x-1)/(\tan\frac{1}{2}x+1)+2\log(\cos\frac{1}{2}x+\sin\frac{1}{2}x)$.

(*iv*) $x\tan\frac{1}{2}x$. (*v*) $x^{-1}-\tan(x^{-1})$.

(*vi*) $(2/n)\tan^{-1}\sqrt{(x^n-1)}$. (*vii*) $-\cos(x/2)(2\sin x+\cos x)$.

(*viii*) $x\log[\sqrt{(x-a)}+\sqrt{(x-b)}]-\frac{1}{2}\sqrt{(x-a)(x-b)}$
$-\frac{1}{2}(a+b)\cosh^{-1}\{[2x-(a+b)]/(a-b)\}$.

(*ix*) $\pi/6$. (*x*) $1/4$.

6

Definite Integral as the Limit of a Sum

6·1 We have so far regarded integration as inverse of differentiation. The definite integral

$$\int_a^b f(x)\,dx$$

has been defined to be equal to

$$F(b)-F(a)$$

where $F(x)$ is a function such that

$$F'(x)=f(x).$$

The application of the process of integration to determining areas, lengths of arcs, volumes and surfaces of solids of revolution has also been based on this anti-derivative aspect of the notion of definite integral.

It will now be shown that a definite integral can also be represented as the limit of the sum of certain number of terms, when the number of terms tends to infinity and each term tends to zero. This aspect of a definite integral is more fundamental and of far more reaching character that that of its anti-derivative character and it is this aspect of integration which really lies at the basis of all its applications. On the theoretical side also, it has led to very extensive and refined developments of the subject. But it is not within the scope of this elementary book to go into these developments. Here the purpose is just to introduce the reader to this aspect. But the beginner should not minimise the importance of this *summation aspect of the integral*, as he may feel inclined to do in view of the comparatively short space devoted to it in this book.

6·2. Fundamental Theorem of Integral Calculus. Definite integral as the limit of a sum. *Let the interval* $[a, b]$, *be divided into* n *equal parts and let the length of each part be,* h, *so that* $nh = b-a$; *then*

$$\int_a^b f(x)dx = \lim h[f(a+h)+f(a+2h)+\ldots\ldots+f(a+nh)],$$

when

$$n \to \infty, h \to 0 \text{ and } nh = b-a$$

Here

$$a, a+h, a+2h, \ldots\ldots, a+(r-1)h, \ldots\ldots, a+(n-1)h, a+nh$$

are the points of division obtained when the interval $[a, b]$ is divided into n equal parts ; h being the length of each part.

We suppose that the function $f(x)$ monotonically increases from a to b.

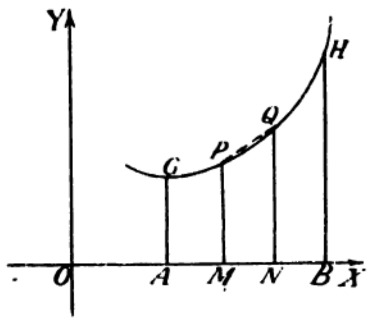

Fig. 9.

Let G, H be points on the curve $y = f(x)$ with abscissae a, b respectively,

Draw GA, HB $\perp$ X-axis.

Let M, N be the points of division of the rth strip MN corresponding to the division of the interval $[a, b]$ into n equal parts so that

$OM = a+(r-1)h$, $ON = a+rh$;

we have

$MP = f[a+(r-1)h]$, $NQ = f(a+rh)$;

We write

$$S_n = h[f(a+h)+f(a+2h)+\ldots\ldots+f(a+nh)].$$

Complete the rectangles MQ, NP.

The sum of the areas of the rectangles of the type NP which are *inscribed* in the area $GABH$

$$= h\sum_{r=1}^{r=n} f[a+(r-1)h]$$

$$= h[f(a)+f(a+h)+\ldots\ldots+f(a+(n-1)h)]$$

$$= S_n - h[f(a+nh)-f(a)]$$

$$= S_n - h[f(b)-f(a)], \text{ for } nh = b-a. \qquad \ldots(i)$$

The sum of the areas of the rectangles of the type MQ which are *circumscribed* to the area $GABH$

$$= \sum_{r=1}^{r=n} h f(a+rh)$$

$$= h\,[f(a+h)+f(a+2h)+\ldots\ldots+f(a+nh)]$$
$$= S_n \qquad \ldots(ii)$$

The area of the region $GABH = \int_a^b f(x)\,dx.$ $\qquad \ldots(iii)$

From the figure, we see that the area of the region $GABH$ lies between the sums of the areas of the two sets of rectangles (i) and (ii).

Thus we have

$$S_n - h[f(b)-f(a)] < \int_a^b f(x)dx < S_n$$

$$\Rightarrow \qquad 0 < S_n - \int_a^b f(x)\,dx < h[f(b)-f(a)].$$

Let $h \to 0$ so that the two extremes tend to 0. Therefore

$$\lim \left[S_n - \int_a^b f(x)\,dx \right] = 0$$

$$\Rightarrow \qquad \lim S_n = \int_a^b f(x)\,dx.$$

Thus we have proved the result for the case when $f(x)$ *increases* as x increases from a to b. It may similarly be shown that the result remains true in the case when $f(x)$ monotonically *decreases* as x increases from a to b. In this case the directions of the inequalities in (iv) and (v) wiil be reversed.

Note 1. The above proof is *geometrical* in nature, the *rigorous analytical* proof being beyond the scope of this book.

Note 2. The result of the theorem can be concisely stated as follows :—

$$\int_a^b f(x)\,dx = \lim \sum_{r=1}^{r=n} hf(a+rh),$$

when $h \to 0$, $n \to \infty$; $nh = b-a$.

Note 3. It should be noted that, since each term in $\Sigma\; hf(a+rh)$ tends to 0, the addition or omission of a certain finite number of terms of the similar type will not alter the limit.

Examples

1. *Evaluate* $\int_a^b x^2\,dx$ *as the limit of a sum.*

Here $f(x) = x^2$.

$$\therefore \quad \int_b^a x^2\,dx$$

$$= \lim h[f(a+h)+f(a+2h)+\ldots\ldots+f(a+nh)]$$

when $h \to 0$, $n \to \infty$ and $nh = b-a$,

$$= \lim h[(a+h)^2+(a+2h)^2\ldots\ldots+(a+nh)^2]$$

$$= \lim h[na^2+2ah(1+2+\ldots\ldots+n)+h^2(1^2+2^2+\ldots+n^2)]$$

$$= \lim h[na^2+n(n+1)ah+\tfrac{1}{6}n(n+1)(2n+1)h^2]$$

$$= \lim [nha^2+nh(nh+h)a+\tfrac{1}{6}nh(nh+h)(2nh+h)]$$

$$= \lim [(b-a)a^2+(b-a)(b-a+h)a + \tfrac{1}{6}(b-a)(b-a+h)\{2(b-a+h)\}]$$

$$= (b-a)a^2+(b-a)^2\,a+\tfrac{1}{3}(b-a)^3$$

$$= \tfrac{1}{3}\,(b^3-a^3).$$

2. *Evaluate* $\int_a^b e^x\,dx$ *as the limit of a sum.*

Here $f(x) = e^x$.

$$\therefore \quad \int_a^b e^x\,dx$$

$$= \lim h[e^{a+h}+e^{a+2h}+\ldots+e^{a+nh}],$$

when $h \to 0$, $n \to \infty$ and $nh = b-a$

$$= \lim \frac{he^{a+h}\,(1-e^{nh})}{1-e^h}$$

$$= \lim \left[\,e^a\,.\,(1-e^{b-a})e^h\,.\,\frac{h}{1-e^h}\right]$$

$$= (e^a-e^b)\,.\,\lim e^h\,.\,\lim \frac{h}{1-e^h}$$

$$= (e^a-e^b)(-1) = e^b-e^a,$$

for, $\lim \{h/(1-e^h)\} = -1$, as $h \to 0$.

3. *Evaluate* $\int_a^b \cos x\,dx$ *as the limit of a sum.*

Here $f(x) = \cos x$. Therefore

$$\int_a^b \cos x\,dx = \lim h[\cos(a+h)+\cos(a+2h)+\ldots\ldots+\cos(a+nh)].$$

Let

$$S = \cos(a+h)+\cos(a+2h)+\ldots\ldots+\cos(a+nh).$$

Multiplying both sides by $2 \sin \tfrac{1}{2}h$, we get

$$2\sin\tfrac{1}{2}h.\,S = 2\sin\tfrac{1}{2}h\cos(a+h)+2\sin\tfrac{1}{2}h\cos(a+2h)+\ldots\ldots$$
$$\ldots\ldots+2\sin\tfrac{1}{2}h\cos(a+nh)$$

$$= \sin(a+\tfrac{3}{2}h)-\sin(a+\tfrac{1}{2}h)+\sin(a+\tfrac{5}{2}h)-\sin(a+\tfrac{3}{2}h)$$
$$+\ldots\ldots+\sin[a+\tfrac{1}{2}(2n+1)h]-\sin[a+\tfrac{1}{2}(2n-1)h]$$

$$= \sin [a + \tfrac{1}{2}(2n+1)h] - \sin [a + \tfrac{1}{2}h]$$

$$= \sin [b + \tfrac{1}{2}h] - \sin [a + \tfrac{1}{2}h], \text{ for } nh = b - a.$$

Thus

$$\int_a^b \cos x\,dx = \lim \frac{h[\sin (b+\frac{1}{2}h) - \sin (a+\frac{1}{2}h)]}{2 \sin \frac{1}{2}h}$$

$$= \lim \frac{\frac{1}{2}h}{\sin \frac{1}{2}h} [\sin (b+\tfrac{1}{2}h) - \sin (a+\tfrac{1}{2}h)]$$

$$= 1 \,.\, (\sin b - \sin a) = \sin b - \sin a.$$

Exercises

Evaluate the following definite integrals as limits of sums :

1. (i) $\int_a^b x\,dx$, (ii) $\int_2^3 x^3\,dx$.
2. (i) $\int_a^b \sin x\,dx$, (ii) $\int_a^b \sin^2 x\,dx$.
3. (i) $\int_a^b \frac{1}{\sqrt{x}}dx$, (ii) $\int_a^b \frac{1}{x^2}\,dx$.
4. (i) $\int_a^b \sinh x\,dx$, (ii) $\int_a^b \cosh 2x\,dx$.

Answers

1. (i) $\frac{1}{2}(b^2-a^2)$. (ii) $65/4$.
2. (i) $\cos a - \cos b$. (ii) $\frac{1}{2}(b-a)+(\sin a \cos a - \sin b \cos b)$.
3. (i) $2(\sqrt{b}-\sqrt{a})$. (ii) $(b-a)/ab$.
4. (i) $\cosh b - \cosh a$. (ii) $\sinh b \cosh b - \sinh a \cosh a$.

6·3. Summation of Series. It is possible to express the limits of sums of certain types of series as definite integrals, and thus to evaluate them. We shall now consider the characteristics of the series the limit of whose sum can be so expressed and also learn how to obtain the corresponding definite integrals. It will also be shown that *it is always possible to so transform such a series that the lower and upper limits of the corresponding definite integral are 0 and 1 respectively.*

We have seen that when $h \to 0$, $n \to \infty$ and $nh=b-a$

$$\int_a^b f(x)\,dx = \lim h[f(a+h)+f(a+2h)+\ldots+f(a+rh)+\ldots+f(a+nh)].$$

Changing h to $(b-a)/n$, we see that when $n \to \infty$, we have

$$\int_a^b f(x)\,dx$$

$$= (b-a) \lim \frac{1}{n}\left\{f\left[a+(b-a)\frac{1}{n}\right]+f\left[a+(b-a)\frac{2}{n}\right]+\dots\right.$$

$$\left.+f\left[a+(b-a)\frac{r}{n}\right]+\dots+f\left[a+(b-a)\frac{r}{n}\right]\right\}$$

$$= (b-a) \lim \frac{1}{n} \sum_{r=1}^{r=n} f\left[a+(b-a)\frac{r}{n}\right].$$

From § 9·2, we easily see that when $n \to \infty$,

$$\lim \frac{1}{n} \sum_{r=1}^{r=n} f\left[a+(b-a)\frac{r}{n}\right] = \int_0^1 f\left[a+(b-a)x\right]dx$$

$$\therefore \quad \int_a^b f(x)\,dx = (b-a)\int_0^1 f\left[a+(b-a)x\right]dx.$$

We now notice the following :

The general term $(1/n)f[a+(b-a)(r/n)]$ of the sum

$$(1/n)\Sigma f[a+(b-a)(r/n)]$$

is a function of (r/n) such that the various terms in the series are obtained by giving values 1, 2, 3,..., n to r and also the new integrand $f[a+(b-a)x]$ is obtained by changing (r/n) to x.

Also $(1/n)$ is a factor of each term.

From above we deduce that in the case of a series the limit of whose sum can be expressed as a definite integral, *the general term is the product of $1/n$ and a function $\varphi(r/n)$ of r/n,* so that the various terms of the series can be obtained from it by changing r to 1, 2, 3, ..., n, successively.

The equivalent definite integral, then, is obtained by *changing* (r/n), *in the general term, to x and taking 0, 1 as the two limits.*

Rule. Thus it is important to remember that

$$\lim_{n\to\infty}\left[\sum_{r=1}^{r=n} \frac{1}{n}\,\varphi\left(\frac{r}{n}\right)\right] = \int_0^1 \varphi(x)\,dx.$$

Note. Here the number of terms is n but, since each term tends to 0, the addition or omission of a *finite* number of terms will not affect the required limit.

Examples

1. *Find the limit when n tends to infinity of the series*

$$\frac{n^2}{(n^2+1)^{3/2}}+\frac{n^2}{(n^2+2^2)^{3/2}}+\frac{n^2}{(n^2+3^2)^{3/2}}+\ldots\ldots+\frac{n^2}{[n^2+(n-1)^2]^{3/2}}$$

Here the general rth term

$$=\frac{n^2}{(n^2+r^2)^{3/2}}=\frac{1}{n}\left\{\frac{1}{\left[1+\left(\frac{r}{n}\right)^2\right]^{3/2}}\right\},$$

which is the product of $1/n$ and the function

$$1\Bigg/\left[1+\left(\frac{r}{n}\right)^2\right]^{3/2}$$

of r/n.

Thus, by the above rule, the required limit is equal to

$$\int_0^1\frac{1}{(1+x^2)^{3/2}}\,dx.$$

Putting $x=\tan\theta$, it can be seen that this integral is equal to $1/\sqrt{2}$.

2. *Find the limit, when n tends to infinity, of the series*

$$\frac{\sqrt{n}}{\sqrt{n^3}}+\frac{\sqrt{n}}{\sqrt{(n+4)^3}}+\frac{\sqrt{n}}{\sqrt{(n+8)^3}}+\frac{\sqrt{n}}{\sqrt{(n+12)^3}}+\cdots+\frac{\sqrt{n}}{\sqrt{[n+4(n-1)]^3}}$$

Here, rth term $=\dfrac{\sqrt{n}}{\sqrt{[n+4(r-1)]^3}}=\dfrac{1}{n}\,\dfrac{1}{\sqrt{\left[1+\dfrac{4(r-1)}{n}\right]^3}}.$

As the rth term contains $(r-1)$, we consider the $(r+1)$th term.

The $(r+1)$th term $=\dfrac{1}{n}\,\dfrac{1}{\sqrt{\left(1+\dfrac{4r}{n}\right)^3}}.$

Changing (r/n) in $1\Bigg/\sqrt{\left(1+4\dfrac{r}{n}\right)^3}$ to x, we see that the required limit

$$=\int_0^1\frac{dx}{\sqrt{(1+4x)^3}}=\int_0^1(1+4x)^{-3/2}\,dx=\frac{1}{10}(5-\sqrt{5}).$$

3. *Find the value of*

$$\lim_{n\to\infty}\frac{[(n+1)(n+2)\ldots\ldots(n+n)]^{1/n}}{n}.$$

Let $y=\dfrac{[(n+1)(n+2)\ldots\ldots(2n)]^{1/n}}{n}.$

We have

$$\log y = \frac{1}{n}\left\{\log(n+1)+\log(n+2)+\ldots\ldots+\log 2n\right\}-\log n$$

$$= \frac{1}{n}\Big\{[\log(n+1)-\log n]+[\log(n+2)-\log n]+$$

$$\ldots\ldots-+[\log(2n)-\log n]\Big\}$$

$$= \frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\ldots\ldots\right.$$

$$\left.+\log\left(1+\frac{n}{n}\right)\right\}$$

The rth term $= \dfrac{1}{n}\log\left(1+\dfrac{r}{n}\right)$.

Therefore, when n tends to infinity, we have

$$\lim \log y = \int_0^1 \log(1+x)\,dx$$

$$= \Big| x\log(1+x)\Big|_0^1 - \int_0^1 \frac{x}{1+x}\,dx$$

$$= \log 2 - \Big| x - \log(1+x)\Big|_0^1$$

$$= \log 2-(1-\log 2)$$

$$= 2\log 2-1 = \log 4-\log e = \log(4/e),$$

$$\therefore \quad \lim y = 4/e.$$

Exercises

Find the limit, when n tends to infinity, of the series

1. $\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dfrac{1}{n+3}+\ldots+\dfrac{1}{n+n}$.

2. $\dfrac{1}{n}+\dfrac{n^2}{(n+1)^3}+\dfrac{n^2}{(n+2)^3}+\ldots+\dfrac{1}{8n}$.

3. $\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{(n^2-1^2)}}+\dfrac{1}{\sqrt{(n^2-2^2)}}+\ldots+\dfrac{1}{\sqrt{[n^2-(n-1)^2]}}$.

4. $\dfrac{\sqrt{n}}{(3+4\sqrt{n})^3}+\dfrac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^3}+\dfrac{\sqrt{n}}{\sqrt{3}(3\sqrt{3}+4\sqrt{n})^3}+\ldots\ldots+\dfrac{1}{49n}$

5. $\dfrac{n+1}{n^2+1^2}+\dfrac{n+2}{n^2+2^2}+\dfrac{n+3}{n^2+3^2}+\ldots\ldots+\dfrac{1}{n}$.

6. $\dfrac{1}{\sqrt{(2n-1^2)}}+\dfrac{1}{\sqrt{(4n-2^2)}}+\dfrac{1}{\sqrt{(6n-3^2)}}+\dots+\dfrac{1}{n}$.

7. $\dfrac{1}{n}+\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dots\dots+\dfrac{1}{4n}$.

8. Show that

$$\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\dots+\frac{1}{2n}\right]=\frac{\pi}{4}.$$

Find the limit, when n tends to infinity, of the following sums :—

9. $\displaystyle\sum_{r=1}^{n-1}\frac{1}{\sqrt{(n^2-r^2)}}$.

10. $\displaystyle\sum_{r=1}^{n}\frac{\sqrt{n}}{(9n+40r)^{3/2}}$.

11. $\displaystyle\sum_{r=1}^{n}\frac{r^3}{r^4+n^4}$.

12. $\displaystyle\sum_{r=1}^{n}\frac{n^3}{(n^2+r^2)(n^2+2r^2)}$.

13. $\displaystyle\sum_{r=1}^{n-1}\frac{1}{n}\sqrt{\left(\frac{n+r}{n-r}\right)}$.

14. $\displaystyle\sum_{r=1}^{n}\frac{n}{(n+r)\sqrt{\{r(2n+r)\}}}$

15. $\displaystyle\sum_{r=1}^{n}\frac{r}{n^2}\sec^2\frac{r^2}{n^2}$.

16. $\displaystyle\frac{1}{n}\sum_{r=1}^{n}\sin^{2k}\frac{r\pi}{2n}$.

17. $\displaystyle\sum_{r=1}^{3n}\frac{n^2}{(3n+r)^3}$.

18. Find the limiting value of

$$\left[\frac{(n^3+1^3)(n^3+2^3)(n^3+3^3)\dots\dots(n^3+n^3)}{n^{3n}}\right]^{1/n},$$

when n tends to infinity.

19. Find the limit of

$$\frac{(n\,!)^{1/n}}{n}$$

when n tends to infinity.

20. Prove that the limit, when n tends to infinity, of

$$\left[1+\left(\frac{1}{n}\right)^4\right]\left[1+\left(\frac{2}{n}\right)^4\right]^{\frac{1}{2}}\left[1+\left(\frac{3}{n}\right)^4\right]^{\frac{1}{3}}\dots\dots\left[2\right]^{1/n}$$

is

$$e^{\pi^2/48}$$

21. Show that the limit, when n tends to infinity, of

$$\left[\phi(a)\,.\,\phi\left(a+\frac{h}{n}\right).\phi\left(a+\frac{2h}{n}\right)\ldots\ldots\phi\left(a+\frac{nh}{n}\right)\right]^{1/n},$$

is e^t where t is

$$\frac{1}{h}\int_a^{a+h}\log\phi(x)\,dx.$$

Deduce the limit of

$$\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\ldots\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}.$$

22. If $na = 1$ always and n tends to infinity, find the limiting value of

$$\prod_{r=1}^{n}[1+(ra)^2]^{1/r}.$$

Answers

1. $\log 2$ 2. $3/8$. 3. $\pi/2$ 4. $1/14$.
5. $\frac{1}{4}\pi+\frac{1}{2}\log 2$. 6. $\frac{1}{3}\pi$. 7. $\log 4$. 9. $\pi/2$.
10. $1/105$. 11. $\frac{1}{2}\log 2$. 12. $\sqrt{2}\tan^{-1}\sqrt{2}-\frac{1}{4}\pi$.
13. $\frac{1}{3}\pi+1$. 14. $\frac{1}{3}\pi$. 15. $\frac{1}{2}\tan 1$.
16. $(2k)!/(2^{2k})(k!)^2$. 17. $1/24$. 18. $4e^{\pi/\sqrt{3}}\ e^{-3}$.
19. $1/e$. 21. $2e^{\frac{1}{2}(\pi-4)}$. 22. $e^{\pi^2/24}$.

APPLICATIONS

7

Areas of Plane Regions

Quadrature

7·1. It has been shown in § 1·8 p. 11, that the *area bounded by a curve* $y = f(x)$, *the axis of* x *and two ordinates* $x = a$ and $x = b$ *is given by the definite integral*

$$\int_a^b f(x)\, dx.$$

It can similarly be shown that the *area bounded by a curve* $x = f(y)$, *the axis of* y *and the two abscissae*, $y = c$ *and* $y = d$, *is given by the definite integral*

$$\int_c^d f(y)\, dy.$$

The areas which are not situated in any of these two ways can sometimes be expressed as combination of areas which are thus situated.

The process of determining the area of a plane region is known as *Quadrature*.

We shall now consider some examples.

Note. It is always necessary to trace the curve with reference to which the region whose area is to be determined is given. The trace of the curve will enable the area to be expressed as a definite integral or a suitable combination of such integrals.

In connection with this and the following chapters on the determination of lengths of arcs and volumes and surfaces of solids of revolution, the student is strongly advised to familiarise himself with elementary aspects of the subject of curve tracing and specially with* the following curves which will occur very frequently in the following.

Cartesian Equations :

Ellipse : $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. *Hyperbola* : $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

Parabola : $y^2=4ax$, *Catenary* : $y=c\cosh(x/c)$.

Cissoid : $y^2(a-x)=x^3$. *Strophoid* : $(x^2+y^2)\,x=a^2(x^2-y^2)$

Folium : $x^3+y^3=3axy$. *Lemniscate* : $(x^2+y^2)^2=a^2(x^2-y^2)$.

Astroid : $x^{2/3}+y^{2/3}=a^{2/3} \Leftrightarrow x=a\cos^3\theta,\ y=a\sin^3\theta$.

Cycloid : $x=a(\theta-\sin\theta),\ y=a(1-\cos\theta)$

The equations of a cycloid are also often given in the forms

I $\quad x=a(\theta+\sin\theta),\quad y=a(1+\cos\theta)$,

II $\quad x=a(\theta+\sin\theta),\quad y=a(1-\cos\theta)$.

These cycloids with different equations are differently situated with respect to the co-ordinate axes.

Tractrix : $x=a(\cos t+\frac{1}{2}\log\tan^2\frac{1}{2}t),\ y=a\sin t$.

Polar Equations :

Cardioide : $r=a(1\pm\cos\theta)$, *Lemniscate* : $r^2=a^2\cos 2\theta$.

Equiangular Spiral : $r=ae^{b\theta}$. *Conic* : $l/r=1+e\cos\theta$.

Three leaved Rose : $r=a\sin 3\theta$.

Four leaved Rose : $r=a\sin 2\theta$.

Examples

1. *Find the area bounded by the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

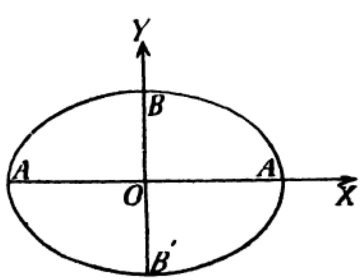

Fig. 10.

The ellipse is symmetrical about both the co-ordinate axes, so that the two axes divide it into four portions whose areas are equal. We will determine the area of one of the portion OAB lying in the first quadrant and multiply the same by four to get the area of the ellipse.

Solving the equation of the ellipse for y in terms of x, we see

*Reference may be made to the chapter (*Some important curves*) and the chapter '*Curve Tracing*' in the Author's Differential Calculus.

that for any point (x, y) on the arc AB, we have

$$y = \frac{b}{a}\sqrt{(a^2-x^2)}.$$

$$\therefore \quad \text{area } OAB = \int_0^a y\,dx = \frac{b}{a}\int_0^a \sqrt{(a^2-x^2)}\,dx$$

$$= \frac{b}{a}\left|\frac{x\sqrt{(a^2-x^2)}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{0}\right|_0^a$$

$$= \frac{b}{a}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right]=\frac{\pi}{4}ab$$

$\therefore$ area of the ellipse $= \pi ab$.

2. *Prove that the area of the region bounded by the curve*

$$a^4y^2 = x^5(2a-x),$$

is to that of the circle whose radius is, a, is 5 to 4.

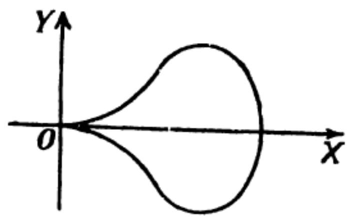

Fig. 11.

The curve consists of a loop lying between the line $x = 0$ and $x = 2a$ and is symmetrical about the x-axis.

The required area

$$= \frac{2}{a^2}\int_0^{2a} x^{5/2}\sqrt{(2a-x)}\,dx.$$

To evaluate this integral, we put $x = 2a\sin^2\theta$.

When $x = 0$, $\theta = 0$ and when $x = 2a$, $\theta = \frac{1}{2}\pi$.

$\therefore$ the required area

$$= \frac{2}{a^2}\int_0^{\frac{1}{2}\pi} (2a)^{5/2}\sin^5\theta \,.\, \sqrt{(2a)} \,.\, \cos\theta \,.\, 4a\sin\theta\cos\theta\,d\theta$$

$$= 64a^2\int_0^{\frac{1}{2}\pi} \sin^6\theta\cos^2\theta\,d\theta$$

$$= 64a^2\,\frac{5.3.1.1}{8.6.4.2.}\cdot\frac{\pi}{2}$$

$$= \frac{5a^2\pi}{4} = \frac{5}{4} \times \text{area of the circle whose radius is } a.$$

3. *Find the area enclosed between one arch of the cycloid*

$$x = a(\theta-\sin\theta),\ y = a(1-\cos\theta),$$

and its base.

To describe the first arc *OPA* of the cycloid, θ varies from 0 to 2π. The coordinates of *A* are $(2a\pi, 0)$

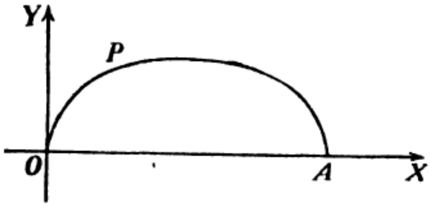

Fig. 12.

The required area

$$= \int_0^{2\pi a} y\,dx.$$

We have

$$dx = a(1-\cos\theta)\,d\theta.$$

$$\therefore \text{ the required area } = \int_0^{2\pi} a\,(1-\cos\theta)\,.\,a(1-\cos\theta)\,d\theta$$
$$= a^2\int_0^{2\pi} (1-2\cos\theta+\cos^2\theta)\,d\theta$$
$$= a^2\int_0^{2\pi} \left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= a^2\left|\theta-2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right|_0^{2\pi}$$
$$= 3a^2\pi.$$

Note. The integral $\int_0^{2\pi} a^2(1-\cos\theta)^2\,d\theta$ could also be evaluated differently as follows. We have

$$\int_0^{2\pi} a^2(1-\cos\theta)^2\,d\theta = 2a^2\int_0^{\pi} (1-\cos\theta)^2\,d\theta$$
$$= 8a^2\int_0^{\pi} \sin^4\frac{\theta}{2}\,d\theta$$
$$= 16a^2\int_0^{\frac{1}{2}\pi} \sin^4\phi\,d\phi, \text{ where } \frac{\theta}{2}=\phi,$$
$$= 16a^2\,.\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 3a^2\pi.$$

4. *Find the area of the region lying above x-axis, and included between the circle $x^2+y^2=2ax$ and the parabola $y^2=ax$.*

Solving the two equations simultaneously, we see that the two curves intersect at $(0, 0)$, (a, a) and $(a, -a)$.

We have to find the area of the region *OABP*, where *P* is the point of intersection (a, a). It is the difference of the areas of the regions *OBPCO* and *OAPCO*.

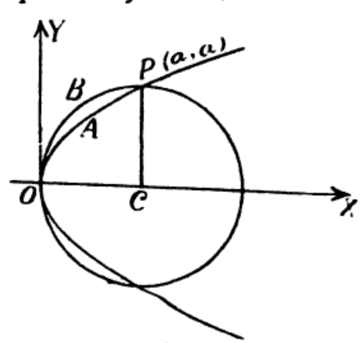

Fig. 13.

Area of the region *OBPCO*

$$= \int_0^a \sqrt{(2ax-x^2)}\, dx$$

Area of region $OAPCO = \int_0^a \sqrt{(ax)}\, dx$

$\therefore$ the required area $= \int_0^a \sqrt{(2ax-x^2)}\, dx - \int_0^a \sqrt{(ax)}\, dx.$

Now $\int_0^a \sqrt{(2ax-x^2)}dx = \int_0^a \sqrt{[a^2-(a-x)^2]}\, dx$

To evaluate this integral, we put

$$a-x = a \sin \theta$$

and obtain

$$\int_0^a \sqrt{(2ax-x^2)}\, dx = \int_{\frac{1}{2}\pi}^0 (a \cos \theta)(-a \cos \theta)\, d\theta$$

$$= \int_0^{\frac{1}{2}\pi} a^2 \cos^2 \theta \, d\theta = a^2 \cdot \frac{1}{2} \frac{\pi}{2} = \frac{\pi a^2}{4}.$$

Also $\int_0^a \sqrt{(ax)}\, dx = \left|\sqrt{a} \cdot \frac{2}{3} x^{3/2}\right|_0^a = \frac{2a^2}{3}$

$\therefore$ the required area $= a^2 \left(\frac{\pi}{4} - \frac{2}{3}\right).$

5. *Find the area between the curve*

$$x(x^2+y^2) = a(x^2-y^2),$$

and its asymptote.

Also find the area of its loop.

The curve is symmetrical about x-axis. The loop is situated between the lines $x = 0$ and $x = a$.

The line $x = -a$ is the asymptote of the curve

We have $y = \pm x \sqrt{\left(\frac{a-x}{a+x}\right)}$.

For any point on the arc OLA,

$$y = x\sqrt{\left(\frac{a-x}{a+x}\right)},$$

and for any point on the arc OMB,

$$y = -x\sqrt{\left(\frac{a-x}{a+x}\right)}.$$

The area of the loop

$$= 2\int_0^a y dx = 2\int_0^a x \sqrt{\left(\frac{a-x}{a+x}\right)} dx \qquad \ldots(1)$$

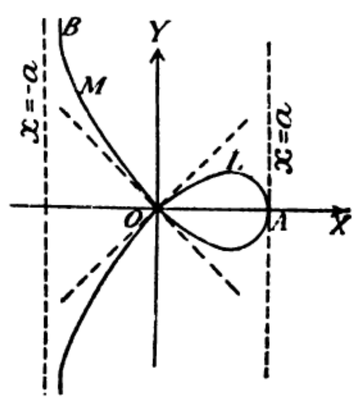

Fig. 14.

The area between the curve and its asymptote

$$= 2\int_{-a}^{0} y\,dx = 2\int_{-a}^{0} -x\sqrt{\left(\frac{a-x}{a+x}\right)}\,dx \qquad ...(2)$$

To evaluate the integral (1), we put $x = a \sin\theta$, so that we have

$$2\int_0^a x\sqrt{\left(\frac{a-x}{a+x}\right)}dx = 2\int_0^a \frac{x(a-x)}{\sqrt{(a^2-x^2)}}dx$$

$$= 2\int_0^{\frac{1}{2}\pi} a^2 (\sin\theta - \sin^2\theta)d\theta$$

$$= 2a^2\left(1 - \frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{4-\pi}{2}a^2.$$

To evaluate the integral (2), we put $x = -a\sin\theta$, so that we obtain

$$2\int_{-a}^{0} -x\sqrt{\left(\frac{a-x}{a+x}\right)}dx = -2\int_{-a}^{0} \frac{x(a-x)}{\sqrt{(a^2-x^2)}}\,d\theta$$

$$= -2\int_{\frac{1}{2}\pi}^{0} a^2 (\sin\theta + \sin^2\theta)\,d\theta$$

$$= 2\int_0^{\frac{1}{2}\pi} a^2 (\sin\theta + \sin^2\theta)\,d\theta$$

$$= \frac{4+\pi}{2}a^2.$$

Note. The area of the region between the curve and its asymptote which extends to infinity is given by the improper integral

$$-2\int_{-a}^{0} x\sqrt{[(a-x)/(a+x)]}dx\ ;$$

the integrand being infinite for $x = -a$. We have seen here that this integral is finite so that the area in question is also finite.

7·2. Quadrature of hyperbola. *If A is the vertex, O the centre and P (x, y) a point on the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

prove that

$$x = a\cosh\frac{2S}{ab}, \quad y = b\sinh\frac{2S}{ab},$$

where S is the sectorial area OPA.

If PM be ordinate of $P(x, y)$, we have

$$S = \text{Area } OAP$$
$$= \text{Area } OMP - \text{area } AMP$$
$$= \tfrac{1}{2}xy - AMP$$

For a point P (x, y) on the arc AP of the hyperbola, we have

$$y = \frac{b}{a}\sqrt{(x^2-a^2)}.$$

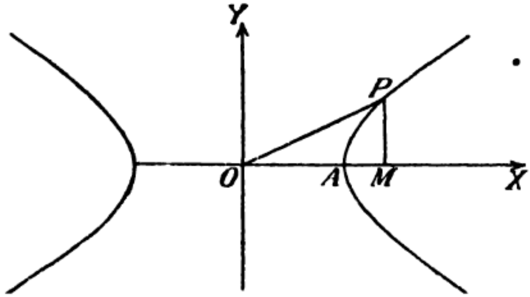

Fig. 15.

Also, $\quad OA = a,$

$$\therefore \text{ area } AMP = \int_a^x y\,dx$$

$$= \frac{b}{a}\int_a^x \sqrt{(x^2-a^2)}\,dx$$

$$= \frac{b}{a}\left|\frac{x\sqrt{(x^2-a^2)}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right|_0^x$$

$$= \frac{b}{a}\left[\frac{x\sqrt{(x^2-a^2)}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right]$$

$$\therefore S = \tfrac{1}{2}x.\frac{b}{a}\sqrt{(x^2-a^2)} - \frac{b}{a}\left\{\frac{x\sqrt{(x^2-a^2)}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}\right\}$$

$$= \frac{ab}{2}\cosh^{-1}\frac{x}{a} \Rightarrow x = a\cosh\frac{2S}{ab}.$$

Hence $\quad y = \dfrac{b}{a}\sqrt{(x^2-a^2)} = b\sinh\dfrac{2S}{ab}.$

Exercises

1. Show that the area of a loop of the curve

$$x^4 = a^2(x^2-y^2) \text{ is } 2a^2/3.$$

2. Show that the area of the loop of the curve

$$3ay^2 = x(x-a)^2 \text{ is } 8a^2/15\sqrt{3}.$$

3. Find the area enclosed by the curve $xy^2 = 4(2-x)$ and Y-axis.

4. Show that the area enclosed between the parabolas

$$y^2 = 4a(x+a),\ y^2 = -4a(x-a),$$

is $16a^2/3$.

5. Find the whole area included between the curve

$$x^2y^2 = a^2(y^2-x^2),$$

and its asymptotes.

6. Show that the area of the infinite region enclosed between the curve $x^2(1-y)y = 1$ and its asymptote is 2π.

7. Calculate the area of a loop of the curve

$$a^4y^2 = x^4(a^2-x^2).$$

8. Show that the area of a loop of the curve

$$y^2 = x^2(4-x^2) \text{ is } 16/3.$$

9. Find the area of the infinite region between the curve

$$y^2(2a-x) = x^3$$

and its asymptote.

10. Find the area enclosed by the curves

$$x^2 = 4ay \text{ and } x^2+4a^2 = 8a^3/y.$$

11. Find the area of the curvilinear triangle, with one vertex at the origin lying in the first quadrant and bounded by the curves

$$y^2 = 4ax,\ x^2 = 4ay,\ x^2+y^2 = 5a^2.$$

12. Find the area enclosed by the curve $xy^2 = a^2(a-x)$ and y-axis.

13. Show that the area enclosed by the curves

$$xy^2 = a^2(a-x) \text{ and } (a-x)y^2 = a^2x,$$

is $(\pi-2)a^2$.

14. In the case of the cycloid

$$x = a(\theta+\sin\theta),\ y = a(1-\cos\theta),$$

find the area included between the curve and its base.

15. Show that the whole area of the curve $a^2x^2 = y^3(2a-y)$ is πa^2.

16. Find the area enclosed by the curve given by the equations

$$x = a\cos^3\theta,\ y = b\sin^3\theta.$$

17. Find the area of the smaller portion enclosed by the curves

$$x^2+y^2 = 9,\ y^2 = 8x.$$

18. Find the area between the curve

$$y^2(a+x) = (a-x)^3$$

and its asymptote.

19. Find the area of the loop of the curve

$$y^2x+(x+a)^2(x+2a) = 0.$$

20. Find the whole area of the curve

$$x^2(x^2+y^2) = a^2(x^2-y^2).$$

21. Trace the curve

$$y^2 = x(x-2)^2/(4-x).$$

Find the area of the loop of the curve and also the area between the curve and its asymptote.

22. Show that the area of the curve

$$a^6y^2 = x^6(a^2-x^2)$$

is $8a^2/15$.

23. Show that the area of the loop of the curve

$$ay^2 = (x-a)(x-5a)^2$$

is $256a^2/15$.

24. Show that the area of the loop of the curve

$$a^2y^2 = x^2(2a-x)(x-a)$$

is $3a^2\pi/8$.

25. Show that the area of the loop of the curve

$$y^2(a+x) = x^2(3a-x)$$

is equal to the area between the curve and its asymptote.

26. Show that the total area included between the two branches of the curve

$$y^2 = x^2/(4-x)(x-2)$$

and the two asymptotes is 6π.

27. Show that the area of the loop of the curve

$$c^2y^2 = (x-a)(x-b)^2$$

is

$$8(b-a)^{5/2}/15c.$$

Trace the curve $y = \tanh x$ and show that the area of the region in the first quadrant enclosed between the curve and its asymptote $y = 1$ is log 2.

Answers

3. 4π. **5.** $4a^2$. **7.** $\pi a^2/8$.

9. $3a^2\pi$. **10.** $\frac{3}{8}(3\pi-2)a^2$. **11.** $\frac{1}{8}[4+15\sin^{-1}\frac{4}{5}]a^2$.

12. πa^2. **14.** $3a^2\pi$. **16.** $\frac{3}{8}\pi ab$.

17. $2\left(\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\frac{1}{3}\right)$. **18.** $3a^2\pi$.

19. $\frac{1}{2}a^2(4-\pi)$. **20.** $a^2(\pi-2)$. **21.** $2(4-\pi)$, $2(\pi+4)$.

7·3. Sectorial Area. *If $r = f(\theta)$ be the equation of a curve in polar co-ordinates, then the area of the sector enclosed by the curve and the two radii vectors $\theta = \alpha$ and $\theta = \beta$ is*

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta.$$

Let $\angle XOA = \alpha$, $\angle XOB = \beta$. Let $P(r, \theta)$ be *any* point on the curve. We denote the area of the sector AOP by A so that 'A' is a function of θ.

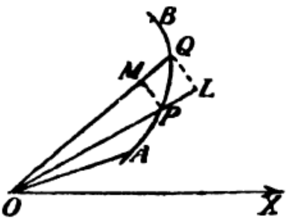

Fig. 16.

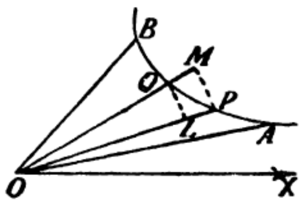

Fig 17.

Let $Q(r+\Delta r,\ \theta+\Delta\theta)$ be another point on the curve which lies so near P that, as a point on the curve moves from P to Q, the radius vector either constantly increases as in Fig. 16, or constantly decreases as in Fig. 17. Area POQ is the increment in A consequent to the increment $\Delta\theta$ in θ. We denote it by ΔA.

With O as centre and OP, OQ as radii, draw arcs of circles cutting OQ and OP at M and L respectively. Clearly ΔA lies between the areas of the circular sectors OPM and OQL.

Area of the circular sector $OPM = \frac{1}{2} r^2 \Delta\theta$, and the area of the circular sector $OQL = \frac{1}{2}(r+\Delta r)^2 \Delta\theta$.

For Fig. 16, we have

$$\tfrac{1}{2} r^2 \Delta\theta < \Delta A < \tfrac{1}{2}(r+\Delta r)^2 \Delta\theta$$

$$\Rightarrow \quad \tfrac{1}{2} r^2 < \frac{\Delta A}{\Delta\theta} < \tfrac{1}{2}(r+\Delta r)^2$$

Let $Q \to P$ so that $\Delta\theta$ and Δr both tend to zero.

In the limit, we obtain

$$\frac{dA}{d\theta} = \tfrac{1}{2} r^2$$

For Fig. 17, we have

$$\tfrac{1}{2}(r+\Delta r)^2 \Delta\theta < \Delta A < \tfrac{1}{2} r^2 \Delta\theta$$

$$\Rightarrow \quad \tfrac{1}{2}(r+\Delta r)^2 < \frac{\Delta A}{\Delta\theta} < \tfrac{1}{2} r^2.$$

In this case also, we have

$$\frac{dA}{d\theta} = \frac{1}{2}\, r^2.$$

Now,

$$\int_\alpha^\beta \frac{1}{2}\, r^2\, d\theta = \int_\alpha^\beta \frac{dA}{d\theta}\, d\theta = \,.$$

= value of A for θ equal to β − value of A for θ equal to α
= Area of the sector $AOB - 0$ = Area of the sector AOB.

Examples

1. *Find the area of the Cardioide* $r = a(1-\cos\theta)$.

The curve is symmetrical about the initial line.

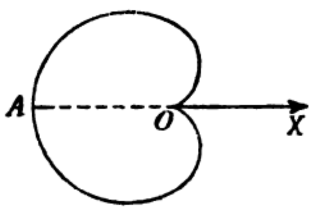

Fig. 18.

$$\text{The required area} = 2\int_0^\pi \tfrac{1}{2} r^2\, d\theta$$

$$= a^2\int_0^\pi (1-\cos\theta)^2\, d\theta$$

$$= 4a^2\int_0^\pi \sin^4\frac{\theta}{2}\, d\theta$$

$$= 8a^2\int_0^{\frac{1}{2}\pi} \sin^4\varphi\, d\varphi, \text{ where } \frac{\theta}{2} = \varphi$$

$$= 8a^2\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3a^2\pi}{2}.$$

2. *Find the area of the loop of the curve*

$$x^5+y^5 = 5ax^2y^2.$$

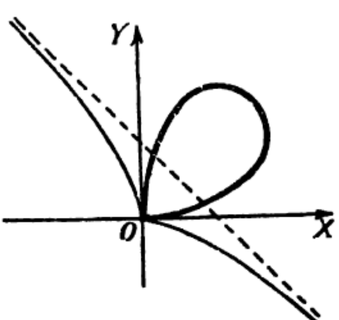

Fig. 19.

The area of the loop is that of a sectorial area bounded by the curve and the radii vectors $\theta = 0$, $\theta = \pi/2$. In fact the area of the loop is swept out as the radius vector moves from

$$\theta = 0 \text{ to } \theta = \pi/2.$$

The polar equation of the curve is

$$r = \frac{5a\cos^2\theta\sin^2\theta}{\cos^5\theta+\sin^5\theta}.$$

Thus the area of the loop

$$= \frac{1}{2}\int_0^{\pi/2} \frac{25a^2\cos^4\theta\sin^4\theta}{(\cos^5\theta+\sin^5\theta)^2}\, d\theta$$

[Dividing the numerator and denominator by $\cos^{10}\theta$]

$$= \frac{25a^2}{2}\int_0^{\pi/2} \frac{\tan^4\theta \sec^2\theta \, d\theta}{(1+\tan^5\theta)^2} .$$

$$= \frac{25a^2}{2}\int_1^{\infty} \frac{1}{5} \cdot \frac{dt}{t^2}, \text{ where } 1+\tan^5\theta = t,$$

$$= \frac{5a^2}{2}\left|-\frac{1}{t}\right|_1^{\infty} = \frac{5a^2}{2} .$$

3. *Find the area between the ellipses*

$$x^2+2y^2 = a^2,\ 2x^2+y^2 = a^2.$$

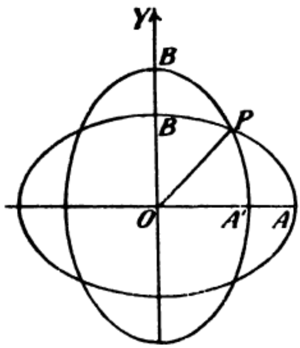

Fig 20.

Rewriting the given equations as

$$\frac{x^2}{a^2}+\frac{y^2}{\frac{1}{2}a^2} = 1, \qquad \ldots(1)$$

$$\frac{x^2}{\frac{1}{2}a^2}+\frac{y^2}{a^2} = 1, \qquad \ldots(2)$$

we see that the major axis lies along x-axis for the ellipse (1) and along y-axis for (2).

Also the required area is four times the area $OA'PB$ where P is a point of intersection. Solving the two equations (1) and (2), we see that P is the point $(a/\sqrt{3}, a/\sqrt{3})$.

Thus $\angle XOP = \pi/4$ and the equation of the line OP is $y = x$.

If we interchange x and y, we see that the equation of one ellipse is transformed into that of the other and accordingly the area of the region $OA'PB$ is twice that of $OA'P$. Now the area $OA'P$ can be thought of as the sectorial area bounded by the curve $A'P$ and the two radii vectors $\theta = 0$, $\theta = \pi/4$.

Changing to polar co-ordinates by writing

$$x = r\cos\theta,\ y = r\sin\theta,$$

we see that the polar equation of the ellipse (2) of which the arc $A'P$ is a part is

$$r^2 = \frac{a^2}{2\cos^2\theta+\sin^2\theta},$$

$$\therefore \quad \text{area } OA'P = \tfrac{1}{2}\int_0^{\pi/4} \frac{a^2}{2\cos^2\theta+\sin^2\theta}\, d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi/4} \frac{a^2\sec^2\theta}{2+\tan^2\theta}\, d\theta$$

$$= \tfrac{1}{2}\int_0^1 \frac{a^2\,dt}{2+t^2}\text{, where } t = \tan\theta$$

$$= \frac{a^2}{2}\cdot\frac{1}{\sqrt{2}}\left|\tan^{-1}\frac{t}{\sqrt{2}}\right|_0^1 = \frac{a^2}{2\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}$$

$$\therefore \text{ required area} = 8\times\frac{a^2}{2\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}$$

$$= 2\sqrt{2}a^2\sin^{-1}\sqrt{\tfrac{1}{3}}.$$

Note. The student may also find the area by drawing PM perpendicular to x-axis and applying the formula of § 6·1 to the two regions $OBPM$, MPA'.

4. *Prove that the area included between the folium*

$$x^3+y^3 = 3axy,$$

and its asymptote is equal to the area of its loop.

The equation of the asymptote is

$$x+y+a = 0.$$

In polar co-ordinates, the equations of the curve and the asymptote are

$$r = \frac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta},$$

and
$$r = \frac{a}{\sin\theta+\cos\theta},$$

respectively.

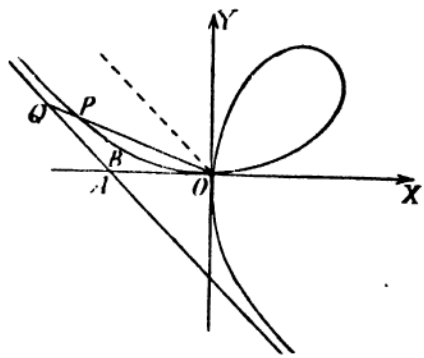

Fig. 21.

The area of the loop

$$= \tfrac{1}{2}\int_0^{\pi/2}\frac{9a^2\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}\,d\theta$$

[Dividing numerator and denominator by $\cos^6\theta$].

$$= \frac{9}{2}a^2\int_0^{\pi/2}\frac{\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}\,d\theta$$

$$= \frac{3a^2}{2}\int_0^{\infty}\frac{dt}{t^2}\text{, where } 1+\tan^3\theta = t$$

$$= \frac{3a^2}{2}\left|-\frac{1}{t}\right|_1^{\infty} = \frac{3a^2}{2}.$$

We shall now find the area between the curve and its asymptote.

The line (shown dotted), drawn parallel to the asymptote makes an angle $3\pi/4$ with OX.

We draw a line through O whose vectorial angle θ lies between $3\pi/4$ and π. Let it cut the curve and the asymptote in P and Q respectively.

We shall first find the area between the curve and its asymptote lying in the second quadrant.

This area is the limit of the area of the curvilinear region *OBPQAO* as the line *OP* starting from *OA* moves towards the dotted line.

The area of the $\triangle$ *OAQ* $= \dfrac{1}{2}\displaystyle\int_0^{\pi} r^2\, d\theta$

$$= \frac{1}{2}\int_0^{\pi} \frac{a^2}{(\sin\theta+\cos\theta)^2}\, d\theta.$$

The area of the region bounded by the curve and the line *OP*

$$= \frac{1}{2}\int_0^{\pi} \frac{9a^2 \sin^2\theta \cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}\, d\theta.$$

Thus the curvilinear area *OBQPAO*

$$= \tfrac{1}{2}\int_0^{\pi} \left[\frac{a^2}{(\sin\theta+\cos\theta)^2} - \frac{9a^2\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}\right] d\theta.$$

Now, $\displaystyle\int \frac{\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}\, d\theta = \int \frac{\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}\, d\theta$

$$= \int \frac{dt}{3t^2} \text{ where } t = 1+\tan^3\theta$$

$$= -\frac{1}{3t} = -\frac{1}{3(1+\tan^3\theta)}.$$

Also, $\displaystyle\int \frac{1}{(\sin\theta+\cos\theta)^2}\, d\theta = \int \frac{\sec^2\theta}{(1+\tan\theta)^2}\, d\theta$

$$= \int \frac{dt}{t^2} \text{ where } t = 1+\tan\theta$$

$$= -\frac{1}{t} = -\frac{1}{1+\tan\theta}.$$

The curvilinear area *OBQPAO*

$$= \frac{a^2}{2}\left| -\frac{1}{1+\tan\theta} + \frac{3}{1+\tan^3\theta}\right|_0^{\pi}$$

$$= \frac{a^2}{2}\left[2 - \left(-\frac{1}{1+\tan\theta} + \frac{3}{1+\tan^3\theta}\right)\right].$$

We have to find its limit as $\theta \to 3\pi/4$.

