STATICS

(Formerly known as *Intermediate Statics*)

Revised for Three-Year Degree Course

(Pass and Honours)

BY

B. C. DAS, M. Sc.

PROFESSOR OF MATHEMATICS, PRESIDENCY COLLEGE,
CALCUTTA : (RETD.),
LECTURER IN APPLIED MATHEMATICS, CALCUTTA UNIVERSITY

AND

B. N. MUKHERJEE, M. A.

Premchand Roychand Scholar

PROFESSOR OF MATHEMATICS,
SCOTTISH CHURCH COLLEGE, CALCUTTA (RETD.)

U. N. DHUR & SONS, PRIVATE LTD.
BOOKSELLERS & PUBLISHERS
15, BANKIM CHATTERJEE STREET, CALCUTTA 12

Rs. 10·00 only

Published by
DWIJENDRANATH DHUR, LL.B.
For U. N. DHUR & SONS, PRIVATE LTD.,
15, Bankim Chatterjee Street, Calcutta 12

Printed by
TRIDIBESH BASU, B.A.
THE K. P. BASU PRINTING WORKS,
11, Mohendra Gossain Lane, Calcutta 6

PREFACE TO THE FIRST EDITION

THIS book, as its name indicates, is meant to be a text-book for the Intermediate students, both Arts and Science, of the Indian Universities, and various Education Boards. Regarding the subject-matter, we have tried to make the exposition clear and concise without going into unnecessary details. Varied types of examples have been worked out by way of illustrations in each chapter and the examples set for exercise have been carefully selected and properly graded.

Questions of the University of Calcutta and some other Universities are given at the end, to give the students an idea of the standard of the examination.

It is hoped that the book will meet the requirements of those for whom it is intended and we shall deem our labour amply rewarded if the book is found to be a suitable text-book both by the teachers and the students.

Any criticism, correction and suggestion towards improvement from teachers and students will be thankfully received.

CALCUTTA
June, 1947

B. C. D.
B. N. M.

INDIAN UNIVERSITIES

SYLLABUS FOR STATICS

Three-Year Degree Course

PASS :

Forces (various types) : Geometrical representation of a force, parallelogram of forces. composition and resolution of forces. Conditions of equilibrium of coplanar forces meeting at a point. Triangle of forces and polygon of forces Like and unlike parallel forces. Moment of forces, theorems on moments. Couples, General conditions for equilibrium of coplanar forces.

Centre of Gravity : Centre of a system of parallel forces. Centre of mass and centre of gravity of a system of particles, of a rigid body, a thin rod of constant or variable density, uniform wire bent in the form of an arc of a circle, a parabola ; homogeneous lamina in the form of a triangle, a parallelogram, a circle, a quadrant of a circle and of an ellipse, portion of a parabola bounded by a double ordinate ; a uniform hemispherical and conical surface, a homogeneous solid hemisphere and a right circular cone. Simple problems involving the above. (Use of Calculus advised)

Friction : Laws of statical and limiting friction. Equilibrium of a particle on a rough plane, angle of friction. Applications to simple problems.

Simple Machines : Levers, system of pulleys. Mechanical advantage. Velocity ratio.

HONOURS :

Force : Nature and representation. Forces acting on a particle ; theorems on composition and resolution, analytical method, conditions for equilibrium.

Forces acting in one plane : Parallel forces, moments, couples, equilibrium of three forces, general conditions for equilibrium.

Centre of mass . Centre of a system of parallel forces. Centre of mass and centre of gravity of a system of particles and of a system of rigid bodies. C.G. of a uniform rod, uniform lamina in the form of a triangle, a parallelogram, a quadrilateral. Related problems : C.G. of a part and of a complete body. Concept of stability of equilibrium of a body under gravity.

Friction : Laws of statical and limiting friction. Equilibrium of a particle on a rough plane. Angle of friction. Applications to simple problems.

Machines : Levers ; balance, systems of pulleys, screws ; mechanical advantage and velocity ratio.

SYLLABUS FOR STATICS OF GAUHATI AND DIBRUGARH UNIVERSITIES

PASS :

Mass, force, parallelogram of forces; composition and resolution of a force; Triangle of forces and Polygon of forces; Lami's theorem; parallel forces; moments; couples; general conditions of equilibrium of coplanar forces; centre of gravity—centre of a system of parallel forces; centre of mass and centre of gravity of a system of particles, of a rigid body, a thin rod of constant or variable density, uniform wire bent in the form of an arc of a circle, a parabola; homogeneous lamina in the form of a triangle, a parallelogram, a circle, a quadrant of a circle and of an ellipse.

Laws of statical and limiting friction; equilibrium of a particle on a rough plane; angle of friction. Applications to simple problems.

Levers, system of pulleys; mechanical advantage, velocity ratio.

HONOURS :

Fuller study of pass course topics only (Calculus to be used where possible).

CONTENTS

CHAP.				PAGE
I.	Introduction	...	...	1
II.	Composition and Resolution of Forces		...	8
III.	Equilibrium of Concurrent Forces		...	33
IV.	Parallel Forces	...	...	51
V.	Moment of a Force	...	...	62
VI.	Couples	...	...	83
VII.	Reduction of Coplanar Forces		...	100
VIII.	Equilibrium of Coplanar Forces		...	118
IX.	Friction	...	...	144
X.	Centre of Gravity	...	...	169
	Centre of Gravity (Contd.)		...	212
X(A).	Constrained Body and Stability		...	248
XI.	Work and Power	...	...	255
XII.	Machines	...	...	262
	Appendix—A	...	...	303
	Appendix—B	...	...	309
	University Questions	...	...	311

STATICS

CHAPTER I

INTRODUCTION

1·1. Definitions.

Matter is anything that occupies space and can be perceived by our senses.

A **body** is a portion of matter limited in all directions, having a definite shape and size, and occupying some definite space.

A **rigid body** is one whose size and shape do not alter when acted on by any forces whatsoever, so that the different parts of it keep invariable positions with respect to one another.

In nature there is no body which is perfectly rigid, for however hard a body may be, it will be deformed more or less when the acting forces are sufficiently large. But ordinary solid bodies like stone, wood, iron etc. when acted on by finite forces yield so slightly, that the deformation is not in general appreciable to the eye, and thus for practical purposes they may be treated as rigid. In Statics and Dynamics we are mainly concerned with such rigid bodies.

A **force** is that which changes, or tends to change, the state of rest, or of uniform motion of a body.

Statics deals with bodies at rest when acted on by forces, or more properly, with the relations between the forces which acting on a rigid body (or a system of bodies) keep it at rest.

When a number of forces acting on a body keeps it at rest, the forces are said to be in **equilibrium**.

1·2. Classification of Forces.

The forces that we meet with in course of our subject may generally be divided into three types :

(1) Forces of the nature of *thrusts* or *tensions*, *i.e.*, push or pull applied through actual material contact, *e.g.*, by a rod or string etc.

(2) *Attraction* or *repulsion* between two bodies, which are of the nature of action at a distance, *e.g.*, earth's gravitation etc.

(3) Forces like *reaction* or *friction* which are of a sort of passive resistance, coming into existence only when necessary, and adjusting themselves (within a certain range) to be of such magnitude and direction as are just required to maintain equilibrium.

1·3. On Some Special Forces.

(i) Weight.

Weight of a body is the force with which the earth attracts the body. The direction of this force is *vertical*.

It is shown in Dynamics (Arts. 5·1 and 6·6) that the earth attracts everybody to itself with a force which is proportional to the mass of the body, *i.e.*, *the quantity of matter in a body.*

The unit of mass in British (F.P.S.) system is one *pound* (lb.), whereas in C.G.S. system it is one *gramme*. [See *Dynamics, Art.* 1·3]

The amount of force exerted by the earth on a body of mass one pound, *i.e.*, the *weight of one pound* (briefly, 1 lb. wt) is usually used in Statics as the **unit for measurement of magnitude of forces** in F. P. S. system.

Similarly in C.G.S. system, the unit used is the *weight of one gramme.*

Strictly speaking, the forces of attraction on the same body varies slightly from place to place on the surface of the earth, which is nearly but not exactly, a sphere. Accordingly, the units above mentioned, 1 lb. wt. and 1 gm. wt., are not fixed. [See *Dynamics*, § *6 6 & 6·8*]

But as in this Elementary Statics we shall not have occasions to compare forces at different places on earth, we shall neglect this small variation in the units.

In practice, for brevity, we shall speak of a force measuring 20 lbs., or 50 gms., though more accurate expressions would be a force of 20 lbs. wt , or 50 gms. wt.

(ii) Reaction.

When one body rests in contact with another body, pressing against it, it experiences a force at the point of contact which is called the *reaction*, exerted by the second body on the first.*

For example, when a heavy body, (say a book), rests on a horizontal table, the weight of the body which would cause it to fall down to the earth has got its effect nullified due to the presence of the table, which does not allow the body to penetrate through it. Thus the table exerts a force on the body neutralising its weight. This is the reaction of the table. As the weight of the body is vertically downwards, the reaction of the table neutralising its effect must be upwards.

As another example, when a ladder standing on a horizontal floor is leading against a vertical wall, it experiences forces of reaction at its points of contact with the floor as well as with the wall. These two reactions, along with the weight of the ladder, keep the ladder at rest.

Now it is a common experience that if a body be placed in contact with a very smooth surface (*e.g.*, a highly polished table), and is urged with any force to slide over it, it experiences very little resistance tangentially, but the surface, (assumed rigid), does not allow the body to penetrate normally through it. The reaction on such a body is therefore normal to the surface.

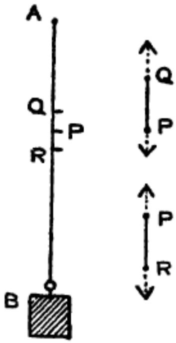
* From Newton's third law of motion [*Dynamics*, § *6·1 & 6·11*] the second body also experiences an equal and opposite force exerted by the first on it, which we may call *action*.

In fact a **perfectly smooth** *surface is one whose reaction on any body in contact with it is along the common normal to the two surfaces at their point of contact.*

The reaction of a rough surface however, on any body pressing against it, is not necessarily along the common normal. [See § *9·1* and *9·3*]

(iii) Tension.

When a string employed to connect two bodies (or two points of a material system) is stretched, for example, when one extremity is tied to a fixed point, and at the other a heavy weight is suspended, the fibres of the string become subject to a certain pull throughout its length, which undergoes by the name of *tension*, and which, if increased beyond a certain limit, will cause the string to break. This tension is a force which at any point P of the string is conceived to be acting in either of the two opposite senses along the string.

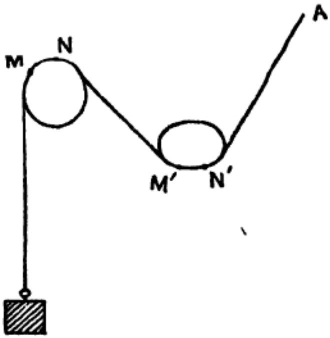


For, considering any small element PQ of the string, this is stretched by forces set up in the fibres pulling it at Q upwards, and at P downwards. Again, considering the element PR, this is stretched by a force pulling it at P upwards and at R downwards. Thus at P the tension acts in either direction, downwards on the portion above it, and upwards on the portion below it. Similar is the case at every point.

If the string be light, the tension is the same throughout its length, and is unchanged even when a portion of the string passes over a smooth surface, say a smooth peg, or pulley.

For, considering the element PQ as before, as the string is of negligible weight, the only forces under which this element is at rest are the two tensions at its extremities Q and P which must accordingly balance one another. Thus tension at Q is equal to that at P. Again, considering the equilibrium of the element PR, the tension at P is equal to that at R. Proceeding in this manner, the tension is the same throughout the length of the string.

Again, when a portion of the string passes over a smooth pulley (or a smooth surface), considering an element MN or $M'N'$ which is



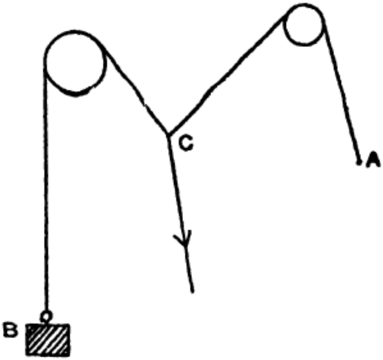


Fig. (i) Fig. (ii)

in contact with the surface, the reaction of the smooth surface is along the normal, and this has no effect in the tangential direction. Hence the only tangential forces, namely the tensions at the extremities M and N must balance one another, and accordingly must be equal and opposite. Thus the magnitude of the tension continues to be the same throughout the string even when it passes over smooth surfaces as in Fig. (i) above.

If however any point C of the string is knotted to other string (or to any other body) as in Fig. (ii), we must regard its continuity as broken, and *the tension will not be the same in the two portions on the two sides of the knot*, though for each separate portion it continues to have a constant value throughout.

1·4. Geometrical representation of a force by a straight line.

A force has a given magnitude, and acts at a particular point of a body in a definite direction ; in other words, it has a definite magnitude, direction, and point of application, the two latter giving the line of action of the force.

Now a straight line has also a length and a direction, and can be drawn through a particular point, thus having a definite position. Thus, a straight line drawn through the point of application of a force can very aptly represent the force completely in magnitude, direction and position, the magnitude of the force being represented on a suitably chosen scale by the length of the line drawn, the direction of the line representing the direction of the force, the *sense* being indicated by an arrow-head on the line, the extremity of the line being at the point of application of the force.

A parallel line of equal length drawn anywhere with an arrow-head indicating the sense will represent the same force equally well in magnitude and direction, but not in position.

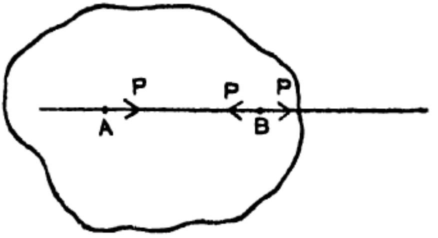
Note. A quantity having magnitude and direction (in a definite sense) is a **vector** quantity, and such quantities are geometrically very aptly represented both in magnitude and direction by straight lines as explained above. [See *Dynamics*, § 2·2] For the sake of convenience, a *force represented in magnitude, direction and sense by AB will usually be denoted by* $\overrightarrow{\mathbf{AB}}$ or $\overline{\mathbf{AB}}$.

1·5. The principle of Transmissibility of a Force.

The effect of a force acting on a rigid body at any point is unaltered if its point of application is transferred to any other point on its line of action, provided the two points are rigidly connected to one another.

This principle follows as an immediate consequence of the **conditions of equilibrium of two forces** acting on a body, which is more or less axiomatic, namely that two equal forces acting along the same line on a rigid body in opposite sense produce equilibrium and will have no effect on the body. In fact *equal forces* are defined as such when they satisfy the above condition.

Thus, P being a force acting at A along AB on a rigid body, if we introduce two equal and opposite forces at B each equal to P along BA and AB, and two latter, being in equilibrium, will neutralise one another and will have no effect on the original force.



Now P at A, and the opposite P at B along the same line, produce equilibrium, and we are left with a force P at B in the sense AB which is thus equivalent to the original force P at A. Hence follows the principle of transmissibility of a force as enunciated above.

CHAPTER II

COMPOSITION AND RESOLUTION OF FORCES

2·1. Resultant and Components.

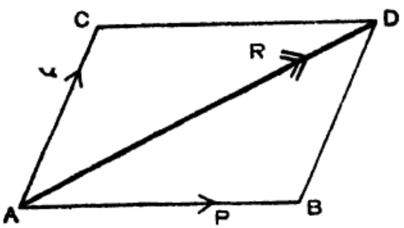
If two or more forces act simultaneously on a rigid body, and if a single force can be obtained whose effect on the body is the same as the joint effect of the given forces (*i.e.*, produces exactly the same motion of the body), then this single force is known as the *resultant* of the given forces, and the given forces in their turn are called the *components* of the single resultant force.

It follows from above that if on a body acted on by two or more forces a force equal and opposite to their resultant is applied, the whole system is in equilibrium and the body remains at rest.

Conversely, if a set of forces acting on a body be in equilibrium, then each force is equal and opposite to the resultant of the other forces.

2·2. Parallelogram of Forces.

If two forces acting at a point on a body be represented in magnitude, direction and sense by the two adjacent sides of a parallelogram drawn from an angular point, then their resultant is represented in magnitude, direction and sense by the diagonal of the parallelogram drawn from that point.



Thus, if two forces P and Q, acting on a body at a point A, be represented (on a chosen scale) in magnitude, direction and sense by the two straight lines AB and AC, both drawn *from A*, and the parallelogram $ABCD$ be completed with AB and AC

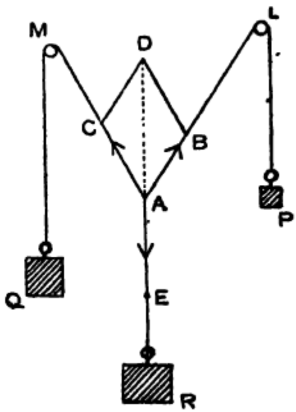
as adjacent sides, then the resultant force (say R) will be represented in magnitude, direction and sense by the diagonal AD drawn *from* A.

Note. If both P and Q be *towards* A, represented by BA and CA, the resultant will be represented by DA *towards* A. If however P is along AB and Q along CA, the resultant will not be along AD or DA, nor represented by it.

A formal theoretical proof of the above theorem is reserved for a later consideration.* Below we give an experimental verification of the principle.

Experimental Verification

Any three weights P, Q, R (of which no one should exceed the sum of the other two) are tied at the extremities of three light flexible strings, the other extremities of which are knotted at a common point A. Two of these strings are placed over two smooth pegs, or two light smooth pulleys (say L and M), fixed against a vertical wall or blackboard, the knot being between the pulleys, and the whole system is allowed to come to the equilibrium position as in the above figure.



Now at A there are tensions acting along the three strings which keep A at rest, and are therefore in equilibrium. These tensions being constant along the respective strings, and supporting the weights P, Q, R at the other extremities, have got their magnitudes equal to P, Q and R respectively along AL, AM and AR.

* See *Appendix*.

Now, on the black-board, along AL, AM and AR, we draw (on any chosen scale) straight lines AB, AC and AE to represent, in magnitude and direction, the forces P, Q and R respectively. The parallelogram $ABDC$ is completed and the diagonal AD joined. *It will be found that AD and AE are in the same straight line and equal in magnitude.*

Since P, Q and R are in equilibrium, R is equal and opposite to the resultant of P and Q. But R is represented by AE, and AD is found equal and opposite to it experimentally, as stated above. Thus, the resultant of P and Q is represented by AD.

By altering P, Q and R in any manner (with the restriction that no one is greater than the sum of the other two), and repeating the experiment, the same result will be verified in every case.

This proves the parallelogram law for finding the resultant of two forces acting at a point.

2·3. Analytical expression for the resultant of two given forces.

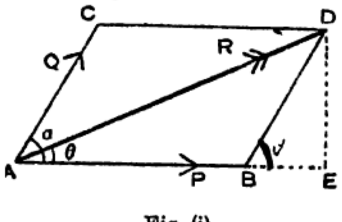


Fig. (i)

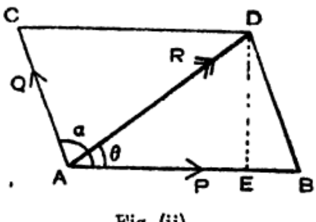


Fig. (ii)

Let P and Q be two given forces acting at the point A at an angle a, and let them be represented by AB and AC respectively. Complete the parallelogram $ABDC$ and join the diagonal AD, which then, by parallelogram of forces, represents the resultant R. Let $\angle DAB = \theta$, which will give the direction of the resultant. Now draw DE perpendicular upon AB, produced if necessary, as in Fig. (i). Then in the right-angled triangle DBE,

$$DE = DB \sin DBE = Q \sin a$$

$$[\text{ In Fig. (ii), } \sin DBE = \sin (180° - a) = \sin a \,]$$

Also, $AE = AB + BE = P + Q\cos\alpha$

[In Fig. (ii), $AE = AB - BE = AB - BD\cos DBE$
$= P - Q\cos(180^\circ - \alpha) = P + Q\cos\alpha$]

Thus, $AD^2 = AE^2 + DE^2$ gives

$$R^2 = (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$$
$$= P^2 + 2PQ\cos\alpha + Q^2.$$

Also, $\tan\theta = \dfrac{DE}{AE} = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$.

Hence, $$\mathbf{R = \sqrt{P^2 + 2PQ\cos\alpha + Q^2}}$$

and $$\boldsymbol{\theta = \tan^{-1}\frac{Q\sin\alpha}{P + Q\cos\alpha}},$$

giving the magnitude and direction of the resultant.

Cor. 1. If $\alpha = 0$, $R = P + Q$ and if $\alpha = \pi$, $R = P - Q$. ($P > Q$)

Hence, *the resultant of two given forces acting along the same line is their algebraic sum.*

Cor. 2. Two forces P and Q acting at a point being given in magnitude, *their resultant is greatest* when $\cos\alpha$ is *greatest i.e.*, $\cos\alpha = 1$ *i.e.*, $\alpha = 0$; and the *greatest resultant* is $P + Q$; and the *resultant* is *least* when $\cos\alpha$ is *least i.e.*, $\cos\alpha = -1$ *i.e.*, $\alpha = \pi$ and the *least resultant is* $P - Q$.

Cor 3. When $P = Q$, it is easily seen that

$$R = 2P\cos\tfrac{1}{2}\alpha \text{ and } \theta = \tfrac{1}{2}\alpha.$$

Thus, *the resultant of two equal forces* P, P *at an angle* α, *is* $2P\cos\frac{1}{2}\alpha$, *in a direction bisecting the angle between them.*

If $\alpha = 120^\circ$, $R = 2P\cos 60^\circ = P$.

Hence, *the resultant of two equal forces acting at an angle* 120°, *is equal to each of the forces.* [$R = P = Q$]

Cor. 4. If $\alpha = 90^\circ$, $R = \sqrt{P^2 + Q^2}$, $\theta = \tan^{-1} Q/P$.

Cor. 5. If $P > Q$, the resultant is nearer to P.

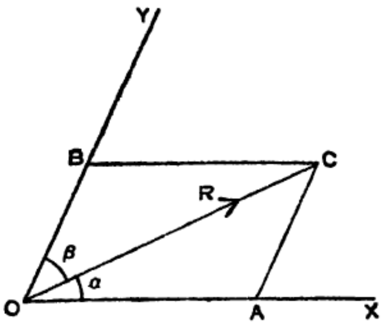
2·4. Breaking up a given force into two components.

A given force may be resolved in two components in an infinite number of ways, for by parallelogram of forces, if with the straight line representing the given force as diagonal we construct *any* parallelogram, the two adjacent sides of this parallelogram will represent the two component forces having the given force as their resultant.

Again, if we want the component of a given force, in a given direction at any inclination to it, the component is not determinable, in as much as the direction of the other component may be chosen to be anyone, and the parallelogram constructed with the given force as diagonal.

If however, with a given force, both the directions are definitely mentioned in which we are required to break it up into components, these components can be determined.

Let OC represent the given force R, and OX and OY two given directions making angles α and β respectively with OC, on opposite sides of it, along which we are to find the components of R.



Complete the parallelogram $OACB$ with diagonal OC, and sides along OX and OY. Then, by parallelogram of forces, OA and OB represent the required components P and Q, having R as their resultant.

Now from the triangle OAC, by Trigonometry,

$$\frac{OA}{\sin OCA} = \frac{AC}{\sin COA} = \frac{OC}{\sin OAC},$$

i.e., $$\frac{OA}{\sin \beta} = \frac{OB}{\sin \alpha} = \frac{OC}{\sin \{180^\circ - (\alpha+\beta)\}} = \frac{R}{\sin (\alpha+\beta)};$$

$$\therefore \quad P = OA = \frac{R \sin \beta}{\sin (\alpha+\beta)}, \checkmark \qquad Q = OB = \frac{R \sin \alpha}{\sin (\alpha+\beta)}. \checkmark$$

Note. The result $\frac{\mathbf{P}}{\sin\beta}=\frac{\mathbf{Q}}{\sin\alpha}=\frac{\mathbf{R}}{\sin(\alpha+\beta)}$ shows that the two components and the resultant are so related that each is proportional to the sine of the angle between the other two.

2·5. Resolving a given force into perpendicular components.

The most important case of resolution of a given force into two components is when the directions of the components are at right angles to one another. In this case the components are referred to as the *resolved parts* of the force in the corresponding directions.

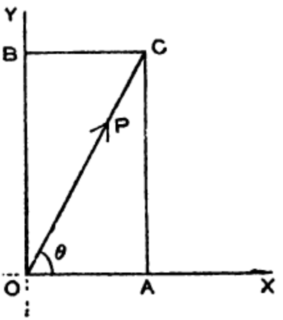


Fig. (i)

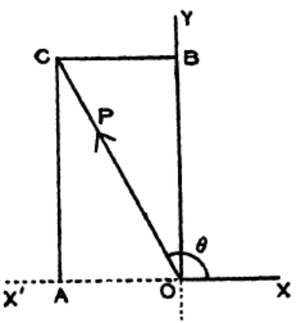


Fig. (ii)

Let the given force P be represented by OC, and let the direction OX make an angle $COX=\theta$ with it, OY being perpendicular to OX.

Complete the parallelogram $OACB$ (which is a rectangle in this case) with OC as diagonal, and sides along OX and OY [produced backwards, if necessary as in Fig. (ii)]. Then by parallelogram of forces, the resolved parts along OX and OY are given by

$$OA=OC\cos XOC=\mathbf{P}\cos\theta$$

$$\text{and}\qquad OB=AC=OC\sin XOC=\mathbf{P}\sin\theta.$$

Note. In Figure (ii), strictly speaking, $OA=OC\cos COA=P\cos(180°-\theta)=-P\cos\theta$, and is positive along OX'. Now *mathe-*

matically, a force F along OX′ is identical with a force −F along OX. Hence, $-P \cos \theta$ along OX' may be described as $P \cos \theta$ along OX.

Thus, the resolved part of P along OX is mathematically $P \cos \theta$, and perpendicular to OX, it is $P \sin \theta$, whether θ is obtuse or acute or of any magnitude.

Hence, *any given force P is mathematically equivalent to (and accordingly can be replaced, whenever needed, by) two simultaneously acting resolved parts, one $P \cos \theta$ along a direction OX at an angle θ to it, and another, $P \sin \theta$ perpendicular to OX,* whatever the angle θ may be. This mode of replacing a given force by its two equivalent resolved parts in two suitable perpendicular directions is particularly useful in finding the resultant of several forces simultaneously acting at a point, as is shown in Article 2·7.

2·6. Theorem. *The algebraic sum of the resolved parts of any two forces acting at a point, along any direction, is equal to the resolved part of their resultant, in the same direction.*

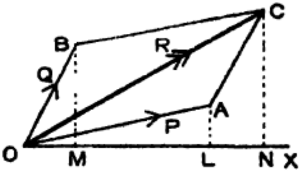


Fig. (i)

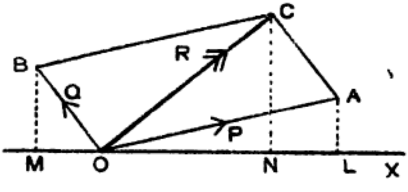


Fig. (ii)

Let OA and OB represent the two forces P and Q acting at the point O. Then OC, the diagonal of the parallelogram $OACB$, represents their resultant R in magnitude and direction.

Let OX be a line drawn in any direction through O and AL, BM and CN the perpendiculars drawn on it from A, B and C respectively, so that OL, OM and ON represent the resolved parts of P, Q and R along OX, in magnitude and sign. In Fig. (i) all three are positive, and in Fig. (ii) OM is negative.

Now, OB and AC being equal and parallel, their projections OM and LN on OX are equal in magnitude.

Hence in Fig. (i), $ON = OL + LN = OL + OM$

and in Fig. (ii), $ON = OL - NL = OL - MO$

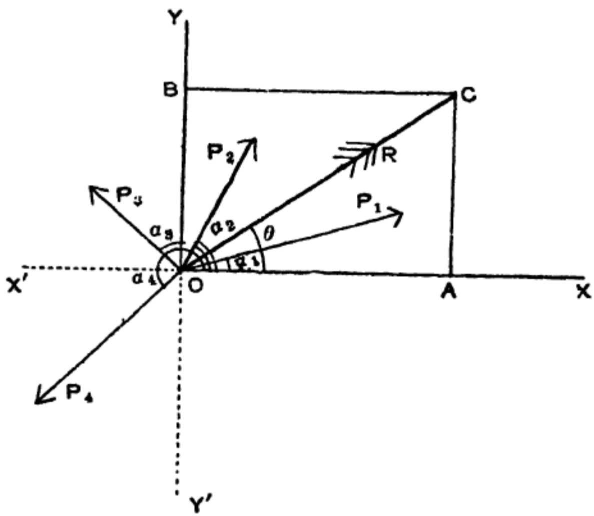
$$= OL - (-OM) = OL + OM.$$

Thus, resolved part of the resultant R is equal to the algebraic sum of the resolved parts of P and Q along OX.

Cor. By a repeated application of the above theorem, we can easily extend the theorem as follows :

If any number of forces act at a point, the algebraic sum of their resolved parts in any direction is equal to the resolved part of their resultant in the same direction.

2·7. Resultant of several coplanar forces simultaneously acting at a point.



Let a number of given coplanar forces P_1, P_2, P_3, etc. be simultaneously acting at the point O, and let their directions make angles $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots$ with any suitably chosen direction OX in the plane, OY being perpendicular to OX.

We can replace the force P_1 by its resolved parts $P_1 \cos \alpha_1$ along OX, and $P_1 \sin \alpha_1$ along OY. Similarly, P_2 may be replaced by $P_2 \cos \alpha_2$ along OX, and $P_2 \sin \alpha_2$ along OY, and so for each one of the given forces.

Now R (represented by OC) being the resultant of the given forces, and θ the angle it makes with OX, its resolved parts along OX and OY being equal to the algebraic sum of the resolved parts of the component forces along the same two directions, we get

$$\mathrm{R} \cos \theta = \mathrm{P}_1 \cos \alpha_1 + \mathrm{P}_2 \cos \alpha_2 + \mathrm{P}_3 \cos \alpha_3 + \cdots \equiv \Sigma\mathrm{X} \text{ (say)},$$

$$\mathrm{R} \sin \theta = \mathrm{P}_1 \sin \alpha_1 + \mathrm{P}_2 \sin \alpha_2 + \mathrm{P}_3 \sin \alpha_3 + \cdots \equiv \Sigma\mathrm{Y} \text{ (say)}.$$

Hence, $\quad R^2 = (\Sigma X)^2 + (\Sigma Y)^2$, or $\mathrm{R} = \sqrt{(\Sigma \mathrm{X})^2 + (\Sigma \mathrm{Y})^2}$

and $\quad \tan \theta = \dfrac{\Sigma Y}{\Sigma X}$, $\qquad$ or, $\theta = \tan^{-1} \dfrac{\Sigma \mathrm{Y}}{\Sigma \mathrm{X}}$.

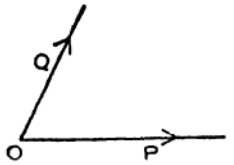
$$\begin{aligned} R^2 &= \{\Sigma P_r \cos \alpha_r\}^2 + \{\Sigma P_r \sin \alpha_r\}^2 \\ &= \Sigma\{P_r^2 (\cos^2 \alpha_r + \sin^2 \alpha_r)\} + 2\Sigma\{P_r P_s (\cos \alpha_r \cos \alpha_s \\ &\qquad\qquad + \sin \alpha_r \sin \alpha_s)\} \\ &= \Sigma P_r^2 + 2\Sigma P_r P_s \cos(\alpha_r - \alpha_s). \end{aligned}$$

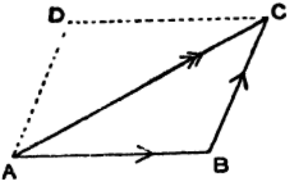
Note. If no suitable direction is apparent, we may take OX along the direction of one of the given forces, say P_1, in which case $\alpha_1 = 0$ and $P_1 \cos \alpha_1 = P_1$, and $P_1 \sin \alpha_1 = 0$.

2·8. Graphical method of construction of the resulttant of concurrent forces. (*Force diagram*)

Let the two given forces P and Q act at O. In their plane, draw a straight line AB parallel to P, on a suitably chosen scale, to represent P in magnitude, direction and sense, and then on the same scale, draw BC parallel to Q, to represent Q.

Two forces :—



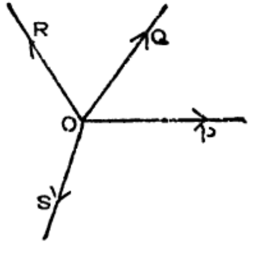


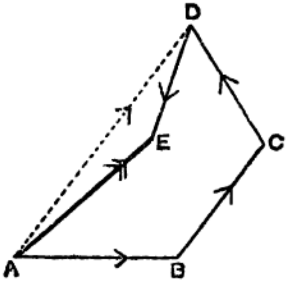
Thus, *AB*, *BC* taken in order represent successively *P* and *Q*. *Then the third side AC (in opposite order) will represent the magnitude, direction and sense of the resultant,* which will however act at *O*.

In vector notation, $\overline{\mathbf{AB}}+\overline{\mathbf{BC}}=\overline{\mathbf{AC}}$.

The proof is easily seen to depend on the parallelogram of forces, for, completing the parallelogram *ABCD*, *AD*, which is equal and parallel to *BC*, represents *Q* equally well in magnitude, direction and sense.

Any number of forces :—





Let *P*, *Q*, *R*,...etc. be any number of given coplanar forces acting simultaneously at *O*.

In their plane, on any chosen scale, draw successively the lines *AB*, *BC*, *CD*,... etc. parallel to the directions of

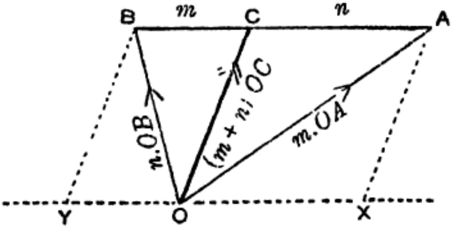
P, Q, R, etc. to represent those forces in magnitude, direction and sense. Then *the last line to close up the polygon, in opposite sense* (say AE, as in the above figure), *represents the magnitude, direction and sense of the resultant*, which will however act at O.

For, from the case of two forces, resultant (R_1 say) of P and Q represented by AB and BC is represented by AC. Then the resultant of R_1 and R, which are given by AC and CD respectively, is represented by AD ; and so on.

In vector notation, $\bar{AB} + \bar{BC} + \bar{CD} + \bar{DE} = \bar{AE}$.

2·9. The (m, n) theorem.

The resultant of two forces represented in magnitude by **m.OA** *and* **n.OB,** *acting at O along OA and OB respectively, is represented in magnitude and direction by* **(m+n).OC,** *where C is a point on AB such that* **AC : CB = n : m,** *i.e.*, $m.AC = n.CB$.



Let two forces act at O along OA and OB whose magnitudes are represented by $m.OA$ and $n.OB$ respectively.

Join AB and let C be the point on it such that $m.AC = n.CB$. Join OC.

Through O draw XOY parallel to ACB, and complete the parallelograms $OCAX$ and $OCBY$.

By parallelogram of forces, the force represented by OA can be replaced by the components OX and OC, and hence a force represented by $m.OA$ can be replaced by the components $m.OX$ and $m.OC$. Similarly, the force represented by $n.OB$ can be replaced by its components $n.OY$ and $n.OC$.

Hence, the two given forces $m.OA$ and $n.OB$ are equivalent to a total component $(m+n).OC$, along OC, a component $m.OX$ along OX and one $n.OY$ along OY. But since $m.OX = m.CA = n.CB = n.OY$ in magnitude, the last two components being equal and opposite along the same line, balance one another.

Hence, the final resultant is the single force represented by $(m+n).OC$ along OC.

Cor. 1. The resultant of two forces OA and OB is represented by $2OC$, where C is the mid-point of AB.

Cor. 2. The resultant of three forces represented by OA, OB, OC is $3OG$, where G is the centroid of the triangle ABC.

2·10. Illustrative Examples.

Ex. 1. *If the resultant of two forces acting on a particle be at right angles to one of them, and its magnitude be one-third of the magnitude of the other, show that the ratio of the larger force to the smaller is* $3 : 2\sqrt{2}$. [*U. P. 1944*]

Let P and Q be the forces, and let the resultant be perpendicular to P, its magnitude being $\frac{1}{3}Q$, as in the figure, when the diagonal of the parallelogram with P and Q as adjacent sides represents the resultant.

Then, from the figure,

$$Q^2 = (\tfrac{1}{3}Q)^2 + P^2, \text{ or, } \tfrac{8}{9}Q^2 = P^2 ;$$

$$\therefore \quad Q^2/P^2 = 9/8,$$

$$\text{or,} \quad Q : P = 3 : 2\sqrt{2}.$$

Ex. 2. *Two forces acting at a point have got their resultant* 10 *when acting at right angles, and their least resultant is* 2. *Find their greatest resultant, and also the resultant when they act at an angle* 60°.

Let P and Q be the forces, P being the greater.

Then, while acting perpendicularly, their resultant,

$$\sqrt{P^2+Q^2}=10, \text{ or, } P^2+Q^2=100.$$

Also their least resultant,

$$P-Q=2. \qquad \therefore \quad P^2+Q^2-2PQ=4.$$

Hence, $\quad PQ=48.$

Now the greatest resultant

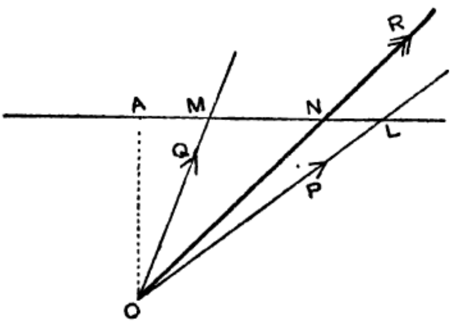
$$=P+Q=\sqrt{P^2+Q^2+2PQ}$$
$$=\sqrt{100+96}=14.$$

Also, when they act at an angle $60°$, their resultant

$$R=\sqrt{P^2+Q^2+2PQ\cos 60°}$$
$$=\sqrt{100+2\times 48\times \tfrac{1}{2}}=\sqrt{148}.$$

Ex. 3. *Forces P and Q, whose resultant is R, act at a point O. If any transversal cut the lines of action of the forces P, Q, R at the points L, M, N respectively, show that*

$$\frac{P}{OL}+\frac{Q}{OM}=\frac{R}{ON}. \qquad [\text{ B. H. U. 1944 ; C. U. 1917 }]$$



Let OA be drawn perpendicular from O on the transversal LNM.

Equating the algebraic sum of the resolved parts of P and Q along OA to that of their resultant R, we get

$$P\cos LOA+Q\cos MOA=R\cos NOA,$$

$$\text{or, } P\frac{OA}{OL}+Q\frac{OA}{OM}=R\frac{OA}{ON}$$

$$i.e., \quad \frac{P}{OL}+\frac{Q}{OM}=\frac{R}{ON}.$$

Alternative method :

P along OL can be written as $\frac{P}{OL}\cdot OL = m.OL =$ where $m = \frac{P}{OL}$.

Similarly, Q along OM can be written as $n.\ OM$ where $n = \frac{Q}{OM}$.

Now the resultant of the forces represented by $m.OL$ and $n.OM$ is $(m+n)\ .\ ON$ along ON, where N is a point on LM such that $LN : NM = n : m$. Thus, ON being the direction of the resultant, intersecting LM at N,

$$R = (m+n).ON, \quad \text{or,} \quad \frac{R}{ON} = m+n = \frac{P}{OL} + \frac{Q}{OM}.$$

Ex. 4. *Two forces P, Q act at a point along two straight lines making an angle α with each other, and R is their resultant. Two other forces P', Q' acting along the same two lines have a resultant R'. Prove that if θ be the angle between the lines of action of the resultants, then*

$$RR' \cos\theta = (PQ' + P'Q)\cos\alpha + PP' + QQ',$$

$$\textit{and} \quad RR' \sin\theta = (PQ' \sim P'Q)\sin\alpha.$$

If ϕ be the angle which the resultant R makes with the line of action of P, resolving along and perpendicular to this line, and equating the resolved part of the resultant to the algebraic sum of the resolved parts of the components,

$$R\cos\phi = P + Q\cos\alpha, \quad R\sin\phi = Q\sin\alpha.$$

Similarly, ϕ' being the angle made by R' with the same line,

$$R'\cos\phi' = P' + Q'\cos\alpha,\ R'\sin\phi' = Q'\sin\alpha.$$

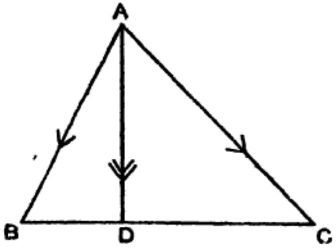
Now, $\quad \theta = \phi \sim \phi'. \quad \therefore \quad \cos\theta = \cos(\phi \sim \phi').$

$$\begin{aligned}\therefore \quad RR'\cos\theta &= RR'(\cos\phi\cos\phi' + \sin\phi\sin\phi')\\ &= (P + Q\cos\alpha)(P' + Q'\cos\alpha) + (Q\sin\alpha)(Q'\sin\alpha)\\ &= PP' + (PQ' + P'Q)\cos\alpha + QQ'(\cos^2\alpha + \sin^2\alpha)\\ &= (PQ' + P'Q)\cos\alpha + PP' + QQ'.\end{aligned}$$

Similarly, the second result follows.

Ex. 5. *Show that the resultant of two forces $\sec B$ and $\sec C$ acting along AB, AC respectively of any triangle ABC is a force $(\tan A + \tan C)$ along AD, where D is the foot of the perpendicular from A on BC.*

We note that the forces sec B and sec C along AB and AC can be written as



$$\frac{\sec B}{AB}\cdot AB \text{ and } \frac{\sec C}{AC}\cdot AC$$

along these lines.

$$\text{Again, } \frac{BD}{DC}=\frac{AB\cos B}{AC\cos C}$$

$$=\frac{\sec C}{AC}:\frac{\sec B}{AB}.$$

Thus, the resultant of $\frac{\sec B}{AB}\cdot AB$ along AB and $\frac{\sec C}{AC}\cdot AC$ along AC is $\left(\frac{\sec B}{AB}+\frac{\sec C}{AC}\right)\cdot AD$ along AD, since D divides BC in the ratio $\frac{\sec C}{AC}:\frac{\sec B}{AB}$. [See § 2·9]

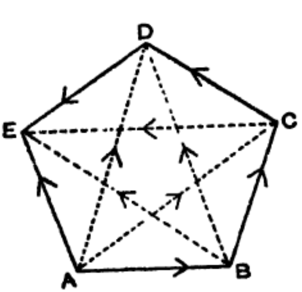
$$\text{Also, } \left(\frac{\sec B}{AB}+\frac{\sec C}{AC}\right)\cdot AD=\sec B.\sin B+\sec C.\sin C$$

$$=(\tan B+\tan C).$$

Hence the result.

Ex. 6. *ABCDE is a regular pentagon and forces acting at a point are represented in magnitude and direction by the lines AB, AC, AD, AE, BC, BD, BE, CD, CE and DE. Prove that their resultant is represented by 4AE+2BD.* [C. U. 1939]

We first of all see that forces acting at a point and represented in magnitude and direction by the lines AB, BC, CD, DE are equivalent to a single force represented by AE [See § 2·8] Written in vector notation,



$$\bar{AB}+\bar{BC}+\bar{CD}+\bar{DE}=\bar{AE}.$$

Similarly, $\bar{AC}+\bar{CE}=\bar{AE}$.

Also, $\bar{AD}+\bar{DE}=\bar{AE}$

and $\bar{BE}+\bar{ED}=\bar{BD}$.

As two forces represented by DE and $\bar{ED}$ acting at a point cancel one another, and as the forces all act at one point, we get by combining the above,

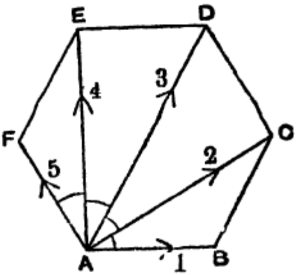
$$AB+BC+CD+DE+AC+CE+AD+BE=3AE+BD.$$

Hence, adding two more forces represented by AE and BD, we get

$$AB+AC+AD+AE+BC+BD+BE+CD+CE+DE$$
$$=4AE+2BD.$$

Ex. 7. *Forces of magnitude* 1, 2, 3, 4 5 *respectively act at an angular point of a regular hexagon towards the other angular points taken in order ; find their resultant.*

$ABCDEF$ being a regular hexagon, forces 1, 2, 3, 4, 5 act along AB, AC, AD, AE and AF.



In the regular hexagon, it is easily seen from Geometry that $\angle BAC = \angle CAD = \angle DAE = \angle EAF = 30°$, and so AB and AE are perpendicular to one another.

If R be the required resultant and θ the angle it makes with AB, we get by equating the resolved parts of the resultant along AB and AE to the algebraic sum of the resolved parts of the components,

$$R\cos\theta = 1+2\cos 30°+3\cos 60°+4\cos 90°+5\cos 120°$$

$$=1+2.\frac{\sqrt{3}}{2}+3.\ \frac{1}{2}+4.0+5\left(-\frac{1}{2}\right)=\sqrt{3}\ ;$$

and $\quad R\sin\theta = 2\sin 30°+3\sin 60°+4\sin 90°+5\sin 120°$

$$=2.\ \frac{1}{2}+3.\ \frac{\sqrt{3}}{2}+4.1+5.\ \frac{\sqrt{3}}{2}=5+4\sqrt{3}.$$

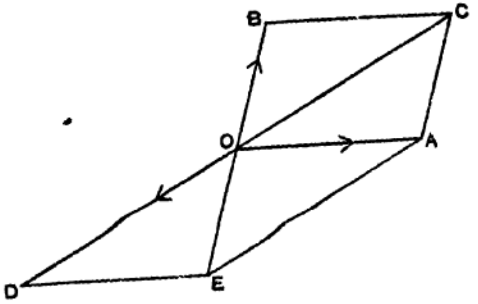
$$\therefore \quad R^2=(\sqrt{3})^2+(5+4\sqrt{3})^2=76+40\sqrt{3}.$$

Also, $\tan\theta=\dfrac{5+4\sqrt{3}}{\sqrt{3}}$.

Hence, $R=2\sqrt{19+10\sqrt{3}}$ and $\theta=\tan^{-1}\left(4+\dfrac{5}{\sqrt{3}}\right)$

giving the magnitude and direction of the resultant.

Ex. 8. *Assuming that the parallelogram law of forces is true so far as the magnitude only of the resultant is concerned, prove it for direction.*



Let OA and OB represent two forces acting at O in magnitude and direction. Complete the parallelogram $OACB$ and join the diagonal OC. It is given that the resultant of OA and OB is represented in magnitude only by OC. We are to show that the direction of the resultant must be along OC.

Let OD be a force exactly equal and opposite to the unknown resultant of OA and OB, so that OD is equal to OC in magnitude, but we do not as yet know the direction of OD. Complete the parallelogram $ODEA$, and join the diagonal OE.

Now, since OD is equal and opposite to the resultant of OA and OB, the three forces OA, OB, OD acting at O are in equilibrium. Hence, OB is exactly equal and opposite to the resultant of OA and OD. But from the given condition, the magnitude of this resultant is given by the diagonal OE. Hence, OB must be equal to OE in magnitude.

Thus, $OE = OB = AC$. Also, $AE = OD = OC$. Hence the figure $OEAC$ is a parallelogram. Therefore EA is parallel to OC. But by construction, AE is parallel to OD. Hence OD and OC must be along the same straight line. Thus, the direction of the resultant of OA and OB being exactly opposite to OD from construction, is along the diagonal OC.

Examples on Chapter II

1. Show that the greater the angle between the lines of action of two forces acting at a point, the less will be their resultant.

2. Two unequal forces inclined at a certain angle act on a particle. Show that the resultant is nearer the greater force.

3. The greatest and least resultants of two forces of given magnitudes acting at a point are 16 lbs. wt. and 4 lbs. wt. respectively. Find their resultant when they are at an angle of 60° with one another.

4. The resultant of two forces P and $2P$, acting at a point, is perpendicular to P. Find the angle between the forces.

5. Find the angle between two equal forces P, when their resultant is a third equal force P. [*P. U. 1930*]

6. Two equal forces act on a particle; find the angle between them when the square of their resultant is equal to three times their product. [*P. U. 1933*]

7. The resultant of two forces acting at an angle of 45° is $\sqrt{10}$ lbs. wt.; one of the components being $\sqrt{2}$ lbs. wt., find the other.

8. Find the components of a force P along two directions making angles of 45° and 60° with P on opposite sides.

9. Two forces of magnitudes $3P$, $2P$ respectively have a resultant R. If the first force is doubled, the magnitude of the resultant is doubled. Find the angle between the forces. [*C. U. 1932*]

10. Two forces given in magnitude and direction act on a particle. Find the direction in which a third force of given magnitude should act on it, so that the resultant of the three may be the least possible in magnitude.

11. Two forces act at a point and are such that if the direction of one is reversed, the direction of the resultant

is turned through a right angle. Prove that the two forces must be equal in magnitude.

12. If the resultant of two equal forces inclined at an angle 2θ is twice as great as when they are inclined at an angle 2ϕ, prove that $\cos\theta = 2\cos\phi$.

•13. If the resultant R of two forces P and Q inclined to one another at any given angle makes an angle θ with the direction of P, show that the resultant of the forces $P+R$ and Q acting at the same angle will make an angle $\frac{1}{2}\theta$ with the direction of $P+R$.

[*B. U. 1926, '29 ; B. E. 1932*]

14. If the resultant of the forces P and Q be equal to that of the forces $P+S$ and $Q-S$ acting at the same angle $(S \neq Q-P)$, find the magnitude of the resultant.

15. Two forces P and Q acting on a particle at an ngle α have a resultant $(2k+1)\sqrt{P^2+Q^2}$. When they act at an angle $90^\circ-\alpha$, the resultant becomes $(2k-1)\sqrt{P^2+Q^2}$; prove that

$$\tan\alpha = \frac{k-1}{k+1}. \qquad [\ B.\ H.\ U.\ 1946\ ]$$

•16. Two forces $P+Q$, $P-Q$ make an angle 2α with one another, and their resultant makes an angle θ with the bisector of the angle between them. Show that

$$P\tan\theta = Q\tan\alpha. \qquad [\ P.\ U.\ 1931\ ]$$

7. The angle of inclination between two forces P and is θ. If P and Q be interchanged in position, show that the resultant will be turned through an angle ϕ, where

$$\tan\frac{\phi}{2} = \frac{P-Q}{P+Q}\tan\frac{\theta}{2}. \qquad [\ P.\ U.\ 1929\ ]$$

•18. Two forces P and Q act at an angle α and have a resultant R. If each force is increased by R, prove that the new resultant makes with R an angle whose tangent is

$$\frac{(P-Q)\sin\alpha}{P+Q+R+(P+Q)\cos\alpha}.$$

[*P. U. 1943 ; B. H. U. 1943*]

° **19.** Two forces P and Q acting respectively along two different straight lines OA and OB have resultant perpendicular to OA. If two forces P' and Q' acting respectively along the same two straight lines have a resultant perpendicular to OB, show that

$$PP' = QQ'.$$

° **20.** Two forces P and Q acting respectively along the straight lines OA and OB which are inclined at an angle α to one another $(\alpha \neq \pi)$, have a resultant R making an angle θ with OA. If Q be changed to Q', the resultant changes to R' making an angle θ' with OA. Show that

$$\frac{R'}{R} = \frac{\sin(\alpha - \theta)}{\sin(\alpha - \theta')}.$$

° **21.** The resultant of two forces P, Q acting at a certain angle is X, and that of P, R acting at the same angle is also X. The resultant of Q, R $(Q \neq R)$ acting at the same angle is Y. Show that if $P + Q + R = 0$, then $X = Y$.

° ***22.** Two forces R and S act at a point along two straight lines inclined at an angle θ, and F is their resultant. Two other forces R' and S' acting along the same lines have a resultant F'. If ϕ be the angle between the lines of action of F and F', prove that

$$(1 - \cos\phi)(1 + \cos\phi)$$
$$= \frac{(R^2 S'^2 - 2RR'SS' + R'^2 S^2)(1 - \cos\theta)(1 + \cos\theta)}{F^2 F'^2}.$$

[*C. U. 1946*]

(i) Two forces P and Q acting at a point have got resultant R; if Q be doubled, R is doubled. Again, if Q be reversed in direction, then also R is doubled. Show that $P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$. [*Bombay, 1934*]

(ii) Show that three concurrent forces lying in a plane cannot produce equilibrium for any arrangement of their directions, if the sum of the magnitudes of two of them be less than that of the third.

24. If one of the two forces acting on a particle be double that of the other, and if θ be the angle between the direction of the resultant and the greater force, show that $\theta \ngtr \frac{1}{6}\pi$

***25.** Three forces P, Q, R in one plane act on a particle, the angles between Q and R, R and P and P and Q being α, β, γ respectively. Show that their resultant is equal to

$$\{P^2 + Q^2 + R^2 + 2QR \cos \alpha + 2RP \cos \beta + 2PQ \cos \gamma\}^{\frac{1}{2}}.$$

[*Delhi, 1931*]

26. Equal forces P act at a point parallel to the sides BC, CA, AB of the triangle ABC. Prove that their resultant is given by

$$P \sqrt{3 - 2 \cos A - 2 \cos B - 2 \cos C}.$$

27. Forces act through the angular points of a triangle perpendicular to the opposite sides, and are proportional to the cosines of the corresponding angles ; show that their resultant is proportional to

$$\sqrt{(1 - 8 \cos A \cos B \cos C)}.$$

28. Prove that any force in the plane of a triangle ABC can be resolved into three components acting along the sides of the triangle.

In particular if E, F are the feet of the perpendiculars from B and C upon the opposite sides of the triangle ABC, show that a force P acting along EF can be replaced by $P \cos A$, $P \cos B$, $P \cos C$ acting along the sides of the triangle. [*B. E. 1935*]

***29.** Show how a force given in magnitude and line of action can be broken up into two equal components passing through two given points, which are (i) on the same side, (ii) on opposite sides of the line of action of the given force.

***30.** Two given forces, which are not parallel, act at two given points of a body. If they be turned through the same angle in the same sense about their respective points of application, prove that the resultant is constant in magnitude and passes through a fixed point.

*31. Two forces P and Q act upon a particle, P is given in magnitude and direction, and Q in magnitude only. Find the locus of the extremity of the resultant.

*32. A, B, C are three fixed points and P is a point such that the resultant of forces PA, PB always passes through C. Show that the locus of P is a straight line.

*33. If the resultant of forces represented by lines drawn from a point P to the vertices of a quadrilateral be of constant magnitude, show that the locus of P is a circle.

*34. Forces are represented in magnitude, direction and sense by the sides AB, AC of the triangle ABC. If their resultant passes through the circum-centre of the triangle, show that the triangle is either right-angled, or isosceles.

*35. A, B, C are three points on the circumference of a circle. Forces act along AB and BC inversely proportional to these lines in magnitude; show that their resultant acts along the tangent to the circle at B. [*U. P. 1911*]

36. AB and CD denote any two equal and parallel chords of a circle; P is a point on the circumference equidistant from A and B. Show that the resultant of forces acting at P and represented by PA, PB, PC, PD is constant. [*C. U. 1943*]

37 $PQRS$ is a quadrilateral. Prove that the resultant of the forces completely represented by the lines PQ, QR, PS, SR is represented in magnitude and direction by $2PR$, and that its line of action bisects QS. [*C. U. 1941*]

38. If H be the ortho-centre and O the circum-centre of a triangle ABC, show that the resultant of the forces represented by

(i) OA, OB, OC is represented by OH;

(ii) HA, HB. HC is represented by $2HO$;

(iii) AH, HB, HC is represented in magnitude and direction by the diameter through A of the circum-circle of the triangle.

39. Three forces PA, PB, PC diverge from the point P, and three forces AQ, BQ, CQ converge to the point Q. Show that the resultant of the six forces is represented in magnitude and direction by $3PQ$ and that it passes through the centroid of the triangle.

40. If O be the circum-centre of the triangle ABC and if forces act along OA, OB, OC respectively proportional to BC, CA, AB, show that their resultant passes through the in-centre.

41. O is any point in the plane of the triangle ABC; D, E, F are the middle points of the sides. Prove that the resultant of the forces OE, OF, DO is represented by OA.

42. Two chords AB, CD of a circle intersect at right angles at P; show that the resultant of the forces PA, PB, PC, PD is $2PO$, where O is the centre of the circle.

43. $ABCDEF$ is a regular hexagon and O is any point. Prove that the resultant of the forces represented by OA, OB, OC, OD, OE, OF is $6OG$, where G is the centre of the circum-circle of the hexagon.

***44.** Eight points are taken on the circumference of a circle at equal distances, and from one of the points straight lines are drawn to the others. If these lines represent forces acting on a particle at the point, show that the direction of the resultant coincides with the diameter through the point, and its magnitude is four times the diameter.

45. Forces each equal to P act along the sides AB, CB, AD, DC of the square $ABCD$; find their resultant.

46. ABC is a triangle right-angled at A, and AD is perpendicular on BC. Show that the resultant of the forces $\frac{\mu}{AB}$ acting along AB and $\frac{\mu}{AC}$ acting along AC is $\frac{\mu}{AD}$ acting along AD.

47. Show that the resultant of the forces $OA \tan A$ and $OB \tan B$ acting along the sides OA and OB of the triangle OAB is $AB \tan A \tan B$, acting in the direction of the perpendicular from O on AB

***48.** Two forces act along the sides CA, CB of a triangle ABC, their magnitudes being proportional to $\cos A$, $\cos B$. Prove that their resultant is proportional to $\sin C$, and its direction divides the angle C into two portions $\frac{1}{2}(C + B - A)$, $\frac{1}{2}(C + A - B)$.

***49.** P is a point in the plane of the triangle ABC, and I is the in-centre. Show that the resultant of the forces represented by $PA \sin A$, $PB \sin B$, $PC \sin C$ along PA, PB, PC respectively is

$$4PI . \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C,$$

along PI.

50. Four horizontal wires are attached to a telephone post and exert the following tensions on it ; 20 lbs. wt. N., 30 lbs, wt. E., 40 lbs. wt. S. W., and 50 lbs. wt. S. E. Calculate the resultant pull on the post.

51. Forces of 2, $\sqrt{3}$, 5, $\sqrt{3}$ and 2 lbs. wt. respectively act at one of the angular points of a regular hexagon towards the five others in order. Find the magnitude and direction of the resultant.

52. Find the magnitude of the resultant of the forces 7, 1, 1 and 3 lbs. wt. acting from one angle of a regular pentagon towards the other angles taken in order.

53. $ABCDEF$ is a regular hexagon of side a, and at A forces act, represented in magnitude and direction by AB, $2AC$, $3AD$, $4AE$, $5AF$, show that the magnitude of the resultant is $\sqrt{351}a$.

***54.** Three forces P, Q, R meet at a point, and the resultant of P and Q is 7 lbs. wt. acting at an angle $\cos^{-1}\left(\frac{11}{14}\right)$ with P. The resultant of P and R is also 7 lbs. wt. at an angle $\cos^{-1}\left(-\frac{1}{7}\right)$ with P, and that of Q and R is

$\sqrt{129}$ lbs. wt. at an angle $\tan^{-1}\left(-\frac{13}{\sqrt{3}}\right)$ with P. Find P, Q, R in magnitude and direction, it being given that Q and R are on the same side of the line of action of P.

*55. At any point of a parabola, forces represented in magnitude and direction by the tangent and normal at the point (up to their intersection with the axis), both towards the axis, act. Show that the resultant passes through the focus.

*56. Two forces are represented by two semi-conjugate diameters of an ellipse; prove that their resultant is a maximum when the diameters are equal and include an acute angle, and their resultant is a minimum when they are equal and include an obtuse angle.

57. P is any point on an ellipse of centre C and foci S and S'. Two equal constant forces act at C parallel to SP and PS'. Show that the end of the straight line which represents their resultant lies on a circle passing through the centre.

*58. Assuming the parallelogram law of forces to be true for direction only, prove it for magnitude.

ANSWERS

3. 11 lbs. wt. 4. 120°. 5. 120°. 6. 60°.

7. 2 lbs. wt. 8. $\frac{2}{\sqrt{3}+1}P$, $\frac{\sqrt{6}}{\sqrt{3}+1}P$. 9. 120°.

10. In the direction opposite to that of the resultant of the given forces.

14. $P+Q$.

31. A circle with centre at the extremity of P and radius equal to the magnitude of Q.

45. $2P$ acting along DC. 50. 57·2 lbs. wt.

51. 10 lbs. wt., towards the opposite vertex. 52. $\sqrt{71}$ lbs. wt.

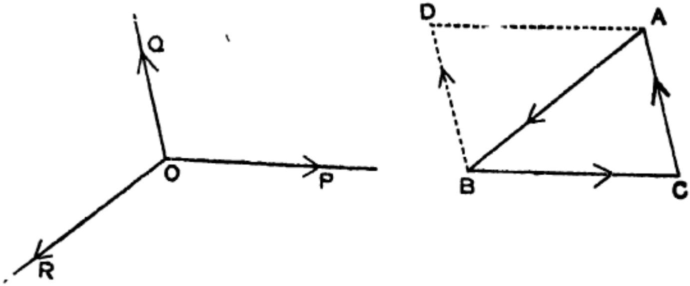
54. $P=3$ lbs. wt., $Q=5$ lbs. wt at angle 60° with P,
$R=8$ lbs. wt., at angle 120° with P.

CHAPTER III

EQUILIBRIUM OF CONCURRENT FORCES

3·1. Triangle of forces.

If three forces acting at a point, be such as can be represented in magnitude, direction, and sense, (but not in position) by the three sides of a triangle taken in order, then the forces are in equilibrium.



Let the three forces P, Q, R acting at the point O be represented in magnitude, direction and sense by the sides BC, CA, AB in order respectively of the triangle ABC. It is required to prove that they shall be in equilibrium.

Complete the parallelogram $BCAD$. Since BD is equal and parallel to CA, the force Q which is represented by CA can as well be represented in magnitude and direction by BD.

Now by parallelogram of forces, the two forces P and Q, represented in magnitude, direction and sense by BC and BD, have got a resultant represented in magnitude, direction and sense by BA. This resultant of P and Q acts however at O, and being equal and opposite to R which is represented by AB, balances the latter force.

Hence the three forces are in equilibrium.

In vector notation, $\overline{AB}+\overline{BC}+\overline{CA}=0$, when referring to forces acting at a point.

Note. The three forces, in this case, though represented in magnitude and direction by the sides of a triangle, *act at a point*, and *do not act along the sides of the triangle*. It will be seen in a later chapter (§ 6·8), what happens if three forces act along the sides of a triangle. Three *forces acting on a particle* mean the same thing as three forces acting *at a point*.

3·2. Modification of the triangle of forces. (*Perpendicular triangle of forces*).*

If three forces acting at a point, be such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards, or all outwards, then also the forces shall be in equilibrium.

For in this case, if we rotate the triangle through one right angle in its own plane in the proper sense, we get a triangle whose sides in order are parallel to the given forces, and will represent those forces in both magnitude and direction. Accordingly the forces are in equilibrium.

Note. The result will also hold if in the above case, the directions of the forces, instead of being perpendicular to the corresponding sides, make *any equal angle* with them, measured the same way round. The proof is exactly similar.

3·3. The converse of the triangle of forces.

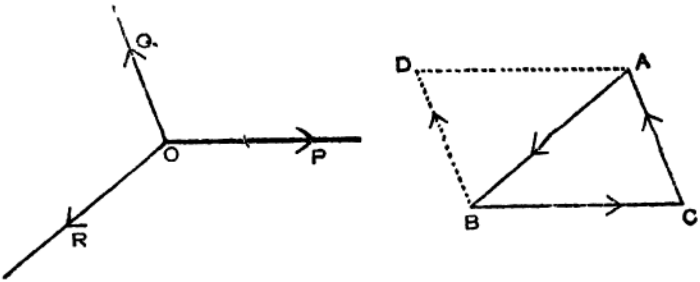
If three forces acting at a point be in equilibrium, they can be represented in magnitude, direction and sense by the three sides of a triangle, taken in order.

Let the three forces P, Q, R acting at O be in equilibrium. Draw the lines BC, CA in succession, parallel to

*For a note on the method of proof of this Art. see *Appendix (B)*.

the directions of P and Q, to represent those forces respectively in magnitude, direction and sense, on any chosen scale. Complete the parallelogram $BCAD$, and join the diagonal BA.

Then BD being equal and parallel to CA, represents Q as well in magnitude, direction and sense. Now, P and Q being represented by BC and BD, by parallelogram of



forces, their resultant is represented by BA. But since P, Q, R are in equilibrium, R is equal and opposite to the resultant of P and Q, and accordingly R is represented in magnitude, direction and sense by AB.

Thus, we get a triangle ABC whose sides BC, CA, AB taken in order, represent the forces P, Q, R in this case which proves the theorem.

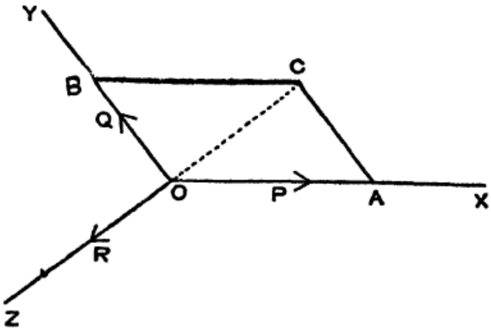
Note. If we draw *any other triangle with the sides parallel to the lines of action of the given forces*, this triangle will evidently be similar to ABC, and accordingly having the corresponding sides proportional, *the three forces in this case may as well be represented in magnitude, direction and sense by the sides of that triangle taken in order.*

Cor. Three forces acting at a point being such that the sum of any two is less than the third, they can never be in equilibrium, for they cannot be represented by the sides of a triangle.

3·4. Lami's Theorem. (V.V.I)

If three forces acting at a point be in equilibrium, then each is proportional to the sine of the angle between the other two.

Let the three forces P, Q, R acting at O along the lines OX, OY, OZ be in equilibrium.



