DIFFERENTIAL CALCULUS

[For B.A. and B.Sc. Students]

SHANTI NARAYAN
Dr. P.K. MITTAL

S. CHAND

THE BOOK WAS DRENCHED

DIFFERENTIAL CALCULUS

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(*Review published in Mathematical Gazette, London, December, 1953*

THE book has reached its 5th edition in 9 years and it can be assumed that it meets all demands. Is it the reviewer's fancy to discern the influence of G. H. Hardy in the opening chapter on real numbers, which are well and clearly dealt with ? Or is this only to be expected from an author of the race which taught the rest of the world how to count ?

* * *

The course followed is comprehensive and thorough, and there is a good chapter on curve tracing. The author has a talent for clear exposition, and is sympathetic to the difficulties of the beginner.

* * *

Answers to examples, of which there are good and ample selections, are given.

* * *

Certaianly Mr. Narayan's command of English is excellent......Our own young scientific or mathematical specialist, grumbling over French or German or Latin as additions to their studies, would do well to consider their Indian confreres, with English to master before their technical education can begin.

DIFFERENTIAL CALCULUS

FOR

B. A. & B. Sc. STUDENTS

By

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TENTH REVISED EDITION

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Preface to the Tenth Edition

The book has been revised. A few more exercises drawn from the recent university papers have been given.

30th April, 1962. SHANTI NARAYAN

PREFACE

This book is meant for students preparing for the B.A. and B.Sc. examinations of our universities. Some topics of the Honours standard have also been included. They are given in the form of appendices to the relevant chapters. The treatment of the subject is rigorous but no attempt has been made to state and prove the theorems in generalised forms and under less restrictive conditions as is the case with the Modern Theory of Function. It has also been a constant endeavour of the author to see that the subject is not presented just as a body of formulae. This is to see that the student does not form an unfortunate impression that the study of Calculus consists only in acquiring a skill to manipulate some formulae through 'constant drilling'.

The book opens with a brief outline of the development of Real numbers, their expression as infinite decimals and their representation by points along a line. This is followed by a discussion of the graphs of the elementary functions x^n, $\log x$, e^x, $\sin x$, $\sin^{-1}x$, etc. Some of the difficulties attendant upon the notion of inverse functions have also been illustrated by precise formulation of Inverse trigonometrical functions. It is suggested that the teacher in the class need refer to only a few salient points of this part of the book. The student would, on his part, go through the same in complete details to acquire a sound grasp of the basis of the subject. This part is so presented that a student would have no difficulty in an independent study of the same.

The first part of the book is analytical in character while the later part deals with the geometrical applications of the subject. But this order of the subject is by no means suggested to be rigidly followed in the class. A different order may usefully be adopted at the discretion of the teacher.

An analysis of the 'Layman's' concepts has frequently been made to serve as a basis for the precise formulation of the corresponding 'Scientist's' concepts. This specially relates to the two concepts of *Continuity* and *Curvature.*

Geometrical interpretation of results analytically obtained have been given to bring them home to the students. A chapter on '*Some Important Curves*' has been given before dealing with geometrical applications. This will enable the student to get familiar with the names and shapes of some of the important curves. It is felt that a student would have better understanding of the properties of a curve if he knows how the curve looks like. This chapter will also serve as a useful introduction to the subject of *Double points* of a curve.

Asymptote of a curve has been defined as a line such that the distance of any point on the curve from this line tends to zero as the point tends to infinity along the curve. It is believed that, of all the definitions of an asymptote, this is the one which is most natural. It embodies the idea to which the concept of asymptotes owes its importance. Moreover, the definition gives rise to a simple method for determining the asymptotes.

The various principles and methods have been profusely illustrated by means of a large number of solved examples.

I am indebted to Prof. Sita Ram Gupta, M.A., P.E.S., formerly of the Government College, Lahore who very kindly went through the manuscript and made a number of suggestions. My thanks are also due to my old pupils and friends Professors Jagan Nath M.A., Vidya Sagar M A., and Om Parkash M A., for the help which they rendered me in preparing this book.

Suggestions for improvement will be thankfully acknowledged.

January, 1942 SHANTI NARAYAN

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CHAPTER I

REAL NUMBERS

FUNCTIONS

Introduction. The subject of Differential Calculus takes its stand upon the aggregate of numbers and it is with numbers and with the various operations with them that it primarily concerns itself. It specially introduces and deals with what is called *Limiting operation* in addition to being concerned with the Algebraic operations of Addition and Multiplication and their inverses, Subtraction and Division, and is a development of the important notion of *Instantaneous rate of change* which is itself a limited idea and, as such, it finds application to all those branches of human knowledge which deal with the same. Thus it is applied to Geometry, Mechanics and other branches of Theoretical Physics and also to Social Sciences such as Economics and Psychology.

It may be noted here that this application is essentially based on the notion of measurement, whereby we employ numbers to measure the particular quantity or magnitude which is the object of investigation in any department of knowledge. In Mechanics, for instance, we are concerned with the notion of time and, therefore, in the application of Calculus to Mechanics, the first step is to correlate the two notions of Time and Number, *i.e.*, to measure time in terms of numbers. Similar is the case with other notions such as Heat, Intensity of Light, Force, Demand, Intelligence, etc. The formulation of an entity in terms of numbers, *i.e.*, measurement, must, of course, take note of the properties which we intuitively associate with the same. This remark will later on be illustrated with reference to the concepts of Velocity, Acceleration, Curvature, etc.

The importance of numbers for the study of the subject in hand being thus clear, we will in some of the following articles, see how we were first introduced to the notion of number and how, in course of time, this notion came to be subjected to a series of generalisations.