Now, $\displaystyle\frac{1}{1+\tan\theta} - \frac{3}{1+\tan^3\theta} = \frac{\tan^2\theta-\tan\theta-2}{1+\tan^3\theta}$

$$= \frac{(\tan\theta+1)(\tan\theta-2)}{(\tan\theta+1)(\tan^2\theta-\tan\theta+1)}$$

$$= \frac{\tan\theta - 2}{\tan^2\theta - \tan\theta + 1},$$

which $\to -1$ as $\theta \to 3\pi/4 \Rightarrow \tan\theta \to -1$.

Hence the area $= (a^2/2)(2-1) = a^2/2$.

Because of symmetry about the line $y = x$, $a^2/2$ is also the area between the curve and the asymptote lying in the fourth quadrant. Also the area in the third quadrant, being that of a triangle, is $\frac{1}{2}a^2$.

Thus the area between the curve and its asymptote

$$= \tfrac{1}{2}a^2 + \tfrac{1}{2}a^2 + \tfrac{1}{2}a^2 = \tfrac{3}{2}a^2.$$

Hence the result.

5. *Find the area of loop of the curve*

$$x^4 + 3x^2y^2 + 2y^4 = a^2x.$$

The curve is symmetrical in opposite quadrants and the two axes $x = 0$, $y = 0$ are the two tangents at the origin. The curve has no asymptote.

Transforming to polar co-ordinates, we get

$$r^2 = \frac{a^2\cos\theta\sin\theta}{\cos^4\theta + 3\cos^2\theta\sin^2\theta + 2\sin^4\theta}.$$

As θ varies from 0 to $\pi/2$, the point (r, θ) on the curve describes the loop $OAPBO$.

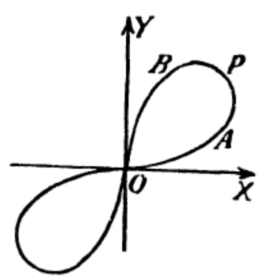

Fig. 22.

Thus the area of the loop

$$= \frac{1}{2}\int_0^{\frac{1}{2}\pi} r^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{1}{2}\pi} \frac{\sin\theta\cos\theta}{\cos^4\theta + 3\cos^2\theta\sin^2\theta + 2\sin^4\theta}\, d\theta$$

(Dividing the numerator and denominator of the integrand by $\cos^4\theta$)

$$= \frac{a^2}{2}\int_0^{\frac{1}{2}\pi} \frac{\tan\theta\sec^2\theta\, d\theta}{1 + 3\tan^2\theta + 2\tan^4\theta}.$$

Putting $\tan^2\theta = t$, we see that the required area

$$= \frac{a^2}{4}\int_0^\infty \frac{dt}{1 + 3t + 2t^2}$$

$$= \frac{a^2}{4}\int_0^\infty \frac{dt}{(2t+1)(t+1)}$$

$$= \frac{a^2}{4}\int_0^\infty \left(-\frac{1}{t+1} + \frac{2}{2t+1}\right) dt$$

$$= \frac{a^2}{4}\log\frac{2t+1}{t+1}\bigg|_0^\infty = \frac{a^2}{4}\log 2,$$

for, as $t \to \infty$, $\log \dfrac{2t+1}{t+1} = \log \dfrac{2+1/t}{1+1/t} \to \log 2.$

Exercises

1. Find the area of a loop of the curve $r = a \sin 2\theta$.

2. Find the area of a loop of the curve $r = a \sin 3\theta$.

3. Find the area bounded by the limacon $r=a+b \cos \theta$, where $a > b$.

4. Prove that the area of the loop of the folium

$$x^3+y^3=3\,axy$$

is three times the area of one of the loops of the Lemniscate

$$r^2 = a^2 \cos 2\theta.$$

5. Show that the area of the loop of the curve

$$r = a\,\theta \cos \theta$$

lying in the first quadrant is $a^2\pi(\pi^2-6)/96$.

6. Show that the area of the region included between the cardioides

$$r = a(1+\cos \theta),\ r = a(1-\cos \theta)$$

is $a^2(3\pi-8)/2$.

7. Show that the area of a loop of the curve

$$r \cos \theta = a \cos 2\theta$$

is $a^2(4-\pi)/2$.

8. Find the area common to the circles

$$r = a\sqrt{2},\ r = 2\,a \cos \theta.$$

9. Find the area of a loop of the curve

$$x^4+y^4 = 2a^2xy$$

10. Find the area bounded by the curve

$$(x^2+y^2)^2 = a^2x^2+b^2y^2$$

11. Prove that the area of a loop of the curve

$$x^6+y^6=a^2x^2y^2 \text{ is } \pi a^2/12.$$

12. Show that the area of a loop of the curve

$$(x^2+y^2)^2-4axy^2=0$$

in the positive quadrant is $\pi a^2/4$.

13. Show that the area of the loop of the curve

$$(x+y)(x^2+y^2) = 2axy$$

is $a^2(1-\pi/4)$; also show that the area of the portion of the curve between the curve and its asymptote is $a^2(1+\pi/4)$.

14. Find the ratio of the areas of the regions in which the line $x+y = 2a$ divides the loop of the folium

$$x^3+y^3 = 3axy.$$

15. Show that the area of the region lying in the second quadrant and bounded by the curve

$$x^5+y^5 = 5ax^2y^2,$$

its asymptote and y-axis is a^2.

16. Show that the area of a loop of the curve

$$r = \sqrt{3} \cos 3\theta + \sin 3\theta$$

is $\frac{1}{3}\pi$.

17. Show that the ratio of the area of the larger to the area of the smaller loop of the curve

$$r = \tfrac{1}{2} + \cos 2\theta$$

is $(4\pi+3\sqrt{3})/(2\pi-3\sqrt{3})$.

18. Show that the area of the region enclosed between the two loops of the curve

$$r = a(1+2 \cos \theta)$$

is $a^2(\pi+3\sqrt{3})$.

19. Find the area of the ellipse $l/r = 1+e \cos \theta$.

20. Show that the area of the loop of the curve

$$r^2 \cos \theta = a^2 \sin 3\theta.$$

lying in the first quadrant is

$$\tfrac{1}{2} a^2 \log (e^3/4).$$

21. Trace the curve

$$r = a\theta/(\theta+1),$$

as θ increases from 0 to 2π and show that the area of the region included between the curve and the initial line is

$$\tfrac{1}{2}a^2[2\pi-(2\pi+1)^{-1}-2 \log (2\pi+1)+1].$$

22. Find the area lying between the cardioide

$$r = a(1-\cos \theta).$$

and its double tangent $x = a/4$.

Answers

1. $\pi a^2/8$.	**2.** $\pi a^2/12$.	**3.** $\pi(2a^2+b^2)/2$.
8. $a^2(\pi-1)$.	**9.** $\pi a^2/4$.	**10.** $\pi(a^2+b^2)/2$.
14. 2 : 1.	**19.** $\pi l^2/(1-e^2)^{3/2}$.	**22.** $a^2\left[\frac{15}{16}\sqrt{3}-\frac{\pi}{2}\right]$.

7·4. Area bounded by a closed curve. We know that the definite integral

$$\int^b ydx$$

gives the area of the region which is bounded by the curve $y = f(x)$, the axis of x, and the two ordinates $x = a$, $x = b$. A region, which is not so bounded, has to be first expressed as a combination of regions each of which is bounded in this manner and the area of the given region is then, equal to the sum of the areas of the component regions. Similar limitations hold for the integrals.

$$\int_c^d x\,dy,\ \frac{1}{2}\int_\alpha^\beta r^2 d\theta.$$

We now consider a formula which gives the area enclosed by any closed curve whatsoever, provided only, that it does not intersect itself; there being no restriction as to the manner in which the curve is situated relative to the co-ordinate axes.

Consider a closed curve represented by the parametric equations

$$x = f(t),\ y = \varphi(t)$$

't' being the parameter. We suppose that the curve does not intersect itself. Also suppose that as the parameter 't' increases from a

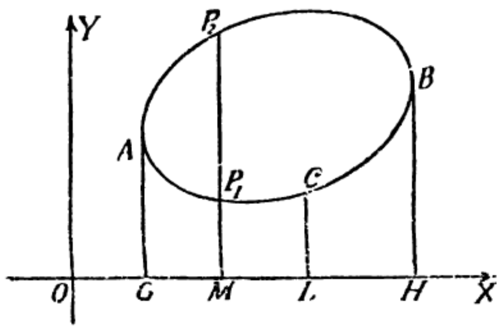

Fig. 23.

value t_1 to the value t_2, the point $P(x, y)$ describes the curve completely in the counter clockwise sense. The curve being closed, the point on it corresponding to the value t_2 of the parameter is the same as the point corresponding to the value t_1 of the parameter. Let this point be C.

It will now be shown that the area of the region bounded by such a curve is

$$\frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt.$$

Let the points on the curve with greatest and least abscissae be A, B.

We suppose that the curve is such that every line parallel to y-axis and lying between the ordinates $x = a$, $x = b$ of the points A, B, meets it in two and only two points P_1 and P_2 ; P_1 lying on the arc ACB and P_2 on the arc BP_2A.

The area of the region

$$= \text{Area } AGHBP_2AG - \text{Area } AGHBCAG$$

$$= \int_a^b MP_2\, dx - \int_a^b MP_1\, dx$$

$$= -\int_b^a MP_2\, dx - \int_a^b MP_1\, dx.$$

We now express these integrals in terms of the variable t.

Let t_a, t_b denote the values of the parameter t for the points A, B.

As $P(x, y)$ moves along the arc BP_2A from B to A, the parameter t increases from t_b to t_a ;

as $P(x, y)$ moves along the arc AC from A to C, the parameter t increases from t_a to t_2 ;

as $P(x, y)$ moves along the arc CB from C to B, the parameter t increases from t_1 to t_b.

$$\therefore \qquad \int_b^a MP_2\, dx = \int_{t_b}^{t_a} y \frac{dx}{dt}\, dt,$$

and

$$\int_a^b MP_1\, dx = \int_a^{OL} MP_1\, dx + \int_{OL}^b MP_1\, dx$$

$$= \int_{t_a}^{t_1} y \frac{dx}{dt}\, dt + \int_{t_1}^{t_b} y \frac{dx}{dt}\, dt.$$

area of the region

$$= -\int_{t_b}^{t_a} y \frac{dx}{dt}\, dt - \int_{t_a}^{t_1} y \frac{dx}{dt}\, dt - \int_{t_1}^{t_b} y \frac{dx}{dt}\, dt$$

$$= -\int_{t_1}^{t_b} y \frac{dx}{dt}\, dt - \int_{t_b}^{t_a} y \frac{dx}{dt}\, dt - \int_{t_a}^{t_1} y \frac{dx}{dt}\, dt$$

$$= -\int_{t_1}^{t_2} y \frac{dx}{dt}\, dt \qquad \ldots(i)$$

It can similarly be shown that the area of the same region is also equal to

$$\int_{t_1}^{t_2} x \frac{dy}{dt}\, dt. \qquad \ldots(ii)$$

Adding the two expressions, we see that the area of the region is equal to

$$\frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt. \qquad ...(iii)$$

The result also holds for the curve which is met by the lines parallel to either axis in more than two points but the details of this extension will not be given here.

Note 1. Because of symmetry, the expression (*iii*) proves convenient in practice than the expressions (*i*) and (*ii*).

Note 2. Suppose that the curve intersects itself once as shown in Fig. 24.

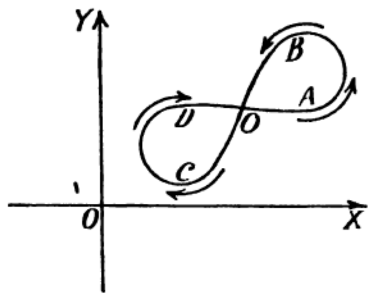

Fig. 24.

Let t vary from t_1 to t_1' as $P(x, y)$ moves along the curve OAB from O back to O. Also, let t vary from t'_1 to t_2 as $P(x, y)$ moves along the curve OCD from O back to O.

The arc $OABO$ is described in the counter clock-wise but the arc OCD in the clock-wise sense.

The area of the loop $OABO = \frac{1}{2}\int_{t_1}^{t_1'}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt,$

The area of the loop $OCDO = -\frac{1}{2}\int_{t_1'}^{t_2}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt,$

Area $OABO -$ area $OCDO = \frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt,$

so that the formula gives the difference of the areas of the two loops and not their sum.

In order to find the whole area, we have to find each area separately and then to obtain their sum.

Note 3. The definite integral

$$\frac{1}{2}\int\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$$

is sometimes briefly written as

$$\frac{1}{2}\int(xdy-ydx).$$

Examples

1. *Find the area of the ellipse*

$$x = a\cos t,\ y = b\sin t.$$

The ellipse is a closed curve and is completely described while t varies from 0 to 2π.

We have

$$x\frac{dy}{dt}-y\frac{dx}{dt}=ab(\cos^2 t+\sin^2 t)=ab.$$

Therefore the required area

$$=\frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt=\frac{1}{2}\int_0^{2\pi}ab\,dt=\pi ab.$$

2. *Sketch the curve*

$$x=\frac{1-t^2}{1+t^2},\; y=\frac{t-t^3}{1+t^2} \qquad -1\leqslant t\leqslant 1$$

and calculate the area enclosed by the loop of the curve.

The curve can be traced on eliminating t. We shall, however, *directly trace the part of the curve which is described when t varies from -1 to $+1$.

We have

$$\frac{dx}{dt}=-\frac{4t}{(1-t^2)^2},\quad \frac{dy}{dt}=-\frac{t^4+4t^2-1}{(t^2+1)^2}.$$

t is negative $\Rightarrow dx/dt$ is positive

t is positive $\Rightarrow dx/dt$ is negative.

Thus we see that

t increases from -1 to $0 \Rightarrow x$ increases

t increases from 0 to $1 \Rightarrow x$ decreases.

To examine the change of sign in dy/dt, we consider the equation

$$t^4+4t^2-1=0$$

which gives

$$t^2=\sqrt{5}-2,\; -\sqrt{5}-2.$$

Thus

$$\begin{aligned}(t^4+4t^2-1)&=[t^2-(\sqrt{5}-2)][t^2-(-\sqrt{5}-2)]\\&=[t^2-(\sqrt{5}-2)][t^2+(\sqrt{5}+2)]\end{aligned}$$

Also

$$\begin{aligned}t^2&=\sqrt{5}-2 \text{ gives}\\ t&=\pm\sqrt{(\sqrt{5}-2)}=\pm\cdot 4\ldots\ldots\end{aligned}$$

We, therefore, have

$$\frac{dy}{dt}=-\frac{(t^2+\sqrt{5}+2)(t+\cdot 4\ldots)(t-\cdot 4\ldots)}{(t^2+1)^2}.$$

---

*This method proves useful for drawing curves whose parametric equations are given. In particular, Cycloid and Astroid may be easily traced in this manner.

Thus

t varies from -1 to $-\cdot 4\ldots \Rightarrow dy/dt$ is negative

t varies from $-\cdot 4\ldots$ to $\cdot 4\ldots \Rightarrow dy/dt$ is positive

t varies from $\cdot 4\ldots$ to $1 \Rightarrow dy/dt$ is negative

Hence

t increases from -1 to $-\cdot 4\ldots \Rightarrow dy/dt < 0 \Rightarrow y$ decreases

t increases from $-\cdot 4\ldots$ to $\cdot 4\ldots \Rightarrow dy/dt > 0 \Rightarrow y$ increases

t increases from $\cdot 4\ldots$ to $1 \Rightarrow dy/dt < 0 \Rightarrow y$ decreases.

Also

$$t = -1 \Rightarrow x = 0,\ y = 0,\ dy/dx = -1$$
$$t = 1 \Rightarrow x = 0,\ y = 0,\ dy/dx = 1.$$

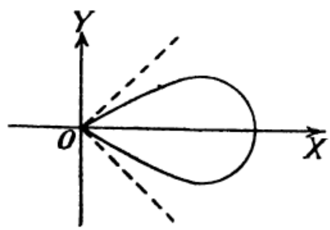

Fig. 25.

Hence we have the loop as traced.

$$x\frac{dy}{dt} - y\frac{dx}{dt} = \frac{1-t^2-t^4+t^6}{(1+t^2)^3} = \frac{(1+t^2)(1-t^2)^2}{(1+t^2)^3} = \frac{(1-t^2)^2}{(1+t^2)^2}$$

The required area

$$= \frac{1}{2}\int_{-1}^{1}\left(x\frac{dy}{dt} - y\frac{dy}{dt}\right)dt$$
$$= \frac{1}{2}\int_{-1}^{1}\frac{(1-t^2)^2}{(1+t^2)^2}\,dt.$$

Now, $\dfrac{(1-t^2)^2}{(1+t^2)^2} = 1 - \dfrac{4t^2}{(1+t^2)^2}$

$$= 1 - \frac{4(t^2+1-1)}{(t^2+1)^2} = 1 - \frac{4}{t^2+1} + \frac{4}{(t^2+1)^2}$$

The required area

$$= \frac{1}{2}\int_{-1}^{1} 1.dt - 2\int_{-1}^{1}\frac{dt}{t^2+1} + 2\int_{-1}^{1}\frac{dt}{(t^2+1)^2}$$
$$= 1 - \pi + 2\int_{-1}^{1}\frac{dt}{(t^2+1)^2}.$$

To evaluate the integral on the right, we put

$$t = \tan\theta.$$

$$\therefore \quad \int_{-1}^{1} \frac{dt}{(1+t^2)^2} = \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \cos 2\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\Big| \theta+\sin\theta\cos\theta \Big|_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} = \frac{1}{2}\left\{\frac{1}{2}\pi+1\right\}$$

$$\therefore \quad \text{area} = 1-\pi+(\tfrac{1}{2}\pi+1) = 2-\tfrac{1}{2}\pi.$$

Note. On eliminating t from the given parametric equations, we obtain $x(x^2+y^2) = x^2-y^2$ as the implicit cartesian equation of the given curve. The curve may also be easily traced with the help of this form of the equation. [Refer Fig. Ex. 5, p. 167].

Exercises

1. Sketch the curve

$$x = a(1-t^2),\ y = at(1-t^2), \qquad -1 \leqslant t \leqslant 1,$$

and show that the area of the loop obtained is $8a^2/15$.

2. Trace the curve

$$x = t-t^3,\ y = 1-t^4$$

for all values of t and prove that it forms a loop of area $16/35$.

3. Show that the curve

$$x = (t-1)e^{-t},\ y = tx$$

has a loop and find its area.

4. Find the area enclosed by the curve

$$x = a\cos^3 t,\ y = b\sin^3 t.$$

5. Sketch the curve

$$x = a\sin 2t,\ y = a\sin t$$

and find the area of one of its loops.

6. Find the area enclosed by the curve

$$x = a\cos t+b\sin t+c,$$
$$y = a'\cos t+b'\sin t+c'.$$

7. Trace the curve

$$x = a(3\cos\theta-\cos^3\theta),\ y = a(3\sin\theta-\sin^3\theta)$$

and find the area enclosed by it.

8. Find the area of the curve

$$x = a(3\sin\theta-\sin^3\theta),\ y = a\cos^3\theta.$$

9. Show that the area enclosed by the curve

$$x = a \sin 2\theta\,(1+\cos 2\theta)$$
$$y = a \cos 2\theta(1-\cos 2\theta)$$

is $\frac{1}{2}a^2\pi$.

10. Trace the curve

$$x = a(\sin\theta+\tfrac{1}{3}\sin 3\theta),$$
$$y = a(\cos\theta-\tfrac{1}{3}\cos 3\theta)$$

and show that its area is $\frac{2}{3}a^2\pi$.

Answers

3. $1/e$. 4. $3\pi ab/8$. 5. $4a^2/3$. 6. $\pi(ab'-a'b)$.
7. $39a^2\pi/8$. 8. $15a^2\pi/8$.

7·5. Simpson's rule for approximate evaluation of definite integrals and Areas. The method given above of finding an area requires

(*i*) that we know the equation $y = f(x)$ of the curve and

(*ii*) that it is possible to determine a function whose derivative is the given function $f(x)$ *i.e.*, it is possible to find the indefinite integral of the given function.

In mechanical work, a curve is often plotted from a finite number of isolated observations so that we do not know its equation. Moreover, even if the equation of a curve be known, it is not always possible to find the indefinite integral required.

For example, let the value of $\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\, dx$ be required.

As we cannot find the indefinite integral of $\sqrt{\sin x}$ in terms of elementary functions we, therefore, cannot find the exact value of the definite integral. But by Simpson's and other similar rules, it is possible to calculate approximately its value.

Simpson's rule only will be consideredhere.

Lemma. Consider the parabola

$$y = l+mx+nx^2,$$

and three points P_1, P_2, P_3 on it such that the ordinates P_1A_1 and P_3A_3 of P_1 and P_3 are equidistant from the ordinate P_2A_2 of P_2.

Let $A_1A_2 = A_2A_3 = h$.

Also, let $OA_2 = a$ so that $OA_1 = a-h$, $OA_3 = a+h$.

Area of the region $P_1A_1A_3P_3P_1$

$$= \int_{a-h}^{a+h} (l+mx+nx^2)\,dx$$

$$= \left| lx+\frac{m}{2}x^2+\frac{n}{3}x^3 \right|_{a-h}^{a+h}$$

$$= 2h\,(l+ma+na^2+\tfrac{1}{3}nh^2) \quad ..(i)$$

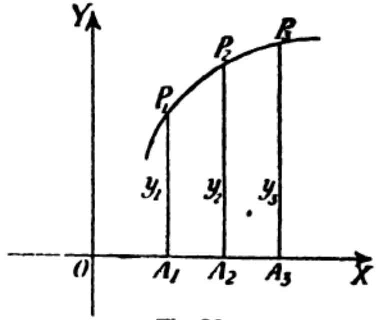

Fig. 26.

We express this area in terms of the ordinates y_1, y_2, y_3 of the points P_1, P_2, P_3.

We have

$$y_1 = P_1A_1 = l+m\,(a-h)+n\,(a-h)^2, \quad ...(ii)$$
$$y_2 = P_2A_2 = l+ma+na^2, \quad ...(iii)$$
$$y_3 = P_3A_3 = l+m\,(a+h)+n\,(a+h)^2. \quad ...(iv)$$

$$\therefore \quad y_1+y_3 = 2\,(l+ma+na^2+nh^2) = 2\,(y_2+nh^2),$$

$$\Rightarrow \quad \frac{y_1+y_3-2y_2}{2} = nh^2 \quad ...(v)$$

From (i), (iii) and (v), we see that the area of the region is

$$2h\left[y_2+\frac{y_1+y_3-2y_2}{6}\right] = \tfrac{1}{3}h\,(y_1+y_3+4y_2).$$

Simpson's rule. Consider the region

$$P_1A_1BP_{2n+1}$$

bounded by a curve $y = f(x)$, X-axis and the two ordinates P_1A and $P_{2n+1}B$.

We divide the interval AB into $2n$ equal parts, n being a sufficiently large positive integer. Let $P_1, P_2, P_3, \dots, P_{2n+1}$ be the points on the curve corresponding to the points of division of AB. We replace the arc $P_1P_2P_3$ of the curve by an arc of the parabola through these points. As the points are very close to each other, we can take the area bounded by the arc $P_1P_2P_3$ of the given curve as approximately equal to the area bounded by the parabola through $P_1P_2P_3$. The arcs $P_3P_4P_5$, $P_5P_6P_7$, ..., $P_{2n-1}P_{2n}P_{2n+1}$, are to be similarly replaced by arcs of parabolas.

Applying the result of the lemma to the areas between the ordinates y_1 and y_3, y_3 and y_5, y_5 and y_7,......, y_{2n-1} and y_{2n+1} we obtain as an approximate value for the whole area

$$\tfrac{1}{3}h(y_1+4y_2+y_3)+(y_3+4y_4+y_5)+(y_5+4y_6+y_7)+\dots\dots$$
$$+(y_{2n-1}+4y_{2n}+y_{2n+1})]$$
$$= \tfrac{1}{3}h[(y_1+y_{2n+1})+2(y_3+y_5+\dots\dots+y_{2n-1})+4(y_2+y_4+\dots+y_{2n})].$$

This is Simpson's rule which may be stated as follows :—

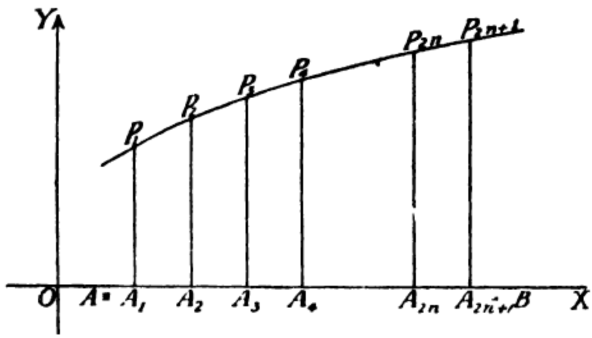

Fig. 27.

To obtain an approximate value of an area (or of a definite integral), divide it into an even number of strips by drawing equidistant ordinates and multiply one-third of the distance between consecutive ordinates by the sum of

(*i*) *the first and the last ordinates,*

(*ii*) *twice all the other odd ordinates, and*

(*iii*) *four times all the even ordinates.*

Example

1. *Obtain an approximate value of, log 2, by calculating the definite integral*

$$\int_1^2 \frac{dx}{x},$$

by Simpson's rule, using eleven ordinates.

We have

$$\int_1^2 \frac{dx}{x} = \log 2.$$

We divide the interval] 1, 2 [into 10 equal parts.

Let $f(x) = 1/x$. We have

$y_1 = f(1 \cdot 0) = 1 \cdot 00000$; $\quad y_2 = f(1 \cdot 1) = \cdot 90909$,

$y_3 = f(1 \cdot 2) = \cdot 83333$; $\quad y_4 = f(1 \cdot 3) = \cdot 76923$,

$y_5 = f(1 \cdot 4) = \cdot 71429$; $\quad y_6 = f(1 \cdot 5) = \cdot 66667$,

$y_7 = f(1 \cdot 6) = 62500$; $\quad y_8 = f(1 \cdot 7) = \cdot 58824$,

$y_9 = f(1 \cdot 8) = 55556$; $\quad y_{10} = f(1 \cdot 9) = \cdot 52632$,

$y_{11} = f(2 \cdot 0) = \cdot 50000$;

S_1 = Sum of the first and last ordinates $= 1 \cdot 50000$,

S_2 = Sum of the remaining odd ordinates $= 2 \cdot 72818$,

S_3 = Sum of the even ordinates. $= 3 \cdot 45955$,

$$\therefore \quad \log 2 = \tfrac{1}{3} \cdot \tfrac{1}{10}(S_1+2S_2+4S_3) = \tfrac{1}{30}\,(20{\cdot}79456) = {\cdot}69315.$$

$$\log 2 = {\cdot}69315,$$

Actually, referring to logarithmic tables, we may see that

$$\log 2 = {\cdot}693147.$$

Exercises

1. Find the area of a curve in which successive ordinates at intervals of ·2 inch are 3·8, 3·5, 3·2, 2·8, 3·3, 3·6, 4 inches.

2. A curve is drawn through the points

(1, 2) (1·5, 2·4), (2, 2·7), (2·5, 2·8), (3, 3), (3·5, 2·6), (4, 2·1).

Estimate the area between the curve, the axis of x and the ordinates $x = 1$, $x = 4$.

3. The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the speed in miles per hour.

Minutes	2	4	6	8	10	12	14	16	18	20
Miles per hour	10	18	25	29	32	20	11	5	2	0

Estimate approximately the total distance run in 20 minutes.

4. A river is 80 feet wide. The depth d in feet at a distance x feet from one bank is given by the following table :—

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3.

Find approximately the area of the cross-section.

5. Calculate

$$\int_2^{10} \frac{dx}{1+x}$$

by the Simpson's rule, using nine ordinates.

6. From the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2},$$

calculate π, using Simpson's rule with $h = 0{\cdot}1$.

7. Find an approximate value of π from

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}},$$

using five ordinates.

8. Calculate $\int_4^8 \dfrac{dx}{x}$, using four equal intervals.

9. Explain Simpson's rule for approximate integration.

Use the rule, taking five ordinates, to find an approximation to two decimal places to the value of the integral.

$$\int_1^2 \sqrt{(x-1/x)}\,dx.$$

10. Given that

$$e^0 = 1,\ e^1 = 2{\cdot}72,\ e^2 = 7{\cdot}39,\ e^3 = 20{\cdot}09,\ e^4 = 54{\cdot}60\ ;$$

verify Simpson's rule by finding an approximate value of

$$\int_0^4 e^x\,dx,$$

and compare it with the exact value.

11. Prove that

$$\int_a^{a+3h} y\,dx = \tfrac{3}{8}h\,(y_1+3y_2+3y_3+y_4),$$

where y is a polynomial of the third degree and y_1, y_2, y_3, y_4 are the values of y corresponding to the values, a, $a+h$, $a+2h$, $a+3h$, of x.

Hence obtain an approximate value of

$$\int_0^{0\cdot3} (1-8x^3)^{1/2}\,dx.$$

12. Obtain Simpson's rule for three equidistant ordinates, *viz.*,

$$\tfrac{1}{3}h(y_1+4y_2+y_3).$$

If, in this method, the middle ordinate y_2 is at unequal distances h, k, from y_1 and y_3 respectively, then show that the formula is

$$\frac{1}{6}\,(h+k)(y_1+4y_2+y_3)+\frac{1}{6}\,(h^2-k^2)\left(\frac{y_1-y_2}{h}+\frac{y_2-y_3}{k}\right)$$

Answers

1. 4·03 square inches.	2. 7·78.	3. 232/45 miles.
4. 710 square feet.	5. 1·299.	8. ·693.
9. 0·84.	10. 53·87, 54·60.	11. 0·29159.

EXERCISES ON CHAPTER 7

1. The coordinates of a point on a cycloid are given by $x = a\,(\theta+\sin\theta)$, $y = a\,(1+\cos\theta)$ and the points corresponding to $\theta = -\frac{1}{2}\pi$ and $\theta = \frac{1}{2}\pi$ are denoted by P, Q respectively. Calculate the area enclosed by the arc PQ of the cycloid and the straight lines joining P and Q to the origin.

2. Find the area of the smallest portion which is bounded by the curve $r = a(\theta + \sin \theta)$ and by a radius vector which is inclined to the initial straight line at a right angle.

3. Prove that the area of either loop of the curve

$$x^4 - 2xya^2 + a^2y^2 = 0$$

is $\frac{2}{3}a^2$.

4. Trace the curve $r \cos \theta = a \sin 3\theta$ and show that the area of a loop is $\frac{1}{8}\, a^2\, (9\sqrt{3} - 4\pi)$.

5. Prove that the area between the curve

$$r = a\,(\sec \theta + \tan \theta),$$

and its asymptote is equal to $(\pi/2 + 2)a^2$. Also prove that the area of the loop of the curve is $(2 - \pi/2)a^2$.

6. Show that the curve $r = a(1 + 2 \sin \frac{1}{2}\theta)$ consists of three loops and find the area of each loop.

7. Find the ratio of the two parts into which the parabola $2a = r(1 + \cos \theta)$ divides the area of the cardioide $r = 2a(1 + \cos \theta)$.

8. Prove that the area of the curve

$$4y^2 + x^2(x^2 - 6x + 8) = 0$$

is equal to $3\pi/2$.

9. Prove that the area of the curve

$$x^4 - 3ax^3 + a^2(2x^2 + y^2) = 0$$

is $\frac{3}{8}\pi a^2$.

10. The loop of the curve $ay^2 = x(x - a)^2$ is cut by the line $x + 2y = a$. Determine the areas of the two parts into which the loop is divided.

11. Determine the area bounded by the parabola

$$x^2 = 4ay + 4a^2,$$

and the line $3x + 4y = 0$.

12. Find the area in the first quadrant bounded by the curves $r = a(1 + \cos \theta)$, $r = a \sin \theta$ and $r = 2a \cos \theta$.

13. O is the pole of the Lemniscate $r^2 = a^2 \cos 2\theta$ and PQ is a common tangent to its two loops. Find the area bounded by the line PQ and the OP and OQ of the curve.

14. Show that the area common to the ellipses

$$a^2x^2 + b^2y^2 = 1,\ b^2x^2 + a^2y^2 = 1,\ (0 < a < b)$$

is

$$4/ab \tan^{-1} (a/b)$$

15. Show that the areas of the successive loops of the curve

$$r = ae^{\theta} \sin \theta$$

form a geometrical progression whose common ratio is $e^{2\pi}$.

16. Prove that the area common to the two curves

$$r = a \sin \theta, \ r = a \sin 2\theta$$

is $(4\pi - 3\sqrt{3})a^2/16$.

17. Trace the curve $r = a \log \theta$, as θ varies from 0 to 1 and show that the area of the region enclosed between the curve and its asymptote is a^2.

18. Prove that the whole area between the four infinite branches of the tractrix

$$x = a\left(\cos t + \tfrac{1}{2} \log \tan^2 \tfrac{1}{2}t\right), \ y = a \sin t$$

is πa^2.

19. Show that the area of any closed curve

$$f\left(\frac{x}{m}, \frac{y}{n}\right) = 0,$$

is mn times the area of the curve

$$f(x, y) = 0.$$

Apply this result to find the areas of the following curves :—

(*i*) $(m^2x^2 + n^2y^2)^2 = a^2x^2 + b^2y^2$.

(*ii*) $(m^2x^2 + n^2y^2)^5 = (a^2x^3 + b^2y^3)^2$.

(*iii*) $(x^2 + 2y^2)^3 = axy^4$.

Answers

1. $(\pi+3)\, a^2$.
2. $a^2(\pi^3 + 6\pi + 48)/48$.
6. $a^2(3\pi+8)$, $\frac{1}{2}a^2(3\pi-8)$, $\frac{1}{2}a^2(3\pi-8)$.
7. $(9\pi+16)/(9\pi-16)$.
10. $53a^2/960$, $153a^2/320$.
11. $125a^2/24$.
12. $a^2[\frac{1}{3} + \frac{1}{4}\pi - \frac{1}{2} \tan^{-1} 2]$.
13. $a^2(3\sqrt{3}-4)/8$.
19. (*i*) $(a^2n^2 + b^2m^2)\pi/2m^3n^3$.
(*ii*) $4(a^2n^3 + b^2m^3)/3m^4n^4$.
(*iii*) $7\sqrt{2}\pi a^2/2^{13}$.

8

Rectification. Length of Plane Curves

8·1. Introduction. In this chapter, we shall be concerned with the determination of the lengths of arcs of plane curves whose equations are given in the Cartesian, Parametric cartesian or Polar form. The process is known as *Rectification.*

It is known to the reader that in order to obtain analytical expressions for an area, we first proved a corresponding formula giving the derivation of the area function. For example, we have shown that $dA/dx = y$, $dA/d\theta = \frac{1}{2}r^2$. The same thing has to be done in order to determine the lengths of curves. These formulae for the derivatives of the arcs have already been proved in Chapter XIII of the author's Differential Calculus. Here we will only refer to them and will not reproduce their proofs.

8·2. Cartesian Equations y = f(x). *The length of the arc of the curve $y = f(x)$ included between two points whose abscissae are a and b is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx = \int_a^b \sqrt{[1+f'^2(x)]}\,dx$$

Let AB be two points with abscissae a, b on the curve

$$y = f(x).$$

If 's' denotes the length of the arc of the curve included between a fixed point A and a variable point P whose abscissa is x so that it is a function of x, we have

$$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}.$$

$$\Rightarrow \quad \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx = \int_a^b \frac{ds}{dx}dx = \Big| s \Big|_a^b$$

= value of, s, for x equal to b − value of, s, for x equal to a
= Arc $AB - 0$ = Arc AB.

Hence the result.

8·3. Other Expressions for lengths of arcs. From the formulae

$$\frac{ds}{dy} = \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]};$$

$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]};$$

$$\frac{ds}{d\theta} = \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]};$$

proved in Differential Calculus, we obtain the following results for the determination of the lengths of arcs :—

8·31. Cartesian Equations x = f(y). *The length of the arc of the curve $x = f(y)$, included between two points whose ordinates are c, d, is*

$$\int_c^d \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}dy = \int_c^d \sqrt{[1+f'^2(y)]}dy.$$

8·32. Parametric Cartesian Equations x = f(t), y = φ (t). *The length of the arc of the curve $x = f(t)$, $y = \varphi(t)$ included between two points whose parametric values are α, β is*

$$\int_\alpha^\beta \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}d\theta = \int_\alpha^\beta \sqrt{[f'^2(t)+\varphi'^2(t)]}dt.$$

8·33. Polar Equations r = f(θ). *The length of the arc of the curve $r = f(\theta)$ included between two points whose vectorial angles are α, β is*

$$\int_\alpha^\beta \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}d\theta = \int_\alpha^\beta \sqrt{[f^2(\theta)+f'^2(\theta)]}\,d\theta.$$

Examples

1. *Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus rectum.*

The abscissa of the extremity L of the latus rectum is $2a$.

Now, $y = \dfrac{x^2}{4a}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{2a}.$$

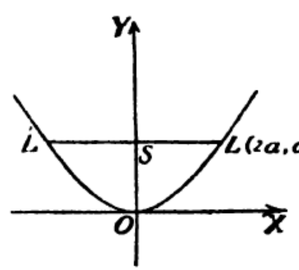

Fig. 28.

$\therefore$ the required length

$$= \int_0^{2a} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{(4a^2+x^2)}dx$$

$$= \frac{1}{2a}\left|\frac{x\sqrt{(x^2+4a^2)}}{2}+2a^2\sinh^{-1}\frac{x}{2a}\right|_0^{2a}.$$

$$= \frac{1}{2a}[2\sqrt{2}a^2+2a^2\sinh^{-1}1]$$

$$= a[\sqrt{2}+\log(1+\sqrt{2})], \text{ for, } \sinh^{-1}x = \log[x+\sqrt{(1+x^2)}].$$

2. *Find the perimeter of the loop of the curve*

$$9ay^2 = (x-2a)(x-5a)^2.$$

The loop lies between the limits $x = 2a$ and $x = 5a$. The curve is symmetrical about x-axis and, therefore, the perimeter of the loop is double of the length of its part lying about x-axis

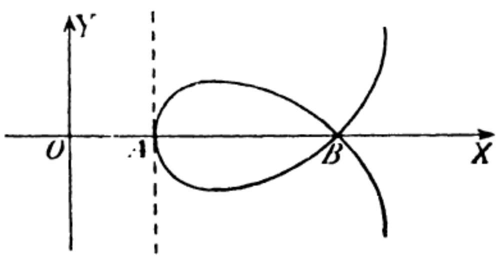

Fig. 29

For any point on the arc lying above X-axis, we have

$$y = -\frac{(x-5a)\sqrt{(x-2a)}}{3\sqrt{a}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{(x-2a)}+\dfrac{x-5a}{2\sqrt{(x-2a)}}}{3\sqrt{a}} = -\frac{x-3a}{2\sqrt{a}\sqrt{(x-2a)}}$$

$\therefore$ the required perimeter

$$= 2\int_{2a}^{5a} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$$

$$= 2\int_{2a}^{5a} \sqrt{\left[1+\frac{(x-3a)^2}{4a(x-2a)}\right]}dx$$

$$= 2\int_{2a}^{5a} \frac{x-a}{2\sqrt{a}\sqrt{(x-2a)}}\,dx$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a} \frac{x-2a+a}{\sqrt{(x-2a)}}\,dx$$

$$= \frac{1}{\sqrt{a}}\left|\frac{2}{3}(x-2a)^{3/2}+2a\sqrt{(x-2a)}\right|_{2a}^{5a}$$

$$= 4\sqrt{3a}.$$

3. *Rectify the curve*

$x = a(\theta+\sin\theta),\ y = a\ (1-\cos\theta).$

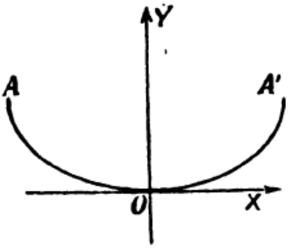

Fig. 30.

As a point moves from one end A' to other end A of the one arc, the parameter θ increases from $-\pi$ to π. The parameter θ is 0 for the vertex O. As the arc is symmetrical about OY,

$$\text{arc } AOA' = 2 \text{ arc } OA.$$

We have

$$\frac{dx}{d\theta} = a(1+\cos\theta), \frac{dy}{d\theta} = a\sin\theta.$$

$$\therefore \quad \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right]} = \sqrt{[2a^2(1+\cos\theta)]}$$

$$= \sqrt{\left(4a^2\cos^2\frac{\theta}{2}\right)} = 2a\cos\frac{\theta}{2}$$

$$\therefore \text{ the required arc } = 2\int_0^{\pi} 2a\cos\frac{\theta}{2}\,d\theta$$

$$= 2\left|4a\sin\frac{\theta}{2}\right|_0^{\pi} = 8a.$$

(Note. Here we take $\sqrt{[4a^2\cos^2(\theta/2)]} = 2a\cos(\theta/2)$, and not $-2a\cos(\theta/2)$ as $\cos(\theta/2)$ remains positive when θ increases from 0 to π).

4. *Find the perimeter of the cardioide*

$r = a(1-\cos\theta).$

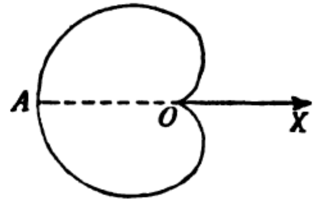

Fig. 31.

The curve is symmetrical about the initial line, and, therefore, its perimeter is double the length of the arc of the curve lying above the same.

Now $\frac{dr}{d\theta} = a\sin\theta.$

$$\therefore \quad \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]} = \sqrt{[a^2(1-\cos\theta)^2+a^2\sin^2\theta]}$$

$$= \sqrt{[2a^2(1-\cos\theta)]} = 2a\sin(\theta/2)$$

$\therefore$ the required perimeter

$$= 2\int_0^{\pi} 2a\sin\frac{\theta}{2}d\theta = 2\left|-4a\cos\frac{\theta}{2}\right|_0^{\pi} = 8a.$$

Exercises

1. Find the length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \pi/3$.

2. Find the length of the arc of the curve
$$y = \log \tanh (x/2)$$
from $x = 1$ to $x = 2$.

3. Find the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum.

4. Find the length of the arc of the catenary
$$y = c \cosh (x/c)$$
measured from the vertex $(0, c)$ to any point (x, y).

5. Prove that the length of the arc of the curve
$$x = a \sin 2\theta\,(1+\cos 2\theta),$$
$$y = a \cos 2\theta\,(1-\cos 2\theta),$$
measured from $(0, 0)$ to (x, y) is equal to $\frac{4}{3}a \sin 3\theta$.

6. (*a*) Find the entire length of the astroid
$$x^{2/3}+y^{2/3} = a^{2/3} \text{ or } x = a \cos^3\theta,\ y = a \sin^3\theta.$$
(*b*) Show that the length s of the curve,
$$x^{2/3}+y^{2/3} = a^{2/3},$$
measured from $(0, a)$ to the point (x, y) is given by
$$s = \tfrac{3}{2}\sqrt[3]{ax^2}.$$

7. A curve is given by the equations
$$x = a\,(\cos \theta+\theta \sin \theta),\ y = a\,(\sin \theta-\theta \cos \theta)\ ;$$
find the length of the arc from $\theta = 0$ to $\theta = \alpha$.

8. Show that the arc of the upper half of the curve
$$r = a(1-\cos \theta)$$
is bisected by $\theta = 2\pi/3$.

9. Find the length of an arc of the curve
$$r = ae^{\theta \cot \alpha}$$
taking $s = 0$ when $\theta = 0$.

10. Find the length of the curve $r = a \cos^3 (\theta/3)$.

11. Find the length of the loop of the curve
$$r = a\,(\theta^2-1).$$

12. Show that the length of the loop of the curve
$$3ay^2 = x(x-a)^2 \text{ is } 4a/\sqrt{3}.$$

13. Prove that the whole length of the curve
$$x^2(a^2-x^2) = 8a^2y^2$$
is $\pi a\sqrt{2}$.

14. Find the length of the arc of the curve

$$x = t^2 \cos t,\ y = t^2 \sin t$$

from the origin to the point t.

15. Sketch the curve $y = -\log(1-x^2)$, and show that the length of the arc measured from the origin to the point whose abscissa is x, is

$$\log[(1+x)/(1-x)]-x.$$

16. Find the length of the arc of the curve $y = \log \dfrac{e^x-1}{e^x+1}$ from $x = 1$ to $x = 2$.

17. Find the length of the arc of the curve $y = x(2-x)$ as x varies from 0 to 2.

18. Prove that the loop of the curve

$$x = t^2,\ y = t - \tfrac{1}{3}t^3$$

is of length $4\sqrt{3}$.

19. Prove that the length of the arc of the hyperbolic spiral $r\theta = a$, taken from the point $r = a$ to $r = 2a$, is

$$a\left\{\sqrt{5}-\sqrt{2}+\log\frac{2+\sqrt{8}}{1+\sqrt{5}}\right\}.$$

20. Find the length of the arc of the parabola

$$l/r = 1+\cos\theta$$

cut off by its latus rectum.

21. Show that the length of the arc of the curve given by

$$x = a(3\sin\theta - \sin^3\theta),\ y = a\cos^3\theta$$

measured from $(0, a)$ to any point (x, y) is $\frac{3}{2}a(\theta+\sin\theta\cos\theta)$.

22. Find the length of the curve

$$x = e^\theta \sin\theta,\ y = e^\theta\cos\theta$$

from $\theta = 0$ to $\theta = \pi/2$.

23. Show that the length of the curve

$$x = e^\theta\left(\sin\theta/2+2\cos\theta/2\right),\ y = e^\theta\left(\cos\theta/2-2\sin\theta/2\right)$$

measured from $\theta = 0$ to $\theta = \pi$ is $5(e^\pi-1)/2$.

24. If 's' be the length of the arc of the curve

$$x = a(\theta+\sin\theta\cos\theta),\ y = a(1+\sin\theta)^2$$

measured from the point $\theta = -\pi/2$ to a point θ, show that s^4 varies as y^3.

25. Trace the curve

$$x = a(\theta-\sin\theta),\ y = a(1-\cos\theta),$$

as θ varies from 0 to 2π and show that the point $\theta = \frac{2}{3}\pi$ divides it in the ratio 1 : 3.

Answers

1.	$\log(2+\sqrt{3})$.	**2.**	$\log[(e^2+1)/e]$.
3.	$2a[\sqrt{2}+\log(1+\sqrt{2})]$.	**4.**	$c \sinh(x/c)$.
6.	$6a$.	**7.**	$\frac{1}{2}a\alpha^2$.
9.	$a \sec\alpha\,(e^{\theta\cot\alpha}-1)$.	**10.**	$\frac{8}{3}a\pi$.
11.	$8a/3$.	**14.**	$\frac{1}{3}[(4+t^2)^{3/2}-8]$.
16.	$\log(e+e^{-1})$.	**17.**	$\frac{1}{2}\log(2+\sqrt{5})+\sqrt{5}$
20.	$l[\sqrt{2}+\log(1+\sqrt{2})]$.	**22.**	$\sqrt{2}(e^{\pi/2}-1)$.

8·4. Intrinsic Equations of a curve

Def. *If, s, denotes the length of the arc of a curve measured from some fixed point A to a variable point P, and ψ denotes the angle between the tangents at A and P, then a relation between, s, and, ψ, is called an* **Intrinsic Equation** *of the curve.*

Also, s and ψ are called the **intrinsic co-ordinates**.

The name '*Intrinsic*' arises from the fact that s and ψ for a point depend only upon the *form* of the curve and not on its *position* in the plane, so that they are inherently associated with the curve.

Intrinsic co-ordinates of a point on a curve will not change if the curve changes its position in the plane. This is not the case for the ordinary cartesian or polar co-ordinates.

8·41. Derivation of Intrinsic Equations from Cartesian Equations.

Let $y = f(x)$ be the cartesian equation of a given curve.

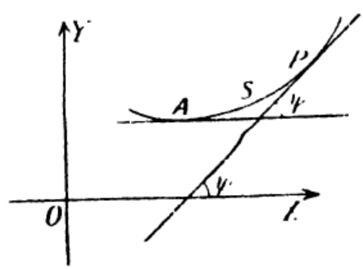

Fig. 32.

We suppose that the abscissa of the fixed point A is a and the tangent at A is parallel to x-axis. Let $P(x, y)$ be any variable point on the curve. Let arc $AP = s$. Let the tangent at P make angle ψ with x-axis. We have

$$s = \int_a^x \sqrt{[1+f'^2(x)]}\, dx, \qquad \ldots(i)$$

$$\tan \psi = f'(x). \qquad ...(ii)$$

Eliminating x between (i) and (ii), we obtain the required intrinsic equation.

Examples

1. *Obtain the intrinsic equation of the catenary*

$$y = a \cosh (x/a),$$

taking the vertex $(0, a)$ *as the fixed point.*

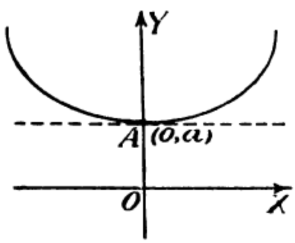

Fig. 33.

The tangent at A $(0, a)$ is parallel to X-axis.

We have

$$s = \int_0^x \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$$

$$= \int_0^x \cosh \frac{x}{a}\, dx$$

$$= a \sinh \frac{x}{a} \Big|_0^x$$

$$= a \sinh \frac{x}{a} \qquad ...(i)$$

Also, $\qquad \tan \psi = \dfrac{dy}{dx} = \sinh \dfrac{x}{a} \qquad ...(ii)$

From (i) and (ii), we obtain

$$\mathbf{s = a \tan \psi},$$

as the required intrinsic equation of the given curve.

2. *Find the intrinsic equation of the parabola* $y^2 = 4ax$; *origin being taken as the fixed point.*

Let $P(x, y)$ be a point on the curve. Y-axis is the tangent at O. Let arc $OP = s$.

From the given equation, we have

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \quad \tan \psi = \frac{dx}{dy} = \frac{y}{2a} \qquad ...(i)$$

ψ being the angle which the tangent at any point makes with Y-axis.

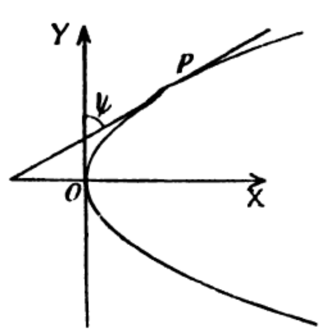

Fig. 34.

Also,

$$s = \int_0^y \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}dy$$

$$= \frac{1}{2}\int_0^y \sqrt{(4a^2+y^2)}\, dy$$

$$= \frac{1}{2a}\left[\frac{y\sqrt{(4a^2+y^2)}}{2} + \frac{4a}{2}\log\frac{\{y+\sqrt{(y^2+4a^2)}\}}{2a}\right]. \qquad ...(ii)$$

Eliminating y from (i) and (ii), we get

$$s = a\,[\tan\psi\sec\psi + \log(\tan\psi+\sec\psi)],$$

as the required intrinsic equation.