It is required to prove that

$$\frac{P}{\sin YOZ} = \frac{Q}{\sin ZOX} = \frac{R}{\sin XOY}.$$

On any chosen scale cut off OA and OB along OX and OY respectively to represent the forces P and Q in magnitude and direction. Complete the parallelogram $OACB$ and join the diagonal OC. Then by parallelogram of forces, the resultant of P and Q is represented by OC.

Now, since P, Q, R are in equilibrium, R is equal and opposite to the resultant of P and Q, and accordingly R must be represented in magnitude and direction by CO, so that COZ must be along the same straight line. Also, AC being equal and parallel to OB, represents Q equally well in magnitude and direction.

Then in the triangle OAC,

$$\frac{OA}{\sin OCA} = \frac{AC}{\sin COA} = \frac{CO}{\sin OAC}.$$

But, $\sin OCA = \sin COB = \sin(180° - YOZ) = \sin YOZ$

$\sin COA = \sin(180° - ZOX) = \sin ZOX$

and $\sin OAC = \sin(180° - XOY) = \sin XOY$

Also, OA, AC, CO represent P, Q, R respectively.

Thus, $$\frac{P}{\sin YOZ} = \frac{Q}{\sin ZOX} = \frac{R}{\sin XOY},$$

or, $$\frac{P}{\sin(Q, R)} = \frac{Q}{\sin(R, P)} = \frac{R}{\sin(P, Q)}.$$

Alternatively, since the concurrent forces are in equilibrium, the algebraic sum of their resolved parts in any direction, being equal to the resolved part of their resultant, is zero. Therefore, resolving perpendicular to OX and to OY respectively, $Q \sin XOY - R \sin XOZ = 0$, and $P \sin YOX - R \sin YOZ = 0$. Hence, $P / \sin YOZ = R / \sin XOY = Q / \sin ZOX$.

3·4 (1). Converse of Lami's Theorem.

If three forces acting at a point be such that each is proportional to the sine of the angle between the other two (the sense of the forces being such that any one of them lies within the angle opposite to that in which the resultant of the other two lies), then the three forces are in equilibrium.

Let the three forces P, Q, R acting at O along OX, OY, OZ be such that

$$\frac{P}{\sin YOZ} = \frac{Q}{\sin ZOX} = \frac{R}{\sin XOY} \qquad \cdots \text{(i)}$$

the sense of the forces being as indicated by the arrow-heads. [See *fig. of* § *3·4*]

Produce ZO to C such that $OC = R$ in magnitude. Complete the parallelogram $OACB$ with diagonal OC, the sides OA, OB being along OX and OY respectively. Then from the triangle OAC,

$$\frac{OA}{\sin OCA} = \frac{AC}{\sin AOC} = \frac{OC}{\sin OAC} = \frac{R}{\sin OAC};$$

also, $\sin OCA = \sin BOC = \sin(180° - YOZ) = \sin YOZ$

$\sin AOC = \sin(180° - ZOX) = \sin ZOX$

and $\sin OAC = \sin(180° - XOY) = \sin XOY.$

Hence, $$\frac{OA}{\sin YOZ} = \frac{AC}{\sin ZOX} = \frac{R}{\sin XOY}.$$

Comparing this with (i), we get

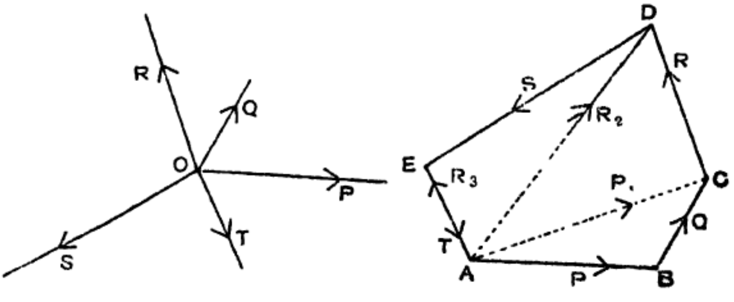
$P = OA$, $Q = AC = OB$.

Hence, by the parallelogram of forces, the resultant of P and Q, which are now represented by OA and OB, is OC, and this by construction is equal and opposite to R.

Thus, P, Q, R are in equilibrium.

3·5. Polygon of forces.

If any number of forces acting at a point be such that they can be represented in magnitude, direction and sense by the sides of a closed polygon taken in order, then they shall be in equilibrium.



Let the forces P, Q, R, S, T acting at O be represented in magnitude, direction and sense by the sides AB, BC,

CD, DE, etc., taken in order, of the closed polygon $ABCDE$. Join the diagonals AC, AD.

The forces P and Q being represented by AB, BC, their resultant (say R_1), is represented by AC. Again, the resultant (R_2, say) of R_1 and R, which are represented by AC, CD, is represented by AD; in other words, AD represents the resultant of P, Q, R. Proceeding in this manner the resultant of the forces P, Q, R, S which are represented respectively by AB, BC, CD, DE is represented by AE [See *Art.* 2·8]. But the last force T is represented by EA, and is thus equal and opposite to the resultant of P, Q, R, S. Both however acting at O, balance one another. Hence the given forces are in equilibrium.

Note. It is not necessary here that the forces acting at O, (and accordingly the polygon) should be coplanar. The result will be equally true in all cases.

3·6. The converse of the polygon of forces.

If any number of forces acting at a point be in equilibrium, then they can be represented in magnitude, direction and sense by the sides, taken in order, of a closed polygon.

Let a number of forces P, Q, R, S, T acting at a point O be in equilibrium. We are to show that they can be represented in magnitude, direction and sense of the sides of a closed polygon.

Let us draw in succession the lines AB, BC, CD, DE, joined end to end, parallel to, and in the sense of the forces P, Q, R, S (all but one), of such lengths that they may represent the corresponding forces on any chosen scale. Join AE, closing up the polygon, and join the diagonals AC, AD. [See *Fig.*, *Art.* 3·5]

Then since P and Q are represented by AB, BC, their resultant (R_1, say) is represented by AC. Similarly, AD represents the resultant (R_2, say) of R_1 and R *i.e.*, of P, Q, R. Proceeding in this manner, the last line AE (from A to E) represents the resultant of P, Q, R, S (all but the last, namely T). As the forces are in equilibrium, T is equal and opposite to the resultant of all the others, and accordingly, from above T is represented by EA.

Hence, the forces are represented in succession by the sides of the closed polygon $ABCDE$ taken in order.

Note. As polygons with their corresponding sides parallel are not necessarily similar, *i.e.*, have not got their corresponding sides proportional always, it follows that the forces in equilibrium acting at O will not necessarily be represented by the sides of *any* polygon drawn with its sides parallel to the forces. [*Cf. Art.* 3·3, *Note*]

3·7. Analytical conditions of equilibrium of any number of concurrent forces.

The two necessary and sufficient conditions that a system of coplanar forces acting at a point may be in equilibrium are that the algebraic sum of the resolved parts of the forces in any two mutually perpendicular directions[*] *may be separately zero.*

Let a system of coplanar forces $P_1, P_2, P_3, \ldots\ldots\ldots$ act at a point O, and let OX and OY be any two perpendicular directions in their plane. Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots$ be the angles made by $P_1, P_2, P_3, \ldots\ldots$ with OX. Now, R being the resultant of the system, and θ the angle which it makes with OX, since the algebraic sum of the resolved parts of any system of concurrent forces is equal to that of the resultant in the same direction, resolving along OX and OY respectively, we get

$$R \cos \theta = \Sigma P_1 \cos \alpha_1 = \Sigma X \text{ (say)},$$

and $$R \sin \theta = \Sigma P_1 \sin \alpha_1 = \Sigma Y \text{ (say)}.$$

Thus, $$R^2 = (\Sigma X)^2 + (\Sigma Y)^2.$$

Now, if the given force system be in equilibrium, $R = 0$, and accordingly $(\Sigma X)^2 + (\Sigma Y)^2$ being zero, each of ΣX and ΣY must be separately zero. Hence, the conditions $\Sigma X = 0$

[*] or any two different directions.

and $\Sigma Y=0$ are necessary for equilibrium of the given system.

Again, if $\Sigma X=0$ and $\Sigma Y=0$, then $R=\sqrt{(\Sigma X)^2+(\Sigma Y)^2}=0$, and so the force system is in equilibrium. Thus the conditions are sufficient.

Thus, the two necessary and sufficient analytical conditions of equilibrium of the given system of concurrent coplanar forces are

$$\mathbf{\Sigma X=\Sigma P_1 \cos \alpha_1=0 \text{ and } \Sigma Y=\Sigma P_1 \sin \alpha_1=0.}$$

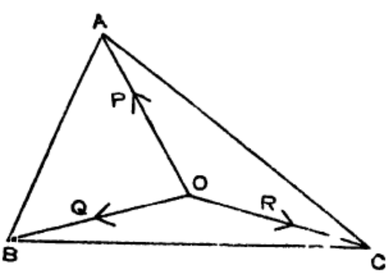
3·8. Illustrative Examples.

Ex. 1. *Three forces P, Q, R acting along OA, OB, OC are in equilibrium. If O be the circum-centre of the triangle ABC, prove that*

$$\frac{P}{\frac{1}{b^2}+\frac{1}{c^2}-\frac{a^2}{b^2c^2}}=\frac{Q}{\frac{1}{c^2}+\frac{1}{a^2}-\frac{b^2}{c^2a^2}}=\frac{R}{\frac{1}{a^2}+\frac{1}{b^2}-\frac{c^2}{a^2b^2}}$$

where a, b, c are the lengths of the sides BC, CA and AB. [*C. U. 1938*]

O being the circum-centre of the triangle ABC, $\angle BOC$ at the centre $=2\angle BAC$ at the circumference$=2A$, and similarly $\angle COA=2B$ and $\angle AOB=2C$.



Now since the forces P, Q, R along OA, OB, OC are in equilibrium, by Lami's theorem,

$$\frac{P}{\sin BOC}=\frac{Q}{\sin COA}=\frac{R}{\sin AOB}$$

i.e., $$\frac{P}{\sin 2A}=\frac{Q}{\sin 2B}=\frac{R}{\sin 2C};$$

or, $$\frac{P}{2\sin A\cos A}=\frac{Q}{2\sin B\cos B}=\frac{R}{2\sin C\cos C}.$$

Now, in the triangle ABC,

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c},\ \text{and}\ \cos A=\frac{b^2+c^2-a^2}{2bc},\ \text{etc.}$$

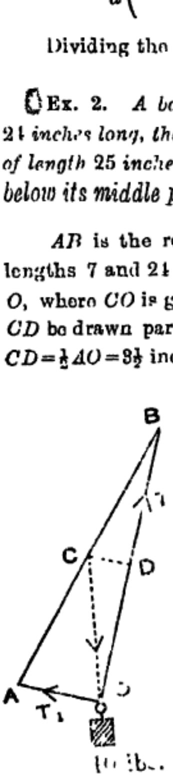
Hence, from above

$$\frac{P}{a\left(\dfrac{b^2+c^2-a^2}{bc}\right)}=\frac{Q}{b\left(\dfrac{c^2+a^2-b^2}{ca}\right)}=\frac{R}{c\left(\dfrac{a^2+b^2-c^2}{ab}\right)}.$$

Dividing the denominator throughout by abc, the result follows.

Ex. 2. *A body of mass 10 lbs. is suspended by two strings, 7 and 24 inches long, their other ends being fastened to the extremities of a rod of length 25 inches. If the rod be so held that the body hangs immediately below its middle point, find the tensions of the strings.* [*U. P. 1943*]

AB is the rod of length 25 inches, OA and OB the strings of lengths 7 and 24 inches by which the weight (10 lbs.) is suspended at O, where CO is given to be vertical, C being the mid-point of AB. If CD be drawn parallel to AO, then D is the middle point of OB, and $CD=\frac{1}{2}AO=3\frac{1}{2}$ inches.



Again, since $25^2=7^2+24^2$ identically, we get $AB^2=AO^2+BO^2$ and so $\angle AOB=1$ rt. $\angle$. Thus, $OC=\frac{1}{2}AB=12\frac{1}{2}$ inches.

Now, if T_1 and T_2 be the required tensions along OA and OB, since the three forces, T_1, T_2 and the vertical weight of 10 lbs. acting at O are in equilibrium, the triangle ODC, whose sides are evidently parallel to the forces, is a triangle of force and its sides will accordingly be proportional to the magnitudes of the forces.

Thus,

$$\frac{T_1}{CD}=\frac{T_2}{OD}=\frac{10}{OC},\ i.e.,\ \frac{T_1}{3\frac{1}{2}}=\frac{T_2}{12}=\frac{10}{12\frac{1}{2}}.$$

Hence, $T_1=2\frac{4}{5}$ lbs. wt. and $T_2=9\frac{3}{5}$ lbs. wt.

Alternatively,

by Lami's theorem in this case,

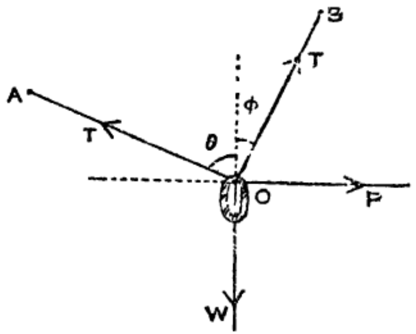
$$\frac{T_1}{\sin COB}=\frac{T_2}{\sin COA}=\frac{10}{\sin AOB}, \text{ or, } \frac{T_1}{\sin CBO}=\frac{T_2}{\sin CAO}=\frac{10}{\sin 90^\circ},$$

or, $$\frac{T_1}{\frac{7}{25}}=\frac{T_2}{\frac{24}{25}}=\frac{10}{1}, \text{ etc.}$$

Ex. 3. *A smooth ring of weight W can slide freely along a string which has its ends attached to two fixed points. The ring is pulled horizontally on one side with a force P. Find P, if in the equilibrium position, portions of the string are inclined at angles θ and φ to the vertical.*

Let T be the tension of the string, which must be same throughout the string, as it passes through a smooth ring.

Now the ring at O is in equilibrium under four forces, namely, the tension T, T on the two sides along OA and OB, the weight vertically downwards, and the horizontal force P.



Resolving horizontally and vertically for equilibrium, we get

$$P+T\sin\phi-T\sin\theta=0, \quad \text{or, } P=T(\sin\theta-\sin\phi)$$

and $$W-T\cos\phi-T\cos\theta=0, \quad \text{or, } W=T(\cos\phi+\cos\theta).$$

$$\therefore \quad \frac{P}{W}=\frac{\sin\theta-\sin\phi}{\cos\theta+\cos\phi}=\frac{2\sin\frac{1}{2}(\theta-\phi)\cos\frac{1}{2}(\theta+\phi)}{2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)}$$

$$=\tan\tfrac{1}{2}(\theta-\phi).$$

$$\therefore \quad P=W\tan\tfrac{1}{2}(\theta-\phi).$$

Examples on Chapter III

1. Three forces of magnitudes 3, 5 and 7 lbs. wt. acting on a particle keep it at rest. Find the angle between the two smaller forces.

2. Can a particle be kept at rest by three forces whose magnitudes are proportional to (i) 4, 5, 9; (ii) 4, 7, 9; (iii) 4, 15, 9 ?

3. (i) Examine whether three forces whose magnitudes are in the proportion 3 : 2 : 1, acting on a particle, can be kept in equilibrium under any circumstances. [*C. U. 1943*]

(iii) A light string suspended from a fixed point O has attached to it three equal masses, one at its lowest point C and the other two at A and B, A being above B. If T_1, T_2, T_3 be the tensions of the parts OA, AB, BC, show that $T_1 : T_2 : T_3 = 3 : 2 : 1$.

4. (i) Three equal forces acting at a point are in equilibrium; show that they are equally inclined to one another, and conversely.

(ii) Show that three concurrent forces lying in a plane cannot produce equilibrium for any arrangement of their directions, if the sum of two of them be less than the third.

5. Three forces acting at a point are in equilibrium; if they are proportional to

(i) 1, 1, $\sqrt{2}$; (ii) $\sqrt{3}+1$, $\sqrt{3}-1$, $\sqrt{6}$

find their inclinations to each other.

6. Find a point within (i) a triangle, (ii) a quadrilateral, such that the forces represented by the lines joining it to the angular points may be in equilibrium.

7. ABC is a triangle; D, E, F are the middle points of the sides BC, CA, AB respectively. Show that the forces acting on a particle and represented by the straight lines AD, BE, CF will maintain equilibrium. [*B.H.U. 1940*]

8. Three forces in equilibrium act perpendicularly to the sides of a triangle through any point in their plane within the triangle. Show that the forces are proportional to the corresponding sides of the triangle.

9. If P be any point in the plane of the triangle ABC, and D, E, F the middle points of its sides BC, CA, AB respectively, show that the forces AP, BP, CP, PD, PE, PF are in equilibrium.

10. Three forces act in given directions at a point O, and are in equilibrium. If a circle is drawn through O to cut the lines of action of the forces in A, B, C respectively, prove that the forces are proportional to the sides of the triangle ABC.

11. OA, OB, OC are three straight lines of equal length in one plane, and they are not all on the same side of any straight line passing through O. Forces P, Q, R act respectively along these lines, such that

$$\frac{P}{\text{area } OBC} = \frac{Q}{\text{area } OCA} = \frac{R}{\text{area } OAB};$$

show that P, Q, R are in equilibrium.

12. Forces P, Q, R acting along IA, IB, IC, where I is the in-centre of the triangle ABC, are in equilibrium; show that $P : Q : R = \cos \frac{1}{2}A : \cos \frac{1}{2}B : \cos \frac{1}{2}C$.

13. Forces P, Q, R acting along OA, OB, OC, where O is the circum-centre of the triangle ABC, are in equilibrium; show that

$$\frac{P}{a^2(b^2 + c^2 - a^2)} = \frac{Q}{b^2(c^2 + a^2 - b^2)} = \frac{R}{c^2(a^2 + b^2 - c^2)}.$$

14. O is the circum-centre of the triangle ABC, and L, M, N are the feet of the perpendiculars from A, B, C respectively on the opposite sides. If forces acting along OA, OB, OC are in equilibrium, show that they are proportional to the sides of the triangle LMN.

15. Forces X, Y act along the sides AB, AD respectively of a cyclic quadrilateral $ABCD$. If they are balanced by a force Z which acts along the diagonal CA from C to A, show that $X : Y : Z = CD : CB : BD$.

16. A transversal cuts the lines of action of three forces P, Q, R which act at the point O, and are in equilibrium, at the points A, B, C: show that (with a convention regarding sign)

$$\frac{P}{OA.BC} = \frac{Q}{OB.CA} = \frac{R}{OC.AB}.$$

17. If forces represented in magnitude, direction and sense by $(m-n)\,OP$, $(n-l)\,OQ$, $(l-m)\,OR$ be such that they are in equilibrium, prove that P, Q, R are collinear.

18. If four forces acting along the sides of a quadrilateral are in equilibrium, prove that the quadrilateral is a plane one.

19. (i) If one of the two intersecting forces be given in magnitude and direction, and the other has its line of action only given, prove that the least force which will produce equilibrium is perpendicular to the second force.

(ii) A particle weighing 10 lbs. is supported by two strings attached to it. If the direction of one string be at 30° to the vertical, find the direction of the other in order that its tension may be as small as possible; find also the magnitude of the tensions in the two strings in this case.

20. OD, OE, OF are drawn perpendiculars from the circum-centre O of the triangle ABC upon the sides BC, CA, AB. Show that the six forces represented by AO, BO, CO, OD, OE, OF are in equilibrium.

21. Forces acting at a point are represented in magnitude, direction and sense by AB, $2BC$, $2CD$, DA, DB where $ABCD$ is a square. Show that the forces are in equilibrium.

22. Forces acting at a point are represented in magnitude and direction by $2AB$, $3BC$, $2CD$, DA, CA and DB where $ABCD$ is a quadrilateral. Show that the forces are in equilibrium. [*C. U. 1937*]

23. A, B, C, X, Y, Z are six points in a plane, no three of which are collinear. Show that the forces BC, CA, AB, acting at X, Y, Z respectively are in equilibrium with forces ZY, XZ, YX acting respectively at A, B, C.

24. Coplanar forces whose magnitudes are proportional to the sides of a closed pologon act perpendicularly to those sides at their middle points, all inwards or all outwards. Prove that they are in equilibrium.

25. (i) Five equal forces so act on a particle that the angles between them in pairs in order are equal; show that the forces are in equilibrium.

(ii) Two given forces act at two given points of a body; if they are turned round those points in the same direction through any two equal angles, show that their resultant will pass through a fixed point.

26. If a transversal cuts the lines of action OA_1, OA_2, OA_3, .. OA_n of the forces P_1, P_2, P_3, ...P_n, which are in equilibrium, at the points A_1, A_2, A_3, ...A_n, then

$$\frac{P_1}{OA_1} + \frac{P_2}{OA_2} + \cdots + \frac{P_n}{OA_n} = 0.$$

27. A uniform plane lamina in the form of a rhombus, one of whose angles is $120°$, is supported by two forces applied at the centre in the directions of the diagonals so that one side of the rhombus is horizontal; show that if P and Q be the forces, and P be the greater, then $P^2 = 3Q^2$.

28. Two light rings slide on a smooth vertical circular wire and a thin string passing through the rings has two weights tied at its extremities. A third weight is attached to a point of the string between the rings, and the system is in equilibrium with the rings resting at points distant $30°$

from the highest point of the circular wire. Find the relation between the weights suspended.

Find also the pressure on the wire at one ring, if the middle weight be 10 lbs.

29. A light string is fastened to two points A, D at the same level, the length of the string exceeding the distance AD, and particles of weights 2 lbs. and 1 lb are fastened to it at two points B and C respectively. If AB, BC, CD make angles α, β, γ respectively with the horizontal, prove that

$$\tan \alpha = 2 \tan \gamma \pm 3 \tan \beta.$$

30. A series of equal weights are knotted at different points of a string, the two extremities of which are tied to two fixed points. Prove that, in the equilibrium position, the tangents of the inclination to the horizontal of the successive portions of the string are in A.P.

31. A string ABC has its extremities tied to two fixed points A and B in the same horizontal line; to a given point C in the string is knotted a given weight W. Prove that the tension in the portion CA is

$$\frac{Wb}{4c\triangle}(c^2 + a^2 - b^2),$$

where a, b, c are the sides and $\triangle$ the area of the triangle ABC.

***32.** (i) Three smooth nails are stuck on a vertical wall so as to form an equilateral triangle with its base horizontal. A light string carrying two equal weights at its extremities passes on them. Find the pressures on the nails.

(ii) Six smooth pegs form a regular hexagon. A loop of string passes round the pegs, fitting tightly against them. Prove that the pressure on each peg is the same.

33. If R be the pressure of a body of weight W on an inclined plane when the supporting force acts horizontally, and R' the pressure when the supporting force acts along the plane, then $RR' = W^2$.

34. Two forces P, Q acting parallel to the length and base of an inclined plane respectively, would each of them singly support a weight W on the plane ; prove that

$$\frac{1}{P^2} - \frac{1}{Q^2} = \frac{1}{W^2}.$$

35. Two planes AB, AC having a common height are inclined to the horizon at angles α and β respectively. Two weights, one in each plane, are kept in equilibrium by a string attached to the weights and passing over A. Show that the weights are as $AB : AC$.

*36. A body is supported on a smooth plane inclined at an angle α to the horizon by a force P_1 acting along the plane, and a horizontal force P_2. The inclination α, as also each of the forces P_1 and P_2 being halved, the body is still found to be at rest. Show that $P_1 : P_2 = 2\cos^2 \frac{1}{2}\alpha : 1$.

37. A weight of 30 lbs. is supported by a string fastened to a point on a smooth plane inclined at an angle 15° to the horizon and the string is only just strong enough to support a weight of 15 lbs. The inclination of the plane to the horizon being gradually increased, find when the string will break.

38. A weight is supported on a smooth plane of inclination α to the horizon by a string inclined to the vertical at an angle γ. If the slope of the plane be increased to β and the slope of the string is unaltered, the tension of the string is doubled to support the weight. Prove that

$$\cot \alpha - \cot \gamma = 2 \cot \beta. \qquad [\,C.\ U.\ 1945\,]$$

*39. A smooth tube in the form of a parabola is placed with its axis vertical and vertex downwards, and a heavy particle is placed within it ; show that the particle can be kept at rest by an outward force along an ordinate which varies as the ordinate, and that the corresponding reaction of the tube varies as the square root of its distance from the focus of the parabola.

*40. A small bead can slide on a smooth elliptic wire, being acted on by forces towards the foci which are proportional to the corresponding focal distances. Prove that the only positions of equilibrium are the extremities of the axes.

ANSWERS

1. $60°$. **2.** (i) Yes. (ii) Yes. (iii) No.

3. (i) Yes, when all the three forces act in the same line, the last two being in the same sense, and the first one opposite.

5. (i) $135°, 135°, 90°$. (ii) $75°, 165°, 120°$.

6. (i) The point of intersection of the medians.

(ii) The mid point of the line joining the middle points of any pair of opposite sides.

19. (ii) At right angles to the first string; $5\sqrt{3}$ lbs. wt. and 5 lbs. wt.

28. The weights are equal; $10\sqrt{3}$ lbs. wt.

32. (i) $W\sqrt{3}$ on the upper, and $\frac{1}{2}W(\sqrt{6}-\sqrt{2})$ on either of the lower.

37. When the inclination is $30°$.

CHAPTER IV

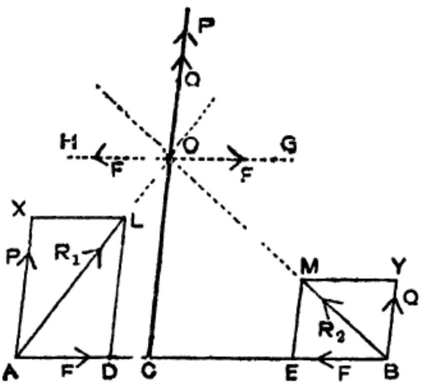
PARALLEL FORCES

4·1. In the previous chapters we have considered forces acting on a particle *i.e*, forces which pass through a point. We shall now consider forces acting on a rigid body. In such cases, it is often necessary to find the resultant of two forces which are parallel.

Two parallel forces are said to be *like* when they act in the same sense and they are said to be *unlike* when they act in opposite senses.

4·2. Resultant of two *like* parallel forces. (४·४·१)

Let two like parallel forces P, Q acting at points A, B respectively of a rigid body be represented by the lines AX, BY. Join AB.



At A apply a force of any magnitude F along AB. At B apply an equal and opposite force F along BA. Since these two forces balance each other, they will not affect the required resultant. Let these forces be represented by AD and BE.

Complete the parallelograms $ADLX$, $BEMY$; let the diagonals AL, BM be produced to meet at O. Through O draw OC parallel to AX or BY to meet AB in C, and draw HOG parallel to AB.

Now, the forces P at A and Q at B are equivalent to the forces P and F at A, and Q and F at B.

But the forces P and F at A are equivalent to their resultant, say R_1, represented by the diagonal AL. Let its point of application be transferred on its line of action to O. Then R_1 at O can be resolved into two component forces, parallel to their original directions, one F along OG, parallel to and in the sense AB, and the other P along CO.

Similarly, the forces Q and F at B are equivalent to their resultant, say R_2, represented by BM. Let its point of application be also transferred to O. Then R_2 at O can be resolved into two component forces, one F along OH parallel to BA, and the other Q along CO.

Thus the given forces are equivalent to two forces P and Q along CO, and two more forces each equal to F, acting in opposite directions OG and OH. The first two forces are equivalent to a single force $(P+Q)$ along CO, and the last two forces balance one another.

Hence, the resultant R of two like parallel forces P and Q is a like parallel force $(P+Q)$ acting through a point C in AB between the points of application of P and Q.

Position of the point C through which R acts.

Since $\triangle^s ACO$, ADL are similar,

$$\therefore \qquad \frac{AC}{CO}=\frac{AD}{DL}=\frac{AD}{AX}=\frac{F}{P}. \qquad \cdots \quad (1)$$

Again, since $\triangle^s BCO$, BEM are similar,

$$\therefore \quad \frac{BC}{CO} = \frac{BE}{EM} = \frac{F}{Q}. \qquad \cdots \quad (2)$$

Dividing (1) by (2), we get

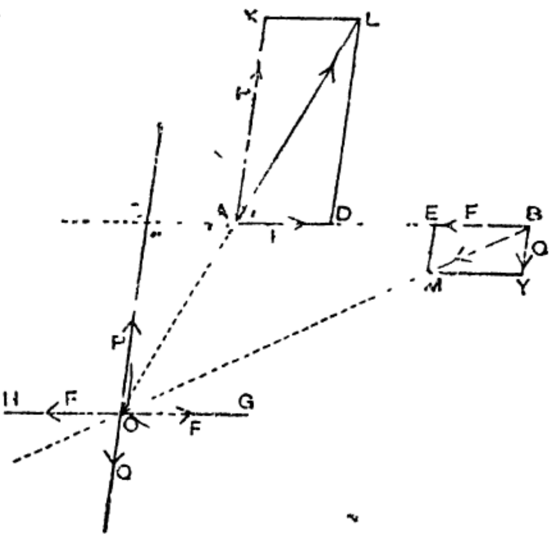
$$\therefore \quad \frac{AC}{CB} = \frac{Q}{P}, \qquad \cdots \quad (3)$$

i.e., C divides the line AB *internally* in the inverse ratio of the forces

Note 1. *The position of the point C remains unaltered whatever be the common direction of the forces P and Q.*

Note 2. When $P=Q$, C is the *mid-point* of AB.

4 3. Resultant of two *unlike* (unequal) parallel forces.



Let two unlike unequal parallel forces P, Q $(P > Q)$ acting at points A, B respectively of a rigid body be represented by the lines AX, BY. Join AB.

At A apply a force of any magnitude F along AB.

At B apply an equal and opposite force F along BA.

Since these two forces balance each other, they will not affect the resultant. Let these forces be represented by AD, BE.

Complete the parallelograms $ADLX$, $BEMY$; let their diagonals AL, BM when produced, meet at O. (Since the given forces are not equal, the diagonals are not parallel, and hence they always meet.)

Draw OC parallel to AX or BY to meet BA produced at C, and draw HOG parallel to AB.

Now, the forces P at A and Q at B are equivalent to forces P and F at A, and Q and F at B.

But the forces P and F at A are equivalent to their resultant, say R_1, represented by the diagonal AL. Let its point of application be transferred on its line of action to O. Then R_1 at O can be resolved into two component forces paralled to their original directions, one F along OG parallel to and in the sense AB, and the other P along OC.

Similarly, the forces Q and F at B are equivalent to their resultant, say R_2, represented by BM. Let its point of application be also transferred on its line of action to O. Then R_2 at O can be resolved into two component forces, one F along OH, parallel to and in the sense BA, and the other Q along CO.

Thus, the given forces are equivalent to two forces, P along OC and Q along CO, and two more forces, each equal to F acting in the opposite directions OG and OH.

The first two forces are equivalent to a single force $(P-Q)$ acting along OC, and the last two forces balance one another.

Hence, the resultant R of the two unlike unequal parallel forces P and Q $(P > Q)$ is a parallel force $(P-Q)$ acting in the direction of the greater force, through a point C, outside the points of application of the forces.

Position of the point C where R acts.

Since Δ^s OCA, AXL are similar,

$$\therefore \quad \frac{OC}{AO} = \frac{AX}{XL} = \frac{AX}{AD} = \frac{P}{F}.$$

$$\therefore \quad P.AC = F.OC. \qquad \cdots \quad (1)$$

Similarly, since Δ^s OCB, MEB are similar,

$$\therefore \quad \frac{OC}{CB} = \frac{ME}{EB} = \frac{BY}{BE} = \frac{Q}{F}.$$

$$\therefore \quad Q.CB = F.OC. \qquad \cdots \quad (2)$$

From (1) and (2), $P.AC = Q.CB$, $\cdots$ (3)

or, $\dfrac{AC}{CB} = \dfrac{Q}{P}$,

i.e., C divides AB *externally* in the inverse ratio of the forces.

Note. *When the parallel forces P and Q are unlike and equal,* Δ^s ADL, BEM being identically equal, $\angle DAL = \angle EBM$. Therefore, AL, BM are parallel; and hence they cannot meet at any finite distance. Hence, the geometrical construction for finding the resultant fails in such a case. Thus, we see that two equal unlike parallel forces cannot be compounded into a single force; in other words *there is no single force of which the effect on a body will be equivalent to the joint effect of two equal and unlike parallel forces.* Such a pair of forces is said to constitute a *couple* [see *Chap. VI*]. This case accordingly is called a *case of failure* for finding the resultant of two unlike parallel forces.

In case of two like parallel forces however, they always have a single resultant whether they are equal or unequal, for in this case AL and BM [*Fig., Art. 4 2*] will never be parallel, as can be easily seen.

4·4. Summing up.

If the parallel forces P and Q (whether like or unlike) have a resultant R, then (*i*) R is parallel to P and Q in the sense of the greater force ;

(*ii*) $R = P \pm Q$ (algebraic sum of P and Q) ;

(*iii*) $\dfrac{P}{Q} = \dfrac{BC}{CA}$.

$\therefore$ If $P > Q$, $BC > CA$. Hence the resultant passes nearer the greater force, dividing AB internally in case of like, and externally in case of unlike forces.

(*iv*) Again, from the above ratio, we get

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{P \pm Q}{BC \pm CA} = \frac{R}{AB}.$$

Hence it follows that if three parallel forces are in equilibrium, one is equal and opposite to the resultant of the other two, and each is proportional to the distance between the other two.

(*v*) The position of the point C is independent of the directions of P and Q. This point is usually referred to as the centre of the parallel forces P and Q, whatever be their common direction.

4·5. Resultant of a system of parallel forces.

(*i*) When the forces are *all like*.

Let $P_1, P_2, P_3, \ldots\ldots$ be a system of like parallel forces. First find the resultant R_1 of P_1 and P_2. Then R_1 is a like parallel force, and is equal to $P_1 + P_2$. Next obtain the resultant R_2 of R_1 and P_3 ; then $R_2 = R_1 + P_3 = P_1 + P_2 + P_3$, and R_2 is a like parallel force. In this way the final resultant R would be obtained, which will be a like parallel force and

$$R = P_1 + P_2 + P_3 + \cdots\cdots$$

(*ii*) When the forces are *not all like*.

Divide the forces into two sets of like parallel forces and let R_1, R_2 be the resultants of the two sets, which are obviously unlike parallel forces.

If $R_1 \neq R_2$, suppose $R_1 > R_2$; then $R_1 - R_2$ is the required resultant, which is parallel to the system of forces and is equal to their algebraic sum.

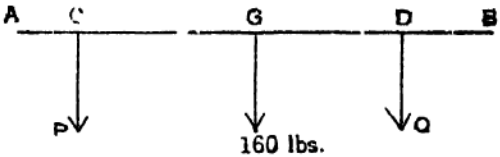
If $R_1 = R_2$, and their lines of action are coincident, the system is in equilibrium ; but if their lines of action are not coincident, they form a couple.

Note. For the point of application of the resultant see *Art. 10·1.*

4·6. Illustrative Examples.

Ex. 1. *Two men are carrying a straight uniform bar* 16 *ft. long and weighing* 160 *lbs. One man supports it at a distance of* 2 *ft. from one end, and the other man at a distance of* 3 *ft. from the other end. What weight does each man bear ?* [*C. U. 1945*]

Let AB be the uniform bar 16 ft. long and G be its middle point, so that its wt. 160 lbs acts at G, [because the weight of a uniform bar acts at its middle point, see *Art 10·5*].



Let the two men support the bar at C and D, so that $AC = 2$ ft., and $BD = 3$ ft., and let P and Q be the downward pressure on their shoulders. Then P, Q are the parallel components of the weight 160 lbs. of the bar acting at G. Hence by Art 4·2,

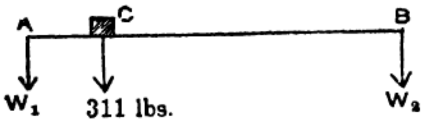
$$P + Q = 160 \cdots (1) \quad \text{and} \quad P.CG = Q.DG, \quad i.e., \quad 6P = 5Q \cdots (2)$$

whence, $P = 72\frac{8}{11}$, $Q = 87\frac{3}{11}$.

Hence the men bear the weights of $72\frac{8}{11}$ lbs. and $87\frac{3}{11}$ lbs. respectively.

Ex. 2. *Two men have to carry a block of stone of weight* 311 *lbs. on a light plank. How must the block be placed so that one of the men should bear the weight* 205 *lbs. more than the other ?* [*C. U. 1935*]

Let AB be the plank and C the point on AB where the stone is to be placed. Let W_1, W_2 ($W_1 > W_2$) be the weights which the two men at A and B have to bear. Then the weight of the stone acting at C is the resultant of the parallel forces W_1, W_2.



$\therefore \qquad W_1 + W_2 = 311; \qquad \cdots \quad (1)$

and $\qquad W_1.AC = W_2.BC. \qquad \cdots \quad (2)$

Also, it is given $W_1 - W_2 = 205. \qquad \cdots \quad (3)$

From (1) and (3), $W_1 = 258$, $W_2 = 53$.

$\therefore$ from (2), $\dfrac{AC}{CB} = \dfrac{W_2}{W_1} = \dfrac{53}{258}$.

Thus, the stone must be placed on the plank at a point dividing it in the ratio 53 : 258.

Examples on Chapter IV

1. A horizontal rod AB which is 4 ft. long (whose weight is negligible) rests on two props at its extremities ; a body of mass 60 lbs. is suspended from a point C such that $AC = 1$ ft. Show that the pressure at A is three times that at B.

2. The extremities of a straight bamboo pole 8 ft., long rest on two smooth pegs P and Q in the same horizontal line. A heavy load hangs from a point R of the pole. If $PR = 3RQ$, and the pressure at Q be 325 lbs. more than that at P, find the weight of the load. [*C. U. 1941*]

3. A heavy uniform rod rests on two pegs in the same horizontal line, 1 foot apart. If the pressure on the pegs are in the ratio 1 : 2, find the distances of the pegs from the middle point of the rod.

4. A uniform see-saw plank, 16 ft. long, weighs 1 cwt.

Find the position of the support when two children weighing 44 lbs. and 68 lbs. respectively, sit at the two ends.

[*P. U. 1945*]

5. Two men, one stronger than the other, have to remove a block of stone weighing 300 lbs. with a light pole whose length is 6 ft. ; the weaker man cannot carry more than 100 lbs. Where must the stone be fastened to the pole so as just to allow him his full share of weight.

[*B. E. Allahabad*]

6. A light horizontal plank of length 8 ft., on which is placed a load of 32 lbs. at a point 1 foot from one end, rests on supports at its ends. If the load be removed from its position and placed at the middle of the plank, find by how much the pressure on each support is altered.

7. If the position of the resultant of two like parallel forces P and Q is unaltered, when the positions of P and Q are interchanged, show that $P = Q$.

8. Two like parallel forces P and Q act at given points of a body ; if Q be changed to P^2/Q, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

9. A man carries a bundle at the end of a stick which is placed horizontally over his shoulder ; if the distance between his hand and his shoulder be changed, how does the pressure on his shoulder change ?

10. A man carries a bundle at the end of a stick 6 ft. long, which is placed on his shoulder. What should be the distance between his hand and shoulder, in order that the pressure on the shoulder may be three times the weight of the bundle ?

11. Show that the algebraic sum of the resolved parts of a pair of parallel forces, (not forming a couple), along any line in their plane is equal to the resolved part of their resultant along the same line.

12. (i) Show that the resultant of three equal like parallel forces acting at the angular points of a triangle passes through the centroid of the triangle.

(ii) Three equal like parallel forces act at the mid-points of the sides of a triangle; show that their resultant passes through the centroid of the triangle.

13. Three like parallel forces P, Q, R act at the angular points of a triangle. If their resultant passes through the centroid of the traingle, whatever be the common direction of the forces, then

$$P = Q = R.$$

14. Three like parallel forces P, Q, R act at the vertices A, B, C of the triangle ABC, and are respectively proportional to a, b, c. Show that their resultant passes through the in-centre of the triangle.

15. A force P acts along AO, where O is the circum-centre of the triangle ABC. Show that the parallel components of P acting at B and C are in the ratio $\sin 2B : \sin 2C$.

16. Three like parallel forces P, Q, R act at the vertices of the triangle ABC. If their resultant passes through the circum-centre in all cases, whatever be the common direction of the forces, show that

$$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}.$$

17. ABC is a triangle, and O any point within it; like parallel forces act at A, B, C, which are proportional to the areas BOC, COA, AOB respectively. Show that the resultant acts at O.

18. A line AB is divided into two parts at C. The resultant of two like parallel forces P and Q acting through the mid-points of AC and CB passes through C. If P and Q be interchanged in position, show that their resultant will pass through the mid-point of AB.

19. The resultant of two parallel forces P, Q at A, B, acts at C when like, and at D when unlike. Prove that if parallel forces whose magnitudes are equal to these resultant forces, act simultaneously at C, D, then A, B will be the points at which their resultant will act in the two cases of like and unlike directions.

20. If the magnitudes of two unlike parallel forces P, Q $(P > Q)$ be increased by the same amount, show that the line of action of the resultant will move further off from P.

21. P, Q are like parallel forces. If P is moved parallel to itself through a distance x, show that the resultant of P, Q moves through a distance $Px/(P+Q)$.

22. If the two like parallel forces P and Q acting on a rigid body at A and B be interchanged in position, show that the point of application of the resultant will be displaced along AB through a distance d where

$$d = \frac{P-Q}{P+Q} \cdot AB. \quad (P > Q)$$ [*C. U 1954*]

23. There are two like parallel forces P, Q. If two equal and unlike parallel forces S, S having their lines of action parallel to those of P and Q and distant b from one another be introduced anywhere in the plane, show that the resultant is displaced through a distance $bS/(P+Q)$.

24. The resultant of two like parallel forces P, Q passes through a point O; when P is increased by R and Q by S, the resultant still passes through O, and also when Q, R replace P, Q respectively; show that

$$S = R - \frac{(Q-R)^2}{P-Q}.$$

ANSWERS

2. 650 lbs. **3.** 4 inches, 8 inches. **4.** $7\frac{1}{2}$ ft. from the heavier child.

5. 4 ft. from the weaker man. **6.** 12 lbs.

9. The pressure varies inversely as the distance between his hand and shoulder.

10. 2 ft.

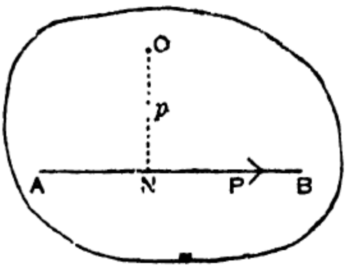
CHAPTER V

MOMENT OF A FORCE

5·1. Forces acting on a particle can produce a motion of translation only ; but forces acting upon a rigid body may produce either a motion of translation or of rotation or of translation and rotation both. The case of rotation introduces the idea of the *turning effect* or *moment* of a force which is defined as follows :

Def. *The* **moment** *of a force about a point is the product of the force and the perpendicular distance of the point from the line of action of the force.*

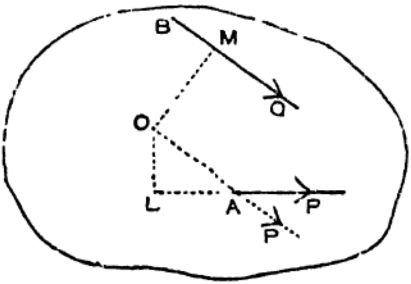
Thus, if P be a force, and p the length of the perpendicular ON drawn from a point O upon AB, the line of action of the force, then the moment of P about O is $P \times ON$ *i e.*, Pp.



Note. It is clear from above that if the line of action of P passes through O, its moment about that point is zero.

5·2. Physical significance of a Moment.

Let a body be capable of turning in a plane about a point ; for instance, let a plane lamina resting on a smooth horizontal table be pinned at O, about which it can turn. Let a force P be applied in its plane at a point A, say with the help of a string, one extremity of which is tied at A. Now, if it be pulled in a direction which when produced passes through O, it is common experience that the lamina does not move. If however the force be applied in a different direction, say LA, as in the figure, we can see that the body will turn about O in an anti-clockwise direction.



If instead, a force Q be applied at B in a direction BM, as in the figure, the lamina will turn about O in a clockwise direction. If both the forces be applied simultaneously, the direction of rotation about O due to a joint effect of these will depend not simply on the magnitude of P and Q, but also on the distances OL and OM of their lines of action from the point O. It will be experimentally observed that if $P.OL = Q.OM$, the body will not turn at all. On the other hand, it will rotate anti-clockwise or clockwise according as $P.OL$ or $Q.OM$ is the greater.

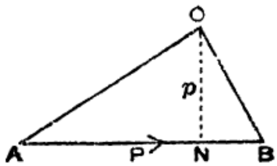
Thus, it is experimentally found that the magnitude of the tendency of rotation about O due to a force depends on the moment of the force about the point, and not on the magnitude of the force only. *The moment of a force about a point, therefore, is a fitting measure of the tendency of rotation of the body about the point caused by the application of the force.*

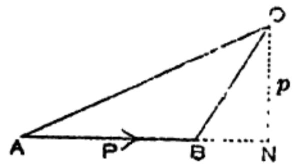
5·3. Sign of a Moment.

As mentioned above, the moment of a force about a point in a body represents the tendency of rotation of the body about the point due to the application of the force on it. Now, as already pointed out, on account of the situation of the point with respect to the line of action of the force, in some cases, the application of the force may cause the body to rotate anti-clockwise (as in the case of P along LA) and in other cases (as in the case of Q along BM), the rotation may be clockwise. The moments of the forces in the two cases about O are to be regarded as of opposite signs.

Although either direction of rotation may be chosen as positive, the usual convention is to regard *the moment in case of anti-clockwise tendency of rotation as positive*, and *in case of clockwise tendency of rotation, the moment is negative.*

5·4. Graphical representation of a Moment.





Let the force P be represented in magnitude, direction and line of action by AB. Let O be any point, and p the length of the perpendicular ON from O upon AB or AB produced. Join OA, OB. The moment of P about O is Pp *i.e.*, $AB \times ON = 2\triangle OAB$. Thus, *the magnitude of the moment of a force about a point is represented by twice the area of the triangle formed by joining the point to the extremities of the line representing the force.*

When proper sign is given to this expression as explained in the previous article, we get the moment completely in magnitude and sign.

5·5. Unit of Moment.

The moment of a unit force about a point at a unit perpendicular distance from the line of action of the force is defined as the unit for the measurement of moments. If the unit of force be a pound weight, and unit of distance be one foot, the unit of moment is a *foot-pound*. Similarly, if the unit of force be a gramme weight, and unit of distance be one centimetre, the unit of moment is a *centimetre-gramme*.

5·6. Varignon's Theorem.

The algebraic sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about that point.*

There are two cases to be considered.

Case (i). When the forces meet at a point.

Let the two forces P and Q act at a point A [as shown in figures (i) and (ii)] along AX and AY respectively, and let O be any point in their plane.

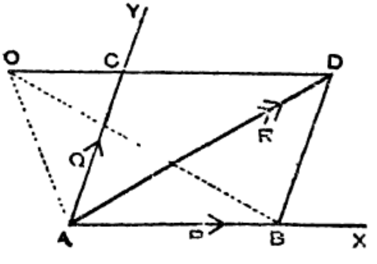


Fig. (i)

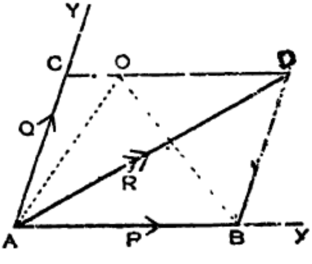


Fig. (ii)

Draw OC parallel to P to meet the line of action of Q in C. Now choose scale so that the length AC may represent the magnitude of Q, and on the same scale let AB represent P.

Complete the parallelogram $ABCD$, and join AD, OA, and OB. Then AD represents the resultant R of P and Q.

* Which do not form a couple.

Now in either figure, the moments of P, Q and R about O are represented by $2\triangle OAB$, $2\triangle OAC$, and $2\triangle OAD$ respectively.

In fig. (i), where O lies outside the $\angle BAC$, the moments of P and Q about O are both of the same sign, (positive in this figure), and their algebraic sum is represented by

$$\begin{aligned} 2\triangle OAB + 2\triangle OAC &= 2\triangle DAB + 2\triangle OAC \\ &= 2\triangle CAD + 2\triangle OAC = 2\triangle OAD \\ &= \text{moment of } R. \end{aligned}$$

In fig. (ii), where O lies within the $\angle BAC$, the moment of P being positive and that of Q being negative, their algebraic sum is equal to

$$\begin{aligned} 2\triangle OAB - 2\triangle AOC &= 2\triangle DAB - 2\triangle AOC \\ &= 2\triangle CAD - 2\triangle AOC \\ &= 2\triangle OAD \\ &= \text{moment of } R. \end{aligned}$$

Case (ii). When the forces are parallel.

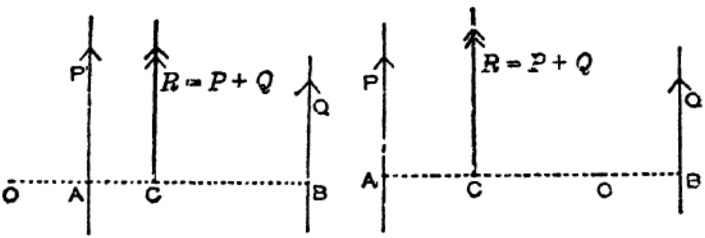


Fig. (i) Fig. (ii)

Let P, Q be two like parallel forces, and let O be any point in their plane.

Through O draw a line perpendicular to the lines of action of the forces P and Q to meet them in A, B respectively. Then by Art. 4·2, their resultant is the like parallel force $R = P + Q$, acting through C on AB, such that $P.AC = Q.BC$.

In fig. (i), the algebraic sum of the moments of P and Q about O is

$$\begin{aligned} & P.OA + Q.OB \\ & \quad = P(OC - AC) + Q(OC + CB) \\ & \quad = (P+Q)\,OC - P.AC + Q.CB \\ & \quad = (P+Q)\,OC = R.OC \\ & \quad = \text{moment of } R \text{ about } O \end{aligned}$$

In fig. (ii), where O is within AB, the algebraic sum of the moments of P and Q about O

$$\begin{aligned} & = -P.OA + Q.OB \\ & = -P.(OC + AC) + Q(BC - OC) \\ & = -(P+Q)\,OC - P.AC + Q.BC \\ & = -(P+Q)\,OC = -R.OC \\ & = \text{moment of } R \text{ about } O \\ & \quad \text{(taking into account its sign as in the figure)} \end{aligned}$$

Note. If the parallel forces are unlike and unequal, the theorem can be proved exactly in the same way.

Cor. It easily follows from above that the algebraic sum of the moments of any two forces about any point on the line of action of their resultant is zero, and conversely, if the algebraic sum of the moments of any two coplanar forces (which are not in equilibrium) about any point in their plane is zero, their resultant passes through that point.

5·7. Generalized theorem of Moments.

If any number of coplanar forces acting on a rigid body have a resultant, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant. [*Extension of Varignon's Theorem*]

Let P_1, P_2, P_3,... be the forces acting in a plane, and let O be the point in it about which the moments are taken. Further let the resultant of P_1 and P_2 be R_1, and the resultant of R_1 and P_3 be R_2 ; then R_2 is the resultant of P_1, P_2, P_3.

Similarly, let the resultant of R_2 and P_4 be R_3 and so on, till the final resultant R is obtained.

Now, by Art. 5·6, the algebraic sum of the moments of P_1 and P_2 about O is equal to the moment of R_1 about O.

Again, the algebraic sum of the moments of R_1 and P_3 *i.e.*, of P_1, P_2, P_3 about O is equal to the moment of R_2 about O: and so on, till all the forces have been taken. If we denote the perpendiculars from O on the lines of action of the forces $P_1, P_2, P_3, \ldots$ by $p_1, p_2, p_3, \ldots$ and if d be the perpendicular distance from O of the line of action of the final resultant R, then we have

$$\Sigma Pp = Rd.$$

Cor. 1. *It follows from above that if a system of coplanar forces be in equilibrium, the algebraic sum of their moments about any point in the plane is zero.*

Cor. 2. If O lies on the line of of action of the resultant, $d=0$ and hence $\Sigma Pp=0$. Hence, *the algebraic sum of the moments of any number of coplanar forces about any point in the line of action of their resultant is zero.*

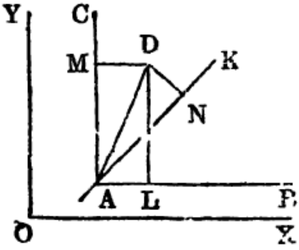
Cor. 3. Again, if $\Sigma Pp=0$, then $Rd=0$; hence either $d=0$, or $R=0$. Thus, *if the algebraic sum of the moments of any number of coplanar forces about any point in their plane be zero, either the resultant passes through the point, or the forces are in equilibrium.*

The above property enables us to determine the line of action of the resultant of a number of coplanar forces by determining the points through which the resultant passes.

5·7(A). Moment of a force about the point (x, y).

Let P be the given force acting along AK and D be the given point. Through any point A on the line of action of the force P, draw two perpendicular lines AB, AC parallel to the axes of co-ordinates OX, OY.

Let D be the point (x_1, y_1) about which the moment is to be taken. Join AD. Draw DN, DL, DM perpendiculars to AK, AB, AC respectively.



Let $\angle KAB = \theta$ and $\angle DAK = \alpha$.

The components of the force P, along AB, AC are $P\cos\theta$, $P\sin\theta$ respectively.

The sum of the moments of the components of the force P about the point D

$$= P\cos\theta . DL - P\sin\theta . DM \qquad \cdots \quad (1)$$

$$= P\cos\theta . AD\sin(\theta+\alpha) - P\sin\theta . AD\cos(\theta+\alpha)$$

$$= P . AD\{\sin(\theta+\alpha)\cos\theta - \cos(\theta+\alpha)\sin\theta\}$$

$$= P . AD\sin\alpha$$

$$= P . DN$$

= the moment of the force P about D.

Thus, *the moment of a force about a point is equal to the algebraic sum of the moments of its components about that point.*

Let X, Y be the components of the force P parallel to the co-ordinate axes and (h, k) be the co-ordinates of A. *The moment of the force P about the point* $D(x_1, y_1)$ is

$$\mathbf{X(y_1 - k) - Y(x_1 - h).} \qquad \cdots \quad (2)$$

If A coincides with O *i.e.*, if the line of action of the force P passes through the origin, h, k become zero. In this case, *the moment of the force P about* $D(x_1, y_1)$ is

$$\mathbf{Xy_1 - Yx_1.} \qquad \cdots \quad (3)$$

Note. In the Examples on Chapter V, for Ex. 36, apply formula (2) and for Ex. 37, apply formula (3).

5·8. Moment of a force about an axis.

So far we had confined ourselves to the consideration of two-dimensional cases only, where forces are confined to act in one plane, and the body is capable of turning in the same plane about some point in it. Now let us consider the more general case of a solid body capable of turning about a fixed line as an axis (a door capable of turning about the line of hinges being an example). A force acting on the body at any point in any manner, it is seen that if the line of action of the force passes through the axis of rotation, or else is parallel to that axis, the body will not turn. On

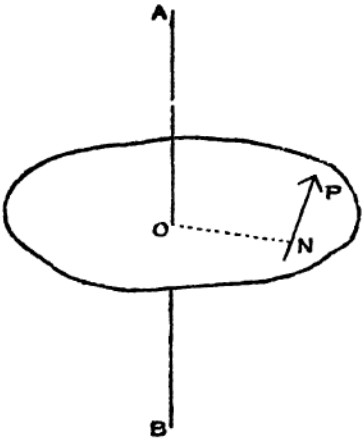


Fig. (i)

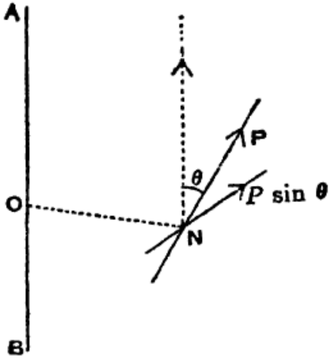


Fig. (ii)

the other hand, if the line of action of the force does not intersect the axis of rotation, nor is parallel to it, the body will turn about the axis. The measure of the tendency of rotation in this case necessitates the definition of the moment of a force about a line as follows :

When a force P acts on a body in a direction perpendicular to a line AB in the body, but not intersecting it, *i.e.*, when P acts in a plane perpendicular to AB, as in

Fig. (i), the moment of the force P about the line AB is defined to be **P.ON,** where ON is the perpendicular distance between the line of action of P and the line AB about which the moment is to be taken.

When P acts in any direction (not necessarily perpendicular to AB) as in Fig. (ii), let ON be the shortest distance between AB and the line of action of P. If now P, assumed to be acting at N on its line of action, be resolved into two perpendicular components, one parallel to AB and the other perpendicular to it, the moment of P about AB is the product of the resolved part of P perpendicular to AB and the shortest distance ON between AB and the line of action of P; in other words, the moment of P about AB in this case is **P sin θ.ON.**

Note 1. The moment of P about AB is zero, if either (i) P is parallel to AB or else (ii) if the line of action of P intersects AB.

Note 2. It must be borne in mind that when in the two-dimensional case we speak of a body in the form of a lamina rotating about a point in its plane, it really rotates about an axis perpendicular to the plane through the point in question. Moment of a force about a point in its plane in two-dimensions is therefore nothing but the particular case of the moment about an axis perpendicular to the plane of the force through the point.

Note 3. As in case of Varignon's theorem in two dimensions, we can show in the general case of a solid body acted on by a system of forces, that *if a system of forces acting on a body have a resultant, the algebraic sum of their moments about any line in the body is equal to that of their resultant.*

Hence, *if a system of forces, acting on a body generally keeps it at rest, the algebraic sum of their moments about any line in the body is zero.*

5·9. Illustrative Examples.

Ex. 1. *Three forces P, Q, R act along the sides BC, CA, AB of a triangle ABC. Their resultant lies in the line joining in-centre and centroid of the $\triangle ABC$. Show that*

$$\frac{P}{a(b-c)}=\frac{Q}{b(c-a)}=\frac{R}{c(a-b)}.$$

Let I and G be the in-centre and centroid of $\triangle ABC$ and r the in-radius. Then the perpendiculars from I on the sides are each equal to r. Let GL, GM, GN be the perpendiculars from G and p_1, p_2, p_3 be the perpendiculars from A, B, C on BC, CA, AB; then $GL=\frac{1}{3}p_1$, and $ap_1=2\triangle ABC$. $\therefore\ GL=\frac{2}{3}\triangle.\frac{1}{a}$. Similarly, $GM=\frac{2}{3}\triangle.\frac{1}{b}$, $GN=\frac{2}{3}\triangle.\frac{1}{c}$. Since the resultant passes through I and G, hence the algebraic sum of the moments of the forces about each of the two points is zero;

$$\therefore \quad P.r+Q.r+R.r=0, \quad i.e., \quad P+Q+R=0, \qquad \cdots (1)$$

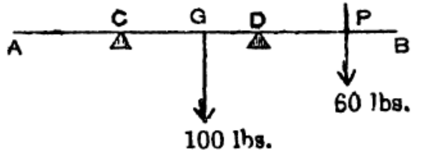
and $\quad P.GL+Q.GM+R.GN=0,$

$$i.e., \quad P.\tfrac{2}{3}\triangle.\frac{1}{a}+Q.\tfrac{2}{3}\triangle.\frac{1}{b}+R.\tfrac{2}{3}\triangle.\frac{1}{c}=0,$$

$$\text{or,} \quad P.\frac{1}{a}+Q.\frac{1}{b}+R.\frac{1}{c}=0, \ i.e., \ P.bc+Q.ca+R.ab=0. \quad \cdots (2)$$

From (1) and (2) by cross-multiplication, we get the required result.

Ex. 2. *A narrow uniform plank 20 ft. long weighing 100 lbs. is supported in a horizontal position on two posts, one 5 ft. from one end and the other 8 ft. from the other end of the plank. A boy weighing 60 lbs. walks on it starting from the latter post towards the corresponding end. Find how far it is safe for him to walk. What are reactions of the posts when he is furthest from the starting point without upsetting the plank?* [*C. U. 1935*]



Let AB be the plank placed upon two posts C and D, so that $AC=5$ ft. and $BD=8$ ft. The wt. of 100 lbs. of the plank acts at G, the mid-point of AB; then $DG=2$ ft.

Let P be the position of the boy between D and B, beyond which he cannot walk safely without upsetting the plank, and let $DP=x$.

In this position as the plank is on the point of being upset about D, the contact with the support at C is just broken and the reaction at C is zero then. Now taking moment about D, we have

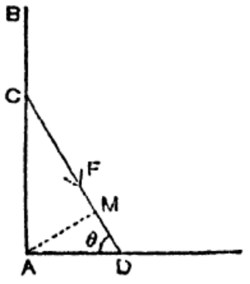
$$60.PD = 100\ GD, \quad i.e., \quad 60x = 100 \times 2, \quad \therefore \quad x = 3\tfrac{1}{3} \text{ ft.}$$

For the position, as already remarked, reaction of the post C is zero and reaction of the point D balancing the resultant of the weights at G and P (which are like parallel forces) is equal to

$$100 + 60 = 160 \text{ lbs. wt.}$$

Ex. 3. *One end of a stout rope of length 20 ft is fixed to a vertical telegraph post standing on the ground, and a man pulls at the other end with a given force. Find the point of the post at which the rope is to be fixed in order that the man will have the best chance of over-turning the post.* [*C. U. 1944*]

Let AB be the telegraph post, A being the base and C the point to which the rope CD must be fixed where D is the position of the man on the ground.



Then $CD = 20$ ft.

From A draw AM perp. to CD and let $\angle ADC = \theta$. Let F be the force acting along CD.

The moment of F about A

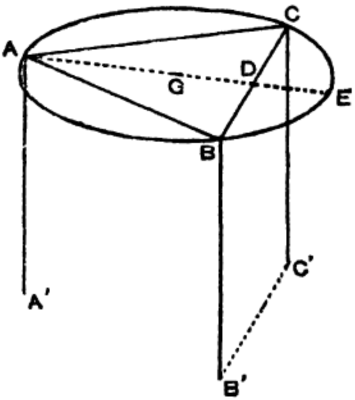
$$= F \times AM = F \times AD \sin\theta = F \times CD \cos\theta \sin\theta$$
$$= F \times \tfrac{1}{2} CD \sin 2\theta$$
$$= 10F.\sin 2\theta.$$

This is greatest when $\sin 2\theta = 1$ *i.e.*, $2\theta = 90°$, *i.e.*, $\theta = 45°$.

Then $AC = CD \sin 45° = 20 \times \frac{1}{\sqrt{2}} = 10\sqrt{2}$ ft.

Thus, the rope is to be fixed at a height of $10\sqrt{2}$ ft. from the ground.

Ex. 4. *A round table of weight W stands on three legs, of which the upper ends are attached to its rim so as to form an equilateral triangle. Show that a body whose weight does not exceed W may be placed anywhere on the table without the risk of toppling it over.* [*C. U. 1913*]



Let ABC be a circular table and let A, B, C be the upper ends of the legs attached to the rim such that ABC is an equilateral triangle and let A', B', C' be the points of contact of the legs with the floor.

Let G be the centre of the table through which its weight W acts. There is a chance of overturning if any weight is placed on the portion of the table outside the triangle ABC, say in the portion BEC, and the table will, if it turns at all in this case, turn about the line $B'C'$, and when it is on the point of being overturned, A' just loses contact with the floor and the weight placed and the weight of the table have equal moments about $B'C'$, *i.e.*, about BC. Now, the weight will clearly have the greatest turning effect when placed farthest away from BC *i.e.*, when placed at E, the mid-point of the arc BEC.

Since ABC is an equilateral triangle, $AGDE$ (D being the mid-point of BC) is perp. to BC. Let X be the weight placed. Then taking moment about BC,

$$X.ED = W.GD. \quad \cdots \quad (1)$$

Join GC, then from $\triangle GCD$, $GD = GC \sin 30° = \frac{1}{2}GC = \frac{1}{2}GE$,

$$\therefore \quad GD = DE.$$

$\therefore$ from (1), $\qquad X = W$,

i.e., the greatest value of X when the table just not overturns is W.

The same value of X would be obtained when placed in the portion on the side of AB or AC, opposite to the triangle ABC.

Hence W is the greatest weight that can be placed anywhere on the table without toppling it over.

It may be noted that if the weight be placed within the triangle ABC, its moment about BC or CA or AB being of the same sign as that of the weight of the table, there is no chance of the table being overturned whatever the weight may be.

Examples on Chapter V

1. AB is a diameter of a circle and AC, AD are chords at right angles to one another. Show that the moments of the forces represented by AC, AD about B are equal.

2. Forces 2, 4, 6, 8, 10 and 12 lbs. wt. act respectively along the sides AB, BC, CD etc. in order, of a regular hexagon each of whose sides is $\sqrt{3}$ feet. O is the centre of the hexagon and on AB an equilateral triangle $O'AB$ is drawn on the side opposite to the hexagon. Find the algebraic sum of the moments of the forces about O, A and O'.

3. A uniform beam AB 16 feet long and weighs 50 lbs ; masses of 20 and 50 lbs. are suspended from A, B respectively. At what point must the beam be supported so that it may rest horizontally ?

4. A metre rule of negligible weight carries weights 1, 2, 3,......100 gms. attached to marks 1, 2, 3,...... 100 cm. Find the point about which it will balance.

5. Masses of 1 lb. 2 lbs., 3 lbs., 4 lbs., and 5 lbs. are suspended from a uniform horizontal rod AB, 10 ft. long, weighing 3 lbs. and supported at its ends, at distances of 1 foot, 2 feet, 3 feet, 4 feet and 5 feet from A. Find the pressure on the supports.

6. The horizontal roadway of a bridge AB is 36 ft. long, weighs 5 tons and rests on two supports at its ends. What is the pressure on each support when a lorry of weight 3 tons starting from A is two-thirds of the way across the bridge ?

7. Prove that if four forces acting along the sides of a square are in equilibrium, they must be equal in magnitude.

8. (i) Show that the sum of the moments of the forces represented in magnitude, direction, sense and line of action by AD, BE, CF where D, E, F are the mid-points of the sides BC, CA, AB of $\triangle ABC$, about each of the points A, B, C is zero.

(ii) If D, E, F be points on the sides BC, CA, AB of a triangle ABC such that $BD : DC = CE : EA = AF : FB$, prove that the algebraic sum of the moments of the forces represented by AD, BE, CF, about each of the points A, B, C are equal.