It is, however, not intended to give here any logically connected amount of the development of the system of real numbers, also known as *Arithmetic Continuum* and only a very brief reference to some well known salient facts will suffice for our purpose. An excellent account of the Development of numbers is given in '*Fundamentals of Analysis*' by Landau.

It may also be mentioned here that even though it satisfies a deep philosophical need to base the theory part of Calculus on the notion of number alone, to the entire exclusion of every physical basis, but a rigid insistence on the same is not within the scope of this book and intuitive geometrical notion of Point, Distance, etc., will sometimes be appealed to for securing simplicity.

1·1. Rational numbers and their representation by points along a straight line.

1·11. Positive Integers. It was to the numbers, 1, 2, 3, 4, etc., that we were first introduced through the process of *counting* certain objects. The totality of these numbers is known as the aggregate of *natural numbers, whole numbers* or *positive integers.*

While the operations of addition and multiplications are unrestrictedly possible in relation to the aggregate of positive integers, this is not the case in respect of the inverse operations of subtraction and division. Thus, for example, the symbols

$$2-3,\ 2\div 3$$

are meaningless in respect of the aggregate of positive integers.

1·12. Fractional numbers. At a later stage, another class of numbers like p/q (*e.g.*, $\frac{1}{2}$, $\frac{3}{4}$) where p and q are natural numbers, was added to the former class. This is known as the class of fractions and it obviously includes natural numbers as a sub-class ; q being equal to 1 in this case.

The introduction of Fractional numbers is motivated, from an abstract point of view, to render Division unrestrictedly possible and, from concrete point of view, to render numbers serviceable for measurement also in addition to counting.

1·13. Rational numbers. Still later, the class of numbers was enlarged by incorporating in it the class of negative fractions including negative integers and zero. The entire aggregate of these numbers is known as the *aggregate of rational numbers.* Every rational number is expressible as p/q, where p and q are any two integers, positive and negative and q is not zero.

The introduction of Negative numbers is motivated, from an abstract point of view, to render Subtraction always possible and, from concrete point of view, to facilitate a unified treatment of oppositely directed pairs of entities such as, gain and loss, rise and fall, etc.

1·14. Fundamental operations on rational numbers. An important property of the aggregate of rational numbers is that the operations of addition, multiplication, subtraction and division can be performed upon any two such numbers, (with one *exception* which is considered below in § 1·15) and the number obtained as the result of these operations is again a rational number.

This property is expressed by saying that the aggregate of rational numbers is *closed* with respect to the four fundamental operations.

1·15. Meaningless operation of division by zero. It is important to note that the only exception to the above property is *'Division* *by zero'* which is a meaningless operation. This may be seen as follows :—

To divide a by b amounts to determining a number c such that

$$bc = a,$$

and the division will be intelligible only, if and only if, the determination of c is *uniquely possible.*

Now, there is no number which when multiplied by zero produces a number other than zero so that $a/0$ is *no* number when $a \neq 0$. Also *any* number when multiplied by zero produces zero so that $0/0$ may be *any* number.

On account of this *impossibility* in one case and *indefiniteness* in the other, the operation of division by zero *must be always avoided.*

A disregard of this exception often leads to absurd results as is illustrated below in (*i*).

(*i*) Let $\qquad x = 6.$

Then

$$x^2 - 36 = x - 6,$$

or $\qquad (x-6)(x+6) = x-6.$

Dividing both sides by $x-6$, we get

$$x + 6 = 1.$$

$$6 + 6 = 1, \text{ i.e., } 12 = 1.$$

which is clearly absurd.

Division by $x-6$, which is zero *here*, is responsible for this absurd conclusion.

(*ii*) We may also remark in this connection that

$$\frac{x^2-36}{x-6} = \frac{(x-6)(x+6)}{(x-6)} = x+6, \text{ only when } x \neq 6. \qquad \dots (1)$$

For $x = 6$, the left hand expression, $(x^2-36)/(x-6)$, is meaningless whereas the right hand expression, $x+6$, is equal to 12 so that the equality ceases to hold for $x = 6$.

The equality (1) above is proved by dividing the numerator and denominator of the fraction $(x^2-36)/(x-6)$ by $(x-6)$ and this operation of division is possible only when the divisor $(x-6) \neq 0$, *i.e.*, when $x \neq 6$. This explains the *restricted* character of the equality (1).

Ex. 1. Show that the aggregate of natural numbers is not closed with respect to the operations of subtraction and division. Also show that the aggregate of positive fractions is not closed with respect to the operations of subtraction.

Ex. 2. *Show that every rational number is expressible as a terminating or a recurring decimal.*

To decimalise p/q, we have first to divide p by q and then each remainder, after multiplication with 10, is to be divided by q to obtain the successive figures in the decimal expression of p/q. The decimal expression will be terminating if, at some stage, the remainder vanishes. Otherwise, the process will be unending. In the latter case, the remainder will always be one of the finite set of numbers $1, 2, \dots, q-1$ and so must repeat itself at some stage.

From this stage onward, the quotients will also repeat themselves and the decimal expression will, therefore, be recurring.

The student will understand the argument better if he actually expresses some fractional numbers, say 3/7, 3/13, 31/123, in decimal notation.

Ex. 3. For what values of x are the following equalities *not* valid :

(*i*) $\dfrac{x}{x}=1.$ (*ii*) $\dfrac{x^2-a^2}{x-a}=x+a.$

(*iii*) $\dfrac{1-x}{1-\sqrt{x}}=1+\sqrt{x}.$ (*iv*) $\dfrac{1-\cos x}{\sin x}=\tan\dfrac{x}{2}.$

1·16. Representation of rational numbers by points along a line or by segments of a line. The mode of representing rational numbers by points along a line or by segments of a line, which may be known as the *number-axis,* will now be explained.