3. *Obtain the intrinsic equation of the cycloid*

$$x = a(\theta+\sin\theta),\ y = a(1-\cos\theta),$$

the fixed point being the origin.

(Refer Fig. 30, p. 186).

X-axis is the tangent at the fixed point O. Let, $P(\theta)$ be a variable point on the cycloid. We have

$$\frac{dx}{d\theta} = a(1+\cos\theta),\ \frac{dy}{d\theta} = a\sin\theta.$$

$$\therefore\ \tan\psi = \frac{dy}{dx} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2},$$

$$\Rightarrow\quad \psi = \frac{\theta}{2}. \qquad ...(i)$$

$$\text{Also, } s = \int_0^\theta \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right]}\,d\theta$$

$$= 2a\int_0^\theta \cos\frac{\theta}{2}\,d\theta = 4a\left|\sin\frac{\theta}{2}\right|_0^\theta = 4a\sin\frac{\theta}{2} \qquad ...(ii)$$

From (i) and (ii), we obtain

$$s = 4a\sin\psi,$$

as the required intrinsic equation of the cycloid.

Exercises

1. Show that the intrinsic equation of the semi-cubical parabola

$$ay^2 = x^3,$$

taking its cusp as the fixed point is

$$27s = 8a(\sec^3\psi - 1).$$

2. Show that the intrinsic equation of the astroid

$$x^{2/3}+y^{2/3} = a^{2/3},$$

taking $(a, 0)$ as the fixed point is

$$s = 3/2a\sin^2\psi.$$

8·42. Derivation of Intrinsic Equations from Polar Equations.

Let $r = f(\theta)$ be the polar equation of a given curve. Let the vectorial angle of the fixed point A be α. We suppose that the tangent at A is parallel to the initial line. We take any point P on the curve whose vectorial angle is θ. Let arc $AP = s$.

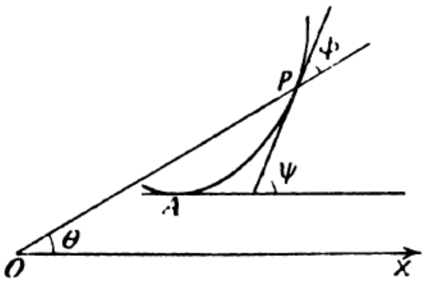

Fig. 35.

We have

$$\tan \varphi = r \frac{d\theta}{dr} = \frac{f(\theta)}{f'(\theta)} \qquad \ldots(i)$$

φ being the angle between the radius vector and the tangent.

Also $$\psi = \theta + \varphi. \qquad \ldots(ii)$$

$$s = \int_{\alpha}^{\theta} \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta$$

$$= \int_{\alpha}^{\theta} \sqrt{[f^2(\theta) + f'^2(\theta)]}\, d\theta \qquad \ldots(iii)$$

Eliminating θ and φ from (i), (ii), (iii), we obtain the required intrinsic equation.

Example

1. *Obtain the intrinsic equation of cardioide*

$$r = a(1 - \cos \theta),$$

taking pole as the fixed point.

(Refer Fig. 31, page 186).

Initial line is the tangent at the pole.

We have

$$\frac{dr}{d\theta} = a \sin \theta$$

$$\therefore \qquad \tan \varphi = r \frac{d\theta}{dr} = \frac{1 - \cos \theta}{\sin \theta}$$

$$= \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \tan \frac{\theta}{2},$$

$$\Rightarrow \qquad \varphi = \frac{\theta}{2}$$

Again, $\psi = \varphi + \theta = \dfrac{3\theta}{2}$ ...(*i*)

Also, $s = \displaystyle\int_0^{\theta} \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta$

$$= 2a \int_0^{\theta} \sin \frac{\theta}{2}\, d\theta$$

$$= -4a \left| \cos \frac{\theta}{2} \right|_0^{\theta} = 4a\left[1 - \cos \frac{\theta}{2}\right] \qquad ...(ii)$$

From (*i*) and (*ii*), we obtain

$$s = 4a\left[1 - \cos \frac{\psi}{3}\right]$$

as the required intrinsic equation.

Ex. Show that the intrinsic equation of the cardioide

$$r = a(1 + \cos \theta)$$

is $s = 4a \sin \frac{1}{3} \psi$, taking $\theta = 0$ as the fixed point.

8·5. Rectification of ellipse. $x = a \cos \theta$, $y = b \sin \theta$.

We have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta$$

$$= a^2 \sin^2 \theta + a^2(1 - e^2) \cos^2 \theta$$

$$= a^2 (1 - e^2 \cos^2 \theta)\ ;$$

e being the eccentricity of the ellipse. Thus the length of the ellipse is given by

$$4a \int_0^{\frac{1}{2}\pi} \sqrt{(1 - e^2 \cos^2 \theta)}\, d\theta.$$

Now it is not possible to express this integral in terms of a combination of a finite number of algebraic, logarithmic, exponential, trigonometric or inverse trigonometric functions. This can be formulated in terms of a new type of functions known as *Elliptic Functions* only. We may, however, proceed as follows :

Now by binomial theorem, we have

$$^*(1 - e^2 \cos^2 \theta)^{1/2} = 1 - \tfrac{1}{2}e^2 \cos^2 \theta - \tfrac{1}{8} e^4 \cos^4 \theta - \tfrac{1}{16} e^6 \cos^6 \theta \ldots$$

---

*It may be remembered that the theorem *the integral of a sum is equal to the sum of the integrals* may not be true in the case of the sum of an infinite series. In the present case, however, this is true but the justification thereof is beyond the scope of this book.

$$\therefore \int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2 \cos^2 \theta)}\, d\theta$$

$$= 1 - \frac{1}{2}e^2 \frac{1}{2} \frac{\pi}{2} - \frac{1}{8}e^4 \frac{3.1.\pi}{4.2.2} - \frac{1}{16}e^6 \frac{5.3.1.\pi}{6.4.2.2}$$

$$= 1 - \frac{1}{8}e^2 \pi - \frac{3}{128} e^4 \pi - \frac{5}{512} e^6 \pi \ldots\ldots$$

Thus we can find the length of the ellipse up to any number of decimal places.

EXERCISES ON CHAPTER 8

1. Find the length of the cardioide $r = a\,(1-\cos\theta)$, lying outside the circle $r = a \cos \theta$.

2. Prove that the cardioide $r = a\,(1+\cos\theta)$ is divided by the line $4r \cos \theta = 3a$ into two parts such that the lengths of the arcs on either side of this line are equal.

3. Show that the length of a loop of the curve

$$r = a\,(1+\cos 2\theta)$$

is

$$\tfrac{2}{3}\sqrt{3}[2\sqrt{3}+\log(\sqrt{3}+2)]a.$$

4. Show that the point $\theta = \frac{1}{6}\pi$ divides the arc of the curve

$$x = a \cos^3 \theta,\ y = a \sin^3 \theta,$$

lying in the first quadrant in the ratio 1 : 3.

5. Find the length of the curve defined by the equations

$$x \cos \theta = a \cos(\tan \theta - \theta),$$
$$y \cos \theta = a \sin(\tan \theta - \theta)$$

between the points for which $\theta = 0$ and $\theta = \alpha < \frac{1}{2}\pi$.

6. Show that the length of the curve whose equation is

$$4\,(x^2+y^2)-a^2 = 3a^{4/3}\, y^{2/3}$$

is equal to $6a$.

$$\left[\text{It may be shown that } \left(\frac{ds}{dy}\right)^2 = \frac{a^{4/3}}{4y^{2/3}\,(a^{2/3}-y^{2/3})}\cdot\right]$$

7. Trace the curve

$$x = a\,(\sin\theta+\tfrac{1}{3}\sin 3\theta),\ y = a\,(\cos\theta-\tfrac{1}{3}\cos 3\theta)$$

and show that its length is $8a$.

8. Trace the curve

$$x = a\,(\theta+\cos\theta \sin\theta-\sin\theta),\ y = a\,(\cos^2\theta-\cos\theta)$$

and find the length of its one span which is obtained as θ varies from 0 to 2π.

9. Find the length of the arc of the curve $y = e^x$ measured from $x = 0$ to $x = 1$.

10. Find the length of the arc of the curve $r = a/\theta^2$, as θ ranges from 1 to 2.

11. Find the length of the arc and the area cut off by $y = mx$ from the curve $y^3 = ax^2$.

12. If s be the length of the curve

$$r = a \tanh \tfrac{1}{2}\theta,$$

between the origin and $\theta = 2\pi$, and $\triangle$ be the area under the curve between the same two points, prove that

$$\triangle = a(s - a\pi).$$

13. If 's' be the length of the curve

$$a\theta = \sqrt{(r^2 - a^2)} - a \sec^{-1}(r/a)$$

enclosed between the points $r = a$ and $r = 2a$ and, A the area of the sector subtended by it at the pole, then $\sqrt{3}A = as$.

14. A curve is given by

$$x = a \sin \theta - b \sin 2\theta, \; y = a \cos \theta - b \cos 2\theta\,;$$

show that its perimeter is equal to that of an ellipse with semi-axes $a + 2b$, $a - 2b$.

15. Show that the intrinsic equation of the curve

$$x = e^\theta \sin \theta, \; y = e^\theta \cos \theta$$

is

$$se^{(-\pi/4)} + \sqrt{2} = \sqrt{2}\,(\cosh\psi - \sinh\psi),$$

where $\theta = \frac{1}{4}\pi$ is the fixed point.

16. Find the intrinsic equation of the curve $y = \log x$, taking the point $(1, 0)$ as the fixed point.

Answers

1. $4a\sqrt{3}$. 5. $\frac{1}{2} a \tan^2 \alpha$. 8. $a(4\sqrt{3} + \frac{8}{3}\pi)$.

9. $\sqrt{(e^2+1)} - \log\{[1+\sqrt{(1+e^2)}]/(\sqrt{2}+1)\} - (\sqrt{2}-1)$

10. $\frac{1}{4} a \{2\sqrt{5} - \sqrt{2} + \log[(\sqrt{2}-1)/(\sqrt{5}-2)]\}$.

11. $\dfrac{a}{27\,m^3}\,[(4m^2+9)^{3/2} - 8m^3]$, $\dfrac{a^2}{10\,m^5}$.

16. $s = \dfrac{\sqrt{2}}{\cos\psi - \sin\psi} - \sqrt{2} + \log(\sqrt{2}+1) - \log\dfrac{\sqrt{2}+\cos\psi - \sin\psi}{\cos\psi + \sin\psi}$.

9

Volumes and Surfaces of Revolution

9·1. Let AB be an arc of a curve and let CD be a straight line which does not intersect the curve.

Draw AL and BM perpendiculars to the line CD.

A solid will be obtained if the region $ALMBA$ revolves about the line CD. This solid is said to be obtained by the revolution of the arc AB about the line CD.

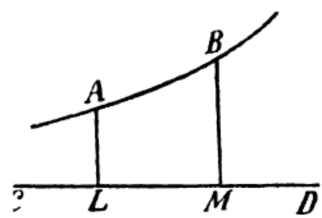

Fig. 36.

In this chapter we shall learn how to obtain the volume of this solid and also the area of its surface.

The line CD about which the curve rotates is called the *axis of revolution.*

To start with, we take X-axis as axis of revolution, and obtain expressions for the volume and the surface of the solid. Later on, we shall obtain expressions for the volume and the surface when any line is the axis of revolution.

9·2. Volume of a solid of revolution. *To show that the volume obtained by revolving about X-axis the arc of the curve* $y = f(x)$, *intercepted between the points whose abscissae are a, b, is*

$$\int_a^b \pi y^2\, dx, \text{ i.e., } \int_a^b \pi[f(x)]^2\, dx\,;$$

it being assumed that the arc does not cut X-axis.

Let G, H, be points on the curve $y = f(x)$ with abscissae a and b. Let $P(x, y)$ be a variable point on the curve. Let V be the

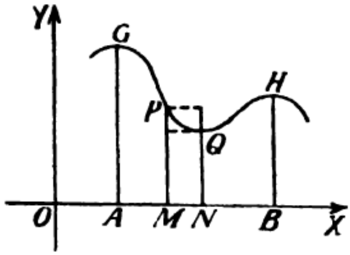

Fig. 37.

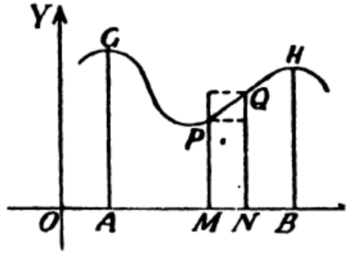

Fig. 38.

volume of the solid obtained by revolving the arc GP about X-axis so that V is a function of x. We take another point

$$Q\ (x+\triangle x,\ y+\triangle y)$$

on the curve so near P that, as a point moves on the curve from P to Q, its ordinate either constantly increases as in Fig. 37 or constantly decreases as in Fig. 38. Complete the rectangles NP, MQ.

Now $\triangle V$, which is the volume obtained by revolving the arc PQ, lies between the volumes of the two discs obtained by revolving the two rectangles NP, MQ about X-axis.

For Fig. 37, we have

$$\pi y^2\ \triangle x < \triangle V < \pi(y+\triangle y)^2\ \triangle x$$

$$\Rightarrow \qquad \pi y^2 < \frac{\triangle V}{\triangle x} < \pi(y+\triangle y)^2$$

$$\Rightarrow \qquad \frac{dV}{dx} = \pi y^2$$

For Fig. 38, we have

$$\pi(y+\triangle y^2)\ \triangle x < \triangle V < \pi y^2\ \triangle x$$

$$\Rightarrow \qquad \pi(y+\triangle y)^2 < \frac{\triangle V}{\triangle x} < \pi y^2$$

$$\Rightarrow \qquad \frac{dV}{dx} = \pi y^2, \text{ as before.}$$

Thus $\displaystyle\int_a^b \pi y^2\, dx = \int_a^b \frac{dV}{dx}\, dx = \Big|\, V \Big|_a^b$

= value of V for x equal to b—value of V for x equal to a

= the value of V for x equal to $b-0$

= the volume of the solid obtained by revolving the arc GH about X-axis.

Note. It follows from above that the volume obtained on revolving about y-axis, the arc of a curve $x = f(y)$ intercepted between the points whose ordinates are a, b is

$$\int_a^b \pi x^2 \, dy.$$

it being assumed that the arc does not cut y-axis.

Examples

1. *Find the volume of the solid obtained by revolving the ellipse $x^2/a^2+y^2/b^2 = 1$ about the axis of x.*

(Refer Fig. 10, page 154).

It is easy to see that the solid obtained by revolving the arc ABA' about X axis is the same as the solid obtained by revolving the whole ellipse. Also, the volume of the solid is double the volume of the solid obtained by revolving the arc AB. The required volume, therefore,

$$= 2\pi \int_0^a y^2 dx = 2\pi \frac{b^2}{a^2} \int_0^a (a^2-x^2)\, dx$$

$$= 2\pi \frac{b^2}{a^2} \left| a^2 x - \frac{x^3}{3} \right|_0^a = \frac{4}{3}\pi a b^2.$$

2. *Find the volume of the solid obtained by revolving one arc of the cycloid*

$$x = a(\theta+\sin\theta),\ y = a(1+\cos\theta)$$

about X-axis.

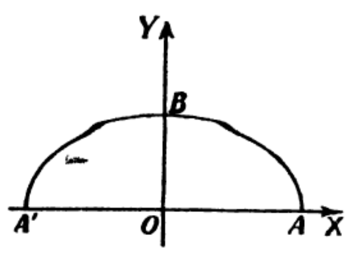

Fig. 39.

The volume of the solid obtained by revolving the arc $A'BA$ is double the volume of the solid obtained by revolving the arc AB. As a point moves from B to A, the value of the parameter θ increases from 0 to π. Therefore the required volume.

$$= 2\pi \int_0^{\pi a} y^2 \, dx, \text{ where } OA = \pi a.$$

Changing the variable x to θ, we see that the volume

$$= 2\pi\int_0^{\pi} a^2(1+\cos\theta)^2\, a(1+\cos\theta)\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} 8\cos^6\frac{\theta}{2}\, d\theta$$

$$= 32\pi a^3 \int_0^{\frac{1}{2}\pi} \cos^6\varphi\, d\varphi, \text{ where } \varphi = \frac{\theta}{2};$$

$$= 32\pi a^3 . \frac{5}{6} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2} = 5\pi^2 a^3.$$

3. *Find the volume of the solid obtained by revolving the cardioide* $r = (l + \cos\theta)$ *about the initial line.*

The required volume

$$= \pi\int_0^{2a} y^2\, dx, \text{ for } OA = 2a.$$

We change the variables x and y to θ.

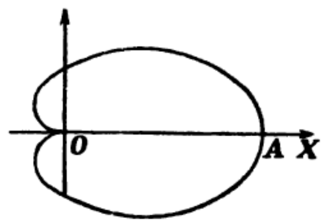

Fig. 40.

We have

$$x = r\cos\theta = a\cos\theta\,(1+\cos\theta)$$

$$y = r\sin\theta = a\sin\theta\,(1+\cos\theta)$$

$\therefore$ $$dx = -a\,(\sin\theta + 2\sin\theta\cos\theta)\, d\theta$$

Also, $\theta = \pi$ when $x = 0$ and $\theta = 0$, when $x = 2a$

$$\therefore \quad V = -\pi\int_{\pi}^{0} a^2\sin^2\theta\,(1+\cos\theta)^2\, a\sin\theta\,(1+2\cos\theta)\, d\theta$$

$$= \pi a^3\int_0^{\pi} \sin^3\theta\,(1+\cos\theta)^2\,(1+2\cos\theta)\, d\theta$$

$$= \pi a^3\int_0^{\pi} 8\sin^3\frac{\theta}{2}\cos^3\frac{\theta}{2} \cdot 4\cos^4\frac{\theta}{2}\left(4\cos^2\frac{\theta}{2} - 1\right) d\theta$$

$$= 128\,\pi a^3\int_0^{\pi} \sin^3\frac{\theta}{2}\cos^9\frac{\theta}{2}\, d\theta$$

$$-32\,\pi a^3\int_0^{\pi} \sin^3\frac{\theta}{2}\cos^7\frac{\theta}{2}\, d\theta$$

$$= 256\,\pi a^3\int_0^{\frac{1}{2}\pi} \sin^3\varphi\cos^9\varphi\, d\varphi$$

$$-64\,\pi a^3\int_0^{\frac{1}{2}\pi} \sin^3\varphi\cos^7\varphi\, d\varphi, \text{ where } \varphi = \theta/2$$

$$= 256\,\pi a^3 \frac{2.8.6.4.2}{12.10.8\,6.4.2} - 64\,\pi a^3 \frac{2.6.4.2}{10.8.6.4.2}$$

$$= \frac{64\pi a^3}{25} - \frac{8\pi a^3}{5} = \frac{8\pi a^3}{5}.$$

4. *Find the volume of the solid obtained by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.*

The volume of the solid obtained by revolving the whole curve about the initial line is double the volume of the solid obtained by revolving the arc in the first quadrant about the same line.

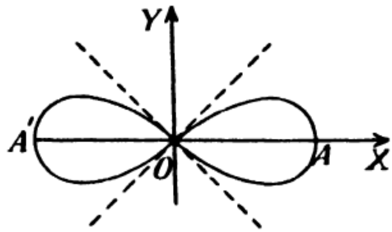

Fig. 41.

The required volume, therefore,

$$= 2\pi \int_0^a y^2\, dx \text{ for } OA = a.$$

We change the variables x and y to θ. We have

$$x = r\cos\theta = a\sqrt{(\cos 2\theta)}\cos\theta$$

$$\Rightarrow \quad dx = \left(-a\sqrt{\cos 2\theta}.\sin\theta - \frac{a\sin 2\theta}{\sqrt{\cos 2\theta}}\cos\theta\right)d\theta$$

$$= -a\frac{\sin 3\theta}{\sqrt{\cos 2\theta}}\,d\theta$$

Also,

$$y^2 = r^2\sin^2\theta = a^2\cos 2\theta\sin^2\theta.$$

Again $\theta = \pi/4$ when $x = 0$ and $\theta = 0$ when $x = a$.

$$\therefore \text{ Volume} = -2\pi a\int_{\frac{1}{4}\pi}^{0} a^2\cos 2\theta\sin^2\theta\,\frac{\sin 3\theta}{\sqrt{\cos 2\theta}}\,d\theta$$

$$= 2\pi a^3\int_0^{\frac{1}{4}\pi}\sin^2\theta\sin 3\theta\sqrt{\cos 2\theta}\,d\theta$$

$$= 2\pi a^3\int_0^{\frac{1}{4}\pi}\sin^2\theta\,(3\sin\theta - 4\sin^3\theta)\sqrt{\cos 2\theta}\,d\theta$$

We put

$$\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$$

$$\therefore \text{ Volume} = -2\pi a^3\int_1^{\sqrt{\frac{1}{2}}}(1-t^2)(4t^2-1).\sqrt{(2t^2-1)}\,dt$$

We write

$$(1-t^2)(4t^2-1)\sqrt{(2t^2-1)} = \frac{(1-t^2)(4t^2-1)(2t^2-1)}{\sqrt{(2t^2-1)}}$$

$$= \frac{-8t^6+14t^4-7t^2+1}{\sqrt{(2t^2-1)}}.$$

Let $\int \frac{-8t^6+14t^4-7t^2+1}{\sqrt{(2t^2-1)}}\,dt$

$=(at^5+bt^4+ct^3+dt^2+et+f)\sqrt{(2t^2-1)}+g\int \frac{dt}{\sqrt{(2t^2-1)}}$.

Differentiating and multiplying with $\sqrt{(2t^2-1)}$, we obtain

$$-8t^6+14t^4-7t^2+1 = (5at^4+4bt^3+3ct^2+2dt+e).(2t^2-1)$$
$$+2t\,(at^5+bt^4+ct^3+dt^2+et+f)+g.$$

Equating the coefficients of like powers of t, we get

$$12a=-8\,; \qquad 10b=0\,; \qquad 8c-5a=14\,;$$
$$6d-4b=0\,; \qquad 4e-3c=-7\,; \qquad -2d+2f=0\,;$$
$$-e+g=1.$$

$\therefore \quad a=\frac{2}{3}, b=0, c=\frac{4}{3}, d=0, e=-\frac{3}{4}, f=0, g=\frac{1}{4}.$

$$\therefore \quad \int_1^{\sqrt{\frac{1}{2}}} \frac{-8t^6+14t^4-7t^2+1}{\sqrt{(2t^2-1)}}\,dt$$

$$=\left|\left(-\frac{2}{3}t^5+\frac{4}{3}t^3-\frac{3}{4}t\right)\sqrt{(2t^2-1)}\right|_1^{\sqrt{\frac{1}{2}}}+\frac{1}{4}\int_1^{\sqrt{\frac{1}{2}}}\frac{dt}{\sqrt{(2t^2-1)}}$$

$$=\left|\left(-\frac{2}{3}t^5+\frac{4}{3}t^3-\frac{3}{4}t\right)\sqrt{(2t^2-1)}\right|_1^{\sqrt{\frac{1}{2}}}$$
$$+\frac{1}{4}\left|\frac{\log\left[\sqrt{2}t+\sqrt{(2t^2-1)}\right]}{\sqrt{2}}\right|_1^{\sqrt{\frac{1}{2}}}$$

$$=\frac{1}{12}-\frac{1}{4\sqrt{2}}\log(\sqrt{2}+1).$$

$$\therefore \qquad V=\frac{\pi a^3}{2}\left[\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)-\frac{1}{3}\right].$$

Exercises

1. The loop of the curve $2ay^2=x(x-a)^2$ revolves about X-axis ; find the volume of the solid so generated.

2. Find the volume of the solid obtained by revolving the loop of the curve $a^2y^2=x^2(2a-x)(x-a)$ about X-axis.

3. Find the area enclosed by the curve $xy^2=4(2-x)$ and Y-axis and also the volume of the solid formed by the revolution of the curve through four right angles about the X-axis.

4. Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis, is a mean proportional between that generated by the revolution of the ellipse and of its auxiliary circle round the major axis.

5. Find the volume of the spindle shaped solid generated by revolving the hypocycloid $x^{2/3}+y^{2/3} = a^{2/3}$ about X-axis.

6. Prove that the volume of the solid generated by the revolution of the curve $y = a^3/(a^2+x^2)$ about its asymptote is $\pi^2a^3/2$.

7. Show that the volume of the solid obtained by revolving the area included between the curves $y^2 = x^3$ and $x^2 = y^3$ about X-axis is $5\pi/28$.

8. Find the volume formed by the revolution of the loop of the curve $y^2 = x^2 \, (a-x)/(a+x)$ about the X-axis.

9. Prove that the volume of the reel formed by the revolution of the cycloid $x = a \, (\theta+\sin \theta)$, $y = a \, (1-\cos \theta)$ about the X-axis is π^2a^3.

10. Show that the volume of the solid generated by the revolution of the cycloid

$$x = a \, (\theta+\sin \theta), \; y = a \, (1-\cos \theta)$$

about the y-axis is $\pi a^3 \, (\frac{3}{2}\pi^2-\frac{8}{3})$.

11. Find the volume generated by the portion of the arc

$$y = \sqrt{(1+x^2)},$$

lying between $x = 0$ and $x = 4$, as it revolves about the axis of x.

12. Trace the curve $y = e^x \sin x$ as x varies from 0 to π and show that the volume of the solid obtained by revolving it about X-axis is $\pi \, (e^{2\pi}-1)/8$.

13. Trace the curves $y = \sin x$ and $y = \cos x$ as x varies from 0 to $\pi/2$ and show that the volume of the solid obtained by revolving about X-axis the region enclosed by them and the X-axis is $\pi(\pi-2)/4$.

14. Find the volume of the solid generated by rotating completely about X-axis the area enclosed between $y^2 = x^3+5x$ and the lines $x = 2$ and $x = 4$.

15. Find the volume generated by the revolution of an arc of the catenary

$$y = c \cosh \frac{x}{c},$$

about the axis of x between $x = a$ and $x = b$.

16. Show that the volume of the solid obtained by revolving about X-axis the area enclosed by the parabola $y^2 = 4ax$ and its evolute

$$27ay^2 = 4(x-2a)^3$$

is $80\pi a^3$.

17. The figure bounded by a parabola and the tangents at the extremities of its latus rectum revolves about the axis of the parabola ; show that the volume of the solid thus obtained is $2a^3\pi/3$; $4a$ being the latus rectum of the parabola.

18. Find the volume generated by $y = x \sin mx$, as the area lying between $x = 0$ and $x = 2\pi/m$ revolves about X-axis.

19. The loop of the curve $r = a \cos 3\theta$ lying between $\theta = -\pi/6$ and $\theta = \pi/6$ revolves about the initial line ; show that the volume of the solid thus obtained is $19\pi a^3/960$.

20. Prove that the volume generated by the revolution about the initial line of the limacon $r = a+b \cos \theta$, $a > b$, is $4\pi a(a^2+b^2)/3$.

Answers

1. $\pi a^3/24$.	**2.** $23a^3\pi/60$.	**3.** 4π, $4\pi^2$
5. $32\pi a^3/105$.	**8.** $\frac{8}{3}\pi a^3 \log (8/e^2)$.	**11.** $76\,\pi/3$.
14. 90π.	**15.** $\left[\frac{\pi c^2}{4}\left(2x + \sinh \frac{2x}{c} \right)\right]_a^b$	
18. $\pi^2(8\pi^2 \quad 3)/6m^3$.		

9·3. Any axis of revolution. We interpret the result obtained in §9·2 in the following manner :—

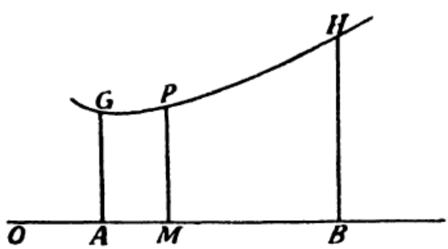

Fig. 42.

y, is the length of the perpendicular PM of a point on the curve from the axis of revolution, M being the foot of this perpendicular ; x denotes the distance of the foot of the perpendicular M from a fixed point O on the axis, a and b are the distances from the fixed point O of the feet A, B of the perpendicular from the extreme ends G, H of the given arc.

From this we deduce that the volume obtained by revolving the arc GH about the line AB is

$$\int_{OA}^{OB} \pi(\mathrm{MP}^2)\, \mathrm{d}\,(\mathrm{OM}).$$

Examples

1. *Find the volume of the solid obtained by the revolution of the cissoid*

$$y^2(2a-x) = x^3$$

about its asymptote.

The line $x = 2a$ is the asymptote of the curve. The perpendicular distance MP of any point $P(x, y)$ on the curve from the asymptote is

$$2a-x.$$

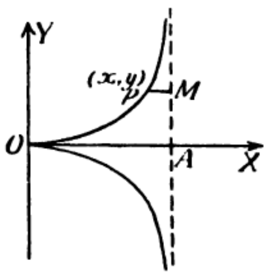

Fig. 43.

We take the point A, where the asymptote meets the X-axis as the fixed point on the axis of revolution. By symmetry, the volume of the solid obtained by revolving the whole curve about the asymptote is double of the volume obtained by revolving the part of it lying in the first quadrant.

The required volume

$$= 2\pi\int_0^{\infty} (MP)^2\, d(AM) = 2\pi \int_0^{\infty} (2a-x)^2\, dy$$

We change the independent variable from y to x.

Since $\quad y = \dfrac{x^{3/2}}{\sqrt{(2a-x)}}$

we have, $dy = \dfrac{(3a-x)\sqrt{x}\,\sqrt{(2a-x)}}{(2a-x)^2}\, dx.$

Thus the volume $= 2\pi \displaystyle\int_0^{2a} (3a-x)\, \sqrt{(2a-x)}\sqrt{x}\, dx.$

To evaluate the integral, we put $x = 2a \sin^2 \theta$.

We thus see that the integral

$$= 16\pi a^3\int_0^{\frac{1}{2}\pi} (3-2 \sin^2 \theta) \sin^2 \theta \cos^2 \theta \, d\theta$$

$$= 16\pi a^3\left[3 \int_0^{\frac{1}{2}\pi} \sin^2 \theta \cos^2 \theta \, d\theta - 2 \int_0^{\frac{1}{2}\pi} \sin^4 \theta \cos^2 \theta \, d\theta\right]$$

$$= 16\pi a^3\left[\frac{3\cdot1\cdot1}{4\cdot2} \cdot \frac{\pi}{2} - 2 . \frac{3\cdot1}{6\cdot4\cdot2} \cdot \frac{\pi}{2} \right] = 2\pi^2a^3.$$

2. *The smaller segment of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

cut off by the chord

$$\frac{x}{a}+\frac{y}{b}=1,$$

revolves completely about this chord. Show that the volume generated is

$$\tfrac{1}{6}\pi(10-3\pi)\,a^2b^2(a^2+b^2)^{-\frac{1}{2}}$$

The given chord $x/a+y/b=1$ joins the points A, B.

Take any point

$$P(a\cos\theta,\ b\sin\theta)$$

on the ellipse.

The length of the perpendicular MP is

$$\frac{ab(\cos\theta+\sin\theta-1)}{\sqrt{(a^2+b^2)}}.$$

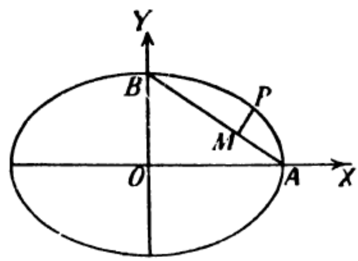

Fig. 44.

We take the point A as the point fixed on the axis of revolution AB. We now require the length AM.

Now
$$\begin{aligned}AM^2 &= AP^2-PM^2\\ &= (a\cos\theta-a)^2+(b\sin\theta)^2-\frac{a^2b^2(\cos\theta+\sin\theta-1)^2}{a^2+b^2}\\ &= a^2(\cos\theta-1)^2+b^2\sin^2\theta\\ &\quad-\frac{a^2b^2}{a^2+b^2}\left[(\cos\theta-1)^2+\sin^2\theta+2\sin\theta(\cos\theta-1)\right]\\ &= \frac{a^4(\cos\theta-1)^2+b^4\sin^2\theta-2a^2b^2\sin\theta(\cos\theta-1)}{a^2+b^2}\\ &= \frac{[a^2(\cos\theta-1)-b^2\sin\theta]^2}{a^2+b^2}.\end{aligned}$$

$$AM=\frac{a^2(1-\cos\theta)+b^2\sin\theta}{\sqrt{(a^2+b^2)}}.$$

$$\Rightarrow\quad d(AM)=\frac{a^2\sin\theta+b^2\cos\theta}{\sqrt{(a^2+b^2)}}\,d\theta.$$

Thus the required volume is

$$=\pi\int_0^{AB}(MP)^2\,d(AM)$$

$$= \pi \int_0^{\frac{1}{2}\pi} \frac{a^2b^2 (\cos\theta + \sin\theta - 1)^2}{(a^2+b^2)} \cdot \frac{a^2 \sin\theta + b^2 \cos\theta}{\sqrt{(a^2+b^2)}} d\theta$$

$$= \frac{\pi a^2 b^2}{(a^2+b^2)^{3/2}} \int_0^{\frac{1}{2}\pi} (a^2 \sin\theta + b^2 \cos\theta)$$
$$\times (2 + 2\sin\theta\cos\theta - 2\sin\theta - 2\cos\theta)\, d\theta$$

$$= \frac{2\pi a^2 b^2}{(a^2+b^2)^{3/2}} \int_0^{\frac{1}{2}\pi} \Big[(a^2 \sin\theta + b^2\cos\theta) + a^2 \sin^2\theta\cos\theta$$
$$+ b^2 \cos^2\theta \sin\theta - (a^2 \sin^2\theta + b^2 \cos^2\theta) - (a^2+b^2)\sin\theta\cos\theta \Big] d\theta$$

$$= \frac{2\pi a^2 b^2}{(a^2+b^2)^{3/2}} \left[(a^2+b^2) + \frac{a^2}{3} + \frac{b^2}{3} - \left(\frac{a^2\pi}{4} + \frac{b^2\pi}{4} \right) - \frac{(a^2+b^2)}{2} \right]$$

$$= \frac{\pi a^2 b^2}{\sqrt{(a^2+b^2)}} \left(\frac{5}{3} - \frac{\pi}{2} \right) = \tfrac{1}{6}\pi(10 - 3\pi) a^2 b^2 (a^2+b^2)^{-\frac{1}{2}}.$$

3. *The loop of the curve $2ay^2 = x(x-a)^2$ revolves about the straight line $y = a$; find the volume of the solid generated.*

The loop of the curve lies between the limits $x = 0$, $x = a$.

The volume obtained by revolving the loop about $y = a$ is the difference of the two volumes obtained by revolving the arcs OBA and OCA about it.

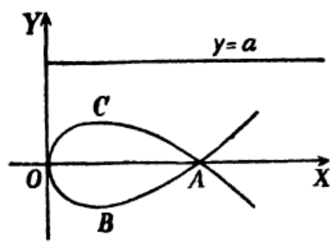

Fig. 45.

For any point on the arc OBA

$$y = \frac{\sqrt{x}(x-a)}{\sqrt{(2a)}} = -\frac{\sqrt{x}(a-x)}{\sqrt{(2a)}}.$$

For any point on the arc OCA,

$$y = -\frac{\sqrt{x}(x-a)}{\sqrt{(2a)}} = \frac{\sqrt{x}(a-x)}{\sqrt{(2a)}}.$$

The volume obtained by revolving the arc OBA

$$= \pi \int_0^a \left(a + \frac{\sqrt{x}(a-x)}{\sqrt{(2a)}} \right)^2 dx.$$

The volume obtained by revolving the arc OCA

$$= \pi \int_0^a \left(a - \frac{\sqrt{x}(a-x)}{\sqrt{(2a)}} \right)^2 dx.$$

the required volume which is the difference of the above two volumes $= \dfrac{4a\pi}{\sqrt{(2a)}} \displaystyle\int_0^a \sqrt{x}(a-x)\, dx$

$$= \frac{4a\pi}{\sqrt{(2a)}}\left|\frac{2}{3}\ ax^{3/2} - \frac{2}{5}\ x^{5/2}\right|_0^a = \frac{8\sqrt{2}\pi a^3}{15}.$$

Exercises

1. A figure is bounded by the axis of y and the arc of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ on which x is negative. A solid is generated by the revolution of this figure about the line $x = -2a$; prove that its volume is $2\pi(\pi - \frac{4}{3})a^2b$.

2. The ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is divided into two parts by the line $x = \frac{1}{2}a$, and the smaller part is rotated through four right angles about this line. Prove that the volume generated is

$$\pi a^2b(\tfrac{3}{4}\sqrt{3} - \tfrac{1}{3}\pi).$$

3. Show that the volume obtained on revolving about $x = a/2$ the area enclosed between the curves

$$xy^2 = a^2(a-x),\ (a-x)\,y^2 = a^2x$$

is $\pi a^3(4-\pi)/4$.

4. The area enclosed by the parabolas

$$x^2 = 4ay \text{ and } x^2 = 4a(2a-y)$$

revolves about the line $y = a$; find the volume of the solid so generated.

5. A quadrant of a circle of radius, a, revolves about its chord. Show that the volume of the spindle thus generated is

$$(\pi/6\sqrt{2})\ (10-3\pi)a^3.$$

6. The area cut off from the parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through four right angles about the chord. Show that the volume of the solid so formed is $(2\sqrt{5})\pi a^3/75$.

7. The ellipse $x^2/a^2 + y^2/b^2 = 1$, $(a > b)$ revolves about the tangent at one extremity of its minor axis; show that the volume of the solid obtained is $2\pi^2ab^2$.

8. Find the volume of the solid generated by the revolution of the curve

$$(a-x)y^2 = a^2x$$

about its asymptote.

9. Show that the volume of the spindle formed by revolving the arc of the parabola $y^2 = 4ax$ joining the vertex to one extremity of the latus rectum, about the tangent at the extremity is $7\sqrt{2}\pi a^3/60$.

10. Find the volume of the spindle formed by the revolution of a parabolic arc about the line joining the vertex to one extremity of the latus rectum.

11. A surface is formed by the revolution of the curve $y = x^2e^{-x}$ about the axis of x; show that the volume enclosed by the part of the surface which corresponds to the positive values of x is $3\pi/4$.

12. Show that the volume of the solid generated by the revolution of the curve $r = a+b \sec \theta$ about its asymptote is

$$2\pi a^2 \left(\tfrac{2}{3}a+\tfrac{1}{2}b\pi\right).$$

Answers

4. $32\pi a^3/15$. 8. $\frac{1}{2}\pi^2a^3$. 10. $(2/15\sqrt{5})\pi a^3$.

9·4. Area of the surface of the frustum of a cone. We can easily see that the area of the surface of right circular cone, the radius of whose circular base is r, and whose slant height is l, is πrl.

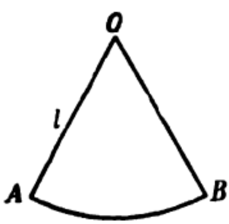

Fig. 46.

If we tear a right circular cone along one of its generators, we get a circular sector whose radius OA is equal to the slant height and whose arc AB is equal to the circumference of the circular base of the cone.

The area of this sector and, therefore, also the surface of the cone is equal to πrl.

If α be the semi-vertical angle of the cone, we have $r/l = \sin \alpha$. Therefore the surface of the cone is also equal to $\pi l^2 \sin \alpha$.

Consider, now, the frustum $CABD$ of a cone VAB. Let the radii $O'A$, and OC of its circular bases be r_1 and r_2 respectively and let its slant height CA be l_1. *The area of the surface of this frustum*

$$= \pi(AV^2-CV^2) \sin \alpha$$
$$= \pi(AV-CV)(AV \sin \alpha+CV \sin \alpha)$$
$$= \pi AC(O'A+OC)$$
$$= \pi l(r_1+r_2)$$

$= \pi \times$ *slant height* $\times$ *sum of the radii of the two bases.*

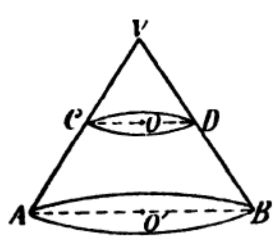

Fig. 47.

9·5. Surface of revolution. *To show that the area of the surface of the solid obtained on revolving about x-axis, the arc of the curve*

$y = f(x)$ intercepted between the points whose abscissae are a, b is,

$$\int_a^b 2\pi y \frac{ds}{dx}\, dx = 2\pi \int_a^b f(x) \sqrt{[1+f'^2(x)]}\, dx$$

Let $G\,[a, f(a)]$ be a fixed point and $P\,(x, y)$, a variable point on the arc. Let H be the point $[b, f(b)]$. Let σ be the area of the

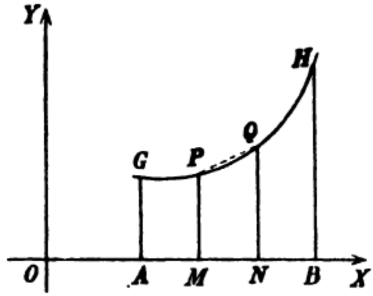

Fig. 48.

surface of the solid obtained by revolving the area GP about x-axis so that, σ is a function of x.

We take a point

$$Q(x+\Delta x,\ y+\Delta y)$$

on the curve near P.

Let arc $GP = s$ and arc $PQ = \Delta s$

Let $\Delta \sigma$ denote the area of the surface of the solid obtained by revolving the arc PQ.

By revolving the chord PQ about x-axis, we get a frustum of the cone whose slant height is PQ and the radii of whose circular ends are MP, NQ. The area of surface of this frustum

$$= \pi(PM+QN)\, PQ = \pi(y+y+\Delta y)\, PQ.$$

Let it be denoted by $\Delta \Sigma$

We take it as an axiom that

$$\lim \frac{\Delta \Sigma}{\Delta x} = \lim \frac{\Delta \sigma}{\Delta x} \Leftrightarrow \frac{d\Sigma}{dx} = \frac{d\sigma}{dx}$$

Now $$\frac{\Delta \Sigma}{\Delta x} = \pi(2y+\Delta y)\, \frac{PQ}{\Delta x}$$

$$= \pi(2y+\Delta y)\, \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\text{arc } PQ}{\Delta x}$$

in the limit, we have

$$\frac{d\Sigma}{dx} = 2\pi y \,.\, 1 \,.\, \frac{ds}{dx} = 2\pi y \frac{ds}{dx},$$

$$\therefore \qquad \frac{d\sigma}{dx} = 2\pi y \frac{ds}{dx}.$$

Now, $\displaystyle\int_a^b 2\pi y \frac{ds}{dx}\, dx = \int_a^b \frac{d\sigma}{dx}\, dx = \Big|\sigma\Big|_a^b$

= the value of σ for x equal to b—the value of σ for x equal to a

= Area of the surface obtained by revolving the arc GH.

Note. Taking s as the independent variable, we see that the surface $= \int 2\pi y ds$. In the case of a polar curve $r = f(\theta)$, where θ is the independent variable, the surface is equal to

$$\int 2\pi y \frac{ds}{d\theta}\, d\theta.$$

Also, in the case of the curve $x = f(t)$, $y = \varphi(t)$, where t is the parameter, the surface is equal to $\displaystyle\int 2\pi y \frac{ds}{dt}\, dt$.

Examples

1. *Find the surface of the solid formed by revolving the cardioide*

$$r = a(1+\cos\theta)$$

about the initial line.

(*See* Fig. 31, page 186)

The required surface

$$= \int_0^{\pi} 2\pi y \frac{ds}{d\theta}\, d\theta.$$

Now $\quad y = r\sin\theta = a\sin\theta(1+\cos\theta)$

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}$$

$$= \sqrt{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]}$$

$$= 2a\cos\frac{\theta}{2}.$$

$$\therefore \quad \text{surface} = \int_0^{\pi} 2\pi a \sin\theta(1+\cos\theta)\, 2a\cos\frac{\theta}{2}\, d\theta$$

$$= 4\pi a^2 \int_0^{\pi} \sin\theta(1+\cos\theta)\cos\frac{\theta}{2}\, d\theta$$

$$= 4\pi a^2 \int_0^{\pi} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \,.\, 2\cos^2\frac{\theta}{2} \,.\, \cos\frac{\theta}{2}\, d\theta$$

$$= 32\pi a^2 \int_0^{\frac{1}{2}\pi} \sin\varphi\cos^4\varphi\, d\varphi, \text{ where } \varphi = \frac{\theta}{2},$$

$$= 32\pi a^2 \left| -\frac{\cos^5\varphi}{5} \right|_0^{\frac{1}{2}\pi} = \frac{32}{5}\pi a^2.$$

2. ***Evaluate the surface area of the solid generated by revolving the cycloid***

$$x = a(\theta - \sin\theta),\ y = a(1-\cos\theta),$$

about the line $y = 0$.

We have

$$\frac{ds}{d\theta} = \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]}$$
$$= \sqrt{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]}$$
$$= 2a\sin\frac{\theta}{2}.$$

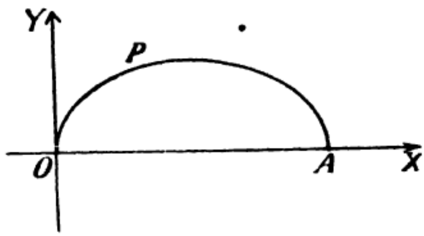

Fig. 49.

For the part OPA, θ varies from 0 to 2π.

The required surface

$$= \int_0^{2\pi} 2\pi y \frac{ds}{d\theta}.\, d\theta$$

$$= \int_0^{2\pi} 2\pi a(1-\cos\theta)\,.\,2a\sin\frac{\theta}{2}\,d\theta$$

$$= 8\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^3\varphi\, d\varphi, \text{ where } \frac{\theta}{2} = \varphi$$

$$= 32\pi a^2 \int_0^{\frac{1}{2}\pi} \sin^3\varphi\, d\varphi = 32\pi a^2\,\frac{2}{3} = \frac{64\pi a^2}{3}.$$

9·6. Theorems of Pappus.

9·61. Volume of Revolution. *If a closed plane curve revolves about a straight line in its plane, (the straight line not intersecting the curve), then the volume of the solid of revolution thus formed is obtained on multiplying the area of the region enclosed by the curve with the length of the path described by the centroid of the region.*

We take x-axis as the axis of revolution. Suppose that the curve is such that every line parallel to y-axis and lying between the co-ordinates $x = a$, $x = b$ of two points A, B meets the curve in two and only two points P_1, P_2.

Let $MP_1 = y_1$, and $MP_2 = y_2$ so that y_1, y_2 are functions of x; x varying between a and b.

Now the volume, V, of the solid of revolution is given by

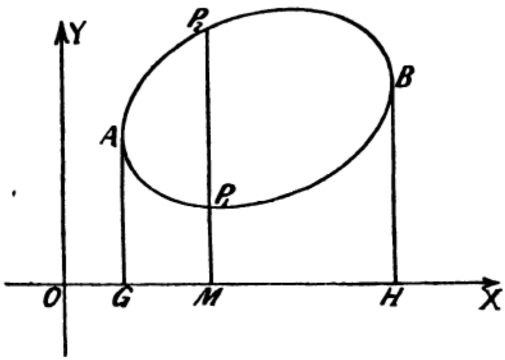

Fig. 50.

$$V = \pi\int_a^b y_2^2 dx - \pi\int_a^b y_1^2 dx$$

$$= \pi\int_a^b (y_2^2 - y_1^2)\, dx \qquad \ldots(1)$$

Also the ordinate, $\bar{y}$, of the centroid of the region is given by

$$\bar{y} = \frac{\int_a^b \frac{1}{2}(y_1+y_2)(y_2-y_1)\, dx}{A}$$

$$= \frac{1}{2}\frac{\int_a^b (y_2^2 - y_1^2)\, dx}{A}, \qquad \ldots(2)$$

A, being the area of the region.

From (1) and (2), we have

$$V = 2\pi\bar{y}A$$

so that the result follows ; $2\pi\bar{y}$ being the length of the path described by the centroid.

9·7. Surface of Revolution. *If a closed plane curve revolves about a straight line in its plane (the straight line not intersecting the curve), then the surface of the solid of revolution thus formed is obtained on multiplying the length of the curve with that of the path described by the centroid of the curve.*

We take x-axis as the axis of revolution

[Refer Fig. 50, above]

The surface, S, of the solid of revolution is given by

$$S = \int_0^l 2\pi y\, ds, \qquad \ldots(i)$$

where, l, denotes the length of the arc of the curve. We have here looked upon y as a function of s.

The centroid, $\bar{y}$, of the curve is given by

$$\bar{y} = \frac{\int_0^l y \, ds}{l}. \qquad \ldots(2)$$

From (1) and (2),

$$S = 2\pi\bar{y}l,$$

so that the result follows ; $2\pi\bar{y}$ being the length of the path described by the centroid.

Exercises

1. Find the surface of a sphere of radius a.

2. Find the surface of the solid generated by revolving the arc of the parabola $y^2 = 4ax$ bounded by its latus rectum about x-axis.

3. The part of the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the tangent at the vertex ; find the area of the curved surface of the reel thus generated.

4. Find the surface of the solid generated by the revolution of the Astroid.

$$x = a \cos^3 t, \qquad y = a \sin^3 t,$$

about the axis of x.

5. Find the surface of the solid generated by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.

6. Find the surface of the solid obtained by revolving the cardioide $r = a\,(1-\cos\theta)$ about the initial line.

7. Find the area of the surface of revolution formed by revolving the curve $r = 2a\cos\theta$ about the initial line.

8. Show that the ratio of the areas of the surface formed by the rotation of the arc of the cycloid

$$x = a\,(\theta+\sin\theta), \qquad y = a\,(1+\cos\theta)$$

between two consecutive cusps about the axis of x to the area enclosed by the cycloid and the axis of x is 64/9.

9. The arc of the curve $8a^2y^2 = x^2\,(a^2-2x^2)$, in the first quadrant, revolves about the axes of X and Y and generates the surfaces A and B. Show that the surface of A is to the surface of B as $3\sqrt{2}$ is to 2, and that the volume of B is that of a right cylinder standing

on a circular base of radius $\frac{1}{2}a$ and of altitude equal to the length of the arc.

10. Prove that the surface of the solid obtained by revolving the ellipse $b^2x^2+a^2y^2 = a^2b^2$ about the axis of x is

$$2\pi ab[\sqrt{(1-e^2)}+(1/e) \sin^{-1} e]$$

e, being the eccentricity of the ellipse.

11. An arc of a circle of radius a revolves about its chord. If the length of the arc is $2a\alpha$, $(\alpha < \pi/2)$, show that the area of the surface generated is $4\pi a^2$ (sin $\alpha - \alpha$ cos α).

12. Show that the surface of the solid obtained by revolving the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ about x-axis is

$$\pi^2 [\sqrt{2}+\log (1+\sqrt{2})]$$

13. Prove that the surface generated by the revolution of the tractrix.

$$x = a \cos t+\tfrac{1}{2}a \log \tan^2 \frac{t}{2}, \quad y = a \sin t$$

about its asymptote is equal to the surface of a sphere of radius a.

Answers

1. $4a^3\pi$.	2. $8a^2\pi\,(2\sqrt{2}-1)/3$.
3. $a^2\pi\,[3\sqrt{2}-\log(\sqrt{2}+1)]$.	4. $12\pi a^2/5$.
5. $2\sqrt{2}\pi a^2\,(\sqrt{2}-1)$.	6. $32\pi a^2/5$.
7. $4\pi a^2$.	

EXERCISES ON CHAPTER IX

1. Prove that the surface and the volume of the solid generated by the revolution, about the x-axis, of the loop of the curve

$$x = t^2, \qquad y = t - \tfrac{1}{3}t^3$$

are respectively 3π and $3\pi/4$.