9. Three forces P, Q, R acting at the vertices A, B, C respectively of a triangle, each perp. to the opposite side, keep it in equilibrium. Prove that

$$P : Q : R = a : b : c.$$

10. If three forces P, Q, R acting along the bisectors of the angles of a triangle, at the angular points A, B, C respectively, keep the triangle in equilibrium, show that

$$P : Q : R = \cos \tfrac{1}{2}A : \cos \tfrac{1}{2}B : \cos \tfrac{1}{2}C.$$

11. Three forces acting along the medians of a triangle, all from the vertices, are in equilibrium. Show that the forces are proportional to the lengths of the medians.

12. Forces P, Q, R act from the angular points of a triangle ABC, perpendicular to the opposite sides. Prove that if their resultant passes through the circum-centre,

$$P(b\cos C - c\cos B) + Q(c\cos A - a\cos C) + R(a\cos B - b\cos A) = 0.$$

13. Forces $l.BC$, $m.CA$, $n.AB$ act along the sides of a triangle ABC taken in order; show that their resultant passes through the centroid of the triangle if $l+m+n=0$.

14. Three forces act along the sides of a triangle taken in order. If the sum of two of the forces be equal in magnitude but opposite in sense to the third force, then their resultant passes through the in-centre of the triangle.

15. If four forces, each acting along a side of a cyclic quadrilateral, be in equilibrium, show that each force is proportional to the opposite side.

16. Three forces P, Q, R act in the same sense along the sides BC, CA, AB of a triangle ABC; show that if their resultant passes through

(i) the *in-centre*, $P+Q+R=0$;

(ii) the *centroid*, $P\operatorname{cosec} A + Q\operatorname{cosec} B + R\operatorname{cosec} C = 0$;

(iii) the *circum-centre*, $P\cos A + Q\cos B + R\cos C = 0$;

(iv) the *ortho-centre*, $P\sec A + Q\sec B + R\sec C = 0$.

17. Forces P, Q, R act along the sides BC, CA, AB of the triangle ABC. If the line of action of their resultant passes through the *in-centre* and the *circum-centre* of the triangle, prove that

$$\frac{P}{\cos B - \cos C} = \frac{Q}{\cos C - \cos A} = \frac{R}{\cos A - \cos B},$$

or, $$\frac{P}{(b-c)(b+c-a)} = \frac{Q}{(c-a)(c+a-b)} = \frac{R}{(a-b)(a+b-c)}.$$

*18. Three forces P, Q, R act in the same sense along the sides BC, CA, AB of the triangle ABC. If their resultant passes through

(i) the *ortho-centre* and the *centroid*,

$$\frac{P}{\sin 2A \sin (B-C)} = \frac{Q}{\sin 2B \sin (C-A)} = \frac{R}{\sin 2C \sin (A-B)};$$

(ii) the *ortho-centre* and the *circum-centre*,

$$\frac{P}{(b^2-c^2)\cos A} = \frac{Q}{(c^2-a^2)\cos B} = \frac{R}{(a^2-b^2)\cos C},$$

or,
$$\frac{P}{\sin 2A \sin (B-C)} = \frac{Q}{\sin 2B \sin (C-A)} = \frac{R}{\sin 2C \sin (A-B)};$$

(iii) the *in-centre* and the *ortho-centre*,

$$\frac{P}{\cos A (\cos B - \cos C)} = \frac{Q}{\cos B (\cos C - \cos A)} = \frac{R}{\cos C (\cos A - \cos B)}.$$

(iv) the *centroid* and the *circum-centre*,

$$\frac{P}{\sin 2A \sin (B-C)} = \frac{Q}{\sin 2B \sin (C-A)} = \frac{R}{\sin 2C \sin (A-B)}.$$

19. A uniform beam 10 ft. long and weighing 60 lbs. rests on two props at equal distances from the ends. Find the maximum value of this distance so that a man weighing 10 stones may stand anywhere on the beam without upsetting it.

20. Eight feet of a plank. 24 ft. long and weighing 200 lbs. project over the side of a quay. What least weight must be placed on the end of the plank so that a man weighing 150 lbs. may be able to walk to the other end without the plank tilting over?

21. A uniform rod of length 6 ft. and weight 2 lbs. rests horizontally on two props at its extremities, each of which will bear a maximum weight of 13 lbs. Find on what part of the rod a weight of 16 lbs. can be placed without breaking either support.

22. A non-uniform rod 16 inches long rests on two pegs 9 inches apart, with its centre midway between them. The greatest masses that can be suspended in succession from the two ends without disturbing the equilibrium are 4 lbs. and 5 lbs. respectively. Find the weight of the rod and the position of the point at which its weight acts.

***23.** A uniform plank of length $2a$ and weight W is supported horizontally on two vertical props at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respectively. Show that

$$\frac{W_1}{W+W_1}+\frac{W_2}{W+W_2}=\frac{b}{a}.$$

24. A heavy carriage wheel of weight W and radius r, is to be dragged over an obstacle of length h, by a horizontal force P applied in the centre of the wheel. Show that P must be slightly greater than

$$W\cdot\frac{\sqrt{2hr-h^2}}{r-h}.$$

25. A man tries to uproot a tree with the help of a rope of length 30 feet, by fastening one extremity at some point of the vertical stem and pulling at the other end from the ground. The least moment about the foot of the tree necessary to uproot it is 1200 ft.-lbs. Find the least force that the man has to apply.

***26.** A smooth bamboo pole just stands vertically on the ground, and a horizontal rope which is once wrapped at its top has the two portions at right angles to one another.

The pole is kept in position by pulling it with a rope attached at one-third the height of the pole. If this latter rope be inclined at an angle 45° with the horizon, prove that the tension in it must be six times that of the rope at the top.

27. If the moments of two given intersecting forces about a point in their plane be equal and of the same sense, prove that the point must be on a certain straight line.

28. The magnitude of a force and also its moments (of the same sign) about two given points are given. Find its line of action.

29. Forces are represented in magnitude, direction and line of action and sense by the perpendiculars drawn from the angular points of a triangle to the opposite sides. If their sum of moments about each of the angular points is zero, show that the triangle is equilateral.

30. If three forces represented in magnitude and direction by the bisectors of the angles of a triangle, all acting from the vertices, be in equilibrium, the triangle must be equilateral.

31. The sums of the moments of a system of forces acting at a point about two given points are equal in magnitude. Show that their resultant is parallel to a fixed line or passes through a fixed point.

32. Of four coplanar forces in equilibrium, one is given completely, a second and a third, which are not parallel, have their lines of action given, while the fourth has its magnitude only given. Prove that the line of action of the fourth force must touch a fixed circle. [*C. U. 1934*]

33. *ABC* is a right-angled triangle, the sides *BC*, *CA*, *AB* being 13, 12 and 5 units of length respectively. The moments of a force *F* about *A*, *B*, *C* are,

(i) 0, 25 and 144 units of moment respectively ; [*C. U. 1936*]

(ii) 0, -25 and 144 units of moment respectively ; find in each case the magnitude, direction and line of action of *F*.

34. ABC is an isosceles right-angled triangle whose equal sides AB, AC are 4 ft. in length ; the moments of a force about the points A, B, C are respectively 8, 8 and 16 units in the same sense ; find the magnitude and the line of action of the force.

35. Forces 1, 2, 4, 5 lbs. wt. act, all in the same sense, along the sides of a square taken in order. Prove that their resultant is parallel to a diagonal and find where it cuts the side along which the first force acts. [*C. U. 1937*]

36. The moments of a force about the points (0, 0), (10, 0), (0, 5) are 184, -46 and 249 foot-pounds. Find where the force meets the axis of x and find its components parallel to the co-ordinate axes.

***37.** OX and OY are two straight lines at right angles, and a force acting in their plane at O has moments G and G' about the two points whose co-ordinates are (x, y) and (x', y') respectively with respect to the lines OX and OY as axes of co-ordinates. If $(xy' - x'y)$ is not zero, prove that the magnitude R of the force and the angle θ between the line of action and OX are given by

$$R^2 = \frac{(xG' - x'G)^2 + (yG' - y'G)^2}{(xy' - x'y)^2}$$

and $$\tan\theta = \frac{yG' - y'G}{xG' - x'G}.$$ [*C. U. 1946*]

***38.** Like parallel forces P, Q, R act at the vertices of a triangle ABC perpendicular to its plane. If the resultant passes through

(i) the *in-centre* of the triangle,

$$P : Q : R = \sin A : \sin B : \sin C ;$$

(ii) the *circum-centre* of the triangle,

$$P : Q : R = \sin 2A : \sin 2B : \sin 2C ;$$

(iii) the *ortho-centre* of the triangle,

$$P : Q : R = \tan A : \tan B : \tan C.$$

***39.** A square table stands on four legs placed at the mid-points of its sides. If the total weight of the table and legs be W, find the greatest weight which can be put at one of the corners of the table without upsetting it.

***40.** A circular table of weight W has four legs spaced at equal distances round its edge. Show that the least weight sufficient to overturn the table is $(\sqrt{2}+1)\,W$.

ANSWERS

2. 63, 63 and 81 ft.-lbs. respectively.

3. 6 ft. from B. **4.** 67th mark. **5.** 7 lbs. wt.; 11 lbs. wt.

6. $3\frac{1}{2}$ tons wt. at A, $4\frac{1}{2}$ tons wt. at B. **19.** $1\frac{1}{2}$ ft. **20.** 25 lbs.

21. Within a distance $1\frac{1}{3}$ ft. from the middle point on either side.

22. $3\frac{1}{3}$ lbs., $\frac{1}{2}$ inch from the mid-point. **25.** 80 lbs. wt.

33. (i) $F=13$, acting along the tangent at A to the circum-circle of $\triangle ABC$. (ii) $F=13$, acting along the perpendicular from A on BC.

34. 2 units acting parallel to AB at a distance 4 ft. from it on the opposite side of C.

35. Divides it externally in the ratio 2 : 3.

36. At a distance 8 ft. from the origin; 13 lbs. wt. along x-axis and 23 lbs. wt. along y-axis,

39. W.

CHAPTER VI

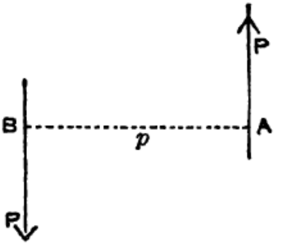
COUPLES

6·1. We have seen in Chapter IV that the general method of finding the resultant of two equal and unlike parallel forces fails, *i.e.*, there is no single force whose effect is the same as the joint effect of two equal and unlike parallel forces. Hence such a pair of forces acting upon a rigid body cannot produce a motion of translation.

Two equal and unlike parallel forces (whose lines of action are not the same) are said to constitute a **couple.**

The **arm** of a couple is the perpendicular distance between the lines of action of the two forces forming the couple.

The **moment** *of a couple is the product of either of the forces forming the couple, and the perpendicular distance between their lines of action (i.e., the arm).*



A couple, each of whose forces is P and whose arm is p, as in the above figure, is very often denoted by (P, p).

The whole effect of a couple acting on a rigid body is to produce rotation without imparting to it any motion of translation.

The moment of a couple is considered *positive* or *negative* according as the couple tends to rotate the body in the anti-clockwise or clockwise direction.

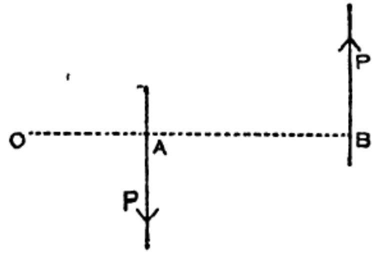
Examples of a couple are the forces applied to the key of a clock in winding it up, or the forces applied by the hand to the handle of a door in opening it.

Couple is called by some writers *Torque.*

6·2. Theorem.

The algebraic sum of the moments of the two forces forming a couple about any point in their plane is a non-zero constant and equal to the moment of the couple.

Let each of the two forces forming the couple be P and O be any point in their plane. Through O draw a line OAB perpendicular to the lines of action of the forces meeting them in A and B.



The algebraic sum of the moments of the forces about O

$$= P.OB - P.OA = P.(OB - OA)$$
$$= P.AB$$

which is constant (*i.e.*, independent of the position of O) and which is equal to the moment of the couple.

Note. The moment of a couple can never be zero, for then the two forces cancel each other.

6·2 (A). *If the algebraic sum of the moments of any two forces acting on a rigid body about any point in their plane is a constant* ($\neq 0$), *then the two forces form a couple.*

Let P, Q be two given forces ; they cannot meet at a point, for then the algebraic sum of the moments of the two forces about that point would be zero, which is contrary to hypothesis. So they must be parallel. Now, if they be like, or unlike and unequal, they will have a resultant and the sum of their moments about any point on the resultant will be zero, which is also contrary to hypothesis. Thus, the two forces cannot meet at a point, nor can they be parallel and like and parallel, unlike and unequal. Hence the two forces must be parallel, unlike and equal and hence they form a couple.

Cor. Here *three forces* acting upon a body form a couple. We easily see that the algebraic sum of the moments of the forces about each of the point A, B, C is equal and equal to the moment of the couple (*viz.*, twice the area of the triangle).

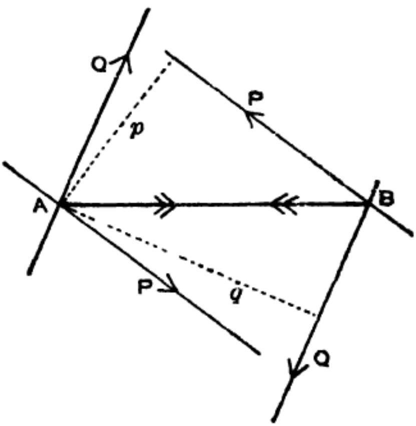
6·3. Equilibrium of two couples.

Theorem. *If two couples, whose moments are equal and opposite, act in the same plane upon a rigid body, they balance one another.*

Let (P, p) and (Q, q) be the given couples, so that

$$P.p = Q.q \text{ in magnitude.} \qquad \cdots \qquad (1)$$

Case I. When the forces forming the couples are not all parallel.



Let one of the forces P of one of the couples intersect one of the forces Q of the other in A, and let the other two forces meet in B:

Now, the sum of the moments about A of P and Q acting at $B = Pp - Qq = 0$ by (1).

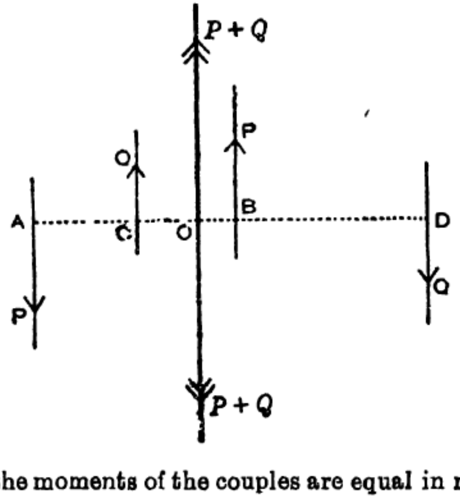
$\therefore$ the resultant of P and Q at B passes through A, and hence acts along BA from B to A. Similarly, the resultant of P and Q at A acts along AB, from A to B.

Since the two forces at A are respectively equal and opposite to those at B and since both pair act at the same angle, their resultants must be equal in magnitude and opposite in sense ; and as they act along AB and BA, they cancel each other.

Thus, the resultant of the forces forming the two couples is nil, and hence the two couples balance one another.

Case II. When the forces forming the couples are all parallel.

Draw a straight line perpendicular to the lines of action of the forces, meeting them at A, C, B, D.



Since the moments of the couples are equal in magnitude, we have

$$P.AB = Q.CD. \qquad \cdots \quad (1)$$

Let the resultant $(P+Q)$ of the like parallel forces P at B and Q at C act at O; then

$$P.BO = Q.CO. \qquad \cdots \quad (2)$$

Subtracting (2) from (1),

$$P(AB - BO) = Q(CD - CO),$$

$$i.e., \quad P.AO = Q.DO.$$

Thus, the resultant $(P+Q)$ of the like parallel forces P at A and Q at D also acts at O.

Since these two resultants are equal in magnitude and opposite in directions and act at the same point, they are in equilibrium, and hence the two couples balance each other.

6·4. Equivalence of two couples.

As a corollary to the above theorem we get the following :

Two couples in the same plane whose moments are equal and of the same sign, are equivalent to one another.

For, by reversing the constituent forces of any couple, all the forces will be in equilibrium.

It follows therefore, that a couple acting in any manner in a plane can be replaced by any other couple in the same plane, provided the moment of the latter is equal to that of the former, and of the same sign. It is immaterial what the direction of the constituent forces of the second couple may be, or their magnitude, or the arm.

Thus, a couple (P, p) may be replaced by a couple $\left(F, \dfrac{Pp}{F}\right)$ in the same plane with its constituent forces each equal to F, the arm being such that the moment remains unaltered. Also one force F may be taken to be acting in any line and sense, the other at the distance $\dfrac{Pp}{F}$ being on that side so as to make the sign of the moment same as that of (P, p).

Similarly, a couple (P, p) may be replaced by a couple $\left(\frac{Pp}{x}, x\right)$ with a given arm x anywhere in the plane.

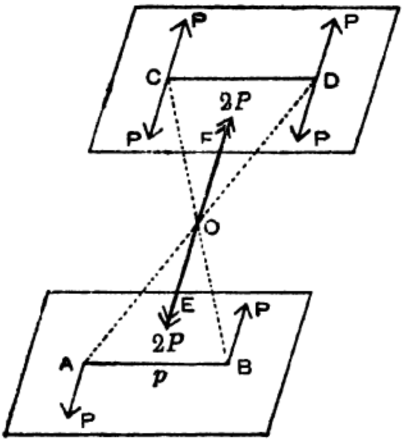
6·5. Couples in parallel Planes.

The effect of a couple is not altered if it is transferred to a parallel plane, provided its moment is unchanged in magnitude and sign.

Let AB be the arm of the couple (P, p) and let CD be a straight line equal and parallel to AB, lying in a plane parallel to the plane of the couple.

Join AD, BC and let O be their point of intersection. Then O is the middle point of both AD, BC.

At each of the points C and D introduce two equal and opposite forces, each being equal and parallel to P.



Now, like parallel forces P at A and P at D may be replaced by their resultant $2P$ acting at O, along OE parallel to them.

Again, like parallel forces P at B and P at C may be replaced by their resultant $2P$ acting at O, along OF parallel to them.

Being equal, opposite and collinear, these two resultant forces balance, and we are left with two unlike parallel forces, one P acting at C in the same sense and direction as P at A, and the other, P acting at D in the sense and direction of P at B.

Thus, the given couple (P, p) with the arm AB is equivalent to the couple (P, p) of the same moment in a parallel plane, having its arm CD equal and parallel to AB.

Now, the couple (P, p) with arm CD can be replaced in its plane by any other couple, provided the moment is unchanged in magnitude and sign, as in Art. 6·4. Hence, a couple in any plane can be replaced by any other couple in a parallel plane, provided its moment remains unchanged in magnitude and sign.

Note. From above it is clear, that the effect of a couple remains unaltered so long as its moment remains the same in magnitude and sense, whatever be the magnitude of its constituent forces, the length of its arm, and its position in any one of a set of parallel planes in which it may be supposed to act.

A couple is therefore completely specified if we know (i) the direction of the set of parallel planes, (ii) the magnitude of its moment, (iii) the sense in which it acts.

These three characteristics of a couple can be aptly represented by a straight line drawn,

(i) perpendicular to the set of parallel planes, to indicate the direction ;

(ii) of a measured length to indicate the magnitude of the moment ;

and (iii) in a definite sense, to indicate the sense of the moment.

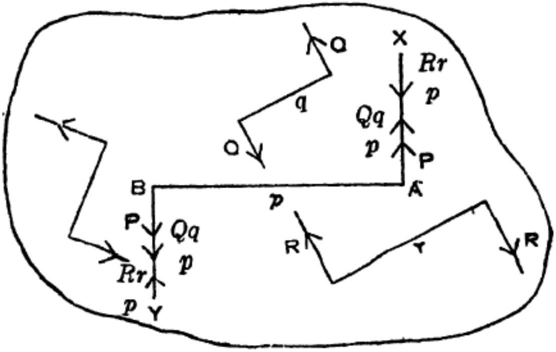
A line so drawn to represent a couple is called the **axis of the Couple.**

6·6. Resultant of coplanar couples.

Any number of coplanar couples acting on a body is equivalent to a single couple whose moment is equal to the algebraic sum of the moments of the couples.

Let (P, p), (Q, q), (R, r),... be a number of couples acting in the same plane upon a body.

Let AB represent the arm p of the couple (P, p) whose component forces P, P act along AX and BY.



The moment of the couple $(Q, q) = Q.q = \dfrac{Qq}{p}.p$. Hence, the couple (Q, q) may be replaced by another couple whose arm coincides with AB and whose component forces of magnitude $\dfrac{Qq}{p}$ act along AX and BY.

Similarly, the couple (R, r) may be replaced by another couple whose arm coincides with AB and whose component forces of magnitude $\dfrac{Rr}{p}$ act along AX and BY.

Replacing all the other couples in this way we get a single couple with the arm AB, each of whose component forces

$$=\left(P+\frac{Qq}{p}+\frac{Rr}{p}+\cdots\right).$$

Hence, the given system of couples is equivalent to a single couple whose moment

$$=\left(P+\frac{Qq}{p}+\frac{Rr}{p}+\cdots\right)\times p$$

$$=Pp+Qq+Rr+\cdots$$

= the algebraic sum of the moments of the different couples.

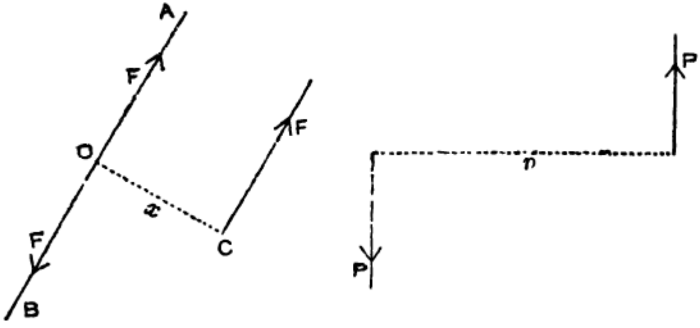
Note. If the moment of any of the couples, say Rr, be negative, as in the figure, corresponding component along AX will be negative *i e.*, opposite to the sense of P there, and similarly for the component at B. Hence, the resultant single force along AX or BY is the algebraic sum of these component forces.

6·7. Resultant of a couple and a force.

A force and a couple in the same plane are equivalent to a single force, equal and parallel to the given single force.

Let F be the given force acting at O along OA and (P, p) the given couple.

Replace the given couple by another couple having its forces each equal to F. If x be the length of the arm of



this new couple, its moment $= xF = Pp$, the moment of the original couple.

Hence, $$x = \frac{Pp}{F}.$$

Place the couple such that one of its component forces F acts at O along the line of action of the given force F but in the opposite sense *i.e.*, acts along OB.

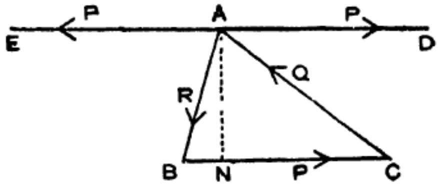
Now the two equal and opposite forces acting at O along the same line balance, and we are left with a single force F at C which is in the direction of the original force, and at a distance Pp/F from it.

Thus, a force F and a couple of moment G are equivalent to a single parallel force F, displaced to a distance G/F from its original position.

Cor. *A force and a couple acting in the same plane cannot produce equilibrium*

6·8. Theorem. *If three forces acting upon a rigid body be represented in magnitude, direction, sense and line of action by the sides of a triangle, taken in order, they are equivalent to a couple whose moment is equal to twice the area of the triangle.*

Let three forces P, Q, R acting upon a body be represented in magnitude, direction, line of action and sense by the sides BC, CA, AB respectively of the triangle ABC. Draw EAD parallel to BC, and introduce at A two equal and opposite forces equal to P, acting in the directions AD and AE. Draw AN perp. to BC. The three forces P along AD, Q along CA, and R along AB, acting at A, and being represented in magnitude, direction and sense by the sides of a triangle taken in order, are in equilibrium, by the triangle of forces.



We are thus left with forces P along AE and P along BC, which form a couple of moment $P \times AN$, *i.e.*, $BC \times AN$, *i.e.*, equal to twice the area of the $\triangle ABC$.

Alternative method :

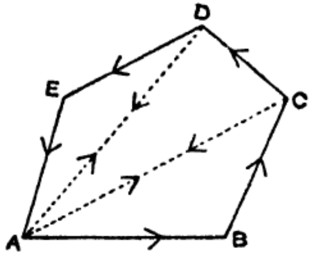
The forces Q and R acting at A and represented by CA and AB are equivalent to a force acting at A represented in magnitude and direction by CB, *i.e.*, equivalent to a force P acting at A parallel and opposite in sense to the given force P. Hence, the three forces are equivalent to a couple of moment $P \times AN$, *i.e.*, $BC \times AN$, *i.e.*, twice the area of $\triangle ABC$, where AN is the perpendicular from A on BC.

6·9. Theorem. *If a system of coplanar forces acting upon a rigid body be represented in magnitude, direction, sense and line of action by the sides of a polygon taken in order, they are equivalent to a couple whose moment is represented by twice the area of the polygon.*

Let the forces be completely represented by the sides AB, BC, CD, DE, EA of the polygon $ABCDE$. Join AC and AD.

Let us introduce in the body two pairs of equal and opposite forces represented by AC, CA and AD, DA acting along these lines. These do not affect the given system.

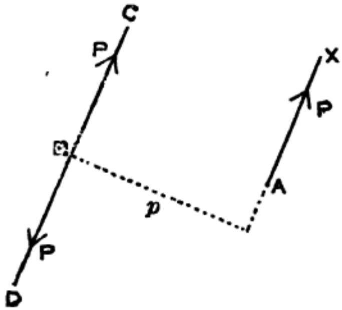
Now, the forces represented by the sides AB, BC, CA of the $\triangle ABC$, actually acting along these lines, are equivalent to a couple of moment $2\triangle ABC$.



Similarly, the forces represented by the sides of $\triangle ACD$ and $\triangle ADE$ are respectively equivalent to couples of moment $2\triangle ACD$ and $2\triangle ADE$.

Now, these three couples are equivalent to a single couple whose moment is equal to $2(\triangle ABC + \triangle ACD + \triangle ADE)$ = twice the area of the polygon $ABCDE$.

6·10. Theorem. *A force acting at any point A of a body is equivalent to an equal and parallel force acting at any other arbitrary point B of the body, together with a couple.*



Let P be a force acting at A along AX, and B any arbitrary point, and let p be the distance of B from AX. At B apply two equal and opposite forces, each equal and parallel to P, along BC, BD. These two forces will have no effect on the body, and the three forces P may now be regarded as P along AX and P along BD forming a couple of moment Pp, and a force P at B along BC, parallel to the original force and in the same sense.

Note. The moment of the couple is equal to the moment of the original force at A about B.

6·11. Theorem. *If a system of coplanar forces reduces to a couple, the algebraic sum of the moments of the forces about any point in their plane is constant, and equal to the moment of the couple.*

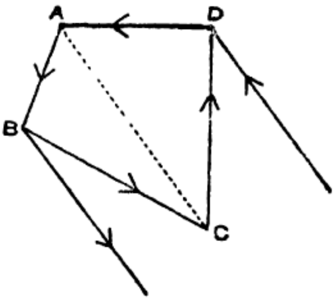
Let P, Q, R, S,... be a system of coplanar forces, and O any arbitrary point in their plane.

As in the previous article, we can replace the force P by an equal and parallel force at O, together with a couple whose moment is equal to the moment of P about O. Dealing with each of the other forces in the same manner, we get the given system of forces equivalent to a set of concurrent forces at O, together with a number of couples, which later can be compounded into a single couple, whose moment being equal to the algebraic sum of the moments of the couples, is ultimately equal to the algebraic sum of the moments of the given forces about O. The concurrent forces at O must in this case be in equilibrium for otherwise they would combine into a single resultant force, which along with the couple would give us a single force as our resultant, and not a couple.

Thus, when the given system of forces reduces to a couple, the algebraic sum of the moments of the forces about O, which is arbitrary, is always the same, namely, equal to the moment of the resultant couple.

6·12. Illustrative Examples.

Ex. *Four forces are completely represented by the sides AB, BC, CD, DA of a quadrilateral $ABCD$; show that they are equivalent to a couple, consisting of two equal forces through B and D.*



Forces AB and BC are equivalent to a force at B represented in magnitude, direction and sense by AC.

Similarly, forces CD and DA are equivalent to a force at D represented in magnitude, direction and sense by CA.

Thus, the four forces are equivalent to two equal, parallel and unlike forces at B and D and hence they are equivalent to a couple.

Examples on Chapter VI

1. Forces equal to 3, 5, 3 and 5 lbs. wt. respectively act along the sides of a square taken in order; find their resultant. [*C. U. 1932*]

2. Show that the forces 3, 8, 7, 11 and 5 lbs. wt. acting respectively along AB, BC, CD, DA and AC of rectangle $ABCD$ are equivalent to a couple, if $AB=6$ ft. and $BC=4\frac{1}{2}$ ft., and show that the moment of the couple is $79\frac{1}{2}$ ft.-lbs.

3. Forces of magnitudes 1, 2, 3, 4, $2\sqrt{2}$ act respectively along the sides AB, BC, CD, DA and the diagonal AC of the square $ABCD$. Show that their resultant is a couple, and find its moment. [*C. U. 1947*]

4. Forces P, $2P$, $-P$, $2P$ act along the sides AB, BC, CD, DA of the square $ABCD$, and a force $P\sqrt{2}$ acts along each of BD and CA. Show that the forces reduce to a couple of moment $2aP$, where a is the side of the square.

5. Unlike parallel forces each equal to 4 lbs. wt. act along a pair of opposite sides of length 2 ft. of a rectangle. Find the magnitude of the forces which, acting along the other sides of length 6 inches, will form with these a system in equilibrium.

6. Two couples with forces acting along the sides of a parallelogram are in equilibrium; find the ratio of the forces of the couples.

7. Four forces acting along the sides of a parallelogram are equivalent to a couple. Show that the forces along the opposite sides are equal in magnitude and opposite in sense.

8. Three forces acting along the sides of a triangle taken in order, are equivalent to a couple. Show that they are proportional to the sides of the triangle.

9. In a tetrahedron $PABC$, show that the resultant of the couples whose moments are represented by the areas of triangles PBC, PCA, PAB is a couple whose moment can be represented by the area of the triangle ABC.

10. Prove that forces represented in magnitude and line of action by the sides of two triangles, taken opposite ways round, are in equilibrium, provided the triangles are of equal area.

11. Three forces proportional to the sides of a triangle act perpendicularly to these sides, all inwards. Show that they are in equilibrium or they form a couple.

12. P and Q are like parallel forces. An unlike parallel force $P+Q$ acts in the same plane at perpendicular distances a, b respectively from the forces and between them. Find the moment of the resultant couple.

13. Find the resultant of a force of 7 lbs. wt. and a couple in the same plane whose arm is $3\frac{1}{2}$ ft. and whose forces are each 4 lbs. wt.

***14.** Three parallel forces P, Q, R acting at the angular points of a triangle ABC are in equilibrium when they are perpendicular to the side BC. If their lines of action are turned through a given angle in the same sense, show that they are equivalent to a couple.

15. D, E, F divide the sides BC, CA, AB respectively of an equilateral triangle ABC of side a in the ratio $5:1$. Three forces each equal to P act at D, E, F perpendicular to the sides and outwards from the triangle. Show that they are equivalent to a couple of moment Pa.

[*C. U. 1943*]

16. $ABCD$ is a rectangle such that $AB=CD=a$ and $BC=DA=b$. Forces P act along AD and CB and forces Q act along AB and CD. Prove that the perpendicular distance between the resultant of the forces P, Q at A and the resultant of the forces P, Q at C is

$$\frac{Pa-Qb}{\sqrt{P^2+Q^2}}.$$

***17.** If three forces P, Q, R acting at the angular points of a triangle ABC along the tangents to the circum-circle, the same way round, are equivalent to a couple, show that

$$P:Q:R=\sin 2A:\sin 2B:\sin 2C.$$

[*Moments about the vertices of the triangle formed by three tangents are equal.*]

*18. P and Q are two like parallel forces. If a couple, each of whose forces is F, and whose arm is a, in the plane of P and Q, is combined with them, show that the resultant is displaced through a distance

$$\frac{Fa}{P+Q}.$$

*19. The constituent forces of a couple of moment G act at A and B; if their lines of action are turned through a right angle, they form a couple of moment H. When they both act at right angles to AB, show that they form a couple of moment

$$\sqrt{G^2+H^2}.$$

*20. $ABCD$ and $A'B'C'D'$ are any two coplanar parallelograms. If forces act along AA', $B'B$, CC', $D'D$ represented by these respective lengths, show that they reduce to a couple.

*21. Forces Ka, Kb, Kc parallel to the sides of a triangle ABC, act at I_1, I_2, I_3, the centres of the escribed circles. Show that they are equivalent to a couple of moment $2KR(a+b+c)$, where R is the radius of the circumcircle.

*22. X, Y, Z are points on the sides BC, CA, AB of a triangle ABC, such that

$$\frac{BX}{XC}=\frac{CY}{YA}=\frac{AZ}{ZB}=\frac{q}{p}.$$

Prove that the system of forces represented by AX, BY, CZ is equivalent to a couple of moment $\frac{p-q}{p+q}\cdot 2\Delta$, where Δ is the area of the triangle.

*23. H is the orthocentre of the triangle ABC and three forces μa, μb, μc act along AH, BH, CH. If the forces are rotated through the same angle α, about A, B, C respectively, show that they become equivalent to a couple whose moment is $4\Delta\mu\sin\alpha$, where Δ is the area of the triangle ABC.

*24. If three forces completely represented by the sides of a triangle taken in order, are in equilibrium with three equal forces acting at the corners of the triangle along the tangents to the circumcircle the same way round, prove that the triangle must be equilateral.

ANSWERS

1. A couple of moment $8a$, where a is a side of the square.

3. $5a$, where a is the side of the square.

5. 1 lb. wt.

6. Proportional to the sides of the parallelogram.

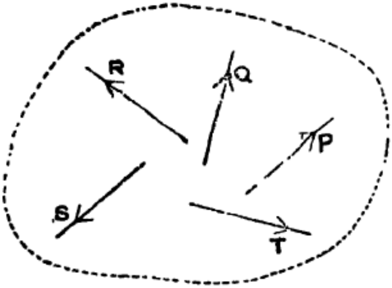
12. $Pa \sim Qb$.

13. 7 lbs. wt. acting parallel to the given force 7 lbs. wt. and at a distance 2 ft. from it.

CHAPTER VII

REDUCTION OF COPLANAR FORCES

7·1. Theorem I. *Any system of coplanar forces acting on a rigid body can be reduced ultimately to either a single force, or a single couple, unless it is in equilibrium.*



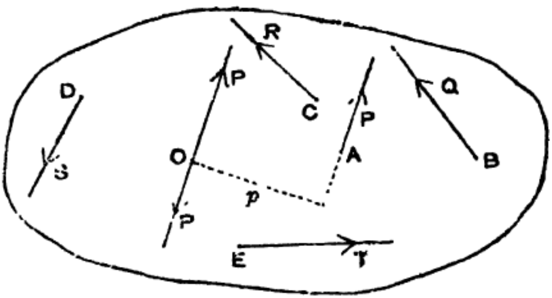
Let P, Q, R, S, T,... etc. be a system of coplanar forces acting on a rigid body.

Take any three forces P, Q, R of the system. The two forces P and Q can be combined into a single resultant by parallelogram of forces if they meet, or by the method of combining parallel forces when they are parallel, like or unlike, excepting in the case when they form a couple. In case P and Q form a couple, we can combine P and R into a single force, unless P forms a couple with R also. Now, if P forms a couple with Q, as well as with R, Q and R must be like parallel forces which can be combined into a single force. In any case therefore, the three forces can be reduced to two.

With these two, take another force S of the system. These three forces again, just as before, can be reduced to two. Proceeding in this way, when all the forces of the system are exhausted, we get ultimately two forces. These two, if they are equal and opposite acting along the same line will produce equilibrium. Otherwise, if they are equal and unlike parallel forces, they form a couple. In case they do not form a couple, we can finally combine them into a single resultant.

7·2. Theorem II. *Any system of coplanar forces acting on a rigid body can ultimately be reduced to a single force acting at any arbitrarily chosen point in the plane, together with a couple.*

Also, the resolved part in any direction of the single force obtained above, is equal to the algebraic sum of the resolved parts of the given forces in that direction, and the moment of the couple is equal to the algebraic sum of the moments of the given forces about the chosen point.



Let P, Q, R, S,... be a system of coplanar forces acting at A, B, C, D,... etc. of a rigid body, and let O be any arbitrary point in the plane.

Consider a force P of the system. If we introduce at O, two equal and opposite forces, each equal and parallel to P, these two forces, balancing one another, will not affect the given system. Now the given force P, along with the equal and unlike parallel force P at O, form a couple whose moment is equal to Pp, where p is the perpendicular distance from O on the line of action of P, and we get in addition a force P acting at O, which is equal and parallel to the original force P at A. Exactly in the same manner the force Q at B is equivalent to an equal and parallel force Q at O in the same sense, together with a couple of moment Qq, which is equal to the moment of the given force Q at B about O; and similarly for every force of the system.

Thus, the given system of forces is ultimately reduced to a system of concurrent forces acting at O, equal and parallel to the original forces, together with a number of couples whose moments are respectively equal to the moments of the individual forces of the given system about O. The concurrent forces at O can ultimately be combined into a single resultant force acting at O, and the couples can be combined into one single couple.

Also, the resolved part of the single resultant in any direction, is equal to the algebraic sum of the resolved parts in the same direction of the constituent forces at O, *i.e.*, of the given forces which are equal and parallel to them.

Again, the moment of the resultant couple, being equal to the algebraic sum of the moments of the individual couples, is equal to the algebraic sum of the moments of the given forces about O.

Hence the theorem.

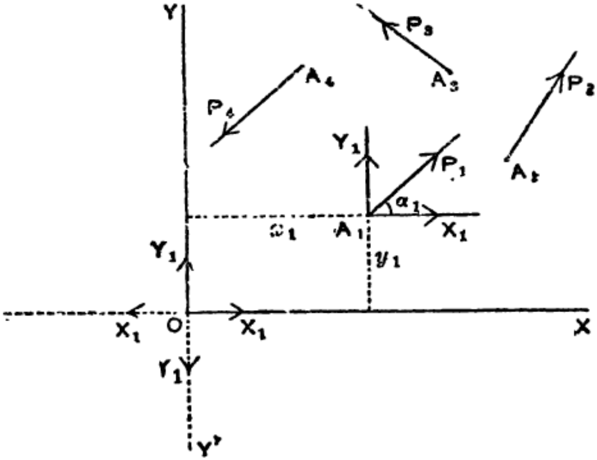
7·3. Analytical reduction of a system of coplanar forces.

Let OX and OY be any two perpendicular lines which are chosen as the axes of co-ordinates in the plane of a given system of coplanar forces which consists of forces $P_1, P_2, P_3, \ldots$ etc., acting at the points $A_1, A_2, A_3, \ldots$ etc, whose co-ordinates are $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ etc.

Let the direction of P_1 make an angle α_1 with OX, and let X_1, Y_1 be the resolved parts of P_1 along OX and OY respectively, so that $X_1 = P_1 \cos \alpha_1$ and $Y_1 = P_1 \sin \alpha_1$.

At O, introduce a pair of equal and opposite forces X_1, X_1 acting in the line OX, and a pair of equal and opposite forces Y_1, Y_1 acting in the line OY. These forces, balancing one another, will have no effect on the given system.

Now, the component force X_1 at A_1, and the equal and unlike parallel force X_1 at O, form a couple whose moment is clearly equal to $-y_1X_1$, since its tendency of rotation



is clockwise. Similarly, the component Y_1 at A_1, and the equal and unlike force Y_1 at O form a couple of moment x_1Y_1 as is easily seen. Also there are left a force X_1 along OX and a force Y_1 along OY at O.

Thus, the force P_1 at A_1 (*i.e.*, x_1, y_1), having the resolved parts X_1, Y_1 parallel to the axes, is equivalent to the components X_1 and Y_1 along the axes at O, together with a single couple of moment $(x_1Y_1 - y_1X_1)$.

Exactly in the same manner, the force P_2 at A_2 can be replaced by the resolved components X_2 and Y_2 along the axes at O, together with a couple of moment $(x_2Y_2 - y_2X_2)$; and similarly for every force of the system.

Combining all the components along OX and OY separately, and combining all the couples, we ultimately get the given system of forces reduced to

a component force $X = \Sigma X_1 (\equiv \Sigma P_1 \cos \alpha_1)$ along OX,
a component force $Y = \Sigma Y_1 (\equiv \Sigma P_1 \sin \alpha_1)$ along OY,
and a single couple of moment $G \equiv \Sigma(x_1 Y_1 - y_1 X_1)$.

The two components $X = \Sigma X_1$ and $Y = \Sigma Y_1$ along OX and OY will give rise to a single resultant force R acting at O in a direction θ with OX, where

$$R \cos \theta = X = \Sigma X_1 \text{ and } R \sin \theta = Y = \Sigma Y_1$$

so that $$R = \sqrt{(\Sigma X_1)^2 + (\Sigma Y_1)^2} = \sqrt{X^2 + Y^2}.$$

Thus, the given system of coplanar forces is reduced to a single resultant force R at the origin O (which may be chosen arbitrarily in the plane), together with a single couple G.

Note 1. The moment of the force P_1 about any point $D(h, k)$, being equal to the algebraic sum of the moments of its components about the point is easily seen to be

$$Y_1 (x_1 - h) - X_1 (y_1 - k),$$

$$\textit{i.e.,} \quad (x_1 Y_1 - y_1 X_1) - hY_1 + kX_1. \qquad \cdots \quad (1)$$

$\therefore$ if G' be the algebraic sum of the moments of the system of forces (*i.e.,* if G' be the moment of the resultant R) about D, then

$$G' = \Sigma(x_1 Y_1 - y_1 X_1) - h\Sigma Y_1 + k\Sigma X_1 \qquad \cdots \quad (2)$$

$$\textit{i.e.,} \quad G' = G - hY + kX. \qquad \cdots \quad (3)$$

If the resultant R passes through D, then $G' = 0$. $\therefore$ $hY - kX - G = 0$. This shows that (h, k) any point on the resultant R, lies on the line

$$xY - yX - G = 0 \qquad \cdots \qquad \cdots \quad (4)$$

which is thus the equation to the line of action of the resultant.

Again, if $y = mx + c$ be the equation of the line of action of the resultant and R be the magnitude of the resultant,

$$G' = \frac{mh - k - c}{\sqrt{(1 + m^2)}} \cdot R. \qquad \cdots \qquad \cdots \quad (5)$$

Note 2. If $R=0$, the given system of forces reduces to a *couple* only, which in this case will be the same, whatever point is chosen as origin.

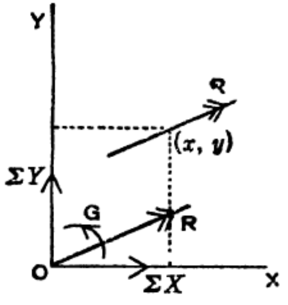
If $G=0$, the given system reduces to a *single resultant* force R at O.

If $R\neq 0$, $G\neq 0$, then also, the single force and single couple can be combined into *a single resultant* force, same in magnitude and direction as R at O, but shifted in position.

If $R=0$, $G=0$, the system will be in *equilibrium*.

7·4. Equation to the line of action of the resultant.

We have seen that the given system of forces can be reduced to a single force R acting at the origin O, having components ΣX and ΣY along the axes, together with a couple $G\equiv\Sigma(x_1Y_1-y_1X_1)$. In case these can be combined



into a single resultant force, the magnitude and direction of it will be the same as those of R at O. To get its position, let x, y be the co-ordinates of any point on its line of action. Then the algebraic sum of the moments of the given forces about this point, being equal to that of the resultant, must be zero. In other words, the algebraic sum of the moments of the components ΣX along OX, ΣY along OY, and of the couple G about the point x, y must also be zero, for this set is equivalent to the given system.

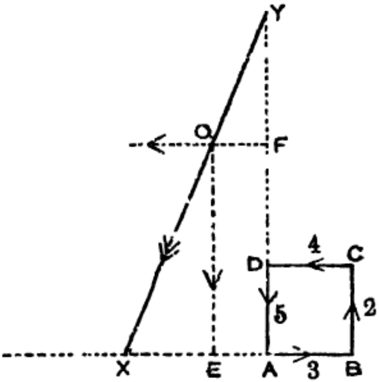
Thus, $y\Sigma X_1 - x\Sigma Y_1 + G = 0$, or, $x\Sigma Y_1 - y\Sigma X_1 - G = 0$, *i.e.*, $xY - yX - G = 0$, is the relation which must be satisfied by the co-ordinates (x, y) of any point on the line of action of the resultant.

Hence, *the equation to the line of action of the resultant* is

$$\mathbf{xY - yX - G = 0.}$$

7·5. Illustrative Examples.

Ex. 1. *Forces 3, 2, 4, 5 lbs. wt. act respectively along the sides AB, BC, CD, DA of a square. Find the magnitude of their resultant and the points where its line of action meets AB and AD.*



The resultant of the two unlike parallel forces 5 lbs. wt. along DA and 2 lbs. wt. along BC are equivalent to a forces $5-2=3$ lbs. wt. parallel to DA along some line OE say, external to AB, but nearer to AD, along which the greater force acts. Similarly, the resultant of 4 lbs. wt. along CD and 3 lbs. wt. along AB have a resultant 1 lb. wt. parallel to CD along some line FO. Now, the resultant of 3 lbs. wt. along OE and 1 lb. wt. along FO which are mutually perpendicular will give a resultant

$$R = \sqrt{3^2 + 1^2} = \sqrt{10} \text{ lbs. wt.}$$

along some line YOX, meeting BA and AD, let us suppose, at X and Y respectively. Let a be the side of the square $ABCD$, and let $AX = x$.

Then equating the algebraic sum of the moments of the given forces about X to the moment of the resultant, we get

$$2(a+x)+4a-5x=0,$$

or, $\quad x=2a$, *i.e.*, $AX=2AB$.

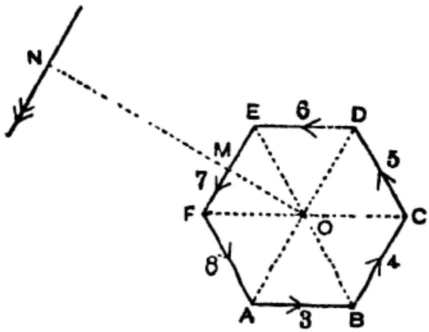
Similarly, assuming $AY=y$, and considering moment about Y,

$$3y+2a-4(y-a)=0,$$

whence $\quad y=6a$, *i.e.*, $AY=6AD$.

Thus the line of action of the resultant is obtained.

Ex. 2. *Forces of magnitude* 3, 4, 5, 6, 7, 8, *act in order along the sides* AB, BC, CD, DE, EF, FA *of a regular hexagon. Find their resultant completely.*



Let O be the centre of the regular hexagon. It is known from Geometry that the opposite sides AB, ED and the diagonal FOC are parallel, and similarly for other sets. Also the angles AOB, BOC, etc. are each equal to 60°.

Now, reducing each force to a parallel force at O, together with a couple, we get the given system of forces equivalent to forces 3, 4, 5, 6, 7, 8, along OC, OD, OE, OF, OA, OB respectively together with a couple moment $p(3+4+5+6+7+8)=33p$, where p is the perpendicular from O on any side of the hexagon.

Combining the force at O in pairs we get the forces 3 along OF, 3 along OA and 3 along OB. Now 3 along OF and 3 along OB give $2\times3\times\cos 60°=3$ along OA as resultant. Hence, we ultimately get a single resultant 6 along OA, together with a couple of moments $33p$. The single force and the couple combine finally into a single force equal and parallel to 6 along OA, but shifted from O towards the left through a distance x, where, considering moment about O,

$$6\times x=33p, \text{ or, } x=\tfrac{11}{2}p.$$

Hence, OM being perpendicular on EF, resultant meets OM produced at N, where $ON=\frac{11}{2}\,OM$,

or, $$ON:NM=11:9.$$

Thus, the magnitude, direction and line of action of the resultant are completely obtained.

Ex. 3. *Forces L, M, N act along the sides of the triangle formed by the lines $x+y-1=0$, $x-y+1=0$, $y=2$.*

Find the magnitude and the line of action of the resultant.

Let ABC be the triangle formed by the given lines, the equation of BC, CA and AB being

$$x+y-1=0,\ x-y+1=0,\ y=2 \text{ respectively.}$$

The point C has co-ordinates $(0, 1)$, so it is situated on the y-axis and AB is parallel to x-axis and at a distance 2 from it,

The algebraic sum of the resolved parts of the forces along OX, (the x-axis) is denoted by X.

$$\therefore\quad X=L\cos 45^\circ+M\cos 45^\circ-N=\frac{L+M-\sqrt{2}N}{\sqrt{2}}. \qquad \cdots \quad (1)$$

The algebraic sum of the resolved parts of the forces along OY (the y-axis) is denoted by Y.

$$\therefore\quad Y=M\sin 45^\circ-L\sin 45^\circ=\frac{1}{\sqrt{2}}(M-L). \qquad \cdots \quad (2)$$

Let R be the resultant of the forces.

$$\therefore\quad R^2=X^2+Y^2=\left(\frac{L+M-\sqrt{2}N}{\sqrt{2}}\right)^2+\left(\frac{M-L}{\sqrt{2}}\right)^2.$$

$$\therefore\quad R=\sqrt{\{L^2+M^2+N^2-\sqrt{2}N(L+M)\}}.$$

The algebraic sum of the moments of the forces about the origin O is denoted by G.

$$\therefore\quad G=-L\cdot\frac{1}{\sqrt{2}}-\quad\frac{1}{\sqrt{2}}+N.2=\frac{2\sqrt{2}N-L-M}{\sqrt{2}}.$$

By Art. 7·4, the equation of the line of action of the resultant R is

$$xY-yX-G=0.$$

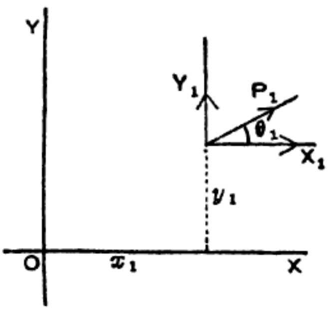
$\therefore$ here *the equation of the line of action* of the resultant

$$x\left(\frac{M-L}{\sqrt{2}}\right)-y\left(\frac{L+M-\sqrt{2}N}{\sqrt{2}}\right)-\frac{2\sqrt{2}N-L-M}{\sqrt{2}}=0,$$

i.e., $\quad x(M-L)-y(L+M-\sqrt{2}N)-2\sqrt{2}N+L+M=0.$

Ex. 4. *If a system of coplanar forces reduces to a couple whose moment is G, and when each force is turned round its point of application through a right angle, it reduces to a couple H; prove that when each force is turned through an angle α, the system is equivalent to a couple whose moment is*

$$G\cos\alpha+H\sin\alpha.$$



Let P_1 acting at the point (x_1, y_1) at an angle θ_1 to the x-axis be any one force of the system. The components parallel to the axes are $P_1\cos\theta_1$ and $P_1\sin\theta_1$, and the algebraic sum of their moments about O is

$$x_1P_1\sin\theta_1-y_1P_1\cos\theta_1.$$

ence, since the system reduces to a couple G, we must have, $\Sigma P_1\cos\theta_1=0$, $\Sigma P_1\sin\theta_1=0$,

and $\qquad G=\Sigma(x_1P_1\sin\theta_1-y_1P_1\cos\theta_1).$

When each force is turned through an angle α, the system reduces to a force component $\Sigma P_1\cos(\theta_1+\alpha)$

$$=\cos\alpha\Sigma P_1\cos\theta_1-\sin\alpha\Sigma P_1\sin\theta_1=0 \text{ along } Ox,$$

a force component $\Sigma P_1\sin(\theta_1+\alpha)$

$$=\cos\alpha\Sigma P_1\sin\theta_1+\sin\alpha\Sigma P_1\cos\theta_1=0 \text{ along } Oy,$$

and a couple $G'=\Sigma\{x_1P_1\sin(\theta_1+\alpha)-y_1P_1\cos(\theta_1+\alpha)\}$

$$=\cos\alpha\Sigma(x_1P_1\sin\theta_1-y_1P_1\cos\theta_1)$$
$$+\sin\alpha\Sigma(x_1P_1\cos\theta_1+y_1P_1\sin\theta_1).$$

In other words, the system in this case reduces to a couple

$$G' = G \cos \alpha + \sin \alpha \Sigma(x_1 P_1 \cos \theta_1 + y_1 P_1 \sin \theta_1).$$

Now, when $\alpha = 90°$, we are given $G' = H$.

$\therefore \qquad H = \Sigma(x_1 P_1 \cos \theta_1 + y_1 P_1 \sin \theta_1).$

Thus, $\qquad G' = G \cos \alpha + H \sin \alpha.$

Examples on Chapter VII

1. Prove that a force acting in the plane of the triangle ABC can be replaced uniquely by three forces acting along the sides of the triangle.

2. Show that a system of coplanar forces can be reduced to

(i) two forces acting through two given points ;

(ii) two forces, one of which acts through a given point, and the other along a given straight line ;

(iii) three forces acting along the sides of a given triangle, in the same plane.

3. (i) If two coplanar systems of forces have equal algebraic sum of moments about each of three non-collinear points, they are equivalent to each other.

(ii) A system of forces P, Q, R acting along the sides of the triangle ABC is equivalent to a system X, Y, Z along the sides of the pedal triangle. Show that

$$2X = Q/\cos B + R/\cos C.$$

4. The sum of the moments of a system of coplanar forces about each of three non-collinear points in the plane of the forces is the same (without being equal to zero). Prove that the system is equivalent to a couple.

Hence show that three forces represented in magnitude, direction and position by the three sides of a triangle taken in order are equivalent to a couple. [*C. U. 1937*]

5. The algebraic sum of the moments of a system of coplanar forces not in equilibrium, is zero about each of two points A and B. Show that the algebraic sum of the resolved parts of the force system in the direction perpendicular to AB is zero.

6. Forces P, Q, R act along the sides BC, CA, AB of the triangle ABC. If F be the magnitude of their resultant, then

$$F^2 = P^2 + Q^2 + R^2 - 2QR\cos A - 2RP\cos B - 2PQ\cos C.$$

7. The moments of a system of coplanar forces (not in equilibrium) about three collinear points A, B, C in their plane are G_1, G_2, G_3; prove that (with due regard to the sign)

$$G_1.BC + G_2.CA + G_3.AB = 0. \qquad [\textit{P. U. 1939}]$$

*8. The moments of a force lying in the plane of the triangle ABC about A, B, C are L, M, N respectively. If the force is the resultant of three forces P, Q, R acting in the same sense along BC, CA, AB respectively, then

$$P : Q : R = aL : bM : cN.$$

*9. A system of forces acts in the plane of an equilateral triangle of side 2 units. The algebraic sum of the moments of the forces about the three angular points are G_1, G_2, G_3. Prove that the magnitude of their resultant is

$$[\tfrac{1}{3}(G_1{}^2 + G_2{}^2 + G_3{}^2 - G_2G_3 - G_3G_1 - G_1G_2)]^{\frac{1}{2}}.$$

*10. The algebraic sum of the moments of a system of coplanar forces about three non-collinear points A, B, C in their plane are L, M, N respectively. Prove that their resultant R is given by $R^2 = \Sigma a^2(L-M)(L-N)/4\Delta^2$. where a, b, c are the sides of the triangle ABC, and Δ is its area.

*11. Two systems of forces P, Q, R and P', Q', R' act along the sides BC, CA, AB of a triangle ABC; prove that their resultants will be parallel if

$$(QR' - Q'R)\sin A + (RP' - R'P)\sin B$$
$$+ (PQ' - P'Q)\sin C = 0. \qquad [\textit{Lucknow, 1929}]$$

***12.** Three forces each equal to P act along the sides of a triangle ABC in order. Prove that the resultant R is given by $R = P\,(1-8 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C)^{\frac{1}{2}}$
and find the distance of its line of action from A. Examine the case when the triangle is equilateral.

***13.** Forces $l.BC$, $m.CA$, $n.AB$, where l, m, n are positive, act along the sides BC, CA, AB respectively of a triangle ABC, in the senses indicated by the order of the letters. Show that the line of action of their resultant divides BC, CA, AB externally in the ratios $m : n$, $n : l$, $l : m$.

What happens if $l = m = n$?

14. ABC is an equilateral triangle; forces of 4, 2 and 1 lb. wt. act along the sides AB, AC, BC respectively, in the senses indicated by the order of the letters. Find the magnitude, direction and the line of action of the resultant. [*P. U. 1932*]

15. The algebraic sum of the moments of a system of forces about the three vertices A, B, C of an equilateral triangle whose sides are 2 ft. long are $+10$, $+20$ and -10 foot-pounds. Find the magnitude of the resultant force, and the points where its line of action intersects AB and AC.

16. Forces proportional to 1, 2, 3, 4 act along the sides AB, BC, AD, DC respectively of a square $ABCD$, the length of whose sides is 2 ft. Find the magnitude and the line of action of the resultant. [*Bombay, 1934*]

***17.** $ABCDEF$ is a regular hexagon of which O is the centre. Forces of magnitudes 1, 2, 3, 4, 5, 6 act along AB, CB, CD, ED, EF, AF in the senses indicated by the order of the letters. Reduce the system to a force at O and a couple, and find the point in AB through which the single resultant passes. [*I. C. S. 1938*]

***18.** If six forces of relative magnitudes 1, 2, 3, 4, 5 and 6 act along the sides of a regular hexagon, taken in order, show that the single equivalent force is of relative magnitude 6 and that it acts along a line parallel to the force 5, at a distance from the centre of the hexagon $3\frac{1}{2}$ times the distance of a side from the centre. [*M. T. 1908*]

19. Six coplanar forces act on a body along the sides AB, BC, CD, DE, EF, FA of a regular hexagon $ABCDEF$, in which AB is one foot long; their magnitudes are 10, 20, 30, 40, P and Q lbs. wt. respectively. Find P and Q so that the system reduces to a couple. [*P. U. 1930*]

***20.** ABC is an equilateral triangle and D, E, F are the mid-points of the sides BC, CA, AB. Forces P, $2P$, $3P$ act along BC, CA, AB and forces $4P$, $5P$, $6P$ act along FE, ED, DF. Find the line of action of the resultant.

21. Forces 1, 2, 4, 5 lbs. wt. act along the sides AB, BC, CD, DA of a square $ABCD$ and a force P acts at the centre of the square. If the five forces are equivalent to a couple, find the magnitude and direction of P.

22. Forces 1, 2, 3, 4, 5, 6 lbs. wt. act along the sides of a regular hexagon, taken in order, and a force acts at the centre of the hexagon. If the several forces are equivalent to a couple, find the moment of the couple and the magnitude and the direction of the force at the centre.

23. (i) $ABCD$ is a quadrilateral in which the sides BC and AD are parallel. If forces $p.AB$, $q.BC$, $r.CD$, $s.DA$ acting along AB, BC, CD, DA are equivalent to a couple, show that $p=r$ and $(p-s)\,AD=(r-q)\,BC$.

(ii) Forces act along the sides AB, BC, CD, DA of a plane quadrilateral, taken in order and their magnitudes are p, q, r, s times the lengths of the sides in which they act. Prove that if they are equivalent to a couple, then

$$(p-q)\,OB=(r-s)\,OD, \text{ and } (q-r)\,OC=(s-p)\,OA.$$

24. $ABCD$ is a square whose side is 2 units in length. Forces a, b, c, d act along the sides AB, BC, CD, DA, taken in order, and forces $p\sqrt{2}$, $q\sqrt{2}$ act along AC and BD respectively. Show that if $p+q=c-a$, and $p-q=d-b$, the forces are equivalent to a couple of moment $a+b+c+d$.

25. A force has moments 6 units, -26 units and 36 units, about the origin, the point (8, 0) and the point (0, 10) respectively. Find the magnitude and the line of action of the force.

***26.** Find the intercepts made on the rectangular axes OX, OY by the line of action of the resultant of a force of 7 units along OP, where the co-ordinates of P are (3, 4) and a counter-clockwise couple of moment 21 units.

27. Forces 2, 4, 6, 8, $8\sqrt{2}$ act along the sides AB, BC, CD, DA and the diagonal BD of a square of side 2 units in the senses indicated by the order of the letters. Taking AB, AD as axes of x and y respectively, find the magnitude of the resultant force, and the equation of its line of action.

***28.** A system of coplanar forces $P_1, P_2, P_3, \ldots$ acting at the points (x_1, y_1), (x_2, y_2), $(x_3, y_3)\ldots$ are equivalent to a couple. The components of $P_1, P_2, \ldots$ parallel to the axes are (X_1, Y_1), $(X_2, Y_2), \ldots$ etc. Prove that by turning the forces about their respective points of application through a certain common angle, the system can be reduced to equilibrium.

***29.** Moments of the resultant R of a system of coplanar forces about three points O, A and B lying in the plane of the forces are G, $G+J_1$ and $G+J_2$ respectively. If referred to O as origin the polar co-ordinates of A and B be (r_1, θ_1) and (r_2, θ_2), show that

$$R^2 \sin^2(\theta_1-\theta_2)=\frac{J_1{}^2}{r_1{}^2}+\frac{J_2{}^2}{r_2{}^2}-2\frac{J_1J_2\cos(\theta_1-\theta_2)}{r_1r_2}.$$

[*C. H. 1954*]

***30.** The algebraical sums of the moments of a system of coplanar forces about points whose co-ordinates are (1, 0), (0, 2) and (3, 3) referred to rectangular axes, are G_1, G_2, G_3 respectively. Find the tangent of the angle which the direction of the resultant force makes with the axis of x.

[*C. H. 1956*]

***31.** The straight line $4x+3y=5$ meets the rectangular axes OX and OY at the points A and B respectively. If forces P, Q, R act along the lines OB, OA and AB, find the magnitude of the resultant and the equation of the line of action. [*C. H. 1958*]

ANSWERS

12. $2P\Delta/aR$.

14. $3\sqrt{3}$ lbs. wt. in a direction perpendicular to BC, dividing BC internally in the ratio 1 : 2.

15. $\frac{10}{3}\sqrt{21}$ lbs. wt. through the mid.-pt. of CA, intersecting BA produced 2 ft. from A.

16. $5\sqrt{2}$ parallel to AC dividing AD internally in the ratio 2 : 3.

17. Forces at O is $2\sqrt{3}$ perpendicular to EF; couple is $-3p$, where p is the perpendicular distance from O on a side. The final single resultant passes through the middle point of AB.

19. $P=-10$ lbs. wt., $Q=60$ lbs wt.

20. Parallel to CB, dividing DA in the ratio 1 : 5.

21. $3\sqrt{2}$ lbs. wt. along AC.

22. $\dfrac{21\sqrt{3}}{2}a$, where a is a side of the hexagon; 6 parallel to the force 2.

25. 5, in a direction joining the points $(0, -2)$ to $(\frac{4}{3}, 0)$.

26. $3\frac{3}{4}$, 5. **27.** $4\sqrt{10}$; $x+3y=9$.

30. $(5G_2-2G_3-G_1)/(G_2+G_3-2G_1)$.

31. $\sqrt{P^2+Q^2+R^2+\frac{8}{5}PR-\frac{6}{5}QR}$;
$x(5P+4R)-y(5Q-3R)-6R=0$.

32. (ii) $2G$. **33.** (i) $2W$, $-W$; $x+2y+a=0$.

34. $(3a, 0)$ and $(0, 3a)$.

35. $\sqrt{\{P^2+Q^2+R^2-2QR\sin\theta+2RP\cos\theta\}}$,
$Px-Qy+R(x\cos\theta+y\sin\theta-p)=0$.

37. $\dfrac{\{(M_1x_2-M_2x_1)^2+(M_1y_2-M_2y_1)^2\}^{\frac{1}{2}}}{x_2y_1-x_1y_2}$; $\theta=\tan^{-1}\left\{\dfrac{M_1y_2-M_2y_1}{M_1x_2-M_2x_1}\right\}$.

38. $\dfrac{aG_1}{G_1-G_2}$ where $AB=a$; the magnitude of the resultant vanishes.

EQUILIBRIUM OF COPLANAR FORCES

8·1. Equilibrium of three coplanar* forces.

Theorem. *If three coplanar forces acting on a rigid body be in equilibrium, they must either all three meet at a point, or else all must be parallel to one another.*

Let the three coplanar forces P, Q, R acting on a rigid body, be in equilibrium.

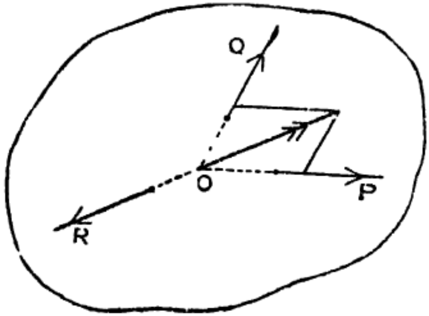


Fig. (i)

Let P and Q meet at O. Then by parallelogram of forces they can be combined into a single resultant at O. Since P, Q, R are in equilibrium, R must balance the resultant of P and Q, and thus must be equal and opposite to it, acting along the same line. Thus R must pass through O. Hence P, Q, R all meet at O.

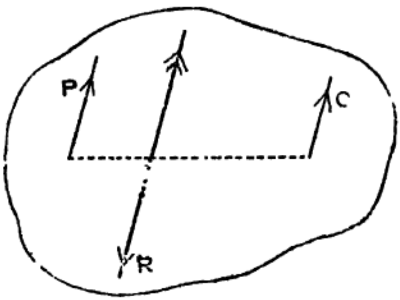


Fig. (ii)

If P and Q be parallel (like or unlike), their resultant is a parallel force, and R, balancing their resultant, must be acting in the same line in opposite sense. Hence P, Q, R are all three parallel to one another.

*See Note 3.