We start with marking an arbitrary point O on the number-axis and calling it the *origin* or zero point. The number zero will be represented by the point O.

The point O divides the number axis into two parts or sides. Any one of these may be called positive and the other, then negative.

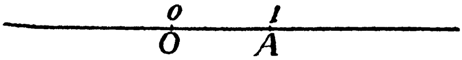


Fig. 1.

Usually, the number-axis is drawn parallel to the printed lines of the page and the right hand side of O is termed positive and the left hand side of O negative.

On the positive side, we take an arbitrary length OA, and call it the *unit* length.

We say, then, that the number 1 is represented by the point A.

After having fixed an *origin*, *positive sense* and a *unit length* on the number axis in the manner indicated above, we are in a position to determine a point representing any given rational number as explained below :—

Positive integers. Firstly, we consider any positive integer, m. We take a point on the positive side of the line such that its distance from O is m times the unit length OA. This point will be reached by measuring successively m steps each equal to OA starting from O. This point, then, is said to represent the positive integer, m.

Negative integers. To represent a negative integer, $-m$, we take a point on the negative side of O such that its distance from O is m times the unit length OA.

This point represents the negative integer, $-m$.

Fractions. Finally, let p/q be any fraction ; q being a positive integer. Let OA be divided into q equal parts ; OB being one of them. We take a point on the positive or negative side of O according as p is positive or negative such that its distance from O is p times (or, $-p$ times if p is negative) the distance OB.

The point so obtained represents the fraction, p/q.

If a point P represents a rational number p/q, then the measure of the length OP is clearly p/q or $-p/q$ according as the number is positive or negative.

Sometimes we say that the number, p/q, is represented by the segment OP.

1·2. Irrational numbers. Real numbers. We have seen in the last article that every rational number can be represented by a point of a line. Also, it is easy, to see that we can cover the line with such points as closely as we like. The natural question now arises, "Is the converse true ?" Is it possible to assign a rational number to every point of the number-axis ? A simple consideration, as detailed below, will clearly show that it is not so.

Construct a square each of whose sides is equal to the unit length OA and take a point P on the number-axis such that OP is equal in the length to the diagonal of the square. It will now be shown that the point P cannot correspond to a rational number *i.e.*, the length of OP cannot have a rational number as its measure.

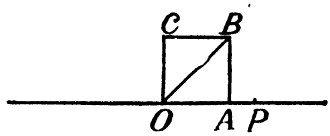


Fig. 2.

If possible, let its measure be a rational number p/q so that, by Pythagoras's theorem, we have

$$\left(\frac{p}{q}\right)^2 = 1^2+1^2=2, \text{ i.e., } p^2=2q^2. \qquad \dots(i)$$

We may suppose that p and q have no common factor, for, such factors, if any, can be cancelled to begin with.

Firstly we notice that

$$(2n)^2=4n^2,\ (2n+1)^2=(4n^2+4n)+1$$

so that the square of an even number is even and that of an odd number is odd.

From the equation (i), we see, that p^2 is an even number. Therefore, p itself must be even.

Let, then, p be equal to $2n$ where n is an integer.

$$\therefore \qquad 4n^2=2q^2 \text{ or } q^2=2n^2.$$

Thus, q^2 is also even and so q is even.

Hence p and q have a common factor 2 and this conclusion contradicts the hypothesis that they have no common factor. Thus the measure $\sqrt{2}$ of OP is not a rational number. There exists, therefore, a point on the number-axis not corresponding to any rational number.

Again, we take a point L on the line such that the length OL is any rational multiple say, p/q, of OP.

The length OL cannot have a rational measure. For, if possible, let m/n be the measure of OL.

$$\therefore \qquad \frac{p}{q}\sqrt{2}=\frac{m}{n} \text{ or } \sqrt{2}=\frac{mq}{np},$$

which states that $\sqrt{2}$ is a rational number, being equal to mq/np.

This is a contradiction. Hence L cannot correspond to a rational number.

Thus we see that there exist an unlimited number of points on the number-axis which do not correspond to rational numbers.

If we now require that our aggregate of numbers should be such that after the choice of unit length on the line, every point of the line should have a number corresponding to it (or that *every length should be capable of measurement*), we are forced to extend our system of numbers further by the introduction of what are called *irrational* numbers.

We will thus associate an irrational number to every point of the line which does not correspond to a rational number.

A method of representing irrational numbers in the decimal notation is given in the next article 1·3.

Def. Real number. *A number, rational or irrational, is called a real number.*

The aggregate of rational and irrational number is, thus, the aggregate of real numbers.

Each real number is represented by some point of the number-axis and each point of the number-axis has some real number, rational or irrational, corresponding to it.

Or, we might say, that each real number is the measure of some length OP and that the aggregate of real numbers is enough to measure every length.

1·21. Number and Point. If any number, say x, is represented by a point P, then we usually say that *the point P is x.*

Thus the terms, number and point, are generally used in an indistinguishable manner.

1·22. Closed and open intervals. set a, b be two given numbers such that $a<b$. Then the set of numbers x such that $a\leqslant x\leqslant b$ is called a **closed** interval denoted by the symbol $[a, b]$.

Also the set of numbers x such that $a<x<b$ is called an **open** interval denoted by the symbol (a,b).

The number $b-a$ is referred to as the length of $[a, b]$ as also of (a, b).

1·3. Decimal representation of real numbers. Let P be any given point of the number-axis. We now seek to obtain the decimal representation of the number associated with the point P.