2. The figure bounded by a quadrant of a circle of radius a and the tangents at its extremities revolves about one of these tangents ; prove that

(*i*) the volume of the solid thus generated is

$$(10-3\pi)\,\pi a^3/6.$$

(*ii*) the area of the curved surface so generated is

$$\pi\,(\pi-2)\,a^2.$$

3. The arc of the cycloid

$$x = a\,(\theta-\sin \theta), \qquad y = a\,(1-\cos \theta)$$

joining the points $(0, 0)$, $(a\pi, 2a)$ revolves about its chord ; show that the surface of the solid obtained is

$$\frac{32\, a^2\pi\, (\pi-2)}{3\sqrt{(\pi^2+4)}}\ .$$

4. Show that the volume generated by the revolution of the loop of the curve $x^2y^2 = a\,(x-a)\,(x-b)^2$ about the x-axis is

$$\pi a\, \{\tfrac{1}{2}a^2+2ab-\tfrac{5}{3}b^2+b\,(2a+b)\log\,(b/a)\}.$$

5. A solid spheroid formed by the revolution of the ellipse $b^2x^2+a^2y^2 = a^2b^2$ about the major axis has a cylindrical hole of circular section, having the major axis drilled through it. Prove that the volume of the solid which remains is $4\pi b^2l^3/3a^2$, where $2l$ is the length of the hole.

6. Show that the surface of the solid obtained by revolving the arc of the curve $y = c\cosh\,(x/c)$ joining $(0, c)$ to (x, y) about x-axis is

$$\pi c\left(x+c\,\sinh\frac{x}{c}\,\cosh\frac{x}{c}\right)\cdot$$

7. The arc of the ellipse $x^2/a^2+y^2/b^2 = 1$ lying in the first quadrant revolves about the line $y = b$, show that the volume of the solid thus obtained is $(\frac{5}{3}-\frac{1}{2}\pi)\,\pi ab^2$.

8. Determine the volume generated by revolving the curve

$$y^2 = b^2\log\,(a/x),$$

lying between $x = a$ and $x = b$ about the axis of x, where $b < a < 0$.

9. Show that if the area lying within the cardioide

$$r = 2a(1+\cos\theta),$$

and without the parabola

$$2a = r(1+\cos\theta),$$

revolves about the initial line, the volume generated is $18\pi a^3$.

10. Show that the volume of the solid formed by revolving one loop of the curve

$$r^2 = a^2\cos 2\theta$$

about the line $\theta = \frac{1}{2}\pi$ is $\pi^2a^3/4\sqrt{2}$.

11. Prove that the volume generated by revolving the tractrix

$$x = a\,[\cos t+\tfrac{1}{2}\log\tan^2\tfrac{1}{2}t],\quad y = a\sin t$$

about its asymptote is $\frac{2}{3}\pi a^3$.

12. Show that the area of the surface of the solid generated by revolving the tractrix

$$x = a\,(u-\tanh u),\qquad y = a\,\mathrm{sech}\,u$$

about OX is equal to the area of the surface of a sphere of radius a.

13. The region enclosed by the curve $y = \tanh x$, its asymptote $y = 1$, and y-axis revolves about the asymptote; show that the volume of the solid generated is $\pi \log (4/e)$.

14. Show that the curve $r = a\,(1+2\cos\theta)$ consists of an outer and an inner loop. If the area of the inner loop is rotated through two right angles about the initial line show that the volume of the solid so formed is $\pi/12$.

15. The arc of the cardioide $r = a\,(1+\cos\theta)$, specified by $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, is rotated about the line $\theta = 0$; prove that the volume and the area of the surface of revolution generated are respectively

$$\tfrac{5}{2}a^3\pi \text{ and } \tfrac{4}{5}\,(8-\sqrt{2})\,a^2\pi.$$

16. The arc of the cardioide $r = a\,(1+\cos\theta)$ included between $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, is rotated about the line $\theta = \frac{1}{2}\pi$; find the area of the surface generated.

Answers

8. $\pi b^2\,[a-b-b\log(a/b)]$. **16.** $48\sqrt{2}\pi a^2/5$.

MISCELLANEOUS EXERCISE III

1. Prove that the area between the curve $y = c\cosh(x/c)$, the axis of x, and the ordinates of two points on the curve varies as the length of the intervening arc.

2. Show that the co-ordinates of a point on the curve

$$y^2\,(a+x) = x^2\,(3a-x)$$

may be taken as

$$x = a\sin 3\theta\,\operatorname{cosec}\theta, \qquad y = a\sin 3\theta\sec\theta,$$

and prove that the area of the loop of the curve and the area between the curve and its asymptote are both equal to $3\sqrt{3}a^2$.

Find also the volume generated by the loop when the curve revolves about the axis of x.

3. In an astroid prove that a tangent divides the portion of the curve between two ends into arcs whose lengths are to each other as the segments of the portion of the tangent intercepted by the axes.

4. $A\,(0, a)$ and $P\,(x, y)$ are two points on the curve whose equation is $y = a\cosh(x/a)$ and s is the length of the arc AP. If the curve makes a complete revolution about the x-axis, prove that the area S of the curved surface bounded by planes through A and P perpendicular to X-axis and the corresponding volume V are given by

$$aS = 2V = \pi a\,(ax + sy).$$

5. Find the length of the arc of the curve

$$6xy = x^4 + 3$$

between the points where $x = 1$ and $x = 4$; find also the area of curved surface generated when this portion of the curve is given a complete turn about OY.

6. Show that if, s, is the arc of the curve

$$9y^2 = x\,(3-x)^2$$

measured from the origin to the point (x, y), then

$$3s^2 = 3y^2 + 4x^2$$

The perimeter of the loop of the curve is S, its area is A and maximum breadth is B ; prove that

$$A = \tfrac{3}{10}\,BS.$$

7. Find the length of the arc of the curve

$$25\,a^3xy^2 = (x^3 + 5a^3)^2,$$

measured from $x = a$ to $x = 4a$.

8. Find the sum of the areas of all the loops included between $y = e^{-kx} \sin px$ and $y = 0$.

9. Find the area A bounded by the curve

$$y = a\,(\sin x + \tfrac{1}{3} \sin 3x + \tfrac{1}{5} \sin 5x),$$

and the axis of x between the limits 0 and π ; and the volume V obtained by rotating the area about the axis of x. Prove that $4V = \pi^2 aA$.

10. Prove that area of a parabolic segment is two-third of that of the triangle formed by the base and the tangents at its extremities.

11. If A be the area of the segment of a parabola cut off by a focal chord of length c, show that the latus rectum is $36A^2/c^3$.

12. Show that the area of the loop of the curve

$$x^5 + 2a^2x^2y - a^3y^2 = 0$$

is $32a^2/105$.

13. Show that the area of the loop of the curve

$$r = 3 \operatorname{cosec} \theta - 5$$

is

$$12 - 30 \log 3 + 25\,(\tfrac{1}{2}\pi - \tan^{-1} \tfrac{3}{4}).$$

14. Find the volume and the surface of the solid formed by the revolution about the initial line of the curve

$$r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta. \qquad (a^2 > b^2).$$

15. The part of the ellipse $x^2/a^2 + y^2/b^2 = 1$, cut off by a latus rectum, revolves about the tangent at the nearer end ; find the volume of the reel thus formed.

16. Find the volume generated by revolving about OX the areas bounded by the following loci :

(*i*) $x^2 + y^2 = 25$; $3x - 4y = 0$, $y = 0$; lying in the first quadrant.

(*ii*) $y^2 = 4ax$, $x^2 + y^2 - 12a^2 = 0$.

17. Show that the volume of the solid generated by revolving about X-axis the region bounded by $y = \log x$, $y = 0$ and $x = 2$ is $2\pi (1 - \log 2)^2$.

18. Draw a rough sketch of the curve whose polar equation is

$$r = a (2 \cos \theta + \cos 3\theta).$$

Show that the radius vector has maximum values $3a$ and $a/3\sqrt{3}$ and that the area of the larger loop of the curve is

$$(\tfrac{5}{8}\pi + \tfrac{9}{4}\sqrt{3})\, a^2$$

and the area of a smaller loop is

$$(\tfrac{5}{24}\pi - \tfrac{9}{8}\sqrt{3})\, a^2.$$

19. Trace the curve $y^4 - 2axy^2 + x^4 = 0$ and show that the area of the curve and the volume generated by revolving it round the axis of x are respectively

$$\frac{\pi}{2\sqrt{2}}\, a^2 \text{ and } \frac{2\pi}{3}\, a^3.$$

20. If $r = a (\sec \theta - \cos \theta)$, find the area between the curve and the straight line $r = a \sec \theta$.

21. Show that the area of the loop of the curve

$$x = a\sqrt{(1 - t^2)}, \quad y = at\sqrt{(1 - t^2)}$$

is $2a^2/3$; square root is to be taken with positive sign.

22. Find the area of the region which is defined by the inequalities $x^2 + y^2 \leqslant 9$, $y^2 \leqslant 8x$.

23. Find the intrinsic equation of the curve

$$\sqrt{x} + \sqrt{y} = \sqrt{a},$$

taking $(a, 0)$ as the fixed point.

24. Show that the perimeter of the limacon $r = a + b \cos \theta$ is the same as that of the ellipse whose semi-axes are $a + b$ and $a - b$.

25. The curve

$$y = \frac{h}{\sqrt{\pi}} \cdot e^{-h^2x^2}$$

is rotated round the axis of y; show that the volume of revolution is $\sqrt{\pi}/h$.

26. Show that the intrinsic equation of the tractrix

$$x = a \cos t + a \log \tan \tfrac{1}{2}t, \; y = a \sin t,$$

taking $t = \frac{1}{2}\pi$ as the fixed point is $s = a \log \cos \psi$.

27. Show that the intrinsic equation of the curve

$$y = a \log \sec \left(\frac{x}{a} \right).$$

is

$$s = a \log (\tan \psi + \cos \psi).$$

28. Find the intrinsic equation of the equiangular spiral

$$r = ae^{\theta \cot \alpha}$$

taking $(a, 0)$ as the fixed point.

29. Find the cartesian parametric representation of the cardioide $r = a\,(1+\cos \theta)$ when the length of the arc measured from $\theta = 0$ is used as parameter.

30. The lemniscate $r^2 = a^2 \cos 2\theta$ revolves about a tangent at the pole; show that the volume and the surface of the solid generated are respectively $\frac{1}{4}\pi^2a^3$ and $4\pi a^2$.

31. Show that the volume generated by the revolution of the loop of the curve

$$r \cos \theta = a \cos 2\theta$$

about the initial line is

$$2\pi a^3 (\log_e 2 - \tfrac{2}{3})$$

32. Show that the area of the loop is equal to the area between the infinite branch and the asymptote of the curve

$$x^7 + y^7 = 7a x^3 y^3.$$

33. Trace the curve $x^4 = ay^3 - axy^2$ and show that the area of its loop is $a^2/210$

34. Show that the area of either of the loops of the curve

$$y^4 - 2c^2y^2 + a^2x^2 = 0$$

is $4\sqrt{2}c^3/3a$.

35. In the curve

$$x = a\,(\cos \varphi + \log \tan \tfrac{1}{2}\varphi), \; y = a \sin \varphi.$$

if 's' denotes the length of the arc measured from the axis of y, prove that, as $x \to \infty$

$$\lim\,(s - x) = a\,(1-\log 2).$$

36. Find the whole area included between the curve

$$x^2y^2 = a^2 (x^2+y^2)$$

and its asymptotes.

37. Find the whole area contained between the curve

$$x^2 (x^2+y^2) = a^2 (y^2-x^2)$$

and its asymptotes.

38. If for a curve

$$x \sin \theta + y \cos \theta = f'(\theta)$$
$$x \cos \theta - y \sin \theta = f''(\theta)$$

show that

$$s = f(\theta) + f''(\theta) + c.$$

Answers

2. $a^2 \log (256/e^3)$.

5. $87/8$, $(255+8 \log 2)/4$.

7. $67a/10$.

8. $p(e^{k\pi/p}+1)/(p^2+k^2)(e^{k\pi/p}-1)$.

9. $518a^2/525$; $259\pi^2a^3/450$

14. $\pi\left[\frac{b^2a}{2}+\frac{a^3}{3}+\frac{b^4}{4\sqrt{(a^2-b^2)}} \log \frac{2a^2-b^2+2a\sqrt{(a^2-b^2)}}{b^2}\right]$;

$\pi\left[a^2+\frac{b^4}{\sqrt{(a^4-b^4)}} \log \frac{a^2+\sqrt{(a^4-b^4)}}{b^2}\right]$.

15. $(2b\pi/3a)\,[6a^2b-b^3-3ab\sqrt{(a^2-b^2)}-3a^3 \sin^{-1} (b/a)]$.

16. $50\pi/3$, $\frac{8}{3}(6\sqrt{3}-5)\,\pi a^3$,

20. $3a^2\pi/4$.

22. $2\left(\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\frac{1}{3}\right)$.

23. $s = \frac{a}{2}-\frac{a}{2}\cdot\frac{\cos\psi-\sin\psi}{(\cos\psi+\sin\psi)^2}+\frac{\sqrt{2}a}{4}\log(1+\sqrt{2})$

$-\frac{\sqrt{2}a}{4}\sinh^{-1}\frac{\cos\psi-\sin\psi}{\cos\psi+\sin\psi}$.

28. $s = a \sec \alpha\left[e^{\psi \cot \alpha}-1\right]$.

29. $x = 2a\left[1-\frac{s^2}{8a^2}\right]\left[1-\frac{s^2}{16a^2}\right]$, $y = s\left[1-\frac{s^2}{16a^2}\right]^{3/2}$

36 $4a^2$.

37. $a^2(\pi+2)$.

10

Centre of Gravity. Moment of Inertia

10·1. Introduction. In chapters 7, 8 and 9 we were concerned with the applications of Integral Calculus to **Geometry** inasmuch as we dealt with the determination of the three types of geometrical magnitudes, *viz.*, *Lengths, Areas* and *Volumes*. In this chapter we shall be concerned with applications to **Mechanics** and in this regard consider the concepts of *Centre of Gravity* (C. G.) and *Moment of Inertia* (M. I.).

The notion of centre of gravity of a rigid material system is of importance inasmuch as we often find it useful to replace both in Statics as well as in Dynamics a given rigid system by a single particle whose position coincides with that of the C. G. and whose mass is the same as that of the given system. In view of the fact that we have only been concerned in this book with ordinary integrals and not with double, triple integrals, etc., we are not in a position to develop general formulae for the C. G. of *arbitrary* systems and shall as such consider only special types of systems.

The notion of *Moment of Inertia about a given line of a rigid material system* is of importance in the study of Dynamics of rigid bodies. In this case also, we shall be considering only a few special types of systems.

10·2. Centre of Gravity. A given rigid material system may involve a Discrete or a Continuous distribution of matter. For a discrete distribution, it is proved in statics that if we have a system of n particles with weights

$$w_1, w_2, \ldots\ldots, w_r, \ldots\ldots, w_n$$

on a plane with co-ordinates

$$(x_1, y_1), \ldots, (x_2, y_2), \ldots\ldots, (x_r, y_r), \quad \ldots, (x_n, y_n)$$

then the C. G. $(\bar{x}, \bar{y})$ of the system is given by

$$\bar{x} = \frac{\sum_{r=1}^{n} w_r x_r}{\sum_{r=1}^{n} w_r}, \quad \bar{y} = \frac{\sum_{r=1}^{n} w_r y_r}{\sum_{r=1}^{n} w_r}. \qquad \ldots(1)$$

In case gravity at each of the points is the same and is denoted by g so that

$$w_r = m_r g,$$

m_r denoting the mass of the particle with weight w_r, we have

$$\bar{x} = \frac{\sum_{r=1}^{n} m_r x_r}{\sum_{r=1}^{n} m_r}, \quad \bar{y} = \frac{\sum_{r=1}^{n} m_r y_r}{\sum_{r=1}^{n} m_r}. \qquad \ldots(2)$$

Because of the relations (2), the centre of gravity is also often referred to as *Centre of mass* or *Centre of Inertia.*

We shall now proceed to adopt the above formulae for the C. G. of a finite discrete distribution of matter to that of continuous distributions.

It will be seen that the C. G. of a continuous distribution of matter will be obtained through the limiting process from that of a discrete distribution. The process will consist in *replacing* the given continuous distribution by an appropriately selected finite system of particles and to have recourse to a limiting process.

10·3. Centre of gravity of a continuous distribution of matter.

10·31. Centre of gravity of a uniform plane curve. Consider a portion of a curve $y = f(x)$ intercepted between points A and B with abscissae a and b. Actually we have to imagine a thin rigid wire in the form of this arc.

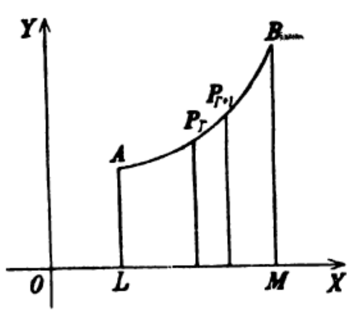

Fig. 51.

Draw AL, $BM \perp OX$ and divide the interval $[a, b]$ and the corresponding line segment LM into n equal parts each of length h so that we obtain points with abscissae

$$a, a+h, \ldots, a+(r-1)h, a+rh\ldots,$$

$a+nh$ which we may re-write as

$$x_0, x_1, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n,$$

Erect ordinates at these points of division meeting the arc AB at points.

$$P_0 = A, P_1, P_2, \ldots\ldots, P_{r-1}, P_r, \ldots\ldots, P_n = B$$

we write

$$y_r = f(x_r) = f(a+rh)$$

Let

arc $AP_r = s_r$.

We replace the arc $P_r P_{r+1}$ by a particle of equal mass $\rho(s_{r+1}-s_r)$ at its initial point P_r ; ρ being the density, *i.e.*, the mass per unit length. Thus we obtain a system of n particles with masses.

$$\rho(s_1-s_0), \ldots\ldots, \rho(s_{r+1}-s_r), \ldots\ldots, \rho(s_n-s_{n-1})$$

at points with co-ordinates

$$(x_0, y_0), \ldots\ldots, (x_r, y_r), \ldots\ldots, (x_{n-1}, y_{n-1})$$

and the C. G. of this system is the point with co-ordinates

$$\left[\frac{\Sigma\rho(s_{r+1}-s_r)x_r}{\Sigma\rho(s_{r+1}-s_r)}, \frac{\Sigma\rho(s_{r+1}-s_r)y_r}{\Sigma\rho(s_{r+1}-s_r)}\right]$$

or

$$\left[\frac{\Sigma(s_{r+1}-s_r)x_r}{\Sigma(s_{r+1}-s_r)}, \frac{\Sigma(s_{r+1}-s_r)y_r}{\Sigma(s_{r+1}-s_r)}\right]$$

summation extending from $r = 0$ to $n-1$.

The denominator in each case is l which denotes the length of the complete arc.

We write

$$(s_{r+1}-s_r)x_r = x_r\left(\frac{s_{r+1}-s_r}{x_{r+1}-x_r}\right)(x_{r+1}-x_r) = x_r\frac{s_{r+1}-s_r}{x_{r+1}-x_r}\cdot h.$$

In the limit, we have

$$\lim_{x_{r+1}\to x_r}\left(\frac{s_{r+1}-s_r}{x_{r+1}-x_r}\right) = \left(\frac{ds}{dx}\right)_{x=x_r}$$

Thus as an approximation we write

$$(s_{r+1}-s_r)x_r = x_r\left(\frac{ds}{dx}\right)_{x=x_r}\cdot h$$

so that the C. G. is the point with co-ordinates

$$\left[\frac{h\sum_{r=0}^{n-1} x_r\left(\frac{ds}{dx}\right)_{x=x_r}}{l}, \frac{h\sum_{r=0}^{n-1} y_r\left(\frac{ds}{dy}\right)_{y=y_r}}{l}\right]$$

By the fundamental theorem of Integral Calculus, we see that these expressions tend to the limits

$$\frac{\int_a^b x\frac{ds}{dx}\,dx}{l}, \frac{\int_a^b y\frac{ds}{dy}\,dx}{l}$$

which we take as the co-ordinates of the C. G. of the given curve.

Note 1. It may be seen that if we amend the above procedure by locating the mass of the arc $P_r P_{r+1}$ at the point P_{r+1} instead of at P_r, we shall obtain the same limits as obtained above.

Note 2. If we write $x \dfrac{ds}{dx} = \phi(x)$, and $y \dfrac{ds}{dy} = \psi(x)$,

we see that

$$h\sum_{r=0}^{n-1} x_r \left(\frac{ds}{dx}\right)_r = h\sum_{r=0}^{n-1} \phi(a+rh)$$

$$h\sum_{r=0}^{n-1} y_r \left(\frac{ds}{dy}\right)_r = h\sum_{r=0}^{n-1} \psi(a+rh)$$

so that we obtain expressions which are exactly of the same form as we were concerned with in the formulation of the fundamental theorem of Integral Calculus.

10·32. Centre of gravity of a uniform plane area. Consider the plane area bounded by a curve $y = f(x)$, x-axis and the two ordinates $x = a$, $x = b$. Actually we have to imagine a thin plane lamina bounded as given.

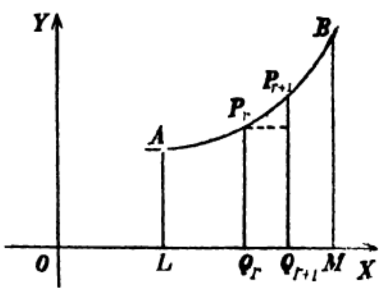

Fig. 52

We divide the segment LM into n equal segments each of length h by points with abscissae,

$$a, a+h, \ldots, a+rh, \ldots, a+nh = b$$

which we may re-write as

$$x_0, x_1, \ldots\ldots, x_r, \ldots\ldots, x_n$$

and erect ordinates at these points meeting the curve at points.

$$A = P_0, P_1, \ldots\ldots, P_r, \ldots\ldots, P_n = B$$

respectively. We write $y_r = f(x_r) = f(a+rh)$, and replace the area below $P_r P_{r+1}$ by the rectangle $P_r Q_{r+1}$ with area

$$hy$$

and C. G. at point

$$[\tfrac{1}{2}(x_r + x_{r+1}), \tfrac{1}{2}y_r].$$

Again we replace this rectangle by a particle of mass

$$\rho h y_r \text{ at point } [\tfrac{1}{2}(x_r + x_{r+1}), \tfrac{1}{2}y_r]$$

so that we obtain n particle with C. G.

$$\left[\frac{\Sigma\, \rho h y_r \,.\, \frac{1}{2}(x_r + x_{r+1})}{\Sigma\, \rho h y_r},\ \frac{\Sigma\, \rho h y_r \,.\, \frac{1}{2} y_r}{\Sigma\, \rho h y_r}\right].$$

We have when $n \to \infty$ and $h \to 0$,

$$\lim \Sigma\, h y_r = \int_a^b y\, dx$$

$$\lim \Sigma\, h y_r \,.\, \tfrac{1}{2}(x_r + x_{r+1}) = \int_a^b yx\, dx$$

$$\lim \Sigma\, \tfrac{1}{2} h y_r^2 = \frac{1}{2}\int_a^b y^2\, dx$$

Thus in the limit we obtain a point with co-ordinates

$$\frac{\int_a^b xy\, dx}{\int_a^b y\, dx},\ \frac{\frac{1}{2}\int_a^b y^2\, dx}{\int_a^b y\, dx}$$

which we take as the C.G. of the given area or the given lamina.

Note 1. It may similarly be shown that the C.G. of a plane area bounded by a curve $x = f(y)$, y-axis and the two ordinates $y = c, y = d$ is

$$\frac{\frac{1}{2}\int_c^d x^2\, dy}{\int_c^d x\, dy},\ \frac{\int_c^d yx\, dy}{\int_c^d x\, dy}$$

Note 2. Centre of gravity of a sectorial area. Consider a sectorial area bounded by a curve $r = f(\theta)$ and two radii vectors $\theta = \alpha, \theta = \beta$.

We divide the angle AOB into n equal parts; each part being equal to say h. Let OP_r, OP_{r+1} be two lines through O making angles $\alpha + hr$, $\alpha + h(r+1)$ respectively with OX.

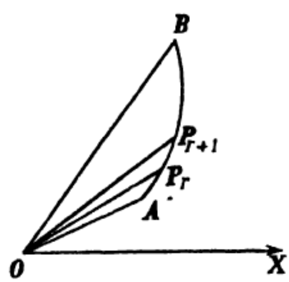

Fig. 53.

We have $OP_r = f(\alpha + hr)$

We replace the area bounded by the arc $P_r P_{r+1}$ by that of the triangle $OP_r P_{r+1}$. The C.G. of the triangle $OP_r P_{r+1}$ is a point on the median through O. We replace the same by the point on OP_r at a distance $\frac{2}{3} OP_r$ from O. Finally we replace the area bounded by the arc

$P_r P_{r+1}$ by a particle of mass equal to that of the triangle at a point on OP_r at a distance equal to $\frac{2}{3} OP_r$ from O.

Thus we have a mass

$$\tfrac{1}{2}OP_r\, OP_{r+1} \sin h$$

at the point with co-ordinates

$$\tfrac{2}{3}\, OP_r \cos(\alpha+hr),\ \tfrac{2}{3}\, OP_r \sin(\alpha+hr)$$

i.e., at the point with co-ordinates

$$\tfrac{2}{3} f(\alpha+hr)\cos(\alpha+hr),\ \tfrac{2}{3} f(\alpha+hr)\sin(\alpha+hr)$$

Thus the abscissa of the C.G. of the system of n particles is

$$\frac{\Sigma\, \rho\, \frac{1}{2} f(\alpha+hr) f(\alpha+h\overline{r+1}) \sin h \,.\, \frac{2}{3} f(\alpha+hr)\cos(\alpha+hr)}{\Sigma\, \rho\, \frac{1}{2} f(\alpha+hr) f(\alpha+\overline{r+1}\, h) \sin h}$$

... ...

We re-write this as an approximation

$$\frac{\frac{2}{3}\Sigma f^3(\alpha+hr)\cos(\alpha+hr)}{\Sigma f^2(\alpha+hr)\, h},$$

which in the limit becomes

$$\frac{\dfrac{2}{3}\displaystyle\int_\alpha^\beta f^3(\theta)\cos\theta\, d\theta}{\displaystyle\int_\alpha^\beta f^2(\theta)\, d\theta} = \frac{\dfrac{2}{3}\displaystyle\int_\alpha^\beta r^3\cos\theta\, d\theta}{\displaystyle\int_\alpha^\beta r^2\, d\theta}$$

It may similarly be seen that the ordinate of the C.G. is

$$\frac{\dfrac{2}{3}\displaystyle\int_\alpha^\beta r^3\sin\theta\, d\theta}{\displaystyle\int_\alpha^\beta r^2\, d\theta}.$$

10·33. Centre of gravity of a volume of Revolution. Let a curve $y = f(x)$ intercepted between two points with abscissae a and b revolve about x-axis. We have to find the position of the C.G. of the solid thus obtained. By considerations of symmetry, it is obvious that C.G. lies on x-axis.

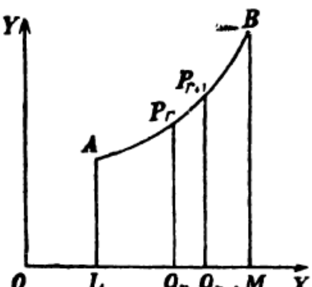

Fig. 54.

We divide LM into n equal parts each of length h and thus obtain parts with abscissae

$$a,\ a+h,\ \ldots\ldots,\ a+rh, \ldots\ldots,\ a+nh = b$$

which we may denote by

$$x_0, x_1, \ldots, x_r, \ldots, x_n$$

respectively. Let the ordinates through these points meet the curve at points

$$P_0 = A,\ P_1, \ldots\ldots, P_r, \ldots \ldots, P_n = B.$$

The ordinate of $P_r = f(a+rh) = y_r$, say.

Consider the part of the volume obtained on revolving the arc $P_r P_{r+1}$ by that obtained on revolving the rectangle $P_r Q_{r+1}$ about x-axis. This latter volume is

$$\rho\pi\, hy_r^2$$

with its C.G. at the mid-point of $Q_r Q_{r+1}$ whose abscissa is $\frac{1}{2}(x_r+x_{r+1})$. Again we replace this latter volume by a particle of equal mass at the mid-point of $Q_r Q_{r+1}$. The abscissa of the C.G. of this system of n particles is

$$\frac{\sum\limits_{r=0}^{n-1} \rho\pi\, hy_r^2 \cdot \frac{1}{2}(x_r+x_{r+1})}{\sum\limits_{r=1}^{n-1} \rho\pi\, hy_r^2}.$$

In the limit when $n \to \infty$ and $h \to 0$ we obtain

$$\frac{\int_a^b xy^2\, dx}{\int_a^b y^2\, dx}$$

which is the required abscissa.

10·34. Centre of gravity of a surface of revolution. (Refer Fig. 54, previous article).

We find the C.G. of the surface of revolution obtained on revolving the arc AB about x-axis.

We replace the part of the surface generated by the arc $P_r P_{r+1}$ by that generated by the chord $P_r P_{r+1}$ whose area as shown in § 9·5 is

$$\pi(y_r+y_{r+1})\, P_r P_{r+1}$$

Its C.G. is at the mid-point $Q_r Q_{r+1}$. Again, we replace this surface by a particle of equal mass at the mid-point of $Q_r Q_{r+1}$. The centre of gravity of this system of particles is a point on x-axis with abscissa

$$\frac{\Sigma\rho\pi(y_r+y_{r+1})\, P_r P_{r+1} \cdot \frac{1}{2}(x_r+x_{r+1})}{\Sigma\rho\pi(y_r+y_{r+1})\, P_r P_{r+1}}$$

which may be taken as

$$\frac{\sum\limits_{r=0}^{n-1} h(y_r+y_{r+1})\left(\frac{ds}{dx}\right)_{x=x_r} \frac{1}{2}(x_r+x_{r+1})}{\sum\limits_{r=0}^{n-1} h(y_r+y_{r+1})\left(\frac{ds}{dx}\right)_{x=x_r}}.$$

which, in the limit, gives

$$\frac{\int_a^b xy\frac{ds}{dx}\,dx}{\int_a^b y\frac{ds}{dx}\,dx}$$

Examples

1. *Find the centre of gravity of the arc of the curve*

$$x = a\sin^3\theta,\ y = a\cos^3\theta$$

lying in the first quadrant.

We have (§ 10·31)

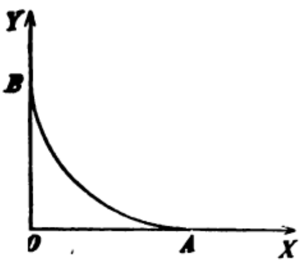

Fig. 55.

$$\bar{x} = \frac{\int_0^a x\frac{ds}{dx}\,dx}{\int_0^a \frac{ds}{dx}\,dx}\ ;\ \bar{y} = \frac{\int_0^a y\frac{ds}{dx}\,dx}{\int_0^a \frac{ds}{dx}\,dx}$$

Also

$$\int_0^a x\frac{ds}{dx}\,dx = \int_0^{\pi/2} x\frac{ds}{dx}\frac{dx}{d\theta}\,d\theta$$

$$= \int_0^{\pi/2} x\frac{ds}{d\theta}\,d\theta = \frac{3a^2}{5}$$

$$\int_0^a \frac{ds}{dx}\,dx = \int_0^{\pi/2}\frac{ds}{dx}\frac{dx}{d\theta}\,d\theta = \int_0^{\pi/2}\frac{ds}{d\theta}\,d\theta = \frac{3a}{2}$$

$$\therefore \qquad \bar{x} = \frac{2a}{5}.$$

We may similarly show that

$$\bar{y} = \frac{2a}{5}$$

2. *Find the centre of gravity of a uniform lamina bounded by the co-ordinate axes and the arc of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

in the first quadrant.

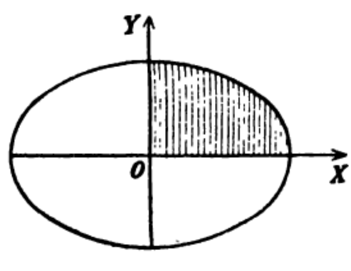

Fig. 56.

We have (§ 10·32)

$$\bar{x} = \frac{\int_0^a xy\,dx}{\int_0^a y\,dx} = \frac{\int_0^a x.\ \frac{b}{a}\sqrt{(a^2-x^2)}\,dx}{\int_0^a \frac{b}{a}\sqrt{(a^2-x^2)}\,dx} = \frac{4a}{3\pi}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^a y^2\,dx}{\int_0^a y\,dx} = \frac{\frac{1}{2}\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx}{\int_0^a \frac{b}{a}\sqrt{(a^2-x^2)}\,dx} = \frac{4b}{3\pi}.$$

9. *Find the C. G. of a uniform solid hemisphere of radius a.*

We can regard the hemisphere as generated by revolving through x-axis the arc of the circle $x^2+y^2=a^2$ lying in the first quadrant.

We have

$$\bar{x} = \frac{\int_0^a xy^2\,dx}{\int_0^a y^2\,dx} = \frac{\int_0^a x(a^2-x^2)\,dx}{\int_0^a (a^2-x^2)\,dx} = \frac{3a}{8}.$$

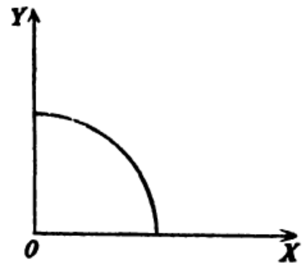

Fig. 57.

10·4. Moment of Inertia. By definition, the particle of mass about any given line is mr^2 where r is the distance of the particle from the line. Again the moment of inertia about a given line of a finite number of particles with masses

$$m_1, \ldots\ldots, m_i, \ldots\ldots, m_n$$

is

$$\sum_{i=1}^{n} m_i r_i^2$$

where

$$r_1, \ldots\ldots, r_i, \ldots\ldots, r_n$$

are the distances of the particles from the given line.

The moment of inertia of a continuous distribution of matter is obtained through a limiting process. The process will be indicated by the following examples.

Examples

1. *Find the moment of inertia of a thin uniform rod of length 2a about the line through its one extremity perpendicular to the rod.*

Let AB be the rod. We take A as origin and AB as x-axis. We require moment of inertia about the line AY through A perpendicular to AB.

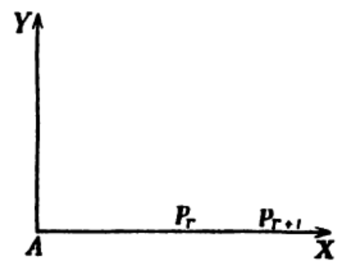

Fig. 58.

We divide AB into n equal parts each of length h by points

$$A = P_0, P_1, \ldots\ldots, P_r, \ldots\ldots, P_n = B$$

we have
$$AP_r = \frac{2a}{n} r = hr.$$

We replace the part $P_r P_{r+1}$ of the rod by a particle at P_r of mass equal to that of the part. If ρ be the density of the rod, the mass of this particle is

$$\rho h$$

so that the moment of inertia about AY is

$$(hr)^2 \rho h.$$

The moment of inertia of the system of particles is, therefore,

$$\sum_{r=0}^{n-1} (hr)^2 \rho h = \frac{1}{n} \sum_{r=0}^{n-1} \rho \left(\frac{r}{n}\right)^2$$

whose limit when $n \to \infty$ and $h \to 0$ and $nh = 2a$ is

$$\int_0^{2a} \rho x^2 \, dx = \frac{8a^3}{3} \rho = M \frac{4a^2}{3}$$

where M denotes the mass of the whole rod.

Ex. Show that the M. I. of a uniform thin rod of length $2a$ and mass M about the line through its middle point perpendicular to the rod is

$$Ma^2/3.$$

2. *Find the moment of inertia of a thin uniform circular disc of radius a about any diameter thereof.*

We take the given diameter as x-axis and the centre O as origin.

We divide the diameter AB into n equal parts each of length h and through the points of division draw $\perp$ AB. We replace the part of the disc contained between the ordinates through Q_r and Q_{r+1} by a rod of length $2y_r$ whose moment of inertia about OX is

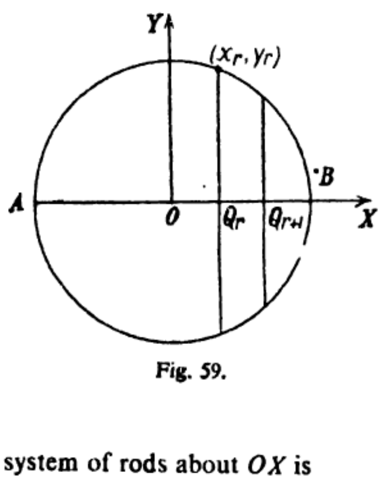

Fig. 59.

$$2\rho y_r h \cdot \frac{y_r^2}{3}$$

so that the moment of inertia of the system of rods about OX is

$$\sum_{r=0}^{n-1} 2\rho y_r h \cdot \frac{y_r^2}{3}.$$

We have $y_r = f(-a+hr)$, where $f(x) = \sqrt{(a^2-x^2)}$.

When $n \to \infty$, $h \to 0$, we obtain in the limit

$$\int_{-a}^{+a} \frac{2\rho}{3}(a^2-x^2)^{3/2}\,dx = \frac{\rho\pi a^4}{4} = \frac{Ma^2}{4},$$

where M is the mass of the disc.

Ex. Show that the moment of inertia of a uniform circular disc of mass M and radius a about the line through its centre perpendicular to its plane is $Ma^2/2$.

3. *Find the moment of inertia of a uniform solid sphere of radius a about a diameter.*

Dividing the diameter into n equal parts and drawing through the points of division planes perpendicular to the axis and replacing each part of the sphere bound between two consecutive planes by a thin circular disc we may see that the required moment of inertia is

$$= \int_{-a}^{+a} \pi\rho\,(a^2-x^2)\,\tfrac{1}{2}\,(a^2-x^2)\,dx$$

$$= \frac{8\rho\pi a^5}{15} = \frac{2a^2}{5}\cdot\frac{4a^3}{3}\rho = \frac{2a^2}{5}M$$

where M is the mass of sphere.

EXERCISES ON CHAPTER 10

1. Find the centroid of a uniform circular wire of radius a in the form of

(*i*) a semi-circle. (*ii*) Quadrant of a circle

2. Find the centre of gravity of the arc of a circle of radius a subtending an angle 2θ at its centre.

3. Show that the centroid of the arc of the cycloid

$$x = a(\theta+\sin\theta), \quad y = a(1-\cos\theta)$$

in the positive quadrant is

$$\left[\left(\pi-\frac{4}{3}\right)a,\ \frac{2}{3}a\right].$$

4. Find the centre of gravity of the arc of the parabola $y^2 = 4ax$ bounded by the vertex and an extremity $(a, 2a)$ of its latus rectum.

5. Show that the centre of gravity of the arc of the catenary

$$y = c \cosh \frac{x}{c}$$

included between its vertex $(0, c)$ and a point (x, y) is

$$\left[x - \frac{c\,(y \quad c)}{s},\ \frac{y}{2} + \frac{cx}{2s} \right]$$

where s is the length of the arc.

6. Find the centroid of a plane lamina bounded by x-axis and the part of the ellipse $x^2/a^2 + y^2/b^2 = 1$ for which y is positive.

7. Show that the centroid of the area bounded by the parabola $x^2 = 4ay$, x-axis and the ordinate $x = b$ is

$$\left(\frac{3}{4} b,\ \frac{3}{40} \cdot \frac{b^2}{a} \right).$$

8. Show that the centroid of the area bounded by the cardioide $r = a\,(1 + \cos\theta)$ is on the initial line at a distance $\frac{5}{6}a$ from the pole.

9. What are the co-ordinates of the C.G. of the area under one arch of the sine curve $y = \sin x$.

10. Determine the C.G. of the area bounded by the parabolas $y^2 = ax$ and $x^2 = by$.

11. Find the centre of gravity of the area in the first quadrant bounded by the axes and the curve

$$y = 2 + x - x^2.$$

12. A flat thin plate of uniform density is bounded by the two curves $y = x^2$, $y = -x^2$ and the line $x = 2$. Find the co-ordinates of its C. G.

13. Show that the centroid of the area bounded by the loop of the curve

$$y^2\,(a + x) = x^2\,(a - x)$$

is $\frac{1}{3}a\,(3\pi - 8)/(4 - \pi)$.

14. Show that the abscissa of the C. G. of the area of one loop of $r^2 = a^2 \cos 2\theta$ is $\frac{1}{8}\pi\, a\sqrt{2}$.

15. Where is the C. G. of the solid formed by revolving about x-axis the area bounded by the lines $y = 0$, $x = \pi/4$ and the curve $y = \sin x$.

16. Find the C. G. of the solid formed by the revolution about x-axis of the parabola $y^2 = 4ax$ cut off by the ordinate $x = b$.

17. Find the centroid of

(*i*) surface, (*ii*) volume

of the solid formed by the revolution of $r = a\,(1+\cos\theta)$ about the initial line.

18. Find the centroid of

(*i*) hemispherical surface,

(*ii*) curved surface

of a right circular cone.

19. A quadrant of the ellipse $x^2/a^2+y^2/b^2 = 1$ revolves about the major axis ; find the C. G. of the solid thus obtained.

20. Find the moment of inertia about the y-axis of a uniform lamina of density ρ bounded by those parts of the x-axis and the curve $y = \cos x$ which lie between $x = -\frac{1}{2}\pi$ and $x = \frac{1}{2}\pi$.

21. Find the moment of inertia about x-axis of the solid obtained by revolving $x^2/a^2+y^2/b^2 = 1$ about x-axis.

22. Find the moment of inertia of a uniform solid right cylinder of height h and radius r about its axis

23. Find the moment of inertia of a thin uniform hollow sphere of radius a about and diameter thereof.

Answers

1. (*i*) $\bar{x} = 0,\ \bar{y} = 2a/\pi$. (*ii*) $(2\sqrt{2}\,a/\pi,\ 2\sqrt{2}\,a/\pi)$.

2. At a distance $a \sin\theta/\theta$ from the centre on the bisecting radius.

4. $\left[\dfrac{a}{4}\cdot\dfrac{3\sqrt{2}-\log(1+\sqrt{2})}{\sqrt{2}+\log(1+\sqrt{2})},\ \dfrac{4a}{3}\ \dfrac{\sqrt{8}-1}{\sqrt{2}+\log(1+\sqrt{2})}\right]$.

6. $(0,\ 4b/3\pi)$. **9.** $\left(\dfrac{\pi}{2},\ \dfrac{\pi}{8}\right)$.

10. $\left(\dfrac{9}{20}\ a^{\frac{1}{3}}\ b^{\frac{5}{3}},\ \dfrac{9}{20}\ a^{\frac{2}{3}}\ b^{\frac{1}{3}}\right)$. **11.** $\left(\dfrac{4}{5},\ \dfrac{24}{25}\right)$.

12. $\left(\dfrac{3}{2},\ 0\right)$. **15.** $\bar{x} = (\pi^2-4\pi+8)/8(\pi-2)$.

16. $\bar{x} = \dfrac{2}{3}\,b$. **17.** (*i*) $\bar{x} = \dfrac{50}{63}a$. (*ii*) $\dfrac{4}{5}\,a$.

18. (*i*) At the mid-point of the bisecting radius.
(*ii*) At the point dividing the axis in the ratio 2 : 1.

19. $\bar{x} = 3a/8$. **20.** $\frac{1}{2}\rho\,(\pi^2-8)$.

21. $2b^2 M/5$ where M is the mass of the solid.

22. $\frac{1}{2}Mr^2$ where M is the mass and r the radius of the base.

23. $\frac{2}{3}Ma^2$ where M is the mass and r the radius.

11

Differential Equations of First Order and First Degree

11·1. A Differential equation *is an equation that involves independent and dependent variables and the derivatives of the dependent variables.*

The following are some examples of differential equations :

(1) $(2x+3y)\dfrac{dy}{dx}+(7x^2+8y)=0.$ (2) $\dfrac{dy}{dx}+y\cos x=\sin x.$

(3) $\dfrac{d^2y}{dx^2}+a^2x=0.$ (4) $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}=\dfrac{d^2y}{dx^2}$

(5) $x^2\left(\dfrac{d^2y}{dx^2}\right)^3+y\left(\dfrac{dy}{dx}\right)^4+y^4=0.$

(6) $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz$ (7) $\dfrac{\partial^2 z}{\partial x^2}-a^2\dfrac{\partial^2 z}{\partial y^2}=0.$

A differential equation is said to be *Ordinary*, if the differential coefficients have reference to a *single* independent variable only and it is said to be *Partial* if there are *two or more* independent variables.

Thus, the differential equations (1) to (5) are ordinary, but (6) and (7) are partial.

In the following, we shall be concerned with ordinary differential equations only.

The *order of a differential equation* is the order of the derivative of highest order derivative occurring in it.

The *degree of a differential equation* is the degree of the derivative of the highest order occurring in it, after it has been expressed in a form free from radicals and fractions so far as derivatives are concerned.

Thus, of the above differential equations

(1), (2) and (6) are of the first order and the first degree ;

(3) and (7) are of the second order and the first degree ;

(4) is of the second order and the second degree ;

(5) is of the second order and the third degree.

A *solution or integral of a differential equation* is a relation between the variables, not involving the differential coefficients such that this relation and the derivatives obtained from it satisfy the given differential equation. This also implies that a differential equation can be derived from its solution by the process of differentiation and other algebraic processes of elimination, etc. On this account the solution of a differential equation is also called its *primitive.*

Examples

1. *Show that, $y = A \cos x + B \sin x$, is a solution of the differential equation*

$$d^2y/dx^2 + y = 0.$$

We have

$$dy/dx = -A \sin x + B \cos x$$
$$d^2y/dx^2 = -A \cos x - B \sin x = -y,$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} + y = 0,$$

which is the given differential equation. ...(*i*)

2. *Show that, $y = A \cos x + \sin x$, is a solution of*

$$\cos x \frac{dy}{dx} + y \sin x = 1. \qquad \text{...}(ii)$$

Differentiating (*i*), we get

$$dy/dx = -A \sin x + \cos x. \qquad \text{...}(iii)$$

Substituting the values of y and dy/dx in the left hand side of (*ii*), we see that

$$\cos x \frac{dy}{dx} + y \sin x$$
$$= \cos x (-A \sin x + \cos x) + (A \cos x + \sin x) \sin x = 1,$$

so that the given differential equation is satisfied.

3. *Show that*

$$Ax^2 + By^2 = 1 \qquad \text{...}(i)$$

is the solution of

$$x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-y\frac{dy}{dx}=0. \qquad \text{...}(ii)$$

We note that the differential equation is free of the constants A, B and is of second order.

Differentiating (i) twice successively, we get

$$Ax+By\frac{dy}{dx}=0, \qquad \text{...}(iii)$$

$$A+B\left[\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right]=0. \qquad \text{...}(iv)$$

On eliminating A and B from (iii) and (iv), we get

$$x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-y\frac{dy}{dx}=0,$$

which is the given differential equation.

4. *By the elimination of the constants h and k, find the differential equation of which* $(x-h)^2+(y-k)^2=a^2$, *is a solution.*

Three relations are necessary to eliminate two constants. Thus, besides the given relation, we require two more and they will be obtained by differentiating the given relation twice successively. Thus we have

$$(x-h)+(y-k)\frac{dy}{dx}=0, \qquad \text{...}(i)$$

$$1+(y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0. \qquad \text{...}(ii)$$

From (i) and (ii), we obtain

$$y-k=-\frac{1+(dy/dx)^2}{d^2y/dx^2},$$

$$x-h=\frac{[1+(dy/dx)^2]\,dy/dx}{d^2y/dx^2}.$$

Substituting these values in the given relation, we obtain

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2,$$

which is the required differential equation.

Exercises

1. By the elimination of the constants a, b, obtain the differential equation of which $xy=ce^x+be^{-x}+x^2$ is a solution.

2. Find the equation of which $y = Ae^{x}+Be^{3x}+Ce^{5x}$ is a solution.

3. By the elimination of the constant, a obtain the differential equation of which $y^2 = 4a(x+a)$ is the solution.

4. Find the differential equation of the family of curves $y = e^{x}(A \cos x+B \sin x)$ where A and B are arbitrary constants.

Answers

1. $x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}-xy+x^2-2=0.$

2. $\dfrac{d^3y}{dx^3}-9\dfrac{d^2y}{dx^2}+23\dfrac{dy}{dx}-15y=0.$

3. $y\left[1-\left(\dfrac{dy}{dx}\right)^2\right]=2x\dfrac{dy}{dx}.$

4. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y=0.$

11·2. Number of arbitrary constants. In order to obtain a differential equation whose solution is

$$f(x, y, c_1, c_2, \ldots\ldots, c_n) = 0, \qquad \ldots(i)$$

where $c_1, c_2, \ldots\ldots, c_n$ are n arbitrary constants, we have to eliminate the n constants for which we require $(n+1)$ equations. The given relation along with n more, obtained by successively differentiating it n times, provide us with the required $(n+1)$ relations. The differential equation thus obtained is clearly of the nth order. The solution (i) contains n arbitrary constants, $c_1, c_2, \ldots\ldots, c_n$.

11·21. General and particular solutions. A solution of a differential equation which contains arbitrary constants as many as the order of the differential equation is called *General solution.* Other solutions, obtained by giving particular values to the arbitrary constants in the general solution, are called *Particular solutions.*

Also, we know that the general integral of a function contains an arbitrary constant. Therefore, the solution of a differential equation, resulting as it does from the operations of integration, must contain arbitrary constants, equal in number to the number of times the integration is involved in obtaining the solution, and this latter is equal to the order of the differential equation.

Thus we see that *the general solution of a differential equation of the nth order must contain n and only n independent arbitrary constants.*

Note 1. We have already seen that it is not possible to obtain the integral of every function *i.e.*, an integral which is obtained as an algebraic combination of a finite number of algebraic, trigonometric, inverse trigonometric, logarithmic and exponential functions. This possibility is still more limited in the case of differential equations. However, as in the case of integration, there exist some standard forms such that a differential equation belonging to any one of them can always be solved. A few only of these standard forms will be considered in this book.

Now in this chapter, we consider differential equations of the first order only.

Note 2. A convenient notation. The most general differential equation of the first order and first degree is

$$\frac{dy}{dx} = \frac{f(x, y)}{\varphi(x, y)} \qquad ...(i)$$

It is sometimes, found convenient to write this equation in the form $\varphi(x, y)\, dy = f(x, y)\, dx$.

11·3. Equations in which the variables are separable are those equations which can be so expressed that the coefficient of dx is only a function of x and that of dy is only a function of y.

Thus the general form of such an equation is

$$f(x)\, dx + \varphi(y)\, dy = 0. \qquad ...(i)$$

To solve it, we write

$$f(x) + \varphi(y) \frac{dy}{dx} = 0.$$

Integrating with respect to x, we get

$$\int f(x)\, dx + \int \varphi(y) \cdot \frac{dy}{dx}\, dx = c,$$

$$\Rightarrow \qquad \int f(x)\, dx + \int \varphi(y)\, dy = c$$

which is the solution of (*i*).

Thus the solution of (*i*) is obtained by adding the integrals of $f(x)$ and $\varphi(y)$ with respect to x and y respectively and equating their sum to a constant.

Examples

1. *Solve*

$$y(1+x)\, dx + x(1+y)\, dy = 0$$

Here, we have

$$y(1+x)\, dx + x(1+y)\, dy = 0$$

$$\Rightarrow \qquad \frac{1+x}{x}\,dx+\frac{1+y}{y}\,dy = 0$$

so that the given equation is one in which the variable are separable.