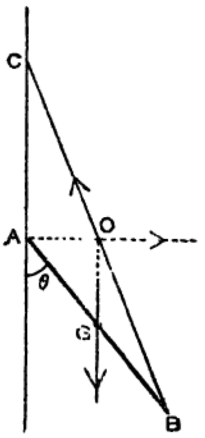
Note 1. P and Q can never form a couple in this case, for then P, Q, R (*i.e.*, a couple and a force) can never be in equilibrium.

Note 2. The above theorem gives a *necessary* condition of equilibrium of three coplanar forces, but *not sufficient*. For sufficient conditions, (i) when the three forces meet, they must also satisfy Lami's theorem, or the converse of the triangle of forces, (ii) when they are all parallel, one being equal and opposite to the resultant of the other two, their algebraic sum must be zero, and the moments of any two about a point on the third must be equal and opposite.

Note 3.* *If three forces acting on a rigid body be in equilibrium, they must automatically be coplanar*; for in this case the algebraic sum of their moments about any line in space must be zero. We can first of all consider lines drawn through any point of P intersecting Q, whereby it will be shown that each of them intersects R, so that Q and R are coplanar. Then P, balancing the resultant of Q and R, must be in that plane.

8·2. Illustrative Examples.

Ex. 1. *A heavy uniform rod of length a rests with one end against a smooth vertical wall, the other end being tied to a point of the wall by a string of length l. Prove that the rod may remain in equilibrium at an angle θ to the wall, given by*



$$\cos^2\theta = \frac{l^2 - a^2}{3a^2}. \qquad [\text{C. U. 1941}]$$

AB is the rod of length a, BC is the string of length l. The three forces which keep the rod in equilibrium are the weight of the rod acting vertically downwards through the middle point G, the tension along the string BC, and the reaction of the smooth wall at A which must be normal to the wall and therefore horizontal. The three forces in equilibrium, not being all parallel, must meet at a point O, as shown in the figure.

*For a note on the method of proof see *Appendix B*.

Now θ being the inclination to the vertical at which the rod rests, from the figure,

$$AO = AG \sin\theta = \frac{a}{2}\sin\theta, \qquad GO = AG\cos\theta = \frac{a}{2}\cos\theta.$$

Also, GO being parallel to AC through the mid-point G of AB,

$$CO = \tfrac{1}{2}CB = \tfrac{1}{2}l \text{ and } AC = 2GO = a\cos\theta.$$

Hence, from the triangle ACO, $CO^2 = AC^2 + AO^2$,

i.e., $\left(\frac{l}{2}\right)^2 = (a\cos\theta)^2 + \left(\frac{a}{2}\sin\theta\right)^2$,

or, $l^2 = 4a^2\cos^2\theta + a^2\sin^2\theta = 3a^2\cos^2\theta + a^2$.

$\therefore \quad \cos^2\theta = \frac{l^2 - a^2}{3a^2}$.

Note. For the above equilibrium position to be possible, $\cos^2\theta$ must be positive and not greater than unity. Hence, $l^2 > a^2$ and $l^2 - a^2 \not> 3a^2$. Therefore, $l > a$ but $\not> 2a$.

Ex. 2. *A uniform square lamina rests in equilibrium under gravity in a vertical plane with two of its sides in contact with smooth pegs in the same horizontal line at a distance c apart. Show that the angle θ made by a side of the square with the horizontal in a non-symmetrical position of equilibrium is given by*

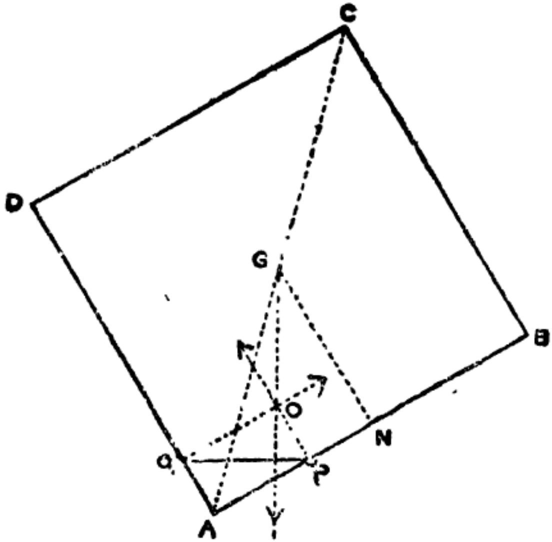
$$c(\sin\theta + \cos\theta) = a,$$

where 2a is the length of a side of the square. [C. U. 1946]

$ABCD$ is the square lamina (of side $2a$) whose weight acts vertically downwards along GO through its centre G. P and Q are the smooth pegs whose reactions are normal to the edges APB and AQD. As these are the only three forces which keep the lamina in equilibrium, they meet at a point O as shown in the figure. GN being drawn perpendicular to AB, N is the mid-point of AB, so that $AN = NG = a$. PQ is horizontal and equal to c. Let θ be the inclination of the side AB of the square to the horizon.

Then $\quad AP = c\cos\theta,\ OP = AQ = c\sin\theta$.

Now from the Geometry of the figure, since GO is vertical, the horizontal distance of G from A = the horizontal distance of O from A,



i.e., $AN \cos\theta - GN \sin\theta = AP \cos\theta - OP \sin\theta$,

or, $a(\cos\theta - \sin\theta) = c(\cos^2\theta - \sin^2\theta)$,

whence $c(\cos\theta + \sin\theta) = a$.

The other possibility, $\cos\theta - \sin\theta = 0$, *i.e.*, $\tan\theta = 1$, or, $\theta = \frac{1}{4}\pi$ gives the symmetrical position of equilibrium.

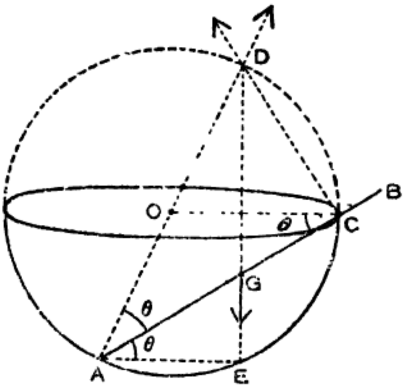
Ex. 3. *Show that the greatest inclination to the horizon at which a uniform rod can rest partly inside and partly outside a fixed smooth hemispherical bowl placed with its rim horizontal, is* $\sin^{-1}(\frac{1}{3}\sqrt{3})$.

AB represents the rod, G its middle point. The reaction of the smooth bowl at A being along the normal AOD passes through the centre O of the bowl. The reaction at C where the rod is in contact with the rim is along the normal CD to the rod and the rim. The weight of the rod vertical downwards through G and the two reactions at A and C, keeping the rod in equilibrium, must meet at a common point D. Let DG produced meet the bowl at E; join AE.

Now AO being along a diameter, and ACD a right angle, the point D must be the extremity of the diameter. Hence AED is also a right angle and so AE is horizontal.

Thus θ being the inclination of the rod to the horizon,

$$\theta = \angle EAB = \angle ACO = \angle OAC. \qquad [\because OA = OC]$$



Thus, if a be the radius of the bowl, and l the length of the rod,

$$2a \cos 2\theta = AE = AG \cos\theta = \frac{l}{2}\cos\theta, \quad \text{or,} \quad l = \frac{4a \cos 2\theta}{\cos\theta}.$$

But part of the rod being out, $l \not< AC$,

or, $\quad \dfrac{4a \cos 2\theta}{\cos\theta} \not< 2a\cos\theta, \quad$ or, $\quad 2\cos 2\theta \not< \cos^2\theta.$

$\therefore \quad 2(1-2\sin^2\theta) \not< (1-\sin^2\theta).$

$\therefore \quad \sin^2\theta \not> \dfrac{1}{3}, \quad \therefore \quad \sin\theta \not> \dfrac{1}{\sqrt{3}};$

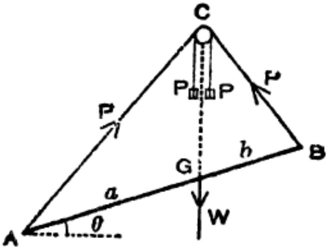
in other words, $\theta \not> \sin^{-1}(\frac{1}{3}\sqrt{3})$.

Ex. 4. *Equal weights P and P are attached to two strings ACP and BCP passing over a smooth peg C. AB is a heavy beam of weight W, whose centre of gravity is a feet from A and b feet from B; show that AB is inclined to the horizon at an angle*

$$\tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right].$$

The rod AB is in equilibrium under the two tensions along AC and BC, and the weight W vertically downwards through its centre of gravity G. These three forces then must meet at a common point C. The tensions in the strings, supporting the equal weights P, P at the other ends, are equal. The resultant of the two equal tensions balancing the weight along CG, CG must bisect the angle ACB. Thus,

$$\angle ACG = \angle BCG = 90^\circ - a \text{ (say)}.$$



Then,

$$W = 2P \cos(90^\circ - a) = 2P \sin a,$$

or, $\quad a = \sin^{-1} \dfrac{W}{2P}$.

Again, θ being the required inclination of the rod to the horizon, $\angle CGB = 90^\circ - \theta$. Hence, $\angle CAG = (90^\circ - \theta) - (90^\circ - a) = a - \theta$ and $\angle CBG = (90^\circ + \theta) - (90^\circ - a) = a + \theta$.

Now, $\dfrac{AG}{BG} = \dfrac{AG}{GC} \times \dfrac{GC}{BG} = \dfrac{\sin ACG}{\sin CAG} \times \dfrac{\sin CBG}{\sin BCG}$

$$= \frac{\cos a}{\sin(a - \theta)} \times \frac{\sin(a + \theta)}{\cos a} = \frac{\sin(a + \theta)}{\sin(a - \theta)},$$

and as $AG = a$ and $GB = b$ (given),

we get, $\quad \dfrac{a}{b} = \dfrac{\sin(a + \theta)}{\sin(a - \theta)}$. $\quad \therefore \quad \dfrac{a - b}{a + b} = \dfrac{\cos a \sin \theta}{\sin a \cos \theta} = \tan \theta \cot a.$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{a - b}{a + b} \tan a\right) = \tan^{-1}\left\{\frac{a - b}{a + b} \tan\left(\sin^{-1} \frac{W}{2P}\right)\right\}.$$

Examples on Chapter VIII(a)

(*Three forces in equilibrium*)

1. A heavy rod is suspended from a point O by two strings OA and OB. Show that the plane OAB is vertical.
[*C. U. 1925*]

2. If a uniform heavy rod be supported by a string fastened at its ends and passing over a smooth peg, show that it can only rest in a horizontal or vertical position.

3. Show that it is impossible for a heavy rod to rest in equilibrium with its ends on two smooth planes, one of which is horizontal and the other inclined to the horizontal at any angle.

4. Prove that a uniform rod cannot rest entirely within a smooth spherical bowl, except in a horizontal position.

5. A uniform rod has its lower end fixed to a hinge, and its other end attached to a string which is tied to a point vertically above the hinge; show that the direction of the action at the hinge bisects the string.

6. A uniform rod can turn freely about one of its ends, and is pulled aside from the vertical by a horizontal force acting at the other end of the rod, equal to half its weight. Prove that the rod will rest at an inclination of $45°$ to the vertical. [*C. U. 1951*]

7. A uniform rod AB is suspended with its end in contact with a smooth vertical wall AC by a string CE; if $AE = \frac{1}{3}AB$, show that CB will be horizontal. [*P. U. 1928*]

8. A uniform rod of weight W and length $2l$ has one end against a smooth vertical wall and rests at an inclination of $45°$ with the vertical upon a smooth rail parallel to the wall. Find the distance of the rail from the wall, and the reactions. [*C. U. 1914*]

9. A heavy uniform rod of length $2a$ rests in equilibrium, having one end against a smooth vertical wall, and being placed upon a peg at a distance b from the wall. Show that the inclination of the rod to the horizon is

$$\cos^{-1}(b/a)^{\frac{1}{3}}.$$

10. A heavy uniform rod is in equilibrium with one end resting against a smooth vertical wall, and the other against a smooth plane inclined to the wall at at an angle θ. Prove that if α be the inclination of the rod to the horizon, then

$$\tan\alpha = \tfrac{1}{2}\tan\theta. \qquad [\ P.\ U.\ 1932\ ]$$

11. Two strings have each one of their ends fixed to a peg, and the others to the ends of a uniform rod. When the rod is hanging in equilibrium, show that the tension of the strings are proportional to their lengths.

12. A uniform beam of length l and weight W hangs from a fixed point by two strings of lengths a and b. Prove that the inclination of the rod to the horizon is

$$\sin^{-1}\frac{a^2-b^2}{l\sqrt{2(a^2+b^2)-l^2}}.$$

Find also the tensions of the strings.

13. A uniform beam is hinged at A, and is kept in equilibrium at an angle of $60°$ to the horizontal plane through A by a string BC which connects B to a point C in this plane behind A. If $AC = AB$, find the direction of the reaction at A. [*C. U. 1943*]

14. A uniform rod AB of weight W can turn freely about a hinge at A, and to the end B is attached a string which passes over a small smooth pulley at C, vertically about A, and carries a weight w hanging freely. Prove that in the position of equilibrium

$$BC : AC = 2w : W.$$

15. One end of a uniform rod of weight 40 lbs. is attached to a hinge, and it is supported by a string attached to the other end and to a point at the same level as the hinge, the rod and the string being inclined at the same angle $30°$ to the horizontal. Find the tension in the string, and the action at the hinge.

***16.** A uniform rod 4 inches long, is free to turn in a vertical plane about its upper end which is hinged. To a point of the rod 3 inches from the hinge is attached a string which running perpendicular to the rod passes over a pulley and supports a weight P. The rod is in equilibrium at an angle of $60°$ to the horizontal. Prove that P is one-third of the weight of the rod, and the reaction at the hinge makes an angle $\tan^{-1}(\frac{5}{3}\sqrt{3})$ with the horizontal. [*Allahabad*]

***17.** A uniform beam AB which is 6 feet long and weighs 40 lbs. can turn freely about its end A which is attached to a vertical wall, and the beam is kept in a horizontal position by a rope attached to a point of the beam $1\frac{1}{4}$ feet from A and to a point of the wall vertically above A. If the tension of the rope is not to exceed 120 lbs. wt., show that the height above A of the point of attachment of the string to the wall must not be less than $1\frac{2}{3}$ ft.

18. A uniform rod AB is in equilibrium at an angle α with the horizontal, with its upper end A resting against a smooth peg, and its lower end B attached to a light cord which is fastened to a point C on the same level as A. If the cord is inclined to the horizontal at an angle β, then

$$\tan\beta = 2\tan\alpha + \cot\alpha.$$

19. A uniform rod of weight W rests with its ends in contact with two smooth planes, inclined at angles α and β respectively to the horizon and intersecting in a horizontal line. If θ be the inclination of the rod to the vertical, show that

$$2\cot\theta = \cot\beta - \cot\alpha.$$

Also find the reactions at the ends of the rod.

[*P. U. 1933*]

20. A uniform rod of length $2l$ rests with its lower end in contact with a smooth vertical wall. It is supported by a string of length a, one end of which is fastened to a point in the wall and the other end to a point in the rod at a distance b from its lower end. If the inclination of the string to the vertical be θ, show that

$$\cos^2\theta = \frac{b^2(a^2-b^2)}{a^2 l\,(2b-l)}. \qquad [\ C.\ U.\ 1940\ ]$$

21. A uniform rod whose weight is W is supported by two fine strings one attached to each end, which, after passing over small fixed smooth pulleys, carry weights W_1 and W_2 respectively at the other ends. Show that the rod is inclined to the horizon at an angle

$$\sin^{-1}\frac{W_1{}^2 - W_2{}^2}{W\sqrt{2(W_1{}^2 + W_2{}^2) - W^2}}.$$

22. A uniform bar of length a rests suspended by two strings of lengths l and l' fastened to the end of the bar and to two fixed points in the same horizontal line at a distance b apart. If the directions of the strings produced meet at right angles, and if T_1 and T_2 be the tensions of the strings, then

$$\frac{T_1}{T_2} = \frac{al + bl'}{al' + bl}.$$

23. A picture of weight 5 lbs. is hung from a nail by a cord 5 feet long fastened to two rings 3 feet apart. Find the tension in the cord.

24. A heavy equilateral triangle hung upon a smooth peg by a string, the ends of which are attached to two of its angular points, rests with one of its sides vertical. Show that the length of the string is twice the altitude of the triangle.

25. A square of side $2a$ is placed with its plane vertical between two smooth pegs which are in the same horizontal line and at a distance d. Show that it will be in equilibrium when the inclination of one of its edges to the horizon is either

$$\frac{\pi}{4} \text{ or } \frac{1}{2}\sin^{-1}\frac{a^2-d^2}{d^2}.$$

26. A uniform square lamina of side $2a$ rests in a vertical plane on two smooth pegs in a horizontal line. Show that if the sum of the distances of the pegs from the lowest corner is equal to a, there is equilibrium.

27. A beam whose centre of gravity divides it into two portions a and b is placed inside a smooth sphere. Show that if θ be its inclination to the horizon in the position of equilibrium, and 2α be the angle subtended by the beam at the centre of the sphere,

$$\tan\theta = \frac{b-a}{b+a}\tan\alpha. \qquad [\textit{C. U. 1924}]$$

[*The centre of gravity of a body is the point at which its weight may be assumed to act.*]

28. A rod of length l rests wholly inside a fixed smooth hemispherical bowl of radius a placed with its axis vertical. The centre of gravity of the rod divides its length in the ratio $m : n$. Show that the inclination of the rod to the horizon is

$$\sin^{-1}\frac{(n-m)l}{2\sqrt{(m+n)^2a^2-mnl^2}}.$$

29. A fixed smooth hemispherical bowl of radius a is placed with its axis vertical, and a uniform rod of length l rests with one end inside the bowl, and the other projecting over the rim. Prove that the length of the rod outside the bowl is

$$\tfrac{1}{8}\left(7l - \sqrt{l^2+128a^2}\right)$$

and hence deduce the shortest length of a rod that can rest in this manner.

30. A smooth hemispherical bowl of radius r is placed on the ground with its rim in contact with a smooth vertical wall. A heavy uniform rod is placed with one end inside the bowl, and the other in contact with the wall. If θ be the inclination to the horizon at which the rod rests, prove that the length of the rod is

$$r \sec\theta \left\{1 + \frac{1}{\sqrt{1+4\tan^2\theta}}\right\}.$$

31. A smooth bowl in the form of a part of a sphere is placed with its axis vertical, and a rod rests with one end within it, and a part of it projecting out over the rim. If α be the angle made by any radius to the rim with the vertical axis, and β, that made with the same axis by the radius to the lower extremity of the rod in the position of equilibrium, prove that the length of the rod is

$$4a \sin\beta \sec\tfrac{1}{2}(\alpha-\beta).$$

32. A solid cone of height h and semi-vertical angle α is placed with its base against a smooth vertical wall and is supported by a string attached to the vertex and to a point in the wall. Show that the greatest possible length of the string is

$$h\sqrt{1+\tfrac{16}{9}\tan^2\alpha}.$$

[*The centre of gravity of a solid right circular cone is on the axis at a distance* $\tfrac{3}{4}$*th from the vertex.*]

***33.** Inside a smooth hollow vertical right circular cylinder of which the external radius is R and the internal radius is r, two spheres whose radii are a and b rest. If the sphere of radius b is upper and if w be its weight, show that the cylinder will not overturn if its weight exceeds $w(2r-a-b)/R$.

ANSWERS

8. $\frac{1}{2}l\sqrt{2}$; W and $W\sqrt{2}$.

12. $\dfrac{Wa}{\sqrt{2(a^2+b^2)-l^2}}$, $\dfrac{Wb}{\sqrt{2(a^2+b^2)-l^2}}$.

13. At an angle $\tan^{-1}(\frac{2}{3}\sqrt{3})$ to the horizon.

15. 20 lbs. wt. ; $20\sqrt{3}$ lbs. wt. at an angle $60°$ with the horizon.

19. $\dfrac{W\sin\alpha}{\sin(\alpha+\beta)}$, $\dfrac{W\sin\beta}{\sin(\alpha+\beta)}$. 23. $3\frac{1}{3}$ lbs. wt. 29. $2a\sqrt{\frac{2}{3}}$.

8·3. General conditions of Equilibrium of any system of coplanar forces.

(A) *The necessary and sufficient conditions that a system of coplanar forces acting on a rigid body may be in equilibrium are that*

(i) & (ii) the algebraic sum of the resolved parts of the forces in any two mutually perpendicular* directions should be separately zero,

and **(iii) the algebraic sum of the moments of the forces about any point in their plane shauld also be zero.**

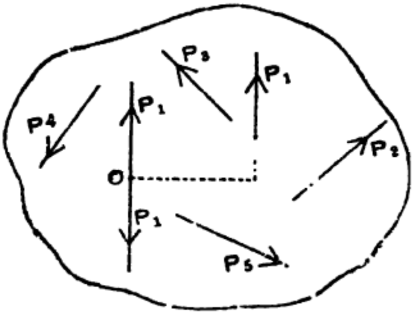
To prove that the conditions are sufficient :

Let $P_1, P_2, P_3, \ldots$ be a system of coplanar forces acting on a rigid body, and O any point in their plane.

By introducing at O two equal and opposite forces each equal and parallel to P_1 (which neutralise each other) the given force P_1 may be taken as equivalent to an equal and

*or any two different directions in the plane.

parallel force P_1 at O, together with a couple. Dealing with each of the other forces in a similar manner, and then recombining, the given system of forces can be reduced ultimately to a single resultant force R at O together with



a couple G (See *Art.* 7·2). Moreover, the resolved part of R in any direction, say OX, is equal to the algebraic sum of the resolved parts of the given forces in that direction, and the moment of the couple G is equal to the algebraic sum of the moments of the given forces about O.

Now, if the algebraic sum of the resolved parts of the given forces in any two perpendicular directions OX and OY, namely ΣX and ΣY, be separately zero, these being also the resolved parts of the resultant R in these directions, $R^2 = (\Sigma X)^2 + (\Sigma Y)^2 = 0$. If in addition, the algebraic sum of the moments of the given forces about O be zero, we get $G = 0$. Hence, both R and G being zero, the force system is in equilibrium.

Thus, the above three conditions being given, the force system will be in equilibrium. Hence the conditions are sufficient.

To prove that the conditions are necessary :

Let the given force system be in equilibrium. As proved above, the given system is reducible to a single force R at O, together with the couple G. In this case R and G must be separately zero, for a couple and a single force can never produce equilibrium. Now the algebraic sum of the moments of the given forces about O, being equal to the moment of G, must therefore be zero. Again, the algebraic sum of the resolved parts of the given forces in any two perpendicular directions, being equal to the resolved parts of R in those directions, must be separately zero, since R is zero here.

Thus, the force system being in equilibrium, the three above conditions follow as a necessary consequence.

Note. Analytically, if OX and OY be any two perpendicular directions in the plane of a system of coplanar forces, and ΣX, ΣY, the algebraic sum of the resolved parts of the given forces along OX and OY respectively, and $G = \Sigma(xY - yX)$ be the algebraic sum of the moments of the given forces about O, the conditions of equilibrium are

$$\Sigma X = 0, \ \Sigma Y = 0, \ G = \Sigma(xY - yX) = 0.$$

(B) *Another set of necessary and sufficient conditions of equilibrium of a given system of coplanar forces is that*

(i), (ii) & (iii) the algebraic sum of the moments of the given forces about any three non-collinear points in their plane should separately vanish.

Let a given system of coplanar forces acting on a rigid body be such that the algebraic sum of the moments of the forces about three non-collinear points A, B, C in their plane are separately zero.

Now, we know that a given system of coplanar forces can always be reduced either to a single force, or to a single couple. In this case, the force system cannot reduce to a couple, for then the moment of this couple, being equal to the algebraic sum of the moments of the given forces about any point A, is zero, and so the couple vanishes. Again, if the force system reduces to a single resultant force R, its moment about A, being equal to the algebraic sum of the moments of the given forces about that point, is zero. Thus the resultant, if it be not zero, must pass through A. Similarly, it will pass through B and C. Now, A, B, C being not in the same straight line, the resultant, which is in a definite direction, cannot pass through all three simultaneously. Thus the resultant must vanish. Hence the given system of forces, being neither reducible to a single resultant, nor to a couple, must be in equilibrium.

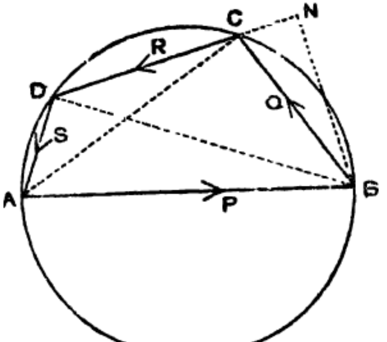
Thus the three conditions above being given, they are sufficient to ensure equilibrium of the given system.

Conversely, if the given system be in equilibrium, reducing it to a single force at any arbitrary point A together with a couple, which must both be zero, we can conclude that the algebraic sum of the moments of the given forces about A must vanish; and similarly for B and C. Thus the conditions follow necessarily.

8·4. Illustrative Examples.

Ex. 1. *Forces P, Q, R, S act along the sides AB, BC, CD, DA of the cyclic quadrilateral $ABCD$, taken in order, where A and B are the extremities of a diameter. If they are in equilibrium, prove that*

$$R^2 = P^2 + Q^2 + S^2 + 2PQS/R. \qquad [\text{ C. U. 1945 }]$$



Here the resultant of P and Q acting at B at an angle $180° - B$ balances the resultant of R and S at D acting at an angle $180° - D$ *i.e.* at an angle B. [$\because$ the quadrilateral is cyclic]

Hence, $P^2 + Q^2 - 2PQ \cos B = R^2 + S^2 + 2RS \cos B$. $\quad \cdots \quad$ (i)

Again, for equilibrium of the whole system, taking moment about B,

$$R.BC \sin C + S.AB \sin A = 0, \quad \text{or,} \quad \frac{BC}{AB} = -\frac{S}{R}.$$

[$\because A = 180^\circ - C$]

Thus, since ACB is a right angle for AB is a diameter,

$$\cos A = \frac{BC}{AB} = -\frac{S}{R}.$$

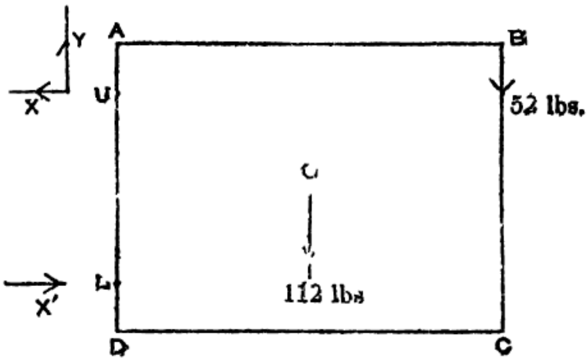
Now from (i),

$$P^2 + Q^2 + 2\frac{PQS}{R} = R^2 + S^2 - 2S^2,$$

i.e., $\quad R^2 = P^2 + Q^2 + S^2 + 2PQS/R.$

Ex. 2. *A gate 6 feet high and 8 feet wide weighs 112 lbs. and is supported by two hinges, one foot from the bottom and top respectively. The lower hinge can only exert a horizontal reaction. Find the reactions at both hinges, if a boy of weight 52 lbs. is sitting on the end of the gate.*

[*C. U. 1912*]



Let X, Y be the horizontal and vertical components of reaction at the upper hinge U, and X' the reaction of the lower hinge L, which is given to be horizontal only. These reactions at the hinges, together with the weight of the gate at its centre G, and the weight of the boy at the end B of the gate, keep it in equilibrium.

For equilibrium, resolving horizontally and vertically, we get

$$X' - X = 0, \quad \text{or,} \quad X' = X$$

$$Y - 112 - 52 = 0, \quad \text{or,} \quad Y = 164 \text{ lbs. wt.}$$

Also, taking moment about the lower hinge, (since $UL = 6 - 2 = 4$ ft., $AB = 8$ ft., distance of G from the line AD is 4 ft.)

$$X.4 - 112.4 - 52.8 = 0,$$

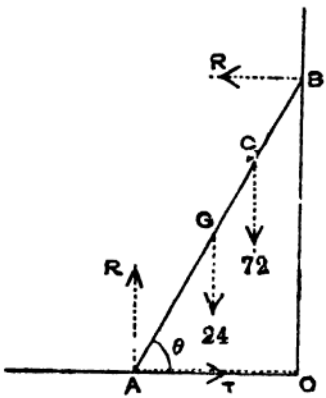
whence $\quad X = 216$ lbs. wt.

Thus total reaction at the upper hinge

$$= \sqrt{X^2 + Y^2} = \sqrt{216^2 + 164^2} = 271{\cdot}2 \text{ lbs. wt. nearly.}$$

The reaction at the lower hinge $= X' = X = 216$ lbs. wt.

Ex. 3. *A ladder of weight 24 lbs. rests on a smooth horizontal ground leaning against a smooth vertical wall at an inclination $\tan^{-1} 2$ with the horizon and is prevented from slipping by a string attached at its lower end, and to the junction of the wall and the floor. A boy of weight 72 lbs. begins to ascend the ladder. If the string can bear a tension of 30 lbs. wt., how far along the ladder can the boy rise with safety ?*



AB being the ladder at an inclination $\theta = \tan^{-1} 2$ to the horizon, let T be the tension in the string AO, when the boy rises a distance $AC = x$ along the ladder. Let $AB = a$, and let R' and R be the reactions at A and B which, since the floor and wall are both smooth, are vertical and horizontal respectively.

Now for equilibrium, resolving horizontally, and taking moment about A, we get

$$R-T=0, \qquad i.e., \quad R=T,$$

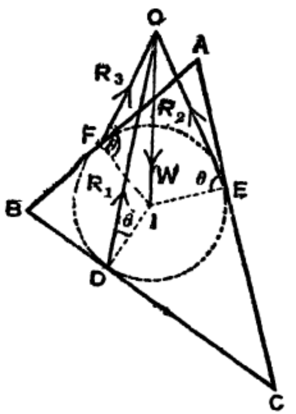
and $$R\,a\sin\theta-24\cdot\frac{a}{2}\cos\theta-72.x\cos\theta=0,$$

whence, $$\frac{x}{a}=\frac{R}{72}\cdot\tan\theta-\frac{1}{6}=\frac{T}{72}\cdot2-\frac{1}{6}=\frac{T}{36}-\frac{1}{6}.$$

Now maximum value of T that the string can bear is 30 lbs. wt. Hence the greatest possible value of x/a consistent with equilibrium is $\frac{30}{36}-\frac{1}{6}$ i.e. $\frac{2}{3}$, or the greatest value of x possible is $\frac{2}{3}a$.

Thus the boy can rise a distance $\frac{2}{3}$ of the ladder with safety.

Ex. 4. *A triangle formed of three rods is fixed in a horizontal position, and a homogeneous smooth sphere rests on it; show that the reaction of each rod is proportional to its length.*



Let D, E, F be the points where the sphere touches the rods BC, CA, AB. The reactions of the rods on the sphere being along the normals, must pass through the centre O of the sphere.

Let R_1, R_2, R_3 be these reactions. The section of the sphere by the plane of the triangle is a circle which touches the sides of the triangle ABC at D, E, F, and is accordingly the in-circle of the triangle. Its centre I is at the foot of the perpendicular from the centre O of the sphere on the plane; thus OI is vertical, and along this line the weight of the sphere acts.

Now, $DI=EI=FI$, and so $\dfrac{OI}{DI}, \dfrac{OI}{EI}, \dfrac{OI}{FI}$ which represent the tangents of the angles ODI, OEI, OFI are also equal. Thus DO, EO, FO are inclined at the same angle (θ say) to the horizon.

The reactions R_1, R_2, R_3 along DO, EO and FO, together with the weight of the sphere are in equilibrium. Hence resolving horizontally, the horizontal components of R_1, R_2, R_3 which are evidently along DI, EI, and FI are in equilibrium.

Thus, by Lami's theorem,

$$\frac{R_1 \cos\theta}{\sin EIF} = \frac{R_2 \cos\theta}{\sin FID} = \frac{R_3 \cos\theta}{\sin DIE}.$$

Now, $\sin EIF = \sin(180° - A) = \sin A$. [$\because$ $AEIF$ is cyclic]

Similarly, $\sin FID = \sin B$, $\sin DIE = \sin C$.

Thus, $R_1 : R_2 : R_3 = \sin A : \sin B : \sin C$

$= a : b : c$,

i.e., the reactions of the rods are proportional to their lengths.

Examples on Chapter VIII(b)

1. A uniform beam whose weight is 200 lbs. and which is 12 ft. long is hinged to a vertical wall. A string attached to the other end keeps the beam horizontal and is fixed to the wall 9 feet above it. A weight of 300 lbs. is hung from this end. Find the tension of the string and the thrust on the beam.

2. A ladder of length $2l$ and weight W rests against a smooth vertical wall. Its lower end is in contact with the floor which is smooth and is prevented from slipping by a string of length a, connecting it with the junction of the wall and the floor. If a person of weight $2W$ stands on the rung of the ladder distant $\frac{1}{2}l$ from its lower end, determine the reactions at the two ends of the ladder, and the tension of the string. [*C. U. 1941*]

***3.** A ladder resting on a smooth floor and against a smooth vertical wall is prevented from slipping by a rope tied to a point on it with its other extremity fixed at the junction of the floor and the wall. If the centre of gravity of the ladder divides it in the ratio $m : n$, and the ladder and the rope be inclined at angles θ and ϕ respectively to the horizon, show that the tension of the rope is

$$W \frac{m}{m+n} \cdot \frac{\cos \theta}{\sin (\theta - \phi)},$$

where W is the weight of the ladder.

4. A heavy rod of weight W is hung from a point by two equal strings, one attached to each extremity of the rod. A weight w is suspended half-way between the mid-point and one end of the rod. If T_1 and T_2 be the tensions in the strings, show that

$$\frac{T_1}{T_2} = \frac{2W+3w}{2W+w}.$$

5. A uniform beam of length $2a$ and weight W rests with its ends on two smooth planes inclined at angles $30°$ and $60°$ respectively to the horizon. A ring of weight $2W$ can slide along its length. Find the position of the ring when the beam rests in a horizontal position.

6. A square lamina $ABCD$ of weight W is hinged to a vertical wall at A with its plane vertical. A weight W is suspended from its corner C and it is supported with AC horizontal by means of a horizontal string joining B to the wall. Find the tension of the string and the reaction at the hinge.

7. A door $7\frac{1}{2}$ ft. high is hung from two hinges placed 9 inches from the top and the bottom. If the weight of the door be 36 lbs. wt., and its C. G. is at a distance $2\frac{1}{4}$ ft. from the line of hinges, show that the total force on each hinge is $22\frac{1}{2}$ lbs. wt., it being assumed that the weight of the door is supported by each hinge.

8. A gate is supported by two hinges in such a way that the action of the upper hinge is entirely horizontal. The distance between the hinges is 3 ft., and the weight of the gate, 60 lbs., acts along a vertical line $3\frac{1}{2}$ ft. from the line of the hinges. Find the force exerted by each hinge

[*C. U. 1944*]

9. Forces P, Q, R, S acting along AB, BC, CD, DA of a quadrilateral $ABCD$ are in equilibrium, show that

$$\frac{P \times R}{AB \times CD} = \frac{Q \times S}{BC \times DA}.$$

***10.** Forces P, Q, R act along the sides BC, CA, AB of a triangle ABC and forces P', Q', R' act along AG, BG, CG, where G is the centroid of the triangle. If the six forces are in equilibrium, show that

$$\frac{PP'}{AG.BC} + \frac{QQ'}{BG.CA} + \frac{RR'}{CG.AB} = 0.$$

***11.** Forces P, Q, R act along the sides BC, CA, AB of the triangle ABC and forces P', Q', R' act along OA, OB, OC, where O is the circum-centre, in the senses indicated by the order of the letters. If the six forces are in equilibrium, show that

$$P \cos A + Q \cos B + R \cos C = 0$$

and
$$\frac{PP'}{a} + \frac{QQ'}{b} + \frac{RR'}{c} = 0.$$

12. Forces P_1, P_2, P_3, P_4, P_5, P_6 act along the sides of a regular hexagon taken in order. Show that they will be in equilibrium, if

$$\Sigma P = 0 \text{ and } P_1 - P_4 = P_3 - P_6 = P_5 - P_2.$$

13. Three uniform rods AB, BC, CD whose weights are proportional to their lengths a, b, c are jointed at B and C, and are resting in a horizontal position on two smooth pegs, the distance between which is x. Show that

$$x = \frac{a^2}{2a+b} + \frac{c^2}{2c+b} + b.$$

14. A weight W is attached to an endless string of length l which hangs over two smooth pegs distant c apart in a horizontal line. Prove that the pressure on each peg is

$$W\sqrt{\left\{\frac{l-c}{2(l-2c)}\right\}}.$$

15. A uniform rod of weight W is supported in equilibrium by a string of length l, attached to its ends and passing over a smooth peg. If now a weight W' be suspended from one end of the rod, prove that the system may be placed in equilibrium by sliding a length $\dfrac{lW'}{2(W+W')}$ of the string over the peg.

***16.** Two uniform rods AB, AC, each of length $2a$, are smoothly jointed at A and rest symmetrically in a vertical plane on two smooth pegs in the same horizontal level at a distance $2x$ apart. Show that they are in equilibrium if each rod makes with the vertical an angle $\sin^{-1}\sqrt[3]{(x/a)}$.

17. Two equal uniform heavy rods are connected at one extremity by a thin string, and the system is placed symmetrically in a vertical plane with the rods resting on two smooth pegs in the same horizontal line. If a be the length of each rod, b the distance between the pegs and l $(< b)$ is the length of the connecting string, show that in equilibrium position, the inclination of each rod to the horizon is

$$\cos^{-1}\left(\sqrt[3]{\frac{b-l}{a}}\right).$$

***18.** A triangular lamina of weight W is supported by three vertical strings attached to its angular points, so that the plane of the triangle is horizontal. A particle of weight W is placed on the triangle at its ortho-centre. Prove that if T_1, T_2, T_3 be the tensions in the three strings, then

$$\frac{T_1}{1+3\cot B\cot C}=\frac{T_2}{1+3\cot C\cot A}=\frac{T_3}{1+3\cot A\cot B}.$$

***19.** A circular lamina is hung up from three points A, B, C on its rim by equal strings attached to a fixed point. If T_1, T_2, T_3 be the tensions in the strings, then

$$\frac{T_1}{\sin 2A}=\frac{T_2}{\sin 2B}=\frac{T_3}{\sin 2C}.$$

20. A heavy triangular lamina is suspended with its plane horizontal from a fixed point by strings attached to its corners ; show that the tension of each string is proportional to its length.

Prove that the same result is true even if the plane of the triangle is not horizontal.

*__21.__ A sphere rests on three smooth pegs, which lie in a horizontal plane, and are at distances a, b, c from one another. Prove that the pressure on the pegs are in the ratios

$$a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2).$$

22. A light table stands on three equal vertical legs and a weight is placed at the in-centre of the triangle formed by the points of intersection of the legs with the table. Show that the pressures on the legs are proportional to the lengths of the opposite sides of the triangle.

23. Two equal smooth spheres, each of weight W and radius r, are placed within a thin hollow vertical cylinder of radius a ($< 2r$), open at both ends, and resting on a horizontal table. Prove that the least weight of the cylinder so that it may not be upset, is

$$2W\left(1-\frac{r}{a}\right).$$

24. Two equal uniform ladders of length l, freely jointed at A, are connected by a horizontal rope PQ and rest on a smooth horizontal plane ; a man of weight W climbs a distance x up one of the ladders. If w be the weight of each ladder, $2b$ = length of the rope, and $AP = AQ = a$, show that the tension of the rope is

$$\frac{Wx+wl}{2a} \cdot \frac{b}{\sqrt{a^2-b^2}}.$$

25. AB, BC, CD are three equal rods jointed at B and C. The rods AB, CD rest on two smooth pegs in the same horizontal line, so that BC is horizontal. If α be the inclination of AB and β that of the reaction at B to the horizon, prove that

$$3 \tan \alpha \tan \beta = 1.$$

26. A solid hemisphere of weight W and radius a is placed with its curved surface on a smooth horizontal table and a string of length l ($< a$) is attached to a point on its rim and to a point on the table; show that the tension of the string is

$$\tfrac{3}{8}W\cdot\frac{a-l}{\sqrt{2al-l^2}}.$$

27. Two smooth balls of the same material of radii a and b are placed inside a hemispherical bowl of radius R; prove that the line joining the centres of the balls will be horizontal, if

$$R=\frac{(a+b)(a^2+b^2)}{a^2-ab+b^2}.$$

28. Inside a fixed vertical ring of radius R, there are placed symmetrically two equal small rings of radius r, and a third equal ring is placed symmetrically on them. Prove that the rings will remain in contact, provided

$$R < r\,(1+2\sqrt{7}).$$

29. Two uniform spheres of equal weight but unequal radii a, b are connected by a cord of length l attached to a point on each surface. They rest in contact, the string hanging over a smooth peg. Show that the two portions of the string make equal angles

$$\sin^{-1}\frac{a+b}{a+b+l}$$

with the vertical.

***30.** Two smooth spheres of radii a and b, of equal density, are connected by a light string of length l, the ends of the string being attached to points on the surface of the spheres The string is slung over a smooth fixed peg and the spheres hang freely in contact with one another. Show that in the position of equilibrium, the peg divides the length of the string in the ratio

$$b^3\cdot(b+l)-a^4 : a^3\,(a+l)-b^4.$$

31. The height of a solid homogeneous right circular cone is h and the radius of its base is r; a string is fastened to the vertex and to a point on the circumference of the circular base, and is then put over a smooth peg; if the cone rests in equilibrium with its axis horizontal, show that the length of the string is $\sqrt{(h^2+4r^2)}$.

[*C. H. 1960*]

***32.** A smooth plane is inclined at an angle θ to the horizon. A smooth rod of length $2l$ has one end resting on the plane and is supported by a horizontal beam which is parallel to the plane and at a distance d from it. If α be the inclination of the rod to the inclined plane, then

$$d \sin\theta = l \sin^2\alpha \cos(\alpha-\theta).$$

***33.** ABC is an equilateral triangle formed by three smoothly jointed uniform rods, each of weight w. If D be the middle point of the rod AB which is the uppermost and C be the lowest vertex of the triangle, find the action at C and at each of A and B.

***34.** Two rods OA, OB are fixed in a smooth vertical plane with O uppermost, each rod making an acute angle α with the vertical. Two smooth rings of equal weight w, which can slide one upon each rod, are connected by a light inextensible string, upon which slides a third smooth ring of weight $2w$. Show that in the symmetrical position of equilibrium, the ratio of the distance from O of the third ring to the distance from O of either of the other strings is $\frac{1}{2}(\sec\alpha+\cos\alpha)$.

***35.** The ends of a light string are attached to two smooth rings Q, R of weights w_1, w_2 and the string carries a third smooth ring P of weight W which can slide upon it. The rings w_1, w_2 are free to slide on two fixed rods inclined at angles α and β to the vertical. If γ be the angle which either part of the string makes with the vertical, then in equilibrium, show that

$$\cot\gamma : \tan\beta : \tan\alpha = W : W+2w_2 : W+2w_1.$$

***36.** Two circular discs, each of radius ρ, with smooth edges, are placed on their flat sides in the corner between two mooth vertical planes, which are inclined at an angle 2θ. The discs touch each other in the line bisecting the angle 2θ. Show that the radius of the smaller disc that can be pressed between them without causing them to separate is $\rho(\sec\theta - 1)$.

*37. Four equal uniform rods jointed together form a square figure $LMNP$ and the system is suspended from the joint L, and kept in the form of a square by a string connecting L and N. If T be the tension of the string, and W, the sum of the weights of the four rods, show that $T = \frac{1}{2}W$; and if R be the reaction at either of the joints B or D, find the magnitude and direction of R.

*38. $ABCD$ is a square board, which is hung flat against a smooth vertical wall by means of a string APB, which is fastened to the two extremities A and B of the upper edge AB and which passes over a smooth nail at P. When the string APB is less than the diagonal AC of the board, show that there are three positions of equilibrium.

*39. A right circular cone of height h and vertical angle 2α is placed with its vertex in contact with a smooth vertical wall and its slant side resting against a smooth horizontal rail fixed parallel to the wall and at a distance C from it. Show that if in the position of equilibrium, the axis makes an angle θ with the horizon, then

$$3h = 4c\sec^2(\theta - \alpha)\sec\theta.$$

ANSWERS

1. $666\frac{2}{3}$ lbs. wt.; $\frac{100}{3}\sqrt{265}$ lbs. wt.

2. Tension = reaction of the wall = $\dfrac{aW}{\sqrt{4l^2 - a^2}}$;

   reaction of the floor = $3W$.

5. $\frac{1}{4}a$ from the end on $30°$ plane.

6. $3W$; $W\sqrt{13}$ making an angle $\tan^{-1}\frac{2}{3}$ with the horizon.

8. 70 lbs. wt., and $10\sqrt{85}$ lbs. wt.

33. At C, $\dfrac{\sqrt{3}}{6}w$; at each of A and B, $\sqrt{\dfrac{13}{12}}w$.

37. $R = \frac{1}{2}W$ and acts in the horizontal direction.

CHAPTER IX

FRICTION

9·1. Hitherto we have considered examples of bodies acted on by forces, which in some cases had been in contact with other *smooth* bodies or surfaces. Such a surface, from the very definition, can exert a reaction in the normal direction only, and is incapable of exerting any force in the tangential direction. A perfectly smooth body or surface is however an ideal one, as is never to be met with in nature, and all bodies or surfaces which are experienced by us are more or less *rough*.

When a body is in contact with a rough surface (or some other rough body), and is acted on by external forces whereby it is urged to slide over that surface, it experiences a tangential resistance at its point of contact, which is known as the force of **friction** *between the body and the surface (or between the two bodies).*

As an illustration, let us consider a book resting on a rough horizontal table. Evidently, the reaction of the table, which is vertically upwards, balances the weight of the body. Now, suppose we apply a horizontal pull on the body by a string; for instance, a string attached at one extremity of the book, after passing over the table, has a weight hanging from it at the other end. If the weight be not sufficiently large, it is found that the book does not move. This shows that the table (in addition to the vertical reaction balancing the weight of the body), exerts a horizontal force opposing the tension, and keeps the book in equilibrium. This is the force of friction. If now the hanging weight is increased a little, provided it does not exceed a certain limit, the body is still found to remain at rest. The force of friction must accordingly have increased. If the weight be diminished or removed, the body continues to remain at rest, which shows that the frictional force also diminishes or disappear simultaneously, or otherwise the body would move in the opposite direction. The direction of the horizontal pull may be altered and the experiments repeated. The results in all cases will indicate that the force of friction is a self-adjusting force, of the nature of a passive resistance, appearing only when necessary and being always (up to a certain limit) just sufficient to prevent motion, and in the requisite direction.

Now suppose the external force urging the body to slide over the rough surface (the tension in the above case) be gradually increased. A time comes when the urging force is sufficiently large, and friction is no longer able to keep the body in equilibrium, and the body begins to move. There is thus a limit to which friction can rise. The value of this limiting friction depends on the weight of the body on the table, (or more generally, on the normal reaction between the body and the rough surface), as can be verified by placing different additional weights on the book, and finding the limiting friction in each case by increasing the tension and seeing when the body just begins to move.

All three experiments performed suitably and repeated under different circumstances ultimately lead to certain laws satisfied by the force of friction which are given below. For experiments, the students are referred to any book on General Physics.

As we have already mentioned, perfectly smooth bodies are never to be met with in nature. In fact friction plays a very important part in our everyday life. If there were no friction of the ground, walking would have been impossible. Screws or nails would not stick to wood. Nothing would rest on any slope, and would slide down. Ladders would not rest on the ground leaning against a wall. Wheels and carriages would not roll. No heat could be generated without friction, and our everyday life practically would be upset.

9·2. The laws of Statical Friction.

When a body rests in contact with a rough surface (or another rough body), and is acted on by forces urging it to slide on the surface, the force of friction at the point of contact satisfies the following laws.

5 **Law I.** *The direction of the force of friction is always opposite to that in which the point of contact has the tendency to slide.*

Law II. *The magnitude of the force of friction is such as would be just sufficient to prevent the sliding motion of the point of contact, subject to a certain maximum limit.*

When the forces acting on the body urging the sliding motion are sufficiently large, such that the limiting value of the friction is reached, and the body is on the point of sliding, the law satisfied by this limiting friction is as follows ·

Law III. *The magnitude of the limiting friction always bears a constant ratio to the normal pressure between the two bodies in contact, and this ratio depends only on the nature of roughness of the materials of which the bodies are composed, but not on the shape or extent of the surfaces in contact.*) 5

The law of Dynamical Friction.

When a body in contact with a rough surface is acted on by forces such that it actually slides on the surface, the force of friction at the point of contact is in a direction opposite to that in which the point of contact slides, and its magnitude bears a constant ratio to the normal pressure between the bodies in contact.

In fact in this case the maximum amount of friction that can be called into play between the bodies is exerted at the point of contact.

9·3. Definitions.

Limiting Equilibrium and Limiting Friction.

5(k) (*When a body in contact with a rough surface (or another rough body) is acted on by forces, and is in such a condition* *that it is about to slide on the surface, the force of friction at the point of contact having reached a maximum limit, the body is said to be in limiting equilibrium. and the force of friction then called into play is known as limiting friction.*) 5

Coefficient of friction.

(b)(*When a body in contact with another rough body or surface is about to slide on it, the constant ratio which the limiting friction bears to the normal reaction between the two bodies in contact is defined as the coefficient of friction between the bodies.*

Thus, in case of limiting equilibrium of a body on a rough surface, if F be the limiting friction at the point of contact, R the normal reaction between the bodies, then

$$\frac{\mathbf{F}}{\mathbf{R}} = \mu, \text{ or, } \mathbf{F} = \mu\mathbf{R},$$

where μ is a constant which represents the coefficient of friction in this case.) 5 (b)

Note. The value of μ is different for different pairs of bodies in contact. Even if one of the pair of bodies in contact has got its roughness altered, say by rubbing or otherwise, the value of μ alters.

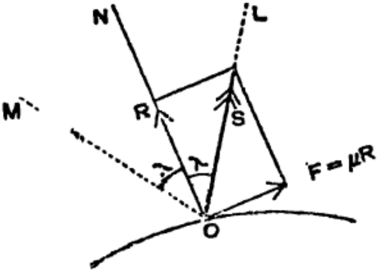
As far as is known, in no case μ has been found to be greater than unity.

The coefficient of dynamical friction, though very nearly equal to the coefficient of statical friction, is strictly speaking, between the same pair of bodies, slightly less than it, for experiments show that the least tangential force necessary to start into motion a body, resting on a rough surface, is slightly greater than that required to continue the motion once started, though the normal pressure between the bodies is the same in the two cases.

(c) **Angle of friction.**

5 (*In case of limiting equilibrium of a body on a rough surface, the angle made by the resultant of the forces of*

limiting friction and the normal reaction (i.e., the resultant reaction of the rough surface) with the normal to the surface at the point of contact is defined as the angle of friction. §(c)



Thus R being the normal reaction, the limiting friction is $F = \mu R$, where μ is the coefficient of friction, and their resultant S making an angle λ with the normal, λ is defined as the angle of friction.

Now from the figure it is clear that

$$\tan\lambda = \frac{F}{R} = \frac{\mu R}{R} = \mu.$$

Thus we may give another definition for the angle of friction as follows :

The angle of friction is that angle of which the tangent is equal to the coefficient of friction.

It may be noted that for a body in contact with a rough surface, if ON represents the normal at the point of contact, and if OL and OM be drawn on either side of ON making the same angle λ with ON, then in the case of limiting equilibrium of the body, the direction of total reaction of the surface (resultant of normal reaction, and friction which is limiting) will be along OL or OM according as the body is on the point of sliding one way or the other.

In case of *non-limiting equilibrium* of the body, the force of friction F being less than μR, where R is the normal reaction, the angle θ made by the resultant reaction with the normal is given by

$$\tan\theta = \frac{F}{R} < \frac{\mu R}{R} \text{ i.e., } < \tan\lambda.$$

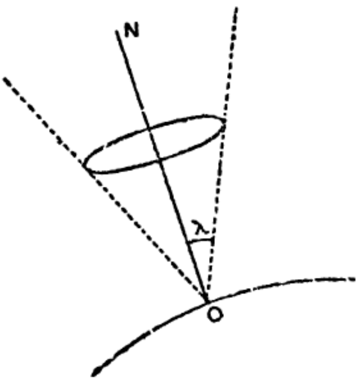
Thus $\theta < \lambda$.

Hence, *in any case of equilibrium of the body on the rough surface, the total reaction of the surface must be within the angle LOM.*

Accordingly, if the resultant of the external forces acting on the body be outside the angle vertically opposite to LOM, the body can never be in equilibrium, for the total reaction of the surface in the case cannot balance the resultant of the external forces. On the other hand, if the resultant of the external forces be within the said angle, the force of friction will adjust itself so that the resultant reaction will balance the resultant of the external forces and will keep the body in equilibrium.

Cone of friction.

5 (*When a body is in contact with a rough surface, and with the common normal at the point of contact as axis, we describe a right circular cone whose semi-vertical angle is $\lambda = \tan^{-1}\mu$, where μ is the coefficient of friction, this cone is defined as the cone of friction.*) 5(d)



Every generator of the cone of friction, therefore, makes an angle equal to the angle of friction with the normal.

If the body is capable of sliding in any direction on the surface, it is clear that the resultant reaction of the surface can never have a direction outside the cone of friction. Accordingly, for equilibrium of the body it is essential that the resultant of the external forces on the body should be within the vertically opposite cone.

9·4. Rolling of a body on a rough surface.

Let a body having a point in contact with a rough surface be acted on by any system of external forces. If this force system reduces to a single resultant through the point of contact, then if this resultant force makes an angle not exceeding the angle of friction with the produced direction of the normal at the point of contact, the total reaction of the surface will adjust itself to neutralise the above resultant, and the body will remain at rest; otherwise the body will slide.

If, however, the external force system acting on the body does not reduce to a single resultant through the point of contact, the total reaction of the surface, by any adjustment whatever, cannot keep the body in equilibrium. In this case we can always reduce the external force system into a single force acting at the point of contact, together with a couple which does not vanish. Now, if the total reaction of the surface can balance the single resultant (which will be the case when the resultant does not make with the normal an angle greater than the angle of friction), the point of contact will have no sliding motion, but the couple will produce a turning effect on the body which will turn about the point of contact, in other words, the body will roll. This is the case of pure rolling.

If on the other hand, the single resultant makes an angle with the normal greater than the angle of friction, the point of contact will slide, while the body will turn due to the couple.

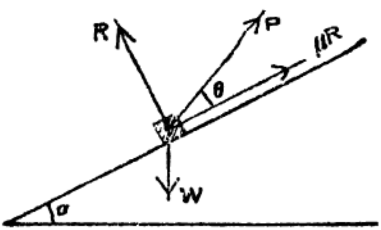
In the case of pure rolling, the force of friction at the point of contact is, in general, less than the limiting friction.

If the body instead of having a point of contact with a rough surface, has an area in common with it, then in case where the external forces reduce to a single resultant intersecting the area of contact, the body will either be in equilibrium, or will slide, according as this single resultant makes with the normal to the surface at its point of intersection, an angle, not exceeding, or exceeding the angle of friction. If however, the single resultant does not intersect the area of contact, the body must topple.

In working out examples where a body with an area in contact with a rough surface, and acted on by external forces, has got the equilibrium broken, and it is not known whether it will begin by rolling or by sliding, it is always advisable to assume rolling first, and work out the magnitude of the friction necessary to prevent the sliding motion. If it is found to exceed the limiting friction, then sliding will take place before this rolling stage is reached. Otherwise the body will roll, and the necessary friction, not exceeding the limiting friction, will come into play to prevent sliding.

9·5. Equilibrium of a heavy body on a rough inclined plane under any force.

Let a heavy body of weight W be placed on a rough inclined plane of inclination α to the horizon, and be acted on by a force P at an angle θ to the plane. Let μ be the coefficient of friction



Case I. *Let the body be just supported, i.e., just on the point of slipping down.*

If R be the normal reaction of the plane, the friction in this case, which is limiting, is μR up the plane

Hence for equilibrium, resolving along and perpendicular to the plane,

$$P\cos\theta+\mu R=W\sin\alpha$$

$$P\sin\theta+R=W\cos\alpha,$$

whence, eliminating R,

$$P(\cos\theta-\mu\sin\theta)=W(\sin\alpha-\mu\cos\alpha).$$

$$\therefore\quad P=W\frac{\sin\alpha-\mu\cos\alpha}{\cos\theta-\mu\sin\theta}.$$

If λ be the angle of friction, so that $\mu = \tan\lambda$,

we get $$P = W\frac{\sin\alpha - \cos\alpha\tan\lambda}{\cos\theta - \sin\theta\tan\lambda} = W\frac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$$

giving the necessary value of P in the given direction just to support the weight on the plane.

Case II. *Let the body be on the point of sliding up.*

In this case the limiting friction μR is down the plane, and hence the equations for equilibrium give

$$P\cos\theta - \mu R = W\sin\alpha$$

and $$P\sin\theta + R = W\cos\alpha,$$

whence, $$P = W\frac{\sin\alpha + \mu\cos\alpha}{\cos\theta + \mu\sin\theta} = W\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}.$$

Alternative method.

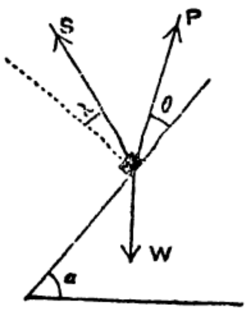


Fig. (i)

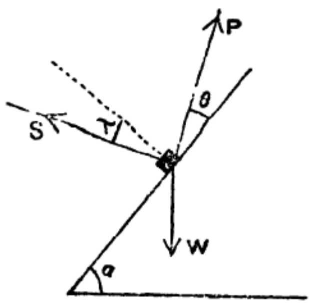


Fig. (ii)

In case I, the total reaction S of the inclined plane is making an angle λ with the normal to the right as in fig. (i), when the body is in limiting equilibrium, about to slide down. Thus angle between P and S is $90° - \theta - \lambda$, and that between W and S is $90° - \alpha + 90° + \lambda$. Now, for equilibrium of the three forces P, S and W, by Lamis' theorem,

$$\frac{W}{\sin(90° - \theta - \lambda)} = \frac{P}{\sin(180° - \alpha + \lambda)}, \text{ whence } P = W\frac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}.$$

In case II, when the body is on the point of sliding up, λ being on the opposite side as in fig. (ii), we get, as before

$$\frac{W}{\sin(90^\circ-\theta+\lambda)}=\frac{P}{\sin(180^\circ-\alpha-\lambda)}, \text{ whence } P=W\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}.$$

Cor. 1. If $\alpha<\lambda$, it is found that for case I, $P=W\dfrac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$ is negative or zero. In other words, no force is required to support the body on the plane, which will rest of itself (in non limiting equilibrium in case $\alpha<\lambda$). The maximum value of α for P not to be positive is λ. Thus, *the greatest inclination of the rough plane to the horizon so that the body will rest on it without support is the angle of friction*, and in this extreme case the body will be in limiting equilibrium, just on the point of slipping down.

Cor. 2. If $\alpha>\lambda$, and $\theta=0$, the extreme values of P applied parallel to the plane when the body is on the point of slipping down, and when it is on the point of slipping up, are respectively $W\sin(\alpha-\lambda)\sec\lambda$ and $W\sin(\alpha+\lambda)\sec\lambda$. If P has any value between these, the body will rest on the plane in non-limiting equilibrium.

Cor. 3. In general, when $\alpha>\lambda$, the least force that will just support the body on the plane is $P=W\sin(\alpha-\lambda)$, when applied in a direction given by $\theta=-\lambda$, and the least force necessary to drag the body up the plane is $P=W\sin(\alpha+\lambda)$ applied in a direction given by $\theta=\lambda$.

The last result can be put in the form, 'the least angle of traction up a rough inclined plane is the angle of friction'.

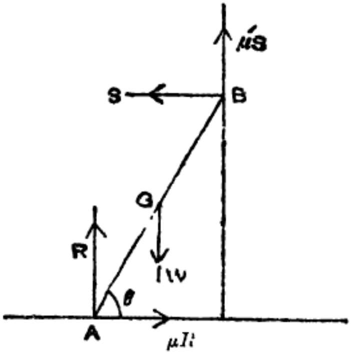
9·6. Illustrative Examples.

Ex. 1. *A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall; if the ground and wall be both rough, the coefficients of friction being μ and μ' respectively, and if the ladder be on the point of slipping at both ends, show that the inclination of the ladder to the horizon is given by*

$$\tan\theta=\frac{1-\mu\mu'}{2\mu}.$$

[*C. U. 1936 ; B. H. U. 1942 ; U. P. 1947*]

Let the length AB of the ladder be $2a$. R and S denoting the normal reaction of the ground and the wall, the limiting friction at these points are μR and $\mu' S$ in directions shown in the figure.



Now for equilibrium, resolving horizontally and vertically, and taking moment about A, we get

$$S=\mu R \qquad \cdots \qquad \cdots \quad \text{(i)}$$

$$R+\mu' S=W \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

and $S.2a \sin\theta+\mu' S.2a\cos\theta-W.a\cos\theta=0.$ $\cdots$ (iii)

From (i) and (ii), $S\left(\frac{1}{\mu}+\mu'\right)=W$

and then, from (iii),

$$2S(\tan\theta+\mu')=W=S\left(\frac{1}{\mu}+\mu'\right).$$

$$\therefore \quad \tan\theta=\frac{1}{2}\left(\frac{1}{\mu}+\mu'\right)-\mu'=\frac{1-\mu\mu'}{2\mu}.$$

Ex. 2. *A straight uniform beam of length $2h$ rests in limiting equilibrium, in contact with a rough vertical wall of height h, with one end on a rough horizontal plane and with the other end projecting beyond the wall. If both the wall and the plane be equally rough, prove that λ, the angle of friction, is given by*

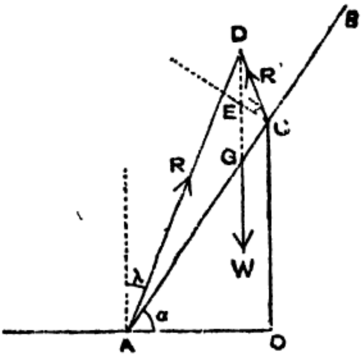
$$\sin 2\lambda=\sin\alpha\sin 2\alpha,$$

where α is the inclination of the beam to the horizon. [C. U. 1944]

AB is the beam, G its middle point, so that $AG = GB = h$.

CO is the wall $= h$. Hence, $AC = h \operatorname{cosec} \alpha$.

Thus, $GC = h(\operatorname{cosec} \alpha - 1)$.



Now the equilibrium being limiting, the total reaction R at A is along AD making an angle λ with the vertical, and the total reaction R' at C is along CD, making an angle λ with the normal CE to the rod at C. As the three forces R, R' and the weight W of the rod at G are in equilibrium, they must meet at a common point D, so that DG is vertical, intersecting CE at E say.

Clearly, $\angle ADG = \lambda = \angle ECD$, $\angle GEC = \alpha$, and so $\angle EDC = \alpha - \lambda$.

Now, $$\frac{GC}{AG} = \frac{GC}{GD} \cdot \frac{GD}{AG} = \frac{\sin GDC}{\sin GCD} \cdot \frac{\sin DAG}{\sin ADG},$$

i.e., $$\frac{h(\operatorname{cosec} \alpha - 1)}{h} = \frac{\sin(\alpha - \lambda)}{\sin(90^\circ + \lambda)} \cdot \frac{\sin(90^\circ - \alpha - \lambda)}{\sin \lambda},$$

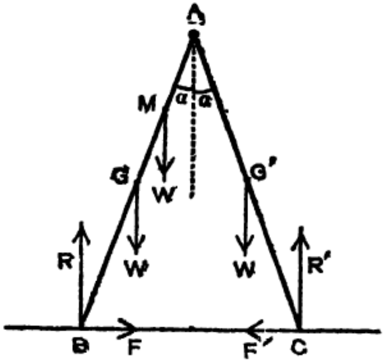
or, $$(\operatorname{cosec} \alpha - 1) = \frac{\sin(\alpha - \lambda)\cos(\alpha + \lambda)}{\sin \lambda \cos \lambda} = \frac{\sin 2\alpha - \sin 2\lambda}{\sin 2\lambda}.$$

$\therefore$ $\sin 2\lambda \operatorname{cosec} \alpha = \sin 2\alpha$, or, $\sin 2\lambda = \sin \alpha \sin 2\alpha$.

Ex. 3. *Two equal uniform ladders are jointed at one end and stand with the other ends on a rough horizontal plane. A man whose weight is equal to that of one of the ladders ascends one of them. Prove that* *the other will slip first. Supposing that it slips when he has ascended a distance x, prove that the coefficient of friction is*

$$(a+x) \tan \alpha/(2a+x),$$

a being the length of each ladder, and α the angle which each makes with the vertical.



Let M be the position of the man on the ladder AB at any instant when $BM=x$, R, R' the normal reactions of the ground, and F, F' the frictions at the instant at B and C respectively.

Now, considering the two ladders AB and AC as forming one system, action and reaction at A on the two neutralise each other. Hence, resolving horizontally and vertically and taking moment about B, for the equilibrium of the combined system, we get

$$F=F' \qquad \cdots \qquad \cdots \quad \text{(i)}$$

$$R+R'=3W \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

$$R'.2a \sin \alpha = W \cdot \frac{a}{2} \sin \alpha + W.x \sin \alpha + W \cdot \frac{3a}{2} \sin \alpha. \quad \cdots \quad \text{(iii)}$$

Again, considering the equilibrium of the ladder AC separately, taking moment about A whereby the unknown action at A of AB on AC will not enter the equation, we get

$$F'.a \cos \alpha = R'.a \sin \alpha - W \cdot \frac{a}{2} \sin \alpha. \qquad \cdots \quad \text{(iv)}$$

From (iii), $R' = W \cdot \dfrac{2a+x}{2a}$, and then from (ii), $R = W \cdot \dfrac{4a-x}{2a}$.

Hence, from (i) and (iv),

$$F=F'=\tan\alpha\left\{W\frac{2a+x}{2a}-\frac{W}{2}\right\}=W\tan\alpha\cdot\frac{a+x}{2a}.$$

Thus, $\frac{F}{R}=\frac{a+x}{4a-x}\cdot\tan\alpha$, and $\frac{F'}{R'}=\frac{a+x}{2a+x}\tan\alpha.$

Now, $(4a-x)-(2a+x)=2(a-x)$ is positive, for $x<a$.