To start with, we suppose that the point P lies on the positive side of O.

Let the points corresponding to integers be marked on the number-axis so that the whole axis is divided into intervals of length one each.

Now, if P coincides with a point of division, it corresponds to an integer and we need proceed no further. In case P falls between two points of division, say a, $a+1$, we sub-divide the interval $(a, a+1)$ into 10 equal parts so that the length of each part is $\frac{1}{10}$. The points of division, now, are,

$$a,\ a+\tfrac{1}{10},\ a+\tfrac{2}{10},\ldots\ldots\ldots,\ a+\tfrac{9}{10},\ a+1.$$

If P coincides with any of these points of division, then it corresponds to a rational number. In the alternative case, it falls between two points of division, say

$$a+\frac{a_1}{10},\ a+\frac{a_1+1}{10},$$

i.e.,

$$a.a_1,\ a.(a_1+1),$$

where, a_1, is any one of the integers 0, 1, 2, 3,........., 9.

We again sub-divide the interval

$$\left[a+\frac{a_1}{10},\quad a+\frac{a_1+1}{10}\right]$$

into 10 equal parts so that the length of each part is $1/10^2$.

The points of division, now, are

$$a+\frac{a_1}{10},\ a+\frac{a_1}{10}+\frac{1}{10^2},\ a+\frac{a_1}{10}+\frac{2}{10^2},\ldots,\ a+\frac{a_1}{10}+\frac{9}{10^2},\ a+\frac{a_1+1}{10}.$$

The point P will either coincide with one of the above points of division (in which case it corresponds to a rational number) or will lie between two points of division say

$$a+\frac{a_1}{10}+\frac{a_2}{10^2},\ a+\frac{a_1}{10}+\frac{a_2+1}{10^2}$$

i.e.,

$$a.a_1a_2,\ a.a_1(a_2+1),$$

where a_2 is one of the integers 0, 1, 2,......, 9.

We again sub-divide this last interval and continue to repeat the process. After a number of steps, say n, the point P will either be found to coincide with some point of division (in this case it corresponds to a rational number) or lie between two points of the form

$$a+\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots\ldots+\frac{a_n}{10^n},\ a+\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots+\frac{a_n+1}{10^n},$$

i.e.,

$$a.a_1a_2\ldots\ldots a_n,\ a.a_1a_2a_3\ldots\ldots(a_n+1),$$

the distance between which is $1/10^n$ and which clearly gets smaller and smaller as n increases.

The process can clearly be continued indefinitely.

The successive intervals in which P lies go on shrinking and will clearly close up to the point P.

This point P is then represented by the infinite decimal

$$a.a_1a_2a_3\ldots\ldots$$

Conversely, consider any infinite decimal

$$a.a_1a_2a_3\ldots\ldots a_n\ldots\ldots$$

and construct the series of intervals

$$[a,\ a+1],\ [a.a_1,\ \overline{a.a_1+1}],\ [a.a_1a_2,\ \overline{a.a_1a_2+1}],\ldots\ldots$$

Each of these intervals lies within the preceding one ; their lengths go on diminishing and by taking n sufficiently large we can make the length as near to zero as we like. We thus see that these intervals shrink to a point. This fact is related to the intuitively perceived aspect of the continuity of a straight line.

Thus there is one and only one point common to this series of intervals and this is the point represented by the decimal

$$a.a_1a_2a_3\ldots\ldots$$

Combining the results of this article with that of Ex. 2, §1·1, p. 3, we see that *every decimal, finite or infinite, denotes a number which is rational if the decimal is terminating or recurring and irrational in the contrary case.*

Let, now, P lie on the negative side of O. Then the number representing it is

$$-a.a_1a_2\ldots a_n\ldots\ldots$$

where

$$a.a_1a_2\ldots\ldots a_n\ldots\ldots$$

is the number representing the point P' on the positive side of O such that PP' is bisected at O.

Ex. 1. *Calculate the cube root of 2 to three decimal places.*

We have $\qquad 1^3=1<2$ and $2^3=8>2.$

$\therefore \qquad 1<\sqrt[3]{2}<2.$

We consider the numbers

$$1,\ 1{\cdot}1,\ 1{\cdot}2,\ldots\ldots,\ 1{\cdot}9,\ 2,$$

which divide the interval [1, 2] into 10 equal parts and find two successive numbers such that the cube of the first is <2 and of the second is >2. We find that

$$(1{\cdot}2)^3=1{\cdot}728<2 \text{ and } (1{\cdot}3)^3=2{\cdot}197>2.$$

$\therefore \qquad 1{\cdot}2<\sqrt[3]{2}<1{\cdot}3,$

Again consider the numbers

$$1{\cdot}2,\ 1{\cdot}21,\ 1{\cdot}22\ldots\ldots,\ 1{\cdot}29,\ 1{\cdot}3,$$

which divide the interval [1·2, 1·3] into 10 equal parts.

We find that

$$(1{\cdot}25)^3=1{\cdot}953125<2 \text{ and } (1{\cdot}26)^3=2{\cdot}000376>2.$$

$$\therefore \qquad 1{\cdot}25<\sqrt[3]{2}<1{\cdot}26.$$

Again, the numbers

$$1{\cdot}25,\ 1{\cdot}251,\ 1{\cdot}252 \ldots\ldots\ldots,\ 1{\cdot}259,\ 1{\cdot}26$$

divide the interval [1·25, 1·26] into 10 equal parts

We find that

$$(1{\cdot}259)^3=1{\cdot}995616979<2 \text{ and } (1{\cdot}26)^3=2{\cdot}000376>2$$

$$\therefore \qquad 1{\cdot}259<\sqrt[3]{2}<1{\cdot}26.$$

Hence

$$\sqrt[3]{2}=1{\cdot}259\ldots\ldots\ldots,$$

Thus to three decimal places, we have

$$\sqrt[3]{2}=1{\cdot}259.$$

Ex. 2. Calculate the cube root of 5 to 2 decimal places.