The solution, therefore, is

$$\int \frac{1+x}{x}\,dx+\int \frac{1+y}{y}\,dy = 0$$

$$\Rightarrow \qquad \log x+x+\log y+y = a$$

$$\Rightarrow \qquad x+y+\log xy = a$$

which is the required general solution containing the arbitrary constant 'a'.

2. *Find a curve for which the tangent at each point makes a constant angle, α, with the radius vector.*

If φ denotes the angle between the radius vector and the tangent at a point of the curve, we have

$$\tan\varphi = r\frac{d\theta}{dr}.$$

As $\varphi = \alpha$, we have

$$\tan\alpha = \frac{rd\theta}{dr},$$

$$\Rightarrow \qquad \frac{dr}{r} = \cot\alpha\, d\theta.$$

$$\Rightarrow \qquad \log r = \theta\cot\alpha+c,$$

$$\Rightarrow \qquad r = e^{c}\,.\,e^{\theta\cot\alpha} = ae^{\theta\cot\alpha},$$

$$\Rightarrow \qquad r = ae^{\theta\cot\alpha}$$

is the required curve ; 'a' being a constant whatsoever.

Exercises

Solve the following differential equations :

1. $(xy^2+x)\,dx+(yx^2+y)\,dy = 0.$
2. $x\sqrt{(1+x^2)}\,dx+x\sqrt{(1+y^2)}\,dy = 0.$
3. $(x^2-yx^2)\,dy+(y^2+xy^2)\,ydx = 0.$
4. $\sqrt{[(1+x^2)(1+y^2)]}\,dx+xydy = 0.$
5. $y-x\dfrac{dy}{dx} = a\left(y^2+\dfrac{dy}{dx}\right).$
6. $\operatorname{cosec} x\log y\,dy+x^2y^2\,dx = 0.$
7. $\dfrac{dy}{dx} = \dfrac{\sin x+x\cos x}{y\,(2\log y+1)}.$

8. $x^{-1} \cos^2 y \, dy + y^{-1} \cos^2 x \, dx = 0.$

9. $x \sqrt{y} \, dx + (1+y) \sqrt{(1+x)} \, dy = 0.$

10. $\cos y \log (\sec x + \tan x) \, dx = \cos x \log (\sec y + \tan y) \, dx.$

11. $(3+2 \sin x + \cos x) \, dy = (1+2 \sin y + \cos y) \, dx.$

12. $(e^x+1) \, y \, dy = (y+1) \, e^x \, dx.$

13. $\dfrac{dy}{dx} = e^{x+y} + x^2 e^y.$

14. Find the equations of the curves for which the
 (*i*) cartesian sub-tangent is constant.
 (*ii*) cartesian sub-normal is constant.
 (*iii*) polar sub-tangent is constant.
 (*iv*) polar sub-normal is constant.

15. Find the equation of the curve for which the cartesian sub-tangent varies as the reciprocal of the square of the abscissa.

16. Find the curve which is such that the portion of the x-axis cut off between the origin and the tangent at a point is twice the abscissa and which passes through the point (1, 2).

17. Find the curves for which the sum of the reciprocals of the polar sub-normal and the radius vector is constant.

18. Find the curve in which the angle between the radius vector and the tangent is m times the vectorial angle and which passes through the point $(a, \pi/2m)$.

Answers

1. $(x^2+1)(y^2+1) = c.$
2. $\sqrt{(1+x^2)} + \sqrt{(1+y^2)} = \log \{[\sqrt{(1+x^2)}+1][\sqrt{(1+y^2)}+1]/xy\}$
3 $2y^2 \log x - 2y(y-x)/x + 2cy^2 = 1.$
4. $\sqrt{(1+x^2)} + \sqrt{(1+y^2)} - \log \{[\sqrt{(1+x^2)}+1]/x\} = c.$
5. $(1-ay)(a+x) = cy.$
6. $(1+\log y)/y + x^2 \cos x - 2x \sin x - 2 \cos x = c.$
7. $y^2 \log y = x \sin x + c.$
8. $(x^2+y^2) + x \sin 2x + y \sin 2y + \frac{1}{2} \cos 2x + \frac{1}{2} \cos 2y = c.$
9. $(x-2)\sqrt{(1+x)} + (y+3)\sqrt{y} = c.$
10. $[\log (\sec x + \tan x)]^2 - [\log (\sec y + \tan y)]^2 = c.$
11. $2 \tan^{-1} (1 + \tan \frac{1}{2}x) = c + \log (1 + 2 \tan \frac{1}{2}y).$
12. $(1+y)(1+e^x) = ce^y.$

13. $e^x + e^{-y} + \frac{1}{3}x^3 = c$.

14. (*i*) $y = ke^{x/c}$. (*ii*) $y^2 = 2cx + k$.

(*iii*) $r(k - \theta) = c$. (*iv*) $r = c\theta + k$.

15. $y = ce^{x^3/3k}$.

16. $xy = 2$.

17. $\theta = cr - \log r + k$.

18. $r^m = a^m \sin m\theta$.

11·4. Linear Equations. *A differential equation is said to be linear if the dependent variable and its differential coefficients occur in the first degree only and are not multiplied together.*

Thus the most general form of a linear equation of the first order is

$$\frac{dy}{dx} + Py = Q \qquad \ldots(i)$$

where P, Q are any functions of x.

To solve this equation, we multiply both sides by

$$e^{\int P\,dx}$$

so that we get

$$e^{\int P\,dx}\left[\frac{dy}{dx} + Py\right] = Qe^{\int P\,dx}$$

The left hand side, now, is the differential coefficient of

$$ye^{\int P\,dx}$$

Thus, on integrating, we get

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$$

$$\Leftrightarrow \qquad y = e^{-\int P\,dx}\left[\int Qe^{\int P\,dx}\,dx + c\right]$$

as the required solution.

Note 1. The factor

$$e^{\int P\,dx}$$

on multiplying by which the left hand side of (*i*) becomes the differential coefficient of a function of x and y, is called the **integrating factor** of the differential equation (*i*).

Note 2. It is very important to remember that on multiplying by the integrating factor, the left hand side becomes the derivative of the product of y and the integrating factor.

Note 3. It will be useful to remember that for every t.

$$e^{\log t} = t.$$

Examples

1. *Solve*

$$x^2(x^2-1)\frac{dy}{dx}+x(x^2+1)y = (x^2-1).$$

We have

$$\frac{dy}{dx}+\frac{x^2+1}{x(x^2-1)}y = \frac{1}{x^2},$$

so that it is linear.

Here, $$P = \frac{x^2+1}{x(x^2-1)},$$

$$\therefore \qquad \int P\,dx = \int\frac{x^2+1}{x(x^2-1)}\,dx$$

$$= \int\left(\frac{1}{x+1}+\frac{1}{x-1}-\frac{1}{x}\right)dx = \log\frac{x^2-1}{x}.$$

Thus the integrating factor is

$$e^{\int P\,dx} = e^{\log\frac{x^2-1}{x}} = \frac{x^2-1}{x}.$$

Multiplying by $(x^2-1)/x$, we obtain

$$\frac{x^2-1}{x}\left[\frac{dy}{dx}+\frac{x^2+1}{x(x^2-1)}y\right] = \frac{1}{x^2}\cdot\frac{x^2-1}{x}.$$

Thus the solution is

$$y\,\frac{x^2-1}{x} = \int\frac{x^2-1}{x^3}\,dx+c = \log x+\frac{1}{2x^2}+c.$$

2. *Solve*

$$(x+2y^3)\frac{dy}{dx} = y.$$

The equation which involves y^3 is not linear, if we take y as the dependent variable, but since this can be written as

$$y\frac{dx}{dy} = x+2y^3$$

$$\Rightarrow \qquad \frac{dx}{dy}-\frac{x}{y} = 2y^2,$$

we see that the equation is linear, if we take x as the dependent variable.

Integrating factor $= e^{-\int dy/y} = e^{-\log y} = y^{-1} = 1/y.$

Multiplying by this integrating factor, we obtain

$$\frac{1}{y}\left(\frac{dx}{dy}-\frac{x}{y}\right) = 2y.$$

Therefore the solution is

$$x\,\frac{1}{y} = y^2+c \text{ or } x = y\,(c+y^2).$$

Exercises

Solve the following equations :

1. $(x^2-1)\dfrac{dy}{dx}+2xy = 1.$

2. $\sin x\dfrac{dy}{dx}+y\cos x = x\sin x.$

3. $\dfrac{dy}{dx}+3x^2y = x^5.e^{x^3}.$

4. $x\log x\dfrac{dy}{dx}+y = 2\log x.$

5. $x\sin x\dfrac{dy}{dx}+(x\cos x+\sin x)y = \sin x.$

6. $\sin x\cos x\dfrac{dy}{dx} = y+\sin x.$

7. $\sin x\dfrac{dy}{dx}+2y+\sin x\,(1+\cos x) = 0.$

8. $(1+x+xy^2)\,dy+(y+y^3)\,dx = 0.$

9. $(2x-10y^3)\dfrac{dy}{dx}+y = 0.$

10. $\sqrt{(x^2+1)}\dfrac{dy}{dx}+y = \sqrt{(x^2+1)}-x.$

11. $(1-x^2)^{3/2}\dfrac{dy}{dx}+y-1 = 0.$

12. $x(x^2+1)\dfrac{dy}{dx} = y\,(1-x^2)+x^3\log x.$

13. $(x^2-1)\sin x\dfrac{dy}{dx}+[2x\sin x+(x^2-1)\cos x]\,y$
$-(x^2-1)\cos x = 0.$

14. $\dfrac{dy}{dx}+y \sec x = \tan x$.

15. $(x+\tan y)\, dy = \sin 2y\, dx$.

16. $\dfrac{dy}{dx}+\dfrac{y}{\sqrt{(1-x^2)^3}}=\dfrac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}$.

17. $(1+y^2)\, dx+(x-e^{-\tan^{-1} y})\, dy = 0$.

18. $\dfrac{dy}{dx} = y \cos x+\cos x \sin^2 x$.

19. $\dfrac{dy}{dx}-\dfrac{1+3x^2}{x(1+x^2)}\, y = x\dfrac{1-x^2}{1+x^2}$ and determine the arbitrary constant so as to make $y = 0$ when $x = 1$.

20. Find the equation of the curve which passes through the point $(2a, a)$ and for which the sum of the cartesian subtangent and the abscissa is equal to the constant a.

21. Find the curves for which the portion of y-axis cut off between the origin and the tangent varies as the cube of the abscissa of the point of contact.

Answers

1. $y(1-x^3) = -x+c$.
2. $(y-1) \sin x+x \cos x = c$.
3. $12y = e^{x^3}(2x^3-1)+ce^{-x^3}$
4. $y \log x = c+(\log x)^2$.
5. $xy \sin x+\cos x = c$.
6. $y \cot x = c+\log \tan \frac{1}{2}x$.
7. $y \tan^2 \frac{1}{2}x = \sin x-x+c$.
8. $xy+\tan^{-1} y = c$.
9. $xy^2 = 2y^5+c$.
10. $y[x+\sqrt{(x^2+1)}] = c+\sinh^{-1} x$.
11. $\log (y-1)+x/\sqrt{(1-x^2)} = c$.
12. $4(x^2+1)y+x^3(1-2 \log x) = cx$.
13. $y(x^2-1) \sin x = (x^2-3) \sin x+2x \cos x+c$.
14. $(y-1)(\sec x+\tan x)+x = c$.
15. $x\sqrt{\cot y} = c+\sqrt{\tan y}$.
16. $y = x \sqrt{(1-x^2)}+ce^{-x/\sqrt{(1-y^2)}}$.
17. $xe^{\tan^{-1} y} = c+\tan^{-1} y$.
18. $y+\sin^2 x+2 \sin x+2 = ce^{\sin x}$.
19. $y = cx(1+x^2)+x^2$; $c = -\frac{1}{2}$.
20. $(x-a)\, y = a^2$.
21. $2y+kx^3 = cx$.

11·5. Equations reducible to the linear form. The equation

$$\frac{dy}{dx}+Py = Qy^n,$$

where P, Q are functions of x, is reducible to the linear form, as will now be seen. On dividing by y^n, we get

$$y^{-n}\frac{dy}{dx}+Py^{-n+1} = Q.$$

Putting $y^{-n+1} = z$ so that $(-n+1)y^{-n}(dy/dx) = dz/dx$, the equation becomes

$$\frac{dz}{dx}+P(1-n)z = Q(1-n)$$

which is linear with, z, as dependent variable.

Example

Solve

$$(x^3y^2+xy)dx = dy$$

We write the equation in the form

$$\frac{dy}{dx}-xy = x^3y^2.$$

Dividing by y^2, we get

$$\frac{1}{y^2}\frac{dy}{dx}-\frac{x}{y} = x^3.$$

On putting,

$$-\frac{1}{y} = z \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx},$$

we obtain

$$\frac{dz}{dx}+xz = x^3,$$

which is linear. The integrating factor, now, is, $e^{\int x\,dx} = e^{\frac{1}{2}x^2}$ and, therefore, the solution is

$$ze^{\frac{1}{2}x^2} = \int x^3 e^{\frac{1}{2}x^2}\,dx+c.$$

To evaluate the integral on the right, we put $x^2/2 = t$ and see that

$$\int x^3 e^{\frac{1}{2}x^2}\,dx = 2\int te^t dt$$

$$= 2(te^t-e^t) = 2e^{\frac{1}{2}x^2}(\tfrac{1}{2}x^2-1).$$

Thus $z = x^2-2+ce^{-\frac{1}{2}x^2}$ where $z = -1/y$, is the required solution.

Exercises

Solve the following equations :

1. $\dfrac{dy}{dx}+\dfrac{y}{x} = y^2x.$ 2. $x\dfrac{dy}{dx}+y = y^2 \log x.$

3. $x\dfrac{dy}{dx}+y = y^2x^3 \cos x.$ 4. $x^3 \dfrac{dy}{dx}-x^2y+y^4 \cos x = 0.$

5. $\dfrac{dy}{dx} = y \tan x-y^2 \sec x.$ 6 $x\dfrac{dy}{dx}+y = x^3y^4.$

7. $\dfrac{1}{y}\dfrac{dy}{dx}+\dfrac{x}{1-x^2} = xy^{-\frac{1}{2}}.$ 8. $(xy-x^2)\dfrac{dy}{dx} = y^2.$

9. $\dfrac{dy}{dx}+\dfrac{y}{x-1} = xy^{\frac{1}{3}}$ 10. $\dfrac{dy}{dx}+xy = y^2e^{\frac{1}{2}x^2}\log x.$

11. $y(x^2y+e^x)\,dx-e^x\,dy = 0.$

Answers

1. $1+x^2y+cxy = 0.$ 2. $y(1+cx+\log x) = 1.$
3. $xy(x \sin x+\cos x+c)+1 = 1.$ 4. $x^3 = (c+3 \sin x)\,y^3.$
5. $\sec x = (c+\tan x)\,y.$ 6. $(-3 \log x+c)\,x^3y^3 = 1.$
7. $\sqrt{y} = \frac{1}{3}(x^2-1)+c\sqrt[4]{(1-x^2)}.$ 8. $y = (\log y+c)\,x.$
9. $y^{\frac{2}{3}} = c(x-1)^{-\frac{2}{3}}+\frac{1}{4}(x-1)^2+\frac{2}{5}(x-1).$
10. $ye^{\frac{1}{2}x^2}(c+x-x \log x) = 1.$ 11. $x^3y+3e^x = cy.$

11·6. Homogeneous Equations. A differential equation of the form

$$\frac{dy}{dx} = \frac{f(x, y)}{\varphi(x, y)},$$

where $f(x, y)$ and $\varphi(x, y)$ are homogeneous functions of x, y and of the same degree, is said to be *homogeneous*. Such equations can be solved by putting

$$y = vx,$$

so that the dependent variable y is changed to another variable v.

Since $f(x, y)$ and $\varphi(x, y)$ are homogeneous functions of the same degree say, n, they can be written as

$$f(x, y) = x^n f_1\left(\frac{y}{x}\right) \text{and } \varphi(x, y) = x^n \varphi_1\left(\frac{y}{x}\right)$$

As $$y = vx,$$

we have

$$\frac{dy}{dx} = v+x\frac{dv}{dx}.$$

The given differential equation, therefore, becomes

$$v+x\frac{dv}{dx} = \frac{f_1(v)}{\varphi_1(v)}$$

$$\Rightarrow \quad \frac{\varphi_1(v)\,dv}{f_1(v)-v\varphi_1(v)} = \frac{dx}{x},$$

so that the variables v and x are now separable.

Examples

Solve

$$x^2\,dy+y\,(x+y)\,dx = 0.$$

Here, we have

$$\frac{dy}{dx}+\frac{y\,(x+y)}{x^2} = 0,$$

so that the equation is homogeneous. Putting $y = vx$, we get

$$v+\frac{xdv}{dx}+v\,(1+v) = 0.$$

$$\Rightarrow \quad \frac{dv}{v^2+2v} = -\frac{dx}{x},$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{1}{v}-\frac{1}{v+2}\right)dv = -\frac{dx}{x}.$$

Thus the solution is

$$\frac{1}{2}\log\frac{v}{v+2} = -\log x+c,$$

$$\Rightarrow \quad \log\sqrt{\left(\frac{y}{y+2x}\right)}+\log x = c,$$

$$\Rightarrow \quad \log\left[x\sqrt{\left(\frac{x}{y+2x}\right)}\right] = c,$$

$$\Rightarrow \quad x\sqrt{\left(\frac{x}{y+2x}\right)} = e^c = a,$$

where, a, is a constant,

$$\Rightarrow \quad x^2y = a^2\,(y+2x).$$

Exercises

Solve

1. $(x+y)\,dy+(x-y)\,dx = 0.$
2. $\dfrac{2dy}{dx} = \dfrac{y}{x}+\dfrac{y^2}{x^2}.$
3. $(x^3+y^3)\,dx = (x^2y+xy^2)\,dy.$
4. $(x^2+y^2)\dfrac{dy}{dx} = xy.$
5. $\dfrac{dy}{dx}+\dfrac{x-2y}{2x-y} = 0.$
6. $x\dfrac{dy}{dx} = y+\sqrt{(x^2+y^2)}.$
7. $(x^2+xy)\,dy = (x^2+y^2)\,dx.$

8. $x^2y\,dy+(x^3+x^2y-2xy^2-y^3)\,dx=0.$

9. $[2\sqrt{(xy)}-x]\,dy+y\,dx=0.$

10. $x^2y\,dx-(x^3+y^3)\,dy=0.$

11. $\dfrac{dy}{dx}+\dfrac{x^2+3y^2}{3x^2+y^2}=0.$

12. $x\sin\dfrac{y}{x}\,dy=\left(y\sin\dfrac{y}{x}-x\right)dx.$

13. $(6x^2+2y^2)\,dx-(x^2+4xy)\,dy=0.$

14. $y^2+x^2\dfrac{dy}{dx}=xy\dfrac{dy}{dx}.$

Answers

1. $\frac{1}{2}\log(x^2+y^2)+\tan^{-1}(y/x)=c.$
2. $y-x=c\sqrt{(xy)}.$
3. $y+x\log[c(x-y)]=0$
4. $y=ce^{x^2/2y^2}.$
5. $(x+y)^3=c\,(y-x).$
6. $c-y=\sqrt{(x^2+y^2)}.$
7. $(x-y)^2=cxe^{-y/x}$
8. $\log\dfrac{c\,(y-x)}{x^4\,(y+x)}=\dfrac{2x}{x+y}$
9. $\log y+\sqrt{(x/y)}=c.$
10. $x^3=3y^3\log cy.$
11. $\log(x+y)+2xy\,(x+y)^{-2}=c.$
12. $\log x=\cos(y/x)+c.$
13. $(2x+y)\,(2y-3x)=cx.$
14. $y/x=c+\log y.$

11·7. Equations reducible to the homogeneous form. The equaations of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C}$$

can be reduced to a homogeneous form by changing the variables x, y, to X, Y related by the equations

$$x=X+h,\ y=Y+k,$$

where h, k are the constants to be chosen so as to make the given equation homogeneous. We have

$$\frac{dy}{dx}=\frac{d\,(Y+k)}{dx}=\frac{dY}{dx}=\frac{dY}{dX}\cdot\frac{dX}{dx}=\frac{dY}{dX}$$

$\therefore$ the equation becomes

$$\frac{dY}{dX}=\frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)}.$$

Let h and k be chosen so as to satisfy the equations

$$ah+bk+c=0,\quad Ah+Bk+c=0.$$

These give

$$h=\frac{bC-Bc}{aB-Ab},\ k=\frac{Ac-aC}{aB-Ab},$$

which are meaningful except when $aB-Ab = 0$, *i.e.*, when $a/A = b/B$.

The homogeneous equation

$$\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$$

can now be solved by means of the substitution

$$Y = vX$$

Exceptional case. Let

$$\frac{a}{A} = \frac{k}{B} = r, \text{ say}$$

$\therefore$ $$a = Ar, b = Br.$$

The equation now becomes

$$\frac{dy}{dx} = \frac{r(Ax+By)+c}{Ax+By+C}.$$

We put $Ax+By = z$ so that $A+B\dfrac{dy}{dx} = \dfrac{dz}{dx}$ and obtain

$$\frac{dz}{dx} = B\frac{rz+c}{z+C}+A,$$

so that we obtain a differential equation with variables separable.

Example

Solve

$$\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}.$$

Putting $X = X+h$, $y = Y+k$, the equation becomes

$$\frac{dY}{dX} = \frac{X+2Y+(h+2k+3)}{2X+3Y+(2h+3k+4)}.$$

To determine h and k, we have

$$h+2k+3 = 0, 2h+3k+4 = 0.$$

$\Rightarrow$ $$h = 1, k = -2.$$

$\therefore$ $$\frac{dY}{dX} = \frac{X+2Y}{2X+3Y}$$

Putting $$Y = vX, \text{ we get}$$

$$v+X\frac{dv}{dx} = \frac{1+2v}{2+3v}$$

$\Rightarrow$ $$\frac{2+3v}{3v^2-1}\,dv = -\frac{dX}{X}$$

$\Rightarrow$ $$\left[\frac{2+\sqrt{3}}{2(\sqrt{3}v-1)} - \frac{2-\sqrt{3}}{2(\sqrt{3}v+1)}\right]dv = -\frac{dX}{X}$$

$\Rightarrow$ $$\frac{2+\sqrt{3}}{2}\log(\sqrt{3}v-1) - \frac{(2-\sqrt{3})}{2}\log(\sqrt{3}v+1)$$

$$= [-\log X+c]\sqrt{3}$$

$$\Rightarrow \qquad \frac{2+\sqrt{3}}{2}\log(\sqrt{3}Y-X)-\frac{2-\sqrt{3}}{2}\log(\sqrt{3}Y+X)$$
$$=\sqrt{3}c=a.$$

This is the required solution where $X=x-1$, $Y=y+2$.

Exercises

Solve the following differential equations :—

1. $\dfrac{dy}{dx}=\dfrac{x-2y+5}{2x+y-1}$.
2. $(3y-7x-3)\,dx+(7y-3x-7)\,dy=0$.
3. $(4x+6y+5)\,dx=(2x+3y+4)\,dy$.
4. $\dfrac{dy}{dx}=\dfrac{ax+by-a}{bx+ay-b}$.
5. $(2x+y+1)\,dx+(4x+2y-1)\,dy=0$.
6. $(2x+3y-6)\,dy=(6x-2y-7)\,dx$.
7. $(2x+3y-8)\,dx=(x+y-3)\,dy$.

Answers

1. $x^2-y^2-4xy+10x+2y=c$. 2. $c\,(y-x-1)^3\,(y+x-1)^5=1$.
3. $\frac{1}{8}(2x+y^3)+\frac{9}{64}\log(16x+24y+23)=x+c$.
4. $(y-x+1)^{a+b}\,(y+x-1)^{a-b}=c$
5. $\log(2x+y-1)+x+2y=c$. 6. $3x^2-\frac{3}{2}y^2-2xy-7x+6y=c$.
7. $\sqrt{3}\log(Y^2-2XY-2X^2)+2\log\{[Y-(1+\sqrt{3})X]/[Y-(1-\sqrt{3})X]\}=c$; where $X=x-1$, $Y=y-2$.

11·8. Exact Differential Equations. A differential equation is said to be exact if it can be derived from its primitive by direct differentiation without any further transformation such as elimination, etc.

Illustration. The differential equation

$$(x^2-ay)\,dx+(y^2-ax)\,dy=0.$$

is exact inasmuch as it can be derived from its primitive

$$x^3-3axy+y^3=c$$

by direct differentiation.

Theorem. *The necessary and sufficient condition for the differential equation*

$$M+N\frac{dy}{dx}=0 \Rightarrow Mdx+Ndy=0 \qquad \ldots(1)$$

be exact is that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

The proof for sufficiency being beyond the scope of this book, we will only prove that the condition is necessary.

Let $$f(x, y) = c \qquad ...(2)$$
be the primitive where, c, is the arbitrary constant.

Differentiating (2) with respect to x, we have

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \qquad ...(3)$$

As the equation (1) is exact, the equation (3) must be identical with (1), so that, we have

$$\frac{\partial f}{\partial x} = M. \qquad ...(4)$$

$$\frac{\partial f}{\partial y} = N. \qquad ...(5)$$

Differentiating (4) and (5) partially with respect to y and x respectively, we obtain,

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial M}{\partial y}, \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial N}{\partial x}.$$

Since $$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 f}{\partial x\partial y},$$

We obtain $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If the equation $Mdx+Ndy = 0$, is exact, then it can be integrated as follows :—

Firstly, integrate M with respect to x regarding y as a constant. Then integrate with respect to y those of the terms in N which do not involve x. The sum of the two expressions thus obtained equated to a constant is the required solution.

A study of the following example will make the validity of this procedure clear.

Example

Show that the equation

$$(x^4-2xy^2+y^4)dx-(2x^2y-4xy^3+\sin y)\,dy = 0$$

is exact and also solve it.

Here $$M = x^4-2xy^2+y^4,$$
$$N = -(2x^2y-4xy^3+\sin y)$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= -4xy+4y^3\\ \frac{\partial N}{\partial x} &= -4xy+4y^3\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the given equation is exact.

Integrating M with respect to x regarding y as constant, we obtain

$$\tfrac{1}{5}x^5 - x^2y^2 + xy^4. \qquad \ldots(1)$$

Now the only term in N which does not involve x is $-\sin y$. Its integral is

$$\cos y. \qquad \ldots(2)$$

The primitive, therefore, is

$$\tfrac{1}{5}x^5 - x^2y^2 + xy^4 + \cos y = c.$$

Exercises

Solve

1. $(ax+hy+g)\,dx+(hx+by+f)\,dy = 0.$
2. $xdy+(x+y)\,dx = 0.$
3. $(e^y+1)\cos x\,dx+e^y \sin x\,dy = 0.$
4. $x(y^2-x^2-a^2x)\,dx+y(y^2+x^2-b^2y)dy = 0.$
5. $[y(1+x^{-1})+\sin y]\,dx+(x+\log x+x\cos y)\,dy = 0.$
6. $(x^2-ay)\,dx+(y^2-ax)\,dy = 0.$

Answers

1. $ax^2+2hxy+by^2+2gx+2fy+c = 0.$
2. $xy+\tfrac{1}{2}x^2 = c.$ 3. $(e^y+1)\sin x = c.$
4. $6x^2y^2-3x^4+3y^4-4a^2x^3-4b^2y^3 = c.$
5. $xy+y\log x+x\sin y = c.$
6. $x^3+y^3-3axy = c.$

11·9. Integrating Factors. Sometimes an equation which is not exact may become so on multiplication by some suitable function known as an *Integrating factor.*

The equation

$$x\,dy - y\,dx = 0$$

which is not exact becomes so on multiplication by $1/y^2$, for we then obtain

$$\frac{x}{y^2}\,dy - \frac{1}{y}\,dx = 0.$$

which is easily seen to be exact.

For the linear equation

$$\frac{dy}{dx} + Py + Q = 0,$$

the function $e^{\int P dx}$ is an integrating factor.

It is not intended to consider general rules for the determination of integrating factors in this book.

Exercises

1. Show that

$$(y-2x^3)\,dx-x\,(1-xy)\,dy=0$$

becomes exact on multiplication by $1/x^2$ and solve it.

2. Solve

$$y(2xy+e^x)\,dx-e^x\,dy=0.$$

3. Solve

$$(1+yx)\,x\,dy+(1-yx)\,y\,dx=0.$$

Answers

1. $xy^2=cx+2y+2x^3$. 2. $2x^3y-cy+2e^x=0$. 3. $\log\dfrac{y}{x}-\dfrac{1}{xy}=c$.

EXERCISES ON CHAPTER 11

Solve the following differential equations :—

1. $x^2\dfrac{dy}{dx}=\dfrac{y(x+y)}{2}$. 2. $\cos^3 x\dfrac{dy}{dx}+y\cos x=\sin x$.

3. $\dfrac{dy}{dx}+y\sec x=\tan x$. 4. $\dfrac{dy}{dx}=\dfrac{x+2y-3}{2x+y-3}$.

5. $\dfrac{dy}{dx}+y\cot x=2\cos x$.

6. $(x^2+y^2)\dfrac{dy}{dx}=xy$.

7. $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$.

8. $2(1-xy)\dfrac{dy}{dx}=y^2$.

9. $\dfrac{dy}{dx}-2y\tan x=y^3\tan^2 x$.

10. $(x^2+y^2)dx-2xy\,dy=0$.

11. $\dfrac{1}{2x}\dfrac{dy}{dx}+\dfrac{x+y}{x^2+y^2}=0$.

12. $\dfrac{dy}{dx}+\dfrac{y}{\sqrt{(x^2+a^2)}}=3x$ such that $y=a^2$ when $x=0$.

13. $x\,dy-y\,dx=\sqrt{(x^2+y^2)}\,dx$.

14. $(x+y)\,dx+(y-x)\,dy=0$.

15. $y^2(y\,dx+2x\,dy)-x^2(2y\,dx+x\,dy) = 0.$

16. $(1+x^2)\dfrac{dy}{dx}+y = \tan^{-1} x.$

17. $\dfrac{dy}{dx}+\dfrac{x}{1-x^2}\,y = x\sqrt{y}.$

18. $\dfrac{dy}{dx}+y\cos x = \sin 2x.$

19. $\cos^2 x\,\dfrac{dy}{dx}+y = \tan x.$

20. $\dfrac{dy}{dx}+\dfrac{y}{x} = \dfrac{y^2}{x^2}.$

21. $(xy^2-e^{1/x^3})dx-x^2y\,dy = 0.$

22. $(2x-y+1)\,dx+(2y-x-1)\,dy = 0.$

23. $xy^2\,(xy_1+y) = a^2.$

24. $(y^3-2yx^2)\,dx+(2xy^2-x^3)\,dy = 0.$

25. $\sec^2 y\dfrac{dy}{dx}+2x\tan y = x^3.$

26. $(x+y+1)\,\dfrac{dy}{dx} = 1.$

27. $(x^2+2xy-y^2)\,dx+(y^2+2xy-x^2)\,dy = 0.$

28. Obtain the differential equation of all circles in a plane in the form

$$y_3(1+y_1^2)-3y_1y_2^2 = 0.$$

29. Find the differential equation of all conics whose axes coincide with the axes of coördinates.

30. Show that the equation

$$(4x+3y+1)\,dx+(3x+2y+1)\,dy = 0$$

represents a family of hyperbolas having as asymptotes the lines $x+y = 0$, $2x+y+1 = 0$.

31. Solve

$$\left\{x\cos\frac{y}{x}+y\sin\frac{y}{x}\right\}y = \left\{y\sin\frac{y}{x}-x\cos\frac{y}{x}\right\}x\,\frac{dy}{dx}.$$

Answers

1. $(y-x)^2 = cxy^2$.
2. $1+y = \tan x+ce^{-\tan x}$.
3. $(y-1)(1+\sin x) = (c-x) \cos x$.
4. $c(x-y)^2 = x+y-2$.
5. $2y \sin x+\cos 2x+c = 0$.
6. $cy=e^{x^2/2y^2}$.
7. $c \cot x \cot y = 1$.
8. $xy^2 = 2y+c$.
9. $y(\sin^2 x+c \cos^2 x)+3 \cos x = 0$.
10. $c(x^2-y^2) = x$.
11. $y^3+3yx^2+2x^3 = c$.
12. $y(x+\sqrt{a^2+x^2}) = x^3+(a^2+x^2)^{3/2}$
13. $c(\sqrt{(x^2+y^2)}-y) = 1$.
14. $c\sqrt{(x^2+y^2)} = e^{\tan^{-1}(y/x)}$.
15. $xy\sqrt{(x^2-y^2)} = c$.
16. $y+1 = \tan^{-1} x+ce^{-\tan^{-1} x}$.
17. $3\sqrt{y+1-x^2} = c(1-x^2)^{1/4}$.
18. $y+2 = 2 \sin x+ce^{-\sin x}$.
19. $y+1 = \tan x+ce^{-\tan x}$.
20. $y-2x = cx^2y$.
21. $x^{-2} y^2 = \frac{1}{2}e^{1/x^2}+c$.
22. $x^2-xy+x+y^2-y = c$.
23. $2x^3y^3 = 3a^2x^2+c$.
24. $xy\sqrt{(y^2-x^2)} = c$.
25. $2 \tan y = x^2-1+ce^{-x^2}$
26. $x+y+2 = ce^y$.
27. $c(x^2+y^2) = x+y$.
29. $xyy_2+xy_1^2 = yy_1$.
30. $xy \cos \frac{y}{x} = c$.

12

Equations of the First Order but not of the First Degree

12·1. In the discussion of equations of the first order which are not of the first degree, it is usual to denote dy/dx by p. The following types of equations, now, present themselves for discussion :—

1. *Equations solvable for p.*
2. *Equations solvable for y.*
3. *Equations solvable for x.*

These will be considered in the following sections :

12·2. Equations solvable for p. The following example will make the procedure clear.

Example

Solve

$$p^2+2py \cot x=y^2.$$

Solving for p, we obtain

$$p=\frac{-2y \cot x \pm \sqrt{(4y^2 \cot^2 x+4y^2)}}{2}$$

$$=-y \cot x \pm y \operatorname{cosec} x$$

$$=y(-\cot x \pm \operatorname{cosec} x).$$

Thus we have

$$\frac{dy}{dx}=y(-\cot x+\operatorname{cosec} x), \qquad \text{...(1)}$$

and

$$\frac{dy}{dx}=-y(\cot x+\operatorname{cosec} x) \qquad \text{...(2)}$$

In (1) and (2), the variables are separable.

(1) gives

$$\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)\, dx.$$

$$\Rightarrow \log y = -\log \sin x + \log \tan \frac{x}{2} + \log c = \log \frac{c \tan \frac{x}{2}}{\sin x}$$

$$\Rightarrow \quad y = \frac{c \tan \frac{x}{2}}{\sin x} = \frac{c}{2 \cos^2 \frac{x}{2}} = \frac{c}{1+\cos x} \qquad \ldots(3)$$

Again (2) gives

$$\frac{dy}{y} = -(\cot x + \operatorname{cosec} x)\, dx$$

$$\Rightarrow \quad \log y = -\left(\log \sin x + \log \tan \frac{x}{2} \right) + \log c$$

$$= \log \frac{c}{\sin x \tan \frac{x}{2}}.$$

$$\Rightarrow \quad y = \frac{c}{\sin x \tan \frac{x}{2}} = \frac{c}{2 \sin^2 \frac{x}{2}} = \frac{c}{1-\cos x}. \qquad \ldots(4)$$

Thus the solutions of (1) and (2) are

$$y - \frac{c}{1+\cos x} = 0,\ y - \frac{c}{1-\cos x} = 0.$$

The composite solution of the given differential equation, therefore, is

$$\left(y - \frac{c}{1+\cos x} \right)\left(y - \frac{c}{1-\cos x} \right) = 0.$$

Exercises

Solve the following differential equations:

1. $x \left(\frac{dy}{dx} \right)^2 + (y-x) \frac{dy}{dx} - y = 0.$
2. $y^2 + xyp - x^2p^2 = 0.$
3. $xyp^2 - (x^2+y^2)p + xy = 0.$

4. $x^2\left(\frac{dy}{dx}\right)^2+xy\frac{dy}{dx}-6y^2=0.$

5. $p^2y+(x-y)p-x=0.$

Answers

1. $(y-c-x)(xy-c)=0$.
2. $(y^2-cx^{1+\sqrt{5}})(y^2-cx^{1-\sqrt{5}})=0.$
3. $(y-cx)(x^2-y^2-c)=0.$
4. $(y-cx^2)(x^2y-c)=0.$
5. $(x^2+y^2-c)(y-x-c)=0.$

12·3. Equations solvable for y. Suppose that the given differential equation, on solving for y, gives

$$y=f(x,p). \qquad ...(1)$$

Differentiating with respect to x, we obtain

$$p=\frac{dy}{dx}=\varphi\left(x,p,\frac{dp}{dx}\right)$$

so that we obtain a new differential equation with variables x and p.

Suppose that it is possible to solve this equation.

Let the solution be

$$F(x,p,c)=0, \qquad ...(2)$$

where, c, is the arbitrary constant.

The solution of (1) may be exhibited in either of the *two* forms. We may either eliminate p between (1) and (2) and obtain $\psi(x,y,c)$ as the required solution or we may solve (1) and (2) for x, y and obtain

$$x=f_1(p,c),\ y=f_2(p,c)$$

as the required solution, where, p, is the parameter.

Example

Solve

$$y+px=p^2x^4.$$

Differentiating with respect to x, we have

$$\frac{dy}{dx}+p+x\frac{dp}{dx}=4x^3p^2+2x^4p\frac{dp}{dx}$$

$$\Rightarrow \qquad 2p-4x^3p^2-2x^4p\frac{dp}{dx}+x\frac{dp}{dx}=0,$$

$$\Rightarrow \qquad 2p(1-2x^3p)+x(1-2x^3p)\frac{dp}{dx}=0.$$

$$\Rightarrow \qquad 2p+x\frac{dp}{dx}=0$$

$$\Rightarrow \qquad \frac{dp}{p}+\frac{2dx}{x}=0$$

$\Rightarrow$ $\log p+2\log x=\text{constant}=\log c$, say

$\Rightarrow$ $px^2=c \Leftrightarrow p=c/x^2$.

Substituting this value of, p, in the given differential equation so as to eliminate, p, we obtain

$$y=-c/x+c^2 \Rightarrow xy=c^2x-c$$

as the required solution.

12·4. Equations solvable for x. Suppose that the given differential equation, on solving for x, gives

$$x=f(y, p). \qquad \ldots(1)$$

Differentiating with respect to y, we obtain

$$\frac{1}{p}=\frac{dx}{dy}=\varphi(y, p, dp/dy), \text{ say}$$

so that we obtain a new differential equation in variables y and p. *Suppose* that it is possible to solve this equation.

Let the solution be ...(2)

$$F(p, y, c)=0$$

As in the preceding article, the elimination of p between (1) and (2) will give the solution, or (1) and (2) may be solved to express x and y in terms of p and c where p is to be regarded as a parameter.

Example

Solve

$$y=2px+y^2p^3.$$

Solving it for x, we obtain

$$x=\frac{y-y^2p^3}{2p}=\frac{y}{2p}-\frac{y^2p^2}{2}.$$

Differentiating with respect to y, we obtain

$$\frac{1}{p}=\frac{dx}{dy}=\frac{1}{2p}-\frac{y}{2p^2}\frac{dp}{dy}-yp^2-py^2\frac{dp}{dy}$$

$$\frac{1}{2p}+p^2y+\left(\frac{y}{2p^2}+py^2\right)\frac{dp}{dy}=0.$$

$$\Rightarrow \qquad \frac{1+2p^3y}{2p}+\frac{y(1+2p^3y)}{2p^2}\cdot\frac{dp}{dy}=0.$$

$$\Rightarrow \qquad 1+\frac{y}{p}\,\frac{dp}{dy}=0.$$

$$\Rightarrow \qquad \frac{dy}{y}+\frac{dp}{p}=0.$$

$$\Rightarrow \qquad \log y+\log p=\text{constant}=\log c, \text{ say,}$$

$$\Rightarrow \qquad py=c \quad \Leftrightarrow \quad p=c/y.$$

Substituting this value of, p, in the given equation, we obtain

$$y=\frac{2cx}{y}+y^2.\ \frac{c^3}{y^3} \quad \Rightarrow \quad y^2=2cx+c^3,$$

which is the required solution.

Exercises

Solve the following differential equations :—

1. $p^2y+2px-y=0.$ 2. $ap^2+py-x=0.$

3. $\left(\dfrac{dy}{dx}\right)^3-4xy\,\dfrac{dy}{dx}+8y^2=0.$

4. $\left(\dfrac{dy}{dx}\right)^3 y^2-2x\,\dfrac{dy}{dx}+y=0.$

5. $4p^3+3xp=y.$

Answers

1. $x=\dfrac{c(1-p^2)}{p^2}$, $y=\dfrac{2c}{p}.$

2. $x=\dfrac{(c-a\cosh^{-1}p)}{\sqrt{(p^2-1)}}p$, $y=\dfrac{c-a\cosh^{-1}p}{\sqrt{(p^2-1)}}-ap.$

3. $64y=c(4x-c)^2.$ 4. $2cx=c^3+y^2.$

5. $y=-\frac{8}{7}p^3+cp^{-\frac{1}{2}}$, $x=-\frac{12}{7}p^2+\dfrac{c}{3}\,p^{-\frac{3}{2}}.$

12·5. Clairut's Equation. The equation

$$y=px+f(p), \qquad \ldots(1)$$

is known as Clairut's equation. Here $f(p)$ is a function of p. To solve (1), we differentiate with respect to x and obtain

$$p=p+x\,\frac{dp}{dx}+f'(p)\,\frac{dp}{dx},$$

$$\left[x+f'(p)\right]\frac{dp}{dx}=0.$$

$$\Rightarrow \quad \begin{cases} \dfrac{dp}{dx}=0. & \dots(2) \\ x+f'(p)=0. & \dots(3) \end{cases}$$

Now, (2) gives

$$p=\text{constant}=c, \text{ say.} \qquad \dots(4)$$

Eliminating p between (1) and (4), we obtain

$$y=cx+f(c) \qquad \dots(5)$$

as a solution of (1).

If we eliminate p between (1) and (3), we will obtain another solution not contained in the general solution (5). This solution is known as the *singular solution*. (Refer § 12·7 p. 263).

Note. As illustrated in Ex. 2 below, change of variables sometimes transforms a given equation to Clairut's.

Examples

1. *Solve*

$$y=px+\log p.$$

The solution of this equation which is Clairut's, is

$$y=cx+\log c,$$

obtained on changing, p, to the arbitrary constant c.

2. *Solve*

$$x^2(y-px)=p^2y.$$

We put

$$x^2=u, \qquad y^2=v.$$

$$\Rightarrow \qquad 2x\ dx=du \text{ and } 2y\ dy=dv.$$

$$\Rightarrow \qquad \frac{ydy}{xdx}=\frac{dv}{du} \Rightarrow p=\frac{dy}{dx}=\frac{x}{y}\cdot\frac{dv}{du}.$$

Substituting this value of p in the given differential equation, we obtain

$$x^2\left(y-\frac{x^2}{y}\cdot\frac{dv}{du}\right)=\frac{x^2}{y}\left(\frac{dv}{du}\right)^2,$$

$$\Rightarrow \qquad \left(y^2-x^2\frac{dv}{du}\right)=\left(\frac{dv}{du}\right)^2,$$

$$\Rightarrow \qquad \left(v-u\frac{dv}{du}\right)=\left(\frac{dv}{du}\right)^2,$$

$$\Rightarrow \qquad v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2,$$

which is Clairut's. The solution, therefore, is

$$v = cu+c^2 \;\Rightarrow\; y^2 = cx^2+c^2$$

3. *Solve*

$$y = xf_1(p)+f_2(p)$$

While this equation is *not* Clairut's it will be seen that the method adopted for solving Clairut's equations will deliver the goods.

Differentiating with respect to x, we obtain

$$p = f_1(p)+xf_1'(p)\frac{dp}{dx}+f_2'(p)\frac{dp}{dx}$$

$$\Rightarrow \qquad \frac{dx}{dp} = \frac{f_1'(p)}{p-f_1(p)}\,x+\frac{f_2'(p)}{p-f_1(p)},$$

which is linear with, p, as independent and, x, as dependent variable. Suppose that on solving this linear equation, we obtain

$$\varphi(x, p, c) = 0.$$

Then, eliminating p between this solution and the given equation, we obtain the required solution.

Exercises

Solve

1. $(y-px)(p-1) = p.$
2. $y = x\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^2.$
3. $y = x\dfrac{dy}{dx}+e^{dy/dx}$
4. $(px-y)(py+x) = a^2p.$ [Put $x^2 = u, y^2 = v$]
5. $y+xp\log p = (2+3\log p)p^3.$
6. $y = 2px-p^2.$

Answers

1. $(y-cx)(c-1) = c.$
2. $y = cx+c^2.$
3. $y = cx+e^c.$
4. $y^2 = cx^2-a^2c/(1+c).$
5. $x = 3p^2+c\,p^{-1}$, $y = 2p^3-c\log p.$
6. $x = \frac{2}{3}p+cp^{-2}$, $y = \frac{1}{3}p^2+2cp^{-1}.$

12·6. Geometrical meaning of a differential equation of the first order. Consider a differential equation

$$\varphi(x, y, dy/dx) = 0 \qquad \ldots(1)$$

and let

$$f(x, y, c) = 0 \qquad \ldots(2)$$

be its general solution ; c, being the arbitrary constant.

For each value of the arbitrary constant c, the equation (2) represents a curve so that the equation (2) represents a family of curves ; c being the parameter for the family. We thus say that *Every differential equation of the first order represents a family of curves.*

Now take a point in the plane and substitute its co-ordinates (x, y) in (1) and (2) and solve the resulting equations in dy/dx and c. Then the values of c, so obtained, are the values of the parameter for the particular curves of the family through (x, y) and the values of dy/dx are the slopes of the tangents to these curves at the point. Thus, in general, we expect that the degree of, c in (2) must be equal to that of dy/dx in (1) which is known as the degree of the differential equation.

12·7. Singular Solutions. In addition to the *General solution* and the *particular solutions*, obtained by giving particular values to the arbitrary constants in the general solution, a differential equation may also possess other solutions. The solutions of differential equations, other than the general and particular, are known as *Singular solutions.* In this connection, we have the following result.

Whenever the family of curves

$$f(x, y, c) = 0 \qquad \ldots(1)$$

represented by the differential equation

$$\varphi(x, y, dy/dx) = 0 \qquad \ldots(2)$$

possesses an envelope, the equation of the envelope is the singular solution of the differential equation (2).

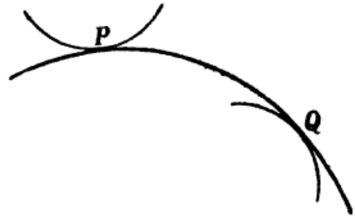

Fig. 51.

Suppose that the family of curves possesses an envelope. Take any point $P(x, y)$ on the envelope. There exists a curve of the family, say,

$$f(x, y, c_1)=0,$$

which touches the envelope at (x, y). The values of x, y, dy/dx for the curve at P satisfy the given differential equation. Also the values of x, y, dy/dx at P for the envelope are the same as for the curve. Thus we see that the values of x, y, dy/dx at every point of the envelope satisfy the given differential equation. Hence the equation of the envelope is a solution of the differential equation.

This solution does not contain an arbitrary constant and, in general, cannot be obtained from the general solution by giving particular values to the arbitrary constants.

12·71. Determination of Singular Solutions. Let

$$\varphi(x, y, dy/dx) = 0$$

be a given differential equation and let $f(x, y, c)=0$ be its general solution. Now it is known that the envelope of any family of curves

$$f(x, y, c)=0 \qquad \text{...(1)}$$

is *contained* in the locus obtained on eliminating c, between (1) and

$$\frac{\partial f(x, y, c)}{\partial c} = 0. \qquad \text{...(2)}$$

Let this eliminant be

$$\psi(x, y) = 0. \qquad \text{...(3)}$$

As the eliminant given by (3) may represent loci other than the envelope ; it is necessary to verify if any part of the locus represented by (3) is or is not a solution of the given differential equation.

Another Method. From the Theory of equations, it is known that the equation (3) represents the locus of points (x, y) such that at least two or the corresponding values of, c, are equal *i.e.*, such that at least two of the curves of the family through (x, y) coincide. As the equation $\varphi(x, y, dy/dx)=0$, *i.e.*, $\varphi(x, y, p)=0$ determines the slopes of the tangents to the curves of the family through (x, y) we see that for a point (x, y) satisfying (3) at least two of the corresponding values of p must coincide. Hence we see that the envelope and hence the singular solution is also contained in the locus obtained on eliminating, p, between

$$\varphi(x, y, p)=0, \quad \frac{\partial \varphi(x, y, p)}{\partial p}=0.$$

Examples

1. *Find the singular solution of the Clairut's equation*

$$y=px+f(p). \qquad \text{...(1)}$$

The general solution is

$$y=cx+f(c) \qquad \text{...(2)}$$

so that the singular solution is obtained on eliminating, c between (2) and

$$0=x+f'(c) \qquad \ldots(3)$$

which is obtained on differentiating (2) partially *w. r.* to c.

Differentiating (1) partially *w. r.* to p, we obtain

$$0=x+f'(p) \qquad \ldots(4)$$

so that the singular solution is also obtained on eliminating, p, between (1) and (4). Clearly the c-eliminant and the p-eliminant are the same.

2. *Solve the equation*

$$x^2p^2+yp(2x+y)+y^2=0,$$

using the substitutions $y=u$, $xy=v$ and find its singular solution.

We have

$$y=u, \qquad x=v/u$$

$$\Rightarrow \qquad \frac{dy}{du}=1, \quad \frac{dx}{du}=\frac{u\dfrac{dv}{du}-v}{u^2}.$$

These give

$$p=\frac{dy}{dx}=\frac{u^2}{u\dfrac{dv}{du}-v}.$$

Making substitutions in the given differential equation, we shall obtain

$$\left(u\frac{dv}{du}-v\right)^2+(u^2+2v)\left(u\frac{dv}{du}-v\right)+v^2=0,$$

$$\Rightarrow \qquad v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2.$$

This is Clairut's equation. Its general solution is

$$v=cu+c^2$$

$$\Leftrightarrow \qquad xy=cy+c^2. \qquad \ldots(1)$$

To obtain the singular solution, we differentiate (1) partially *w.r.* to c, and obtain

$$0=y+2c. \qquad \ldots(2)$$

Eliminating, c, between (1) and (2), we obtain

$$y(y+4x)=0.$$

Consider $y=0$. Differentiating this we obtain $p=0$. Substituting $y=0, p=0$, we see that the given differential equation is satisfied.

Consider now, $y+4x = 0$. This gives $p = -4$. Substituting $y = -4x$ and $p = -4$ in the given differential equation, we see that it is again satisfied. Thus $y = 0$, $y+4x = 0$ are both singular solutions.

Exercises

1. Find the general and singular solution of

$$y^2-2pxy+p^2(x^2-1) = m^2.$$

2. Prove that, for the equation

$$y = px-(1+p^2)^{1/2}$$

the envelope has the equation

$$(x^2+y^2)^2-2(x^2-y^2)+1 = 0.$$

3. Find the general and singular solution of

$$9p^2(1-y)^2 = 4(2-y).$$

4. Show that the equation

$$x^2(y-px) = p^2y$$

has *no* singular solution.

5. Solve and examine for singular solution the differential equation

$$x^3p^2+x^2yp+a^3 = 0.$$

6. Solve

$$y = 2x\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^3,$$

and find the singular solution also.