Hence, $\frac{F'}{R'}>\frac{F}{R}$ for all values of x.

One of the ladders will slip when either $\frac{F}{R}$ or $\frac{F'}{R'}$ is equel to the coefficient of friction μ, and as $\frac{F'}{R'}$ is the greater, this will attain the value μ first. Thus the other ladder AC will slip first, and the coefficient of friction is connected to x by the relation

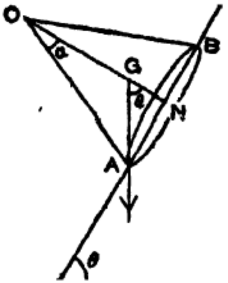
$$\mu=\frac{F'}{R'}=\frac{a+x}{2a+x}\tan\alpha.$$

Note. Initially, when $x=0$, *i.e.*, the man is on the lowest rung of the first ladder, $\frac{F}{R}=\frac{1}{4}\tan\alpha$, and $\frac{F'}{R'}=\frac{1}{2}\tan\alpha$, and μ must be $>\frac{1}{2}\tan\alpha$ in order that the ladders may not slip. Provided this is satisfied, $\frac{F}{R}$ and $\frac{F'}{R'}$ both increase with x, but $\frac{F'}{R'}$ being always $>\frac{F}{R}$, will attain the value μ first.

Ex. 4. *A heavy solid right circular cone is placed with its base on a rough inclined plane, the inclination of which is gradually increased; determine whether the initial motion of the cone will be one of sliding or tumbling over.*

[*The C.G. of a solid right circular cone is on the axis at a height one-fourth of the height of the vertex from the base.*]

The total reaction of the plane (resultant of normal reaction and friction at all points of the base) must be a force, acting somewhere on the base AB. Hence, if the external force, namely the weight of the cone, which acts vertically downwards through its centre of gravity G, falls outside the base AB, the total reaction of the plane will not be able to blance the weight, and the cone will topple. Thus, assuming θ to be the inclination of the inclined plane in the marginal case when the weight passes through the extremity A of the base, and the cone is on the point of tumbling over about A, the base will not slip before this stage is reached provided the external force *i.e.*, the weight makes with normal to the plane an angle, less than the angle of friction λ, *i.e.*, provided $\angle AGN = \theta < \lambda$. This requires



$$\tan\theta < \tan\lambda, \text{ or, } \frac{AN}{GN} < \mu.$$

Now, a being the semi-vertical angle of the cone,

$$\tan a = \frac{AN}{ON} = \frac{AN}{4GN} = \tfrac{1}{4}\tan\theta.$$

Hence, the cone will tumble if $4\tan a < \mu$.

If $4\tan a > \mu$, then before the contemplated position for tumbling is reached, the base will slip.

Hence, the initial motion of the cone will be one of slipping or tumbling over according as

$$\mu < \text{or} > 4\tan a.$$

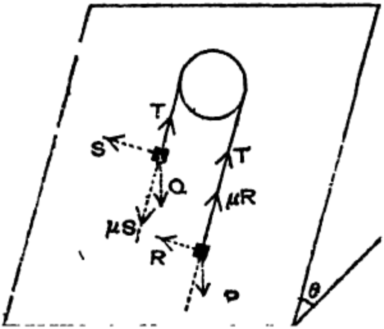
In case $\mu < 4\tan a$, the cone will slip when the inclination of the plane to the horizon, *i.e.*, $\theta = \lambda = \tan^{-1}\mu$. If $\mu > 4\tan a$, the cone will tumble when $\theta = \tan^{-1}(4\tan a)$.

Ex. 5. *Two weights P and Q ($P > Q$) are placed on a rough inclined plane, being connected by a thin string passing over a small smooth pulley on the plane, the parts of the string being parallel to the line of greatest slope. The inclination of the plane to the horizon is gradually increased. Prove that the weights will begin to slip on the plane when its inclination θ to the horizon is given by*

$$\tan\theta = \frac{P+Q}{P-Q}\tan\lambda,$$

where λ is the angle of friction of the plane, assumed same with respect to either weight.

θ being the inclination of the plane when P is on the point of slipping down, and μ denoting the coefficient of friction, if R be the normal reaction on P, the friction is μR up the plane. If S be the normal reaction on Q, as Q is on the point of moving up, the friction μS will act down the plane. Let T be the tension of the string.



Now, for equilibrium of P and Q, resolving along and perpendicular to the plane in each case

$$P \sin \theta = T + \mu R, \qquad P \cos \theta = R$$

$$Q \sin \theta = T - \mu S, \qquad Q \cos \theta = S.$$

From these,

$$P (\sin \theta - \mu \cos \theta) = T = Q (\sin \theta + \mu \cos \theta).$$

$$\therefore \quad \sin \theta \, (P - Q) = \mu \cos \theta \, (P + Q).$$

$$\therefore \quad \tan \theta = \mu \frac{P+Q}{P-Q} = \frac{P+Q}{P-Q} \tan \lambda.$$

Examples on Chapter IX

1. A body of weight W rests on a rough horizontal plane, λ being the corresponding angle of friction. It is desired to move the body on the plane by pulling it with the help of a string. Find the least angle of friction, and the least force necessary.

2. A body of weight 4 lbs. rests in limiting equilibrium on an inclined plane whose slope is 30°. Find the coefficient of friction and the normal reaction.

[*C. U. 1916 ; B. E. Allahabad, 1938*]

3. How high can a particle rest inside a hollow sphere of radius a, if the coefficient of friction be $\frac{1}{\sqrt{3}}$?

[*P. U. 1929, '31, '39*]

4. Prove that the horizontal force which will just sustain a heavy particle on a rough inclined plane, will sustain the same particle on a smooth plane whose inclination is less than that of the rough plane by the angle of friction.

5. A weight 30 lbs. can just rest on a rough inclined plane when its inclination to the horizon is 30°. When the inclination is increased to 60°, find the least horizontal force which will support it.

Find also the least force along the plane that will drag it up.

6. A body of mass 100 lbs. is placed on a rough inclined plane, and is just supported by a force 40 lbs. wt. applied along the plane. When this force is increased to 80 lbs. wt., the body is on the point of sliding up. Find the coefficient of friction.

7. A body of weight W can just be sustained on a rough inclined plane by a force P, and just dragged up the plane by a force Q, P and Q both acting up the line of the greatest slope. Show that the coefficient of friction is

$$\frac{Q-P}{\sqrt{4W^2-(P+Q)^2}}.$$

8. The force P acting along a rough inclined plane just supports a body on the plane, the angle of friction λ being less than α, the inclination of the plane to the horizon. Show that the least force acting along the plane, which is sufficient to drag the body up the plane is

$$P\frac{\sin(\alpha+\lambda)}{\sin(\alpha-\lambda)}.$$

9. The least force which will move a weight up an inclined plane is P. Show that the least force, acting parallel to the plane which will move the weight upwards, is

$$P\sqrt{1+\mu^2},$$

μ being the coefficient of friction of the plane.

10. A body is resting on a rough inclined plane of inclination α to the horizon, the angle of friction being λ, $(\lambda > \alpha)$. If P and Q be the least forces which will respectively drag the body up and down the plane, then

$$\frac{P}{Q}=\frac{\sin(\lambda+\alpha)}{\sin(\lambda-\alpha)}.$$

11. If the force, which acting parallel to a rough plane of inclination α to the horizon is just sufficient to draw a weight up, be n times the force which will just let it be on the point of sliding down, show that

$$\tan\alpha=\mu\frac{n+1}{n-1}.$$

***12.** Two rough particles connected by a light string rest on an inclined plane, the string passing round a smooth pulley on the plane, and the parts of the string being parallel to the line of the greatest slope. If the weights and corresponding coefficients of friction are W_1, W_2, and μ_1, μ_2 respectively, show that the greatest inclination of the plane consistent with equilibrium is

$$\tan^{-1}\left(\frac{\mu_1 W_1+\mu_2 W_2}{W_1 \sim W_2}\right).$$ [*Punjab, 1940*]

***13.** A beam rests with one end on a horizontal ground and the other against a vertical wall. Prove that for equilibrium, there *must* be friction between the beam and the ground, and *need not be* friction between the beam and the wall.

***14.** A uniform ladder rests with one end on the rough horizontal ground and the other against a rough vertical wall. The coefficients of friction at the lower and upper ends are $\frac{3}{7}$ and $\frac{1}{3}$ respectively. Determine the angle which the ladder makes with the ground when it is about to slip.
[*C. U. 1943*]

15. A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and its upper end against a smooth vertical wall. If θ be the inclination of the ladder to the vertical, prove that

$$\tan\theta = 2\mu,$$

where μ is the coefficient of friction.

What happens if the ladder be non-uniform ?

16. A uniform ladder rests with its lower end on a rough horizontal ground and its upper end against a rough vertical wall, the ground and the wall being equally rough, and the angle of friction λ. Show that the greatest inclination of the ladder to the vertical is 2λ.

17. A ladder, 30 feet long, rests with one end against a smooth vertical wall and with the other on the ground, which is rough, the coefficient of friction being $\frac{1}{2}$; find how high a man whose weight is 4 times that of the ladder can ascend before it begins to slip, the foot of the ladder being 6 ft. from the wall. [*B. H. U. 1946*]

18. A uniform ladder rests inclined at 45° to the vertical with one end on rough horizontal ground, the coefficient of friction being $\frac{5}{8}$, and the other end against a smooth vertical wall. Show that a man whose weight is equal to that of the ladder cannot ascend to the top.

What weight must be placed on the bottom of the ladder to enable the man to ascend to the top ?

19. A man weighing 140 lbs. climbs up a uniform ladder 20 ft. long and 70 lbs. in weight, which rests against a rough vertical wall at an angle of 45°. If the coefficient of friction at each end of the ladder is 0·5, how far will the man be able to climb up the ladder before it begins to slip ?

Find also the greatest weight of a creature which can climb to the top. [*B. H. U. 1940*]

20. A uniform ladder rests in limiting equilibrium with one end on a rough horizontal plane and the other against a smooth vertical wall. A man then ascends the ladder.

Show that, whatever his weight, he cannot go more than half-way up. What happens if the horizontal plane also be smooth ? Give reasons for your answer. [*C. U. 1942*]

21. A uniform ladder rests in limiting equilibrium with one end against a rough vertical wall and the other on a rough horizontal plane, the angles of friction being λ and λ' respectively. Show that the inclination θ of the ladder to the horizon is given by

$$\tan\theta = \frac{\cos(\lambda + \lambda')}{2\sin\lambda'\cos\lambda}. \qquad [\textit{C. U. 1947}]$$

***22.** A uniform ladder of weight W, inclined to the horizon at 45°, rests with its upper extremity against a rough vertical wall and its lower extremity on the ground. Prove that the least horizontal force which will move the lower end towards the wall is just greater than

$$\frac{W}{2}\left(\mu + \frac{1+\mu}{1+\mu'}\right),$$

where μ and μ' are the coefficients of friction at the lower and upper end respectively. [*C. U. 1939*]

23. A uniform ladder of weight w rests on a rough horizontal ground and against a smooth vertical wall, inclined at an angle α to the horizon. Prove that a man of weight W can climb to the top of the ladder without the ladder slipping if

$$\frac{w}{W} > \frac{2(1-\mu\tan\alpha)}{2\mu\tan\alpha - 1},$$

μ being the coefficient of the friction. [*C. U. 1933*]

24. Two equal uniform rods AC, CB are freely jointed at C and rest in a vertical plane with the ends A and B in contact with a rough horizontal plane. If the equilibrium is limiting, and the coefficient of friction is μ, show that

$$\sin ACB = \frac{4\mu}{1+4\mu^2}. \qquad [\textit{C. U. 1935}]$$

25. Two equal ladders of weight W are placed so as to lean against each other at an angle 2θ, with their ends resting on a rough horizontal floor, the coefficient of friction of which with respect to either being μ, where $\tan\theta > \mu > \frac{1}{2}\tan\theta$. If W' be the weight which placed on the top causes the ladders to slip, show that

$$W' = W\frac{2\mu - \tan\theta}{\tan\theta - \mu}.$$

Explain the case when $\mu < \frac{1}{2}\tan\theta$ or $> \tan\theta$.

26. A bar rests on two pegs and makes an angle α with the horizontal. The centre of gravity is between the pegs at distances a, b from them. Prove that for equilibrium

$$\tan\alpha \not> \frac{\mu_1 b + \mu_2 a}{a+b},$$

where μ_1, μ_2 are the coefficients of friction at the pegs.

[*Agra, 1940*]

27. A heavy uniform rod is placed over one and under the other of two horizontal pegs, so that the rod lies in a vertical plane; shew that the length of the shortest rod which will rest in such a position is

$$a(1 + \tan\alpha \cot\lambda),$$

where a is the distance between the pegs, α is the inclination to the horizon of the line joining them, and λ is the angle of friction.

28. A straight uniform beam is placed upon two rough planes whose inclinations to the horizon are α_1 and α_2, and coefficients of friction $\tan\lambda_1$ and λ_2. If θ be the inclination of the beam to the horizon in limiting equilibrium,

$$2\tan\theta = \cot(\alpha_2 + \lambda_2) - \cot(\alpha_1 - \lambda_1).$$

***29.** A heavy uniform rod rests in limiting equilibrium within a fixed rough hollow sphere. If λ be the angle of friction, and 2α the angle subtended by the rod at the centre of the sphere, show that the inclination θ of the rod to the horizon is given by

$$2\tan\theta = \tan(\alpha + \lambda) - \tan(\alpha - \lambda).$$

[*P. U. 1934, 1943*]

***30.** A thin uniform rod of length $2l$ rests in limiting equilibrium inside a rough vertical circular hoop of radius a. Prove that the inclination of the rod to the horizontal is

$$\cot^{-1}\left(\frac{a^2 - l^2 - \mu^2 l^2}{a^2 \mu}\right),$$

where μ is the coefficient of friction.

***31.** A glass rod is balanced partly in and partly out of a cylindrical tumbler with the lower end resting against the vertical side of the tumbler. If α and β are the greatest and least angles which the rod can make with the vertical, prove that the angle of friction is

$$\tfrac{1}{2}\tan^{-1}\left(\frac{\sin^3\alpha - \sin^3\beta}{\sin^2\alpha\cos\alpha + \sin^2\beta\cos\beta}\right)$$

***32.** Prove that the least force, which applied to a uniform heavy sphere of weight W will maintain it in equilibrium against a rough vertical wall, is $W\cos\lambda$, provided λ, the angle of friction, is less than

$$\cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right).$$

***33.** A rectangular block of wood with a square base is placed on a rough inclined plane with two sides of the base horizontal, and the inclination of the plane to the horizon is gradually increased. If μ be the coefficient of friction, and a side of the base, find the greatest height of the block so that it may slide down the plane before toppling.

34. A uniform solid circular cylinder is placed with its plane base on a rough inclined plane and the inclnation of the plane to the horizon is gradually increased ; show that the cylinder will topple over before it slides, if the ratio of the diameter of the base of the cylinder to its height is less than the coefficient of friction.

35. A solid right circular cone is placed with its base on a rough inclined plane, the inclination of which to the horizon is gradually increased. If the angle of friction be 30°, find the angle of the cone when it is on the point of both slipping and turning over simultaneously.

ANSWERS

1. λ, $W \sin \lambda$. 2. $\frac{1}{\sqrt{3}}$, $2\sqrt{3}$ lbs. wt.

3. $\frac{1}{2}a(2-\sqrt{3})$ above the lowest point.

5. $10\sqrt{3}$ lbs. wt.; $20\sqrt{3}$ lbs. wt.

6. $\frac{1}{2}$. 14. $45°$. 15. If the c.g. divides the ladder in the ratio $m : n$, $\tan\theta = \mu\,(m+n)/m$.

17. He can rise to the top without the ladder slipping.

18. $\frac{2}{5}$ of the weight of the ladder. 19 13 ft.; $17\frac{1}{2}$ lbs.

25. If $\mu > \frac{1}{2}\tan\theta$, the ladders cannot rest without W' being negative *i.e.*, the top must be pulled upwards, in which case the ladders will be in limiting equilibrium, the lower ends tending to slip outwards. If $\mu > \tan\theta$, the ladders will be in non-limiting equilibrium whatever positive value W' may have, and to put them in limiting equilibrium W' must be negative *i.e.*, the top must be pulled upwards, when the lower ends will tend to approach each other.

33. $\frac{a}{\mu}$. 35. $\cos^{-1}\frac{47}{49}$.

Examples on Chapter IX(a)

*1. A hemispherical shell rests on a rough inclined plane whose angle of friction is λ. If θ be the inclination of the plane base to the horizon, show that θ cannot be greater than $\sin^{-1}(2\sin\lambda)$.

*2. A heavy particle rests on a rough parabola with its axis vertical and vertex downwards. If the latus rectum of the parabola is $4a$ and if the greatest height above the vertex at which the particle can remain at rest be b and if μ be the coefficient of friction, show that $\mu = (b/a)^{\frac{1}{2}}$.

*3. (i) A rough ellipse $x^2/a^2 + y^2/b^2 = 1$ is placed with its x-axis vertical. Find in which position a heavy particle can rest on it, if μ be the coefficient of friction.

(ii) A particle rests on a rough wire in the shape of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is so placed that its x-axis is vertical and y-axis horizontal. If μ be the coefficient of friction, find the depth of the position of the limiting equilibrium of the particle below the tangent at the highest point of the ellipse. [*C. H. 1954*]

*4. A heavy uniform wire in the shape of the ellipse $x^2/a^2+y^2/b^2=1$ is hung over a small rough peg. If the wire can be in equilibrium with any point in contact with the peg and if μ be the coefficient of friction, show that

$$\mu \nless \frac{a^2-b^2}{2ab}.$$

*5. A rough cycloid is placed with its axis vertical and vertex downwards. Show that a particle can rest on it at any height h above the lowest point where $h \ngtr 2a\sin^2\lambda$, λ being the angle of friction and a the generating circle of the cycloid.

*6. A hemisphere rests in equilibrium with its curved surface in contact with a rough plane inclined at an angle θ to the horizon. If θ be less than $\sin^{-1}\frac{3}{8}$ and also less than the angle of friction, show that the plane base of the hemisphere is inclined at an angle $\sin^{-1}(\frac{8}{3}\sin\theta)$ to the horizon.

*7. A uniform hemisphere of radius r, and weight w, rests with its spherical surface on a horizontal plane and a rough particle of weight w' rests on the plane surface at the point P. If C be the centre of the plane base of the hemisphere, show that $CP \ngtr \dfrac{3}{8}\dfrac{w\mu}{w'}r$, where μ is the coefficient of friction.

*8. A rigid framework $ABCD$ in the form of a rhombus, of side 'a' and acute angle θ, rests on rough peg whose coefficient of friction is μ. Show that the distance between the two extreme positions which the point of contact with the peg can have is $a\mu\sin\theta$.

*9. The handles of a drawer $ABCD$ whose length and depth AB, BC are $2a$ and $2b$ respectively, are distant $2d$ from each other. If μ be the coefficient of friction at the sides of the drawer and its base be smooth, show that it is not possible to pull out the drawer by pulling one handle straight outwards if $b<\mu d$.

*10. A uniform cylinder of mass m is supported against a smooth vertical wall by a wedge of mass M and angle α, which can slide on a rough horizontal floor. If slipping is about to take place, calculate the coefficient of friction between the wedge and the floor.

*11. A particle of weight W_1 lies on the rough plane face of a solid homogeneous hemisphere of radius a and weight W resting with its curved surface on a fixed horizontal plane. In the position of the equilibrium of the system, the particle is at a distance x from the centre. Prove that the friction exerted between the particle and the hemisphere is equal to

$$8xW_1^2(9a^2W^2+64x^2W_1^2)^{-\frac{1}{2}}. \qquad [\textit{C. H. 1959}]$$

*12. A solid homogeneous hemisphere rests on *rough* horizontal plane and against a *smooth* vertical wall. If the coefficient of friction μ be greater than $\frac{3}{8}$, the hemisphere can rest in any position and if it be less, the least angle that the base of the hemisphere can make with the horizontal is $\sin^{-1}(\frac{8}{3}\mu)$.

If the wall be rough (coefficient of friction μ') show that the angle is $\quad \sin^{-1}\left(\dfrac{8\mu}{3}\cdot\dfrac{1+\mu'}{1+\mu\mu'}\right)$.

*13. A, B are the lowest points of the wheels of a bicycle. The length of AB is $2l$ and the centre of gravity is at a height h above AB and at a distance d in front of its middle point. Neglecting the axle friction, show that the greatest incline on which the bicycle can rest without slipping, is

$$\tan^{-1}\frac{\mu(l+d)}{2l-\mu h} \text{ or, } \tan^{-1}\frac{\mu(l-d)}{2l+\mu h},$$

according as the front or back wheel is braked. (μ being the coefficient of friction).

ANSWERS

3. (i) Depth of the particle below the highest point of the ellipse is

$$\leqslant a-\frac{a^2}{(a^2+\mu^2b^2)^{\frac{1}{2}}}.$$

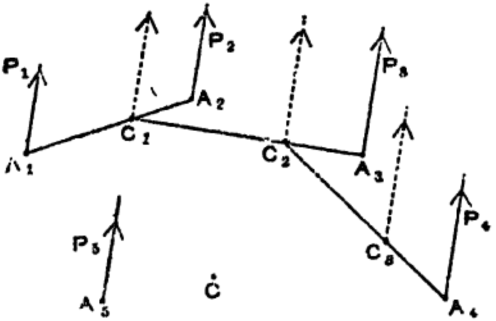
(ii) $a-\dfrac{a^2}{\sqrt{a^2+\mu^2b^2}}$. 10. $\dfrac{m\tan\alpha}{m+M}$.

CHAPTER X

CENTRE OF GRAVITY

10·1. Centre of like parallel forces.

Let P_1, P_2, P_3,...... be a set of like parallel forces acting on a rigid body at A_1, A_2, A_3,...... respectively. Join A_1A_2



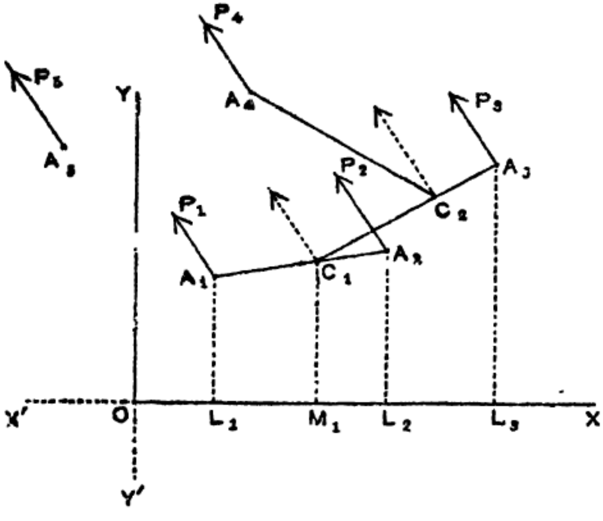
and divide it internally at C_1, such that $A_1C_1 : C_1A_2 = P_2 : P_1$. Then the resultant of P_1 and P_2 is a parallel force P_1+P_2 acting at C_1. Join C_1A_3, and divide it internally at C_2 such that $C_1C_2 : C_2A_3 = P_3 : P_1+P_2$. Then the parallel force $P_1+P_2+P_3$ acting at C_2, is the resultant of P_1+P_2 at C_1 and P_3 at A_3, *i.e.*, of P_1 at A_1, P_2 at A_2 and P_3 at A_3. Next join C_2A_4, and divide it at C_3 such that $C_2C_3 : C_3A_4 = P_4 : P_1+P_2+P_3$. Proceeding in this manner till all the forces are exhausted, we finally arrive at a point C through which passes a force ΣP parallel to the system, which is the resultant of all the given parallel forces.

Now, it is evident that the positions of C_1, C_2, C_3, etc., depend only on the magnitudes of P_1, P_2, P_3,...... and on the positions of the points A_1, A_2, A_3,...... where they act, but have nothing to do with the common direction of the parallel forces. Hence, the point C arrived at, through which the final resultant of the parallel forces passes, is fixed, whatever be the common direction of the parallel forces, so long as their magnitudes and points of application remain unchanged. The fixed point C is defined as the *centre of the given system of like parallel forces.*

We may note that C being the point through which the resultant ΣP always passes, whatever be the common direction of the given parallel forces $P_1, P_2, P_3, \ldots\ldots$ it is a unique point, and will come out to be the same in whatever order we proceed to combine the given forces in succession.

Analytical determination.

We shall confine ourselves to the case of a set of like parallel forces *acting in one plane.*



Let $P_1, P_2, P_3, \ldots\ldots$ be a set of like parallel forces acting at $A_1, A_2, A_3, \ldots\ldots$ on a plane, and let $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\ldots$ be the co-ordinates of $A_1, A_2, A_3, \ldots\ldots$ referred to a set of rectangular axes on the plane.

The resultant of P_1 at A_1 and P_2 at A_2 is P_1+P_2 acting at C_1 on A_1A_2 such that $A_1C_1 : C_1A_2 = P_2 : P_1$. Let ξ_1, η_1 be the co-ordinates of C_1. Then, A_1L_1, A_2L_2 and C_1M_1 being perpendiculars upon OX, since these are parallel lines,

$$\frac{L_1M_1}{M_1L_2}=\frac{A_1C_1}{C_1A_2} \quad i.e., \quad \frac{\xi_1-x_1}{x_2-\xi_1}=\frac{P_2}{P_1}$$

whence we get $\xi_1=\dfrac{P_1x_1+P_2x_2}{P_1+P_2}$. ... (i)

Exactly in a similar manner, the x-co-ordinate (ξ_2 say) of C_2, where the resultant $P_1+P_2+P_3$ of P_1+P_2 at C_2 and P_3 at A_3 acts, is given by

$$\xi_2=\frac{(P_1+P_2)\,\xi_1+P_3x_3}{(P_1+P_2)+P_3}=\frac{P_1x_1+P_2x_2+P_3x_3}{P_1+P_2+P_3}, \text{ from (i).}$$

Proceeding in this manner, when all the forces are exhausted, the centre of the parallel forces through which the final resultant passes having co-ordinates ξ, η, we get

$$\xi=\frac{P_1x_1+P_2x_2+P_3x_3+\cdots}{P_1+P_2+P_3+\cdots}=\frac{\Sigma \mathrm{Px}}{\Sigma \mathrm{P}}$$

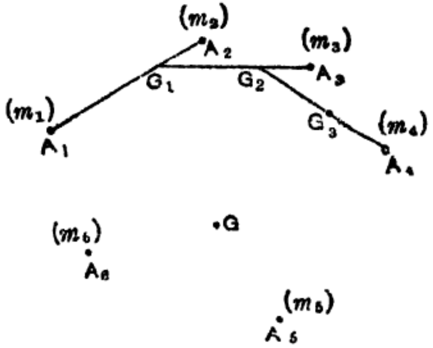
and similarly

$$\eta=\frac{P_1y_1+P_2y_2+P_3y_3+\cdots}{P_1+P_2+P_3+\cdots}=\frac{\Sigma \mathrm{Py}}{\Sigma \mathrm{P}}.$$

10·2. Centre of mass.

Let a system of particles of masses m_1, m_2, m_3,... be placed at the points A_1, A_2, A_3,......

Divide A_1A_2 at G_1 in the inverse ratio of the masses at its extremities, *i.e.*, $A_1G_1 : G_1A_2 = m_2 : m_1$. Assume the total mass m_1+m_2 collected at G_1. Divide G_1A_3 at G_2 in the inverse ratio of the masses at the extremities, such that $G_1G_2 : G_2A_3 = m_3 : (m_1+m_2)$, and assume the total mass at the extremities, *i.e.*, $m_1+m_2+m_3$ collected at G_2. Proceeding in this manner, when all the particles have been exhausted, we arrive at a final point G which is defined as the *centre of mass*, or *centre of inertia of the given system of particles.*



It is evident from the mode of construction, that the point G is identical with the centre of a set of parallel forces proportional to m_1, m_2, m_3, etc. acting at A_1, A_2, A_3, etc., and hence, as proved in the previous article, the point is unique.

Again, if the particles m_1, m_2, m_3, etc. be in one plane, and their co-ordinates referred to a set of rectangular axes be (x_1, y_1), (x_2, y_2), (x_3, y_3) etc., the co-ordinates $(\bar{x}, \bar{y}$ say) of the centre of mass G, exactly as in the previous article, will be given by

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma mx}{\Sigma m}$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma my}{\Sigma m}.$$

The symmetry of the forms of the co-ordinates also shows that the point G is unique, and will be arrived at finally, in whatever order the masses may be combined in succession.

If instead of discreet particles, we have a finite body of any shape, we may consider it as an agglomeration of an infinite number of infinitely small particles, and define its centre of mass as above. It will be a definite point in the body.

Centre of Mean Position.

More generally, let there be a set of points A_1, A_2, A_3,... and let m_1, m_2, m_3,... by any set of numbers which are associated with these points respectively. Now join any two points A_1A_2 and divide it at G_1 in the inverse ratio of the number associated at the ends, such that $A_1G_1 : G_1A_2 = m_2 : m_1$. Assume tho number $m_1 + m_2$ to be associated to G_1. Join G_1A_3 and divide it at G_2 such that $G_1G_2 : G_2A_3 = m_3 : m_1 + m_2$, and suppose G_2 to be associated with $m_1 + m_2 + m_3$. Proceeding in this manner, we finally arrive at a unique point G which is called the *centre of mean position of the given points for the system of given multipliers.*

If the points A_1, A_2,... etc. are coplanar, with co-ordinates (x_1, y_1), (x_2, y_2), etc. we get exactly as in the previous article, the co-ordinates of G given by

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \qquad \bar{y} = \frac{\Sigma my}{\Sigma m}.$$

Note. The centre of mean position is a more general term which includes as special cases such things as the centre of a set of parallel forces, and the centre of mass of a system of particles.

When the given multipliers are all unity (or equal), the centre of mean position is referred to as the **centroid** of the given points. This also means that the centre of mass of a body of uniform density is the same as the centroid of the body.

10·3. Centre of gravity.

We know from the law of gravitation that every material particle is attracted towards the centre of the earth with a force which is proportional to the mass of the particle, and this force, we call its weight.

Now, given any material body, we can consider it to be an assemblage of particles each of which is acted on by earth's attraction, and all these forces passing through the centre of the earth, they have got a single resultant which we call the weight of the body. If the body be held in different positions, the positions of the line of action of this weight relative to the body will be different. Now, in some cases, the line of action of the weight passes always through a fixed point in the body, however the body may be placed ; for instance, in case of a spherical body, by symmetry, the line of action of the resultant weight always passes through the centre of the sphere. In case such a fixed point is available in the body, it is called the centre of gravity of the body.

Def. *The* **centre of gravity** *of a body (or a system of particles rigidly connected to one another) is that point fixed in the body (or with respect to the system of particles), when one such exists, through which the resultant weight of the body or the system always passes, in whatever manner the body may be placed.*

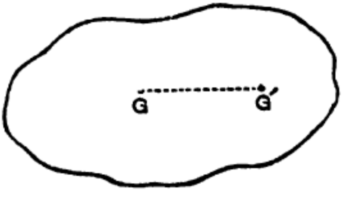
It may be mentioned however, that strictly speaking, in most cases such a point does not exist ; in other words, in a strict sense, the centre of gravity of a body does not exist in all cases.

Now, for bodies of ordinary size which we have to deal with in general, the radius of the earth is so large in comparison, that lines drawn from different points of the body to the centre of the earth may be taken to be practically parallel. Thus, the weights of different elements of which the body is composed of can be taken as like parallel forces, the common direction being the vertical at that point on the earth's surface, and the magnitudes of the forces are proportional to the masses of the elements. If the body be held in a different manner, the magnitudes as well as the points of application of these parallel forces remain unchanged in the body, only the common direction, which is still vertical (*i.e.*, the same in space) changes relative to the body. Hence, there is a fixed point in the body through which the resultant weight always passes, and this we call the centre of gravity of the body. Strictly speaking, as defined in the previous article, this point is the centre of mass of the body. But on the above assumption, when the body is of ordinary finite size, and therefore small compared to the earth, the centre of gravity and the centre of mass coincide.

It is clear from what has already been said that if we proceed to find out the centre of gravity of a big body, a mountain for instance, lines from different points of which to the centre of the earth cannot be treated as parallel, the body may not have a centre of gravity at all, and even if it has, the centre of gravity will not in general be the same as the centre of mass, which latter point will always exist.

In what follows, we shall always assume weights of different elements of a body or a material system to be parallel, and proceed to determine the centre of gravity, which is identical with the centre of mass, and is always available.

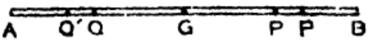
10·4. The centre of gravity of a body is unique.



For, if possible, let a body have two centres of gravity, G and G'. The weight of the body passes through both G and G', by definition, in whatever manner the body may be held. Now, hold the body such that GG' is horizontal. The weight, which is a vertical force, cannot now pass through both G and G' unless they coincide. Hence there cannot be two distinct centres of gravity of the body.

10·5. Determination of centre of gravity in special cases.

I. A thin uniform rod.



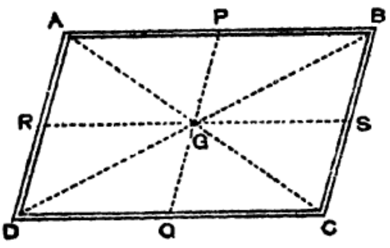
Let AB be a thin uniform rod of any material, G the middle point of AB.

Consider two equal infinitesimal lengths PP' and QQ' of the rod, equidistant from G, so that $GP = GQ$. Since the rod is uniform, the weights of these two elements (which can ultimately, be treated as two particles equidistant from G) are equal, both vertically downwards, and the resultant of these two equal and like parallel forces, act at the middle point G.

Since $AG = GB$, the whole rod can be divided into pairs of such equal infinitesimal elements equidistant from G, and for each pair the resultant of the weights acts at G. Hence the weight of the whole rod acts at G, and so G is the centre of gravity.

Thus the C.G. of a thin uniform rod is at its middle point.

II. Four rods forming a parallelogram.



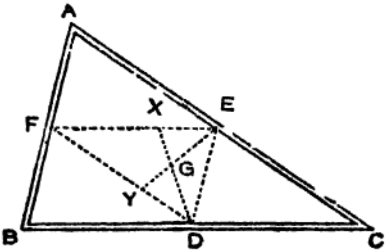
Let four thin uniform rods AB, BC, CD, DA of the same material and thickness form a parallelogram $ABCD$. Let

P and Q be the middle points of AB and CD, and R and S those of AD and BC. PQ and RS intersecting at G, we easily see from Geometry that PQ and RS bisect each other at G, and this is also the point of intersection of the diagonals AC and BD of the parallelogram.

The C.G. of the rods AB and CD are at their middle points P and Q, and at these points the weights of the rods, which are equal, act. The resultant of these two equal weights at P and Q, (which are like parallel forces, both being vertical) passes through the mid-point of PQ, *i.e.*, through G. Similarly, the resultant of the weights of the two equal rods AD and BC also acts through G, the mid-point of RS. Thus the resultant weight of the whole system acts at G.

Thus, the C.G. of the system of four uniform rods forming a parallelogram is at G, the point of intersection of the lines joining the middle points of the opposite pair of sides. This point is also the point of intersection of the diagonals.

III. Three rods forming a triangle.



Let AB, BC, CA be three thin uniform rods of the same material and thickness (or parts of a thin uniform wire) forming a triangle ABC. Let D, E, F be the middle points of BC, CA, AB respectively.

The weights of the uniform rods AB, AC act vertically downwards at their middle points F and E respectively, their magnitudes being proportional to their lengths. The resultant of these two weights which are like parallel forces, acts at a point X on FE, where

$$\frac{FX}{XE}=\frac{\text{wt. at } E}{\text{wt. at } F}=\frac{\text{length } AC}{\text{length } AB}=\frac{2DF}{2DE}=\frac{DF}{DE}.$$

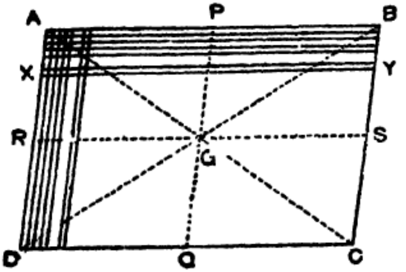
Thus DX bisects the angle EDF.

Now, the resultant of the weight of the rod BC acting at its middle point D, and the joint weight of AB and AC acting at X, will act at some point on DX. Hence, the combined C.G. of the three rods is situated on the line DX.

Similarly, combining the weights of AB, BC first and then considering the weight of AC, the combined C.G. of the three rods is shown to be some point on EY which bisects the angle DEF.

Thus, the combined C.G. of the system of three rods being a common point situated on both DX and EY, is their point of intersection, *i.e., the required C.G. is the in-centre of the triangle DEF formed by joining the middle points of the rods.*

IV. A uniform parallelogram lamina.



Let $ABCD$ be a uniform thin plate or lamina in the form of a parallelogram.

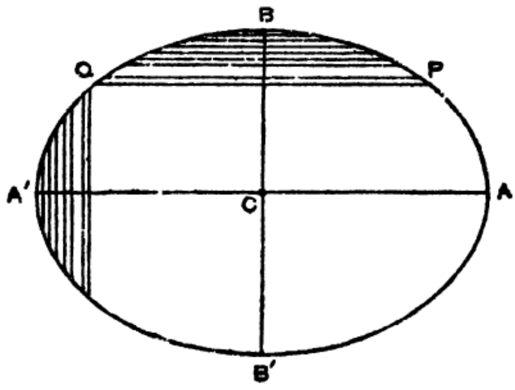
Imagine the lamina to be divided into an infinite number of thin strips by lines parallel to AB or CD, and let

XY represent one such strip. This can be treated as a thin uniform rod whose C.G. is at its middle point O, where its weight acts. Now, if P and Q be the mid-points of AB and CD, from Geometry, PQ bisects every line like XY parallel to AB or CD, and so the middle point of XY is situated on PQ. Similarly, the C.G. of every other strip lies on PQ. Hence, the combined C.G. of the whole lamina lies on PQ.

Again, dividing the lamina into an infinite number of thin strips parallel to AD or BC, we can show exactly in a similar way that the C.G. of the whole lamina also lies on RS joining the mid-points of AD and BC.

Hence, the required C.G. of the lamina is the common point of intersection G of PQ and RS joining the mid-points of the opposite pair of sides, and from Geometry, *this is also the point of intersection of the diagonals of the parallelogram.*

V. A uniform elliptic lamina.



Let ACA' and BCB' be the major and minor axes of a thin uniform elliptic plate.

Divide the lamina into an infinite number of thin strips by lines parallel to the major axis, and let PQ be any such strip. This can be treated as a thin rod whose C.G. is at its middle point, which, since the ellipse is symmetrical about its minor axis, lies on the minor axis. Similarly, the C.G. of every strip parallel to AA' lies on the minor axis. Hence, the C.G. of the whole lamina lies on the minor axis.

Again, by dividing the lamina into strips parallel to the minor axis, the C.G. of the whole lamina can be shown to lie on the major axis as well.

Thus, the required C.G. of the elliptic lamina is the common point of both the major and minor axes, i.e., the centre of the ellipse.

Cor. By making the two axes equal, the ellipse reduces to a circular plate, and we see exactly as before, *that the C.G. of a thin uniform circular plate is at its centre.*

VI. Bodies having an axis of symmetry.

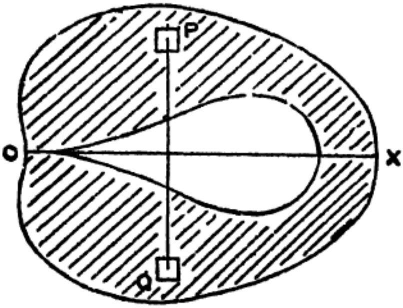


Fig (i)

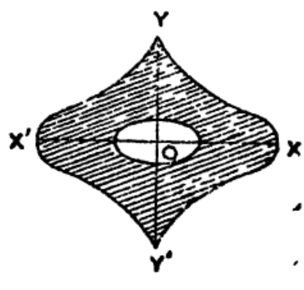


Fig. (ii)

If a material system be symmetrical about an axis OX, as in Fig. (i), then corresponding to any element P on one side of OX, there is an equal and similar element Q situated symmetrically on the other side of OX, so that PQ is bisected at right angles by OX. Now the C.G. of these two equal elements P and Q is at the middle point of PQ

i.e., lies on OX. As the whole body in this case can be divided into pairs of such equal elements symmetrically situated with respect to OX, and for each pair the C.G. is on OX, *the combined C.G. of the whole body lies on* OX.

If a body in the form of a lamina, or a material system in one plane, be symmetrical about two perpendicular axes, say XOX' and YOY' as in Fig. (ii), the C.G. of the system, as shown above, will lie on each of these axes, and so must be the common point O, which is also the centre of symmetry of the system. Similarly for a solid body, if it be symmetrical about three mutually intersecting perpendicular axes, and accordingly has a centre of symmetry, this point is the C.G. of the body.

Thus generally, if a uniform body or a material system has a geometrical centre of symmetry, the C.G. of the body or the system will be at this centre.

Examples of this we get in *uniform circular or elliptic lamina* given above. Among other examples we may cite the cases of (i) *uniform square or rectangular plate*, (ii) *uniform circular or elliptic ring*, (iii) *uniform sphere, solid or hollow*, (iv) *rectangular parallelopiped*, (v) *uniform right circular cylinder solid or hollow*, etc.

In all these cases the centre of gravity is at the geometrical centre of the body, which in case (v) is the middle point of its axis.

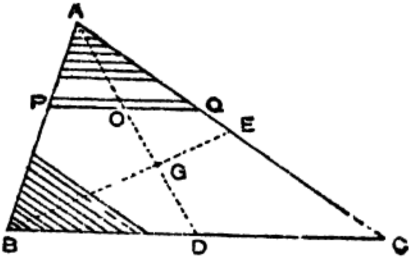
VII. Uniform triangular lamina. 7.

Let ABC be a uniform triangular lamina, D and E the middle points of the sides BC and AC respectively, and let AD and BE intersect at G.

Divide the triangle into an infinite number of thin strips by lines parallel to BC, and let PQ be any such strip. This can be treated as a thin uniform rod whose C.G. is at its middle point. Now if O be the point of intersection of PQ with the median AD, since POQ is parallel to BDC,

$$\frac{PO}{BD} = \frac{AO}{AD} = \frac{OQ}{DC}.$$

But $BD = DC$. Hence $PO = OQ$. Thus the mid-point of the strip PQ, which is its C.G., lies on AD. Similarly the



C.G. of every strip parallel to BC will lie on AD, and so the C.G. of the entire lamina lies on the median AD.

Exactly in the same way, by dividing the lamina into thin strips parallel to CA, we can show that the C.G. of the entire lamina also lies on the median BE.

Thus, the required C.G. of the triangular lamina is the common point of intersection of the medians, i.e., the centroid of the triangle.

From Geometry we easily see that G divides each of the medians in the ratio 2 : 1, *i.e.*, it is the point of trisection of the medians.

Cor. *The centre of gravity of a uniform triangular lamina is identical with that of any three equal particles placed at its vertices.*

For, let w, w, w be the weights of any three equal particles placed at the vertices A, B, C of the triangular lamina. Now the resultant of the equal weights w at B and w at C is $2w$ at D, the middle point of BC. Again, the resultant of $2w$ at D and w at A acts at a point G on AD, where

$$AG : GD = 2w : w = 2 : 1.$$

Thus, the C.G. of the three particles is at G which is the point of trisection of a median, and is thus exactly the same as that of the uniform triangular lamina ABC.

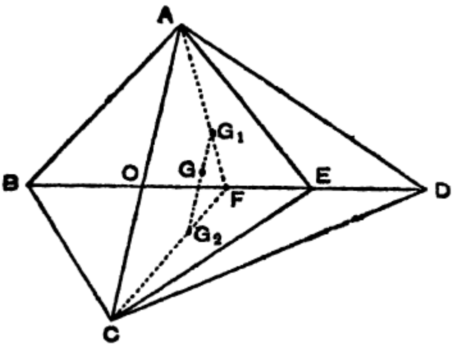
Note. If W be the weight of the uniform triangular lamina, we may replace it by equal weights $\frac{1}{3}W$, $\frac{1}{3}W$, $\frac{1}{3}W$ at the vertices instead of placing any three equal weights there, in which case not only the C.G. would be unaltered, but also the magnitude of the resultant weight.

The weight of a uniform triangular lamina is therefore statically equivalent to that of three equal particles, each of one-third the total weight, placed at the vertices.

10·6. Illustrative Examples.

Ex. 1. *ABCD is a quadrilateral whose diagonals AC, BD intersect at O. If a point E be taken in BD, such that BE = OD, show that the C.G. of the triangle AEC is the same as that of the quadrilateral ABCD.*

[*C. U. 1936*]



Let F be the middle point of BD. Then since $BE = OD$, we have $BO = ED$, and accordingly F is the mid-point of OE as well. Let G_1 be the point of AF such that $AG_1 : G_1F = 2 : 1$. Then G_1 is the C.G. of the triangle ABD, as well as that of the triangle AOE. Similarly,

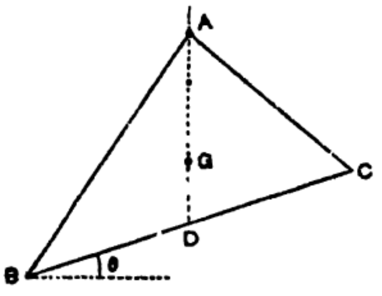
G_2 being the point on CF such that $CG_2 : G_2F = 2:1$, G_2 is the C.G. of the triangle CBD as well as that of COE. Let p_1 and p_2 be the perpendicular distances from A and C on BD. Then the weights of the triangles ABD and CBD, being in the ratio of their areas, are as $\frac{1}{2}p_1.BD : \frac{1}{2}p_2.BD = p_1 : p_2$. Similarly, the ratio of the weights of the triangles AOE ahd COE is also $p_1 : p_2$.

Now, the whole quadrilateral $ABCD$ is composed of the triangles ABD and CBD whose weights act at G_1 and G_2 and are in the ratio of $p_1 : p_2$. Hence, dividing G_1G_2 at G such that $G_1G \cdot GG_2 = p_2 : p_1$, G is the C.G. of the quadrilateral $ABCD$.

Again, the triangle AEC is composed of the triangles AOE and COE whose weights also act at G_1 and G_2 and are in the ratio $p_1 : p_2$. Hence, the C.G. of the whole triangle AEC is exactly the same point G on G_1G_2 where $G_1G : GG_2 = p_2 : p_1$.

Hence the result.

Ex. 2. *A triangular lamina ABC hangs at rest, one of the angles A being supported at a fixed point. Find the angle which the lower side makes with the horizon.*



D being the mid-point of the lower side BC, the C.G. of the triangle lies on AD. Now as the lamina hangs in equilibrium under the weight acting vertically downwards through G, and the reaction at the point of support A, these two forces must be equal and opposite, acting in the same straight line. Thus AGD must be vertical.

If θ be the required inclination of BC to the horizon, the $\angle ADC = (90° - \theta)$.

Now $BD = DC$, $\therefore \dfrac{BD}{AD} = \dfrac{DC}{AD}$, or, $\dfrac{\sin BAD}{\sin ABD} = \dfrac{\sin CAD}{\sin ACD}$,

i.e., $\dfrac{\sin (90° - \theta - B)}{\sin B} = \dfrac{\sin (90° + \theta - C)}{\sin C}$,

or, $\dfrac{\cos (\theta + B)}{\sin B} = \dfrac{\cos (\theta - C)}{\sin C}$.

$\therefore \cos \theta \cot B - \sin \theta = \cos \theta \cot C + \sin \theta$.

$\therefore \tan \theta = \frac{1}{2} (\cot B - \cot C)$, $\therefore \theta = \tan^{-1} \frac{1}{2} (\cot B - \cot C)$.

Examples on Chapter X(a)

1. Show that the C.G. of a uniform triangular lamina is situated at the same point as that of three equal particles placed at the mid-points of its sides.

2. The sides of a uniform triangular lamina are 5, 6 and 7 inches respectively. Find the distances of its C.G. from the shortest and longest sides.

3. The distances of the vertices of a uniform triangular lamina from a straight line in its plane are z_1, z_2, z_3. Find the distance of its C.G. from the line. [*C. U. 1947*]

4. If a particle is placed at each vertex of a triangle, the mass of each particle being proportional to the length of the opposite side, prove that the centre of mass will be the in-centre of the triangle.

5. D, E, F are the mid-points of the sides BC, CA, AB of the triangle ABC. Masses m_1, m_2, m_3 are placed at A, B, C and masses μ_1, μ_2, μ_3 are placed at D, E, F. If the two systems have the same C.G., prove that

$$\frac{m_1}{\mu_2 + \mu_3} = \frac{m_2}{\mu_3 + \mu_1} = \frac{m_3}{\mu_1 + \mu_2}.$$

6. If three men support a heavy triangular board of weight W at its three corners, compare the weight supported by each man.

7. A given weight is placed anywhere on a heavy uniform triangular lamina ; show that the centre of gravity of the system lies within a certain triangle. [*C. U. 1927*]

8. Find the locus of the C.G. of a triangle whose base is fixed and (i) whose vertical angle is given, (ii) whose vertex moves on a given straight line.

9. A uniform wire is bent into the form of a triangle. Show that if its C.G. coincides with that of the area of the triangle, the triangle is equilateral.

10. If the C.G. of a quadrilateral lamina coincides with (i) that of four equal particles placed at its angular points, (ii) the point of intersection of the diagonals, show that the quadrilateral must be a parallelogram.

11. A triangle of uniform rods of different densities has its C.G. at

(i) the circum-centre ;

(ii) the in-centre ;

show that in the first case, the densities are proportional to sec A, sec B, sec C, and in the second case, they are proportional to $\operatorname{cosec}^2 \frac{1}{2}A$, $\operatorname{cosec}^2 \frac{1}{2}B$, $\operatorname{cosec}^2 \frac{1}{2}C$.

12. Three rods of unequal lengths are joined together to form a triangle ABC. If the masses are equal, prove that the C.G. coincides with that of the area. If the masses of the sides a, b, c are proportional to $b+c-a$, $c+a-b$, $a+b-c$, prove that the C.G. is the in-centre.

13. A uniform wire 24 inches long is bent into the shape of a triangle ABC, the sides BC, CA, AB being as $3:4:5$. Particles of weights p, q, r are placed at A, B, C and it is found that the C.G. is unchanged. Prove that

$$p:q:r=9:8:7.$$

14. A thin uniform wire is bent into the form of a triangle ABC and heavy particles of weights P, Q, R are placed at the angular points. Prove that if the centre of mass of the particles coincides with that of the wire, then

$$\frac{P}{b+c}=\frac{Q}{c+a}=\frac{R}{a+b}.$$

15. A thin uniform wire is bent into a triangle ABC. Prove that its C.G. is the same as that of three weights $\frac{b+c}{2}, \frac{c+a}{2}, \frac{a+b}{2}$ placed at A, B, C respectively, where a, b, c are the lengths of the sides BC, CA, AB. [*C. U. 1946*]

16. A uniform wire is bent into the form of a triangle of sides of lengths a, b, c. Prove that the distances of the C.G. of the whole triangle from the sides are as

$$\frac{b+c}{a} : \frac{c+a}{b} : \frac{a+b}{c}.$$ [*P. U. 1940*]

17. AB and AC are two uniform rods of lengths $2a$ and $2b$ respectively. If $\angle BAC = \theta$, prove that the distance from A of the C.G. of the two rods is

$$\frac{(a^4+2a^2b^2 \cos \theta+b^4)^{\frac{1}{2}}}{a+b}.$$ [*C. U. 1939*]

18. ABC is a triangular lamina ; points D, E, F are taken in BC, CA, AB, such that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}.$$

Prove that the C.G. of the triangle DEF is the same as that of the triangle ABC.

19. Masses proportional to $b+c, c+a, a+b$ are placed at the points A, B, C of a triangular lamina, where a, b, c are the sides of the triangle. Show that their C.G. is at the in-centre of the triangle joining the mid-points of the sides of the triangle ABC.

20. Three heavy particles are placed at the angles A, B, C of a triangle, their weights being as $a : b : c$. Show that the distance of the C.G. of the particles from A is

$$\frac{2bc \cos \frac{1}{2}A}{a+b+c}.$$

21. The in-circle of a triangle ABC touches the sides BC, CA, AB in D, E, F respectively. Prove that the C.G. of weights proportional to BC, CA, AB placed at A, B, C respectively coincides with the C.G. of the same weights placed at D, E, F respectively.

22. O is any point within the triangle ABC; another triangle is formed by joining the centres of gravity G_1, G_2, G_3 of the Δ^s BOC, COA, AOB. Show that $\Delta G_1G_2G_3$ is similar to ΔABC, and is one-ninth of it.

23. Prove that the C.G. of four equal particles in any position is the same as that of four other equal particles, each of which is placed at the C.G. of the three of the former.

24. A uniform triangular plate hangs from one angle with the base horizontal; show that the plate is isosceles.

25. The sides of a heavy triangle are 3, 4, 5 respectively; if it be suspended from the in-centre, show that it will rest with the shortest side horizontal.

26. Two uniform heavy rods AB, BC rigidly united at B, are hung up by the end A; show that BC will be horizontal if

$$\sin C = \sqrt{2} \sin \tfrac{1}{2}B.$$

27. A triangle ABC of uniform wire has the side CA removed and is hung up by the point A. Show that for BC to horizontal

$$b^2(c+2a) = c(c+a)^2,$$

where $BC = a$, $CA = b$, $AB = c$. [*C. U. 1941*]

28. A uniform wire is bent in the form of a triangle ABC and is suspended from A. Prove that a plumb-line hung from A will cut BC in the point D, such that

$$BD : DC = a + b : a + c.$$

29. A uniform triangular lamina in the form of a right-angled triangle is suspended by means of a string attached to the right angle. Show that the inclinations of the sides, other than the hypotenuse, to the vertical, are equal to their inclinations to the hypotenuse.

30. A triangular lamina is suspended successively from the angles A and B and the two positions of any side are found to be at right angles to each other. Prove that

$$a^2 + b^2 = 5c^2.$$

31. A triangular lamina having a right angle at C is suspended from the angle A, and the side AC makes an angle α with the vertical. It is then suspended from B, and the side BC makes an angle β with the vertical.

Show that $\cot\alpha \cot\beta = 4$.

32. A triangular lamina ABC of weight W, obtuse-angled at C, stands in a vertical plane with its side BC on a horizontal table. Show that the least weight suspended from A, which will overturn the triangle, is

$$\tfrac{1}{3}W\frac{3a^2 + b^2 - c^2}{c^2 - a^2 - b^2}.$$

Interpret the case when $c^2 > 3a^2 + b^2$.

33. If G be the centre of gravity of two particles of masses m_1 and m_2 at P_1 and P_2, and O be any given point, prove that

$$m_1 \,.\, OP_1^2 + m_2 \,.\, OP_2^2 = m_1 \,.\, GP_1^2 + m_2 \,.\, GP_2^2 + (m_1 + m_2) \,.\, OG^2.$$

Generalise this result for n particles. [*C. U.* 1966, '67]

***34.** A particle P is acted upon by forces towards the points $A_1, A_2, \dots A_n$, which are represented by $\lambda_1 PA_1$, $\lambda_2 PA_2, \dots\dots\dots \lambda_n PA_n$. Show that their resultant is represented by

$$(\lambda_1 + \lambda_2 + \cdots + \lambda_n)\, PG,$$

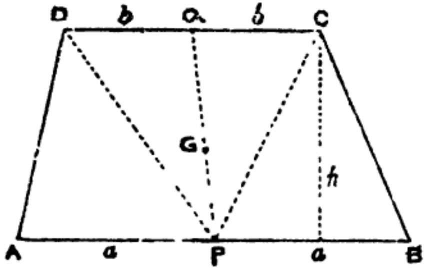
where G is the C.G. of the weights placed at $A_1, A_2, \dots A_n$ proportional to $\lambda_1, \lambda_2, \dots \lambda_n$ respectively.

ANSWERS

2. $\frac{4}{5}\sqrt{6}$ and $\frac{4}{7}\sqrt{6}$ inches. 3. $\frac{1}{3}(s_1+s_2+s_3)$. 6. Each supports $\frac{1}{3}W$. 8. (i) A circle. (ii) A straight line parallel to the given straight line.

10·7. Determination of centre of gravity in special cases *(continued)*.

VIII. A uniform trapezium lamina.



Let $ABCD$ be a uniform lamina in the form of a trapezium, whose parallel sides AB and CD are of lengths $2a$ and $2b$ respectively.

Let h be the height of the trapezium, w the weight per unit area of its surface, and let P and Q be the mid-points of AB and CD respectively.

The trapezium is comdosed of three triangles, DAP, CPB and PCD whose weights are clearly $\frac{1}{2}ahw$, $\frac{1}{2}ahw$ and $\frac{1}{2}.2bbhw$ rospectively.

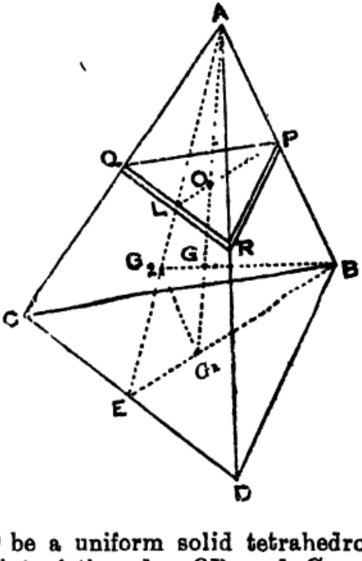
So far as the weight of $\triangle DAP$ is concerned, we can replace it by *t*hree weights $\frac{1}{6}ahw$ each at D, A, P. Similarly, the other *t*wo triangles can be replaced by $\frac{1}{6}ahw$ at each of C, P, B and $\frac{1}{3}bhw$ at P, C, D.

We thus get $\frac{1}{6}ahw+\frac{1}{3}bhw$ at each of D and and C, $\frac{1}{6}ahw$ at each of A and B, and $(\frac{1}{6}ahw+\frac{1}{6}ahw+\frac{1}{3}bhw)$ at P.

The two equal weights at D and C give rise to a resultant $\frac{1}{3}hw(a+2b)$ at Q, the mid-point of CD. Similarly, the two equal weights at A and B give a resultant $\frac{1}{3}ahw$ at P. We thus finally get a weight $\frac{1}{3}hw\,(a+2b)$ at Q and a weight $\frac{1}{3}hw\,(2a+b)$ at P as equivalent to the weight of the given lamina.

The required C.G. therefore is at a point G on PQ such that $PG : GQ = (a+2b) : (2a+b)$.

IX. A uniform solid tetrahedron.



Let $ABCD$ be a uniform solid tetrahedron. Let E be the middle point of the edge CD, and G_1 and G_2, points on BE and AE, such that $BG_1 : G_1E = 2 : 1 = AG_2 : G_2E$. Then G_1 and G_2 are the centroids of the triangular faces BCD and ACD respectively.

Divide the tetrahedron into infinitely thin triangular slices by planes parallel to the face BCD, and let PQR be one such slice which can be treated as a uniform triangular lamina. Now, AE intersecting QR at L, since QLR is parallel to CD,

$$\frac{QL}{CE}=\frac{AL}{AE}=\frac{LR}{ED},$$

and as E is the mid-point of CD, L must be the mid-point of QR.

Again in the plane AEB, which evidently contains PL and AG_1, if O be the point of intersection of PL and AG_1, since PL is parallel to BE (as planes PQR and BCD are parallel),

$$\frac{PO}{BG_1}=\frac{AO}{AG_1}=\frac{OL}{G_1E};$$

so that $$\frac{PL}{OL}=\frac{BG_1}{G_1E}=\frac{2}{1}.$$

Thus, O is the centroid of the triangle PQR.

Hence, the C.G. of the triangular slice PQR lies on AG_1. Similarly, the C.G. of every slice parallel to BCD will lie on AG_1. Thus, the C.G. of the whole tetrahedron lies on AG_1.

Exactly in a similar way, by dividing the tetrahedron into thin triangular slices by planes parallel to the face ACD, it can be shown that the C.G. of the whole tetrahedron also lies on the line BG_2.

Let AG_1 and BG_2, which both lie in the plane AEB, intersect at G. Then G is the required C.G. of the tetrahedron.

Now, $AG_2 : G_2E=2:1=BG_1 : G_1E$,

$\therefore$ G_1G_2 is parallel to AB.

Thus, $AG : GG_1=BG : GG_2=AB : G_1G_2$
$=BE : EG_1=3:1.$

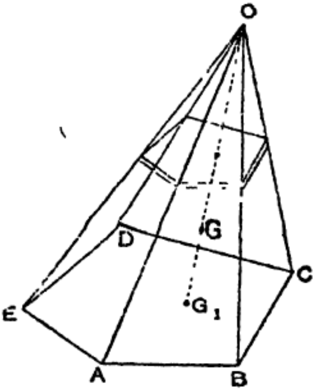
Therefore the C.G. of the tetrahedron lies on the line joining any angular point to the centroid of the opposite triangular face, dividing it in the ratio 3 : 1.

We may note in this case that $GG_1 = \frac{1}{4}AG_1$. Therefore the distance of G from any face is $\frac{1}{4}$ of the distance of the opposite vertex from the face.

Cor. *The C.G. of a tetrahedron is identical with that of four equal particles placed at its vertices.*

The proof is left as an exercise to the student.

X. A uniform solid pyramid on any base.



Let O be the vertex, and $ABCDE$ the polygonal base of a uniform solid pyramid, and let G_1 be the centroid of the base.

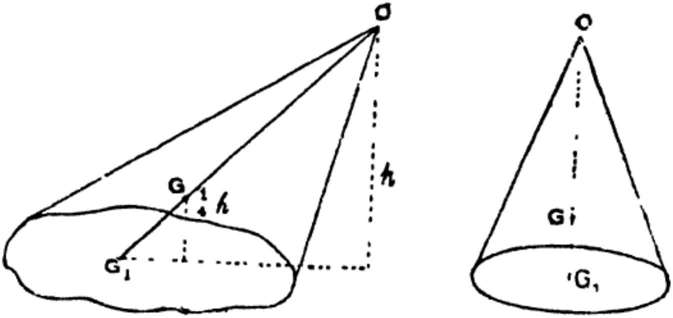
By dividing the pyramid into thin similar and similarly situated polygonal slices by planes parállel to the base, as in the previous article, it can be shown that the C.G. of the whole pyramid lies on OG_1.

Again, joining G_1 to each of the angular points of the base, the whole pyramid is divided into a number of tetrahedrons, for each of which the distance of C.G. from the base is $\frac{1}{4}$ of the distance of the vertex O from the base. Hence, the combined C G. of the pyramid is also at the same distance from the base.

Thus, the C.G. of the pyramid is the point G on OG_1, the line joining the vertex to the centroid of the base, at *a distance from the base equal to $\frac{1}{4}$ of the distance of the vertex from the base.*

It follows that $G_1G = \frac{1}{4}G_1O$, or, $OG : GG_1 = 3 : 1$.

XI. A uniform solid cone of any base, and a uniform solid right circular cone.



The above result for the C.G. of a pyramid is true, whatever be the number of sides of its polygonal base.

By making the number of sides infinitely large, the base can ultimately be made to coincide with any closed curve, and the pyramid reduces to a cone with any base.

A particular and important case is that of a right circular cone. Thus we may state the results :

(i) *The C.G. of a uniform solid right circular cone is on the axis at a height $\frac{1}{4}$ of the height of the vertex from the circular base.*

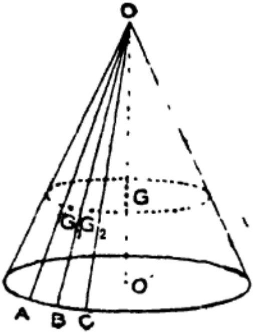
(ii) The C.G. of a uniform solid cone with any closed base is on the line joining the vertex to the centroid of the base, at a height $\frac{1}{4}$ of the height of the vertex from the base.

Thus, $OG : GG_1 = 3 : 1$ in either case.

XII. A hollow right circular cone without base (formed of a thin uniform sheet).

By dividing the conical surface into an infinite number of circular rings by planes parallel to the circular base, since the C.G. of each such ring is at its centre which lies on the axis of the cone, the C.G. of the whole hollow cone lies on the axis OO'.

Again, dividing the conical surface into an infinite number of triangular elements like OAB, OBC etc. with common vertex at O, and infinitesimal arcs of the circular base (which may be treated as practically straight) as bases, we note that for each such triangle, the C.G., being at the point of trisection of the median, is at a height $\frac{1}{3}$ of the height of O above the circular base of the cone, and this is the same for every such triangle. Hence, for the whole hollow cone, the height of the C.G. is $\frac{1}{3}$ of the height of O above the base.

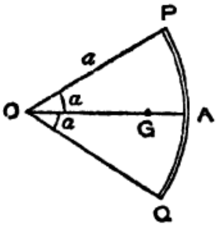


Thus, the C.G. of the hollow right circular cone without base, formed of a thin uniform sheet, is on the axis OO' at a height $\frac{1}{3}$ of the height of the vertex above the circular base.

Thus, $OG : GO' = 2 : 1$.

XIII. Some other special cases.*

We give below, without proof, the positions of the C.G. in some other special cases for ready reference.



(i). A thin uniform circular arc.

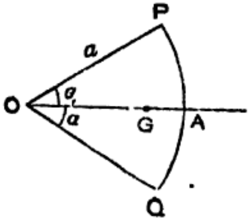
If a be the radius of the arc, 2α the angle subtended by it at the centre,

the C.G. is on the radius bisecting the arc, at a distance $a\dfrac{\sin\alpha}{\alpha}$ *from the centre.*

For a **semi-circular arc** this becomes $\dfrac{2a}{\pi}$.

*Proofs of these special cases will be given in the subsequent chapter.

(ii) A uniform lamina in the form of a sector of a circle.

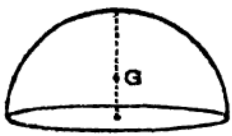


If a be the radius and 2α be the angle at the centre of the sector,

the C.G. is on the radius bisecting the sector, at a distance $\frac{2}{3}a\frac{\sin\alpha}{\alpha}$ *from the centre.*

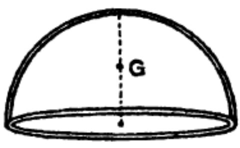
For a **semi-circular lamina** this becomes $\frac{4a}{3\pi}$.