Note. The method described above in Ex. 1 which is indeed very cumbersome, has only been given to illustrate the basic and elementary nature of the problem. In actual practice, however, other methods involving infinite series or other limiting processes are employed.

1·4. The modulus of a real number.

Def. *By the modulus of a real number, x, is meant the number x, $-x$ or 0 according as x is positive, negative, or zero.*

Notation. *The smybol $|\,x\,|$ is used to denote the modulus of x.*

Thus the modulus of a number means the same thing as its *numerical* or *absolute value*. For example, we have

$$|\,3\,|=3\ ;\ |\,-3\,|=-(-3)=3\ ;\ |\,0\,|=0\ ;$$
$$|\,5-7\,|=|\,7-5\,|=2.$$

The modulus of the difference between two numbers is the measure of the distance between the corresponding points on the number-axis.

Some results involving moduli. We now state some simple and useful results involving the moduli of numbers.

1·41. $$|\,\mathbf{a}+\mathbf{b}\,|\leqslant|\,\mathbf{a}\,|+|\,\mathbf{b}\,|,$$

i.e., the modulus of the sum of two numbers is less than or equal to the sum of their moduli.

The result is almost self-evident. To enable the reader to see its truth more clearly, we split it up into two cases giving examples of each.

Case 1. Let a, b have the same sign.

In this case, we clearly have

$$|\,a+b\,|=|\,a\,|+|\,b\,|.$$

e.g., $$|\,7+3\,|=|\,7\,|+|\,3\,|,$$

and $$|\,-7-3\,|=|\,-7\,|+|\,-3\,|.$$

Case II. Let a, b have opposite signs.

In this case, we clearly have

$$|a+b| < |a| + |b|,$$

e.g., $\quad 4 = |7-3| < |7| + |3| = 10.$

Thus in either case, we have

$$|a+b| \leqslant |a| + |b|.$$

1·42. $\quad |\mathbf{ab}| = |\mathbf{a}| \,.\, |\mathbf{b}|,$

e.g., the modulus of the product of two numbers is equal to the product of their moduli,

$$\begin{aligned} \textit{e.g.,} \quad |4 \cdot 3| &= 12 = |4| \,.\, |3|; \\ |(-4)(-3)| &= 12 = |-4| \,.\, |-3|; \\ |(-4).(3)| &= 12 = |-4| \,.\, |3|. \end{aligned}$$

1·43. *If x, a, l, be three numbers such that*

$$|x-a| < l, \qquad \ldots(A)$$

then

$$(a-l) < x < (a+l)$$

i.e., x lies between $a-l$ and $a+l$ or that x belongs to the open interval $(a-l, a+l)$.

The inequality (A) implies that the numerical difference between a and x must be less than l, so that the point x (which may lie to the right or to the left of a) can, at the most, be at a distance l from the point a.

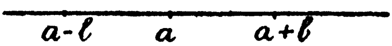


Fig. 3

Now, from the figure, we clearly see that this is possible, if and only if, x lies between $a-l$ and $a+l$.

It may also be at once seen that

$$|x-a| \leqslant l$$

is equivalent to saying that, x, belongs to the closed interval

$$[a-l, a+l].$$

Ex. 1. *If* $|a-b| < l$, $|b-c| < m$, *show that*

$$|a-c| < l+m.$$

We have $|a-c| = |a-b+b-c| \leqslant |a-b| + |b-c| < l+m.$

2. Give the equivalents of the following in terms of the modulus notation :

(*i*) $-1 \leqslant x \leqslant 3$. (*ii*) $2 < x < 5$. (*iii*) $-3 \leqslant x \leqslant 7$. (*iv*) $l-\varepsilon < x < l+\varepsilon$.

3. Give the equivalents of the following by doing away with the modulus notation :

(*i*) $|x-2| < 3$. (*ii*) $|x+1| \leqslant 2$. (*iii*) $0 < |x-1| < 2$.

4. If $y = |x| + |x-1|$, then show that

$$y = \begin{cases} 1-2x, & \text{for } x \leqslant 0 \\ 1, & \text{for } 0 < x < 1. \\ 2x-1, & \text{for } x \geqslant 1. \end{cases}$$

1·5. Variables, Functions. We give below some examples to enable the reader to understand and formulate the notion of a variable and a function.

Ex. 1. *Consider two numbers x and y connected by the relation,*

$$y=\sqrt{(1-x^2)},$$

where we take only the positive value of the square root.

Before considering this relation, we observe that there is no real number whose square is negative and hence, so far as real numbers are concerned, the square root of a negative number does not exist.

Now, $1-x^2$, is positive or zero so long as x^2 is less than or equal to 1. This is the case if and only if x is any number satisfying the relation

$$-1\leqslant x\leqslant 1,$$

i.e., when x belongs to the interval $[-1, 1]$.

If, now, we assign any value to x belonging to the interval $[-1, 1]$, then the given equation determines a unique corresponding value of y.

The symbol x which, in the present case, can take up as its value any number belonging to the interval $[-1, 1]$, is called the *independent variable* and the interval $[-1, 1]$ is called *its domain of variation.*

The symbol y which has a value corresponding to each value of x in the interval $[-1, 1]$ is called the *dependent variable* or a *function of x defined in the interval* $[-1, 1]$.