7. Show that the singular solution of

$$px+y = p^2x^4$$

is

$$4x^2y+1 = 0.$$

Answers

1. $(y-cx)^2 = m^2+c^2$, $y^2+m^2x^2 = m^2$.
3. $(x+c)^2 = (y+1)^2(2-y)$, $y = 2$.
5. $c^2x-cxy+a^3 = 0$, $xy^2 = 4a^3$.
6. $y^2 = 2cx+c^3$, $27y^4+32x^3 = 0$.

EXERCISES ON CHAPTER 12

1. $p^2+py = x(x+y)$.
2. $p = \sin(x-y)$.
3. $p(p-y) = x(x+y)$.
4. $p^2xy+p(4x^2-3y^2)-12xy = 0$.
5. $(1+p^2)y = 2px$.
6. $p^2x^2-2pxy+y^2+4p = 0$.

7. $p^2(p+m) = a(y+mx)$. 8. $y = 2px+f(p^2x)$.

9. $y = 2px+y^{n-1}p^n$.

10. Solve and examine for singular solution the equation

$$xp^2-(x-a)^2 = 0.$$

11. A curve satisfies the differential equation

$$y = p^2(x-p),$$

and also that $p = 0$ when $x = \frac{1}{2}$; determine its equation.

12. Integrate

$$xp^2-py-y = 0,$$

and examine the relationship of the lines

$$y^2+4xy = 0,$$

to the integral curves of the differential equation.

13. Solve the following

$$e^{3x}(p-1)+p^3e^{2y} = 0.$$

14. Show that all integral curves of the equation

$$y+\frac{dy}{dx} = 1+\frac{dx}{dy}+x\frac{dy}{dx},$$

are either parabolas or straight lines.

15. Solve and determine singular solutions :

(*i*) $(px^2+y^2)(px+y) = (p+1)^2$.

[Use the substitution : $u = xy, v = x+y$]

(*ii*) $4p^2x(x-1)(x-2) = (3x^2-6x+2)^2$.

16. Solve completely

$$y^2(y-px) = x^4p^2.$$

[Use the substitution : $u = 1/x, v = 1/y$].

Answers

1. $(2y-x^2-c)(x+y-1-ce^{-x}) = 0$.
2. $x+c = \cot[\frac{1}{4}\pi-\frac{1}{2}(x-y)]$.
3. $(y+x-1-ce^{x})(x^2+2y-c) = 0$.
4. $(y-cx^2)(y^2+4x^2+c) = 0$.
5. $x = c(1+p^2)/p^2, y = 2c/p$.
6. $c^2x^2-2cxy+4c+y^2 = 0$.

7. $x = \dfrac{1}{a}\left\{\dfrac{3}{2}p^2 - mp + m^2 \log(p+m)\right\} + c$

$y = \dfrac{1}{a}\left(p^2\left(p - \frac{1}{2}m\right) + m^2 p - m^3 \log(p+m)\right) - mc.$

8. $y = 2c\sqrt{x} + f(c^2.)$

9. $y^2 = 2cx + c^n.$

10. $9(\pm y + c)^2 = 4x(x-3a)^2,\ x = 0.$

11. $(2x-1)^2 = 8y.$

12. $x = c(1+p)e^p,\ y = cp^2e^p.$

13. $e^y = ce^x + c^3.$

15. (i) $x + y = cxy + c^2,\ x^2y^2 + 4(x+y) = 0.$
(ii) $(y+c)^2 = x(x-1)(x-2)\ ;\ x = 0,\ x = 1,\ x = 2.$

16. $x = c + c^2xy\ ;\ y + 4x^2 = 0.$

13

Trajectories of a Family of Curves

13·1. Def. *A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.*

We shall consider only the case when each trajectory cuts every member of a given family at a *constant* angle. The trajectory will be called *Orthogonal*, if the constant angle is a right angle. For example, every line through the origin of co-ordinates is an orthogonal trajectory of the family of concentric circles with centre at the origin.

13·2. Cartesian Co-ordinates. *To find the trajectories which cut every member of the family $f(x, y, c) = 0$, at a constant angle ; (c parameter).*

Differentiating

$$f(x, y, c) = 0 \qquad ...(1)$$

with respect to x and eliminating, c, between (1) and the derived result, we will obtain the differential equation of the given family. Let it be

$$\varphi\left(x, y, \frac{dy}{dx}\right) = 0. \qquad ...(2)$$

Let (X, Y) be the current co-ordinates of any point on a trajectory so that dY/dX is the slope of the tangent at the point (X, Y) thereof.

At a point of intersection of any member of (2), with the trajectory, we have

$$x = X, \qquad \ldots(3)$$

$$y = Y, \qquad \ldots(4)$$

$$\tan \alpha = \frac{\dfrac{dy}{dx} - \dfrac{dY}{dX}}{1 + \dfrac{dY}{dX}\,\dfrac{dy}{dx}} \qquad \ldots(5)$$

Now, (5) gives

$$\frac{dy}{dx} = \frac{\dfrac{dY}{dX} + \tan \alpha}{1 - \dfrac{dY}{dX}\tan \alpha} \qquad \ldots(6)$$

From (2), (3), (4) and (6), we have, on eliminating x, y, and dy/dx,

$$\varphi\left(X,\ Y,\ \frac{\dfrac{dY}{dX} + \tan \alpha}{1 - \dfrac{dY}{dX}\tan \alpha}\right) = 0, \qquad \ldots(7)$$

as the differential equation of the required family of trajectories. Solving (7), we shall obtain the cartesian equation of the family of trajectories.

Cor. *Orthogonal trajectories.* For orthogonal trajectories, we have to replace the relation (5) by

$$\frac{dy}{dx} \cdot \frac{dY}{dX} = -1 \Leftrightarrow \frac{dy}{dx} = -\frac{dX}{dY}$$

Thus the differential equation of the family of orthogonal trajectories is

$$\varphi\left(X, Y, -\frac{dX}{dY}\right) = 0.$$

In the usual notation, we see that the differential equation of the family of orthogonal trajectories of the family of curves given by

$$\varphi(x, y, dy/dx) = 0,$$

is

$$\varphi(x, y, -dx/dy) = 0,$$

so that it is obtained on replacing, dy/dx, by $-dx/dy$.

13·3. Polar Co-ordinates. Orthogonal Trajectories. *To find the orthogonal trajectories of the family of curves.*

$$f(r, \theta, c) = 0\ ; \qquad c \text{ being a parameter.}$$

Differentiating

$$f(r, \theta, c)=0$$

with respect to θ, and eliminating c between (1) and the derived result, we shall obtain the differential equation of the given family.

Let the eliminant be

$$\psi(r, \theta, dr/d\theta)=0$$

Let r', θ' be the current co-ordinates of any point on a trajectory. At a point of intersection of any number of (2) with the trajectory, let φ, φ' be the angles which the tangents to the two curves make with the common radius vector. We have

$$\tan\varphi=r\frac{d\theta}{dr}, \tan\varphi=r'\frac{d\theta'}{dr'}$$

Also

$$\varphi-\varphi'=\frac{\pi}{2}$$

$$\Rightarrow \qquad \varphi=\varphi'+\frac{\pi}{2}.$$

$$\Rightarrow \qquad \tan\varphi=-\cot\varphi'$$

$$\Rightarrow \qquad \tan\varphi\tan\varphi'=-1,$$

$$\Rightarrow \qquad r\,r'\frac{d\theta}{dr}\frac{d\theta'}{dr'}=-1. \qquad \ldots(3)$$

Also

$$r=r'\Rightarrow\theta=\theta'; \qquad \ldots(4)$$

We re-write (3) as

$$\frac{dr}{d\theta}=-rr'\frac{d\theta'}{dr'} \qquad \ldots(5)$$

From (2), (3), (4) and (5), on eliminating

$$r, \theta, dr/d\theta$$

we obtain

$$\psi\left(r', \theta', -r'^2\frac{d\theta'}{dr'}\right)=0, \qquad \ldots(6)$$

as the differential equation of the required family of trajectories.

In the usual notation, we rewrite (7) as

$$\psi\left(r, \theta, -r^2\frac{d\theta}{dr}\right)=0,$$

which may be obtained from the differential equation (1) of the given family on changing

$$r, \theta, dr/d\theta \text{ to } r, \theta, -r^2\,d\theta/dr$$

respectively.

Examples

1. *Find the orthogonal trajectories of the curves*

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \qquad \text{...(1)}$$

λ being the parameter of the family.

Differentiating (1) with respect to x, we have

$$\frac{x}{a^2+\lambda}+\frac{y}{b^2+\lambda}\frac{dy}{dx}=0.$$

From (1) and (2), we have to eliminate λ.

Now, (2) gives

$$\lambda=-\frac{b^2x+a^2y\dfrac{dy}{dx}}{x+y\dfrac{dy}{dx}}.$$

$$\Rightarrow \qquad a^2+\lambda=\frac{(a^2-b^2)x}{x+y\,dy/dx}, \quad b^2+\lambda=\frac{(a^2-b^2)\,y\,dy/dx}{x+y\,dy/dx}$$

Substituting these values in (1), we get

$$\left(x+y\frac{dx}{dy}\right)\left(x+y\frac{dy}{dx}\right)=a^2-b^2, \qquad \text{...(2)}$$

as the differential equation of the given family.

Changing dy/dx to $-dx/dy$ in (3), we obtain

$$\left(x+y\frac{dy}{dx}\right)\left(x-y\frac{dx}{dy}\right)=a^2-b^2, \qquad \text{...(3)}$$

which is the same as (2). Thus we see that the family (1) is self-orthogonal, *i.e.*, every member of the family (1) cuts every other member of the same family orthogonally.

2. *Find the orthogonal trajectories of the family of co-axial circles*

$$x^2+y^2+2gx+c=0 \qquad \text{...(1)}$$

where, g, is the parameter.

Differentiating (1), we have

$$x+y\frac{dy}{dx}+g=0. \qquad \text{...(2)}$$

Eliminating, g, from (1) and (2) we have

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)+c=0,$$

$$\Rightarrow \qquad y^2-x^2-2xy\frac{dy}{dx}+c=0.$$

Changing dy/dx to $-dx/dy$ we see that the differential equation of the required family of orthogonal trajectories is

$$y^2-x^2+2xy\frac{dx}{dy}+c=0,$$

$$\Rightarrow \qquad 2xy\frac{dx}{dy}-x^2=-c-y^2.$$

Putting $x^2=t$, we will obtain a linear equation with, t, as dependent variable.

The solution can now easily be seen to be

$$x^2+y^2+2fy-c=0,$$

which represents a co-axal system of circles orthogonal to the given system with, f, as parameter.

3. *Find the orthogonal trajectory of the family of cardioides*

$$r=a(1+\cos\theta);$$

a being the parameter.

Differentiating

$$r=a(1+\cos\theta), \qquad ...(1)$$

with respect to θ, we obtain

$$\frac{dr}{d\theta}=-a\sin\theta. \qquad ...(2)$$

Eliminating, a, between (1) and (2), we obtain

$$\frac{r}{dr/d\theta}=-\frac{1+\cos\theta}{\sin\theta}$$

which is the differential equation of the given family. Changing $dr/d\theta$ to $-r^2 d\theta/dr$, we obtain

$$\frac{r}{-r^2d\theta/dr}=-\frac{1+\cos\theta}{\sin\theta}\Rightarrow\frac{1}{r}\frac{dr}{d\theta}=\frac{1+\cos\theta}{\sin\theta} \qquad ...(3)$$

as the differential equation of the family of orthogonal trajectories. To solve (3), we rewrite the same as

$$\frac{dr}{r}=\frac{1+\cos\theta}{\sin\theta}d\theta=(\operatorname{cosec}\theta+\cot\theta)\,d\theta.$$

$$\Rightarrow \qquad \log r=\log\tan\frac{\theta}{2}+\log\sin\theta+\log c$$

$$= \log\left(c \sin\theta \tan\frac{\theta}{2}\right)$$

$$\Rightarrow \qquad r = c \sin\theta \tan\theta/2$$

$$= 2c \sin\frac{\theta}{2} \cos\frac{\theta}{2} \tan\frac{\theta}{2}$$

$$= 2c \sin^2\frac{\theta}{2} = c(1-\cos\theta)$$

Thus the orthogonal trajectories of the family

$$r = a(1+\cos\theta)$$

of cardioides is the family of cardioides

$$r = c(1-\cos\theta),$$

The integration could also be completed by writing

$$\frac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2}\cdot$$

Exercises

1. Find the orthogonal trajectories of the family of semicubical parabolas $ay^2 = x^3$ where, a, is a variable parameter.

2. Find the orthogonal trajectories of the family of parabolas

$$y = ax^2.$$

3. Find the orthogonal trajectories of

$$\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda} = 1,$$

where λ is an arbitrary parameter.

4. Find the equation of the system of orthogonal trajectories of a series of confocal and coaxal parabolas

$$r = 2a/(1+\cos\theta).$$

5. Find the orthogonal trajectories of a system of equal circle with collinear centres.—Sketch a typical curve of the family which is the required family of trajectories.

6. Find the orthogonal trajectories of

(i) $r\theta = a$, (ii) $r = a\theta$,

(iii) $r^n = a^n \cos n\theta$, (iv) $r^n \cos n\theta = a^n$,

(v) $r = a(1-\cos\theta)$. (vi) $r = c\sin^2\theta$.

7. Find the orthogonal trajectories of the following families of curves :

(i) $x^{2/3}+y^{2/3} = \lambda^{2/3}$. (ii) $r\sin 2\theta = \lambda$.

(iii) $r = \tan(\theta+\alpha)$. (iv) $x^2+y^2 = r^2$.

Answers

1. $3y^2+2x^2 = c^2$.

2. $x^2+2y^2 = c^2$.

3. $x^2+y^2+c = 2a^2 \log x$

4. $r = 2b/(1-\cos\theta)$.

5. $x = r(\tanh t - t)+c,\ y = r \operatorname{sech} t$; the given family of the circles being

$$x^2+y^2-2gx+g^2 = r^2.$$

6. (*i*) $r^2 = ce^{\theta^2}$. (*ii*) $r = ce^{-\theta^2/2}$.

(*iii*) $r^n = c^n \sin n\theta$. (*iv*) $r^n \sin n\theta = c^n$.

(*v*) $r = c(1+\cos\theta)$. (*vi*) $r^2 = c\cos\theta$.

7. (*i*) $y^{4/3}-x^{4/3} = c^{4/3}$. (*ii*) $r^4 \cos 2\theta = c^4$.

(*iii*) $\theta + r = c + r^{-1}$. (*iv*) $y = mx$.

14

Linear Equations

14·1. Linear Differential Equations *are the equations in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together.* Thus the general linear differential equation of the nth order is

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots\ldots+P_n y=X,$$

where P_1, P_2, ... , P_n, and X are functions of x only.

The discussion of the general equation is beyond the scope of this book and only linear differential equations with constant coefficients and homogeneous linear equations will be considered in this book. Linear differential equations of the first order have already been considered in § 10·4, P. 229.

14·2. Linear Differential Equations with constant coefficients. *To solve the differential equation*

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots\ldots+a_n y=X, \quad \ldots(\text{A})$$

where a_1, a_2, ..., a_n are constants and X is a function of x.

Consider the differential equation

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots\ldots+a_n y=0, \quad \ldots(\text{B})$$

obtained on equating to zero the left hand expression of (A).

We will, now, show that y_1, y_2 *are any two solutions of* (B), *then,* $c_1y_1+c_2y_2$ *is also a solution of* (B) ; c_1, c_2 *being arbitrary constants.*

Since y_1, y_2 are solutions of (B), we have

$$\frac{d^n y_1}{dx^n}+a_1\frac{d^{n-1}y_1}{dx^{n-1}}+a_2\frac{d^{n-2}y_1}{dx^{n-2}}+\ldots\ldots+a_n y_1=0, \qquad \ldots(i)$$

$$\frac{d^n y_2}{dx^n}+a_1\frac{d^{n-1}y_2}{dx^{n-1}}+a_2\frac{d^{n-2}y_2}{dx^{n-2}}+\ldots\ldots+a_n y_2=0. \qquad \ldots(ii)$$

If c_1, c_2 are any two arbitrary constants, we have

$$\frac{d^n(c_1y_1+c_2y_2)}{dx^n}+a_1\frac{d^{n-1}(c_1y_1+c_2y_2)}{dx^{n-1}}+a_2\frac{d^{n-2}(c_1y_1+c_2y_2)}{dx^{n-2}}+\ldots$$

$$=c_1\left(\frac{d^n y_1}{dx^n}+a_1\frac{d^{n-1}y_1}{dx^{n-1}}+a_2\frac{d^{n-2}y_1}{dx^{n-2}}+\ldots\ldots\right)$$

$$+c_2\left[\frac{d^n y_2}{dx^n}+a_1\frac{d^{n-1}y_2}{dx^{n-1}}+a_2\frac{d^{n-2}y_2}{dx^{n-2}}+\ldots\ldots\right]$$

$$=c_1.0+c_2.0=0. \qquad \text{By } (i) \text{ and } (ii)$$

This proves the statement made above.

Since the general solution of a differential equation of the *n*th order contains, n arbitrary constants, we deduce from above that if $y_1, y_2, \ldots\ldots, y_n$ be any n *independent* solutions of (B), then

$$c_1y_1+c_2y_2+\ldots\ldots+c_ny_n=u, \text{ say}$$

is the *general* solution of (B) ; $c_1, c_2, \ldots\ldots, c_n$ being n arbitrary constants.

Again, let v be any *particular* solution of (A), so that we have

$$\frac{d^n v}{dx^n}+a_1\frac{d^{n-1}v}{dx^{n-1}}+a_2\frac{d^{n-2}v}{dx^{n-2}}+\ldots\ldots=X.$$

It will be shown that

$$c_1y_1+c_2y_2+\ldots\ldots+c_ny_n+v=u+v,$$

is the *general* solution of (A), containing, as it does, n arbitrary constants.

Now we have

$$\frac{d^n(u+v)}{dx^n}+a_1\frac{d^{n-1}(u+v)}{dx^{n-1}}+a_2\frac{d^{n-2}(u+v)}{dx^{n-2}}+\ldots\ldots$$

$$=\frac{d^n u}{dx^n}+a_1\frac{d^{n-1}u}{dx^{n-1}}+a_2\frac{d^{n-2}u}{dx^{n-2}}+\ldots\ldots$$

$$+\frac{d^n v}{dx^n}+a_1\frac{d^{n-1}v}{dx^{n-1}}+a_1\frac{d^{n-2}v}{dx^{n-2}}+\ldots\ldots$$

$$=0+X=X,$$

for, u, is a solution of (B).

Thus we see that $u+v$, is the general solution of (A). It follows that in order to be able to solve the differential equation (A), where

a_1, a_2, etc., are constants and X is a function of x, *we have first to obtain n independent solutions* y_1 y_2, ..., y_n, *of the auxiliary equation* (*B*), *and any solution*, *v*, *not involving any arbitrary constant*, *of* (*A*) *and then*

$$y = c_1y_1 + c_2y_2 + \dots \dots c_ny_n + v,$$

is the general solution of (A).

The part, $c_1y_1 + c_2y_2 + \dots\dots + c_ny_n$ is called the **Complementary Function** (C., F.) and v the **Particular Integral** (P. I.).

Thus it appears that in order to be able to solve the differential equation (A), we have first to learn to solve the equation (B). This is done in § 14·5. As a preparation for this, we introduce the notion of operators in the next section.

The method of finding a particular integral of the equation (A) is given in § 14·7.

Note. It is easy to see that the conclusions arrived at above hold good for general linear differential equations also, *i.e.*, even when the coefficients are functions of x.

14·3. Operators. The part, d/dx of the symbol dy/dx, can be thought of as an *operator* such that, when it operates on y, the result is the derivative of y.

From this point of view, the symbol d/dx is called an *operator* and the function y is called *operand.* We can similarly think of d^2/dx^2, d^3/dx^3, etc., as operators. For the sake of convenience, the operators, d/dx, d^2/dx^2, etc., are denoted by D, D^2, D^3, etc. The index of D indicates the number of times the operation of differentiation is to be carried out. Extending these ideas, we write

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \dots\dots + a_{n-1}\frac{dy}{dx} + a_ny$$
$$= (D^n + a_1D^{n-1} + a_2D^{n-2} + \dots\dots + a_{n-1}D + a_n)\,y = f(D)\,y$$
$$\Rightarrow \quad D^n + a_1D^{n-1} + a_2D^{n-2} + \dots\dots + a_{n-1}D + a_n \quad = f(D)$$

14·31. Product of operators. If $f(D)$ and $\varphi(D)$ be two operators, then $f(D)\varphi(D)$ is also an operator which indicates that the *operator* $f(D)$ *is to operate on the function obtained as a result of the operation of* $\varphi(D)$ *on a given function.*

For example we calculate

$$(2D-3)\,(D^2-4D+5)\sin x$$

Now, $\quad (D^2-4D+5)\sin x$

$$= \frac{d^2\sin x}{dx^2} - 4\frac{d\sin x}{dx} + 5\sin x$$
$$= 4\sin x - 4\cos x$$

Again, $(2D-3)(4\sin x-4\cos x)$

$$= 2\frac{d}{dx}(4\sin x-4\cos x)-3(4\sin x-4\cos x)$$
$$= 20\cos x-4\sin x.$$

14·32. *To prove that*

$$(D-\alpha)(D-\beta)y \equiv (D-\beta)(D-\alpha)y,$$

α, β *being any constants whatsoever,*

$$(D-\beta)y = \frac{dy}{dx}-\beta y.$$

$$\Rightarrow \quad (D-\alpha)(D-\beta)y = (D-\alpha)\left(\frac{dy}{dx}-\beta y\right)$$
$$= \frac{d}{dx}\left(\frac{dy}{dx}-\beta y\right)-\alpha\left(\frac{dy}{dx}-\beta y\right)$$
$$= \frac{d^2y}{dx^2}-\beta\frac{dy}{dx}-\left(\alpha\frac{dy}{dx}-\alpha\beta y\right)$$
$$= \frac{d^2y}{dx^2}-(\beta+\alpha)\frac{dy}{dx}+\alpha\beta y$$
$$= [D^2-(\alpha+\beta)D+\alpha\beta]\,y.$$

We can similarly show that

$$(D-\beta)(D-\alpha)y = [D^2-(\alpha+\beta)D+\alpha\beta]\,y.$$
$$\therefore \quad (D-\alpha)(D-\beta)y = (D-\beta)(D-\alpha)y,$$

so that the *order* of the operational factors is immaterial.

Cor. We have seen that

$$(D-\alpha)(D-\beta)y = [D^2-(\alpha+\beta)D+\alpha\beta]\,y$$

so that the operators

$$(D-\alpha)(D-\beta) \text{ and } D^2-(\alpha+\beta)D+\alpha\beta$$

are the same.

If D were a number, then by the ordinary laws of multiplication of numbers, we see that

$$(D-\alpha)(D-\beta) = D^2-(\beta+\alpha)D+\alpha\beta.$$

In general, let

$$f(D) = D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots\ldots+a_{n-1}D+a_n.$$

Regarding D as a number we may factorize $f(D)$ so that, from this point of view, we may have

$$f(D) \equiv (D-m_1)(D-m_2)\ldots\ldots(D-m_n) \qquad \ldots(1)$$

The result above now shows that the equality (1) remains valid when D is given its operational character and also that, while carrying out the operations on the R.H.S. of (1), we may interchange the order of the factors at will.

14·5. *To solve the differential equation*

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\dots+a_{n-1}\frac{dy}{dx}+a_n y=0.$$

Writing the equation in the symbolic form, we have

$$f(D)y=(D^n+a_1D^{n-1}+a_2D^{n-2}+\dots\dots+a_{n-1}D+a_n)y=0$$

Suppose α_1 is a *non-repeated* root of the equation

$$f(D)\equiv D^n+a_1D^{n-1}+a_2D^{n-2}+\dots\dots+a_{n-1}D+a_n=0. \quad \dots(1)$$

Regarding D as a number, we may write

$$f(D)=\varphi(D)\,(D-\alpha_1),$$

where $\varphi(D)$ is a polynomial in D. We have

$$0=f(D)y=\varphi(D)\,(D-\alpha_1)y. \quad \dots(2)$$

The solution of the equation

$$(D-\alpha_1)y=0 \quad \dots(3)$$

will clearly be a solution of the given equation (2). Now, (3), gives

$$\frac{dy}{dx}-\alpha_1 y=0 \Rightarrow \frac{dy}{y}-\alpha_1 dx=0.$$

$$\Rightarrow \quad \log y=\alpha_1 x \Rightarrow y=e^{\alpha_1 x}$$

where we have omitted the arbitrary constant.

Thus $e^{\alpha_1 x}$, is a solution of the given differential equation (2).

Now if $\alpha_1, \alpha_2, \alpha_3, \dots\dots, \alpha_n$ are different roots of (1), then, as proved above,

$$e^{\alpha_1 x}, e^{\alpha_2 x}, e^{\alpha_3 x}, \dots\dots, e^{\alpha_n x}$$

will be n, different independent solutions of the given equation. Thus, as shown in §14·2,

$$c_1e^{\alpha_1 x}+c_2e^{\alpha_2 x}+\dots\dots+c_ne^{\alpha_n x}$$

is the general solution of the given equation ; $c_1, c_2, \dots\dots, c_n$ being n arbitrary constants.

Case of repeated Roots. In case some roots are equal, then, proceeding as above, we will obtain less than, n, independent solutions and the solution obtained above will not be general. In this case, we proceed as follows :—

Firstly, we suppose that α is a root repeated twice only so that we may write

$$f(D)=\psi(D)\,(D-\alpha)^2$$

$$\Rightarrow \qquad 0 = f(D)y = \psi(D)\,(D-\alpha)^2 y.$$

The solution of

$$(D-\alpha)^2 y = 0. \qquad \ldots(i)$$

will clearly be a solution of

$$f(D)y = 0.$$

We have

$$(D-\alpha)(D-\alpha)y = 0.$$

Let $\qquad (D-\alpha)y = z. \qquad \ldots (ii)$

so that

$$(D-\alpha)z = 0.$$

$$\Rightarrow \qquad \frac{dz}{dx} - \alpha z = 0.$$

$$\Rightarrow \qquad \frac{dz}{z} - \alpha dx = 0 \Rightarrow z = c_1 e^{\alpha x}$$

taking the general solution with, c_1 as an arbitrary constant. Substituting this value of z in (i), we obtain

$$(D-\alpha)y = c_1 e^{\alpha x} \Rightarrow \frac{dy}{dx} - \alpha y = c_1 e^{\alpha x}$$

Multiplying this linear equation with the integrating factor $e^{-\alpha x}$, and integrating we get

$$ye^{-\alpha x} = c_1 \int e^{-\alpha x}\, e^{\alpha x}\, dx + c_2 = c_1 x + c_2$$

$$\Rightarrow \qquad y = (c_1 x + c_2) e^{\alpha x},$$

which is the general solution of (i) in this case ; c_1, c_2 being two arbitrary constants.

Thus we get a solution containing two arbitrary constants corresponding to a root repeated twice.

In general, we may similarly show that corresponding to a root, α, repeated r times, the solution is

$$(c_1 x^{r-1} + c_2 x^{r-2} + \ldots\ldots + c_{r-1} x + c_r)\, e^{\alpha x}$$

containing r, arbitrary constants.

Note. If the coefficients in $f(D)$ be real then the roots of the equation $f(D) = 0$ will be either real or conjugate imaginary in pairs. In case the roots are imaginary ; the solution of the equation

$$f(D) = 0,$$

be real, then the roots will be either real or conjugate imaginary in pairs. In case the roots are imaginary, the solution of the equation

$$f(D)\, y = 0.$$

can be exhibited in a form free from imaginaries by making use of Euler's Theorem which states that

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Let $\alpha+i\beta$, $\alpha-i\beta$ be two conjugate imaginary roots of $f(D) = 0$. Then the corresponding part of the solution is

$$\begin{aligned} & c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}\left[c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right] \\ &= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right]. \\ &= e^{\alpha x}\left[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x\right] \\ &= e^{\alpha x}(p_1\cos\beta x + p_1\sin\beta x), \end{aligned}$$

where we have written

$$c_1 + c_2 = p_1,\ i(c_1 - c_2) = p_2.$$

It can similarly be shown that if $\alpha+i\beta$, $\alpha-i\beta$ are conjugate imaginary roots, each repeated, r times, then the corresponding part of the solution is

$$e^{\alpha x}[(p_1 + p_2 x + \dots + p_{r-1}x^{r-1})\cos\beta x + (q_1 + q_2 x + \dots + q_{r-1}x^{r-1})\sin\beta x].$$

Examples

1. *Solve*

(*i*) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$, (*ii*) $4\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = 0$,

(*iii*) $2\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 4y = 0$.

(*i*) We have $(D^2 - 5D + 6)\, y = 0$.

Since the roots of $D^2 - 5D + 6 = 0$ are 2 and 3, the general solution is

$$y = c_1 e^{2x} + c_2 e^{3x}.$$

(*ii*) We have $(4D^2 + 4D + 1)\, y = 0$.

Since both the roots of $4D^2 + 4D + 1 = 0$ are equal to $-\frac{1}{2}$, the general solution is

$$y = (c_1 x + c_2)\, e^{-x/2}.$$

(*iii*) We have $(2D^2 + 3D + 4)\, y = 0$.

Since the two roots of $2D^2 + 3D + 4 = 0$ are $\frac{1}{4}[-3 \pm i\sqrt{23}]$

i.e., $$-\frac{3}{4} \pm i\,\frac{\sqrt{23}}{4},$$

therefore, the general solution is

$$e^{-3x/4}\left[c_1 \cos\frac{\sqrt{23}}{4}x + c_2 \sin\frac{\sqrt{23}}{4}x\right],$$

where c_1, c_2 are arbitrary constants.

2. *Solve the differential equation*

$$\frac{d^4y}{dx^4}-2\,\frac{d^3y}{dx^3}+5\,\frac{d^2y}{dx^2}-8\,\frac{dy}{dx}+4y=0.$$

We have

$$(D^4-2D^3+5D^2-8D+4)\,y=0.$$

Now

$$D^4-2D^3+5D^2-8D+4=(D-1)^2\,(D^2+4),$$

so that the roots of

$$D^4-2D^3+5D^2-8D+4=0,$$

are $\qquad 1,\ 1,\ 2i,\ -2i\ ;$

the root, 1, being repeated twice. Therefore,

$$y=(c_1+c_2x)\,e^x+(c_3\cos 2x+c_4\sin 2x).$$

is the solution.

3. *Solve*

$$(D^3+1)\,y=0.$$

We have

$$D^3+1=(D+1)\,(D^2-D+1).$$

Now roots of $D^3+1=0$ are

$$-1,\frac{1}{2}\pm\frac{\sqrt{3}i}{2}.$$

$$\therefore\quad y=ce^{-x}+e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}\,x\right)$$

is the required solution.

Exercise

Solve the following differential equations :

1. $\dfrac{d^2y}{dx^2}-3\,\dfrac{dy}{dx}+2y=0.$ 2. $(D^2+D+1)y=0.$

3. $(9D^2+12D\quad 4)y=0.$ 4. $(D^2+7D+10)y=0.$

5. $\dfrac{d^2y}{dx^2}+a^2y=0.$ 6. $\dfrac{d^2y}{dx^2}+4\,\dfrac{dy}{dx}+13y=0.$

7. $(8D^2-6D-5)y=0.$ 8. $16\dfrac{d^2y}{dx^2}+24\,\dfrac{dy}{dx}+9y=0.$

9. $\dfrac{d^4y}{dx^4}-y=0.$ 10. $(D^3+3D^2+3D+1)y=0.$

11. Find the value of y which satisfies the equation

$$\frac{d^2y}{dx^2}+4\,\frac{dy}{dx}-12y=0,$$

given that $y = 0$ and $dy/dx = 1$, when $x = 0$.

12. Solve

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0.$$

given that $y = 1$ and $dy/dx = 2$, when $x = 0$,

13. Solve

$$\frac{d^2x}{dt^2} + \mu x = 0, \mu > 0,$$

given that $x=a$ and $dx/dt = 0$, when $t=\pi/\sqrt{\mu}$.

Answers

1. $y = ae^{x}+be^{2x}$. **2.** $y=e^{-\frac{1}{2}x}\left(a\cos\frac{\sqrt{3}}{2}x+b\sin\frac{\sqrt{3}}{2}x\right)$

3. $y = (a+bx)e^{-2x/3}$. **4.** $y = ae^{-5x}+be^{-2x}$

5. $y = c_1\cos ax + c_2 \sin ax$. **6.** $y = (a\cos 3x+b\sin 3x)e^{-2x}$.

7. $y = ae^{-\frac{1}{2}x}+be^{3x/4}$. **8.** $y = (a+bx)\, e^{-3x/4}$

9. $y = ae^{x}+be^{-x}+c\cos x+ d\sin x$.

10. $y = (a+bx+cx^2)e^{-x}$. **11.** $y = -\frac{1}{8}{}^{-6x}+\frac{1}{8}e^{2x}$.

12. $y = e^{2x}\cos x$. **13.** $x = -a\cos\sqrt{\mu}t$.

14.6. Inverse operators. The operator

$$\frac{1}{D-\alpha}\text{; } \alpha \text{ being a constant}$$

Def. *If X is a function of x, then*

$$\frac{1}{D-\alpha}X$$

stands for the function such that when $(D-\alpha)$ operates on it the result is the function X.

Thus, $\dfrac{1}{D}X$ stands for the integral of X.

To evaluate

$$\frac{1}{D-\alpha}X.$$

We write

$$y=\frac{1}{D-\alpha}X.$$

By definition, y is a function such that when $(D-\alpha)$ operates on it, the result is X, *i.e.*,

$$(D-\alpha)\,y=X \Leftrightarrow \frac{dy}{dx}-\alpha y=X$$

which is a linear differential equation.

Multiplying by the integrating factor, $e^{-\alpha x}$ we see that the solution is

$$ye^{-\alpha x}=\int Xe^{-\alpha x}dx+c,$$

so that

$$y=e^{\alpha x}\{\int Xe^{-\alpha x}dx+c\}.$$

$$\therefore \qquad y=\frac{1}{D-\alpha}X=e^{\alpha x}\left\{\int Xe^{-\alpha x}dx+c\right\}$$

If any *one* value of $\frac{1}{D-\alpha}X$ be required, we omit the constant c, and write

$$\frac{1}{D-\alpha}X=e^{\alpha x}\int Xe^{-\alpha x}\,dx.$$

The symbol, $\frac{1}{(\mathbf{D}-\beta)(\mathbf{D}-\alpha)}X$, *means that* $\frac{1}{\mathbf{D}-\beta}$ *is to operate on* $\frac{1}{\mathbf{D}-\alpha}X$.

Let $\qquad y=\frac{1}{(D-\beta)(D-\alpha)}X.$

Then, by definition,

$$(D-\beta)y=\frac{1}{D-\alpha}X.$$

Again, by definition

$$(D-\alpha)(D-\beta)y=X.$$

Thus the symbol, $\frac{1}{(D-\beta)(D-\alpha)}X$, denotes the function y which satisfies the differential equation

$$(D-\alpha)(D-\beta)y=X.$$

Examples

Evaluate the following, obtaining only particular values :—

(*i*) $\dfrac{1}{D}x^3.$ (*ii*) $\dfrac{1}{D^2}x^4.$

(*iii*) $\dfrac{1}{D-2}\sin x.$ (*iv*) $\dfrac{1}{(D-2)(D-3)}e^{2x}.$

(*i*) $\dfrac{1}{D}x^3 = \displaystyle\int x^3 dx = \dfrac{x^4}{4.}$

(*ii*) $\dfrac{1}{D^2}x^4 = \dfrac{1}{D}\cdot\dfrac{1}{D}x^4 = \displaystyle\int\left[\int x^4 dx\right]dx = \int\dfrac{x^5}{5}dx = \dfrac{x^6}{30}$

(*iii*) $\dfrac{1}{D-2}\sin x = e^{2x}\displaystyle\int e^{-2x}\sin x\,dx$

$$= e^{2x}\,\frac{e^{-2x}}{\sqrt{5}}\,\sin\left(x+\tan^{-1}\frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{5}}\,\sin\left(x+\tan^{-1}\frac{1}{2}\right)$$

(*iv*) $\dfrac{1}{D-3}e^{2x} = e^{3x}\int e^{-3x}.e^{2x}dx = -e^{2x}.$

$$\frac{1}{D-2}\cdot\left(\frac{1}{D-3}e^{2x}\right) = \frac{1}{D-2}(-e^{2x})$$

$$= e^{2x}\int e^{-2x}(-e^{2x})\,dx = e^{2x}\int -1\,.\,dx = -x\,e^{2x}$$

$$\frac{1}{(D-2)(D-3)}e^{2x} = -xe^{2x}.$$

Exercises

Evaluate the following obtaining only particular values :

(*i*) $\dfrac{1}{D^2}e^{x},$ (*ii*) $\dfrac{1}{(D+1)(D-1)}\cos x,$

(*iii*) $\dfrac{1}{D+2}(x+e^{x}),$ (*iv*) $\dfrac{1}{(D+1)(D-1)}e^{x},$

(*v*) $\dfrac{1}{D+2}x^2,$ (*vi*) $\dfrac{1}{(D+3)(D-3)}(\sin 2x+2e^{3x}).$

Answers

(*i*) $e^{-x}.$ (*ii*) $-\frac{1}{2}\cos x.$ (*iii*) $-\frac{1}{4}x-1/16-\frac{1}{3}e^{x}.$

(*iv*) $\frac{1}{2}x^2e^{x}.$ (*v*) $\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{4}.$ (*vi*) $-\frac{1}{13}\sin 2x+\frac{1}{3}xe^{3x}$

14·7. To determine the particular integral of

$$\mathbf{f(D)\,y = X.} \qquad \text{...(1)}$$

By definition

$$\frac{1}{f(D)}X \qquad \text{...(2)}$$

means a function such that when $f(D)$ operates on it, the result is the function X, so that the symbol (2) stands for a solution of the differential equation (1).

Let

$$f(D) = (D-m_1)(D-m_2)\ldots(D-m_n).$$

$$\therefore \text{ P.I.} = \frac{1}{f(D)}X = \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_n)}X$$

$$= \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_{n-1})}\, e^{m_n x}\int Xe^{-m_n x}\,dx.$$

We write

$$X' = e^{m_n x}\int Xe^{-m_n x}\,dx.$$

$$\therefore \text{ P.I.} = \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_{n-1})}X'$$

$$= \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_{n-1})}\, e^{m_{n-1}x}\int X'e^{-m_{n-1}\cdot x}\,dx.$$

Proceeding in this manner we will, after a finite number of steps, obtain the particular integral as required.

Another method. We have, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_n)}$$

$$= \frac{A_1}{D-m_1}+\frac{A_2}{D-m_2}+\ldots+\frac{A_n}{D-m_n}$$

$$\therefore \text{ P.I.} = \frac{1}{f(D)}X = \left[\frac{A_1}{D-m_1}+\frac{A_2}{D-m_2}+\ldots+\frac{A_n}{D-m_n}\right]X$$

$$= A_1 e^{m_1 x}\int Xe^{-m_1 x}\,dx + A_2 e^{m_2 x}\int Xe^{-m_2 x}\,dx+\ldots$$

$$+\ldots+A_n e^{-m_n x}\int Xe^{-m_n x}\,dx.$$

Note 1. The method given here is a general one which can be employed to obtain a particular integral in any given case. Shorter methods depending upon the form of the function X will be given in the following sections.

Note 2. As we require only a *particular integral*, we may not introduce any arbitrary constant in carrying out the integrations in the method explained above.

Examples

1. *Solve*

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{2x}$$

The equation may be re-written as

$$(D^2-3D+2)\,y=e^{2x}.$$

Since, the roots of, $D^2-3D+2=0$, are 1, 2, the complementary function, (C.F.), is $c_1e^x+c_2e^{2x}$.

To find the particular integral, (P.I.), we have to evaluate

$$\frac{1}{(D-2)\,(D-1)}\,e^{2x}.$$

Now $$\frac{1}{(D-1)}\,e^{2x}=e^x\int e^{-x}\,.\,e^{2x}\,dx=e^{2x}.$$

Again $$\frac{1}{D-2}\,\frac{1}{D-1}\,e^{2x}=\frac{1}{D-2}\,e^{2x}$$
$$=e^{2x}\int e^{-2x}\,.\,e^{2x}\,dx=xe^{2x}$$

$$\therefore\qquad y=c_1e^x+c_2e^{2x}+xe^{2x},$$

is the general solution.

2. *Solve the differential equation*

$$\frac{d^2y}{dx^2}+9y=\sec 3\,x.$$

We have the equation

$$(D^2+9)\,y=\sec 3x.$$

Since the roots of $D^2+9=0$, are $\pm\,3i$, we have

$$\text{C.F.}=a_1\cos 3x+a_2\sin 3x.$$

Again $$\text{P. I.}=\frac{1}{D^2+9}\sec 3x$$
$$=\frac{1}{(D+3i)\,(D-3i)}\sec 3x,$$

$$= \frac{1}{6i}\left(\frac{1}{D-3i} - \frac{1}{D+3i}\right) \sec 3x$$

$$= \frac{1}{6i}\left[\frac{1}{D-3i}\sec 3x - \frac{1}{D+3i} \sec 3x\right]$$

$$= \frac{1}{6i}\left[e^{3ix}\int e^{-3ix} \sec 3x\, dx - e^{-3ix}\int e^{3ix} \sec 3x\, dx\right]$$

Now $\int e^{-3ix} \sec 3x\, dx = \int(\cos 3x - i \sin 3x) \sec 3x\, dx$

$$= \int (1 - i \tan 3x)\, dx = x + \frac{i}{3} \log \cos 3x.$$

Also $\int e^{3ix} \sec 3x\, dx = \int(\cos 3x + i \sin 3x) \sec 3x\, dx$

$$= \int(1+i \tan 3x)\, dx = x - \frac{i}{3} \log \cos 3x$$

$$\therefore \quad \text{P.I.} = \frac{1}{6i}\left[e^{3ix}\left(x + \frac{i}{3} \log \cos 3x\right) - e^{-3ix}\left(x - \frac{i}{3} \log \cos 3x\right)\right]$$

$$= \frac{1}{6i}\left[\left(\cos 3x + i \sin 3x\right)\left(x + \frac{i}{3} \log \cos 3x\right) - \left(\cos 3x - i \sin 3x\right)\left(x - \frac{i}{3} \log \cos 3x\right)\right]$$

$$= \frac{1}{6i}\left[2ix \sin 3x + \frac{2i}{3} \cos 3x \log \cos 3x\right]$$

$$= \frac{x}{3} \sin 3x + \frac{\cos 3x}{9} \log \cos 3x$$

$$\therefore \quad y = a_1 \cos 3x + a_2 \sin 3x + \frac{x}{3} \sin 3x + \frac{\cos 3x}{9} \log \cos 3x.$$

Exercises

Solve the following equations :

1. $9 \dfrac{d^2y}{dx^2} - y = e^{-x}$.
2. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 2x$.
3. $(D^2 - 2D + 1)\, y = x^2 - 1$.
4. $(6D^2 - D - 2)\, y = xe^{-x}$.
5. $(D^2 + 4)\, y = \tan 2x$.
6. $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$.

Answers

1. $y = ae^{-x/3} + be^{x/3} + \frac{1}{3}e^{-x}$.
2. $y = ae^{-2x} + be^{x} - (x + \frac{1}{2})$.
3. $y = (a + bx)e^{x} + (x^2 + 4x + 5)$.
4. $y = ae^{2x/3} + be^{-x/3} + \frac{1}{5}(12/5 + x)e^{-x}$.
5. $y = a \cos 2x + b \sin 2x - \frac{1}{4} \cos 2x \log \tan (x + \frac{1}{4}\pi)$.
6. $y = a_1 \cos ax + a_2 \sin ax + \dfrac{1}{a} x \sin ax + \dfrac{1}{a^2} \cos ax \log \cos ax$.

14·81. Rule for finding the particular integral when X is of the form e^{mx}.

We have to find a particular value of y which satisfies the differential equation

$$f(D)\, y = e^{mx},$$

i.e., we have to evaluate the symbol

$$\frac{1}{f(D)} e^{mx},$$

where

$$f(D) \equiv D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n.$$

We know that

$$D^r e^{mx} = m^r e^{mx}.$$

$$\begin{aligned} f(D)e^{mx} &= (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n) e^{mx} \\ &= D^n e^{mx} + a_1 D^{n-1} e^{mx} + a_2 D^{n-2} e^{mx} + \ldots\ldots \\ &\qquad + a_{n-1} D e^{mx} + a_n e^{mx} \\ &= m^n e^{mx} + a_1 m^{n-1} e^{mx} + a_2 m^{n-2} e^{mx} + \ldots\ldots \\ &\qquad + a_{n-1} m e^{mx} + a_n e^{mx} \\ &= (m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_{n-1} m + a_n)\, e^{mx} \\ &= f(m)\, e^{mx}. \end{aligned}$$

Thus we see that

$$f(D)e^{mx} = f(m)e^{mx}.$$

Let

$$f(m) \neq 0.$$

Operating on both sides with $1/f(D)$, we have

$$e^{mx} = \frac{1}{f(D)} f(m)e^{mx} = f(m) \frac{1}{f(D)} e^{mx}$$

Dividing by $f(m) \neq 0$, we obtain

$$\frac{1}{f(m)} e^{mx} = \frac{1}{f(D)} e^{mx}$$

Let $f(m) = 0$, so that m is a root of the equation $f(D) = 0$.

Suppose that m is a root repeated r-times so that $(D-m)^r$ is a factor of $f(D)$.

Let $$f(D) = (D-m)^r \varphi(D),$$
where $$\varphi(m) \neq 0.$$

We have

$$\frac{1}{f(D)} e^{mx} = \frac{1}{(D-m)^r \varphi(D)} e^{mx} = \frac{1}{(D-m)^r} \cdot \frac{1}{\varphi(D)} e^{mx}.$$

As proved in the first part, we have

$$\frac{1}{\varphi(D)} e^{mx} = \frac{1}{\varphi(m)} e^{mx}, \text{ for } \varphi(m) \neq 0.$$

$$\therefore \quad \frac{1}{f(D)} e^{mx} = \frac{1}{(D-m)^r} \frac{1}{\varphi(m)} e^{mx} = \frac{1}{\varphi(m)} \frac{1}{(D-m)^r} e^{mx}.$$

Now $$\frac{1}{D-m} e^{mx} = e^{mx} \int e^{-mx} . e^{mx}\, dx = xe^{mx}$$

$$\frac{1}{(D-m)^2} e^{mx} = \frac{1}{D-m} \frac{1}{D-m} e^{mx}$$
$$= \frac{1}{D-m} xe^{mx}$$
$$= e^{mx} \int e^{-mx} x\, e^{mx}\, dx = \frac{x^2}{2} e^{mx}.$$

$$\frac{1}{(D-m)^3} e^{mx} = \frac{1}{D-m} \frac{1}{(D-m)^2} e^{mx}$$
$$= \frac{1}{D-m} \frac{x^2}{2} e^{mx}$$
$$= e^{mx} \int e^{-mx} . \frac{x^2}{2}\, dx = \frac{x^3}{3\,!} e^{mx}.$$

Continuing in this manner, we may show that

$$\frac{1}{(D-m)^r} e^{mx} = \frac{x^r}{r\,!} e^{mx}.$$

$$\therefore \frac{1}{f(D)} e^{mx} = \frac{1}{\varphi(m)} \frac{x^r}{r\,!} e^{mx},$$

where
$$f(D) = (D-m)^r \varphi(D),$$
and m is *not* a root of $\varphi(D) = 0$.

Examples

Solve :

1. $4\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}-3y=e^{2x}$. 2. $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+3y=e^{-3x}$.

3. $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+9y=2e^{-3x}$.

1. We have

$$(4D^2+4D-3)y=e^{2x}.$$

The roots of $4D^2+4D-3=0$ are $\frac{1}{2}$ and $-\frac{3}{2}$.

$\therefore$ C.F. $=c_1e^{x/2}+c_2e^{-3x/2}$.

$$\text{P.I.}=\frac{1}{4D^2+4D-3}\cdot e^{2x}=\frac{1}{4(2)^2+4(2)-3}e^{2x}=\frac{e^{2x}}{21}$$

$\therefore$ the solution is

$$y=c_1e^{x/2}+c_2e^{-3x/2}+\tfrac{1}{21}e^{2x}.$$

2. We have

$$(D^2+4D+3)y=e^{-3x}.$$

The roots of

$$D^2+4D+3=0 \text{ are } -1,\ -3$$

$$\therefore \qquad \text{C.F.}=c_1e^{-x}+c_2e^{-3x}$$

$$\text{P.I.}=\frac{1}{D^2+4D+3}e^{-3x}.$$

Now D^2+4D+3 becomes 0 for $D=-3$. Therefore, we write

$$\text{P.I.}=\frac{1}{(D+3)(D+1)}e^{-3x}=\frac{1}{D+3}\cdot\frac{1}{(-3+1)}e^{-3x}$$

$$=\frac{1}{D+3}\cdot\frac{1}{-2}e^{-3x}=-\frac{x}{2}e^{-3x}.$$

$$y=c_1e^{-x}+c_2e^{-3x}-\tfrac{1}{2}xe^{-3x},$$

is the general solution.

3. We have

$$(D^2+6D+9)y=2e^{-3x}.$$

The roots of $D^2+6D+9=0$ are both equal to -3.

$$\therefore \qquad \text{C.F.}=(c_1x+c_2)e^{-3x}.$$

$$\text{P.I.}=\frac{1}{D^2+6D+9}\cdot 2e^{-3x}$$

$$=\frac{1}{(D+3)^2}2e^{-3x}$$

$$=2.\frac{1}{(D+3)^2}e^{-3x}=2.\tfrac{1}{2}x^2e^{-3x}=x^2e^{-3x}$$

$$\therefore \qquad y = (c_1 x + c_2)\, e^{3x} + x^2\, e^{-3x},$$

is the general solution.

Exercises

Solve the following differential equations :

1. $6\dfrac{d^2y}{dx^2} + 17\dfrac{dy}{dx} + 12y = e^{-x}$. **2.** $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y = e^{2x} + e^x$.

3. $3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - y = e^{x/2} + 2e^{3x}$. **4.** $(D^2 - 4)y = 3e^{2x} - 4e^{-2x}$

5. $\dfrac{d^2y}{dx^2} - (a+b)\dfrac{dy}{dx} + aby = e^{ax} + e^{bx}$.

6. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{x/2} + e^{-x/2}$.

7. $9\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 4y = e^{-2x/3}$.

8. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = e^{2x}$.

9. $(3D^2 - 4D + 5)y = e^x - 2e^{2x} + 3e^{3x}$.

10. $(D-3)^2\, y = 2e^{3x} + e^{-x}$.

11. $(D^2 - 2D + 1)y = (1 + e^{-x})^2$

12. Solve $(D^2 + 5D + 6)y = e^{2x}$, given that $y = 0$, $dy/dx = 0$, when $x = 0$.

13. Solve $(D^2 - 7D + 6)y = 2e^{3x}$, given that $y = 1$, $dy/dx = 0$, when $x = 0$.

Answers

1. $y = ae^{-3x/2} + be^{-4x/3} + e^{-x}$.

2. $y = \left(a \cos \dfrac{\sqrt{3}}{2} x + b \sin \dfrac{\sqrt{3}}{2} x\right)e^{x/2} + e^x + \dfrac{e^{2x}}{4}$.

3. $y = ae^{x/3} + be^{-x} + \frac{4}{3}e^{x/2} + \frac{1}{16}e^{3x}$.