(iii) A uniform solid hemisphere.



If a be the radius of hemisphere,

the C.G. is on the axis of the hemisphere (*i.e.*, radius perpendicular to the plane base) *at a distance* $\frac{3}{8}a$ *from the centre.*

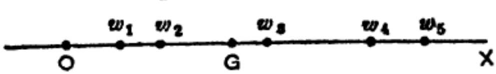


(iv) A hemispherical surface (formed of a thin uniform sheet).

The C.G. is on the axis at a distance $\frac{1}{2}a$ *from the centre,* where a is the radius.

10·8. Analytical determination of C.G. for a system of material particles.

Case I. *When the particles are situated on a straight line.*



Let w_1, w_2, w_3,...... be the weights of a system of particles situated on a straight line, and let their distances (with proper sign) be x_1, x_2, x_3,...... measured from any fixed point on the line chosen as origin.

Let $\bar{x}$ be the distance of their C.G., namely G, from O. Now, since the position of the C.G. is the same, however the straight line containing the particles be held, let us assume the line to be held in a horizontal position. The résultant of the weights of the particles which are all vertically downwards is $w_1+w_2+w_3+\cdots$, and this acts at G. Hence, equating the moment of the resultant about O to the algebraic sum of the moments of the components,

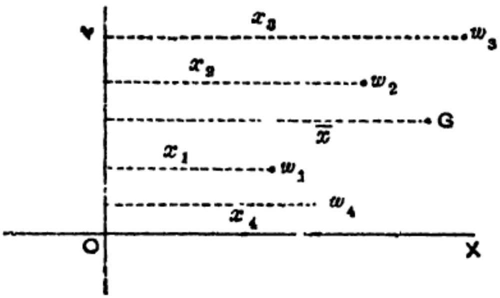
$$(w_1+w_2+w_3+\cdots)\bar{x}=w_1x_1+w_2x_2+w_3x_3+$$

$$\text{or,}\quad \bar{x}=\frac{w_1x_1+w_2x_2+w_3x_3+\cdots}{w_1+w_2+w_3+\cdots}=\frac{\Sigma wx}{\Sigma w}.$$

If m_1, m_2, m_3,... be the masses of the particles, as the weights are proportional to the masses, we may also write

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}.$$

Case II. *When the particles are situated on a plane.*



Let w_1, w_2, w_3,...... be the weights of a system of particles on a plane, whose co-ordinates referred to a set of fixed rectangular axes on the plane are (x_1, y_1), (x_2, y_2), (x_3, y_3),...

Let $(\bar{x}, \bar{y})$ be the co-ordinates of their C.G., namely G. As the position of the C.G. on the plane is definite, however the plane may be held, let us assume the plane to be placed horizontally. The resultant of the weights of the particles, which are like parallel forces, being all vertically downwards, is $w_1+w_2+w_3+\cdots\cdots$, and this acts at G. Now, equating the moment of the resultant about the y-axis to the algebraic sum of the moments of the components,

$$(w_1+w_2+w_3+\cdots)\bar{x}=w_1x_1+w_2x_2+w_3x_3+\cdots$$

or, $$\bar{x}=\frac{w_1x_1+w_2x_2+w_3x_3+\cdots}{w_1+w_2+w_3+\cdots}=\frac{\Sigma wx}{\Sigma w}.$$

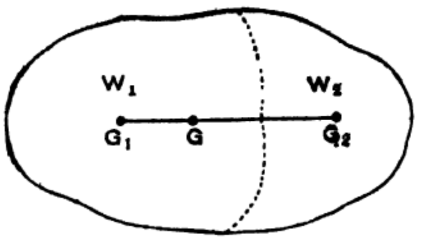
Similarly, considering moments about x-axis,

$$\bar{y}=\frac{w_1y_1+w_2y_2+w_3y_3+\cdots}{w_1+w_2+w_3+\cdots}=\frac{\Sigma wy}{\Sigma w}.$$

If $m_1, m_2, m_3, \ldots$ be the masses of the particles, the weights being proportional to the masses, we may also write

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}, \quad \bar{y}=\frac{\Sigma my}{\Sigma m}.$$

10·9. Given the weights and C.G. of two parts of a body, to find the C.G. of the whole body.



Let W_1 and W_2 be the weights, and G_1 and G_2 the corresponding centres of gravity of the two parts constituting a body.

Join G_1G_2, and divide it internally at G in the inverse ratio of the weights acting at its extremities, so that

$$G_1G : GG_2 = W_2 : W_1.$$

The resultant of the weights W_1 and W_2, which are both vertically downwards, and are therefore like parallel forces, is W_1+W_2 acting at G.

Thus, G is the point where the resultant weight of the whole body acts, however the body, and accordingly the line G_1G_2 is placed.

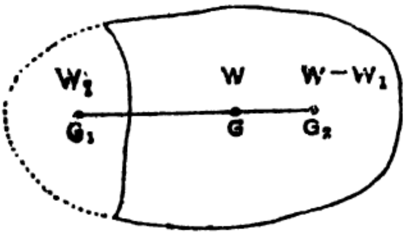
Thus, G is the required C.G.

It may be mentioned that if the distances of G_1 and G_2 from some chosen point O on the line G_1G_2 be x_1 and x_2, then the distance $\bar{x}$ of G from O is given by

$$\bar{x} = \frac{W_1x_1 + W_2x_2}{W_1 + W_2}.$$

Note. When the two parts are portions of a thin uniform sheet, the weights may be taken proportional to their surface areas. If they are parts of a uniform solid, the weights may be taken proportional to their volumes. If they are parts of a uniform thin wire, the weights may be taken proportional to the lengths.

10·10. Given the weight and C.G. of a whole body, and also those of a part of it, to find the C.G. of the remaining part.



Let W be the whole weight and G the centre of gravity of a body, and W_1 the weight and G_1 the C.G. of a part of it. Then the weight of the remaining part is $W - W_1$. Let its C.G. be at G_2.

Then W acting at G being the resultant of W_1 acting at G_1 and $W - W_1$ acting at G_2, which are like parallel forces, G_1, G, G_2 must be on the same straight line, G_2 being on the opposite side of G, with respect to G_1, and

$$G_1G : GG_2 = (W - W_1) : W_1,$$

$$\therefore \quad GG_2 = \frac{W_1}{W - W_1} \cdot G_1G.$$

This gives the position of G_2, the required C.G. of the remaining part.

Analytically, if x_1 and x be the distances of G_1 and G measured from any suitably chosen point O on the line joining them, the required distance x_2 of G_2 from O is obtained as

$$x = \frac{W_1x_1 + (W-W_1)x_2}{W_1 + (W-W_1)} = \frac{W_1x_1 + (W-W_1)x_2}{W}$$

whence, $x_2 = \dfrac{Wx - W_1x_1}{W - W_1}$.

This result for determining x_2, giving the position of G_2, may be interpreted as follows :

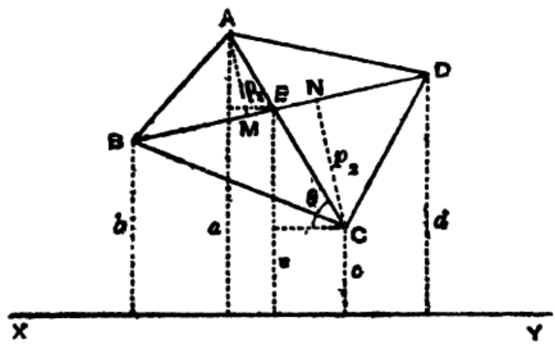
Assume W acting at G and a negative weight W_1 acting at G_1, and use the analytical formula to find out the resultant C.G.

Note. The note given below the previous article applies here also.

10·11. Illustrative Examples.

Ex. 1. *The distances of the angular points and the point of intersection of the diagonals of a plane uniform quadrilateral lamina from any line in its plane are a, b, c, d and e ; show that the distance of its C.G. from the same line is*

$$\tfrac{1}{3}(a+b+c+d-e).$$



The quadrilateral is made up of the two triangles ABD and CBD

whose weights W_1 and W_2 are proportional to their areas, and thus if AM and CN be perpendiculars from A and C on BD,

$$\frac{W_1}{W_2}=\frac{\frac{1}{2}AM \cdot BD}{\frac{1}{2}CN \cdot BD}=\frac{AM}{CN}=\frac{AE}{EC}=\frac{(a-e)\operatorname{cosec}\theta}{(e-c)\operatorname{cosec}\theta}=\frac{a-e}{e-c}.$$

[θ being the angle made by AEC with the given line XY]

$$\therefore \quad \frac{W_1}{a-e}=\frac{W_2}{e-c}. \qquad \ldots \quad \text{(i)}$$

Now, the weight of the triangle ABD can be replaced by weights $\frac{1}{3}W_1$, $\frac{1}{3}W_1$, $\frac{1}{3}W_1$ at A, B and D, and similarly that of the triangle CBD by weights $\frac{1}{3}W_2$, $\frac{1}{3}W_2$, $\frac{1}{3}W_2$ at C, B, D.

Thus, the weight of the quadrilateral is equivalent to those of the particles of weights $\frac{1}{3}W_1$, $\frac{1}{3}(W_1+W_2)$, $\frac{1}{3}W_2$, $\frac{1}{3}(W_1+W_2)$ at A, B, C and D respectively. Thus, z the distance of the required C.G. from the given line XY is

$$z=\frac{a.\frac{1}{3}W_1+(b+d).\frac{1}{3}(W_1+W_2)+c.\frac{1}{3}W_2}{W_1+W_2}$$

$$=\frac{1}{3}\cdot\frac{a(a-e)+(b+d)(a-e+e-c)+c(e-c)}{(a-e)+(e-c)} \quad [\textit{from } (i)]$$

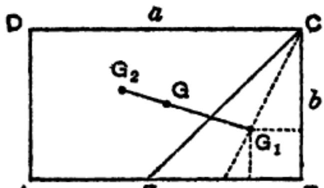
$$=\frac{1}{3}\cdot\frac{(a^2-c^2)-e(a-c)+(b+d)(a-c)}{a-c}$$

$$=\tfrac{1}{3}(a+c-e+b+d)=\tfrac{1}{3}(a+b+c+d-e).$$

Ex. 2. *$ABCD$ is a uniform rectangular lamina in which $AB=a$, $BC=b$, and $a>b$. A triangular portion CBE is removed, where E is a point in AB such that $BE=b$. Show that the distance of the C.G. of the remainder from AB is*

$$\frac{b(3a-b)}{3(2a-b)}$$

and find also its distance from AD.



The whole weight W of the rectangle is $ab\sigma$, where σ is the surface density of the lamina, and the C.G. is at G whose distances from AB and AD are $\frac{1}{2}b$ and $\frac{1}{2}a$ respectively.

Again, the weight W_1 of the removed portion CBE is $\frac{1}{2}b^2\sigma$, and its C.G. is at G_1 whose distances from AB and AD are easily seen to be $\frac{1}{3}b$ and $(a-\frac{1}{3}b)$ respectively.

Now, assuming a whole weight W at G, and a negative weight W_1 at G_1, the distance of the required C.G., namely G_2, from AB is given by

$$\frac{ab\sigma.\frac{1}{2}b-\frac{1}{2}b^2\sigma.\frac{1}{3}b}{ab\sigma-\frac{1}{2}b^2\sigma}=\frac{b\left(\frac{1}{2}a-\frac{1}{6}b\right)}{(a-\frac{1}{2}b)}=\frac{b\,(3a-b)}{3\,(2a-b)}.$$

Also the distance of G_2 from AD is given by

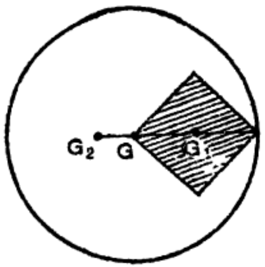
$$\frac{ab\sigma.\frac{1}{2}a-\frac{1}{2}b^2\sigma.(a-\frac{1}{3}b)}{ab\sigma-\frac{1}{2}b^2\sigma}=\frac{\frac{1}{2}a^2-\frac{1}{2}b\,(a-\frac{1}{3}b)}{(a-\frac{1}{2}b)}=\frac{3a^2-3ab+b^2}{3\,(2a-b)}.$$

Ex. 3. *A square hole is punched out of a circular lamina, the diagonal of the square being a radius of the circle. Show that the centre of gravity of the remainder is at a distance*

$$\frac{a}{8\pi-4}$$

from the centre of the circle, where a is its diameter.

[*Allahabad, 1945*]



σ being the surface density of the lamina, the weight of the whole circle is $\frac{1}{4}\pi a^2\sigma$, (a being its diameter), and the C.G. is at the centre G.

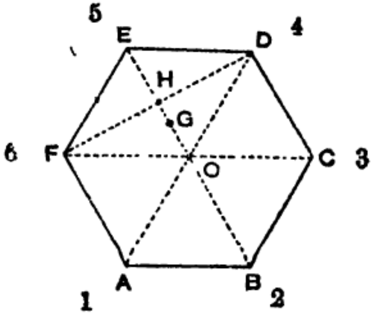
The square portion punched out having a diagonal $\frac{a}{2}$, its side is $\frac{a}{2\sqrt{2}}$, and its weight is therefore $\frac{1}{8}a^2\sigma$, the C.G. being at G_1 where $GG_1=\frac{1}{2}\cdot\frac{a}{2}=\frac{1}{4}a$.

The weight of the remainder is therefore $\frac{1}{4}\pi a^2\sigma - \frac{1}{8}a^2\sigma$, and if its C.G. be at G_2, we get, since the resultant C.G. of the two weights at G_1 and G_2 is at G,

$$\tfrac{1}{8}a^2\sigma(2\pi-1).GG_2 = \tfrac{1}{8}a^2\sigma.GG_1$$

or, $$GG_2 = \frac{GG_1}{2\pi-1} = \frac{a}{4(2\pi-1)} = \frac{a}{8\pi-4}.$$

Ex. 4. *Weights of* 1, 2, 3, 4, 5 *and* 6 *lbs. are placed at the angular points of a regular hexagon of side* 12 *inches, taken in order. Find the distance of their C.G. from the centre of the hexagon.*



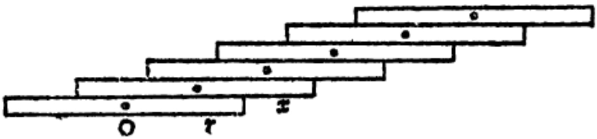
A, B, C, D, E, F being the angular points of a regular hexagon where the weights 1, 2, 3, 4, 5, 6 lbs. are placed respectively; we know from Geometry that the diagonals AD, BE, CF bisect each other at O, the centre of the hexagon, and that OA, OB, OC etc., are all equal, each equal to the side of the hexagon = 12 inches.

Now, wts. 1 lb. at A and 1 lb. at D are equivalent to 2 lbs. wt. at O. Similarly, 2 lbs. wt. at B and 2 lbs. wt. at E are equivalent of 4 lbs. wt. at O, and so also 3 lbs. wt. at C and 3 lbs. wt. at F are equivalent to 6 lbs. wt. at O.

Thus, the given system is equivalent to 12 lbs. wt. at O, and 3 lbs. wt. each at D, E and F. Again, $ODEF$ is a rhombus, and so DF and OE bisect each other at H, and so 3 lbs. wt. at F and 3 lbs. wt. at D are equivalent to 6 lbs. wt. at H.

Thus, ultimately, the given system is equivalent to 12 lbs. wt. at O, 6 lbs. wt. at H and 3 lbs. wt. at E, and so the combined C.G. of the system is on the line OE at a distance $\bar{x}$ from O, where remembering that $OE=12$ inches and $OH=6$ inches,

$$\bar{x}=\frac{12\times0+6\times6+3\times12}{12+6+3}=\frac{72}{21}=3\frac{3}{7}\text{ inches.}$$

Ex. 5. *A pile of six rupees rests on a horizontal table and each rupee projects the same distance beyond one below it. Find the greatest possible horizontal distance between the centres of highest and lowest rupees.* [*P. U. 1937*]

Let r be the radius of a rupee, W its weight, and let x be the distance which each rupee projects beyond the one below it. Then referred to the centre of the lowest rupee as origin, the horizontal distances of the centres of the successive rupees above it are respectively x, $2x$, $3x$, $4x$ and $5x$. Thus, the horizontal distance from O of the combined C.G. of the five rupees above the lowest is

$$\bar{x}=\frac{Wx+W.2x+W.3x+W.4x+W.5x}{5W}=3x$$

and in order that this system may be balanced by the reaction of the lowest rupee, the combined C.G. of the upper five rupees must be vertically above the surface of the lowest, not going beyond it. For this, the conditiom is

$$3x \not> r, \text{ or, } x \not> \tfrac{1}{3}r.$$

Hence, the greatest possible horizontal distance between the centres of the highest and lowest rupees is the greatest value of $5x$ namely $\frac{5}{3}r$.

It may be noted that this condition also ensures that the combined C.G. of any lesser number of rupees from the top will remain vertically over the surface of the next lower rupee, and there is no chance of overturning at any place.

Examples on Chapter X(b)

1. Particles whose masses are 1 lb., 2 lbs., 3 lbs., 4 lbs., and 5 lbs. are placed at the points $(4, 0)$, $(3, 4)$, $(-5, 4)$, $(0, -5)$ and $(-5, 0)$ respectively. Find the co-ordinates of their C.G.

2. Find the C.G. of a uniform square plate $ABCD$ of weight 10 lbs. together with weights of 20, 30, 40, 50 lbs. placed at its four corners A, B, C, D respectively.

[*C. U. 1945*]

3. If masses proportional to 1, 5, 3, 4, 2 and 6 are placed at the vertices of a regular hexagon, taken in order, show that the centre of mass is at the centre of the hexagon.

4. Find the C.G. of particles of weights 2, 5, 7, 1, 6 and 11 lbs. placed successively at the angular points A, B, C, D, E, F of a regular hexagon.

5. Find the C.M. of seven equal particles placed at the angular points of a regular octagon.

6. At each of $n-1$ of the angular points of a regular polygon of n sides, equal particles are placed. Show that the distance of the C.G. from the circum-centre of the polygon is $\frac{r}{n-1}$, where r is the circum-radius.

7. Find the C.M. of three equal rods each of length $2a$ forming the consecutive sides of a square.

8. Find the C.M. of the perimeter of a quadrilateral two of whose sides of lengths 6 inches and 14 inches are parallel to one another, while the other sides are each 8 inches long.

9. Having given the positions of the particles A, B, C and the positions of the C.G.'s of B, C and C, A, find the C.G. of A, B.

10. Show that the C.G. of a quadrilateral is the same as that of four particles of equal weights placed at the four corners, together with a fifth particle of equal but negative weight placed at the intersection of the diagonals.

11. $ABCD$ is a quadrilateral lamina whose diagonals intersect at L; M and N are points on the diagonals AC and BD respectively such that $AM=CL$ and $BN=DL$. Show that the C.G. of the quadrilateral $ABCD$ coincides with that of the triangle LMN.

12. Prove that the C.G. of a uniform triangular lamina of mass M, bordered with a thin uniform rim of mass m, and loaded with a particle of mass $\frac{1}{2}m$ at the in-centre, is at the centroid of the triangle.

13. A uniform rod, 18 inches long, is bent so that the two parts 8 and 10 inches long respectively are at right angles to one another. Find the distance between the C.M. of the new shape and the original.

14. A square $ABCD$ is divided into four equal triangles by its diagonals which intersect at O; if the triangle OAB be removed, find G, the centroid of the remaining portion on the square.

15. The sides of a parallelogram $ABCD$ are bisected at D, E, F, H and the points of bisection of the opposite sides are joined. If these lines meet at O, and if the small parallelogram $ADOH$ be removed, find the C.G. of the remainder.

16. D, E, F are the mid-points of the sides BC, CA, AB of a triangle ABC. If the triangle DEF is removed, show that the C.G. of the remainder will coincide with that of the whole triangle.

17. G is the C.G. of the triangle ABC. If the triangle GBC be removed, find the distance of the C.G. of the remainder from A.

18. From a uniform triangular lamina ABC, a portion PBC is removed. Find the position of P so that it may be the centre of gravity of the remainder.

19. In the triangle ABC, G is the point of intersection of the medians AD, BE, CF. If the portion $AFGE$ is removed, show that the C.G. of the remainder is on DG at a distance $\frac{7}{12}DG$ from D.

20. The middle points of two adjacent sides of a uniform triangular lamina are joined and the lamina is cut in two along the joining line. Find the C.G. of the larger portion. [*C. U. 1942*]

21. $ABCD$ is a trapezium in which AB and CD are parallel and of lengths a and b respectively. Prove that the distance of the C.G. of $ABCD$ from the side AB is

$$\frac{h}{3} \cdot \frac{a+2b}{a+b},$$

h being the height of the trapezium. [*C. U. 1944*]

22. From a triangle ABC, a portion ADE, where DE is parallel to BC, is removed. If a and b be the distances of A from BC and DE respectively, show that the distance of the C.G. of the remainder from BC is

$$\frac{a^2+ab-2b^2}{3(a+b)}.$$ [*C. U. 1938*]

23. If equal triangles be cut from the corners of a given triangle by lines drawn parallel to the corresponding opposite sides, the C.G. of the remainder will coincide with that of the triangle.

24. If from a triangle ABC, three equal triangles ARQ, BPR, CQP be cut off, the C.G.'s of the triangles ABC and PQR will be coincident.

25. G is the C.G. of a uniform quadrilateral plate, G' is the C.G. of four equal particles placed at its corners, and O is the intersection of its diagonals. Prove that O, G, G' are collinear, and $OG' = 3GG'$.

26. A lamina in the form of a regular hexagon $ABCDEF$ has its centre at O. If the triangular portion OAB be removed, find the C.M. of the remainder.

27. A square $ABCD$ is divided into two parts by joining A to E, the mid-point of BC. Prove that the line joining the C.G. of the triangle ABE to that of the quadrilateral $ADCE$ is perpendicular to AE.

28. From a thin uniform triangular board ABC, the portion constituting the inscribed circle is removed. Prove that the distance of the C.G. of the remainder from the side a is

$$\frac{\Delta}{3as} \cdot \frac{2s^3 - 3\pi a \Delta}{s^2 - \pi \Delta},$$

Δ being the area, and s the semi-perimeter of the board.

29. From a uniform circular disc of radius r, is cut out a circle which passes through the centre and whose diameter is $\frac{1}{3}r$. Find the C.G. of the remainder.

[*C. U. 1940*]

30. (i) In a uniform circular disc of radius R, a circular hole of radius r is punched out, the distance between the two centres being c, where $r + c < R$. Show that the C.G. of the remainder is at a distance

$$\frac{cr^2}{R^2 - r^2}$$

from the centre of the disc.

*(ii) If any portion of volume (v) of a body or a system of bodies (of total valume V) be displaced to another position, prove that the displacement GG$'$ of the centre of gravity of the whole is parallel to gg' the displacement of the centre of gravity of the portion and its amount is given by

$$GG' = \frac{v}{V} gg'. \qquad [\ \textit{C. H. 1950}\ ]$$

31. A square is described externally on a side of an equilateral triangle. Find the C.G. of the area of the combined figure.

32. A thin uniform wire is bent into two coplanar circular rings of radii r, r', touching each other externally. Find the distance of its centre of gravity from the point of contact. [*C. U. 1946*]

33. A piece of uniform wire is bent into three sides of a square *ABCD* of which the side *AD* is wanting. Show that if it be hung up by the two points *A* and *B* successively, the angle between the two positions of *BC* is $\tan^{-1}18$.

34. *AB*, *BC*, *CD* are three equal thin uniform rods firmly jointed at *B* and *C*, the angles *ABC* and *BCD* being each 120°. The system is suspended from the point *A*. Show that *CD* is horizontal.

35. The centre of gravity of a hollow right circular cone closed by a base, made of a thin uniform sheet, is the same as if the cone was solid. Prove that its vertical angle is $2 \sin^{-1} \frac{1}{3}$.

36. A uniform solid right circular cone whose height is double of the diameter of the base, is hung up from a point on the rim of the base. Show that its axis makes an angle of 45° with the vertical.

37. A buoy is formed of a uniform thin sheet of metal in the form of a hollow cone standing on a hollow hemisphere with a common base. Find the vertical angle of the cone, so that the combined C.G. may be at the centre of the hemisphere.

What would be the corresponding result if the cone and the hemisphere were both solid ?

38. A solid right circular cylinder is attached to a solid hemisphere of equal base. Find the ratio of the height of the cylinder to the radius of the base so that the combined C.G. may be at the centre of the base.

39. From a solid right circular cylinder, a solid right circular cone on the same base is scooped out. Find the ratio of the height of the cone to that of the cylinder if the C.G. of the remainder is at the vertex of the cone.

40. From a uniform right circular cone whose vertical angle is 60°, the greatest possible sphere is scooped out. Find the ratio in which the C.G. of the remainder divides the axis of the cone.

***41.** A frustum of a cone is formed by cutting off the upper portion of a solid right circular cone by plane parallel to the base. The radii of the parallel circular sections being R and r, and h the height of the frustum, show that the height of the C.G. of the frustum from the base is

$$\frac{h}{4} \cdot \frac{R^2 + 2Rr + 3r^2}{R^2 + Rr + r^2}.$$

***42.** A pack of cards is laid on a table, and each card projects in the direction of the length of the pack beyond the one below it ; if each projects as far as possible, show that the distances between the extremities of successive cards from the top will form a harmonical progression.

***43.** A thin hemispherical bowl of weight W contains a weight W' of water and rests on a rough inclined plane of inclination α to the horizon. Show that the plane of the top of the bowl makes an angle ϕ with the horizontal given by

$$W \sin \phi = 2(W + W') \sin \alpha. \qquad [\text{ C. H. } 1955]$$

ANSWERS

1. -2, 0.

2. The C.G. divides the line joining the middle points of AB and CD in the ratio 19 : 11.

4. On OF, dividing it in the ratio 5 : 27, where O is the centre of the hexagon.

5. If A be the unoccupied angular point, and O the centre, the required C G. is in AO produced at a distance $\frac{1}{5}AO$ from O.

7. At a distance $\frac{1}{3}a$ from the centre of the square, on the line from the centre perpendicular to the middle rod.

8. In the line joining the middle points of the parallel sides, dividing it in the ratio 11 . 7.

9. G_1 and G_2 being the C.G.'s of B, C and C, A, if AG_1 and BG_2 intersect at G, the required C.G. of A, B is at the point of intersection of AB and CG.

13. $\frac{16}{3}\sqrt{2}$ inches.

14. $OG=\frac{2}{3}OE$, where OGE is perpendicular to CD.

15. On OC at a distance $\frac{1}{3}OC$ from O. **17.** $\frac{2}{3}AG$.

18. P is the middle point of the median AD.

20. The C.G. divides the line joining the middle points of the parallel sides in the ratio 4 : 5, being nearer the base.

26. On the perpendicular ON from O on DE, at a distance $\frac{8}{15}ON$ from O.

29. At a distance $\frac{r}{210}$ from the centre of the disc on the line joining the centre of the disc to that of the hole, produced backwards.

31. At a distance $\frac{6\pm5\sqrt{3}}{13}a$ from the vertex of the triangle, on the line from the vertex to the centre of the square, where a is the side of the triangle.

32. $r\sim r'$. **37.** $2\cos^{-1}\left(\frac{\sqrt{37}-1}{6}\right)$; $60°$.

38. $1 : \sqrt{2}$. **39.** $(2-\sqrt{2}) : 1$. **40.** $49 : 11$.

CENTRE OF GRAVITY (Continued)

ANALYTICAL TREATMENT

10·12. Introduction.

(i) We have seen in Art. 10·8 [Case (I)] that if there be a system of particles of masses $m_1, m_2, \ldots\ldots m_n$ lying on a straight line at points whose distances from a fixed point on the straight line are $x_1, x_2, \ldots\ldots x_n$, then $\bar{x}$, the distance of their centre of gravity (C.G.) or centre of mass (C.M.) is given by

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}. \qquad \cdots \qquad \cdots \quad (1)$$

(ii) We have further seen in Art. 10·8 [Case (II)] that if the system of particles, lying in a plane, be situated at the points whose co-ordinates referred to fixed rectangular axes on the plane are $(x_1, y_1), (x_2, y_2), \ldots\ldots (x_n, y_n)$, then $(\bar{x}, \bar{y})$, the co-ordinates of their C.G. are given by

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}, \qquad \bar{y}=\frac{\Sigma my}{\Sigma m} \qquad \cdots \quad (2)$$

where Σmx stands for $m_1x_1+m_2x_2+\cdots\cdots+m_nx_n$.

If, instead of the masses being situated at isolated points as above, there is a continuous distribution of mass in a rigid body, then we can find the pasitions of C.G. for different entities [*viz.* rod, arc of a curve, plane area, volume and surface area of a solid etc.] by the above principle with the help of integration as illustrated in the following articles.

Now, relations (1) and (2) can be written as

$$\bar{x}=\frac{\Sigma x\delta m}{\Sigma \delta m} \qquad \cdots \qquad \cdots \quad (1')$$

and $$\bar{x}=\frac{\Sigma x\delta m}{\Sigma \delta m}, \quad \bar{y}=\frac{\Sigma y\delta m}{\Sigma \delta m} \qquad \cdots \quad (2')$$

where δm is the element of mass at any point P, and these can be written in the notation of Integral Calculus as

$$\bar{x}=\frac{\int x\,dm}{\int dm} \quad \cdots \quad \cdots \quad (3)$$

and $$\bar{x}=\frac{\int x\,dm}{\int dm},\quad \bar{y}=\frac{\int y\,dm}{\int dm} \quad \cdots \quad (4)$$

where integrations extend throughout the whole of the mass of the required portion of the body.

10·13. Important Definite Integrals.

The following results of definite integrals will be constantly required in the evalution of integrals obtained in connection with the determination of C.G. of the different kinds of entities.

(A) $\int_0^{\frac{1}{2}\pi} \sin^n x\,dx=\int_0^{\frac{1}{2}\pi} \cos^n x\,dx$ (*n being a positive integer*)

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

or $$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1,$$

according as n is *even* or *odd*.

(B) $\int_0^{\frac{1}{2}\pi} \sin^m x \cos^n x\,dx$ (*m and n being positive integers*).

If *one of the indices be odd*, say m is odd, then put $z=\cos x$ and then express powers of $\sin x$ in terms of $\cos x$ *i.e.*, z. It would then be expressed as the algebraic sum of several integrals each easily integrable. If n be odd, then put $z=\sin x$.

If *both m and n be even integers*, first express either $\cos x$ in terms of $\sin x$ or vice versa and then it would reduce to several different integrals of the type (A).

(C) Those who are familiar with Beta and Gamma functions may use the following formulæ :

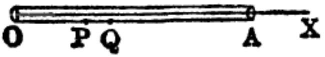
(i) $$\int_0^{\frac{1}{2}\pi} \sin^m x \cos^n x \, dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$$

(ii) $$\int_0^{\frac{1}{2}\pi} \sin^n x \, dx = \int_0^{\frac{1}{2}\pi} \cos^n x \, dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and $\Gamma(n+1) = n\Gamma(n)$, whether n is integer or fraction. Here m and n are positive.

10·14. C.G. of a thin rod.

(i) When the rod is *uniform.*



Let OA be a rod of length a and let us take OAX as the x-axis.

Let P, Q be two neighbouring points on the rod at distances x and $x + \delta x$ from O, so that $PQ = \delta x$. Let ρ be the density and α be the uniform cross-section of the rod. Then the element of mass δm at $P = \alpha . \delta x \rho$, where α and ρ are constants.

Let $\bar{x}$ be the distance of its C.G. from O. Then taking moment about O, we have

$$\bar{x}.\Sigma \alpha \, \delta x \, \rho = \Sigma \alpha \, \delta x \, \rho . x,$$

i.e., $\bar{x}\Sigma \delta x = \Sigma x \, \delta x$ (on dividing both sides by the constants α, ρ).

$$\therefore \quad \bar{x} = \frac{\int_0^a x \, dx}{\int_0^a dx} = \frac{\left[\frac{1}{2}x^2\right]_0^a}{\left[x\right]_0^a} = \frac{1}{2}a. \qquad \cdots \quad (1)$$

The limits of integration are taken as such, since for the whole rod x varies from 0 to a.

Thus, the C.G. of a uniform thin rod is at its mid-point.

(ii) When the rod is of *variable density*.

Suppose the density ρ at the point P be a known function of its distance from one end, say O. Then $\rho = f(x)$.

Here proceeding as above, the element of mass δm at $P = a\,\delta x.\rho = a\,\delta x.f(x)$.

$$\therefore \quad \bar{x}\Sigma a\delta x f(x) = \Sigma a\delta x f(x).x,$$

i.e., $\bar{x}\Sigma f(x)\,\delta x = \Sigma x f(x)\,\delta x$, dividing by the constant a.

$$\therefore \quad \bar{x} = \frac{\int_0^a x f(x)\,dx}{\int_0^a f(x)\,dx}. \qquad \cdots \qquad \cdots \quad (2)$$

Substituting the known value of $f(x)$ in any case, and integrating, the final value of $\bar{x}$ is obtained.

For example, if the density at any point of the rod varies as the distance from the extremity O, then $f(x) = \kappa x$, where κ is a constant, and therefore

$$= \int_0^a x^2 dx \Big/ \int_0^a x\,dx = \frac{2}{3}a. \qquad \cdots \quad (3)$$

Note. If a be the cross section of a rod at a point P on it and ρ be the density there, then $a\rho$ (*i.e.*, mass per unit length) is called the *line-density* of the rod at P. By the single word 'density' is usually meant volume-density *i.e.*, mass per unit volume.

If in the case (ii) it is given that *the line-density* λ *at any point* P *varies as its distance from* O, then δm (the element of mass)* at P would be $\lambda\,\delta x$. Now we can proceed as in (3).

---

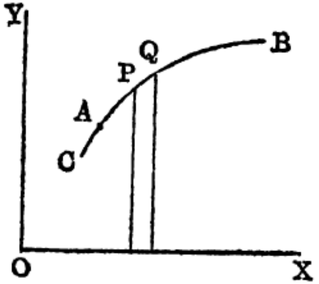
*Strictly speaking, δm (= element of mass in length PQ) lies between $\lambda_1\delta x$ and $\lambda_2\delta x$ when λ_1 and λ_2 are the greatest and least values of λ in PQ. Since we assume λ is continuous and since $\delta x \to 0$, $\delta m \to \lambda\,\delta x$; thus with sufficient accuracy for our purpose, we can write $\delta m = \lambda\,\delta x$.

Similarly, strictly speaking, the distance of C.M. of PQ from O lies between x and $x + \delta x$ and hence is equal to $x + \theta.\delta x$, where $0 < \theta < 1$ which however tends to x as $\delta x \to 0$. Hence, with sufficient accuracy for our purpose we take the distance of the C.M. of mass δm from O as x.

In the following articles, the above principle would be followed in considering the element of mass and the distance of its C.M. from a point or a straight line.

10·15. C.G. of an arc.

Let (x, y) be the co-ordinates of any point P on the arc AB, and ρ be the density at P. Let s be the length of the



arc CP measured from a fixed point C on the arc. Then δs = elementary arc PQ at P, and hence

$\rho.\delta s$ = element of mass at P ($=\delta m$).

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the arc AB.

Then, as in (4) of Art. 10·12, we have

$$\bar{x}=\frac{\int x\,dm}{\int dm}=\frac{\int \rho x\,ds}{\int \rho\,ds},\qquad \bar{y}=\frac{\int y\,dm}{\int dm}=\frac{\int \rho y\,ds}{\int \rho\,ds},\quad \cdots \quad (1)$$

the limits of integration extending from A to B.

When ρ *is constant*, the formula (1) becomes

$$\bar{\mathbf{x}}=\frac{\int \mathbf{x}\,\mathbf{ds}}{\int \mathbf{ds}},\ \bar{\mathbf{y}}=\frac{\int \mathbf{y}\,\mathbf{ds}}{\int \mathbf{ds}}. \quad \cdots \quad \cdots \quad (2)$$

The formulæ (1) and (2) are fundamental formulæ for the determination of the C.G. of an arc and this can be easily transformed when the equation of the curve is given in Cartesian co-ordinates (general or parametric), or in polar co-ordinates.

Note 1. In the application of the above integrals the following results should be noted. When the equation of the curve is

(i) $y=f(x)$, $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$.

(ii) $x=f(y)$, $ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy$

(iii) $x=\phi(t)$, $y=\psi(t)$, $ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$.

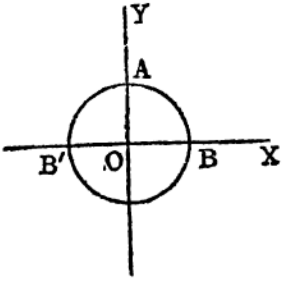
(iv) $f(r, \theta)=0$, $\left.\begin{aligned} ds&=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\cdot d\theta. \\ ds&=\sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\cdot dr. \end{aligned}\right\}$

and $x=r\cos\theta$, $y=r\sin\theta$.

Note 2. The C.G. in such cases is generally not on the arc AB.

10·16. Illustrative Examples.

Ex. 1. *Find the C.G. of an arc of a quadrant of the circle $x^2+y^2=a^2$ in the positive quadrant, ρ being constant.*



Here, $y=\sqrt{a^2-x^2}$, $\therefore \dfrac{dy}{dx}=\dfrac{-x}{\sqrt{a^2-x^2}}$,

$$\therefore\ ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\frac{a}{\sqrt{a^2-x^2}}\,dx.$$

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the arc AB.

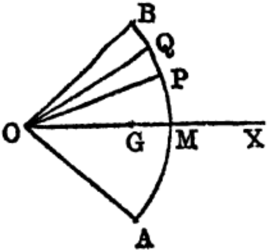
$$\bar{x}=\frac{\int x\,ds}{\int ds}=\frac{\int_0^a \frac{ax}{\sqrt{a^2-x^2}}\,dx}{\int_0^a \frac{a}{\sqrt{a^2-x^2}}\,dx}=\frac{\left[-\sqrt{a^2-x^2}\right]_0^a}{\left[\sin^{-1}\frac{x}{a}\right]_0^a}=\frac{a}{\frac{\pi}{2}}=\frac{2a}{\pi}.$$

To evaluate the indefinite integral in the numerator put $z^2 = a^2 - x^2$ (we can also put $z = a^2 - x^2$ or $x = a \sin\theta$). $\therefore\ z\,dz = -x\,dx$.

$$\therefore\quad I = \int \frac{-z\,dz}{z} = -z = -\sqrt{a^2 - x^2}$$

$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int_0^a \frac{ay}{\sqrt{a^2-x^2}}\,dx}{\int_0^a \frac{a}{\sqrt{a^2-x^2}}\,dx} = \frac{\int_0^a dx}{\left[\sin^{-1}\frac{x}{a}\right]_0^a} = \frac{a}{\frac{\pi}{2}} = \frac{2a}{\pi}.$$

Ex. 2. *Find the centre of mass G, of a circular arc of radius a, subtending an angle* 2α *radians at the centre.*



Let O be the centre of the circular arc AB and let $\angle AOB = 2\alpha$. Let M be the mid-point of the arc AB. Join OM, and produce it to X and take O as origin and OX as x-axis. Now, $\angle AOM = \angle BOM = \alpha$. Thus, from symmetry G is on OMX. Let $OA = OB = a$ and $OG = \bar{x}$. Let $\angle XOP = \theta$.

For a circle we know $r = a$, $s = a\theta$. $\therefore\ \delta s = a\,\delta\theta$. For A and B, $\theta = -\alpha$ and α.

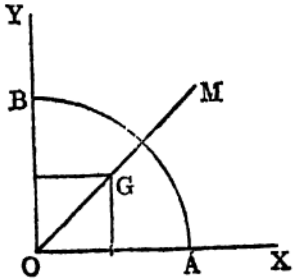
$$\therefore\ \bar{x} = \frac{\int x\,ds}{\int ds} = \frac{\int_{-\alpha}^{+\alpha} a\cos\theta . a\,d\theta}{\int_{-\alpha}^{+\alpha} a\,d\theta} = a \cdot \frac{\left[\sin\theta\right]_{-\alpha}^{+\alpha}}{\left[a\theta\right]_{-\alpha}^{+\alpha}} = a\frac{\sin\alpha}{\alpha} \quad \cdots \quad (1)$$

Cor. 1. For *a semi-circular arc* $2\alpha = \pi$, $\alpha = \frac{1}{2}\pi$.

$$\therefore\ \bar{x} = a\frac{\sin\frac{1}{2}\pi}{\frac{1}{2}\pi} = \frac{2a}{\pi}.$$

Cor. 2. For *an arc of a quadrant of circle*, $\alpha = \frac{1}{4}\pi$.

$$\therefore\ OG = a\frac{\sin\frac{1}{4}\pi}{\frac{1}{4}\pi} = \frac{2a\sqrt{2}}{\pi}.$$



If the two bounding diameters OA, OB of the quadrant be taken as x-axis and y-axis respectively, then

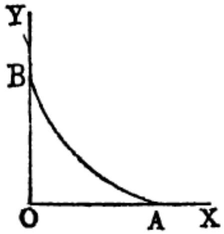
$(\bar{x}, \bar{y})$, co-ordinates of the C.G. with reference to these two as axes are $OG \cos \frac{1}{4}\pi$, and $OG \sin \frac{1}{4}\pi$,

i.e., $\bar{x} = \bar{y} = \dfrac{2a}{\pi}$. [See *Ex. 1 above*]

Ex. 3. *Find the C.G. of an arc of a quadrant of the astroid* $x = a \cos^3\phi$, $y = a \sin^3\phi$ *in the first quadrant.*

For the point A, $y = 0$, $\therefore$ $\phi = 0$.

For the point B, $y = a$, $\therefore$ $\phi = \frac{1}{2}\pi$.



Here, $ds = \sqrt{\left(\dfrac{dx}{d\phi}\right)^2 + \left(\dfrac{dy}{d\phi}\right)^2} \cdot d\phi = 3a \sin\phi \cos\phi \, d\phi$.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the arc AB

$$\therefore \ \bar{x} = \frac{\int x \, ds}{\int ds}, \ \bar{y} = \frac{\int y \, ds}{\int ds}. \quad \cdots \quad \cdots \quad (1)$$

$$\int x \, ds = \int_0^{\frac{1}{2}\pi} a \cos^3\phi \, 3a \sin\phi \cos\phi \, d\phi = 3a^2 \int_0^{\frac{1}{2}\pi} \cos^4\phi \sin\phi \, d\phi$$

$$= 3a^2 \int_0^1 z^4 \, dz \text{ (putting } z = \cos\phi \text{)} = \frac{3}{5} a^2. \quad \cdots \quad (2)$$

$$\int ds = \int_0^{\frac{1}{2}\pi} 3a \sin\phi \cos\phi \, d\phi = 3a \int_0^{\frac{1}{2}\pi} \sin\phi \cos\phi \, d\phi$$

$$= 3a \int_0^1 z \, dz \text{ (putting } z = \sin\phi \text{)} = \frac{3a}{2}. \quad \cdots \quad (3)$$

$$\int y \, ds = \int_0^{\frac{1}{2}\pi} a \sin^3\phi \, 3a \sin\phi \cos\phi \, d\phi$$

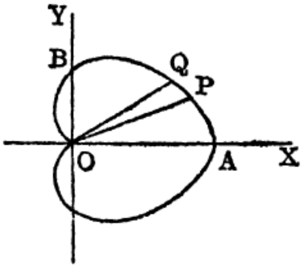
$$= 3a^2 \int_0^{\frac{1}{2}\pi} \sin^4\phi \cos\phi \, d\phi$$

$$= 3a^2 \int_0^1 z^4 \, dz \text{ (putting } z = \sin\phi \text{)} = \frac{3a^2}{5} \quad (4)$$

From (1), (2), (3), (4), $\bar{x}=\dfrac{3a^2/5}{3a/2}=\dfrac{2a}{5}$; $\bar{y}=\dfrac{3a^2/5}{3a/2}=\dfrac{2a}{5}$.

Ex. 4. *Find the position of the C.G. of the arc of a semi-cardioide.*

Let the equation of the cardioide be $r=a\,(1+\cos\theta)$.



Hence, $APBO$ is the arc. At A, $\theta=0$; at O, $\theta=\pi$.

Now, $$ds=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$=\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\;d\theta=2a\cos\tfrac{1}{2}\theta\;d\phi,$$

and $$r=a(1+\cos\theta)=2a\cos^2\tfrac{1}{2}\theta.$$

Let $(\bar{x},\bar{y})$ be the co-ordinates of the C.G. of the arc $APBO$. Then

$$\bar{x}=\frac{\int x\,ds}{\int ds},\quad \bar{y}=\frac{\int y\,ds}{\int ds}. \qquad \cdots \qquad \cdots \quad (1)$$

$$\int x\,ds=\int_0^{\pi} r\cos\theta\,.\,2a\cos\tfrac{1}{2}\theta\;d\theta$$

$$=\int_0^{\pi} a(1+\cos\theta)\cos\theta\,.\,2a\cos\tfrac{1}{2}\theta\;d\theta$$

$$=2a^2\int_0^{\pi} 2\cos^3\tfrac{1}{2}\theta\,(2\cos^2\tfrac{1}{2}\theta-1)\,d\theta$$

$$=4a^2\,.\,2\int_0^{\frac{1}{2}\pi}\cos^3\phi\,(2\cos^2\phi-1)\,d\phi \quad \text{(on putting } \theta=2\phi)$$

$$=8a^2\cdot\left[2\int_0^{\frac{1}{2}\pi}\cos^5\phi\,d\phi-\int_0^{\frac{1}{2}\pi}\cos^3\phi\,d\phi\right]$$

$$=8a^2\,.\,[2.\tfrac{4}{5}.\tfrac{2}{3}-\tfrac{2}{3}]\;\text{[See § } \mathit{10{\cdot}13\ above}\text{]}$$

$$=\tfrac{16}{5}a^2. \qquad \cdots \qquad \cdots \qquad \cdots \quad (2)$$

$$\int ds=2a\int_0^{\pi}\cos\tfrac{1}{2}\theta\;d\theta=4a. \qquad \cdots \qquad \cdots \quad (3)$$

Again, $$\int y\,ds = \int_0^{\pi} r\sin\theta\,.\,2a\cos\tfrac{1}{2}\theta\,d\theta$$
$$= \int_0^{\pi} a(1+\cos\theta)\,.\,2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\,.\,2a\cos\tfrac{1}{2}\theta\,d\theta$$
$$= 8a^2\int_0^{\pi}\cos^4\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta\,d\theta.$$

Putting $z=\cos\frac{1}{2}\theta$, $dz=-\frac{1}{2}\sin\frac{1}{2}\theta\,d\theta$, and when $\theta=0$ and π, $z=1$ and 0.

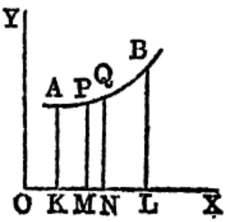
$$\therefore\quad \int y\,ds = 16a^2\int_0^1 z^4\,dz = 16a^2\left[\frac{z^5}{5}\right]_0^1 = \frac{16}{5}a^2. \quad \cdots \quad (4)$$

From (1), (2), (3), (4), $\bar{x}=\dfrac{\frac{16}{5}a^2}{4a}=\frac{4}{5}a$, $\bar{y}=\dfrac{\frac{16}{5}a^2}{4a}=\frac{4}{5}a$.

10·17. C.G. of a plane area.

Case I. Cartesian.

Suppose the area is bounded by the curve $y=f(x)$, the axis of x and the ordinates $x=x_1$, $x=x_2$.



Let (x, y), $(x+\delta x,\ y+\delta y)$ be the co-ordinates of P and a neighbouring point Q on the curve. Divide the whole area into elementary strips like $PMNQ$, by drawing lines parallel to the y-axis. The area of the strip $= y.\delta x$ ultimately, since δx is very small. Let the area be homogeneous and let ρ be the *surface density* of the strip $PMNQ$. Then δm, the element of mass of the strip $PMNQ = y\delta x.\rho$ and the C.G. of the strip $PMNQ$ is ultimately at the point $(x, \frac{1}{2}y)$ (with sufficient accuracy for our purpose). Let $(\bar{x}, \bar{y})$

be the C.G. of the area $AKLB$. Then taking moments about OY and OX respectively we have.

$$\bar{x}.\Sigma\rho y\,\delta x=\Sigma\rho y\,\delta x.x, \qquad \bar{y}.\Sigma\rho y\,\delta x=\Sigma\rho y\,\delta x.\tfrac{1}{2}y.$$

Cancelling out the constant ρ from both sides we get in the limit

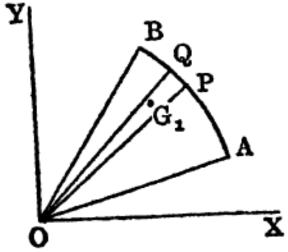
$$\bar{x}=\frac{\int_{x_1}^{x_2}xy\,dx}{\int_{x_1}^{x_2}y\,dx}, \qquad \bar{y}=\frac{1}{2}\frac{\int_{x_1}^{x_2}y^2\,dx}{\int_{x_1}^{x_2}y\,dx},$$

where y has to be expressed in terms of x from the equation of the curve.

Note. The *surface-density* ρ at any point of an area $=\sigma\lambda$ where $\sigma=$ the volume-density and $\lambda=$ the thickness at the point.

Case II. Polar.

Let the area AOB be bounded by the curve $r=f(\theta)$ and the radii vectors OA, OB ($\theta=\alpha$ and $\theta=\beta$) so that $\angle XOA=\alpha$, $\angle XOB=\beta$.



Let O be the origin, OX, the initial line and OY the y-axis.

Let the whole area be divided into elementary triangular strips like OPQ by radii vectors drawn from O. Let the co-ordinates of P, Q be (r, θ), $(r+\delta r,\ \theta+\delta\theta)$. Then $\angle POQ=\delta\theta$.

Now, area of the strip $OPQ=\frac{1}{2}r^2\delta\theta$ ultimately, since $\delta\theta$ is very small. Then the C.G. of the strip OPQ is a point G_1 in OPQ, whose co-ordinates are ultimately $(\frac{2}{3}r\cos\theta, \frac{2}{3}r\sin\theta)$

(with sufficient degree of accuracy for our purpose). Let ρ be the surface-density of the strip. Then elementary mass δm of the strip OPQ is $\frac{1}{2}r^2\,\delta\theta.\rho$, situated at G_1. Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the area AOB.

Therefore, taking moments about y-axis and x-axis respectively, we have

$$\bar{x}.\Sigma\tfrac{1}{2}r^2\,\rho\delta\theta=\Sigma\tfrac{1}{2}r^2\,\rho\,\tfrac{2}{3}r\cos\theta.\delta\theta,$$

$$\bar{y}.\Sigma\tfrac{1}{2}r^2\,\rho\,\delta\theta=\Sigma\tfrac{1}{2}r^2\rho.\tfrac{2}{3}r\sin\theta.\delta\theta.$$

Cancelling out from both sides $\frac{1}{2}\rho$, since ρ is constant, we have finally in the limit

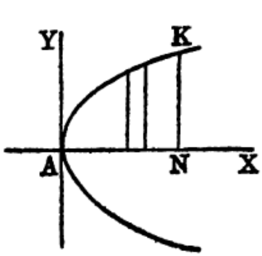
$$\bar{x}=\frac{2}{3}\cdot\frac{\int_\alpha^\beta r^3\cos\theta\,d\theta}{\int_\alpha^\beta r^2\,d\theta},\quad \bar{y}=\frac{2}{3}\cdot\frac{\int_\alpha^\beta r^3\sin\theta\,d\theta}{\int_\alpha^\beta r^2\,d\theta}.$$

where $r=f(\theta)$ from the equation to the bounding curve.

10·18. Illustrative Examples.

Ex. 1. *Find the C.G. of the homogeneous area bounded by the parabola $y^2=4ax$, the x-axis and the ordinate $x=h$.*

Here, $y=2\sqrt{ax}$, and ρ is constant.



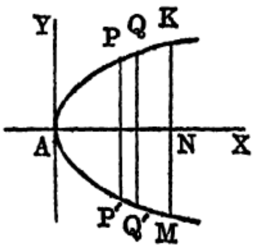
Let $(\bar{x}, \bar{y})$ be the C.G. of the area AKN, bounded by the ordinate KN, where $AN=h$.

$$\bar{x}=\frac{\int_0^h xy\,dx}{\int_0^h y\,dx}=\frac{\int_0^h x.2\sqrt{ax}\,dx}{\int_0^h 2\sqrt{ax}\,dx}=\frac{\int_0^h x^{\frac{3}{2}}\,dx}{\int_0^h x^{\frac{1}{2}}\,dx}=\frac{\frac{2}{5}h^{\frac{5}{2}}}{\frac{2}{3}h^{\frac{3}{2}}}=\frac{3}{5}h.$$

$$\bar{y}=\tfrac{1}{2}\cdot\frac{\int_0^h y^2\,dx}{\int_0^h y\,dx}=\tfrac{1}{2}\frac{\int_0^h 4ax\,dx}{\int_0^h 2\sqrt{ax}\,dx}=\frac{3}{4}\sqrt{ah}\text{ (on integration)}$$

Ex. 2. *Find the C.G. of the homogeneous area bounded by the parabola $y^2=4ax$ and the double ordinate $x=h$.*

Let (x, y) be the co-ordinates of a point P on the parabola. Dividing the whole area into elementary strips by drawing lines parallel to the



double ordinate KM, area of the elementary strip $PP'Q'Q=2y\,\delta x$, and the co-ordinates of its C.G. are $(x, 0)$, since from symmetry the C.G. lies on the x-axis. The elementary mass δm of the strip $=2y\delta x.\rho$, ρ being the surface-density of the strip.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. Then taking moment about AY, we have

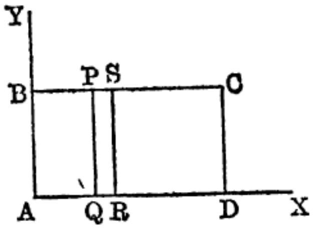
$$\bar{x}=\frac{\int_0^h 2y\,dx.\rho.x}{\int_0^h 2y\,dx.\rho}=\frac{\int_0^h xy\,dx}{\int_0^h y\,dx}=\frac{3}{5}h\text{ as in Ex. 1.}$$

$\bar{y}=0$ (from symmetry).

$\therefore$ the C.G. of the area lies on the x-axis at a distance $\frac{3}{5}h$ from the vertex.

Ex. 8. *Find the C.G. of a uniform rectangular lamina (area),* [By the method of Calculus].

Let $ABCD$ be a rectangular lamina of which the sides AB, AD are b and a respectively.



Let us take AD and AB as axes of x and y.

Let the area be divided into elementary strips by lines drawn parallel to AB and let P be (x, y) and $PQRS$ be an elementary strip whose area is $b.\delta x$ and if ρ be the surface-density, the elementary mass $= b\ \delta x.\rho$. Co-ordinates of the C.G. of the area $PQRS$ are $(x, \frac{1}{2}b)$ ultimately.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C. G. of the given area. Then taking moment about y-axis,

$$\bar{x}.\ \Sigma b\ \delta x\ \rho = \Sigma b\ \delta x.\rho.x.$$

$$\therefore \quad \bar{x} = \frac{\int_0^a x\ dx}{\int_0^a dx} = \frac{\left[\frac{1}{2}x^2\right]_0^a}{\left[x\right]_0^a} = \frac{\frac{1}{2}a^2}{a} = \frac{1}{2}a.$$

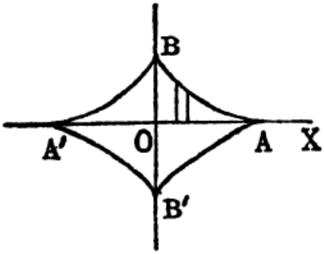
Similarly, taking moment about x-axis,

$$\bar{y}.\Sigma b\ \delta x.\rho = \Sigma b\ \delta x.\rho.\tfrac{1}{2}b.$$

$$\therefore \quad \bar{y} = \frac{\frac{1}{2}b\int_0^a dx}{\int_0^a dx} = \frac{1}{2}b.$$

$\therefore$ the C.G. of the rectangle is $(\frac{1}{2}a, \frac{1}{2}b)$, *i.e., at the middle point of the line joining the mid-points of AB, CD.*

Ex. 4. *Find the C.G. of the part of the four-cusped hypocycloid* $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$ *lying in the positive quadrant.*



The part of the hypocycloid lying in the positive quadrant is $BOAB$, where $OA=a$, $OB=b$.

The parametric representation of the curve is $x=a\cos^3\theta$, $y=b\sin^3\theta$.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of area $BOAB$.

$$\therefore\quad \bar{x}=\frac{\int_0^a xy\,dx}{\int_0^a y\,dx},\quad \bar{y}=\frac{1}{2}\frac{\int_0^a y^2\,dx}{\int_0^a y\,dx}. \qquad (1)$$

Putting $x=a\cos^3\theta$, $y=b\sin^3\theta$, $dx=-3a\cos^2\theta\sin\theta\,d\theta$

when $x=0$, $\theta=\frac{1}{2}\pi$ and when $x=a$, $\theta=0$.

$$\therefore\quad I_1=\int_0^a xy\,dx=3a^2b\int_0^{\frac{1}{2}\pi}\sin^4\theta\cos^5\theta\,d\theta$$

$$=3a^2b\int_0^{\frac{1}{2}\pi}\sin^4\theta\cos^4\theta.\cos\theta\,d\theta.$$

Put $s=\sin\theta$, $\therefore$ $ds=\cos\theta\,d\theta$;

when $\theta=0, \frac{\pi}{2}$, $s=0, 1$.

$$\therefore\quad I_1=3a^2b\int_0^1 s^4(1-s^2)^2\,ds$$

$$=3a^2b\int_0^1 (s^4-2s^6+s^8)\,ds$$

$$=3a^2b\left(\tfrac{1}{5}-\tfrac{2}{7}+\tfrac{1}{9}\right)=\tfrac{8}{105}a^2b. \qquad \cdots \qquad (2)$$

$$I_2=\int_0^a y\,dx=3ab\int_0^{\frac{1}{2}\pi}\cos^2\theta\,\sin^4\theta\,d\theta$$

$$=3ab\int_0^{\frac{1}{2}\pi}(1-\sin^2\theta)\sin^4\theta\,d\theta$$

$$=3ab\left[\int_0^{\frac{1}{2}\pi}\sin^4\theta\,d\theta-\int_0^{\frac{1}{2}\pi}\sin^6\theta\,d\theta\right]$$

$$=3ab\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]=\frac{3\pi ab}{32}. \qquad (3)$$

$$I_3=\int_0^a y^2\,dx=3ab^2\int_0^{\frac{1}{2}\pi}\cos^2\theta.\,\sin^7\theta\,d\theta$$

$$=3ab^2\int_0^{\frac{1}{2}\pi}(1-\sin^2\theta)\sin^7\theta\,d\theta$$

$$=3ab^2\left[\int_0^{\frac{1}{2}\pi}\sin^7\theta\,d\theta-\int_0^{\frac{1}{2}\pi}\sin^9\theta\,d\theta\right]$$

$$=3ab^2\left[\tfrac{6}{7}\cdot\tfrac{4}{5}\cdot\tfrac{2}{3}-\tfrac{8}{9}\cdot\tfrac{6}{7}\cdot\tfrac{4}{5}\cdot\tfrac{2}{3}\right]=\tfrac{16}{105}ab^2.$$

$$\therefore\quad \bar{x}=\frac{I_1}{I_2}=\frac{8}{105}a^2b\times\frac{32}{3\pi ab}=\frac{256}{315\pi}a.$$

$$\therefore\quad \bar{y}=\frac{1}{2}\cdot\frac{I_3}{I_2}=\frac{1}{2}\cdot\frac{16}{105}ab^2\times\frac{32}{3\pi ab}=\frac{256}{315\pi}b.$$

$$\therefore\ \frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{256}{315\pi}.$$

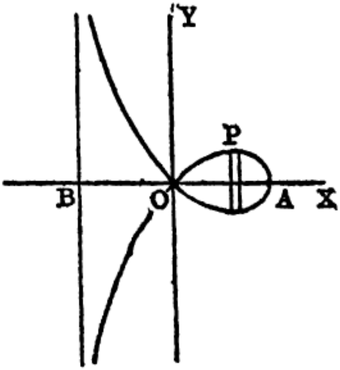
Ex. 5. *Find the C.G. of the uniform area in the form of the loop of the curve* $x^3+xy^2-2x^2+2y^2=0$.

The curve can be written as $y^2(2+x)=x^2(2-x)$,

i.e., $y^2=x^2.\dfrac{2-x}{2+x}$ *i.e.*, $y=\pm x\sqrt{\dfrac{2-x}{2+x}}$. ... (1)

Thus, the curve is symmetrical with respect to the x-axis, there is an asymptote $x+2=0$ and there is a loop $OPACO$ in $0\leqslant x\leqslant 2$.

Divide the loop into elementary strips by lines parallel to the y-axis. The area of an elementary strip at $P=2y\,dx$ and if ρ be the surface-density of the area (supposed homogeneous), the elementary mass $=2y\,dx\,.\,\rho$ and the co-ordinates of the C.G. of the area are $(x, 0)$ since from symmetry, C.G. lies on the x-axis. Let $(\bar{x}, \bar{y})$ be the



co-ordinates of the C.G. of the loop. Then taking moment about y-axis, we have

$$\bar{x}\,.\,\Sigma\, 2y\,\delta x\,\rho=\Sigma\, 2y\,\delta x\,\rho . x.$$

Cancelling out the constant common factor 2ρ from both sides and noting that x varies from 0 to 2, to include the entire loop, we have

$$\bar{x}=\frac{\int_0^2 yx\,dx}{\int_0^2 y\,dx}. \qquad \ldots \qquad \ldots \qquad (2)$$

Writing $y=x\sqrt{\dfrac{2-x}{2+x}}$ from the equation of the curve (1),

$$\text{Numerator}=\int_0^2 x^2\sqrt{\frac{2-x}{2+x}}dx=\int_0^2 \frac{x^2(2-x)}{\sqrt{2^2-x^2}}dx$$

$$=\int_0^2 \frac{2x^2\,dx}{\sqrt{2^2-x^2}}-\int_0^2 \frac{x^3\,dx}{\sqrt{2^2-x^2}}.$$

Put $x=2\sin\theta$. $\therefore$ $dx=2\cos\theta\,d\theta$; when $x=0$, $\theta=0$, and when $x=2$, $\theta=\frac{1}{2}\pi$.

$\therefore$ Numerator $= 2^3 \int_0^{\frac{1}{2}\pi} \sin^2\theta \, d\theta - 2^3 \int_0^{\frac{1}{2}\pi} \sin^3\theta \, d\theta$

$$= 2^3 \cdot \frac{\pi}{4} - 2^3 \cdot \frac{2}{3} = 2\pi - \frac{16}{3} = \frac{2}{3}(3\pi - 8) \qquad (3)$$

Denominator $= \int_0^2 x \sqrt{\frac{2-x}{2+x}}\, dx = \int_0^2 \frac{x(2-x)}{\sqrt{2^2 - x^2}}\, dx$

$$= \int_0^2 \frac{2x\, dx}{\sqrt{2^2 - x^2}} - \int_0^2 \frac{x^2}{\sqrt{2^2 - x^2}}\, dx$$

$$= 2^2 \int_0^{\frac{1}{2}\pi} \sin\theta \, d\theta - 2^2 \int_0^{\frac{1}{2}\pi} \sin^2\theta \, d\theta,$$

(putting $x = 2 \sin\theta$ as above)

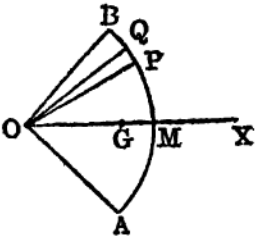
$$= 2^2 . 1 - 2^2 \frac{\pi}{4} = 4 - \pi. \qquad (4)$$

$\therefore$ from (2), (3) and (4),

$$\bar{x} = \frac{2}{3} \cdot \frac{3\pi - 8}{4 - \pi}.$$

From symmetry, the C.G. lies on the x-axis OAX. $\therefore$ $\bar{y} = 0$.

Ex. 6. *Find the C G. of a uniform sector of a circle.*



Let $OAMB$ be a sector of a circle with O as centre and let $\angle AOB = 2\alpha$ and $OA = OB =$ radius of the circle $= a$. Let OM bisect $\angle AOB$.

Taking OM as x axis, let $(\bar{x}, \bar{y})$ be the co ordinates of the C.G. of the sector. From symmetry, G lies on OX and hence $\bar{y} = 0$.

$$\bar{x} = \frac{2}{3} \frac{\int_{-\alpha}^{+\alpha} r^3 \cos\theta \, d\theta}{\int_{-\alpha}^{+\alpha} r^2 \, d\theta} = \frac{2}{3} \frac{\int_{-\alpha}^{+\alpha} a^3 \cos d\theta}{\int_{-\alpha}^{+\alpha} a^2 \, d\theta} \qquad [\because \text{ here } r = a]$$

$$= \frac{2}{3} \frac{a^3 \cdot \left[\sin\theta\right]_{-\alpha}^{+\alpha}}{a^2 \cdot \left[\ \theta\ \right]_{-\alpha}^{+\alpha}} = \frac{2}{3} a \cdot \frac{\sin\alpha}{\alpha}.$$

Hence G, the C.G. is situated on OM, where $OG=\frac{2}{3}a.\frac{\sin\alpha}{\alpha}$.

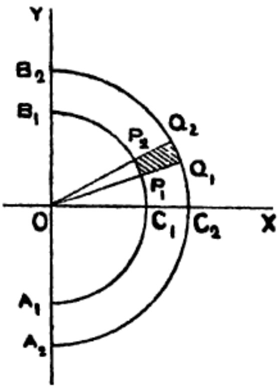
Cor. 1. For *a uniform semi-circular lamina*, $\alpha=\frac{1}{2}\pi$. $\therefore$ $OG=\frac{4a}{3\pi}$.

Cor. 2. For *a complete uniform circular lamina*, $\alpha=\pi$. $OG=0$.

Hence, the C.G. of a circular lamina is at the centre of the circle.