2. *Consider the two numbers x and y connected by the relation,*

$$y=(x-1)/(x-2).$$

Here, the determination of y for $x=2$ involves the meaningless operation of division by zero and, therefore, the relation does not assign any value to y corresponding to $x=2$. But for every other value of x the relation does assign a value to y.

Here, x is the independent variable whose domain of variation consists of the entire aggregate of real numbers excluding the number 2 and y is a function of x defined for this domain of variation of x.

3. *Consider the two numbers x and y with their relationship defined by the equations*

$$y=x^2 \text{ when } x<0, \qquad \ldots(i)$$

$$y=x \text{ when } 0\leqslant x\leqslant 1, \qquad \ldots(ii)$$

$$y=1/x \text{ when } x>1. \qquad \ldots(iii)$$

These relations assign a definite value to y corresponding to every volue of x, although the value of y is not determined by a single formula as in Examples 1 and 2. In order to determine a value of y corresponding to a given value of x, we have to select one of the three equations depending upon the value of x in question. For instance,

for $x=-2$, $y=(-2)^2=4$, [Equation (*i*) $\because -2<0$

for $x=\frac{1}{2}$, $y=\frac{1}{2}$ [Equation (*ii*) $\because 0<\frac{1}{2}<1$

for $x=3$, $y=\frac{1}{3}$ [Equation (*iii*) $\because 3>1$.

Here again, y is a function of x, defined for the entire aggregate of real numbers.

This example illustrates an important point that it is not necessary that only one formula should be used to determine y as a function of x. What is required is simply the existence of a law or laws which assign a value to y corresponding to each value of x in its domain of variation.

4. *Let*

$$y=x\,!.$$

Here y is a function of x defined for the aggregate of positive integers *only*.

5. *Let*

$$y=|x|.$$

Here we have a function of x defined for the entire aggregate of real numbers.

It may be noticed that the same function can also be defined as follows :

$$y=\ \ x \text{ when } x\geqslant 0,$$
$$y=-x \text{ when } x<0.$$

6. *Let*

$y=1/q$, *when x is a rational number p/q in its lowest terms r*

$y=0$, *when x is irrational.*

Hence again y is a function of x defined for the entire aggregate of real numbers.

1·51. Independent variable and its domain of variation. The above examples lead us to the following precise definitions of variable and function. *If x is a symbol which does not denote any fixed number but is capable of assuming as its value any one of a set of numbers, then x is called a* **variable** *and this set of numbers is said to be its* **domain of variation.**

1·52. Function and its domain of definition. *If to each value of an independent variable x, belonging to its domain of variation, there corresponds, by any law, or laws, whatsoever, a value of a symbol, y,* *then y is said to be a function of x defined in the domain of variation of x, which is then called the domain of definition of the function.*

1·53. Notation for a function. The fact that y is a function of a variable x is expressed symbolically as

$$y=f(x)$$

and is read as the 'f' of x.

If x_1 be any particular value of x belonging to the domain of variation of x, then the corresponding value of the function is denoted by $f(x_1)$. Thus, if $f(x)$ be the function considered in Ex. 3, § 1·5, p. 11, then

$$f(-2)=4\ ;\ f(\tfrac{1}{2})=\tfrac{1}{2}\ ;\ f(3)=\tfrac{1}{3}.$$

If functional symbols be required for two or more functions, then it is usual to replace the latter f in the symbol $f(x)$ by other letters such as F, G, etc.

1·6. Some important types of domains of variation.

Usually the domain of variation of a variable x is an interval $[a, b]$, *i.e.*, x can assume as its value any number greater than or equal to a and less than or equal to b, *i.e.*,

$$a\leqslant x\leqslant b.$$

Sometimes it becomes necessary to distinguish between *closed* and *open* intervals.

If x can take up as its value any number greater than a or less than b but *neither a nor b i.e.*, if $a<x<b$, then we say that its domain of variation is an *open* interval denoted by (a, b) to distinguish it from $[a, b]$ which denotes a closed interval where x can take up the values a and b also.

We may similarly have semi-closed or semi-opened intervals

$$[a, b),\ a\leqslant x<b\ ;\ (a, b],\ a<x\leqslant b$$

as domains of variation.

We may also have domains of variation extending without bound in one or the other directions *i.e.*, the intervals

$$(-\infty, b] \text{ or } x\leqslant b\ ;\ [a, \infty) \text{ or } x\geqslant a\ ;\ (-\infty, \infty) \text{ or any } x.$$

Here it should be noted that the symbols $-\infty, \infty$ are no *numbers* in any sense whatsoever. Yet, in the following pages they will be used in various ways (but, of course never as numbers) and in each case it will be explicitly mentioned as to what they stand for. Here, for example, the symbol $(-\infty, b)$ denotes the domain of variation of a variable which can take up as its value any number less than or equal to b.

Similar meanings have been assigned to the symbols

$$(a, \infty),\ (-\infty, \infty).$$

Constants. *A symbol which denotes a certain fixed number is called a constant.*

It has become customary to use earlier letters of the alphabet, like a, b, c ; α, β, γ, as symbols for constants and the latter letters like x, y, z ; u, v, w as symbols for variables.

Note. The following points about the definition of a function should be carefully noted :

1. A function need not be necessarily defined by a formula or formulae so that the value of the function corresponding to any given value of the independent variable is given by *substitution*. All that is necessary is that some rule or set of rules be given which *prescribe* a value of the function for every value of the independent variable which belongs to the domain of definition of the function. [Refer Ex. 6, page 12.]

2. It is not necessary that there should be a *single* formula or rule for the whole domain of definition of the function. [Refer Ex. 3, page 11.]