4. $y = ae^{2x} + be^{-2x} + \frac{3}{4}xe^{2x} + xe^{-2x}$.

5. $y = c_1 e^{ax} + c_2 e^{bx} + \dfrac{x}{a-b}(e^{ax} - e^{bx})$.

6. $y = (a+bx)e^{-x} + e^{x/2} + 4e^{-x/2}$.

7. $y = (a+bx)e^{-2x/3} + \frac{1}{18}x^2 e^{-2x/3}$.

8. $y = ae^x + be^{2x} + ce^{-2x} + \frac{1}{4}xe^{2x}$.

9. $y = e^{2x/3}\left(a \cos \dfrac{\sqrt{11}}{3} x + b \sin \dfrac{\sqrt{11}}{3} x\right) + \dfrac{1}{4}e^x - \dfrac{2}{9}e^{2x} + \dfrac{3}{20}e^{3x}$.

10. $y = (a+bx)e^{3x}+\frac{1}{16}e^{-x}+x^2e^{3x}$.

11. $y = (a+bx)e^{x}+1+\frac{1}{9}e^{-2x}+\frac{1}{4}e^{-x}$.

12. $y = \frac{3}{5}e^{-3x}-\frac{1}{4}e^{-2x}+\frac{1}{10}e^{2x}$.

13. $y = \frac{1}{8}e^{x}-\frac{1}{12}e^{5x}-\frac{1}{2}e^{3x}$.

14·82. To determine the particular integral, when X is of the form

sin mx or cos mx

Here we have to find a particular value of, y, which satisfies the differential equation

$$f(D)\, y = \sin mx,$$

so that we have to evaluate the symbol

$$\frac{1}{f(D)} \sin mx.$$

First we consider the case when $f(D)$ contains even powers of D only so that it is a function of D^2. Let

$$\begin{aligned} f(D) &= \phi(D^2) \\ &= (D^2)^n + a_1(D^2)^{n-1} + a_2(D^2)^{n-2} + \ldots a_{n-1} D^2 + a_n. \end{aligned}$$

We have
$$\begin{aligned} D \sin mx &= m \cos mx, \\ D^2 \sin mx &= (-m^2) \sin mx, \\ D^4 \sin mx &= (-m^2)^2 \sin mx, \\ D^6 \sin mx &= (-m^2)^3 \sin mx, \text{ etc.} \end{aligned}$$

With the help of these results, we may easily see that

$$\varphi(D^2) \sin mx = \varphi(-m^2) \sin mx.$$

Let $\varphi(-m^2) \neq 0$. Operating on both sides with $1/\varphi(D^2)$, we have

$$\begin{aligned} \sin mx &= \frac{1}{\varphi(D^2)} \varphi(-m^2) \sin mx, \\ &= \varphi(-m^2) \frac{1}{\varphi(D^2)} \sin mx. \end{aligned}$$

Dividing by $\varphi(-m^2) \neq 0$, we obtain

$$\frac{1}{\varphi(-m^2)} \sin mx = \frac{1}{\varphi(D^2)} \sin mx.$$

Similarly we may show that

$$\frac{1}{\varphi(D^2)} \cos mx = \frac{1}{\varphi(-m^2)} \cos mx, \text{ if } \varphi(-m^2) \neq 0.$$

The method of procedure for the case of failure which arises when $\varphi(-m^2) = 0$ is indicated in the example 2 on page 297.

Suppose now, that $f(D)$ contains odd powers of D also. In this case, breaking up $f(D)$ into its even and odd parts, we may write

$$f(D) = \varphi_1(D^2) + D\varphi_2(D^2).$$

We have

$$\frac{1}{f(D)} \sin mx$$

$$= \frac{1}{\varphi_1(D^2) + D\varphi_2(D^2)} \sin mx$$

$$= [\varphi_1(D^2) - D\varphi_2(D^2)] \frac{1}{[\varphi_1(D^2) + D\varphi_2(D^2)][\varphi_1(D^2) - D\varphi_2(D^2)]} \sin mx$$

$$= [\varphi_1(D^2) - D\varphi_2(D^2)] \frac{1}{[\varphi_1(D^2)]^2 - D^2[\varphi_2(D^2)]^2} \sin mx$$

$$= [\varphi_1(D^2) - D\varphi_2(D^2)] \frac{1}{[\varphi_1(-m^2)]^2 - (-m^2)[\varphi_2(-m^2)]^2} \sin mx$$

We write

$$[\varphi_1(-m^2)] = p, \; [\varphi_2(-m^2)] = q.$$

$$\therefore \quad \frac{1}{f(D)} \sin mx = \frac{1}{p^2 + m^2q^2} [\varphi_1(D^2) - D\varphi_2(D^2)] \sin mx$$

$$= \frac{1}{p^2 + m^2q^2} [\varphi_1(-m^2) - D\varphi_2(-m^2)] \sin mx$$

$$= \frac{1}{p^2 + m^2q^2} [\varphi_1(-m^2) \sin mx - m\,\varphi_2(-m^2) \cos mx]$$

$$= \frac{p \sin mx - mq \cos mx}{p^2 + m^2q^2}.$$

The following *shorter* procedure has its justification in the result obtained above.

We write

$$\frac{1}{f(D)} \sin mx = \frac{1}{\varphi_1(D^2) + D\varphi_2(D^2)} \sin mx$$

$$= \frac{1}{\varphi_1(-m^2) + D\varphi_2(-m^2)} \sin mx$$

$$= \frac{1}{p + Dq} \sin mx$$

$$= (p - Dq) \frac{1}{(p + Dq)(p - Dq)} \sin mx$$

$$= (p-Dq)\frac{1}{p^2-D^2q^2}\sin mx$$

$$= (p-Dq)\frac{1}{p^2+m^2q^2}\sin mx$$

$$= \frac{1}{p^2+m^2q^2}(p-Dq)\sin mx.$$

$$= \frac{1}{p^2+m^2q^2}(p\sin mx-mq\cos mx),$$

which is the same as obtained above.

The case of, $\cos mx$, can be similarly treated.

Examples

1. *Solve*

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=\sin 3x.$$

We have

$$(D^2-5D+6)\,y=\sin 3x.$$

The roots of $D^2-5D+6=0$ are 2, 3 so that, the complementary function is

$$c_1e^{2x}+c_2e^{3x}.$$

Also

$$\text{P. I.}=\frac{1}{D^2-5D+6}\sin 3x.$$

Replacing D^2 by $(-3^2)=-9$, we obtain

$$\text{P. I.}=\frac{1}{-9-5D+6}\sin 3x$$

$$=\frac{1}{-3-5D}\sin 3x$$

$$=-(3-5D)\frac{1}{(3-5D)(3+5D)}\sin 3x$$

$$=-(3-5D)\frac{1}{9-25D^2}\sin 3x$$

$$=-(3-5D)\frac{1}{9-25(-9)}\sin 3x$$

$$=-\frac{1}{234}(3-5D)\sin 3x$$

$$=-\frac{1}{234}(3\sin 3x-15\cos 3x)$$

$$= -\frac{1}{78}(\sin 3x - 5\cos 3x)$$

Thus the required solution is

$$y = c_1e^{2x} + c_2e^{3x} - \frac{1}{78}(\sin 3x - 5\cos 3x).$$

2. *Solve*

$$(D^2+4)y = \cos 2x.$$

The roots of $D^2+4 = 0$ are $2i$, $-2i$ and therefore

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$$

Also $\text{P.I.} = \dfrac{1}{D^2+4}\cos 2x$

Since $D^2 + 4$ becomes zero on replacing D^2 by $-2^2 = -4$, we have here a case of failure of the procedure outlined in § 14·83.

The procedure, now is as follows :—

The P.I. is the real part of

$$\frac{1}{D^2+4} e^{2ix}$$

Now, D^2+4. vanishes for $D = 2i$. Therefore, we write

$$D^2+4 = (D-2i)(D+2i)$$

$$\therefore \quad \frac{1}{D^2+4}e^{2ix} = \frac{1}{(D-2i)(D+2i)}e^{2ix}$$

$$= \frac{1}{4i} \cdot \frac{1}{D-2i}e^{2ix}$$

$$= \frac{1}{4i}\, xe^{2ix}$$

$$= -\tfrac{1}{4}ix(\cos 2x + i \sin 2x)$$

$$= \tfrac{1}{4}x \sin 2x - i.\, \tfrac{1}{4}x \cos 2x$$

$$\therefore \quad y = c_1 \cos 2x + c_2 \sin 2x + \tfrac{1}{4}x \sin 2x.$$

3. *Solve*

$$(D^3+1)\, y = \cos 2x.$$

It can easily be shown that

$$\text{C. F.} = c_1e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2}\right)$$

Also $\quad \text{P. I.} = \dfrac{1}{D^3+1}\cos 2x$

$$= \frac{1}{D.D^2+1}\cos 2x$$

$$= \frac{1}{D(-2^2)+1} \cos 2x$$

$$= \frac{1}{1-4D} \cos 2x$$

$$= (1+4D) \frac{1}{(1+4D)(1-4D)} \cos 2x$$

$$= (1+4D) \quad \frac{1}{1-16D^2} \cos 2x$$

$$= (1+4D) \frac{1}{1-16(-4)} \cos 2x$$

$$= \tfrac{1}{65} (1+4D) \cos 2x = \tfrac{1}{65} (\cos 2x - 8 \sin 2x)$$

$$\therefore \qquad y = c_1 e^{-x} + e^{x/2} \left(c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2} \right)$$
$$+ \tfrac{1}{65} (\cos 2x - 8 \sin 2x).$$

Exercises

Solve the following differential equations :

1. $\dfrac{d^2y}{dx^2} - 3 \dfrac{dy}{dx} + 2y = \cos 2x.$
2. $(D^2+4D-3)y = 2 \sin 3x.$
3. $\dfrac{d^2y}{dx^2} + a^2 y = \sin ax.$ 4. $\dfrac{d^2y}{dx^2} + 4y = \cos 3x + \sin 3x.$
5. $\dfrac{d^2y}{dx^2} + 9y = e^x - \cos 2x.$ 6. $\dfrac{d^2y}{dx^2} - 3 \dfrac{dy}{dx} + y = 2 \sin 3x.$
7. $2 \dfrac{d^2y}{dx^2} - 3 \dfrac{dy}{dx} + 4y = 3 \cos \dfrac{x}{2}.$
8. $(D^2-8D+9)y = 40 \sin 5x.$
9. $(4D^2+16D-9)y = -4e^{x/2} + 3 \sin \frac{1}{4}x.$
10. $\dfrac{d^2y}{dx^2} - 4y = \cos^2 x.$
11. $(D^2+D+1)y = (1+\sin x)^2.$
12. $(D^4-2D^2+1)y = \cos x.$
13. $(D^3+D^2+D+1)y = \sin 2x.$
14. **Solve $\dfrac{d^2y}{dx^2} + 2 \dfrac{dy}{dx} + 10y + 37 \sin 3x = 0$ and find the value of y when $x = \frac{1}{2}\pi$, if it is given that $y = 3$ and $dy/dx = 0$, when $x = 0$.**

Answers

1. $y = ae^{2x}+be^{x}-\frac{1}{20}\cos 2x-\frac{3}{20}\sin 2x.$

2. $y = ae^{(-2+\sqrt{7})x}+be^{(-2-\sqrt{7})x}-\frac{1}{13}(\cos 3x+\sin 3x).$

3. $y = c_1\cos ax+c_2\sin ax-\frac{x\cos ax}{2a}.$

4. $y = a\cos 2x+b\sin 2x-\frac{1}{5}(\cos 3x+\sin 3x).$

5. $y = a\cos 3x+b\sin 3x+\frac{1}{10}e^{x}-\frac{1}{5}\cos 2x.$

6. $y = ae^{\frac{3+\sqrt{5}}{2}x}+be^{\frac{3-\sqrt{5}}{2}x}+\frac{18}{145}\cos 3x-\frac{16}{145}\sin 3x.$

7- $y = \left(a\cos\frac{\sqrt{23}}{4}x+b\sin\frac{\sqrt{23}}{4}x\right)e^{\frac{3}{4}x}$
$+\frac{8}{29}\left(7\cos\frac{x}{2}-3\sin\frac{x}{2}\right).$

8. $y = ae^{(4+\sqrt{7})x}+be^{(4-\sqrt{7})x}+\frac{5}{58}(5\cos 5x-2\sin 5x).$

9. $y = ae^{x/2}+be^{-9x/2}+\frac{1}{5}xe^{x/2}-\frac{48}{1625}\left(4\cos\frac{x}{4}+\frac{37}{4}\sin\frac{x}{4}\right).$

10. $y = ae^{2x}+be^{-2x}-\frac{1}{8}(1+\frac{1}{3}\cos 2x).$

11. $y = \left(a\cos\frac{\sqrt{3}}{2}x+b\sin\frac{\sqrt{3}}{2}x\right)e^{-x/2}+\frac{3}{2}+\frac{1}{26}(3\cos 2x$
$-2\sin 2x)-2\cos x.$

12. $y = (a+bx)e^{x}+(c+dx)e^{-x}+\frac{1}{4}\cos x.$

13. $y = a\cos x+b\sin x+ce^{-x}+\frac{1}{15}(2\cos 2x-\sin 2x).$

14. $y = e^{-x}(a\cos 3x+b\sin 3x) = 6\cos 3x-\sin 3x.$
$y = 1.$

14·83. *To find particular integral when X is of the form x^m, m being a positive integer.*

Here we have to evaluate

$$\frac{1}{f(D)}x^m.$$

The general method is to expand $1/f(D)$ in ascending integral powers of D, regarding D as a number, and to let each term of the expression operate on x^m. Then the sum of the results of these operations is the required value. Since $(m+1)$th and higher derivatives of x^m are zero, it is enough to expand $1/f(D)$ in ascending powers of, D, up to mth power only.

The justification for this expansion will become clear by the following discussion.

As in § 14·7, p. 287 we decompose $1/f(D)$ into partial fractions and obtain

$$\frac{1}{f(D)}=\frac{A_1}{D-\alpha_1}+\frac{A_2}{D-\alpha_2}+\dots\dots+\frac{A_n}{D-\alpha_n}.$$

We expand $1/(D-\alpha_1)$, regarding D as a number and obtain

$$\frac{1}{D-\alpha_1}=-\frac{1}{\alpha_1}\left(1-\frac{D}{\alpha_1}\right)^{-1}$$

$$=-\frac{1}{\alpha_1}\left(1+\frac{D}{\alpha_1}+\frac{D^2}{\alpha_1^2}+\frac{D^3}{\alpha_1^3}+\dots\right)$$

The above equality has now to be justified when D is given its operational character, *i.e.*, we have to prove that

$$\frac{1}{D-\alpha_1}X=-\frac{1}{\alpha_1}\left(1+\frac{D}{\alpha_1}+\frac{D^2}{\alpha_1^2}+\frac{D^3}{\alpha_1^3}+\dots\right)X,$$

where X is any function of x. By the definition of inverse operations, this will be so if

$$(D-\alpha_1)\left[-\frac{1}{\alpha_1}\left(1+\frac{D}{\alpha_1}+\frac{D^2}{\alpha_1^2}+\dots\right)X\right]=X.$$

which is easily seen to be a fact.

Thus we have seen that the partial fractions of $1/f(D)$ can, with justification, be expanded in ascending integral powers of D. Also we know that the direct expansion of $1/f(D)$ in ascending integral powers of D will be same as the sum of the expansions of its partial fractions.

Examples

Solve

$$(D^3+2D^2+4D+8)y=x^2.$$

We have

$$D^3+2D^2+4D+8=D^2(D+2)+4(D+2)=(D+2)(D^2+4),$$

so that the roots of $D^3+2D^2+4D+8=0$ are $-2, 2i, -2i$

$$\therefore \qquad \text{C.F.}=c_1e^{2x}+c_2\cos 2x+c_3\sin 2x.$$

$$\text{Again P.I.}=\frac{1}{D^3+2D^2+4D+8}x^2$$

$$=\frac{1}{8}\left[1+\frac{4D+2D^2+D^3}{8}\right]^{-1}x^2$$

$$=\frac{1}{8}\left[1-\frac{4D+2D^2+D^3}{8}+\left(\frac{4D+2D^2+D^3}{8}\right)^2+\dots\right]x^2$$

$$=\tfrac{1}{8}[1-\tfrac{1}{2}D+0D^2+\dots]\,x^2=\tfrac{1}{8}[x^2-x].$$

$$y=c_1e^{-2x}+c_2\cos 2x+c_3\sin 2x+\tfrac{1}{8}\,(x^2-x).$$

Exercises

Solve

1. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=2x+x^2$

2. $\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}=1+x^2.$

3. $\dfrac{d^2y}{dx^2}-4y=x^2.$ 4. $(D^4+D^3+D^2-D-2)=x^2.$

Answers

1. $y=(a+bx)e^{-x}+x^2-2x+2.$
2. $y=a+be^{3x}+ce^{-2x}-\frac{1}{6}\left(\frac{25}{18}x+\frac{1}{6}x^2+\frac{1}{3}x^3\right).$
3. $y=ae^{2x}+be^{-2x}-\frac{1}{4}\left(x^2+\frac{1}{2}\right).$
4. $y=ae^{-x}+be^{x}+e^{-\frac{1}{2}x}\left(c\cos\frac{\sqrt{7}}{2}x+d\sin\frac{\sqrt{7}}{2}x\right)-\left(x^2-x+\frac{1}{2}\right).$

14·9. The following two sections will give us formulae which will be helpful in finding particular integrals of the differential equations of the form

$$f(D)\,y=e^{ax}\,V,\ f(D)\,y=xV,$$

where V is a function of x.

14·91. *To show that*

$$\frac{1}{f(D)}e^{ax}\,V=e^{ax}\frac{1}{f(D+a)}\,V,$$

where V is a function of x.

We consider any function V_1 of x to be later on determined in terms of the given function V.

We have

$$D\,(e^{ax}\,V_1)=e^{ax}\,DV_1+ae^{ax}\,V_1=e^{ax}\,(D+a)\,V_1.$$

In general, by the Leibnitz theorem of Differential Calculus,

$$\begin{aligned}D^r\,(e^{ax}\,V_1)&=e^{ax}\,D^rV_1+{}^rC_1\,ae^{ax}\,D^{r-1}\,V_1+{}^rC_2\,a^2\,e^{ax}\,D^{r-2}\,V_1\\&\qquad+\ldots\ldots+\ldots\ldots+a^r\,e^{ax}\,V_1\\&=e^{ax}\,(D^r+{}^rC_1aD^{r-1}+{}^rC_2\,a^2D^{r-2}+\ldots\ldots+a^r)\,V_1\\&=e^{ax}\,(D+a)^r\,V_1.\end{aligned}$$

$$\therefore f(D)\,e^{ax}V_1=(D^n+a_1\,D^{n-1}+a_2\,D^{n-2}+a_3\,D^{n-3}+\ldots\ldots$$

$$+a_r D^{n-r}+\ldots\ldots+a_{n-1} D+a_n)\, e^{ax} V_1$$

$$= e^{ax}\,[(D+a)^n+a_1 (D+a)^{n-1}+\ldots\ldots$$

$$+a_r (D+a)^{n-r}+\ldots\ldots+a_n]\, V_1$$

$$= e^{ax} f(D+a)\, V_1.$$

$$\therefore\ f(D) e^{ax} V_1 = e^{ax} f(D+a)\, V_1. \qquad \ldots(1)$$

Suppose, now that V_1 is given by

$$f(D+a)\, V_1 = V \Rightarrow V_1 = \frac{1}{f(D+a)} V. \qquad \ldots(2)$$

From (1) and (2),

$$f(D)\, e^{ax} \frac{1}{f(D+a)} V = e^{ax} V.$$

Operating by $\dfrac{1}{f(D)}$ on both sides, we obtain

$$e^{ax} \frac{1}{f(D+a)} V = \frac{1}{f(D)} e^{ax} V.$$

Thus we have the given result.

Example

Solve

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+4y = e^x \cos x.$$

We have

$$(D^2-2D+4)\, y = e^x \cos x.$$

Now the roots of $D^2-2D+4 = 0$ are

$$1 \pm \sqrt{3}i,$$

so that,

$$\text{C. F.} = e^x (c_1 \cos \sqrt{3}x+c_2 \sin \sqrt{3}x).$$

$$\text{Again P. I.} = \frac{1}{D^2-2D+4} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+4} \cos x$$

$$= e^x \frac{1}{D^2+3} \cos x = e^x \frac{1}{(-1^2)+3} \cos x = \tfrac{1}{2} e^x \cos x$$

$$\therefore \qquad y = e^x (c_1 \cos \sqrt{3}x+c_2 \sin \sqrt{3}x)+\tfrac{1}{2} e^x \cos x.$$

Exercises

Solve

1. $(D^2+3D+2)y = e^{2x} \sin x.$

2. $\dfrac{d^2y}{dx^2}+2y = x^2e^{3x}+e^x \cos 2x.$

3. $(D^3-D^2+3D+5) = e^x \cos 2x.$

4. $(D^3-7D-6)y = e^{2x}(1+x).$

Answers

1. $y = ae^{-2x}+be^{-x}+\frac{1}{170}(11 \sin x-7 \cos x)e^{2x}.$

2. $y = a \cos \sqrt{2}x+b \sin \sqrt{2}x-\frac{1}{17}(\cos 2x-4 \sin 2x)\, e^x$
$+\frac{1}{11}(x^2-12x/11+50/121)\, e^{3x}.$

3. $y = ae^{-x}+(b \cos 2x+c \sin 2x)\, e^x-\frac{1}{16}xe^x(\cos 2x-\sin 2x).$

4. $y = ae^{-x}+be^{3x}+ce^{-2x}-\frac{1}{12}(17/12+x)e^{2x}.$

14·92. *To show that*

$$\frac{1}{f(D)}\, xV = \left[x-\frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}V.$$

Here also we start with a function V_1 of x to be determined later on in terms of V.

We have

$$D(xV_1) = xDV_1+V_1 = xDV_1+(D)'\,V_1,$$

$$D^2(xV_1) = xD^2V_1+2DV_1 = xD^2V_1+(D^2)'\,V_1, \text{ etc.}$$

In general,

$$D^r(xV_1) = xD^rV_1+rD^{r-1}V_1 = xD^rV_1+(D^r)'\,V_1.$$

$$\therefore \quad f(D)xV_1 = (D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots\ldots+a_rD^{n-r}+\ldots\ldots+a_{n-1}D+a_n)x\,V_1$$

$$= xf(D)\,V_1+f'(D)\,V_1. \qquad \ldots(1)$$

We, now, determine V_1 such that

$$f(D)V_1 = V \Rightarrow V_1 = \frac{1}{f(D)}V. \qquad \ldots(2)$$

From (1) and (2), we have

$$f(D)\,x\,\frac{1}{f(D)}V = xV+f'(D)\,\frac{1}{f(D)}V.$$

Operating with $1/f(D)$, we obtain

$$x\,\frac{1}{f(D)}V = \frac{1}{f(D)}xV+\frac{1}{f(D)}f'(D)\,\frac{1}{f(D)}V.$$

$$\therefore \quad \frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{1}{f(D)}f'(D)\frac{1}{f(D)}V$$

$$= \left[x - \frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}V.$$

Examples

1. *Solve*

$$\frac{d^2y}{dx^2}+4y = x\sin x.$$

We have

$$(D^2+4)y = x\sin x.$$

Since the roots of $D^2+4 = 0$ are $\pm 2i$, we have

$$\text{C.F.} = c_1\cos 2x+c_2\sin 2x.$$

$$\text{P.I.} = \frac{1}{D^2+4}x\sin 2x$$

$$= \left[x - \frac{1}{D^2+4}2D\right]\frac{1}{D^2+4}\sin x$$

$$= \left(x - \frac{1}{D^2+4}2D\right)\frac{1}{3}\sin x$$

$$= \frac{x}{3}\sin x - \frac{2}{3}\cdot\frac{1}{D^2+4}\cos x$$

$$= \frac{x}{3}\sin x - \frac{2}{9}\cos x$$

$$\therefore \quad y = c_1\cos 2x+c_2\sin 2x+\tfrac{1}{3}x\sin x-\tfrac{2}{9}\cos x.$$

2. *Solve*

$$\frac{d^4y}{dx^4}-y = x^2\sin x$$

We have

$$(D^4-1)\,y = x^2\sin x.$$

The C.F., as may easily be seen, is

$$c_1e^{x}+c_2e^{-x}+c_3\cos x+c_4\sin x.$$

Again $\quad \text{P.I.} = \dfrac{1}{D^4-1}x^2\sin x$

$$x^2\sin x = x(x\sin x),$$

and apply the result obtained above by taking $V = x\sin x$. This process, however, becomes very tedious. The following process is comparatively shorter.

We have

$$\text{P. I.} = \text{Imaginary part of } \frac{1}{D^4-1}\, x^2 e^{ix}.$$

Now $\dfrac{1}{D^4-1} x^2 e^{ix}$

$$= e^{ix} \frac{1}{(D+i)^4-1} x^2$$

$$= e^{ix} \frac{1}{D^4+4D^3 i-6D^2-4Di}\, x^2$$

$$= e^{ix} \frac{1}{-4Di}\left[1+\frac{-6D+4D^2 i+D^3}{4i}\right]^{-1} x^2$$

$$= -\frac{e^{ix}}{4i}\,\frac{1}{D}\left[1+\frac{-6D+4D^2 i+D^3}{4i} + \left(\frac{-6D+\ldots}{4i}\right)^2\right] x^2$$

$$= i\,\frac{e^{ix}}{4}\,\frac{1}{D}\left[1-\frac{3D}{2i}-\frac{5}{4}D^2\ldots\ldots\right] x^2$$

$$= \frac{i}{4} e^{ix}\frac{1}{D}\left[x^2+3ix-\frac{5}{2}\right]$$

$$= \frac{i}{4} e^{ix}\left[\frac{x^3}{3}+\frac{3i}{2}x^2-\frac{5}{2}x\right]$$

$$= \frac{i\cos x-\sin x}{4}\left(\frac{x^3}{3}+\frac{3i}{2}x^2-\frac{5}{2}x\right)$$

whose imaginary part is

$$\frac{\cos x}{4}\left(\frac{x^3}{3}-\frac{5}{2}x\right)-\frac{3}{8}x^2\sin x.$$

$$\therefore \quad y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{\cos x}{4}\left(\frac{x^3}{3}-\frac{5}{2}x\right)-\frac{3}{8}x^2 \sin x.$$

Exercises

Solve

1. $(D^2+2D+1)\,y = x \sin x.$
2. $(D^2+1)\,y = x^2 \sin 2x.$

Answers

1. $y = (a+bx)\,e^{-x}+\frac{1}{2}(1-x)\cos x+\frac{1}{2}\sin x.$
2. $y = a\cos x+b\sin x-\frac{1}{3}\left(x^2-\frac{26}{9}\right)\sin 2x-\frac{4}{9}x\cos 2x.$

14·10. Homogeneous linear Equations. *To solve*

$$x^n \frac{d^n y}{dx^n}+a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\ldots\ldots$$
$$+\ldots+a_{n-1}x\frac{dy}{dx}+a_n y = X,$$

where a_1, a_2, etc., are constants and X is a function of x.

The transformation

$$z = \log x \Rightarrow x = e^z,$$

will convert the equation into one with constant coefficients.

We have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\cdot\frac{1}{x^2}-\frac{1}{x^2}\cdot\frac{dy}{dz} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right).$$

Similarly

$$\frac{d^3y}{dx^3} = \left(\frac{d^3y}{dz^3}-3\frac{d^2y}{dz^2}+2\frac{dy}{dz}\right)\frac{1}{x^3}.$$

We, now, write

$$D = \frac{d}{dz},$$

so that the above results can be re-written as

$$x\frac{dy}{dx} = Dy,$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)\,y,$$

$$x^3\frac{d^3y}{dx^3} = (D^3-3D^2+2D)\,y = D(D-1)(D-2)\,y.$$

By mathematical induction, we will obtain

$$x^n\frac{d^ny}{dx^n} = D(D-1)(D-2)\ldots\ldots(D-n+1) = y.$$

Thus the given differential equation becomes

$$[D(D-1)(D-2)\ldots\ldots(D-n+1)$$
$$+a_1 D(D-1)\ldots\ldots(D-n+2)+\ldots\ldots a_n]\,y = Z;$$

or $\qquad \varphi(D)\,y = Z,$

where Z is a function of z obtained from X by the substitution

$$z = \log x.$$

Here z is the independent variable.

The equation may now be solved by the methods already given.

Examples

Solve

$$x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y = 10\left(x+\frac{1}{x}\right).$$

By means of the transformation $z = \log x$, the equation becomes

$$[D(D-1)(D-2)+2D(D+1)+2]y = 10\,(e^z+e^{-z})\,;\ D=d/dz$$

$$\Rightarrow \qquad (D^3-D^2+2)y = 10(e^z+e^{-z}).$$

Here z is the independent and y the dependent variable.

Now, by trial, -1, is a root of the equation

$$D^3-D^2+2 = 0,$$

so that $D+1$ is a factor of D^3-D^2+2.

In fact, we have

$$\begin{aligned} D^3-D^2+2 &= D^3+D^2-2D^2+2 \\ &= D^2(D+1)-2(D^2-1) \\ &= (D+1)(D^2-2D+2). \end{aligned}$$

Now the roots of $D^2-2D+2 = 0$ are $1 \pm i$.

Thus the roots of $D^3-D^2+2 = 0$ are $-1,\ 1+i,\ 1-i$.

$$\begin{aligned} \therefore \qquad \text{C.F.} &= c_1e^{-z}+e^z(c_2\cos z+c_3\sin z) \\ &= c_1x^{-1}+x(c_2\cos\log x+c_3\sin\log x). \end{aligned}$$

$$\begin{aligned} \text{Also} \qquad \text{P.I.} &= \frac{1}{D^3-D^2+2}\,10(e^z+e^{-z}) \\ &= 10\frac{1}{D^3-D^2+2}e^z+10\frac{1}{D^3-D^2+2}e^{-z} \\ &= 10\frac{1}{1^3-1^2+2}e^z+10\frac{1}{(D+1)(D^2-2D+2)}e^{-z} \\ &= 5e^z+10\frac{1}{D+1}\cdot\frac{e^{-z}}{(-1)^2-2(-1)+2} \\ &= 5e^z+2z\,e^{-z} = 5x+2x^{-1}\log x. \end{aligned}$$

$$\therefore\ y = c_1x^{-1}+x(c_2\cos\log x+c_3\sin\log x)+5x+2x^{-1}\log x.$$

Exercises

Solve

1. $x^2\dfrac{d^2y}{dx^2}+7x\dfrac{dy}{dx}+5y = 2x^6.$

2. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-4y = x^3.$

3. $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y = (1-x)^2.$

4. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x + x^2 \log x + x^3.$

5. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (x+1)^2.$

Answers

1. $y = ax^{-5} - bx^{-2} + \frac{1}{77}x^4.$
2. $y = ax^2 + bx^{-3} + \frac{1}{6}x^3.$
3. $y = (a+b \log x)x^{-1} + 1 - \frac{1}{2}x + \frac{1}{9}x^3.$
4. $y = ax^2 + bx + \frac{1}{2}x^3 - x \log x + x^2 [\frac{1}{2} (\log x)^2 - \log x].$
5. $y = c_1x^4 + c_2x^{-5} - \frac{1}{14}x^2 - \frac{1}{9}x - \frac{1}{20}.$

EXERCISES ON CHAPTER 13

Solve :

1. $\dfrac{d^2y}{dx^2} - 2m\dfrac{dy}{dx} + m^2y = \sin nx.$

2. $\dfrac{d^2y}{dx^2} + 4y = \sin 3x \cos x.$

3. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = (e^{2x}-1)^2.$

4. Find the solution of the equation
$$\frac{d^2y}{dx^2} - y = 1,$$
which vanishes when $x = 0$ and tends to a finite limit as $x \to -\infty$.

5. $4\dfrac{d^2y}{dx^2} - y = 2 \sin (x + \frac{1}{4}\pi).$

6. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 13y = e^{x/2} - \sin \frac{1}{2}x.$

7. $4\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 2y = \cancel{\cos^2 (x+\frac{1}{4}x)}.$ $\cos^2(x + 1/6\,\pi)$.

8. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = e^{2x} \sin 2x.$

9. $\dfrac{d^4y}{dx^4} - y = e^x \cos x.$

10. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2 \cos x.$

11. $(D^3-1)y = xe^x + \cos^2 x.$

12. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \cos 2x + x^2.$

13. $\dfrac{d^2y}{dx^2}+y = xe^{2x}$.

14. $(D^2+4)\,y = x\cos 2x$.

15. $(D^2+3D+2)\,y = x^2\cos x$.

16. $(D^2-4D-5)\,y = xe^{-x}$.

17. $(D^2+9)\,y = \cos^2 \dfrac{3x}{2}$.

18. $(D^5+D^4+4D^3+4D^2+4D+4)\,y = \cos 3x \cos x$.

19. $(x^2D^2+xD-4)\,y = x^2$.

20. $(x^3D^3+3x^2D^2+xD+1)\,y = x\log x$.

21. $x\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2} = \dfrac{1}{x}$.

22. $\dfrac{d^2y}{dx^2}-\dfrac{6}{x^2}y = x\log x$.

23. $(x+1)^2\dfrac{d^2y}{dx^2}-3\,(x+1)\dfrac{dy}{dx}+4y = x^2$.

24. $x^2D^2y-3xDy+5y = x^2\sin(\log x)$.

25. $(x+3)^2\dfrac{d^2y}{dx^2}-4\,(x+3)\dfrac{dy}{dx}+6y = x$.

26. $(D^4+D^2+1)\,y = e^{(-x/2)}\cos \dfrac{x\sqrt{3}}{2}$.

Answers

1. $y = (a+bx)\,e^{mx}+\dfrac{(m^2-n^2)\sin nx+2mn\cos nx}{(m^2+n^2)^2}$.

2. $y = a\cos 2x+b\sin 2x-\frac{1}{24}\sin 4x-\frac{1}{8}x\cos 2x$.

3. $y = ae^{4x}+be^{2x}+\frac{1}{8}+\frac{1}{2}\,xe^{4x}+xe^{3x}$.

4. $y = e^x-1$

5. $y = ae^{-x/2}+be^{-x/2}-\frac{1}{8}(\sin x+\sqrt{3}\cos x)$.

6. $y = ae^{(2+\sqrt{17})\,x}+be^{(2-\sqrt{17})\,x}-\frac{4}{59}\,e^{\frac{1}{2}x}-\frac{9}{1034}\,(-59\sin \tfrac{1}{2}x$
$+6\cos \tfrac{1}{2}x)$.

7. $y = \left(a\cos\dfrac{x}{2}+b\sin\dfrac{x}{2}\right)e^{\frac{1}{2}x}+\frac{1}{4}-\dfrac{1}{520}[(7+4\sqrt{3})\cos 2x$
$+(4-7\sqrt{3})\sin 2x]$.

8. $y = ae^{(2+\sqrt{3})\,x}+be^{(2-\sqrt{3})\,x}-\frac{1}{7}\,e^{2x}\sin 2x$.

9. $y = ae^{-x}+be^{x}+c\cos x+d\sin x-\dfrac{e^x}{5}\cos x$.

10. $y = (a+bx)\cos x+(c+dx)\sin x-\frac{1}{48}(9x^2-x^4)\cos x+\frac{1}{12}x^3\sin x$.

11. $y = ae^x+\left[b\cos\dfrac{\sqrt{3}}{2}x+c\sin\dfrac{\sqrt{3}}{2}x\right]e^{-\frac{1}{2}x}$
$-\frac{1}{2}+\frac{1}{3}(\frac{1}{2}x^2-x)\,e^x-\frac{1}{130}(8\sin 2x+\cos 2x)$.

12. $y = \left(a \cos \frac{\sqrt{3}}{2} x + b \sin \frac{\sqrt{3}}{2} x\right) e^{-\frac{1}{2}x} - \frac{1}{13}(3 \cos 2x - 2 \sin 2x)$
$+ (x^2 - 2x)$.

13. $y = (a \cos x + b \sin x) + \frac{1}{5}\left(x - \frac{4}{5}\right) e^{2x}$.

14. $y = (a \cos 2x + b \sin 2x) + \frac{1}{16} x \cos 2x + \frac{1}{8} x^2 \sin 2x$.

15. $y = ae^{-2x} + be^{-3x} + \frac{1}{10}(\cos x + 3 \sin x) x^2 - \frac{1}{50}(17 \sin x - 6 \cos x)$
$+ \frac{1}{500}(81 \sin x - 133 \cos x)$.

16. $y = ae^{5x} + be^{-x} - \frac{1}{36} x (1 + 3x) e^{-x}$.

17. $y = a \cos 3x + b \sin 3x + \frac{1}{36}(3x \sin 3x + 2)$.

18. $y = ae^{-x} + (bx + c) \cos \sqrt{2}x + (dx + e) \sin \sqrt{2}x + \frac{1}{3332}(\cos 4x + 4 \sin 4x)$
$+ \frac{1}{20}(\cos 2x + 2 \sin 2x)$.

19. $y = ax^2 + bx^{-2} + \frac{1}{4} x^2 \log x$.

20. $y = ax^{-1} + \sqrt{x}\left[b \cos\left(\frac{1}{2}\sqrt{3} \log x\right) + c \sin\left(\frac{1}{2}\sqrt{3} \log x\right)\right] + \frac{1}{2} x \log x - \frac{3x}{4}$.

21. $y = a + (b \log x + c)\, x + \frac{1}{6} x (\log x)^2$.

22. $y = ax^3 + bx^{-3} + \frac{1}{36}(5 \log x - 2)\, x^3 \log x$

23. $y = (x+1)^2 [a \log (x+1) + b] + \frac{1}{4}\left[2 (x+1)^2 \{\log (x+1)\}^2 - 8x - 7\right]$.

24. $y = x^2 [a \cos (\log x) + b \sin (\log x)] - \frac{1}{2} x^2 \log x \cos (\log x)$.

25. $y = a (x+3)^2 + b (x+3)^3 + \frac{1}{2}(x+2)$.

26. $y = e^{-\frac{1}{2}x} \left\{\left(\frac{1}{4} x + a\right) \cos \frac{x\sqrt{3}}{2} + \left(b + \frac{x}{4\sqrt{3}}\right) \sin \frac{x\sqrt{3}}{2}\right\}$
$+ ce^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}x}{2} + d\right)$.

MISCELLANEOUS EXERCISES IV

Solve the following differential equations :—

1. $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1 + x)\, e^x \sec y$.

2. $(y \sin x - 1)\, dx + \cos x\, dy = 0$.

3. $(x+y)^2\, dx = xy\, dy$. 4. $(x+y)^2\, dx = dy$.

5. $2xy^2\, dx = e^x (dy - y dx)$.

6. $\dfrac{x+y-a}{x+y-b} \dfrac{dy}{dx} = \dfrac{x+y+a}{x+y+b}$.

7. $\dfrac{dy}{dx} + y \cot x = \sin x$.

8. $x \dfrac{dy}{dx} - 2y = x^2 + \sin \dfrac{1}{x^2}$.

9. $x \dfrac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$.

10. $2\cos x \dfrac{dy}{dx}+4y\sin x = \sin 2x$, given that $y = 0$, when $x = \pi/3$.

11. $x\dfrac{dy}{dx}+3y = x^4y^3 e^{1/x^2}$. 12. $xy-\dfrac{dy}{dx} = y^3 e^{-x^2}$.

13. $\dfrac{dy}{dx}=\dfrac{y(x+y)}{x(x-y)}$. 14. $\dfrac{dy}{dx}=\dfrac{6x-4y+3}{3x-2y+1}$.

15. $\dfrac{dy}{dx}=\dfrac{2x-y+1}{x+2y-3}$. 16. $x^2y\dfrac{dy}{dx} = xy^2-e^{-1/x^3}$

17. $3e^x\tan y+(1-e^x)\sec^2 y\dfrac{dy}{dx} = 0$.

18. $(x+y+1)\,dy = dx$. 19. $x\dfrac{dy}{dx}+y\log y = xye^x$.

20. $(3x-2y+1)\,dx+(2x-3y+4)\,dy = 0$.

21. $(2xy+y-\tan y)\,dx+(x^2-x\tan^2 y+\sec^2 y)\,dy = 0$.

22. $x\dfrac{dy}{dx}-2y = x^2+\tan\dfrac{1}{x^2}$.

23. Solve the equation $\dfrac{dy}{dx}+2y\tan x = \sin x$ and find the minimum value of y if $y = 2$, when $x = 0$.

24. $(x^2+3x+2)\dfrac{dy}{dx}+(2x+1)\,y = (xy+2y)^2$.

25. $(x+2y^3)\dfrac{dy}{dx} = y$.

26. Prove that the orthogonal trajectories to

$$y = \tan x+c$$

are the curves

$$4y+2x+\sin 2x+c = 0.$$

27. Show that each curve of the system $x^2-y^2 = c$; cuts at right angle each curve of the system $xy = a$.

28. Show that the orthogonal trajectories of the curves

$$r\sin^2\theta = A,$$

are the curves

$$r^2\cos\theta = A.$$

29. Prove that

$$(r^2-1)\sin\theta = cr,$$

are the orthogonal trajectories of the curve

$$(r^2+1)\cos\theta = cr.$$

30. Find the curves in which the polar sub-tangent at any point is proportional to the radius vector at that point.

31. Find the curves in which the angle between the tangent at any point and the radius vector to that point is equal to m times the vectorial angle.

32. Find the orthogonal trajectories of the system of curves

$$\left(\frac{dy}{dx}\right)^2 = \frac{a}{x}.$$

33. Prove that the differential equation of all parabolas lying in a plane is

$$\frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right)^{-2/3} = 0.$$

34. Find the cartesian equation of the curve in which the perpendicular from the origin to the tangent is equal to the abscissa of the point of contact.

35. Find a curve such that the area comprised between the curve, the axis of x and any two ordinates is proportional to the arc between those ordinates.

36. Find the solution of

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y = e^{-x},$$

that satisfies the conditions $y = 0$, $dy/dx = 0$ at $x = 0$.

37. Solve

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y = x^2,$$

given that $y = 0$, when, $x = 1$ and that $y = e^2$ when $x = e$.

38. Find the solution of

$$4xy\frac{dy}{dx} = 4y^2-x^2,$$

for which $y = 0$ when $x = 1$.

39. Solve

$$\left(\frac{dy}{dx}\right)^2+(\sin x+\cos x)\,y\frac{dy}{dx}+\frac{1}{2}y^2\sin 2x = 0.$$

40. Solve the equation

$$(D^3+D^2-2)\,y = x^3e^x,$$

subject to the conditions that

$$y = 0 \text{ at } x = 0 \text{ and } y \to 0 \text{ as } x \to -\infty.$$

41. Solve

$$(D^4+2D^2+1)\,y = 24\,x\sin x.$$

42. Solve

$$(D-1)^3\,(D^2+1)^2y = \sin^2 \tfrac{1}{2}x+e^x$$

Answers

1. $(1+x)(e^x+c) = \sin y$.
2. $y = c \cos x+\sin x$.
3. $cx^3 (x+2y) = e^{2y/x}$.
4. $x+y = \tan (x+c)$.
5. $e^x+(c+x^2)\, y = 0$.
6. $2(x-y+c) = (b-a) \log [(x+y)^2-ab]$.
7. $(2y+\cos x) \sin x = c+x$.
8. $2y = cx^2+2x^2 \log x+x^2 \cos (x^{-2})$,
9. $y = cx+x \log \tan x$.
10. $y \sec^2 x = \sec x-2$.
11. $y^2x^6 (c+e^{1/x^2}) = 1$.
12. $y^2 (c+2x) = e^{x^2}$.
13. $xy^{-1}+\log xy = c$.
14. $4x-2y-2 \log (3x-2y+3) = c$.
15. $xy+y^2-3y = x^2+x+c$.
16. $3y^2+2x^2e^{-1/x^3} = cx^2$.
17. $\sqrt[3]{\tan y} = c(1-e^x)$.
18. $x+y+2 = ce^y$.
19. $x \log y = (x-1)e^x+c$.
20. $(y-x-1)(y+x-3)^5 = c$.
21. $x^2y+xy-x \tan y+\tan y = c$.
22. $xy^{-2} = \log x+\frac{1}{2} \cos x^{-2}+c$.
23. $y = \cos x+c \cos^2 x\,;\ -\frac{1}{2}$.
24. $x+1 = y(x+2)^2+cy(x+2)^3$.
25. $x = y^3+cy$.
30. Equiangular spirals.
31. $r^m = a^m \sin m\theta$.
32. $(x^{3/2}+c^{3/2})^2 = \frac{9}{2}\, ay^2$.
34. $x^2+y^2 = 2cx$.
35. The system of catenaries $y = k \cosh \dfrac{x+c}{k}$; k being the given constant of proportionality.
36. $y = (x-1)\, e^{-x}+e^{-2x}$.
37. $x^2 \log x \log (ex) = 2y$.
38. $xe^{2y^2/x^2} = 1$.
39. $(cy-e^{-\sin x})(cy-e^{\cos x}) = 0$.
40. $y = e^x\left(\dfrac{x^4}{20} - \dfrac{4x^3}{25} + \dfrac{33x^2}{125} - \dfrac{144x}{625} \right)$.
41. $y = (c_1x+c_2) \cos x+ (c_3x+c_4) \sin x-(3 \cos x+x \sin x)\, x^2$.
42. $y = (c_1+c_2x) \cos x+(c_3+c_4x) \sin x + (c_5+c_6x+\frac{1}{8}x^2)\, e^x- \frac{1}{8}x^2 \sin x+\frac{1}{2}$.

APPENDIX I

Beta and Gamma Functions

A·1. Convergence of improper integrals. The notion of improper definite integrals has already been introduced in Chapter I. We shall now obtain two simple comparison tests for the existence (also called convergence) of improper integrals for the case of positive integrands. This will be done with a view to the introduction of Beta and Gamma functions defined as definite integrals.

A·21. Case of Infinite Region of Integration. The improper integral

$$\int_a^{\infty} f(x)\, dx \qquad \ldots(1)$$

is said to converge at ∞, if

$$\varphi(X) = \int_a^X f(x)\, dx$$

tends to a finite limit as $X \to \infty$ and the limit, then, is said to be the value of (1).

If $f(x)$ is positive for every $x > a$, it follows from the geometrical interpretation of the definite integral as area, that $\varphi(X)$ is a monotonically increasing function of X. This shows *that the improper integral*

$$\int_a^{\infty} f(x)\, dx$$

is convergent at ∞ if, and only if, $\varphi(X)$ is bounded above, i.e., there exists a positive number k such that

$$\varphi(X) < k \textit{ for all } X.$$

In this connection, we have the following useful sufficient test for convergence :

Theorem. *If $f(x)$ and $F(x)$ are two functions positive for every value of $x \geqslant a$ and if*

$$\lim_{x \to \infty} \frac{f(x)}{F(x)} = l,$$

where, l, is neither zero nor infinity, then the integrals

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty F(x)\,dx$$

either both converge or both do not converge.

Let ε be a positive number. Then there exists a number X_0 such that

$$\left| \frac{f(x)}{F(x)} - l \right| < \varepsilon \text{ for } x \geqslant X_0,$$

$$\Leftrightarrow \qquad (l-\varepsilon)\,F(x) < f(x) < (l+\varepsilon)\,F(x), \text{ for } x \geqslant X_0. \qquad \text{...(2)}$$

*From (2), we have

$$\int_{X_0}^{X} (l-\varepsilon)\,F(x)dx \leqslant \int_{X_0}^{X} f(x)dx \leqslant \int_{X_0}^{X} (l+\varepsilon)F(x)dx \qquad \text{...(3)}$$

Suppose now that

$$\int_a^\infty F(x)\,dx$$

converges. Then

$$\int_{X_0}^\infty F(x)\,dx$$

also converges so that

$$\int_{X_0}^X F(x)\,dx$$

and accordingly

$$(l+\varepsilon)\int_{X_0}^X F(x)\,dx$$

is bounded above. It now follows from (3) that

$$\int_{X_0}^X f(x)\,dx$$

is bounded above. Thus

$$\int_{X_0}^\infty f(x)\,dx$$

*This follows from the geometrical interpretation of definite integrals as areas.

and hence also

$$\int_a^{\infty} f(x)\, dx$$

is convergent.

The other cases may be similarly deduced from (3).

A·22. Case of Infinite Integrand. Suppose now that $f(x)$ is continuous in the interval $]a, b-h[$ and that $f(x) \to \infty$ as $x \to b$. We write

$$\varphi(h) = \int_a^{b-h} f(x)\, dx.$$

Then the improper integral

$$\int_a^b f(x)\, dx \qquad \text{...(4)}$$

is said to converge at, a, if $\varphi(h)$ tends to a finite limit as h tends to 0 and the limit, then, is said to be the value of (4).

If $f(x)$ is positive, then, $\varphi(h)$ monotonically increases as h decreases and accordingly we see that in the case of positive integrand, the integral (4) is convergent if, and only if, $\varphi(h)$ is bounded above, *i.e.*, there exists a number k such that $\varphi(h) < k$ for all positive values of h.

As in the case of improper integrals with infinite range of integration, we may show that *if $f(x)$ and $F(x)$ be positive and*

$$\lim_{x \to b} \frac{f(x)}{F(x)} = l,$$

where, l, is neither zero nor infinite, then the two integrals

$$\int_a^b f(x)\, dx \quad \text{and} \quad \int_a^b F(x) dx$$

either both converge or both do not converge at b.

We may also state and prove a similar test for convergence at a.

A·3. Three important Comparison Integrals.

We shall now consider three important comparison integrals :

(*i*) $\int_a^{\infty} \frac{dx}{x^{\mu}}$, $(a>0)$. (*ii*) $\int_a^b \frac{dx}{(b-x)^{\mu}}$, (*iii*) $\int_a^b \frac{dx}{(x-a)^{\mu}}$.

(*i*) *The integral*

$$\int_a^{\infty} \frac{dx}{x^{\mu}}$$

is convergent if, and only, if $\mu > 1$.

If $\mu \neq 1$, we have

$$\varphi(X) = \int_a^X \frac{dx}{x^\mu} = \frac{1}{1-\mu} \left| \frac{1}{x^{\mu-1}} \right|_a^X = \frac{1}{1-\mu} \left[\frac{1}{X^{\mu-1}} - \frac{1}{a^{\mu-1}} \right]$$

which tends to $1/(\mu-1)a^{\mu-1}$ or ∞ according as $\mu > 1$ or $\mu < 1$ when X tends to infinity.

If $\mu = 1$, we have

$$\varphi(X) = \int_a^X \frac{dx}{x} = \log X - \log a,$$

which tends to infinity as $X \to \infty$.

Hence the result.

(*ii*) *The integral*

$$\int_a^b \frac{dx}{(b-x)^\mu}$$

is convergent if, and only if, $\mu < 1$.

If $\mu \neq 1$, we have

$$\varphi(h) = \int_a^{b-h} \frac{dx}{(b-x)^\mu} = \frac{1}{\mu-1} \left| \frac{1}{(b-x)^{\mu-1}} \right|_a^{b-h}$$
$$= \frac{1}{\mu-1} \left[\frac{1}{h^{\mu-1}} - \frac{1}{(b-a)^{\mu-1}} \right],$$

which tends to $1/(1-\mu)(b-a)^{\mu-1}$ or ∞ according as $\mu < 1$ or $\mu > 1$, when h tends to zero.

If $\mu = 1$, we have

$$\varphi(h) = \int_a^{b-h} \frac{dx}{b-x} = -\Big| \log (b-x) \Big|_a^{b-h} = \log(b-a) - \log h,$$

which tends to infinity as h tends to zero.

Hence the result.