Ex. 7. *Find the C.G. of the area bounded by two semi-circles of radii a and b and their common diameter.*

Let us divide the area into elementary strips by drawing radii vectors like OP_1Q_1 and OP_2Q_2 from O to the two semi circles, O being



their common centre, and $\angle P_1OX=\angle Q_1OX=\theta$, $\angle P_2OP_1=\angle Q_2OQ_1=\delta\theta$. Let $a>b$. Let the line through O perpendicular to the bounding diameter be taken as the x-axis and the bounding diameter as the y-axis.

Then, the elementary area

$$P_1Q_1Q_2P_2=\text{area } OQ_1Q_2-\text{area } OP_1P_2$$
$$=\tfrac{1}{2}a^2\,\delta\theta-\tfrac{1}{2}b^2\,\delta\theta=\tfrac{1}{2}(a^2-b^2)\,\delta\theta.$$

Let ρ be the surface-density of the elementary area; then mass of the elementary area $=\rho.\frac{1}{2}(a^2-b^2)\,\delta\theta$.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the whole area. Then taking moment about the y-axis,

$\bar{x}\Sigma P_1Q_1Q_2P_2$ *i.e.*, $\bar{x}\Sigma\frac{1}{2}(a^2-b^2)\,\delta\theta=\Sigma\frac{1}{2}a^2\,\delta\theta.x_1-\Sigma\frac{1}{2}b^2\,\delta\theta.x_2$,
where x_1, x_2 are the distances of the C.G.'s of the ultimate triangular areas OQ_1Q_2 and OP_1P_2 from the y-axis, *i.e.*,

$$x_1=\tfrac{2}{3}a\cos\theta,\ x_2=\tfrac{2}{3}b\cos\theta.$$

$$\therefore\quad \bar{x}=\frac{\Sigma\frac{1}{2}a^2\delta\theta.\frac{2}{3}a\cos\theta-\Sigma\frac{1}{2}b^2\delta\theta.\frac{2}{3}b\cos\theta}{\Sigma\frac{1}{2}(a^2-b^2)\,\delta\theta},\text{ or, ultimately}$$

$$\bar{x}=\frac{2}{3}\frac{\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(a^3-b^3)\cos\theta\,d\theta}{\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(a^2-b^2)\,d\theta}=\frac{2}{3}\frac{(a^3-b^3).2}{(a^2-b^2)\pi}=\frac{4}{3\pi}.\frac{a^2+ab+b^2}{a+b},$$

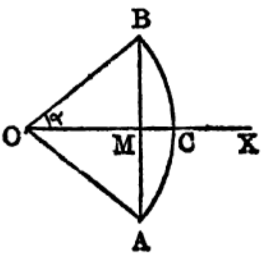
(For the points A_1, A_2, $\theta=-\frac{1}{2}\pi$ and for the points B_1, B_2, $\theta=\frac{1}{2}\pi$.)

From symmetry, C.G. would lie on OX, $\therefore\ \bar{y}=0$.

Note. Putting $b=0$ in the above result we can easily verify that if G be the C.G. of a semi-circular lamina of radius a, then $\bar{x}=OG=\dfrac{4a}{3\pi}$.

Ex. 8. *Find the C.G. of a uniform segment of a circle*

Let AOB be a sector of a circle of radius a bounded by the radii vectors OA, OB, where $\angle AOB=2\alpha$. Let OC be the bisector of the angle AOB. We take OC as the axis of x.



From symmetry, the C.G. of the segment $ACBMA$ lies on OX. If $G_1\equiv(x_1, y_1)$ be the C.G. of the sector, then we have

$$x_1=\frac{2}{3}a\frac{\sin\alpha}{\alpha},\ y_1=0.\quad (\text{ See } \textit{Ex. 6 above})$$

Also if $G_2\equiv(x_2, y_2)$ be the C.G. of the triangle AOB, then

$$x_2=\tfrac{2}{3}a\cos\alpha,\ y_2=0.$$

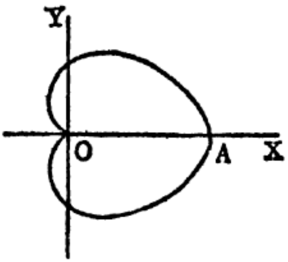
Let $(\bar{x}, \bar{y})$ be the C.G. of the segment $ACBMA$. Then by Art. 10·10 of the book,

$$\bar{x} = \frac{a^2\alpha \cdot \frac{2}{3} a \frac{\sin \alpha}{\alpha} - a \cos \alpha . a \sin \alpha \cdot \frac{2}{3} a \cos \alpha}{a^2\alpha - a^2 \sin \alpha \cos \alpha}$$

$$= \frac{2}{3} a \cdot \frac{\sin \alpha - \sin \alpha \cos^2 \alpha}{\alpha - \sin \alpha \cos \alpha} = \frac{2}{3} a \frac{\sin^3 \alpha}{\alpha - \sin \alpha \cos \alpha}$$

and $\bar{y} = 0$.

Ex. 9. *Find the C.G. of the area of the cardioide $r = a(1 + \cos \theta)$.*



Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the cardioide.

The centroid evidently lies on the axis of symmetry *viz.*, the x-axis.

$\therefore \quad \bar{y} = 0.$

Since the two halves of the cardioide are equal and symmetrical, the abscissa of the C.G. of the whole is the same as that of the upper half.

Thus, $$\bar{x} = \frac{2}{3} \frac{\int_0^\pi \rho r^3 \cos \theta \, d\theta}{\int_0^\pi \rho r^2 \, d\theta} = \frac{2}{3} \frac{\int_0^\pi r^3 \cos \theta \, d\theta}{\int_0^\pi r^3 \, d\theta}, \quad (\rho \text{ being const.})$$

$$= \frac{2}{3} \frac{\int_0^\pi a^3(1 + \cos \theta)^3 \cos \theta \, d\theta}{\int_0^\pi a^2(1 + \cos \theta)^2 \, d\theta}$$

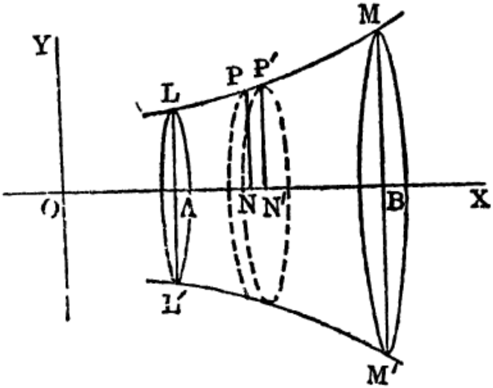
$$= \frac{2a}{3} \cdot \frac{\int_0^\pi 2^3 \cos^6 \frac{\theta}{2} \left(2 \cos^2 \frac{\theta}{2} - 1\right) d\theta}{\int_0^\pi 2^2 \cos^4 \frac{\theta}{2} \, d\theta}$$

$$= \frac{4a}{3} \frac{\int_0^{\pi/2} (2 \cos^8 \phi - \cos^6 \phi) \, d\phi}{\int_0^{\pi/2} \cos^4 \phi \, d\phi}, \quad (\text{putting } \theta = 2\phi)$$

$$= \frac{4a}{3} \cdot \frac{2 \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}}{\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}} = \frac{5a}{6}.$$

10·19. C.G. of the volume and surface of revolution of a uniform solid.

Suppose a solid is formed by the revolution of the curve $y = f(x)$ about the x-axis OX and suppose it is bounded by two ordinates AL, BM corresponding to $x = x_1$ and $x = x_2$.



(i) The volume generated by the element of area $PNN'P'$, where (x, y) are the co-ordinates of P is (the area of the circle described by PN) × (the thickness between the two circles described by PN and $P'N'$) $= \pi y^2\, \delta x$, ultimately [since $PN = y$ and δx is very small]. If ρ be the density of the slice bounded by the two circles, then, δm, the element of mass of the strip $= \rho.\pi y^2\, \delta x$. The C.G. of the element from symmetry lies on OX, and is ultimately at a distance x from O. Hence, if $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the volume generated by the area $ALMB$, then taking moment about y-axis, we have

$$\bar{x}.\Sigma \rho\pi y^2\, \delta x = \Sigma \rho\pi y^2\, \delta x.x.$$

As the solid is of uniform density, cancelling out $\rho\pi$ from both sides, we get

$$\bar{x} = \frac{\Sigma y^2 x\, \delta x}{\Sigma y^2\, \delta x} = \frac{\int_{x_1}^{x_2} y^2 x\, dx}{\int_{x_1}^{x_2} y^2\, dx}$$

and from symmetry, $\bar{y} = 0$.

(ii) The area of the surface generated by the revolution of the arc PP' $(=\delta s)$ about OX is (the circumference of the circle described by PN) $\times$ (length of the arc PP') *i.e.*, $2\pi y.\delta s$ ultimately, since $PN = y$ and δs is small. If ρ be the surface-density then, δm, the element of mass of the belt $= \rho.2\pi y.\delta s$.

The C.G. of the belt from symmetry lies on OX and is ultimately at a distance x from O. Hence, if $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the surface generated by LM, then taking moment about y-axis, we have

$$\bar{x}\,.\,\Sigma\rho\,.\,2\pi y\,\delta s = \Sigma\rho\,.\,2\pi y\,\delta s\,.\,x.$$

As the surface is of uniform density, cancelling out $2\pi\rho$ from both sides, we get

$$\bar{x} = \frac{\Sigma y\, \delta s . x}{\Sigma y\, \delta s} = \frac{\int yx\, ds}{\int y\, ds}.$$

In the integration, the limits for s correspond to $x = x_1$ and $x = x_2$.

Cor. When the equation of the curve is given in polar co-ordinates, say $r = f(\theta)$, the above formulæ can easily be transformed into the following forms by the relation between Cartesian and polar co-ordinates *viz.*, $x=r\cos\theta$, $y=r\sin\theta$,

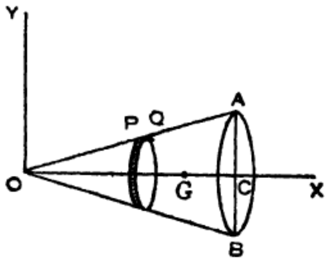
Solid : $$\left\{\begin{array}{l} \bar{x}=\dfrac{\int r^3\sin^2\theta\cos\theta\dfrac{d}{d\theta}(r\cos\theta).d\theta}{\int r^2\sin^2\theta\dfrac{d}{d\theta}(r\cos\theta).d\theta} \\ \bar{y}=0. \end{array}\right.$$

Surface : $$\left\{\begin{array}{l} \bar{x}=\dfrac{\int r^2\sin\theta\cos\theta\cdot\dfrac{ds}{d\theta}\cdot d\theta}{\int r\sin\theta\dfrac{ds}{d\theta}\cdot d\theta} \\ \bar{y}=0 \end{array}\right.$$

taken between proper limits.

10·20. Illustrative Examples.

Ex. 1. *Find the C G of a homogeneous solid cone.*



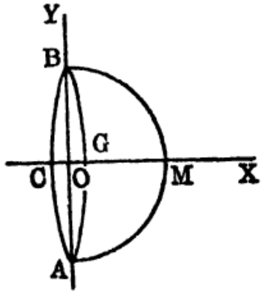
Let $(\bar{x}, \bar{y})$ be the co-ordinates of G, the C G. of the homogeneous solid cone AOB.

$$\therefore\ \bar{x}=\frac{\int y^2x\,dx}{\int y^2\,dx}=\frac{\int_0^h (x^2\tan^2\alpha).x\,dx}{\int_0^h (x^2\tan^2\alpha)\,dx}=\frac{\int_0^h x^3\,dx}{\int_0^h x^2\,dx}=\frac{\left[\frac{1}{4}x^4\right]_0^h}{\left[\frac{1}{3}x^3\right]_0^h}=\frac{3}{4}h.$$

From symmetry, C G. lies on the x-axis, the axis of rotation. $\therefore\ \bar{y}=0$.

$\therefore$ G lies on OC, such that $OG=\frac{3}{4}OC$, *i.e.*, $OG:GC=3:1$.

Ex. 2. *Find the C.G. of a homogeneous solid hemisphere.*



Let $(\bar{x}, \bar{y})$ be the co-ordinates of G, the C.G. of the homogeneous solid hemisphere AMB.

$$\therefore \quad \bar{x} = \frac{\int y^2 x\, dx}{\int y^2\, dx} = \frac{\int_0^a (a^2 - x^2) x\, dx}{\int_0^a (a^2 - x^2)\, dx} = \frac{a^2 . \frac{1}{2}a^2 - \frac{1}{4}a^4}{a^2 . a - \frac{1}{3}a^3} = \frac{3}{8} a.$$

From symmetry, C.G. lies on the x-axis, the axis of rotation. $\therefore \ \bar{y} = 0$.

$\therefore$ G is situated on OM such that $OG : GM = 3 : 5$.

Ex. 3. *Find the C.G. of a homogeneous conical surface.*

Let OAB be a right circular cone, formed by the complete revolution of the line OA round the fixed line OC [See *Fig. of Ex.* 1 above] and let $\angle AOC = \alpha$, and $OC = h$. Suppose it is of uniform density. Let us take the vertex O of the cone as origin and its axis OX as x-axis and OY, perpendicular to OX as y-axis.

The equation of the line OA is $y = x \tan \alpha$.

$$\therefore \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \tan^2 \alpha}\, dx = (\sec \alpha)\, dx.$$

Let $\bar{x}, \bar{y}$ be the co-ordinates of the C.G. of the conical surface.

$$\therefore \quad \bar{x} = \frac{\int yx\, ds}{\int y\, ds} = \frac{\int_0^h (x \tan \alpha) x . (\sec \alpha)\, dx}{\int_0^h (x \tan \alpha)(\sec \alpha)\, dx} = \frac{\int_0^h x^2 dx}{\int_0^h x\, dx} = \frac{\left[\frac{1}{3} x^3\right]_0^h}{\left[\frac{1}{2} x^2\right]_0^h} = \frac{2}{3} h.$$

From spmmetry, C.G. lies on x-axis, the axis of rotation, and hence $\bar{y}=0$.

$\therefore$ G, the C.G. of the conical surface is situated on the axis of the cone such that $OG=\frac{2}{3}OC$ *i.e.*, $OG:GC=2:1$.

Note. Here the *generating curve* is the *straight line* $y=x \tan \alpha$.

Ex. 4. *Find the C.G. of a uniform hemispherical surface.*

Suppose AMB be a hemisphere of radius a, with its plane base $ABCA$, and let O be the centre of the base and let OM be the line perpendicular to the base and suppose it is formed by the complete revolution (*i.e.*, the revolution through 2π radians) of the quadrant of the circle MB round the axis OM [See *Fig. Ex. 2*, above]. Let us take O as origin and OMX as x-axis and OBY as y axis; then the equation of the circle MB is $x^2+y^2=a^2$ *i.e.*,

$$y=\sqrt{a^2-x^2}. \quad \therefore \quad ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,dx=\frac{a}{\sqrt{a^2-x^2}}\,dx=\frac{a}{y}\,dx.$$

$\therefore$ $y\,ds=a\,dx$.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the homogeneous hemispherical surface.

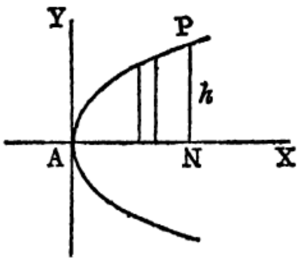
$$\therefore \bar{x}=\frac{\int yx\,ds}{\int y\,ds}=\frac{\int_0^a ax\,dx}{\int_0^a a\,dx} \quad \left[\text{since for } B \text{ and } M,\ x=0 \text{ and } x=a\right]$$

$$=\frac{\left[a\frac{1}{2}x^2\right]_0^a}{\left[a.x\right]_0^a}=\frac{1}{2}a.$$

From symmetry, C.G. lies on the x-axis of rotation and hence $\bar{y}=0$.

Thus, the C.G. is at the mid-point of OM.

Ex. 5. *Find the C.G. of the solid formed by revolution about the x-axis of the parabola $y^2 = 4ax$, bounded by the ordinate $x = h$.*

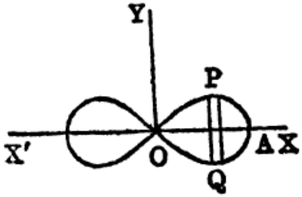


$$\bar{x} = \frac{\int_0^h y^2 x\,dx}{\int_0^h y^2\,dx}$$

$$= \frac{\int_0^h 4ax^2\,dx}{\int_0^h 4ax\,dx} = \frac{\int_0^h x^2\,dx}{\int_0^h x\,dx} = \frac{\frac{1}{3}h^3}{\frac{1}{2}h^2} = \frac{2}{3}h.$$

From symmetry, $\bar{y} = 0$.

Ex. 6. *Find the C.G. of the surface generated by the revolution of a loop of the lemniscate $r^2 = a^2 \cos 2\theta$, about the initial line.*



Let $(\bar{x}, \bar{y})$ denote the C.G. of the surface of revolution of the loop $OQAP$. Then from symmetry, $\bar{y} = 0$.

Here, $r^2 = a^2 \cos 2\theta$. $\therefore$ $r\dfrac{dr}{d\theta} = -a^2 \sin 2\theta$.

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}}.$$

$$\therefore \quad \bar{x}=\frac{\int xy\,ds}{\int y\,ds}=\frac{\int_0^{\frac{1}{4}\pi} r\cos\theta . r\sin\theta\cdot\frac{a}{\sqrt{\cos 2\theta}}\,d\theta}{\int_0^{\frac{1}{4}\pi} r\sin\theta\cdot\frac{a}{\sqrt{\cos 2\theta}}\,d\theta}$$

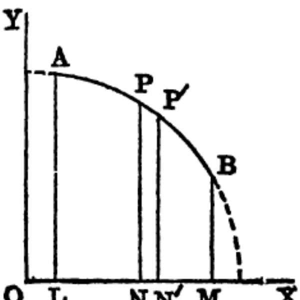
$$=a\frac{\int_0^{\frac{1}{4}\pi}\sin\theta\cos\theta . \sqrt{\cos 2\theta}\,d\theta}{\int_0^{\frac{1}{4}\pi}\sin\theta\,d\theta}$$

$$=\frac{a}{2}\frac{\int_0^{\frac{1}{4}\pi}\sin 2\theta\sqrt{\cos 2\theta}\,d\theta}{\Big|-\cos\theta\Big|_0^{\frac{1}{4}\pi}}$$

$$=\frac{a}{2}\frac{\int_0^1 z^2\,dz}{1-\frac{1}{\sqrt{2}}} \qquad [\text{ putting } \cos 2\theta=z^2\ ]$$

$$=\frac{a}{6}\cdot\frac{\sqrt{2}}{\sqrt{2}-1}=\frac{a(2+\sqrt{2})}{6}.$$

Ex. 7. *Find the centre of gravity of the surface and volume of the part of a sphere, of radius a, included between two parallel planes which are at distances x_1 and x_2 from its centre.* $(x_2 > x_1)$



Let the sphere be generated by the revolution of the circle $x^2+y^2=a^2$ about the x-axis. Let us consider the portion of the sphere bounded by the parallel planes AL, BM, where $OL=x_1$, and $OM=x_2$.

Then, for the surface

$$\bar{x}=\frac{\int xy\,ds}{\int y\,ds}=\frac{\int_{x_1}^{x_2} xy\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}=\frac{\int_{x_1}^{x_2} xy\sqrt{1+\frac{x^2}{y^2}}\,dx}{\int_{x_1}^{x_2} y\sqrt{1+\frac{x^2}{y^2}}\,dx}$$

$$=\frac{\int_{x_1}^{x_2} ax\,dx}{\int_{x_1}^{x_2} a\,dx}=\frac{1}{2}\frac{x_2^2-x_1^2}{x_2-x_1}=\frac{x_1+x_2}{2}$$

and from symmetry, $\bar{y}=0$.

For the volume,

$$\bar{x}=\frac{\int y^2 x\,dx}{\int y^2\,dx}=\frac{\int_{x_1}^{x_2}(a^2-x^2)x\,dx}{\int_{x_1}^{x_2}(a^2-x^2)\,dx}=\frac{\frac{a^2}{2}(x_2^2-x_1^2)-\frac{1}{4}(x_2^4-x_1^4)}{a^2(x_2-x_1)-\frac{1}{3}(x_2^3-x_1^3)}$$

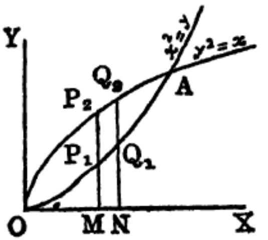
$$=\frac{3}{4}\cdot(x_1+x_2)\,\frac{2a^2-x_1^2-x_2^2}{3a^2-x_1^2-x_1x_2-x_2^2}.$$

From symmetry, $\bar{y}=0$.

Note. If we put $x_2=a$ and $x_1=0$, we get the case of a hemisphere, and for the surface $\bar{x}=\frac{1}{2}a$, $\bar{y}=0$, and for the solid $\bar{x}=\frac{3}{8}a$, $\bar{y}=0$.

Ex. 8. *Find the C. G. of the volume generated by the revolution about the x-axis of the area bounded by the parabolas $y^2=x$, $x^2=y$.*

Clearly, the points of intersection are given by



$y^4=x^2=y$. $\quad\therefore\quad y=0, 1$

and so $x=0, 1$.

$\therefore$ the points of intersection are (0, 0) and (1, 1).

The volume of the strip generated by the revolution of $P_1Q_1Q_2P_2$ about x-axis is $\pi(y_2^2-y_1^2)\,\delta x$, where $y_1=P_1M$ and $y_2=P_2M$, and δx being very small. Then $(\bar{x}, \bar{y})$ being the C. G. of the volume generated, we have

$$\bar{x}=\frac{\int_0^1 (y_2^2-y_1^2)\,dx.x}{\int_0^1 (y_2^2-y_1^2)\,dx}=\frac{\int_0^1 (x-x^4)\,x\,dx}{\int_0^1 (x-x^4)\,dx}=\frac{\frac{1}{3}-\frac{1}{6}}{\frac{1}{2}-\frac{1}{5}}=\frac{5}{9}.$$

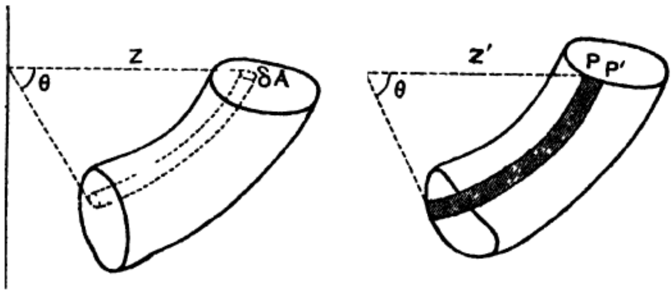
Also from symmetry, $\bar{y}=0$.

10·21. Theorems of Pappus or Guldin.

If a plane area bounded by a closed curve revolves through any angle about a straight line in its own plane, which does not intersect the curve, then

(I) the volume of the solid generated is equal to the product of the revolving area and the length of the arc described by the centroid of the area ;

(II) the surface-area of the solid generated is equal to the product of the perimeter of the revolving area and the length of the arc described by the centroid of that perimeter.



(I) Let δA be any element of the area whose distance from the axis of rotation is z. Then θ being the angle through which the area is rotated, the length of the arc described by δA is $z\theta$, and hence the elementary volume described by the element δA is $z\theta.\delta A$.

The whole volume described by the given area therefore

$$=\Sigma z\theta.\delta A=\theta.\Sigma z.\delta A=\theta\bar{z}A \quad [\text{ From } Art.\ 10{\cdot}8\ ]$$

(where A is the total area of the curve and $\bar{z}$ is the distance of its centroid from the axis of revolution)

$=A\bar{z}\theta=$ area of the closed curve × length of the arc described by its centroid.

(II) Let δs be the length of any element PP' of the perimeter of the given curve, and z' its distance from the axis of revolution. The elementary surface traced out by the element δs is ultimately $z'\theta\,\delta s$.

The total surface-area of the solid generated is therefore

$$=\Sigma z'\theta.\delta s=\theta\Sigma z'.\delta s=\theta\bar{z}'s \text{ [From } Art.\ 10{\cdot}8 \text{]}$$

(where s is the whole perimeter of the curve, and $\bar{z}'$, the distance of the centroid of this perimeter from the axis)

$$=s.\bar{z}'\theta=\text{perimeter}\times\text{length of the arc described by its centroid.}$$

Note. The above results hold even if the axis of rotation touch the closed curve.

Ex. 1. *Find the volume and surface-area of a solid tyre, a being the radius of its section, and b that of the core.*

The tyre is clearly generated by revolving a circle of radius a about an axis whose distance from the circle is b.

The centre of the circle is the centroid of both the area of the circle as also of the perimeter of the circle, and the length of the path described by it is evidently $2\pi b$.

Hence, the required volume $=\pi a^2\times 2\pi b=2\pi^2a^2b$, and the surface-area required $=2\pi a\times 2\pi b=4\pi^2ab$.

Ex. 2. *Use the theorems of Pappus to find the centre of gravity of*
(a) *a semi-circular arc.* (b) *a semi-circular area.*

Let the semi-circle be of radius a and let it revolve about the bounding diameter so that the surface generated is a sphere of radius a.

Let $\bar{x}$ denote the distance of the C.G. from the centre.

(a) For the semi-circular arc,

$$2\pi\bar{x}.\pi a=\text{surface-area of the sphere}=4\pi a^2.$$

$$\therefore\ \bar{x}=\frac{2a}{\pi}.$$

(b) For the semi-circular area,

$$2\pi\bar{x}\tfrac{1}{2}\pi a^2 = \text{volume of the sphere} = \tfrac{4}{3}\pi a^3$$

$$\therefore\ \bar{x} = \frac{4a}{3\pi}.$$

Examples on Chapter X(c)

1. Find the C.G. of a rod AB of length a, the density at any point of which varies as the nth power of the distance from the end A.

2. G is the centre of mass of the rod AB of length l. The line-density at any point of AB varies as the distance from the point O on BA produced, where $OA = a$. Find $AG : GB$.

3. Find the C.G. of the arc which is in the first quadrant of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

4. (i) Find the C.G. of the arc of the parabola $y^2 = 4ax$ from the vertex to an end of the latus rectum.

(ii) Find the C.G. of a uniform wire bent into the form of the cardioide $r = a(1 + \cos\theta)$.

5. Find the centroid of the arc of the catenary $y = c\cosh\dfrac{x}{c}$, from the vertex V to any point $P(x, y)$ on the arc.

6. Find the centroid of the area of the circle $x^2 + y^2 = a^2$ lying in the first quadrant.

7. Find the centroid of the area of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ lying in the first quadrant.

8. Find the centroid of the area between the sine curve $y=\sin x$ and $y=0$ where $0 \leqslant x \leqslant \pi$.

9. Find the centroid of the area between the cosine curve $y=\cos x$ and $y=0$ where $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

10. Find the centroid of the area between the semi-cubical parabola $ay^2=x^3$ and $x=h$.

11. Find the centroid of the area bounded by $y=x^2$, $x=1$, $y=0$, $x=2$.

12. Find the centroid of the area bounded by $x=0$, $y=0$, $y=3+2x-x^2$ and lying in the positive quadrant.

13. Find the centroid of the area of the loop of the curve $y^2(a+x)=x^2(a-x)$.

14. Find the centroid of the area of the loop of the curve $x^3+xy^2-x^2+y^2=0$.

15. Find the centroid of the area between the Cissoid $y^2(2a-x)=x^3$ and its asymptote.

16. Find the C.G. of the area of the parabola $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$ between the curve and the axes.

17. (i) Find the centroid of the area between $y^2=x$ and $y=x$.

(ii) Find the centroid of the area bounded by $y^2=4ax$ and $y=mx$.

18. (i) Find the centroid of the area bounded by $y=x^2$, $y=-x^2$ and $x=3$.

(ii) Find the centroid of the area bounded by $y^2=ax$ and $x^2=by$ $(a, b>0)$.

19. Find the C.G. of the area enclosed by the curves $x^2+y^2-2x=0$, and $x^2+y^2-4x=0$.

20. Find the centroid of the area of half the cardioide $r=a(1+\cos\theta)$ bounded by $\theta=0$.

21. Find the centroid of the area of the right loop of the Lemniscate $r^2=a^2\cos 2\theta$.

22. Find the locus of the centroid of the area of the parabola $y^2=4ax$ cut off by a variable straight line passing through the vertex.

23. Find the C.G. of the segment of a sphere of radius a, cut off a plane at a distance b from the centre.

24. Find the C.G. of the solid formed by the revolution of the quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about its (i) major axis, (ii) minor axis.

25. Find the C.G. of the solids formed by revolving :

(i) $ay^2=x^3$, about the x-axis between $x=0$ and $x=c$;

(ii) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, about the axis of y ;

(iii) $r=a(1+\cos\theta)$ about the x-axis.

26. Find the centroid of the (i) surface and (ii) solid generated by revolving half of the cardioide $r=a(1+\cos\theta)$ bounded by $\theta=0$ about the initial line.

27. Find the C.G. of the surface formed by the revolution of the parabola $y^2=2x$ cut off by the line $x=4$, about the axis of the parabola.

28. Find the centroids of the surfaces formed by the revolution of the following curves :

(i) Cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the axis of y;

(ii) Cardioide $r = a(1 + \cos\theta)$ about its axis.

29. If the distances of the vertices of a triangle from a fixed line on its plane (not intersecting the triangle) are x_1, x_2, x_3, and if S be the area of the triangle, show that the volume generated by the revolution of the triangle about the fixed line is $\dfrac{2\pi S}{3}(x_1 + x_2 + x_3)$.

30. An equilateral triangle of side a revolves round its base which is fixed. Find the volume of the solid generated.

31. Show that the volume of the solid formed by revolving the ellipse $x = a\cos\theta$, $y = b\sin\theta$ about the line $x = 2a$ is $4\pi^2 a^2 b$.

32. Find the volume of the solid formed by the revolution of $y^2 = 4ax$ about the latus rectum.

ANSWERS

1. On the rod at a distance $\dfrac{n+1}{n+2}a$ from A.

2. $(3a+2l)/(3a+l)$. **3.** $\bar{x} = (\pi - \frac{4}{3})a$, $\bar{y} = \frac{2}{3}a$.

4. (i) $\bar{x} = \dfrac{1}{4}a \cdot \dfrac{3\sqrt{2} - \log(\sqrt{2}+1)}{\sqrt{2} + \log(\sqrt{2}+1)}$, $\bar{y} = \dfrac{4}{3}a \dfrac{2\sqrt{3} - 1}{\sqrt{2} + \log(\sqrt{2}+1)}$.

(ii) $\bar{x} = \dfrac{4a}{5}$, $\bar{y} = 0$. **5.** $\bar{x} = x - c(y-c)/s$, $\bar{y} = \frac{1}{2}y + cx/2s$, where $VP = s$.

6. $\bar{x} = \bar{y} = \dfrac{4a}{3\pi}$. **7.** $\bar{x} = \bar{y} = \dfrac{256a}{315\pi}$. **8.** $\bar{x} = \frac{1}{2}\pi$, $\bar{y} = \frac{1}{8}\pi$.

9. $\bar{x} = 0$, $\bar{y} = \frac{1}{8}\pi$. **10.** $\bar{x} = \frac{4}{7}h$, $\bar{y} = 0$. **11.** $\bar{x} = \frac{45}{28}$, $\bar{y} = \frac{93}{70}$.

12. $\bar{x} = \frac{4}{5}$, $\bar{y} = \frac{17}{10}$. **13.** $\bar{x} = \dfrac{a}{3} \cdot \dfrac{3\pi - 8}{4 - \pi}$, $\bar{y} = 0$. **14.** $\bar{x} = \dfrac{1}{3} \cdot \dfrac{3\pi - 8}{4 - \pi}$, $\bar{y} = 0$.

15. $\bar{x}=\frac{5}{3}a,\ \bar{y}=0.$ 16. $\frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{1}{5}.$ 17. (i) $\bar{x}=\frac{3}{5},\ \bar{y}=\frac{1}{2}.$

(ii) $\bar{x}=8a/5m^2,\ \bar{y}=2a/m.$ 18. (i) $\bar{x}=\frac{9}{4},\ \bar{y}=0.$

(ii) $\bar{x}=\frac{9}{20}a^{\frac{1}{3}}b^{\frac{2}{3}},\ \bar{y}=\frac{9}{20}a^{\frac{2}{3}}b^{\frac{1}{3}}.$ 19. $\bar{x}=\frac{7}{3},\ \bar{y}=\frac{28}{9\pi}.$

20. $\bar{x}=\frac{5a}{6},\ \bar{y}=\frac{16a}{9\pi}.$ 21. $\bar{x}=\frac{\pi a\sqrt{2}}{8},\ \bar{y}=0.$ 22. $2y^2=5ax.$

23. $\bar{x}=\frac{3}{4}\frac{(a+b)^2}{2a+b},\ \bar{y}=0.$ 24. (i) $\bar{x}=\frac{3}{8}a,\ \bar{y}=0.$ (ii) $\bar{x}=0,\ \bar{y}=\frac{3}{8}b.$

25. (i) $\bar{x}=\frac{4}{5}c,\ \bar{y}=0.$ (ii) $\bar{x}=0,\ \bar{y}=\frac{a}{6}\cdot\frac{63\pi^2-64}{9\pi^2-16}.$ (iii) $\bar{x}=\frac{4a}{5},\ \bar{y}=0.$

26. (i) For surface $\bar{x}=\frac{89}{83}a,\ \bar{y}=0.$ (ii) For volume $\bar{x}=\frac{4}{5}a,\ \bar{y}=0.$

27. $\bar{x}=\frac{149}{65},\ \bar{y}=0.$ 28. (i) $\bar{x}=0,\ \bar{y}=\frac{2a}{15}\frac{15\pi-8}{3\pi-4}.$

(ii) $\bar{x}=\frac{80}{83}a,\ \bar{y}=0.$ 30. $\frac{\pi a^3}{4}.$ 32. $\frac{32}{15}\pi a^3.$

CHAPTER X(A)

CONSTRAINED BODY AND STABILITY

10(A)·1. Equilibrium of a heavy body supported at a fixed point.

If a body supported at a point be at rest under the action of gravity only (and no other external forces), the C.G. of the body and the point of support must be in the same vertical line.

This is obvious ; for the only forces acting upon the body are (1) its weight acting vertically downwards through its C.G. and (2) the reaction at the point of support. For equilibrium these two forces must be equal and opposite and also must have the same line of action. Hence, the fixed point and centre of gravity must be in the same vertical line.

Note. The above principle can be used in *determining graphically the C.G. of a plane lamina.* Thus, first suspend the body by a string attached to any point A on its boundary and draw the vertical line AD on the lamina through A. We know that the C.G. lies on AD. Again, suspend the body from any other point B on the boundary and draw the vertical line BE, through B, on the lamina. Then the C.G. also lies on BE.

Hence, the reqd. C.G. is the point of intersection of AD, BE.

10(A)·2. Equilibrium of a heavy body with an area in contact with a plane.

Theorem : *A body placed in contact with a horizontal plane will or will not rest in equilibrium, according as the vertical line through its centre of gravity meets the plane inside or outside the base on which it stands.*

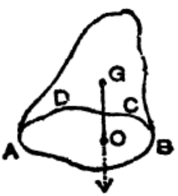


Fig. (i)

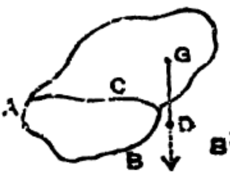


Fig. (ii)

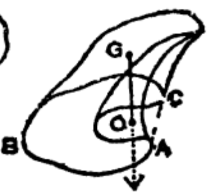


Fig. (iii)

The only forces acting upon the body are (1) the weight of the body acting vertically downwards through its C.G. and (2) the reaction of the plane, which is nothing but the resultant of the total reactions of the several points of contact of the body with the plane, and hence acts through a point inside the base.

For equilibrium, the weight of the body and the reaction of the plane must be equal, opposite and also must have the same line of action. Hence when there is equilibrium, the vertical line through the centre of gravity of the body meets the plane inside the base. If the vertical line meets the plane outside the base, obviously there cannot be equilibrium and the body will topple over.

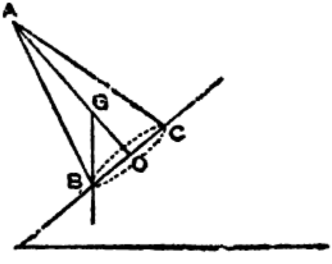
Note 1. By the base is meant the polygon without re-entrant angles, (or more generally the closed curve having no convexity inwards) formed by joining the extreme points of the body in contact with the plane. [See *Fig. (iii)*]

Note 2. A bus for example will overturn if the vertical through its C.G. falls outside the wheel base.

Note 3. Exactly as above it can be shown that a body placed on an *inclined* plane, sufficiently rough to prevent *sliding*, will be in equilibrium or topple over, according as the vertical through the centre of gravity of the body does or does not pass through the base of the body.

10(A)·3. Illustrative Examples.

Ex. *A solid right circular cone whose height is h and radius of whose base is r, is placed on an inclined plane and prevented from sliding. If the inclination of the plane be gradually increased, find when the cone will topple over.*



In the extreme position, *i.e.*, when the cone is on the point of toppling over, the vertical through the C.G. of the cone must pass through the extreme point of the base, *i.e*, would pass through the end B of the base. Let θ be the inclination of the plane at that time. Then obviously $\angle BGO = \theta$.

Now, from the right-angled triangle BGO, $\tan BGO = \dfrac{BO}{OG} = \dfrac{r}{\frac{1}{4}h} = \dfrac{4r}{h}$.

$\therefore$ $\tan \theta = 4r/h$, giving the required inclination.

Note. If α be the semi-vertical angle of the cone, then $r/h = \tan \alpha$.

Hence, the cone will topple if $\tan \theta > 4 \tan \alpha$.

10(A)·4. Stable, Unstable, and Neutral Equilibrium.

Let a body be in equilibrium under a system of external forces and reactions, being supported in any manner. If the body be slightly displaced from its equilibrium position, the external forces and the reactions in the new position of the body will not in general be in equilibrium, so that the body, when left to itself, will begin to move.

Now, according to the way in which the body moves, the original equilibrium position is defined to be *stable*, *unstable*, or *neutral* under different circumstances.

(i) Stable Equilibrium.

A body is said to be in stable equilibrium if, after it is slightly displaced from its position of equilibrium, it has a tendency to return to its original position.

(ii) Unstable Equilibrium.

A body is said to be in unstable equilibrium, provided when slightly displaced from its position of equilibrium, it tends to recede further away from its original position.

(iii) Neutral Equilibrium.

A body is said to be in neutral equilibrium, provided, when slightly displaced from its position of equilibrium, it remains in equilibrium in this new position and tends neither to come back to, nor to go further away from its original position.

10(A)·5. Stability of a body under gravity with one point fixed.

In this case we know that the point of support must be in the same vertical line with the C.G.

If a body suspended from any point O and having its C.G. at G vertically below O, be slightly displaced by being turned through a small angle about O, as in Fig. (i) then the weight of the body W, acting at G, will have a moment about O, which will tend to cause the body to revolve back to its original position. In this case, the equilibrium is *stable*. If, however, G, the C.G. of the body, be vertically above O, the point of suspension, and

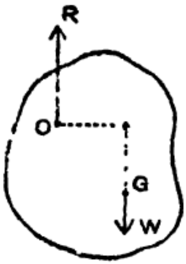


Fig. (i)

the body be slightly displaced through any angle, then the moment of the weight about O will have a tendency to revolve it further away from its original position as in Fig. (ii). In this case, the equilibrium is *unstable*. When the body is suspended from its C.G. it will remain at rest

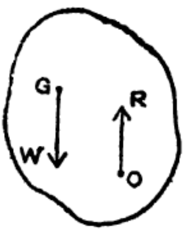


Fig. (ii)

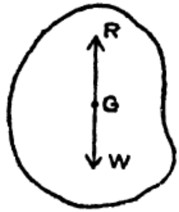


Fig. (iii)

in *any* position, for the weight of the body and the equal and opposite reaction of the support always act at the same point and hence they balance one another. Hence in this case, if the body be displaced, it will not tend either to return to, or to recede further away from its original position of equilibrium.

In such a case, the equilibrium is *neutral.*

10(A)·6. Stability of a body with a portion of it in the form of a sphere resting with spherical portion in contact with a horizontal table.

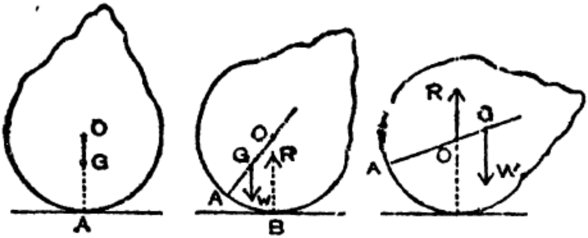


Fig. (i) Fig. (ii) Fig. (iii)

In the position of equilibrium the weight of the body acting vertically downwards through its centre of gravity G is balanced by the reaction of the table at the point of contact A, which must accordingly be perpendicular to the plane of the table and will therefore pass through the centre O of the sphere. Thus OG must be vertical [*Fig.* (*i*)].

Now, let the body be slightly displaced so that B is the new point of contact. The vertical through B being normal at B to the sphere will pass through the centre O. Now if G be below O, the weight of the body acting vertically downwards through G will tend to rotate the body about B back to its equilibrium position as in Fig. (ii) and the equilibrium position is accordingly stable.

If on the other hand G be above O, the weight in the displaced position, as in Fig. (iii), will tend to rotate the body further away from its equilibrium position. The original equilibrium position is accordingly unstable.

If, however, G coincides with O, the weight in the displaced position acting vertically downwards in the line OB will be balanced by the reaction at B, and the body therefore will remain in equilibrium in this displaced position. The equilibrium in this case is neutral.

Thus in this case the equilibrium of the body is stable, unstable or neutral, according as the C.G. of the body in the equilibrium position is vertically below, above, or coincident with the centre of the spherical portion in contact with the table.

Note. In the first illustration [§ *10(A)·5*], the equilibrium is stable or unstable according as G falls below or above the point of suspension. In the second illustration [§*10(A)·6*], the equilibrium is stable or unstable according as G is below or above the point O through which the reaction passes in all positions of the body. In both the cases, a lower position of G ensures stability.

In general, top heavy bodies are unstable and bottom heavy bodies are stable in their equilibrium positions.

Examples on Chapter X(A)

1. A solid right circular cylinder, of height h and radius of cross-section r, is placed on an inclined plane of inclination a and prevented from sliding. Show that the cylinder will topple when

$$\tan a > 2r/h.$$

2. A leaning tower of n equal circular coins, each of radius a and thickness $2b$, is piled over on a horizontal table, so that the centres of gravity of all the coins lie in one straight line. Show that the greatest inclination of the line to the vertical is $\tan^{-1}(a/nb)$.

3. How many equal circular coins, having the thickness of each equal to $\frac{1}{25}$th of its diameter, can stand in a cylindrical pile on an inclined plane, whose height is one-fourth of the base, assuming that there is no slipping ?

4. A frustum of a solid right cone is placed with its base on a rough inclined plane whose inclination is gradually increased ; if R, r be the radii of the larger and smaller sections, and h the height of the frustum, show that the frustum will ultimately either tumble or slide according as the coefficient of friction is greater or less than $4R(R^2+Rr+r^2)/h(R^2+2Rr+3r^2)$. [*C. H. 1960, old*]

5. A solid homogenous body, consisting of right circular cylinder of height h and a hemisphere of radius r, on the same base, rests with its spherical portion in contact with a horizontal table ; show that it will be in stable, unstable or neutral equilibrium according as

$$r \text{ is } > \text{ or } < \text{ or } = h\sqrt{2}.$$

***6.** A cone rests on a rough table and a cord fastened to the vertex of the cone passes over a smooth pulley at the same height as the top of the cone and supports a weight. Show that if the weight be continually increased, the cone will topple or slide according as the coefficient of friction is $>$ or $<$ $\tan\alpha$, where α is the semi-vertical angle of the cone.

***7.** A heavy rod of length $2l$ lies over a rough peg with one extremity leaning against a rough vertical wall. The inclination of the rod to the wall is α and P is the point of contact of the rod with the wall, d is the distance of the peg from the wall and λ the angle of friction both at the peg and the wall.

(i) If P is above the peg, show that the rod is on the point of sliding down when $l\sin^3\alpha = d\cos^2\lambda$.

(ii) If P is below the peg, show that the rod is on the point of slipping *downwards* when $l\sin^2\alpha\sin(\alpha+2\lambda) = d\cos^2\lambda$ and on the point of slipping *upwards* when $l\sin^2\alpha\sin(\alpha-2\lambda) = d\cos^2\lambda$.

***8.** A frustum of a uniform right circular cone whose semi-vertical angle is α, is made by cutting $\frac{1}{n}$th of the axis. Prove that the frustum will rest with a slant side on a horizontal plane if

$$\tan^2\alpha < \frac{1}{4}\,\frac{3n^4-4n^3+1}{n^3-1}. \qquad [\textit{ C. H. 1954 }]$$

ANSWERS

3. 100.

CHAPTER XI

WORK AND POWER

11·1. Work.

A force is said to do work when its point of application moves in the direction of the acting force, and the work done by a force, acting at a point of a body for any time, is measured by the product of the force and the displacement of the point of application of the force in its own direction.

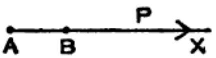


Fig. (i)

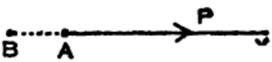


Fig. (ii)

Let a force P be acting on a body at A in the direction AX for any time, and let A move to B during the interval. If AB be in the direction AX, as in fig. (i), the work done $= P.AB$, and is *positive*. If the displacement AB of A is in a direction opposite to that of P as in fig. (ii), the displacement measured in the direction of P is $-AB$, and the work done by the force here $= -P.AB$, which is *negative*.

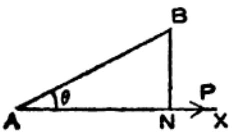


Fig. (iii)

If the displacement AB be in a direction different from the direction of the force, say making an angle θ with AX as in fig. (iii), the displacement measured in the direction of P is $AN = AB \cos\theta$, and in this case we get more generally

Work done by $P = \mathbf{P.AB \cos\theta = AB.P \cos\theta}$

= *Force × component of displacement of its point of application along the line of action of the force,*

or, = *Displacement × component of the force along the direction of displacement.*

Evidently, the work done is *positive* or *negative* according as θ is *acute* or *obtuse*.

Cor. In particular if $\theta = 90°$, the work done is zero, *i.e.*, no work is done by a force if the displacement of its point of application is perpendicular to the line of action of the force.

If the displacement or its component is in a direction opposite to that of the acting force, work is said to be *done against* the force.

11·2. Units for measurement of work.

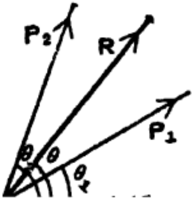
When a force equal to the weight of one pound displaces its point of application through one foot in its own direction, the amount of work done is defined to be one **Foot-pound.**

This is usually the unit of work used in Statics in the English system.

When a force equal to the weight of one gramme displaces its point of application through one centimetre in its own direction, the amount of work done is defined as one **Gramme-centimetre.**

This is the unit of work in Statics is the French system.

11·3. Theorem I. *The algebraic sum of the works done by a number of coplanar forces acting on a particle, for any displacement of the particle, is equal to the work done by their resultant.*



Let the particle be displaced from O to A, and let the forces P_1, P_2,... inclined at angles θ_1, θ_2,... with OA act on it. Let R be the resultant of the forces inclined at θ with OA.

The algebraic sum of the works done by the forces

$$= P_1 \cos\theta_1 \,.\, OA + P_2 \cos\theta_2 \,.\, OA + \cdots$$
$$= OA\,(P_1 \cos\theta_1 + P_2 \cos\theta_2 + \cdots)$$
$= OA \times$ algebraic sum of the resolved parts of the forces along OA

$= OA \times$ the resolved parts of the resultant along OA

$= OA \times R \cos\theta$

$=$ work done by the resultant.

Theorem II. *The work done in raising a number of particles from one position to another is Wh, where W is the total weight of the particles, and h is the distance through which the centre of gravity of the particles has been raised.*

Let $w_1, w_2, \ldots w_n$ be the weights of the particles so that $W = w_1 + w_2 + \cdots + w_n$.

In initial position, let $x_1, x_2, \ldots\ldots x_n$ be the distances of the particles and $\bar{x}$ that of their C.G. from a fixed horizontal plane (measured positive upwards), so that

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w_1 + w_2 + \cdots + w_n} \qquad [\textit{Art. } 10\cdot8]$$

$$\text{or,} \quad w_1x_1 + w_2x_2 + \cdots + w_nx_n = W.\bar{x}. \qquad \cdots \quad (1)$$

In the displaced position let $x'_1, x'_2, \ldots\ldots\ldots x'_n$ be the distances of the particles and $\bar{x}'$ that of their C.G. from the same fixed plane, so that

$$\bar{x}' = \frac{w_1x'_1 + w_2x'_2 + \cdots + w_nx'_n}{w_1 + w_2 + \cdots + w_n}$$

$$\text{or,} \quad w_1x'_1 + w_2x'_2 + \cdots + w_nx'_n = W.\bar{x}'. \qquad \cdots \quad (2)$$

Subtracting (1) from (2),

$$w_1(x'_1 - x_1) + w_2(x'_2 - x_2) + \cdots + w_n(x'_n - x_n)$$
$$= W(\bar{x}' - \bar{x}) = Wh.$$

Since $x'_1 - x_1$, $x'_2 - x_2$,... are the heights through which the particles have been raised, left side represents the total work done in raising the particles.

Hence the result.

Note. It should be noted that in the above result the work done is quite independent of the path by which the particles are displaced from one position to another.

If the C.G. of the system is lowered, instead of being raised, h is negative, and so the work done against the weight is negative, in other words, positive work is done by the weight.

11·4. Power.

When an agent (say, a man or a machine or an engine) *is doing work continuously, the rate at which it does work per unit of time is defined to be its power.*

The unit of power in Statics is a *Foot-pound per second* in the F.P.S. system, and a *Gramme-centimetre per second* in the C.G.S. system.

The above unit, being very small, is not suitable for practical purposes ; so engineers use a higher unit called a Horse-power.

When an agent is doing work at the rate of 550 *foot-pounds per second, it is said to have one* **Horse-power.**

The word Horse-power is usually abbreviated into H.P.

Note. This estimate of the average power of a horse was arrived at by J. Watt by experiment.

11·5. Illustrative Examples.

Ex. *Find the Horse-power of an engine that would empty in* 48 *minutes a cylindrical well full of water, if the diameter of the well is* 14 *ft., its depth* 40 *ft., and if water is raised by pumping to a level ground* 70 *ft. above the surface of the well.* [*C. U. 1946*]

Volume of water = volume of the cylinder

$$= \pi \times 7^2 \times 40 = \tfrac{22}{7} \times 49 \times 40 \text{ cu. ft.}$$

Since one cubic foot of water weighs $62\frac{1}{2}$ pounds,

$\therefore$ its weight $= \frac{22}{7} \times 49 \times 40 \times 62\frac{1}{2}$ lbs wt.

$= 385000$ lbs. wt.

The C.G. of a solid cylinder is at half its height.

Hence, initially the height of the C.G. of the water above the bottom of the well $= 20$ ft., and finally it is $70+40$, *i.e.*, 110 ft.

$\therefore$ the height through which the C.G. of the water has been raised $= 110-20$, *i.e.*, 90 ft.

Hence, the work done $= 385000 \times 90$ ft. lbs.

Let x be the reqd. H.P.; then the work done by the engine in 48 minutes $= x \times 48 \times 60 \times 550$ ft.-lbs.

$$\therefore \quad x \times 48 \times 60 \times 550 = 385000 \times 90.$$

$$\therefore \quad x = \frac{385000 \times 90}{48 \times 60 \times 550} = \frac{175}{8} = 21\frac{7}{8}.$$

Examples on Chapter XI

1. Find how many foot-pounds of work is done in pushing a mass of 10 lbs. through 5 feet up a smooth incline of 1 in 10.

2. Show that the work done in drawing a body up a smooth inclined plane is equal to the work done in lifting the body through the height of the plane.

3. Find the work done in piling over one another five bricks, originally lying flat on the ground, having given that the thikness of a brick is 3 inches and its weight 10 lbs.

4. A load of one ton is suspended by a vertical chain 100 ft. long, the chain itself weighing 6 lbs. per foot. How many foot-pounds of work is done in winding up the load to the top?

5. A shaft, the horizontal section of which is a rectangle 10 ft. by 8 ft. is to be sunk 100 ft. into the earth. If the average weight of the soil is 150 lbs. per cubic foot, find the work done in bringing the soil to the surface.

[*P. U. 1935*]

*6. In digging a circular well of radius 3 ft. and of depth 20 ft., 12 ft. of clay and later 8 ft. of sand were taken out. Find the work done in raising the materials to the surface, assuming that one cubic foot of clay and one cubic foot of sand weigh a lbs. and b lbs. respectively. [*P. U. 1938*]

7. A tower is to be built of brick-work, the base being a rectangle whose external measurements are 20 feet by 10 ft., the height of the tower 132 ft., and the walls $2\frac{1}{2}$ ft. thick. Find the number of hours in which an engine of 3 H.P. would raise the bricks from the ground, the weight of a cubic foot of brick-work being 12 lbs.

[*C. U. 1942*]

8. There are 37 steps in a staircase, and on every step except the highest is placed a marble ball weighing 4 ounces. If each step be 8 inches high, find the work done in carrying all the balls to the top of the staircase.

9. A horse draws a carriage 11 miles along a road with a constant force of 42 lbs. wt. and takes 70 minutes to perform the journey. Compare his power with a horse-power.

10. The Darjeeling Mail has a maximum speed of 60 miles per hour. If the total resistance then be the weight of 1 ton, find the Horse-power of the engine.

[*C. U. 1932*]

11. What is the H.P. required for a motor-car which weighs 3000 lbs. and can run at 30 miles an hour against an air resistance equal to $\frac{1}{30}$th of its own weight ?

[*C. U. 1943*]

12. Calculate the H.P. of an engine which takes 90 minutes to pump out water from a rectangular well of length 20 ft., breadth 15 ft. and depth 100 ft. to the level of the top of the well. [One cubic foot of water weighs 62·5 lbs.] [*C. U. 1938*]

13. A well of which the section is a circle of diameter 14 ft. and depth 206 ft. is half full of water. Find the work done in foot-pounds in pumping out the water to a level 4 ft.

above the top of the well in 10 minutes, and calculate the average horse-power of the pumping machine. [*C. U. 1935*]

14. An engine of 12 H.P. working 8 hours a day supplies 2000 houses with water which it raises to an average height of 40 ft. Find the supply of water to each house.

15. A man whose weight is 11 stones climbs a pole at the rate of 15 inches per second. Show that he is working at just over $\frac{1}{3}$ H.P.

***16.** A cage containing coal of total weight W cwt. is being raised from the bottom of a coal-mine whose depth is d feet, with the help of a wire-rope weighing w lbs. per foot. If the work is done in t minutes, find the H.P. of the engine employed.

***17.** A solid homogeneous right circular cone whose height is h, radius r, specific gravity $s(>1)$ and weight W, is placed inside a vertical right circular cylinder of radius r, their bases being in contact. Water is poured into the cylinder up to the height so that the cone is just immersed. If P be the work done to raise the cone vertically so as to be just clear of the water, then

$$P=\frac{2}{3}Wh\left(1-\frac{7}{8s}\right).$$

ANSWERS

1. 5 ft.-lbs. **3.** 25 ft.-lbs. **4.** 254000 ft.-lbs.

5. 6×10^7 ft.-lbs. **6.** $72\pi(9a+16h)$ ft.-lbs.

7. 22 hours. **8.** 111 ft.-lbs. **9.** 132 ; 125.

10. $358\frac{2}{3}$ H.P. **11.** 8 H.P. **12.** $142\frac{1}{17}$ H.P.

13. $157132937\frac{1}{2}$ ft.-lbs. ; $476\frac{77}{480}$ H.P. **14.** 2376 lbs. daily.

16. $\dfrac{224W+w\dfrac{d}{2}}{66000}\cdot\dfrac{d}{t}$ H.P.

CHAPTER XII

MACHINES

12·1. Machine and its use.

Any contrivance, or arrangement of bodies fitted together, so as to be in a convenient form to apply force at one point in order to overcome a resisting force acting at another point, is called a *Machine.*

The former force is called the *Effort* (or *Power*), usually denoted by P, and the latter, *Resistance* (or *Weight*), usually denoted by W. In Statics we are chiefly concerned with finding the relation between the effort and the resistance when there is equilibrium.

That by using a machine we can counteract one force by another, differing from it in magnitude, point of application, or direction, or in all three, is evident from the following familiar cases.

For example, by using a *single pulley*, a bucket of water can be raised to a great height by a person standing on the ground. Here the effort, though equal to the weight to be raised, is applied more conveniently as a downward force. Again, by means of an *Inclined Plane*, a heavy body can easily be raised through a great distance, the effort required being less than the weight raised. A *pair of Tongues* enables us to apply a force in a more convenient form, though the effort is greater than the resistance in this case.

In the present chapter we shall discuss the working and properties of some simple types of useful machines :

(i) the System of Pulleys,

(ii) the Lever,

(iii) the Common Balance,

(iv) the Steelyards (Roman and Danish),

(v) the Wheel and Axle.

The principle and use of an inclined plane has already been illustrated in many examples in the previous chapters, and accordingly we need not deal with it here separately.

In the following discussions, for the sake of simplicity, we shall suppose that the machines are perfectly smooth and rigid, and all ropes and strings used in their working are light, inextensible and flexible.

12·2. Principle of Work.

In the working of a machine two kinds of resistances are overcome *viz.*, (1) those which the machine is specially designed to overcome and (2) those which are due to the internal adjustment of the different parts of the machine *e g.*, friction and weights of the different parts of the machine. The former are called *useful*, and the latter *wasteful* resistances. It should be noted however that wasteful resistances can never be wholly eliminated even in the case of most delicate and highly finished machines. In elementary investigation of simple machines, the wasteful resistances are usually ignored, and it is the effort which balances the weight in such a machine Hence, the general principle of work in Statics, in this particular case for a machine, can be stated as follows :

If in a machine, friction and weights of component parts are neglected, the work done by the effort for any assumed displacement of the system, is always equivalent to the work done against the resistance.

This principle may sometimes be used to work out the relation between the effort and the resistance in a machine, as will be illustrated later.

12·3. Mechanical Advantage, Velocity-ratio, and Efficiency.

(i) The ratio of the two forces, Resistance and Effort exerted on a machine to balance one another, is called the

Mechanical Advantage (or *Force-ratio*) of the machine. Thus,

$$\textbf{Mechanical Advantage} = \frac{\textbf{Resistance}}{\textbf{Effort}} = \frac{\mathbf{W}}{\mathbf{P}}$$

and Resistance = Effort × mechanical advantage.

Almost all machines are so constructed that the effort exerted is less than the resistance overcome. Hence, mechanical advantage is usually greater than unity. But there are machines, as already mentioned, for which the effort is equal to (as in case of a single pulley) or sometimes greater than (as in case of a pair of tongues) the resistance, and this really amounts to a case of mechanical disadvantage. Mechanical advantage is often abbreviated as M.A.

(ii) When a machine is worked, if u and v are the velocities, and x and y are the displacements of the points of application of the effort and resistance during a given time, then $u : v$ is defined as the *velocity-ratio* of the machine.

Obviously, $u : v = x : y$.

$$\therefore \quad \textit{velocity-ratio} = \frac{\textit{Distance through which } P \textit{ moves}}{\textit{Distance through which } W \textit{ moves}}.$$

From the Principle of work, we have

$P \times$ distance through which P moves

$= W \times$ distance through which W moves.

$$\therefore \quad \frac{W}{P} = \frac{\text{Distance through which } P \text{ moves}}{\text{Distance through which } W \text{ moves}}.$$

Thus, in an ideal machine whose parts are weightless and in which there is no friction,

Mechanical Advantage = Velocity-Ratio.

(iii) In practical machines, where there is friction, or other wasteful resistances, the effort will have to do some work in overcoming these, *i.e.*, the work done by P will exceed that done against W. The work done by the moving forces in overcoming useful resistance is called *useful work*, and the work done in overcoming wasteful resistance is termed *lost work*.

The **Efficiency** of a machine is measured by the ratio

$$\frac{\textit{useful work done by the machine}}{\textit{work supplied to the machine}}.$$

Efficiency is usually less than one, and is often expressed as a percentage, but, in an ideal machine where there is no friction etc., efficiency is unity.

If x and y are the distances moved through by the points of application of P and W respectively,

$$\text{Efficiency} = \frac{Wy}{Px} = \frac{W}{P} \Big/ \frac{x}{y} = \frac{\text{Mechanical Advantage}}{\text{Velocity-ratio}},$$

i.e., in general,

Mechanical Advantage = *Velocity-ratio* × *Efficiency*.

I. PULLEYS

12·4. A **Pulley** consists of a circular plate with a groove cut along its circumference so as to receive a string and to prevent it from slipping off. It can turn round freely about an axle passing through its centre and perpendicular to its plane, the ends of this axle being held by a frame called the *block*. A pulley is said to be *fixed* or *movable* according as the supporting block is fixed or movable. When the weight of a pulley is found very small in comparison with the weight it supports, it is neglected, and in such a case the pulley is often called a weightless pulley. The weight of the string that passes round the pulley, being very small, will always be neglected and the pulley will be considered to be perfectly smooth, so that the tension of the string passing round it is constant throughout its length.

12·5. Single fixed pulley.

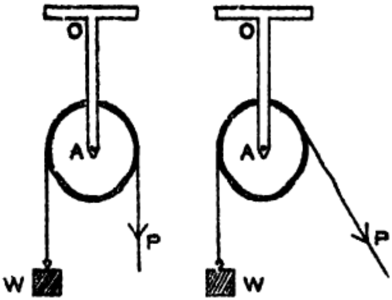


Fig. (i) Fig. (ii)

In this case the weight W is fastened to one end of the string while the effort P is applied at the other end. The portions of the string on the two sides may be parallel as in fig. (i), or inclined to each other as in fig. (ii).

In both the cases, for equilibrium position, $W = P$, since each is equal to the tension of the string.

$$\text{Mechanical advantage} = \frac{W}{P} = 1.$$

Thus, in this case, the effort exerted is equal to the weight overcome ; hence there is no mechanical advantage. The only advantage is that it enables us to apply the force in a convenient direction.

If the pulley be weightless,

in fig. (i), pressure on the fixed support

$$= P + W = W + W = 2W,$$

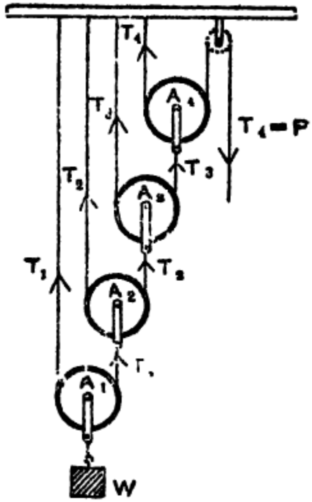
in fig. (ii), pressure on the fixed support

$$= P \cos\theta + W \cos\theta = 2W \cos\theta,$$

where 2θ is angle between the direction of P and W.

Note. That $W = P$ can also be shown by taking moment about the centre of the pulley.

12·6. First System of Pulleys. *(Separate-string system).*



In this system, there is a number of movable pulleys each of which is supported by a separate string passing below it, one end of which is attached to a fixed support, and the other end, except for the string round the highest pulley, is attached to the block of the next higher pulley. Effort is applied to the free end of the last string passing round the highest pulley. The weight is suspended from the block of the lowest pulley.

In order to apply the effort as a downward force, an additional pulley is very often kept fixed in the supporting beam over which the free end of the string passes. It should be noted that this pulley does not form an essential part of the main system in the sense that it does not contribute anything to the mechanical advantage.

Relation between the Effort (or Power) and the Weight.

Let A_1, A_2,.... be the pulleys beginning from the lowest, T_1, T_2,............ be the tensions of the strings passing round them and let W be the weight and P the power. Let us suppose the portions of strings not in contact with the pulleys to be vertical. The string passing round any pulley pulls it vertically *upwards* on either side, and the next higher, vertically *downwards*.

Case I. Let the weights of the pulleys be neglected.

Consider the equilibrium of the pulley A_1; the forces acting on it are the two upward tensions each equal to T_1 on either side of it, and the weight W acting downwards.

$$\therefore \quad 2T_1 = W. \qquad \therefore \quad T_1 = \tfrac{1}{2}W.$$

Since the forces acting on the pulley A_2 are the two upward tensions, each equal to T_2, and a downward tension equal to T_1,

hence as before,

$$2T_2 = T_1. \qquad \therefore \quad T_2 = \tfrac{1}{2}T_1 = \frac{1}{2^2}W.$$

Similarly,

$$2T_3 = T_2. \qquad \therefore \quad T_3 = \tfrac{1}{2}T_2 = \frac{1}{2^3}W.$$

$$2T_4 = T_3. \qquad \therefore \quad T_4 = \tfrac{1}{2}T_3 = \frac{1}{2^4}W.$$

If we have 4 movable pulleys, as in the figure,

$$T_4 = P. \qquad \therefore \quad P = \frac{1}{2^4}W.$$

Similarly, if there be n movable pulleys, we shall have

$$P = \frac{1}{2^n}W.$$

$\therefore$ *mechanical advantage* $= \dfrac{\mathbf{W}}{\mathbf{P}} = 2^n$

which obviously increases with the number of pulleys.

Case II. Weights of the pulleys considered.

Let $w_1, w_2, \ldots$ be the weights of the pulleys $A_1, A_2, \ldots\ldots$ Considering the equilibrium of the pulleys $A_1, A_2, \ldots\ldots\ldots$ if we have n pulleys, then,

$$2T_1 = W + w_1,$$
$$2T_2 = T_1 + w_2,$$
$$2T_3 = T_2 + w_3.$$

$$2T_n = T_{n-1} + w_n$$

and lastly, for the free end of the highest string

$$P = T_n.$$

Multiplying the equations successively by $1, 2, 2^2, 2^3, \ldots 2^n$ and adding, we have ultimately

$$2^n P = W + (w_1 + 2w_2 + 2^2 w_3 + \cdots + 2^{n-1} w_n)$$

which gives the relation between P and W.