Ex. 1. *Show that the domain of definition of the function*

$$1/\sqrt{[(1-x)(x-2)]}$$

is the open interval (1, 2).

For $x=1$ and 2 the denominator becomes zero. Also for $x<1$ and $x>2$ the expression $(1-x)\,(x-2)$ under the radical sign becomes negative.

Thus the function is not defined for $x \leqslant 1$ and $x \geqslant 2$. For $x>1$ and <2 the expression under the radical sign is positive so that a value of the function is determinable. Hence the function is defined in the open interval (1, 2).

2. Show that the domain of definition of the function $\sqrt{[(1-x)\,(x-2)]}$ is the closed interval [1, 2].

3. Show that the domain of definition of the functions

$$1/\sqrt{x},\ 1/\sqrt{(-x)}$$

are $(0, \infty)$ and $(-\infty, 0)$ respectively.

4. Obtain the domains of definition of the functions

(*i*) $\sqrt{(2x+1)}$ (*ii*) $1/(1+\cos x)$ (*iii*) $\sqrt{(1+2\sin x)}$.

1·7. Graphical representation of functions.

Let us consider a function

$$y=f(x)$$

defined in an interval $[a, b]$. ... (*i*)

To represent it graphically, we take two straight lines $X'OX$ and $Y'OY$ at right angles to each other as in *Plane Analytical Geometry*. These are the two co-ordinate axes.

We take O as origin for both the axes and select unit intervals on OX, OY (usually of the same lengths). Also as usual, OX, OY are taken as positive directions on the two axes.

To any number x corresponds a point M on X-axis such that

$$OM=x.$$

To the corresponding number y, as determined from (i), there corresponds a point N on Y-axis such that

$$ON=y$$

Completing the rectangle $OMPN$, we obtain a point P which is said to correspond to the pair of numbers x, y.

Thus to every number x belonging to the interval $[a, b]$, there corresponds a number y determined by the functional equation $y=f(x)$ and to this pair of numbers, x, y corresponds a point P as obtained above.

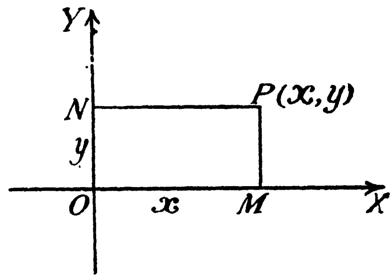


Fig. 4

The totality of these points, obtained by giving different values, to x, is said to be the graph of the function $f(x)$ and $y=f(x)$ is said to be the equation of the graph.

Examples

1. The graph of the function considered in Ex. **3**, § 1·5, page 11, is

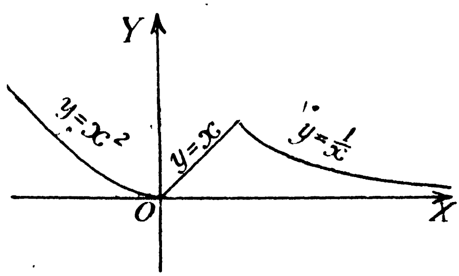


Fig. 5

2. The graph of $y=(x^2-1)/(x-1)$ is the straight line $y=x+1$ excluding the point $P(1, 2)$.

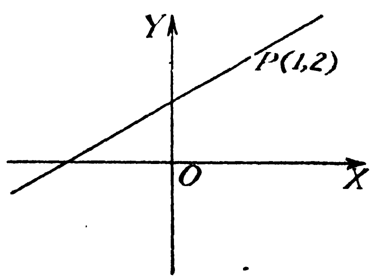


Fig. 6

3. The graph of $y=x\,!$ consists of a discrete set of points

(1, 1), (2, 2), (3, 6), (4, 24), etc.

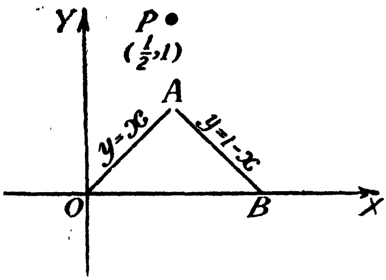


Fig. 7

4. The graph of the function

$$f(x)=\begin{cases} x, \text{ when } 0 \leqslant x < \frac{1}{2} \\ 1, \text{ when } x=\frac{1}{2} \\ 1-x \text{ when } \frac{1}{2} < x \leqslant 1, \end{cases}$$

is as given.

5. *Draw the graph of the function which denotes the positive square root of x^2.*

As $\sqrt{x^2}=x$ or $-x$ according as x is positive or negative, the graph (Fig. 8) of $\sqrt{x^2}$ is the graph of the function $f(x)$ where,

$$f(x)=\begin{cases} x, \text{ when } x \not< 0\,; \\ -x, \text{ when } x<0. \end{cases}$$

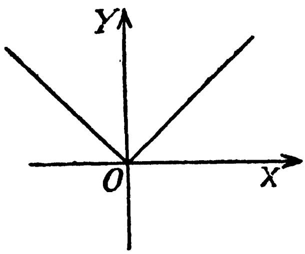


Fig. 8

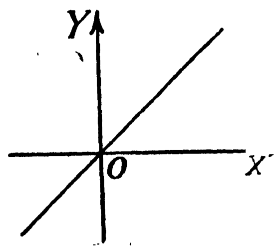


Fig. 9

The student should compare the graph (Fig. 8) of $\sqrt{x^2}$ with the graph (Fig. 9) of x.