(*iii*) *The integral*

$$\int_a^b \frac{dx}{(x-a)^\mu}$$

is convergent if, and only if, $\mu < 1$.

It may be proved like (*ii*) above.

Examples

1. *Examine the convergence of* :

(*i*) $\displaystyle\int_1^\infty \frac{x\,dx}{(1+x)^3}$. (*ii*) $\displaystyle\int_1^\infty \frac{dx}{x^{\frac{1}{3}}(1+x)^{\frac{1}{2}}}$.

(*i*) We have

$$f(x) = \frac{x}{(1+x)^3}.$$

Take

$$F(x) = \frac{x}{x^3} = \frac{1}{x^2}.$$

Then

$$\lim_{x\to\infty} \frac{f(x)}{F(x)} = \lim_{x\to\infty} \frac{x^3}{(1+x)^3} = 1.$$

$$\therefore \qquad \int_1^\infty f(x)\,dx \text{ and } \int_1^\infty F(x)\,dx$$

have identical behaviours. But, by § A·3 (*i*), the latter integral is convergent. Hence the given integral is convergent.

(*ii*) We have

$$f(x) = \frac{1}{x^{\frac{1}{3}}(1+x)^{\frac{1}{2}}}.$$

Take

$$F(x) = \frac{1}{x^{\frac{1}{3}}\,x^{\frac{1}{2}}} = \frac{1}{x^{\frac{5}{6}}}.$$

Then

$$\lim_{x\to\infty} \frac{f(x)}{F(x)} = \lim \frac{x^{\frac{5}{6}}}{x^{\frac{1}{3}}(1+x)^{\frac{1}{2}}} = \lim \frac{x^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}} = 1.$$

Also by § A·3 (*i*), the integral

$$\int_1^\infty F(x)dx = \int_1^\infty \frac{1}{x^{5/6}}\,dx,\ \mu = \tfrac{5}{6} < 1,$$

is *not* convergent. Hence the given integral does *not* converge.

2. *Examine the convergence of* :

(*i*) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$. (*ii*) $\displaystyle\int_0^1 \frac{dx}{x^{\frac{1}{2}}(1+x)^2}$.

(*iii*) $\displaystyle\int_{\frac{1}{2}}^{1} \frac{dx}{x^{1/3}(1-x)^{1/3}}$.

(*i*) The integrand $1/x^{1/3}(1+x^2)$ is positive and continuous for every value of, x, other than the lower limit zero and $\to\infty$, as $x\to 0$. We write

$$f(x) = \frac{1}{x^{1/3}(1+x^2)},$$

and take

$$F(x) = \frac{1}{(x-0)^{1/3}} = \frac{1}{x^{1/3}}.$$

Then

$$\lim_{x\to 0} \frac{f(x)}{F(x)} = 1,$$

and by § A·3 (*iii*)

$$\int_0^1 \frac{1}{x^{1/3}} dx \qquad (\mu = \tfrac{1}{3} < 1)$$

converges. Therefore, the given integral also converges.

(*ii*) We write

$$f(x) = \frac{1}{x^2(1+x)^2},$$

and take

$$F(x) = \frac{1}{x^2}.$$

Then

$$\lim_{x\to 0} \frac{f(x)}{F(x)} = 1,$$

and

$$\int_0^1 \frac{1}{x^2} dx \qquad (\mu = 2 > 1)$$

does not converge. Therefore given integral does not converge.

(*iii*) The integrand is positive and continuous for every value of x in the interval $[\frac{1}{2}, 1]$ except at the upper limit 1 and $\to\infty$ as $x\to 1$. We write

$$f(x) = \frac{1}{x^{1/3}(1-x)^{1/3}}$$

and take

$$F(x) = \frac{1}{(1-x)^{1/3}}.$$

Then

$$\lim_{x\to 1} \frac{f(x)}{F(x)} = 1,$$

and by § A·3 (*ii*),

$$\int_{\frac{1}{2}}^{1} F(x)\, dx. \qquad (\mu = \tfrac{1}{2} < 1)$$

is convergent. Hence the given integral also converges.

Exercises

Examine the convergence of the following :

(*i*) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$. (*ii*) $\int_0^1 \frac{x^{a-1}}{1+x}\,dx$.

(*iii*) $\int_0^{\frac{1}{2}\pi} \frac{\sqrt{x}}{\sin x}\,dx$. (*iv*) $\int_0^1 \frac{dx}{[(x-1)^2(x-2)]^{1/3}}$

Answers

(*i*) Convergent. (*ii*) Convergent if $a > 0$.
(*iii*) Convergent. (*iv*) Not convergent.

A·4. Beta Function. $B(m, n)$. Consider the integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

We see that (*i*) if $m \geqslant 1$, the integrand is continuous at $x = 0$, and if $m < 1$, the integrand tends to infinity as x tends to 0 ;

(*ii*) if $n \geqslant 1$, the integrand is continuous at $x = 1$ and if $n < 1$ the integrand tends to infinity as x tends to 1 ;

(*iii*) the integrand is continuous for every value of x other than 0 and 1.

Thus the integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,$$

is proper if $m \geqslant 1$ and $n \geqslant 1$ and, as it stands, defines a function of two variables m, n ; the domain of definition being defined by

$$m \geqslant 1,\ n \geqslant 1.$$

We now suppose that $m < 1$ or $n < 1$ and examine the convergence of the improper integrals

$$\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx,\ \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx$$

at 0 and 1 respectively.

Convergence at 0. We write

$$f(x) = x^{m-1}(1-x)^{n-1} = (1-x)^{n-1}/x^{1-m},$$

and take $$F(x) = \frac{1}{x^{1-m}},$$

so that $$f(x)/F(x) \to 1 \text{ as } x \to 0.$$

As $$\int_0^{\frac{1}{2}} F(x)\, dx$$

is convergent at 0, if and only if, $1-m < 1 \Leftrightarrow 0 < m$, we deduce that the integral

$$\int_0^{\frac{1}{2}} x^{m-1} (1-x)^{n-1}\, dx$$

is convergent at 0 if and only if m is positive.

Convergence at 1. Writing $f(x) = x^{m-1}(1-x)^{n-1} = x^{m-1}/(1-x)^{1-n}$ and taking $F(x) = 1/(1-x)^{1-n}$, we may show that the integral

$$\int_{\frac{1}{2}}^{1} x^{m-1} (1-x)^{n-1}\, dx$$

is convergent at 1 if, and only if, n is positive.

Thus we see that the symbol

$$\int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

has a meaning when m and n have any positive values.

The function

$$\int_0^1 x^{m-1} (1-x)^{n-1}\, dx,$$

defined for positive values of m and n, is known as Beta function and we write

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx. \qquad (m > 0, n > 0)$$

Putting $x = \sin^2\theta$, we see that

$$B(m, n) = 2 \int_0^{\frac{1}{2}\pi} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta,$$

so that if, $2m-1$ and $2n-1$ are positive integers, we have

$$B(m, n) = 2\, \frac{(2m-2)(2m-4)\ldots\ldots(2n-2)(2n-4)\ldots\ldots}{(2m+2n-2)(2m+2n-4)\ldots\ldots}$$

multiplied by $\pi/2$ if $2m-1$, and $2n-1$, are both even positive integers.

Exercises

1. Show that

(i) $$\frac{B(p, q+1)}{q} = \frac{B(p+1, q)}{p} = \frac{B(p, q)}{p+q}.$$

(ii) $B(p, q) = B(p+1, q)+B(p, q+1)$.

2. Show that

$$\int_0^1 \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \text{ where } p>-1, q>-1.$$

3. Show that

$$\int_0^1 x^{m-1} (1-x^2)^{n-1} \, dx = \tfrac{1}{2}B(\tfrac{1}{2} m, n).$$

4. Evaluate

(i) $\displaystyle\int_0^1 \frac{x \, dx}{\sqrt{(1-x^5)}}$. (ii) $\displaystyle\int_0^2 x^4(8-x^3)^{-1/3}dx$

5. Show that

$$\int_0^p x^m(p^q-x^q)^n \, dx = \frac{p^{qn+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right),$$

if $p > 0, q > 0, m+1 > 0, n+1 > 0$.

6. Evaluate

$$\int_a^b (x-a)^{l-1} (b-x)^{m-1} \, dx. \qquad (l > 0, m > 0)$$

[Put $x=py+q$ where p, q are such that $x=a$ for $y=0$ and $x=b$ for $y=1$].

7. (a) Show that

$$\int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} \, dx = B(p, q).$$

[Put $t = x/(1+x)$ so that as x varies from 0 to ∞, t varies from 0 to 1.]

(b) Show that

$$\int_0^\infty \frac{x^{m-1}dx}{(a+bx)^{m+n}} = \frac{1}{a^n b^m} B(m, n).$$

8. Prove that

$$\int_0^1 \frac{x^{m+1}+x^{n-1}}{(1+x)^{m+n}} \, dx = B(m, n).$$

[We have $\displaystyle B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} \, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx.$

Now put $x = 1/t$ in the second integral on the right].

9. Show that

$$\int_0^{\infty} \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx = 2\, B(m, n).$$

10. By putting,

$$\frac{x}{1-x} = \frac{at}{1-t},$$

where the constant, a, is suitably selected, show that

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}dx = \frac{1}{9^{1/3}} B(\tfrac{2}{3}, \tfrac{1}{3}).$$

11. Show that

$$\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{(b+cx)^{l+m}}\, dx = \frac{B(l, m)}{(b+c)^l\, b^m}$$

12. Evaluate

$$\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\, dx.$$

13. Prove that

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\, d\theta = \frac{B(m, n)}{2a^m b^n}.$$

$$\left[\text{Write } \int_0^{\frac{1}{2}\pi} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\, d\theta = \int_0^{\frac{1}{2}\pi} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{[a+(b-a)\sin^2\theta]^{m+n}}\, d\theta = \frac{1}{2}\int_0^1 \frac{x^{n-1}(1-x)^{m-1}}{[a+(b-a)x]^{m+n}}\, dx\right]$$

14. Show that

$$\int_0^{\pi} \frac{\sin^{n-1}x}{(a+b\cos x)^n}\, dx = \frac{2^{n-1}}{(a^2-b^2)^{n/2}} B(\tfrac{1}{2}n, \tfrac{1}{2}n), \text{ if } a^2 > b^2.$$

15. Show that

$$\int_0^{n} \left(1-\frac{t}{n}\right)^n t^{x-1}\, dt = n^x B(x, n+1) = \frac{n!\, n^x}{x(x+1)\dots(x+n)}$$

where $x > 0$, $n = 1, 2, \dots\dots$

Answers

4. (i) $\frac{1}{5} B\left(\frac{2}{5}, \frac{1}{2}\right)$. (ii) $\frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right)$.

6. $(b-a)^{l+m-1} B(l, m)$. 12. $\frac{1}{4(2)^{1/4}} B\left(\frac{7}{4}, \frac{1}{4}\right)$.

A·5. Gamma function. Consider the integral

$$\int_0^\infty x^{a-1} e^{-x}\, dx.$$

If $a \geqslant 1$, the integrand is continuous as $x = 0$ and if $a < 1$, the integrand tends to infinity as $x \to 0$.

We have now to consider the convergence of the two integrals, *viz.*,

$$\int_0^1 x^{a-1} e^{-x}\, dx \text{ and } \int_1^\infty x^{a-1} e^{-x}\, dx$$

at 0 and ∞ respectively.

Convergence at 0. We write

$$f(x) = x^{a-1} e^{-x} = e^{-x}/x^{1-a}$$

and take

$$F(x) = 1/x^{1-a}.$$

As

$$\lim_{x \to 0} \frac{f(x)}{F(x)} = 1,$$

and the integral

$$\int_0^1 F(x)\, dx = \int_0^1 \frac{1}{x^{1-a}}\, dx$$

is convergent at 0, if and only if, $(1-a) < 1$, *i.e.*, $a > 0$, we see that

$$\int_0^1 x^{a-1} e^{-x}\, dx$$

is convergent at 0, if and only if $a > 0$.

Convergence at ∞. In this case the comparison test in the form stated does not prove useful.

We know that

$$e^x > x^{a+1},$$

whatever value, a, may have,

$$\therefore \qquad e^{-x} < x^{-a-1},$$

and

$$x^{a-1} e^{-x} < x^{a-1} x^{-a-1} = 1/x^2.$$

Thus $\qquad \int_1^X x^{a-1} e^{-x}\, dx$ is bounded above.

Hence $$\int_1^\infty e^{a-1}\, e^{-x}\, dx$$

is convergent at ∞ for every value of a.

From above, it now follows that

$$\int_0^\infty x^{a-1}\, e^{x}\, dx$$

has a meaning when, a, is any positive number.

The function

$$\int_0^\infty x^{a-1} e^{-x}\, dx,$$

defined for positive values of, a, is known as **Gamma Function** and we write

$$\Gamma(a) = \int_0^\infty x^{a-1}\, e^{-x}\, dx.$$

Employing integration by parts, we may show that

$$\Gamma(a+1) = a\,\Gamma(a).$$

From this, we may deduce that

$$\Gamma(a) = (a-1)\,!\,;$$

a, being a positive integer.

A·6. Relation between Beta and Gamma Functions. We shall now prove the following important relation

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)},\ (m>0,\ n>0).$$

We have

$$B(m, n) = \int_0^1 x^{m-1}\, (1-x)^{n-1}\, dx$$

$$= 2 \int_0^{\frac{1}{2}\pi} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, d\theta. \qquad \ldots(1)$$

(Putting $x = \sin^2\theta$)

and $$\Gamma(m) = \int_0^\infty x^{m-1}\, e^{-x}\, dx$$

$$= 2 \int_0^\infty r^{2m-1}\, e^{-r^2}\, dr. \qquad \ldots(2)$$

(Putting $x = r^2$)

Let E denote the square bounded by the lines

$$x = 0,\ x = R\ ;\ y = 0,\ y = R.$$

We have

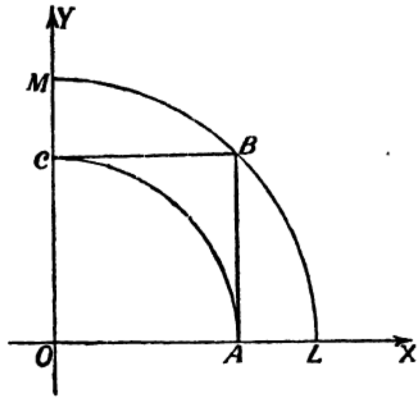

Fig. 52.

$$4\iint_E x^{2m-1} y^{2n-1} e^{-x^2-y^2}\,dx\,dy$$

$$=\left[\int_0^R 2x^{2m-1} e^{-x^2}\,dx\right]\left[\int_0^R 2y^{2n-1} e^{-y^2}\,dy\right].$$

Now the positive quadrant of the circle

$$x^2+y^2 = R^2$$

is a part of the square E and the square R is itself a part of the positive quadrant of the circle

$$x^2+y^2 = 2R^2.$$

We denote the positive quadrants of these circles by E_1 and E_2 respectively.

The integrand being positive, we have

$$4\iint_{E_1} x^{2m-1} y^{2n-1} e^{-x^2-y^2}\,dx\,dy \leqslant 4\iint_R x^{2m-1} y^{2n-1} e^{-x^2-y^2}\,dx\,dy$$

$$\leqslant 4\iint_{E_2} x^{2m-1} y^{2n-1} e^{-x^2-y^2}\,dx\,dy.$$

Changing the variables to r, θ where $x = r\cos\theta$, $y = r\sin\theta$, we have

$$4\iint_{E_1} x^{2m-1} y^{2n-1} e^{-x^2-y^2}\,dx\,dy$$

$$= 4\int_0^{\frac{1}{2}\pi} \cos^{2m-1}\theta \sin^{2n-1}\theta\,d\theta \int_0^R e^{-r^2} r^{2m+2n-1}\,dr$$

$$= 2B(m, n)\int_0^R e^{-r^2} r^{2m+2n-1}\,dr.$$

Similarly

$$4\iint_{E_2} x^{2m-1}\, y^{2n-1}\, e^{-x^2-y^2}\, dx\, dy$$

$$= 2B(m, n)\int_0^{2\sqrt{R}} e^{-r^2}\, r^{2m+2n-1}\, dr.$$

Thus

$$B(m, n)\int_0^R 2e^{-r^2}\, r^{2(m+n)-1}\, dr$$

$$\leqslant \left[\int_0^R 2e^{-x^2}\, x^{2m-1}\, dx\right]\left[\int_0^R 2e^{-y^2}\, y^{2n-1}\, dy\right]$$

$$\leqslant B(m, n)\int_0^{2\sqrt{R}} 2e^{-r^2}\, r^{2(m+n)-1}\, dr.$$

Let, now, $R \to \infty$. We obtain

$B(m, n)\, \Gamma(m+n) \leqslant \Gamma(m)\, \Gamma(n) \leqslant B(m, n)\, \Gamma(m+n)$.

$$\Rightarrow \qquad B(m, n)\, \Gamma(m+n) = \Gamma(m)\, \Gamma(n),$$

$$\Rightarrow \qquad B(m, n) = \frac{\Gamma(m)\, \Gamma(n)}{\Gamma(m+n)} \qquad \ldots(1)$$

Cor. 1. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Taking $m = \frac{1}{2} = n$, we see that

$$B(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{[\Gamma(\frac{1}{2})]^2}{\Gamma(1)}.$$

Also $\qquad B(\frac{1}{2}, \frac{1}{2}) = 2\int_0^{\frac{1}{2}\pi} \cos^{\circ}\theta\, \sin^{\circ}\theta\, d\theta = \pi,$

and $\qquad \Gamma(1) = \int_0^1 e^{-x}\, dx = 1.$

$$\therefore \qquad [\Gamma(\tfrac{1}{2})]^2 = \pi \Rightarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi},$$

for $\Gamma(\frac{1}{2})$ is necessarily positive.

The student may prove this result independently in the manner in which the relation (1) has been proved.

Cor. 2. It may be deduced that

$$\int_0^{\infty} e^{-x^2}\, dx = \tfrac{1}{2}\sqrt{\pi}.$$

Cor 3. Duplication Formula.

$$\sqrt{\pi}\, \Gamma(2m) = 2^{2m-1}\, \Gamma(m)\, \Gamma(m+\tfrac{1}{2}).$$

We have

$$\frac{\Gamma(m)\, \Gamma(n)}{\Gamma(m+n)} = B(m, n) = 2\int_0^{\frac{1}{2}\pi} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta \qquad \ldots(1)$$

Taking $n = m$, we have

$$\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = 2\int_0^{\frac{1}{2}\pi} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta$$

$$= \frac{1}{2^{2m-2}} \int_0^{\frac{1}{2}\pi} \sin^{2m-1} 2\theta \, d\theta$$

$$= \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} \varphi \, d\varphi \qquad (2\theta = \varphi)$$

$$= \frac{1}{2^{2m-2}} \int_0^{\frac{1}{2}\pi} \sin^{2m-1} \varphi \, d\varphi \qquad \ldots(2)$$

In (1), taking $n = \frac{1}{2}$ we obtain

$$\frac{\Gamma(m)\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = 2\int_0^{\frac{1}{2}\pi} \sin^{2m-1}\theta \, d\theta. \qquad \ldots(3)$$

From (2) and (3), we obtain

$$\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

$$\Rightarrow \quad 2^{2m-1}\,\Gamma(m)\,\Gamma(m+\tfrac{1}{2}) = \Gamma(\tfrac{1}{2})\,\Gamma(2m)$$

$$= \sqrt{\pi}\,\Gamma(2m), \text{ for } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Hence the result.

Exercises

1. Express

$$I_n = \int_0^1 x^p (1-x^q)^n \, dx,$$

where p, q, n are positive in terms of Gamma functions.

2. Show that

$$\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \frac{\sqrt{\pi}}{n} \, \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)} = 2^{\frac{2-n}{n}} \, \frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{n\,\Gamma\left(\frac{2}{n}\right)}.$$

3. Show that

(*i*) $B(m, m)\, B(m+\frac{1}{2}, m+\frac{1}{2}) = \pi\, m^{-1}\, 2^{1-4m}$.

(*ii*) $B(x, x) = 2^{1-2x}\, B(x, \frac{1}{2})$.

4. Show that

$$\Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \sqrt{2}\pi.$$

5. Prove that

(*i*) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(\sin\theta)}} \times \int_0^{\frac{1}{2}\pi} \sqrt{(\sin\theta)}\, d\theta = \pi.$

(*ii*) $\int_0^\infty \sqrt{y}\, e^{-y^2}\, dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\, dy = \frac{\pi}{2\sqrt{2}}$.

(*iii*) $\int_0^\infty x\, e^{-x^8}\, dx \times \int_0^\infty x^2\, e^{-x^4}\, dx = \frac{\pi}{16\sqrt{2}}$.

(*iv*) $\left\{\int_0^{\frac{1}{2}\pi} \sin^p x\, dx\right\}\left\{\int_0^{\frac{1}{2}\pi} \sin^{p+1} x\, dx\right\} = \frac{\pi}{2(p+1)}$

6. Show that

$$\int_0^1 \sqrt{(1-x^4)}\, dx = \frac{1}{12}\sqrt{\left(\frac{2}{\pi}\right)}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

7. Show that the perimeter of the lemniscate

$$r^2 = 2a^2 \cos 2\theta$$

is

$$\frac{a}{\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

8. If n is a positive integer, prove that the ratio of the areas enclosed by the curves

$$x^{2n}+y^2 = 1,\ x^{2n}+y^{2n} = 1$$

is

$$n2^{1/n}/(n+1).$$

APPENDIX II

Differentiation Under Integral Sign

A-7. Consider a continuous function $f(x, y)$ of two variables defined in a rectangle bounded by the lines

$$x = a, x = b \,;\, y = c, y = d$$

and the integral

$$\int_a^b f(x, y)\, dx.$$

Clearly this integral is a function of y. We write

$$\varphi(y) = \int_a^b f(x, y)\, dx.$$

It can be shown that $\varphi(y)$ is a continuous function of y.

We further suppose that $f(x, y)$ possesses continuous first order partial derivative $f_y(x, y)$, with respect to y. *Under the supposition of the continuity of $f(x, y)$ and $f_y(x, y)$, it will be shown that*

$$\varphi'(y) = \int_a^b f_y(x, y)\, dx,$$

or in other words,

$$\frac{d}{dy}\int_a^b f(x, y)\, dx = \int_a^b \frac{\partial}{\partial y} f(x, y)\, dx,$$

i.e., the order of the two operations of differentiation and integration can be inverted.

We have

$$\varphi(y) = \int_a^b f(x, y)\, dx.$$

Let $\triangle y$ denote a change in y. We then have

$$\varphi(y+\triangle y) = \int_a^b f(x, y+\triangle y)\, dx.$$

$$\therefore \quad \varphi(y+\triangle y)-\varphi(y) = \int_a^b [f(x, y+\triangle y)-f(x, y)]\, dx.$$

Employing Lagrange's mean value theorem in Differential Calculus, we obtain

$$\varphi(y+\triangle y)-\varphi(y) = \int_a^b \triangle y\, f_y(x, y+\theta \triangle y)\, dx \quad 0 < \theta < 1$$

$$= \triangle y \int_a^b f_y(x, y+\theta \triangle y)\, dx$$

$$\Rightarrow \quad \frac{\varphi(y+\triangle y)-\varphi(y)}{\triangle y} = \int_a^b f_y(x, y+\theta \triangle y)\, dx$$

$$\Rightarrow \quad \frac{\varphi(y+\triangle y)-\varphi(y)}{\triangle y} = \int_a^b f_y(x, y)\, dx$$

$$+\int_a^b [f_y(x, y+\theta \triangle y)-f_y(x, y)]\, dx.$$

Let ε, be any pre-assigned positive number. There, then exists, by virtue of continuity of $f_y(x, y)$, a positive number δ such that

$$|f_y(x, y+\theta \triangle y)-f_y(x, y)| < \varepsilon, \qquad \ldots(2)$$

when $|\triangle y| < \delta$.

*It can be shown that, δ, is independent of x so that the inequality (2) holds for every value of x in $[a, b]$.

†Thus when $|\triangle y| \leqslant \delta$,

$$\left|\int_a^b [f_y(x, y+\theta \triangle y)-f_y(x, y)]\, dx\right| < \varepsilon(b-a),$$

i.e., $\left|\dfrac{\varphi(y+\triangle y)-\varphi(y)}{\triangle y} - \displaystyle\int_a^b f_y(x, y)\, dx\right| < \varepsilon(b-a)$ when $|\triangle y| \leqslant \delta$.

Thus

$$\lim_{\Delta y \to 0} \frac{\varphi(y+\triangle y)-\varphi(y)}{\triangle y} = \int_a^b f(x, y)\, dx,$$

i.e.,

$$\varphi'(y) = \int_a^b f_y(x, y)\, dx.$$

---

*This is a consequence of uniform continuity. The proof is beyond the scope of this book.

†It may be easily seen from the interpretation of definite integral as an area that if

$$|f(x)| \leqslant k,$$

then

$$\left|\int_a^b f(x)\, dx\right| \leqslant k(b-a).$$

Note. The reader may be inclined to deduce the required result by taking the limit, as $\Delta y \to 0$, in (1) It should, however, be seen that in this way, apart from assuming the existence of the limit, we overlook the real point of interest and difficulty.

Thus taking limits in (1), as $\Delta y \to 0$, we have

$$\varphi'(y) = \lim_{\Delta y \to 0} \int_a^b f_y(x, y+\theta \Delta y)\, dx$$

and if we now write

$$\lim_{\Delta y \to 0} \int_a^b f_y(x, y+\theta\, \Delta y)\, dx$$

$$= \int_a^b \lim_{\Delta y \to 0} f_y(x, y+\theta \Delta y)\, dx = \int_a^b f_y(x, y)\, dx$$

we do so on the assumption that *the limit of the integral is equal to the integral of the limit*, *i.e.*, the two operations of taking limit and integral are invertible. In fact the proof, as given, shows that this assumption is actually justifiable.

A·8. The case of variable limits of integration. Consider, now,

$$\varphi(y) = \int_{g(y)}^{h(y)} f(x, y)\, dx,$$

where the limits $g(y)$, $h(y)$ of integration are themselves functions of y. We shall assume that $g(y)$ and $h(y)$ possess continuous first order derivatives with respect to y and prove that

$$\varphi'(y) = \int_{g(y)}^{h(y)} f_y(x, y)\, dx + h'(y)\, f\,[h(y), y] - g'(y)\, f\,[g(y), y].$$

Lemma. We shall need the following result known as **Mean value theorem of the Integral Calculus.**

If $f(x)$ is continuous in an interval $[a, b]$, then there exists a number, ξ, between a and b such that

$$\int_a^b f(x)\, dx = (b-a)\, f(\xi).$$

From the interpretation of definite integral as an area or also from the integral as the limit of a sum, it follows that

$$m(b-a) \leqslant \int_a^b f(x)\, dx \leqslant M(b-a),$$

where m, M are the least and greatest values of $f(x)$ in $[a, b]$. Thus there exists a number, k, between m and M such that

$$\int_a^b f(x) = k\,(b-a).$$

As a continuous function assumes every value between its least and greatest values, it follows that there exists a number ξ such that $k = f(\xi)$. Thus

$$\int_a^b f(x)\,dx = (b-a)\,f(\xi).$$

We now prove the theorem.

We have

$$\varphi(y) = \int_{g(y)}^{h(y)} f(x, y)\,dx,$$

$$\varphi(y+\triangle y) = \int_{g(y+\triangle y)}^{h(y+\triangle y)} f(x, y+\triangle y)\,dx$$

$$= \int_{g(y+\triangle y)}^{g(y)} f(x, y+\triangle y)dx + \int_{g(y)}^{h(y)} f(x, y+\triangle y)dx$$

$$+ \int_{h(y)}^{h(y+\triangle y)} f(x, y+\triangle y)\,dx.$$

$$\therefore\ \varphi(y+\triangle y) - \varphi(y) = \int_{g(y)}^{h(y)} [f(x, y+\triangle y) - f(x, y)]dx$$

$$+ \int_{h(y)}^{h(y+\triangle y)} f(x, y+\triangle y)dx - \int_{g(y)}^{g(y+\triangle y)} f(x, y+\triangle y)dx.$$

Now by the mean value theorem of Integral Calculus,

$$\int_{h(y)}^{h(y+\triangle y)} f(x, y+\triangle y)dx = [h(y+\triangle y) - h(y)]\,f(\xi, y+\triangle y)$$

and $$\int_{g(y)}^{g(y+\triangle y)} f(x, y+\triangle y)\,dx = [g(y+\triangle y) - g(y)]\,f(\eta, y+\triangle y),$$

where ξ lies between $h(y)$, $h(y+\triangle y)$ and η between $g(y)$ and $g(y+\triangle y)$.

$$\therefore\ \varphi(y+\triangle y) - \varphi(y) = \int_{g(y)}^{h(y)} \triangle y\, f_y(x, y+\theta\triangle y)dx$$

$$+[h(y+\triangle y) - h(y)]\,f(\xi, y+\triangle y) - [g(y+\triangle y) - g(y)]\,f(\eta, y+\triangle y),$$

$$\frac{\varphi(y+\triangle y) - \varphi(y)}{\triangle y} = \int_{g(y)}^{h(y)} f_y(x, y+\theta\triangle y)dx$$

$$+ \frac{h(y+\triangle y) - h(y)}{\triangle y} f(\xi, y+\triangle y) - \frac{g(y+\triangle y) - g(y)}{\triangle y} f(\eta, y+\triangle y) \ldots (1)$$

As in the preceding § A·7,

$$\lim_{\triangle y \to 0} \int_{g(y)}^{h(y)} f_y(x, y+\theta\,\triangle y)\,dx = \int_{g(y)}^{h(y)} f_y(x, y)\,dx.$$

Thus proceeding to the limit, when $\triangle y \to 0$, we have, from (1),

$$\varphi'(y) = \int_{g(y)}^{h(y)} f_y(x, y)dx + h'(y) f[h(y), y] = g'(y) f[g(y), y].$$

A·9. Differentiation under Integral sign in the case of improper Integrals. The results obtained above may not be applicable in the case of improper integrals, and the question of the validity of the results to improper integrals requires further investigation. This however, is not within the scope of this book and whenever we shall deal with any improper integral in the following, it will be assumed that the necessary conditions for validity of the results are satisfied.

Examples

1. *Evaluate*

$$\int_0^{\frac{1}{2}\pi} \log(\alpha \cos^2\theta + \beta \sin^2\theta)\, d\theta. \qquad (\alpha > 0, \beta > 0)$$

Let $\varphi(\alpha, \beta) = \int_0^{\frac{1}{2}\pi} \log(\alpha \cos^2\theta + \beta \sin^2\theta)\, d\theta.$

$$\therefore\ \varphi_\alpha(\alpha, \beta) = \int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta}{\alpha \cos^2\theta + \beta \sin^2\theta}\, d\theta$$

$$= \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\alpha + \beta \tan^2\theta} \qquad \ldots(1)$$

$$= \int_0^{\infty} \frac{dt}{(1+t^2)(\alpha+\beta^2 t)}. \qquad \text{putting } \tan\theta = t$$

$$= \frac{1}{\alpha-\beta} \int_0^{\infty} \left(\frac{1}{1+t^2} - \frac{\beta}{\alpha+\beta t^2} \right) dt \qquad \text{if } \alpha \neq \beta$$

$$= \frac{1}{\alpha-\beta} \left| \tan^{-1} t - \frac{\beta}{\sqrt{\alpha\beta}} \tan^{-1} \sqrt{\beta} t \right|_0^{\infty}$$

$$= \frac{1}{\alpha-\beta} \left[\frac{\pi}{2} - \sqrt{\left(\frac{\beta}{\alpha}\right)} \frac{\pi}{2} \right] = \frac{\pi}{2\sqrt{\alpha}(\sqrt{\alpha}+\sqrt{\beta})}.$$

For $\beta = \alpha$, we have from (1),

$$\varphi_\alpha(\alpha, \beta) = \int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta}{\alpha}\, d\theta = \frac{\pi}{4\alpha}.$$

This shows that we have, without exception,

$$\varphi_\alpha(\alpha, \beta) = \frac{\pi}{2\sqrt{\alpha}(\sqrt{\alpha}+\sqrt{\beta})}.$$

Integrating *w.r.* to α, we obtain

$$\varphi(\alpha, \beta) = \pi \log(\sqrt{\alpha}+\sqrt{\beta}) + c, \qquad \ldots(2)$$

where, c, is independent of α.

Also we have

$$\varphi(\alpha, \beta) = \int_0^{\frac{1}{2}\pi} \log(\alpha \cos^2\theta + \beta \sin^2\theta)\, d\theta$$

$$= \int_0^{\frac{1}{2}\pi} \log\left[\alpha \cos^2\left(\frac{\pi}{2} - \theta\right) + \beta \sin^2\left(\frac{\pi}{2} - 2\right)\right] d\theta,$$

(§ 4·92, p. 102)

$$= \int_0^{\frac{1}{2}\pi} \log(\alpha \sin^2\theta + \beta \cos^2\theta) d\theta = \varphi(\beta, \alpha). \qquad \ldots(3)$$

In view of the equality $\varphi(\alpha, \beta) = \varphi(\beta, \alpha)$ we see that in (2), c. is free from β also. Thus, c, is an absolute constant.

Now $\varphi(1, 1) = \int_0^{\frac{1}{2}\pi} \log(\cos^2\theta + \sin^2\theta)\, dx = 0.$

Putting $\alpha = 1 = \beta$ in (2), we obtain

$$\theta = \varphi(1, 1) = \pi \log 2 + c \text{ or } c = -\pi \log 2$$

$$\therefore \qquad \varphi(\alpha, \beta) = \pi \log(\sqrt{\alpha} + \sqrt{\beta}) - \pi \log 2$$

$$= \pi \log\left[\tfrac{1}{2}(\sqrt{\alpha} + \sqrt{\beta})\right].$$

2. *Evaluate*

$$\int_0^a \frac{\log(1+ax)}{1+x^2}\, dx$$

and show that

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx = \frac{\pi}{8} \log 2.$$

We write

$$\varphi(a) = \int_0^a \frac{\log(1+ax)}{1+x^2} dx. \qquad \ldots(1)$$

$$\therefore \quad \varphi'(a) = \int_0^a \frac{\partial}{\partial a}\left(\frac{\log(1+ax)}{1+x^2}\right) dx + 1 . \frac{\log(1+a^2)}{1+a^2}$$

$$= \int_0^a \frac{x}{(1+ax)(1+x^2)} dx + \frac{\log(1+a^2)}{1+a}.$$

Throwing into partial fractions, we obtain

$$\frac{x}{(1+ax)(1+x^2)} = -\frac{a}{(1+a^2)(1+ax)} + \frac{x+a}{(1+a^2)(1+x^2)}.$$

$$\therefore \int_0^a \frac{x}{(1+ax)(1+x^2)} dx = -\frac{1}{1+a^2} \Big|\log(1+ax)\Big|_0^a$$

$$+ \left|\frac{1}{2(1+a^2)} \log(1+x^2)\right|_0^a + \left|\frac{a \tan^{-1} x}{1+a^2}\right|_0^a$$

$$= -\frac{1}{2(1+a^2)} \log(1+a^2) + \frac{a}{1+a^2} \tan^{-1} a$$

$$\therefore \quad \varphi'(a) = \frac{1}{2(1+a^2)} \log(1+a^2) + \frac{a}{1+a^2} \tan^{-1} a.$$

Integrating, we now get

$$\varphi(a) = \frac{1}{2}\left[\int \frac{1}{1+a^2} \log(1+a^2)\, da + \int \frac{a}{1+a^2} \tan^{-1} a\, da\right]$$

$$= \frac{1}{2}\left[\tan^{-1} a \log(1+a^2) - \int \frac{2a}{1+a^2} \tan^{-1} a\, da\right]$$

$$+ \int \frac{a}{1+a^2} \tan^{-1} a\, da + c,$$

where we have applied the rule of integration by parts to the first integral on the right and, c, is any arbitrary constant. Thus

$$\varphi(a) = \tfrac{1}{2} \tan^{-1} a \log(1+a^2) + c \qquad \ldots(2)$$

From (1), we see that $\varphi(0) = 0$. Putting $a = 0$ in (2), we get

$$c = 0.$$

$$\therefore \varphi(a) = \tfrac{1}{2} \tan^{-1} a \log(1+a^2).$$

From this, taking $a = 1$, we get

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx = \varphi(1) = \tfrac{1}{2} \tan^{-1} 1 \log 2 = \frac{\pi}{8} \log 2.$$

3. *Show that*

$$\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\, dx = \tfrac{1}{2}\pi \log(1+a) \text{ if } a \geqslant 0,$$

and find the value of the integral if $a < 0$.

We write

$$\varphi(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\, dx. \qquad \ldots(1)$$

Assuming the validity of the differentiation under integral sign, we obtain

$$\varphi'(a) = \int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)}\, dx$$

$$= \int_0^\infty \frac{1}{1-a^2}\left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right] dx$$

$$= \frac{1}{1-a^2} \left| \tan^{-1} x - a \tan^{-1} ax \right|_0^\infty$$

$$= \frac{1}{1-a^2}\left(\frac{\pi}{2} - a\,\frac{\pi}{2}\right) = \frac{\pi}{2(1+a)} \qquad \ldots(2)$$

Here, a, being positive we have

$$\lim_{x \to \infty} (\tan^{-1} ax) = \tfrac{1}{2}\pi.$$

It is easy to see that (2) is valid for $a = 0$ also.

Integrating (2) *w.r.* to a, we get

$$\varphi(a) = \frac{\pi}{2} \log(1+a)+c \qquad \ldots(3)$$

where, c, is the arbitrary constant.

Also from (1)

$$\varphi(0) = 0$$

so that putting $a = 0$ in (3), we get

$$c = 0$$

$$\therefore \qquad \varphi(a) = \tfrac{1}{2}\pi \log(1+a)$$

Suppose now that, a is negative. We have

$$\varphi'(a) = \frac{1}{1-a^2}\Big| \tan^{-1} x - a \tan^{-1} ax \Big|_0^\infty$$

$$= \frac{1}{1-a^2}\left[\frac{\pi}{2} - a\left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{2(1-a)}. \qquad \ldots(4)$$

for, a, being negative,

$$\lim_{x \to \infty} (\tan^{-1} ax) = -\tfrac{1}{2}\pi.$$

Integrating (4), we get

$$\varphi(a) = -\frac{\pi}{2} \log(1-a)+c.$$

As before it may be shown that $c = 0$ so that we have

$$\varphi(a) = -\frac{\pi}{2} \log(1-a).$$

4. *Evaluate*

$$\int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x}\, dx, \text{ where } \alpha \geqslant 0,$$

and deduce that

$$\int_0^\infty \frac{\sin \beta x}{x}\, dx = \begin{cases} \tfrac{1}{2}\pi, & \text{if } \beta > 0, \\ 0, & \text{if } \beta = 0, \\ -\tfrac{1}{2}\pi, & \text{if } \beta < 0. \end{cases}$$

We write

$$\varphi(\alpha, \beta) = \int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x}\, dx \qquad \ldots(1)$$

Assuming the validity of differentiation under integral sign, we have, on differentiating *w. r.* to β

$$\varphi_\beta(\alpha, \beta) = \int_0^\infty e^{-\alpha x} \cos \beta x \, dx$$

$$= \frac{\alpha}{\alpha^2+\beta^2}, \text{ if } \alpha > 0. \qquad \text{[Refer Ex. 2, page 45]}$$

Integrating *w. r.* to β, we get

$$\varphi(\alpha, \beta) = \tan^{-1} \frac{\beta}{\alpha} + c, \qquad \ldots(2)$$

where, c, is a constant. From (1), we have

$$\varphi(\alpha, 0) = 0. \qquad \ldots(3)$$

so that putting $\beta = 0$ in (2), we obtain $c = 0$. Thus

$$\varphi(\alpha, \beta) = \tan^{-1} \frac{\beta}{\alpha}, \qquad \ldots(4)$$

where $\alpha > 0$.

Also we assume that $\varphi(\alpha, \beta)$ is a continuous function of α for $\alpha \geqslant 0$.

We have, from (1),

$$\varphi(0, \beta) = \int_0^\infty \frac{\sin \beta x}{x} \, dx,$$

and from (4)

$$\lim_{\alpha \to 0} \varphi(0, \beta) = \lim_{\alpha \to 0} \left(\tan^{-1} \frac{\beta}{\alpha} \right) = \begin{cases} \pi/2, & \text{if } \beta > 0, \\ 0, & \text{if } \beta = 0, \\ -\pi/2, & \text{if } \beta < 0. \end{cases}$$

Also, because of continuity,

$$\lim_{\alpha \to 0} \varphi(\alpha, \beta) = \varphi(0, \beta).$$

$$\therefore \quad \int_0^\infty \frac{\sin \beta x}{x} \, dx = \begin{cases} \pi/2, & \text{if } \beta > 0, \\ 0, & \text{if } \beta = 0, \\ -\pi/2, & \text{if } \beta < 0. \end{cases}$$

In particular, we have

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}.$$

5. *Assuming the validity of differentiation under integral sign, show that*

$$\int_0^\infty e^{-x^2} \cos \alpha x \, dx = \tfrac{1}{2}\sqrt{\pi} e^{-\frac{1}{4}\alpha^2}$$

We write

$$\varphi(\alpha) = \int_0^\infty e^{-x^2} \cos \alpha x \, dx. \qquad \ldots(1)$$

$$\therefore \quad \varphi'(\alpha) = -\int_0^\infty xe^{-x^2} \sin \alpha x \, dx.$$

Integrating by parts, we have

$$\varphi'(\alpha) = \tfrac{1}{2} e^{-x^2} \sin \alpha x \Big|_0^\infty - \frac{\alpha}{2} \int_0^\infty e^{-x^2} \cos \alpha x \, dx$$

$$= 0 - \frac{\alpha}{2} \varphi(\alpha) = \frac{-\alpha}{2} \varphi(\alpha),$$

$$\Rightarrow \qquad \frac{\varphi'(\alpha)}{\varphi(\alpha)} = -\frac{\alpha}{2}.$$

Integrating, we get

$$\log \varphi(\alpha) = -\tfrac{1}{4}\alpha^2 + c_1 \Rightarrow \varphi(\alpha) = ce^{-\frac{1}{4}\alpha^2}. \qquad \ldots(2)$$

Putting $\alpha = 0$ in (1), we get

$$\varphi(0) = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

[Refer § A·6, Cor. 2, p. 327] ...(3)

Putting $\alpha = 0$ in (2), we get

$$\varphi(0) = c. \qquad \ldots(4)$$

$$\therefore \quad \varphi(\alpha) = \tfrac{1}{2}\sqrt{\pi} e^{-\frac{1}{4}\alpha^2}$$

6. *Show that*

$$\int_0^\infty \frac{\tan^{-1} \alpha x \tan^{-1} \beta x}{x^2} dx = \frac{\pi}{2} \log \left[\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha \beta^\beta} \right].$$

We write

$$\varphi(\alpha, \beta) = \int_0^\infty \frac{\tan^{-1} \alpha x \tan^{-1} \beta x}{x^2} dx \qquad \ldots(1)$$

Assuming the validity of differentiation under integral sign, we have

$$\varphi_\alpha(\alpha, \beta) = \int_0^\infty \frac{\tan^{-1} \beta x}{x(1+\alpha^2 x^2)} dx. \qquad \ldots(2)$$

$$\varphi_{\beta\alpha}(\alpha, \beta) = \int_0^\infty \frac{1}{(1+\beta^2 x^2)(1+\alpha^2 x^2)} dx$$

$$= \frac{1}{\alpha^2 - \beta^2} \int_0^\infty \left(\frac{\alpha^2}{1+\alpha^2 x^2} - \frac{\beta^2}{1+\beta^2 x^2} \right) dx, \qquad \text{if } \alpha \neq \beta$$

$$= \frac{1}{\alpha^2 - \beta^2} \Big| \alpha \tan^{-1} \alpha x - \beta \tan^{-1} \beta x \Big|_0^\infty$$

$$= \frac{\pi}{2(\alpha+\beta)}; \qquad \ldots(3)$$

α, β being assumed positive.

It is easy to show that (3) remains valid even for $\alpha = \beta$.

Integrating (3) w. r. to β, we get

$$\varphi_\alpha(\alpha, \beta) = \frac{\pi}{2} \log(\alpha+\beta) + f(\alpha), \quad \ldots(4)$$

where $f(\alpha)$ is an arbitrary function of α.

Now, from (2),

$$\varphi_\alpha(\alpha, 0) = 0. \quad \ldots(5)$$

From (4) and (5), we have

$$0 = \frac{\pi}{2} \log \alpha + f(\alpha),$$

$$\Rightarrow \quad f(\alpha) = -\frac{\pi}{2} \log \alpha.$$

$$\therefore \varphi_\alpha(\alpha, \beta) = \frac{\pi}{2} \log(\alpha+\beta) - \frac{\pi}{2} \log \alpha \quad \ldots(6)$$

We could similarly obtain

$$\varphi_\beta(\alpha, \beta) = \frac{\pi}{2} \log(\alpha+\beta) - \frac{\pi}{2} \log \beta \quad \ldots(7)$$

Integrating, (6), *w.r.* to α, we obtain

$$\varphi(\alpha, \beta) = \frac{\pi}{2}[(\alpha+\beta) \log(\alpha+\beta) - (\alpha+\beta)]$$
$$-\frac{\pi}{2}(\alpha \log \alpha - \alpha) + g(\beta), \quad \ldots(8)$$

$g(\beta)$ being any arbitrary function of β.

From (8) we have

$$\varphi_\beta(\alpha, \beta) = \frac{\pi}{2} \log(\alpha+\beta) + g'(\beta). \quad \ldots(9)$$

From (7) and (9), we have

$$g'(\beta) = -\frac{\pi}{2} \log \beta$$

$$\therefore \quad g(\beta) = -\frac{\pi}{2}[\beta \log \beta - \beta] + c. \quad \ldots(10)$$

From (8) and (1),

$$0 = \varphi(\alpha, 0) = g(0). \quad \ldots(11)$$

From (10) and (11), we obtain $c = 0$.

$$\therefore \quad \varphi(\alpha, \beta) = \frac{\pi}{2}[(\alpha+\beta) \log(\alpha+\beta) - (\alpha+\beta)]$$
$$-\frac{\pi}{2}(\alpha \log \alpha - \alpha) - \frac{\pi}{2}(\beta \log \beta - \beta)$$

$$= \frac{\pi}{2}[(\alpha+\beta)\log(\alpha+\beta)-\alpha\log\alpha-\beta\log\beta]$$

$$= \frac{\pi}{2}\log\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^{\alpha}\,\beta^{\beta}}.$$

Exercises

(For improper integrals in the following the validity of differentiation under integral sign may always be assumed).

1. Find the value of

$$\int_0^\pi \frac{dx}{a+b\cos x}, \qquad a > 0, \ |b| < a,$$

and deduce that

$$\int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}} \text{ and } \int_0^\pi \frac{\cos x\,dx}{(a+b\cos x)^2} = -\frac{\pi b}{(a^2-b^2)^{3/2}}.$$

2. Starting from

$$\int_0^\infty e^{-ax}dx = \frac{1}{a},\ a > 0;$$

$$\int_0^1 x^n\,dx = \frac{1}{n+1},\ n > -1,$$

deduce that

$$\int_0^\infty x^m e^{-ax}\,dx = \frac{m!}{a^{m+1}},$$

and

$$\int_0^1 x^n(\log x)^m\,dx = \frac{(-1)^m\,m!}{(n+1)^{m+1}},$$

where, m, is any positive integer.

3. Starting from a suitable integral, show that

$$\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a}\tan^{-1}\frac{x}{a}+\frac{x}{2a^2(x^2+a^2)}.$$

4. Differentiating under integral sign the integrals

$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}},\ \int_0^\infty \frac{dx}{x^2+a} = \frac{\pi}{2\sqrt{2}};\ a > 0.$$

show that

$$\int_0^\infty x^{2n}e^{-ax^2}dx = \frac{\sqrt{\pi}}{2}\cdot\frac{1\cdot3\ldots(2n-1)}{2^n a^{n+\frac{1}{2}}}.$$

$$\int_0^\infty \frac{dx}{(x^2+a)^{n+1}} = \frac{\pi}{2}\cdot\frac{1\cdot3\ldots(2n-1)}{2^n\,n!\,a^{n+\frac{1}{2}}}.$$

5. Show that for $y > 0$.

$$\int_0^\infty e^{-xy} \frac{\sin x}{x}\, dx = \frac{\pi}{2} - \tan^{-1} y.$$

6. If $|a| \leqslant 1$, show that

$$\int_0^\pi \log(1+a\cos x)dx = \pi \log\left[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1-a^2)}\right].$$

7. If $|a| < 1$, prove that

(i) $$\int_0^\pi \frac{\log(1+a\cos x)}{\cos x}\, dx = \pi \sin^{-1} a.$$

(ii) $$\int_0^{\frac{1}{2}\pi} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\, dx = \frac{\pi^2-4\alpha^2}{8}.$$

8. Show that

$$\int_0^{\frac{1}{2}\pi} \log(1-x^2\cos^2\theta)d\theta = \pi\log\{1+\sqrt{(1-x^2)}\} - \pi\log 2,$$

if $x^2 \leqslant 1$.

9. Evaluate

$$\int_0^{\frac{1}{2}\pi} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\operatorname{cosec}\theta\, d\theta. \quad (a > b).$$

10. Evaluate

$$I(y) = \int_0^{\frac{1}{2}\pi} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\, dx$$

by showing that

$$I'(y) = \frac{\pi}{2\sqrt{(1+y)}}.$$

11. Show that

$$y = \frac{1}{k}\int_0^x f(t)\sin k(x-t)\, dt,$$

satisfies the differential equation,

$$\frac{d^2y}{dx^2} + k^2 y = f(x),$$

where, k, is a constant.

12. Let

$$u = \int_0^\infty e^{-x\cos\theta} x^{n-1}\sin(x\sin\theta)\, dx,$$

$$v = \int_0^\infty e^{-x\cos\theta} x^{n-1}\cos(x\sin\theta)\, dx.$$

Prove that

$$\frac{du}{d\theta} = -nv, \frac{dv}{d\theta} = nu,$$

and $$\frac{d^2u}{d\theta^2}+n^2u = \theta, \frac{d^2v}{d\theta^2}+n^2v = \theta.$$

Deduce that

$u = \Gamma(n) \cos n\theta$, $v = \Gamma(n) \sin n\theta$.

13. Prove that

$$\int_0^\infty e^{-ax} x^{m-1} \cos bx \, dx = \frac{\Gamma(m) \cos m\theta}{r^m},$$

where $r^2 = a^2+b^2$ and $\theta = \tan^{-1} (a/b)$.

14. Show that

$$\int_0^\infty \frac{\cos x}{\sqrt{x}} \, dx = \sqrt{\frac{\pi}{2}} = \int_0^\infty \frac{\sin x}{\sqrt{x}} \, dx.$$

15. Prove that

$$\int_{\frac{1}{2}\pi-\alpha}^{\frac{1}{2}\pi} \sin \theta \cos^{-1} (\cos \alpha \operatorname{cosec} \theta) \, d\theta = \frac{\pi}{2}(1-\cos \alpha).$$

16. Obtain the first order differential equation satisfied by

$$\varphi(y) = \int_0^\infty e^{-x^2} \sin 2yx \, dx,$$

and hence show that

$$\varphi(y) = \int_0^y e^{x^2-y^2} \, dx.$$

Answers

9. $\pi \sin^{-1} (b/a)$.

10. $\pi[\sqrt{(1+y)}-1]$.

16. $\dfrac{d\varphi}{dy} +2y \, \varphi(y) = 1$.