If the pulleys be all equal, each of weight w,

$$2^n P = W + (1 + 2 + 2^2 + \cdots + 2^{n-1})\, w$$
$$= W + (2^n - 1)\, w.$$

Hence it follows that the mechanical advantage W/P depends upon the weights of the pulleys.) 4

Note 1. From the above equation, it is clear that the greater the weights of the pulleys, the greater must be P to raise a given weight W, and so the mechanical advantage would be diminished. Hence pulleys should be made as light as possible.

Note 2. This system is called *separate-string system* because each pulley in this case has got a separate string passing round it.

Application of the Principle of Work.

The above relation between P and W can also be deduced from the principle of work.

Suppose the end of the string to which P is applied moves through a distance x in the direction of P. By this, it is easily seen that the uppermost movable pulley would be raised through a height $\frac{1}{2}x$, the next lower pulley through a height $\frac{1}{2^2}x$, and so on, the lowest pulley and weight being raised through a height $\frac{1}{2^n}x$, in case of n movable pulleys. Hence, from the principle of work (when weights of the pulleys are neglected)

$$P.x = W \cdot \frac{1}{2^n}\, x, \quad \textit{i.e.,} \quad \mathbf{P} = \frac{\mathbf{1}}{\mathbf{2^n}} \cdot \mathbf{W}.$$

If the weights of the pulleys are taken into consideration,

$$P.x = W \cdot \frac{x}{2^n} + w_1 \cdot \frac{x}{2^n} + w_2 \cdot \frac{x}{2^{n-1}} + \cdots\cdots + w_n \cdot \frac{x}{2} \cdot$$
$$2^n P = W + (w_1 + 2w_2 + 2^2 w_3 + \cdots\cdots + 2^{n-1} w_n).$$

12·7. Second System of Pulleys. *(Single-string system)*

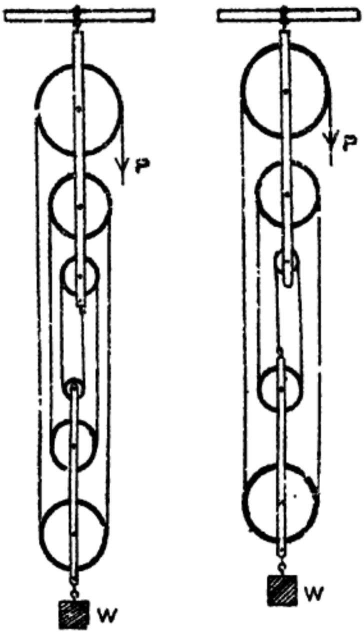


Fig. (i) Fig. (ii)

In this system there are two blocks, each containing pulleys, the upper block being fixed to a support and the lower block, which has the weight to be raised attached to it, being movable. The same string passes round all the pulleys. If the total number of pulleys be even, divided into equal numbers in each block [as in Fig. (i)], the string must be fastened to the upper block ; but if the total number of pulleys be odd, the number in the upper block will be one greater than the number in the lower block [as in Fig. (ii)], and the string must be attached to the lower block. In both the cases, the string passes alternatively over a fixed pulley in the upper block and under a movable pulley in the lower block, the radii of different pulleys being such that the portions of the string not in contact with a pulley are vertical. The effort is applied as a downward force at the free end of the string after it passes over the topmost pulley,

Relation between Effort and Weight.

Let W be the weight supported and w the weight of the lower block with its pulleys.

It is easily seen that if n be the total number of pulleys used in the system, whether n be odd or even, there will be n portions of string supporting the lower block. Since the same string passes round all the pulleys which are smooth. the tension in each portion of the string is the same, being equal to the effort P applied at the free end. Since the lower block is supported by n parallel forces each equal to P, we have

$$W + w = nP.$$

When the weights of the pulleys are neglected,

$$W = nP.$$

Hence, in this case, the *mechanical advantage* $= \frac{\mathbf{W}}{\mathbf{P}} = \mathbf{n}$.

Application of the Principle of Work.

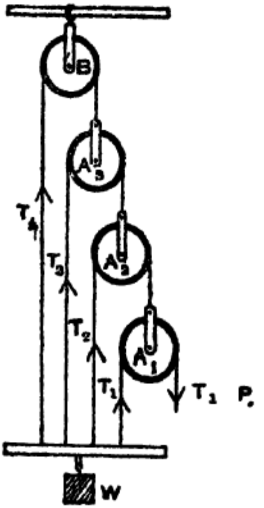
Suppose the weight W (and consequently the whole of the lower block) is raised through a distance x. Then each of the n portions of the string supporting the lower block will be slackened by the length x, and the total length of the string slackened being nx, P must descend through a distance nx to keep the string taut.

$\therefore$ by the principle of work,

$$(W + w)x = P.nx, \quad i.e., \quad W + w = nP.$$

Note. This system is called ***single-string system*** because a single string passes round all the pulleys.

12·8. Third System of Pulleys. (*Inverted separate-string System*).



This system consists of several pulleys of which the uppermost is fixed to a beam, and all the others are movable. The string passing over any pulley has one end attached to a bar from which the weight is suspended, and the other end attached to the next lower pulley. The effort is applied at the free end of the string passing over the lowest pulley.

Relation between Effort and Weight.

Let A_1, A_2, A_3 be the movable pulleys beginning from the lowest, and B the fixed pulley, and let T_1, T_2, T_3, T_4 be the tensions of the strings passing round them. Also let W be the weight, and P the effort. We shall suppose all portions of the strings not in contact with the pulleys to be vertical, and that there is no friction.

Case I. Weights of the pulleys neglected.

From the equilibrium of the system, considering the free end, and also the pulleys A_1, A_2,..., we have

$$T_1 = P$$

and $$T_2 = 2T_1 = 2P$$

$$T_3 = 2T_2 = 2^2 P$$

$$T_4 = 2T_3 = 2^3 P.$$

Again, from the equilibrium of the bar from which the weight is suspended, (the bar being supposed of negligible weight),

$$W = T_1 + T_2 + T_3 + T_4$$
$$= P + 2P + 2^2P + 2^3P$$
$$= (1 + 2 + 2^2 + 2^3)\,P = (2^4 - 1)\,P.$$

If there are n pulleys, of which the upper one is fixed, and $(n-1)$ movable, we have similarly,

$$W = T_1 + T_2 + T_3 + \cdots + T_n$$
$$= P + 2P + 2^2P + \cdots + 2^{n-1}P$$
$$= (1 + 2 + 2^2 + \cdots + 2^{n-1})P$$
$$= (2^n - 1)\,P,$$

by summing the series which is a G.P.

$$\therefore \quad \textit{mechanical advantage} = \frac{\mathrm{W}}{\mathrm{P}} = 2^n - 1$$

which obviously increases with the number of pulleys.

Case II. Weights of the pulleys considered.

Let the weights of the movable pulleys A_1, A_2,... be w_1, w_2,... respectively.

Considering the effort at the free end, and also the equilibrium of the pulleys in succession, we have

$$T_1 = P$$
$$T_2 = 2T_1 + w_1 = 2P + w_1$$
$$T_3 = 2T_2 + w_2 = 2^2P + 2w_1 + w_2$$
$$T_4 = 2T_3 + w_3 = 2^3P + 2^2w_1 + 2w_2 + w_3.$$

From the equilibrium of the bar,

$$W = T_1 + T_2 + T_3 + T_4$$
$$= (1 + 2 + 2^2 + 2^3)P + (1 + 2 + 2^2)w_1 + (1 + 2)w_2 + w_3$$
$$= (2^4 - 1)P + (2^3 - 1)\,w_1 + (2^2 - 1)\,w_2 + w_3.$$

If there are n pulleys of which the upper one is fixed, and the rest movable, we have similarly,

$$W = T_1 + T_2 + \cdots T_n$$
$$= (1+2+2^2+\cdots+2^{n-1})P + (1+2+2^2+\cdots+2^{n-2})w_1$$
$$+ (1+2+2^2+\cdots+2^{n-3})w_2 + \cdots + (1+2)w_{n-2} + w_{n-1}$$
$$= (2^n-1)P + (2^{n-1}-1)w_1 + (2^{n-2}-1)w_2 + \cdots$$
$$+ (2^2-1)w_{n-2} + (2-1)w_{n-1}.$$

If the pulleys be all equal, each being of weight w, so that $w_1 = w_2 = \cdots = w_{n-1} = w$,

$$W = (2^n-1)P + \{2+2^2+\cdots+2^{n-1}-(n-1)\}w$$
$$= (2^n-1)P + \{2^n-n-1\}w$$

by summing the series in G.P.

Note 1. From the above equation it is clear that the greater the weights of pulleys, the smaller is the effort P required for a given weight W.

Note 2. In this system, unless the point in the bar from which the weight is suspended is properly chosen, the bar will not remain horizontal. In any particular case, the point can be easily determined.

Note 3. As in the case of the first and second system of pulleys, in this case also the relation between effort and weight can be obtained by the principle of work.

12·9. Illustrative Examples.

Ex. 1. *A "first system" of pulleys consists of 4 pulleys, each of weight 8 lbs., and the string passing round the top-most pulley passes over a fixed pulley. With what force must a man of weight 220 lbs. pull at the free end of the string in order to balance himself, suspended from the lowest pulley?* [*C. U. 1945*]

The man being suspended from the lowest pulley, and himself pulling at the free end of the string, let P lbs. wt. be the pull exerted at the free end, and W lbs. wt. the downward force exerted by him at the lowest pulley. The reactions at these two ends balance his total weight, so that

$$P+W=220. \qquad \cdots \quad \text{(i)}$$

Again, P and W clearly serve as the effort and the weight balanced by the system of pulleys in this case, and as the pulleys have equal weights, we get as in Art. 12·6, Case II,

$$2^4.P=W+(2^4-1).8,$$

i.e., $$16P=W+120. \qquad \cdots \quad \text{(ii)}$$

From (i) and (ii), we get

$$17P=340, \quad \text{or,} \quad P=20 \text{ lbs. wt.},$$

giving the required pull at the free end.

Ex. 2. *A man whose weight is 154 lbs. raises a body of 3 cwt. by means of a system of pulleys in which the same rope passes round all the pulleys, there being four in each block, and the rope being attached to the upper block. Neglecting the weights of the pulleys, find what will be his thrust on the ground if he pulls vertically downwards* [*C. U. 1914*]

Here we have the "second system" of pulleys.

The number of strings at the lower block $=2\times4=8$.

Since the weights of the pulleys are neglected, if P be the effort,

$$8P=3 \text{ cwt.}=3\times112 \text{ lbs. wt.}$$

$$\therefore \qquad P=42 \text{ lbs. wt.}$$

The thrust of the man on the ground is clearly the difference between his weight and the pull he exerts.

$\therefore$ the reqd. thrust $=154$ lbs. wt. -42 lbs. wt. $=112$ lbs. wt.

Ex. 3. *In the "third system" of pulleys, if the weight supported be 56 lbs., each movable pulley, of which there are 3, weighs 1 lb., and the radius of each pulley including the fixed one be a, find the point in the bar from which the weight must be suspended in order that the bar may remain horizontal.*

Taking the figure of Art. 12·8, let K, L, M, N be the points of attachment of the strings in the bar beginning with the longest (extreme left), and X the point from which the weight is suspended. Obviously, $KL = LM = MN = a$. Now, as in Art. 12·8, Case II,

$$56 \text{ lbs. wt.} = W = T_1 + T_2 + T_3 + T_4$$

$$= (2^4 - 1)P + (2^3 - 1)\,w_1 + (2^2 - 1)\,w_2 + w_3$$

$$= 15P + (7 + 3 + 1) \text{ lbs. wt., since } w_1 = w_2 = w_3 = 1 \text{ lb. wt.}$$

$$\therefore \quad 15P = 45 \text{ lbs. wt.} \qquad \therefore \quad P = 3 \text{ lbs. wt.} \qquad \cdots \quad (1)$$

Thus, $T_1 = P = 3$ lbs. wt., $T_2 = 2T_1 + w_1 = 7$ lbs. wt.,

$$T_3 = 2T_2 + w_2 = 15 \text{ lbs. wt.}$$

Now, for the equilibrium of the rod, taking moment about K,

$$W.XK = T_3.a + T_2.2a + T_1.3a,$$

or, $$56.XK = (15 + 14 + 9).a = 38a.$$

$$\therefore \qquad XK = \tfrac{38}{56}a = \tfrac{19}{28}a.$$

Hence, the weight must be attached to a point in the bar at a distance $\frac{19}{28}a$ from the point of attachment of the longest string.

Examples on Chapter XII(a)

1. If in the first system of pulleys, the number of weightless pulleys be seven, find the weight which can be raised by an effort 16 lbs. weight. [*C. U. 1936*]

2. The number of movable pulleys in a first system is three and the sum of the power and weight is 90 lbs. If the pulleys are weightless, calculate the power. [*C. U. 1947*]

3. If in the first system of pulleys, the power $= 30$ lbs., the weight $= 16\frac{1}{56}$ cwt., and the weight of each pulley $= 2$ lbs., find the number of movable pulleys in the system.

4. In the system of pulleys in which each pulley hangs from a fixed support by a separate string, the weights of the three movable pulleys are 5, 3 and 1 lbs. respectively beginning from the lowest. What weight will a power of 5 lbs. weight support ?

5. In a system of pulleys in which each pulley hangs by a separate string, there are three pulleys of equal weight; the weight attached to the lowest is 32 lbs., and the power is 11 lbs. Find the weight of each pulley.

6. In raising the weight two inches by the first system of pulleys, five feet four inches of string passes through the hand. Find the number of the pulleys, assuming their weights to be negligible.

7. In the first system of pulleys, show that, whatever be the weights of pulleys, the equilibrium will not be affected by increasing the effort, load, and the weight of each pulley by the same amount.

8. In the first system of pulleys, if the weights of the n pulleys, reckoning from the one nearest to W, increase in a geometric progression, the common ratio of which is 2, prove that

$$P = \frac{W}{2^n} + \frac{w}{3}(2^n - 2^{-n}),$$

where w is the weight of the lowest pulley.

9. In the first system of pulleys, in which there are three movable pulleys, the weights of the pulleys beginning from the highest increase in arithmetical progression downwards, and a power P supports a weight W. The pulleys are then arranged in the reverse order, the highest being placed lowest, and it is found that the interchange of P and W maintains equilibrium. Prove that

$$3(W + P) = 2W_1,$$

where W_1 = total weight of the three pulleys. [*C. U. 1941*]

10. In the first system of pulleys, the weights of the n pulleys beginning with the highest are in A.P., and a power P supports a weight W; the pulleys are then reversed, the highest being placed lowest and so on, and now W and P when interchanged are in equilibrium. Show that

$$2W_1 = n(W + P),$$

where W_1 is the total weight of all the pulleys.

11. In the first system in which there are four movable pulleys, each of weight w, if P be the effort (supposed to act upwards) and R the stress on the beam, then

$$R = 15P - 11w.$$

12. If in the first system of pulleys, P is the power (acting upwards), W the weight, and R the stress on the beam from which the pulleys hang, show that

$$(1 - 2^{-n})\,W < R < (2^n - 1)\,P,$$

n being the number of pulleys in the system.

13. In the first system in which there are 4 weightless movable pulleys, a man of weight 10 stones hangs from the lowest pulley and supports himself by pulling at the end of the string which passes over a fixed pulley. With what force does he pull the string ?

If in the above case, the pulleys instead of being weightless, be all of the same weight 8 lbs., what would be the pull on the string ?

14. A man of weight 136 lbs. standing on the floor pulls at the lower end of the first system of 4 weightless pulleys. If the weight suspended be eight times the weight of the man, what is the pressure of his feet on the floor ?

15. If there be twelve pulleys divided equally between the two blocks in the second system of pulleys, find the weight which a power of 10 lbs. wt. will support, the weights of the pulleys being neglected.

16. A second system of pulleys has 5 pulleys in the upper block and 4 in the lower. How many times his own weight can a man raise by this machine, if each block weighs $\frac{1}{10}$th of his own weight ?

17. The cable by which Great Paul, the bell weighing 18 tons, was lifted to its place in the cathedral tower, passed four times through each of two blocks of pulleys of negligible weight. Find the strength of the cable.

18. In the second system of pulleys, a weight of 7 lbs. supports a weight of 30 lbs. and a weight of 9 lbs. just supports a weight of 44 lbs. Find the total number of pulleys in the system, and the weight of the lower block.

19. In the second system of pulleys, unless the ratio of the weight of the lower block to the suspended weight be less than the number of strings in the lower block diminished by unity, show that there is no mechanical advantage.

20. It is required to lift a weight of 10 cwt. with four pulleys each weighing 8 lbs. Would you prefer the first or the second system as being more advantageous ?

[*C. U. 1933*]

21. A man weighing 10 stones raises a load of 6 cwt. by means of a single string system of light pulleys, there being 6 pulleys in each block. Find the thrust of the man on the ground, and the stress on the supporting beam.

[*B. E. 1936, '40*]

22. By the second system of pulleys having three pulleys in the lower block and the string attached to that block a man standing on the ground supports a weight of 6 stones (including that of the lower block and the pulleys), and the pressure on the ground exerted by his feet is 128 lbs. wt. Find the maximum additional weight he can support.

23. A man standing on the ground raises a weight of 1 ton by means of two blocks, each containing three pulleys, and each block, with the pulleys on it, weighs 10 lbs. Find the thrust on the beam from which the upper block is suspended, and the least weight of the man.

24. In the second system of pulleys, a platform is suspended from the lower block. A man of weight W, standing on the platform, supports himself by exerting on the string a force equal to P. If n be the total number of pulleys in the system, and mW the weight of the platform and the lower block together, show that

$$\frac{W}{P}=\frac{n+1}{m+1}.$$

25. Draw a system of pulleys with parallel strings by means of which a force may balance a weight seven times as great. [*C. U. 1923*]

26. In raising a weight by (i) the first system, (ii) the third system of pulleys, which is the more advantageous, to have the pulleys heavy or light ?

27. There is one system of pulleys in which as the weights of the pulleys increase, the mechanical advantage increases. What is that system ?

28. If in the third system there are three movable pulleys such that the weight of each is equal to the power, show that the power will support a weight 26 times as great as itself.

29. In the third system in which there are three movable pulleys of weights 1 lb., 2 lbs., 3 lbs., respectively ; find the greatest and the least weight which can be kept in equilibrium by the power of 10 lbs. wt., the pulleys being arranged in order.

30. In the third system in which there are four pulleys of equal size (of which one is fixed), each of weight 1 lb., find the effort required to support a weight of 161 lbs. Also find to what point of the bar the weight must be attached, so that the beam may remain horizontal.

***31.** In the third system there are n weightless pulleys each of radius a. Show that the distance of the point of application of the weight from the line of action of the effort is $\frac{2^n}{2^n-1}\cdot na$.

32. In the third system of weightless pulleys, if the free end of the string round the lowest pulley be attached to the bar from which the weight is suspended, show that the tension of the string is diminished in the ratio $2^n - 1 : 2^n$.

***33.** If the weight of the lowest pulley, in that system of pulleys in which all the strings, n in number, are attached to the weight, be equal to the power P, of the next lowest, to $3P$, and so on, that of the highest movable pulley being $3^{n-2} P$, prove that $W : P = 3^n - 1 : 2$.

***34.** There are three movable pulleys of weights w_1, w_2, w_3 in the third system, and the force P then balances a load W; when the first and second pulleys are interchanged, then a force P' balances the same load. Show that

$$\frac{P - P'}{w_2 - w_1} = \frac{4}{15}.$$

***35.** A man weighing 126 lbs. supports a weight of 106 lbs. by means of four pulleys of which one is fixed, in the third system. Find his thurst on the ground if the masses of the movable pulleys beginning from the lowest are 1, 2 and 3 lbs. respectively. [*C. U. 1940*]

ANSWERS

1. 2048 lbs. wt. **2.** 10 lbs. wt. **3.** 6. **4.** 25 lbs. wt.

5. 8 lbs. wt. **6.** 5. **13.** $8\frac{4}{17}$ lbs. wt. ; $15\frac{3}{17}$ lbs. wt.

14. 68 lbs. wt. **15.** 120 lbs. **16.** $8\frac{9}{10}$ times his own weight.

17. $2\frac{1}{2}$ tons wt. **18.** 7 ; 19 lbs. wt. **20.** First system.

21. 84 lbs. wt. ; 728 lbs. wt. **22.** 64 stones.

23. 2635 lbs. wt. ; 375 lbs. wt. **25.** Second system, with 7 pulleys, 4 in the upper and 3 in the lower block, or third system with 8 pulleys.

26. Light in the first system, and heavy in the third system.

27. Third system. **29.** 178 lbs., 166 lbs.

30. 10 lbs. wt. ; the point required divides the distance between the first two strings (passing over the two topmost pulleys) in the ratio 5 : 2.

35. 120 lbs. wt.

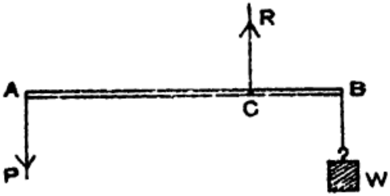
II. LEVERS

12·10. A **Lever** is a rigid rod, straight or curved, movable in one plane about a fixed point in the rod. The fixed point is called the *fulcrum*, and the parts of the lever between the fulcrum and the points of application of the effort and the weight are called the *arms* of the lever.

When the arms are in the same straight line, the lever is called a *straight lever* ; in other cases, it is called a *bent lever.*

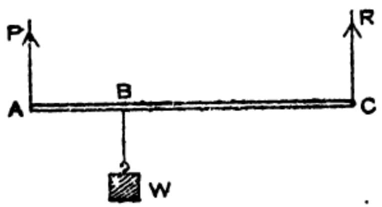
Straight levers are usually divided into three classes according to the positions of the points of application of the effort and the weight with respect to the fulcrum.

Class. I. In levers of the first class the effort P and the weight W act on opposite sides of the fulcrum C.



A crow-bar used to raise a heavy weight, a poke used to raise coals in a grate, etc. are levers of the first class ; and scissors, pincers etc. are double levers of the first class.

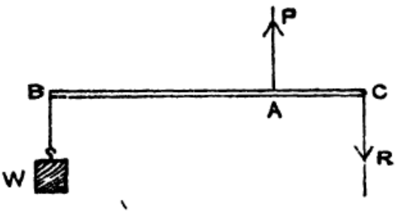
Class II. In levers of the second class the effort P and



the weight W act on the same side of fulcrum C, the weight being nearer the fulcrum.

The oar of a boat is a lever of the second class and a pair of nut-crackers is a double lever of this class.

Class III. In levers of the third class the effort P and the weight W act on the same side of the fulcrum C, the effort being nearer to the fulcrum.



An example of a third class lever is seen in the human forearm raising an object placed on the palm of the hand, the effort being in this case the tenson in the ligament near the joint; a pair of tongues is a double lever of this type.

Equilibrium condition and mechanical advantage of a straight Lever.

If the weight of the lever is neglected, then in each of the above three cases, the lever is in equilibrium under the action of three forces, the effort P, the weight W and the reaction R at the fulcrum. Hence R must be equal and opposite to the resultant of P and W.

In Class I, $\quad R = P + W.$

In Class II, $\quad R = W - P.$

In Class III. $\quad R = P - W.$

Again, as the resultant of the parallel forces P and W acts through C,

$$P.AC = W.BC.$$

$$\therefore \quad \textit{mechanical advantage} = \frac{W}{P} = \frac{AC}{BC}.$$

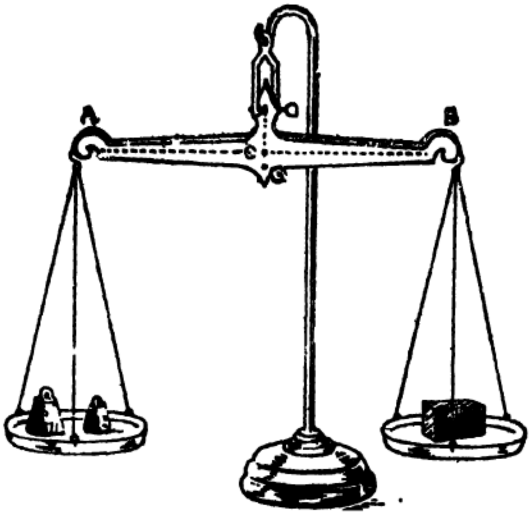
Thus, the levers of Class I generally and those of Class II always have got mechanical advantage, whereas the levers of Class III have got mechanical disadvantage. The levers of third class are used to apply force at a point where the direct application of the force is not convenient.

If the weight of the lever is taken into consideration, the equation for equilibrium may be written by taking moment about the fulcrum, the moment of the weight of the lever being also added.

III. THE COMMON BALANCE

12·11. The Common Balance.

The common balance is an instrument for determining the weights of bodies. It consists of a straight uniform beam AB, having two scale-pans of equal weight suspended from the two ends, and turning freely about a fulcrum O outside the beam but rigidly connected to it.



In a perfect balance the fulcrum and the centre of gravity G of the beam (with its connected parts) both lie on the line which bisects the beam perpendicularly, so that when the beam is horizontal, O, G and the mid-point C of the beam are in the same vertical line. AC and BC are called the arms of the balance.

The beam is horizontal when no weights or equal weights are placed on the scale-pans. The body to be weighed is placed in one of the scale-pans, and weights of known magnitudes are placed in the other till the beam is horizontal. If the balance be true (*i.e.*, perfect), the sum of the known weights gives the weight of the body.

Note. The common balance is a lever of the first class.

12·12. Requisites of a good balance.

The requisites of a good balance are :

(i) it must be *true*, *i.e.*, the beam should remain horizontal when no weight, as well as equal weights are placed in the scale-pans.

For this, it is necessary that

(*a*) the arms of the balance must be exactly equal,

(*b*) the weights of the scale-pans must be equal,

(*c*) the C.G. of the beam including the rigid connections must be on the line through the fulcrum perpendicular to the beam.

To test the truth of a balance, we first see that the beam is horizontal when the pans are empty. Next, a body is placed in one scale-pan and such weights are placed in the other that the beam is horizontal ; now, if the contents of the pans being interchanged, the beam is still found to be horizontal, the balance must be true. If in the second case, the beam is not horizontal, the balance is said to be false.

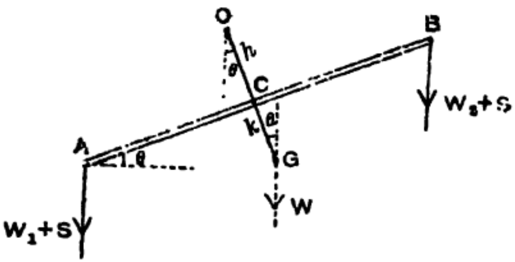
(ii) it must be *sensitive*, *i.e.*, for a very small difference in the weights of the contents of the scale-pans, the beam should be inclined to the horizontal at a perceptible angle ;

(ii) it must be *stable, i.e.*, the beam, with the pans empty, must quickly return to its horizontal position, when disturbed.

Note 1. In a good balance a rod or a pointer called the tongue is attached perpendicular to the beam at its middle point, and as the beam oscillates, the pointer moves against a graduated scale. Since it is vertical when the beam is horizontal, by its help the determination of sensitiveness and stability of a balance becomes easier.

Note 2. A balance is said to be *faulty in every respect*, if it is not true in every respect *i.e.*, (i) if its arms are unequal in length, (ii) its scale-pans are unequal in weight, and (iii) the C.G. of the machine is not on the perpendicular from the fulcrum on the beam.

12·13. Position of equilibrium of a balance with unequal weights in the scale-pans.



Let C be the middle-point of the beam AB, G, the C.G. of the beam with its rigid connections, and O the fulcrum. Let $OC=h$, $CG=k$.

Let $S=$ weight of each scale-pan

$W=$ weight of the beam

W_1, $W_2=$ weights placed in the pans at A and B respectively, and let $W_1 > W_2$.

$a=$ length of each arm, so that $AC=CB=a$.

Let θ be the inclination of the beam to the horizon in the position of equilibrium.

The horizontal distances of A, B and G from O in this position are easily seen from figure to be $a \cos\theta - h \sin\theta$, $a \cos\theta + h \sin\theta$ and $(h+k) \sin\theta$ respectively

The beam is acted upon by the following forces :

$W_1 + S$, $W_2 + S$, vertically downwards at A and B, W vertically downwards at G, and the vertical upward reaction at O.

Hence, for equilibrium, taking moments about O, we have

$$(W_1 + S)(a \cos\theta - h \sin\theta) = W(h+k) \sin\theta$$
$$+ (W_2 + S)(a \cos\theta + h \sin\theta),$$

$$\therefore\ (W_1 - W_2)\, a \cos\theta = \sin\theta\, \{W(h+k) + (W_1 + W_2 + 2S)h\}.$$

$$\therefore\quad \tan\theta = \frac{(W_1 - W_2)a}{W(h+k) + (W_1 + W_2 + 2S)h}.$$

Note 1. The result shows that if $W_1 = W_2$, $\theta = 0$, *i.e.*, the beam can rest only in a horizontal position.

Note 2. It should be noted that if h and k were both zero, *i.e.*, if the C.G. of the beam and the fulcrum coincided in the line AB, the beam could rest in any position when equal weights were put in the pans, and could rest only in a vertical position if the weights were different.

Note 3. For a given difference $W_1 - W_2$ of the weights on the pans, the greater the value of θ, the more sensitive is the balance. Thus, *for a balance to be sensitive*, a must be large, and h and k both small, *i.e., the arm should be long, and the fulcrum and the C.G. of the beam as near the beam as possible*, but not exactly coincident with the centre of the beam (see *Note* 2 *above*).

Note 4. If W_1 and W_2 be removed, while the inclination of the beam to the horizon is θ, the moment about O of the acting forces, tending to restore the beam to its horizontal position is, from the figure,

$$S(a \cos\theta + h \sin\theta) + W(h+k) \sin\theta - S(a \cos\theta - h \sin\theta)$$
$$= \sin\theta\, \{2hS + (h+k)\, W\}$$

and for this to be large, h and k should be large ; in other words, *a balance is stable for which h and k are large.* Thus, if a balance is more stable, it will be less sensitive, and vice versa.

12·14. Double weighing.

Method I. First place the body to be weighed in one scale-pan and in the other put suitable material (such as sand, brick-chips, etc.) sufficient to balance the body. Next remove the body, and in its place put weights of known magnitudes sufficient to balance the brick-chips. The weight of the body is obviously the sum of the weights.

This is known as *Borda's method* of double weighing.

Method II. The weight of a body is observed by placing it successively in the two scale-pans. If the weights are found to be exactly the same in both cases, the observed weight is the true weight of the body and the balance is true. This method enables us to test the truth of a balance.

IV. STEELYARDS

12·15. The **Steelyards** are also levers of the first kind with graduation marks on them, used for weighing goods, in which the necessity of keeping a number of weights is obviated. There are two kinds in common use :

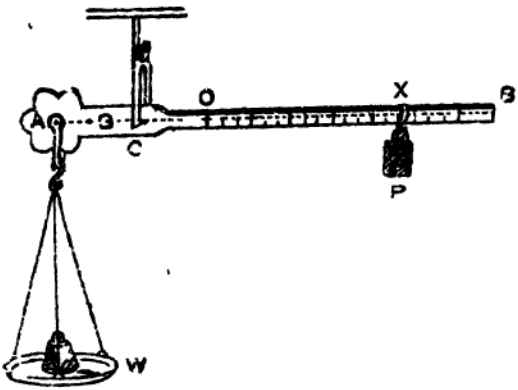
(i) the Common (or Roman) Steelyards, having a fixed fulcrum,

(ii) the Danish Steelyards, having a movable fulcrum.

12·16. The Common (or Roman) Steelyard.

It consists of a straight steel lever AB having a fulcrum at a fixed point C near one end A. At A there is a hook or a scale-pan in which the body to be weighed can be placed, and a movable weight P slides along the arm CB which has graduations marked on it. After an article has been placed in the scale-pan, the movable weight is shifted along CB until the beam is horizontal and the mark at X

where the movable weight rests, indicates the weight of the article.



Graduation.

Let W' be the weight of the steelyard and the scale-pan and let G be the point of the beam through which W' acts. The steelyard is usually constructed in such a way that its C.G. is on the shorter arm. When there is no weight in the scale-pan, let O be the position of the movable weight P for which the beam is horizontal. The mark of the graduation at O is then zero. Taking moment about C for this case,

$$P.OC = W'.GC. \qquad \cdots \quad \text{(i)}$$

Next put a weight W in the scale-pan, and let X be the new position of P for which the beam is horizontal. Then taking moment about C, we have,

$$P.XC = W.AC + W'.GC. \qquad \text{(ii)}$$

Subtracting (i) from (ii), we get

$$P.OX = W.CA.$$

$$\therefore \quad OX = \frac{W}{P}.CA. \qquad \text{(iii)}$$

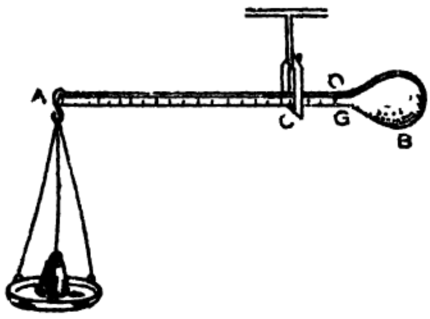
Now, if we measure off distances OX_1, OX_2, OX_3,... along OB, equal to CA, $2CA$, $3CA$,... and write 1, 2, 3,... for X_1, X_2, X_3,... these graduations give the points for which the weight of the body placed in the pan is P, $2P$, $3P$,...

It should be noted that the graduations are of equal length and if the movable weight P is taken as 1 lb. (or 1 kg.), the graduations obtained would indicate pounds (or kilogram). If smaller graduations are required, these divisions can again be divided into suitable sub-divisions.

Note 1. The distances of the successive graduations from the fulcrum are in A.P.

Note 2. *Weigh-Bridge* is a modified form of this machine. It is generally used in railway stations for measuring the weights of heavy

12·17. The Danish Steelyard.



The Danish steelyard consists of a lever AB whose fulcrum C is movable. At one end B, there is a lump of metal as a knob, and at the other end A there is a hook or a pan where the body to be weighed is placed. The beam is graduated and the weight of an article placed in the scale-pan is ascertained by observing the mark of graduation of the point at which the fulcrum must be placed so that the beam should rest horizontally.

Graduation.

Let P be the weight of the beam and the pan, acting through the point G of the steelyard. It is obvious that the zero graduation is at G, since the fulcrum must be at G, when the beam balances in a horizontal position without any weight in the scale-pan.

Let C be the position of the point where the fulcrum must rest when there is a weight $W = nP$ (say) in the scale-pan, and the beam balances horizontally.

Taking moment about C, we have

$$nP.AC = P.GC$$
$$= P.(AG - AC).$$
$$\therefore \quad AC = \frac{AG}{n+1}.$$

Thus, the successive graduations for $n = 1, 2, 3, \ldots$ etc. are at points $C_1, C_2, C_3, \ldots$, whose distances from A are

$$\tfrac{1}{2}AG, \tfrac{1}{3}AG, \tfrac{1}{4}AG, \ldots$$

If we mark 1, 2, 3,... for C_1, C_2, C_3, these graduations give the points for which the weights of the body on the pan are P, $2P$, $3P$,... respectively. If P be equal to 1 lb., the graduations indicate pounds.

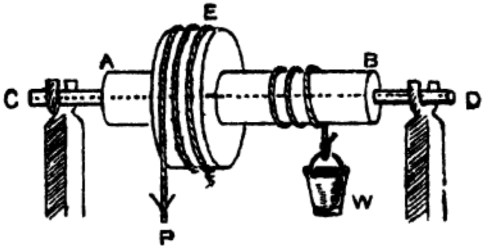
Note. Since, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ are in H.P., *the distances of the successive graduations from the point from which the scale-pan is suspended are in H.P.*

V. WHEEL AND AXLE

12·18. The Wheel and Axle.

This machine consists of the *axle AB*, in the form of a cylinder, capable of rotation about a fixed horizontal axis CD, and a *wheel EF*, rigidly attached to it, and therefore rotating about the same axis which passes through the centre of the wheel and is perpendicular to its plane. At the ends of the axis, there are two pivots C and D

resting in fixed sockets This machine is used for raising heavy bodies.



A string is wound round the circumference of the wheel with one end fixed to it ; effort P is applied at the other end of the string. The wheel is grooved along the circumference to prevent the rope from slipping off. Another string is coiled round the axle in the opposite direction with one end fixed to it ; the weight W is suspended from the other end of this string. When effort is applied, the point of application of P moves down and the string round the wheel is uncoiled, and that round the axle winds up, so that W is raised.

Mechanical Advantage.

Let a and $b\,(b > a)$ be the radii of the axle and the wheel respectively. Since the only forces (except the reaction on the axis) acting on the machine are P and W which tend to rotate it round the axis CD in opposite directions, for equilibrium, the sum of their moments round the axis is zero.

Hence, $\quad W.a - P.b = 0,$

i.e., $\quad W.a = P.b.$

$\therefore$ *mechanical advantage* =

$$\frac{\mathbf{W}}{\mathbf{P}} = \frac{\mathbf{b}}{\mathbf{a}} = \frac{\textit{radius of the wheel}}{\textit{radius of the axle}},$$

which is obviously greater than unity.

Note. Mechanical advantage can easily be increased by making the radius of the wheel larger and that of the axle smaller.

Verification of the Principle of Work.

When the wheel and the axle make one complete revolution, the length of the string uncoiled from the wheel is $2\pi b$ and the length of the string wound up round the axle is $2\pi a$. Hence, the point of application of P moves down through a distance $2\pi b$ and that of W moves up through a distance $2\pi a$.

$\therefore$ work done by the effort $= P \times 2\pi b$

and work done against the weight $= W \times 2\pi a$.

Hence, $W \times 2\pi a = P \times 2\pi b$. $\therefore$ $W a = P.b$

as otherwise shown above.

Note. Windlass, used for drawing water from a well, and Capstan, used on board a ship, are different forms of wheel and axle. In the former the axis is horizontal, and in the latter, the axis is vertical.

12·19. Illustrative Examples.

Ex. 1. *The arms of a balance are of unequal length, but the beam remains in a horizontal position when the scale-pans are not loaded. If a body be weighed, being placed in succession in the two scale-pans, show that its true weight is the geometric mean between its apparent weights.*

Since the beam remains horizontal when the scale pans are empty, the C.G. of the beam with the pans must be vertically below the fulcrum.

Let a and b be the lengths of the arms of the beam and let a body whose true weight is W appear to weigh W_1 and W_2 successively.

Then taking moment about the fulcrum,

for the first weighing, $W.a = W_1.b$ ... (1)

for the second weighing $W.b = W_2\,a.$ ... (2)

Multiplying (1) and (2), we have $W^2 = W_1 W_2$.

$\therefore$ $W = \sqrt{W_1 W_2}$.

Ex 2. *If the scale-pans are unequal in weight, but the balance is otherwise correct, find the real weight of a body which appears to weigh W_1 and W_2 when placed successively in the two scale-pans.*

Let S_1, S_2 be the weights of the scale-pans, a the length of either arm, and W the true weight of the body.

Taking moment about the fulcrum at the 1st weighing,

$$(W+S_1)a=(W_1+S_2)a. \qquad \cdots \quad (1)$$

Similarly, at the 2nd weighing,

$$(W+S_2)a=(W_2+S_1)a. \qquad \cdots \quad (2)$$

Adding (1) and (2), $2W=W_1+W_2$.

$$\therefore \quad W=\tfrac{1}{2}(W_1+W_2).$$

Ex. 3. *The arms of a balance are equal in length but the beam is unjustly loaded (i.e., the C.G. of the whole machine is not on the perpendicular from the fulcrum on the beam). Show that the true weight of the body is the arithmetic mean between its apparent weights when it is weighed being placed in succession in each scale-pan.*

Let a be the length of each arm and x the horizontal distance of the C G. of the machine from the fulcrum on the side of the body at the first weighing. Let W be the true weight of the body, W_1 and W_2 its apparent weights and w be the weight of the machine.

Then taking moment about the fulcrum, at the 1st weighing,

$$W_1a=Wa+wx.$$

Similarly, at the 2nd weighing,

$$W_2a+wx=Wa.$$

Adding, $(W_1+W_2)a=2Wa. \qquad \therefore \quad W=\tfrac{1}{2}(W_1+W_2).$

Ex. 4. *A grocer has a balance whose arms are 30 cm. and 36 cm. respectively, but which is otherwise correct. If he sells 10 Kg. of tea to a customer at Rs. 6 per Kg. by weighing half the quantity in one scale-pan and the other half in the other, find how much does he gain or lose by the transaction.*

Let W_1 and W_2 be the true weights of the quantity of tea which appear to weigh 5 Kg. at the time of first and second weighing.

Then, taking moment about the fulcrum,

$$W_1.30=5.36\,; \qquad W_2.36=5.30$$

$$\therefore \quad W_1=6\,; \qquad \therefore \quad W_2=4\tfrac{1}{6}.$$

$$\therefore \quad W_1+W_2=6+4.\tfrac{1}{6}=10\tfrac{1}{6}.$$

$\therefore$ the grocer really gives the customer $10\frac{1}{6}$ Kg. of tea and receives the price for 10 Kg.

$\therefore$ he loses by the transaction the price of $\frac{1}{6}$ Kg. of tea *i.e.*, he loses $\frac{1}{6} \times Rs.\ 6 =$ Re. 1.

Ex. 5. *A shopkeeper using a common steelyard, alters the movable weight for which it has been graduated. Does he cheat himself or his customers ?* [*P. U. 1938*]

From the fig of Art. 12'16, we have, when the machine is correct,

$$W.CA + W'.CG = P.CX,$$

where W is the weight of the body placed in the scale pan and P the movable weight. If the shopkeeper increases P, the right side of the above equation becomes increased. Hence the left side, and therefore, W is increased. But W was the quantity corresponding to the marked graduation. Hence where P is increased, the shopkeeper cheats himself.

Similarly, if P is decreased, he cheats his customers.

Ex. 6. *If in a Danish steelyard, a_n be the distance of the fulcrum from the extremity from which the weight of n lbs. is suspended, show that*

$$\frac{1}{a_n} + \frac{1}{a_{n+2}} = \frac{2}{a_{n+1}}.$$ [*C. U. 1938*]

Here a_{n+1}, a_{n+2} are distances of the fulcrum from the extremity carrying the scale-pan, when masses of $(n+1)$ lbs. and $(n+2)$ lbs. are placed in the pan. Then from the fig. of Art. 12.17, if P be the weight of the machine, taking moment about the positiom of the fulcrum in the first case,

$$n.a_n = P.GC = P(AG - a_n).$$

$$\therefore \quad a_n = \frac{P}{P+n} \cdot AG.$$

$$\therefore \quad \frac{1}{a_n} = \frac{1}{AG} \cdot \frac{P+n}{P}. \qquad (1)$$

Similarly, $$\frac{1}{a_{n+1}} = \frac{1}{AG} \cdot \frac{P+(n+1)}{P} \qquad (2)$$

$$\frac{1}{a_{n+2}} = \frac{1}{AG} \cdot \frac{P+(n+2)}{P}. \qquad (3)$$

$\therefore$ adding (1) and (3), we get

$$\frac{1}{a_n}+\frac{1}{a_{n+1}}=\frac{1}{AG}\left[\frac{2P+2n+2}{P}\right]=\frac{2}{AG}\left[\frac{P+(n+1)}{P}\right]=\frac{2}{a_{n+1}} \text{ from (2).}$$

Ex. 7. *In a wheel and axle, if the radius of the wheel be 6 times that of the axle, and if by means of an effort equal to 5 lbs. wt. a body be lifted through 50 ft., find the amount of work expended.* [*P. U. 1932*]

Let a = the radius of the axle,

then $6a$ = radius of the wheel.

Since the body is lifted through 50 ft., the circular measure of the angle through which the axle turns $=\frac{50}{a}$, which is also equal to the angle through which the wheel turns during the time, as they are rigidly connected with each other.

Let x be the length of the string uncoiled from the wheel as the body is raised.

Then $\frac{x}{6a}=\frac{50}{a}$. $\therefore$ $x=50\times 6=300$ ft.

$\therefore$ the amount of the work expended $=5\times 300=1500$ ft. lbs.

Examples on Chapter XII(b)

1. The pressure on the fulcrum of a straight lever of first kind is 6 lbs. wt. and the difference of the forces acting at the ends is 2 lbs. wt. Find the ratio of the arms at which they act.

2. Two weights P and $Q(P>Q)$ balance, acting at the ends A and B of a straight lever AB. If P and Q interchange places and additional weights P_1 and Q_1 are added at A and B respectively, the equilibrium is undisturbed. Show that $P^2-Q^2=P_1Q-Q_1P$.

3. A straight light horizontal lever has for fulcrum a hinge at one end C, and from a point B is suspended the weight W. If the pressure on the hinge (either upwards or downwards) must not exceed $\frac{1}{3}W$, show that the effort must act somewhere within a space equal to $\frac{3}{4}BC$.

4. In a lever of the first class, a weight W fastened to one end is supported by a force P at the other ; if the ends are interchanged, the necessary force to balance W is a force Q ; prove that $W=\sqrt{(PQ)}$.

5. A pair of nut-crackers is $4\frac{1}{2}$ inches long, and a nut is placed at a distance $\frac{5}{8}$ in. from the hinge. What pressure applied at the ends of the arms, will crack the nut if a weight of $20\frac{1}{4}$ lbs., when placed on the top of the nut cracks it ?

6. A rectangular block of stone weighing $\frac{1}{2}$ ton, whose weight acts at its centre, is to be raised by a crow-bar 3 ft. long resting against a log of wood in front of it, at a distance 6 inches from the end of the bar in contact with the stone. Find the least force that must be applied to raise the stone.

7. A straight lever AB whose arms AC, BC are a and b, is in equilibrium under the action of the forces P and Q at its ends A and B respectively ; the lines of action of the forces meet at O and $\angle OAB = \alpha$ and $\angle OBA = \beta$. Find the ratio of P to Q, and the pressure on the fulcrum.

8. A straight lever is acted on at its extremities by forces P, Q inclined at angles of 30° and 60° to its length. If $P : Q = \sqrt{3}+1 : \sqrt{3}-1$, show that the reaction at the fulcrum is $2\sqrt{2}$ at 45° to the lever.

9. A lever without weight is of length c, and a weight is supported by two strings of lengths a and b from its ends ; if the lever rests in a horizontal position, show that the arms of the lever are in the ratio

$$(a^2+c^2-b^2) : (b^2+c^2-a^2).$$

10. The arms of a false balance are a and b, and a weight W balances P at the end of the shorter arm b, and Q at the end of the arm a , show that

$$\frac{a}{b} = \frac{P-W}{W-Q}.$$ [*P. U. 1940* ; *U. P. 1947*]

11. A tradesman weighs out to a customer apparently equal quantities of wheat alternately from the two scale-pans of a balance with unequal arms. Does he gain or lose ? [*A. U. 1931*]

12. A substance weighed from the two arms successively of a false balance has apparent weights, 9 and 4 lbs. Find the ratio of the lengths of the arms and the true weight of the body. [*P. U. 1930*]

13. In a false balance the arms being of unequal length, a weight is measured in one scale-pan by P lbs. and in the other by Q lbs. Show that the arms are to one another as $\sqrt{P} : \sqrt{Q}$.

14. In a balance with unequal arms, the apparent weights of a body are $42\frac{1}{4}$ lbs. and 49 lbs. when weighed in succession in the two scale-pans, and the whole length of beam is $2\frac{1}{4}$ ft. Find the length of each arm.

15. A man sitting in one scale of a common balance places his "pugree" on the beam between the fulcrum and the point of suspension of the scale. Will he weigh more or less than if he had pugree on ? Give reasons for your answer. [*C. U. 1930*]

16. A boy sitting in one scale-pan of a balance presses upwards with a rod against the beam at any point between the fulcrum and the point from which the scale-pan in which he is seated is suspended. Show that he will appear to weigh more.

17. The arms of a false balance are in the ratio of 20 . 21. How much does a trader gain or lose if he places articles to be weighed at the end of the shorter arm, when he is asked for 4 Kilograms of potatoes at 50 paise per Kg.

18. A balance has its arms unequal and one scale-pan unjustly loaded. A body of true weight 9 Kg. appears to weigh $8\frac{1}{3}$ and 10 Kg. when placed successively in the two scale-pans. Find the ratio of the arms and the weight with which the pan is loaded.

19. If the scale-pans are unequal in weight but the balance is otherwise correct, find the real weight of the body whose apparent weights are 12 lbs. and 14 lbs., when the body is placed successively in the two pans.

20. One scale-pan of a balance is unjustly loaded. If W_1 and W_2 be the apparent weights of a body when weighed in succession in the two scale-pans, find its true weight and the weight with which the scale is loaded.

21. If a balance be faulty in every respect, and if the apparent weight of a body when weighed from the arms of lengths a and b be W_1 and W_2 respectively, its true weight W is given by

$$W = \frac{W_1 b + W_2 a}{a+b}.$$

***22.** A dealer has a balance faulty in every respect, the arms being 10 and 12 inches long. He weighs out to a customer two bags of rice each of the same weight. If W_1 and W_2 be their apparent weights when weighed from the shorter and longer arms respectivefy, show that the customer loses a quantity equal to $\frac{1}{11}(W_2 - W_1)$.

***23.** If a tradesman weighs out to a customer a quantity of wheat by alternately weighing apparently equal portions of it in the two scale-pans of a balance which is unjustly loaded, has unequal arms, and whose C.G. is in the longer arm, show that he will defraud himself.

***24.** A tradesman has a pair of scales, which do not quite balance and makes them balance by attaching a small weight to one of the pans. Show that if he tries to serve a customer with any weight of commodity by weighing parts of it in succession in each scale-pan against half the weight in the other, he will always cheat himself.

25. A balance is faulty in every respect. A certain article appears to weigh P_1 or P_2 according as it is put into one scale-pan or the other. Similarly, another article appears to weigh Q_1 or Q_2. Show that the true weight of an article which appears to weigh the same in whichever scale-pan it is put, is

$$\frac{P_1 Q_2 - P_2 Q_1}{(P_1 - P_2) - (Q_1 - Q_2)}.$$

***26.** Three bodies of weights P, Q, R appear to weigh P', Q', R' in a balance which is faulty in every respect. Show that $(PQ' - P'Q) + (QR' - Q'R) + (RP' - R'P) = 0$.

27. In a common steelyard, show that the distance between any two graduations is proportional to the difference between corresponding weights. [*C. U. 1923*]

28. If the distance of the C.G. of the beam of a common steelyard from the fulcrum is 2 inches, the movable weight 4 ozs., and the weight of the beam 2 lbs., find the distance of zero graduation from the C.G. [*A. U. 1925*]

29. A uniform beam AB, 2 ft. long and weighing 3 lbs. is used as a steelyard, whose fulcrum is at a distance 3 in. from A. If the movable weight be 1 lb., find the greatest and least weights which can be weighed with the machine.

30. A shopkeeper using a common steelyard alters the movable weight for which it has been graduated. Show that he cheats himself or his customers according as he increases or decreases the movable weight.

31. A common steelyard, correctly graduated when new, has its weight and position of its C.G. slightly changed by the wearing away of the rod. A body of weight 5 lbs. $\frac{1}{2}$ oz. appears to weigh 5 lbs. Find the true weight of a body which appears to weigh 12 lbs.

32. If the beam of a common steelyard be uniform and its weight be m times the movable weight P, and the fulcrum one-nth part of the length of the beam from the end where the weight is suspended, show that the greatest weight that can be weighed is $\frac{1}{2}\{(2n-2)+m(n-2)\}P$. [*P. U. 1938*]

33. When weights P and Q are successively placed in the scale-pan of a common steelyard, the movable weight is at distances a and b from the fulcrum. If the movable weight is equal to that of the machine, show that the distances of the C.G. of the machine from the fulcrum is

$$\frac{Pb-Qa}{P-Q}.$$

34. The weight of a Danish steelyard is 6 lbs. and the fulcrum is at a distance of 3 inches from the end to which the weight is attached, to balance a weight of 8 lbs. Find how far the fulcrum must be shifted in order to balance a weight of 16 lbs.

35. A Danish steelyard loses $\frac{1}{10}$th of its weight by use. If the C.G. remains unchanged, find the real weight of a body whose apparent weight is 20 lbs. as determined by it.

36. In a certain Danish steelyard, it is found that the distances of the fulcrum from the end carrying the scale-pan are a and b if the weights P and Q respectively are placed on the scale-pan. Find the position of the centre of gravity of the instrument and show that its weight is

$$\frac{bQ-aP}{a-b}. \qquad [\textit{C. U. 1944}]$$

37. In a Danish steelyard, show that the sensibility at any point varies as the square of the distance of the point from the end at which the weight is suspended.

[*For a small change in the weight, the greater the shifting of the fulcrum, i.e., the greater the distance between the graduations showing the difference in weights, the more sensitive is the steelyard.*]

38. The radius of the wheel being three times that of the axle, find how far the weight will be lifted when the power is pulled down through the space of one foot. [*C. U. 1922*]

39. A bucket weighing 33 lbs. is raised from well by means of wheel and axle. The radius of the wheel is 21 inches and while it makes 5 revolutions, the bucket rises 10 ft. Find the force which will just raise the bucket.

40. If the difference between the radii of a wheel and axle be eight inches, and the power and the weight be as 6 : 7, find the radii.

41. The radius of the wheel is four times that of the axle, and the string on the wheel is only strong enough to support a tension of 40 lbs. wt.; find the greatest weight which can be raised.

42. Two men, who can exert forces of 200 lbs. wt. and 225 lbs. wt. respectively, work at a wheel and axle, in which two wheels are attached, of 5 feet and 4 feet diameter respectively, the diameter of the axle being 20 inches; find the greatest weight the men can raise by it.

43. The radii of the wheel and axle are a and b respectively, the weight consists of a cage of weight W with a man of weight W' inside it, who supports the system by holding the rope that passes over the wheel. Find the tension he produces in the rope. [*Allahabad*]

***44.** A particle of weight 40 lbs. placed on an inclined plane is supported by a force 24 lbs. wt. acting along the plane. If the same weight were to be supported by a force acting horizontally, show that the force must be increased in the ratio of 5 : 4, while the pressure on the plane will be increased in the ratio of 25 : 16.

45. Show that the smallest force which will keep a body in equilibrium on a smooth inclined plane must act along the plane.

46. Find the inclination of a plane to the horizon on which a power parallel to the plane will support double its own weight.

47. A heavy body rests on a plane inclined to the horizon at an angle α ; if the pressure on the plane be equal to the effort applied. show that the effort is inclined at an angle $\frac{1}{2}\pi - 2\alpha$ to the plane.

***48.** A power P acting parallel to an inclined plane can support W_1, and acting horizontally can support W_2, both resting on the same plane. Prove that $P^2 = W_1{}^2 - W_2{}^2$.

ANSWERS

1. 2 : 1. **5.** $2\frac{1}{3}$ lbs. wt. **6.** 112 lbs. wt.

7. $P : Q = b \sin\beta \,.\, a \sin\alpha$, reaction $= \sqrt{P^2 + Q^2 - 2PQ \cos(\alpha+\beta)}$.

11. Lose. **12.** 2 : 3 ; 6 lbs. wt. **14.** 13 inches ; 14 ins.

15. Less. **17.** Loses 10 paise. **18.** 4 : 5 ; $1\frac{1}{4}$ Kg.

19. 18 lbs. **20.** $\frac{1}{2}(W_1 + W_2)$; $\frac{1}{2}(W_1 - W_2)$. **28.** 18 inches.

29. 16 lbs. ; 8 lbs. **31.** 12 lbs. $\frac{1}{2}$ oz. **34.** $1\frac{1}{11}$ inches.

35. 18 lbs. wt. **36.** Distance of C.G. from the scale-pan is $ab(Q-P)/(bQ-aP)$.

38. 4 inches. **39.** 6 lbs. wt. **40.** 4 ft. ; $4\frac{2}{3}$ ft.

41. 160 lbs. **42.** 1140 lbs. **43.** $(W+W')b/(a+b)$.

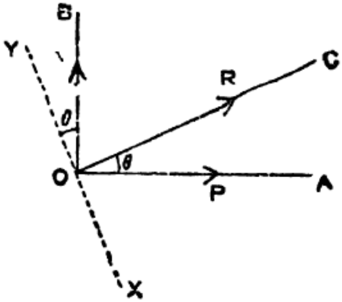
46. 30°.

Appendix A

THEORETICAL PROOF OF THE PARALLELOGRAM OF FORCES

1. Laplace's proof.

We shall first of all consider the case of two perpendicular forces, and then extend the result to the case of any two oblique forces.



Let P and Q be any two perpendicular forces acting at O along OA and OB, and let R be the magnitude of their resultant acting in an unknown direction OC at angle θ to OA. Let XOY be drawn perpendicular to OC.

Then R along OC is equivalent to a force P at an angle θ to it (*i.e.*, along OA), and a force Q perpendicular to P. Hence, a force $\lambda.R$ along OC is equivalent to force $\lambda.P$ at an angle θ to it, together with a force $\lambda.Q$ perpendicular to the latter, for multiplying by the factor λ is essentially the same as an alteration in the scale of representation. Thus, the force P along OA, which can be taken as $\frac{P}{R}\cdot R$ along OA, can be replaced by a force $\frac{P}{R}\cdot P$ at an angle θ to OA, (*i.e.*, along OC), together with force $\frac{P}{R}$ Q in the direction OX perpendicular to OC. In the same manner, the force $Q = \frac{Q}{R} \cdot R$ along OB can be replaced by a force $\frac{Q}{R} \cdot P$ along OY at an angle θ to OB, together with a perpendicular force $\frac{Q}{R} \cdot Q$ along OC. Thus, the two given forces P along OA and Q along OB are equivalent to a force $\frac{P^2}{R} + \frac{Q^2}{R}$ along OC, together with a force $\frac{PQ}{R}$ along OX and a force $\frac{QP}{R}$ along OY, and the two latter, being equal and opposite, cancel one another. Thus, the single force equivalent to the two given forces P and Q is $\frac{P^2 + Q^2}{R}$ along OC, which is thus the required resultant R.

Hence, $R = \frac{P^2 + Q^2}{R}$, or, $R^2 = P^2 + Q^2$, *i.e.*, $R = \sqrt{P^2 + Q^2}$, giving the magnitude of the resultant of two perpendicular forces.

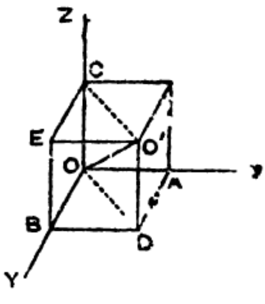


Fig (i)

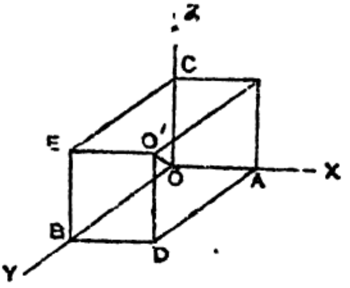


Fig. (ii)

For direction, let us consider first three equal forces F, F, F, acting at O along three mutually perpendicular directions OX, OY, OZ, and let them be represented by OA, OB, OC respectively. Complete the rectangular parallelopiped with OA, OB, OC as adjacent edges (Fig. i).

The resultant of the two equal forces F, F represented by OA and OB must, from symmetry, be equally inclined to them, and therefore must act along the diagonal OD; also its magnitude, from what has been proved above, is $\sqrt{F^2+F^2} = F\sqrt{2}$. Combining with this the force F along OC, the resultant of $F\sqrt{2}$ along OD and F along OC must evidently be along some line in the plane COD. Again, considering the two forces F, F represented by OB and OC first, and then combining their resultant with OA, the final resultant will lie in the plane AOE. Thus, the direction of the final resultant being common to the two planes COD and AOE must be along the diagonal OO'. Hence, we establish that the resultant of two perpendicular forces $F\sqrt{2}$ along OD and F along OC is in the direction of the diagonal OO' of the rectangle $ODO'C$, and its magnitude is

$$\sqrt{F^2+(F\sqrt{2})^2} = F\sqrt{3} = OO'.$$

Next, taking forces F, $F\sqrt{2}$, F along OX, OY, OZ represented by OA, OB, OC (Fig. ii), considering first the resultant of OA, OB, and then combining it with OC, and alternatively, finding the resultant of OB, OC and then combining with OA, we can show exactly in a similar manner as above that the resultant of two perpendicular forces $F\sqrt{3}$ along OD and F along OC is along the diagonal OO', and its magnitude $= F\sqrt{4} = OO'$.

Then take F, $F\sqrt{3}$, F along OX, OY, OZ. Proceeding in this manner, we show finally that the resultant of two perpendicular forces $F\sqrt{n}$ and F is represented by the diagonal in magnitude and direction.

Now, taking F, F, $F\sqrt{n}$ along OX, OY, OZ we extend the above result to the case of two perpendicular forces $F\sqrt{2}$ and $F\sqrt{n}$. Then taking F, $F\sqrt{2}$, $F\sqrt{n}$ the result is extended to $F\sqrt{3}$ and $F\sqrt{n}$. Proceeding thus, we prove the result for the case of two perpendicular forces $F\sqrt{m}$ and $F\sqrt{n}$ where m and n are any two positive integers. Writing $m=p^2$ and $n=q^2$, where p and q are any two positive integers, we finally prove the parallelogram law of forces to hold good for two perpendicular forces pF and qF. We can replace pF and qF by P and Q, where P and

Thus, the two forces P along OA and $Q+R$ along OBD are ultimately equivalent to two forces both acting at E, one along OE and the other along BE. Hence, the resultant of P and $Q+R$ acting at O, represented by OA and OD respectively, must be acting through E and therefore must be along the diagonal OE of the parallelogram $ODEA$.

Now, to start with, take two equal forces F, F along any two directions, represented by OA and OB. From symmetry, their resultant must be equally inclined to OA and OB, and accordingly it is in the direction of the diagonal of the rhombus $OACB$. Hence, from what has been proved above, the parallelogram law for direction of the resultant will hold good for forces F and $F+F$ *i.e.*, F and $2F$ along OA and OB. Again, as the result is true for F, F and for F, $2F$ acting at the same angle, it is true for forces F and $3F$ acting at the same angle. Proceeding in this manner, it can be shown to be true for F and pF. Thus, it is true for $F+F$ and pF, *i.e.*, for $2F$ and pF. Similarly, it will be true for $3F$ and pF and ultimately for qF and pF where p and q are any integers. Replacing pF and qF by P and Q, we see that, so far as the direction is concerned, the parallelogram law for resultant is true for any two commensurable forces P and Q acting at any angle.

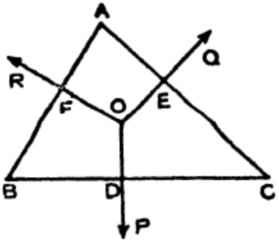
The result then can be extended to incommensurable forces as well, in the limit, as in the previous proof.

Hence, for any two forces P and Q, commensurable or incommensurable, acting at any angle, the parallelogram law is established so far as the direction of the resultant is concerned. Now to establish that the law being true for direction, it will be true for magnitude as well, it is left as an exercise to the student. In this connection, see Art. 2·10, Ex. 8, worked out, and Ex. 58, p. 32 set in the book.

Appendix B

1. Note on Art. 3·2.

Let ABC be the triangle where the forces P, Q, R acting perp. to BC, CA, AB all outwards meet at O, a point inside the triangle ABC. Let the forces cut BC, CA, AB at D, E, F respectively.



By hyp., $\dfrac{P}{a}=\dfrac{Q}{b}=\dfrac{R}{c}$.

But we have,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

$$\therefore \quad \frac{P}{\sin A}=\frac{Q}{\sin B}=\frac{R}{\sin C},$$

$$i.e., \quad \frac{P}{\sin EOF}=\frac{Q}{\sin FOD}=\frac{R}{\sin DOE}.$$

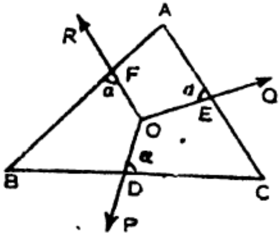
[$\because \angle A=180°-\angle EOF$]

Hence by the converse of Lami's theorem, the forces are in equilibrium.

We can prove the theorem similarly if O be outside, on a side or at any angular point of $\triangle ABC$.

Note. Also in case the directions of the forces are such that forces make equal angles with the corresponding sides of the triangle (instead of being only perpendicular), the theorem can also be proved. The proof is as follows :—

Let ABC be a triangle where the forces P, Q, R through O make the same angle α ($\alpha \neq 0$, or π) with BC, CA, AB.

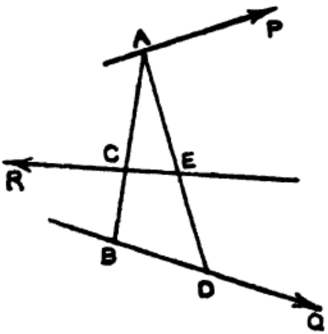


Let the forces cut BC, CA, AB at D, E, F respectively. $\angle ODC=\angle OEA=\angle OFB=\alpha$. Let $\angle EOF=\theta_1$, $\angle FOD=\theta_2$, $\angle DOE=\theta_3$. From the quad. $ODCE$, $\theta_3+\alpha+C+\pi-\alpha=2\pi$, *i.e.*, $\theta_3=\pi-C$. Similarly $\theta_1=\pi-A$ and $\theta_2=\pi-B$. Hence the proof follows as before.

2. On Note 3, Art. 8·1.

Proof: In note 3, Art. 5·8 we have seen that if a system of forces, acting on a rigid body keeps it at rest, the algebraic sum of their moments about any line in the body is zero.

Let three forces P, Q, R acting on a rigid body keep it in equilibrium. Let A be a point in the body on the line of action of P. We take two distinct points B and D on the line of action of Q. Since forces P, Q, R are in equilibrium, the algebraic sum of their moments about the straight lines AB, AD must be zero. But the moments of P and Q about AB and AD vanish, since AB, AD intersect the lines of action of P and Q. Hence, the moment of R about AB and AD must vanish. It therefore follows that R must intersect the straight lines AB and AD. Let the points of intersection be C and E. Now AB and AD are two intersecting straight lines; so they determine a plane π. Hence it follows that the lines of action of Q and R lie in this plane π. The coplanar forces Q and R have a single resultant (say S) in the plane π. Now this force S and the force P keep the body in equilibrium. Hence P and S must have the same line of action.



$\therefore$ forces P, Q, R are co-planar.

Otherwise, we may proceed thus:

Since the plane π passes through A, which is *any* point on the line of action of P, the plane π contains the line of action of P.

Hence the forces P, Q, R are co-planar.