The reader may see that

$$\sqrt{x^2}=|x|.$$

6. *Draw the graph of*

$$y=|x|+|x-1|.$$

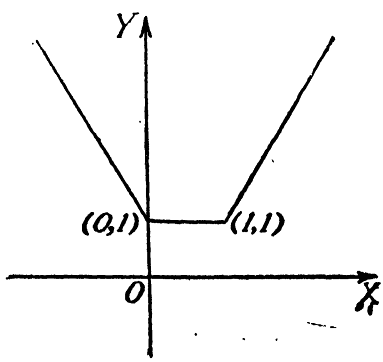


Fig. 10

We have

$$y=\begin{cases} -x+1-x=1-2x \text{ when } x \leqslant 0, \\ x+1-x=1 \text{ when } 0 \leqslant x \leqslant 1, \\ x+x-1=2x-1 \text{ when } x > 1. \end{cases}$$

Thus we have the graph as drawn :

The graph consists of parts of 3 straight lines

$$y=1-2x,\ y=1,\ y=2x-1$$

corresponding to the intervals

$$(-\infty, 0],\ [0, 1],\ [1, \infty).$$

7. *Dawn the graph of*

$$[x],$$

where $[x]$ *denotes the greatest integer not greater than x.*

We have

$$y = \begin{cases} 0, \text{ for } 0 \leqslant x < 1, \\ 1, \text{ for } 1 \leqslant x < 2, \\ 2, \text{ for } 2 \leqslant x < 3 \text{ and so on.} \end{cases}$$

The value of y for negative values of x can also be similarly given.

The right-hand end-point of each segment of the line is not a point of the graph.

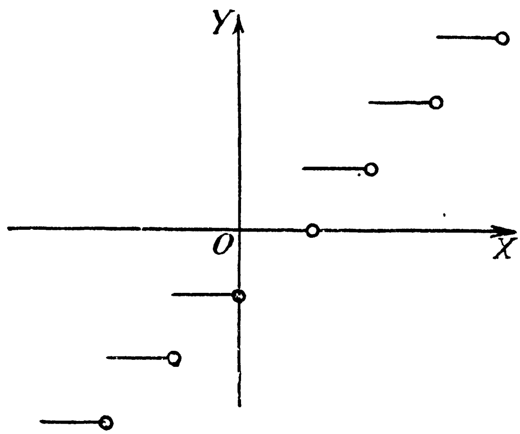


Fig. 11.

Exercises

1. Draw the graphs of the following functions :

(i) $f(x) = \begin{cases} 1, \text{ when } x \leqslant 0 \\ -1, \text{ when } x > 0 \end{cases}$
(ii) $f(x) = \begin{cases} x, \text{ when } 0 \leqslant x < \frac{1}{2}. \\ 1-x, \text{ when } \frac{1}{2} \leqslant x < 1. \end{cases}$

(iii) $f(x) = \begin{cases} x, \text{ when } 0 \leqslant x \leqslant \frac{1}{2} \\ 2-x, \text{ when } \frac{1}{2} < x < 1 \end{cases}$
(iv) $f(x) = \begin{cases} x, \text{ when } 0 \leqslant x < \frac{1}{2}\ ; \\ 1, \text{ when } x = \frac{1}{2}\ ; \\ 1-x, \text{ when } \frac{1}{2} < x < 1\ ; \end{cases}$

(v) $f(x) = \begin{cases} x^2, \text{ when } x \leqslant 0\ ; \\ \sqrt{x}, \text{ when } x > 0\ ; \end{cases}$
(vi) $f(x) = \begin{cases} 1/x, \text{ when } x < 0 : \\ 0, \text{ when } x = 0\ ; \\ -1/x. \text{ when } x > 0. \end{cases}$

2. Draw the graphs of the following functions :

(i) $\dfrac{\sqrt{x^2}}{x}$.
(ii) $x + \dfrac{\sqrt{(x-1)^2}}{x-1}$.

(iii) $x + \dfrac{\sqrt{(x-1)^2}}{x-1} + \dfrac{\sqrt{(x-2)^2}}{x-2}$.

(iv) $x + \dfrac{\sqrt{(x-1)^2}}{x-1} + \dfrac{\sqrt{(x-2)^2}}{(x-2)} + \dfrac{\sqrt{(x-3)^2}}{x-3}$.

The positive value of the square root is to be taken in each case.

3. Draw the graphs of the functions :

(i) $|x|$. (ii) $|x|+|x+1|$. (iii) $2|x-1|+3|x+2|$.

4. Draw the graphs of the functions :

(i) $[x]^2$, (ii) $[x]+[x+1]$

where $[x]$ denotes the greatest integer not greater than x.

CHAPTER II

SOME IMPORTANT CLASSES OF FUNCTIONS AND THEIR GRAPHS

Introduction. This chapter will deal with the graphs and some simple properties of the elementary functions

$$x^n, a^x, \log_a x ;$$

$$\sin x, \cos x, \tan x, \cot x, \sec x, \operatorname{cosec} x ;$$

$$\sin^{-1}x, \cos^{-1}x, \tan^{-1}x, \cot^{-1}x, \sec^{-1}x, \operatorname{cosec}^{-1}x.$$

The logarithmic function is inverse of the exponential just as the inverse trigonometric functions are inverses of the corresponding trigonometric functions. The trigonometric functions being periodic, the inverse trigonometric are multiple-valued and special care has, therefore, to be taken to define them so as to introduce them as single-valued.

2·1. *Graphical representation of the function*

$$\mathbf{y=x^n} ;$$

n being any integer, positive or negative.

We have, here, really to discuss a class of functions obtained by giving different integral values to n.

It will be seen that, from the point of view of graphs, the whole of this class of functions divides itself into four sub-classes as follows :—

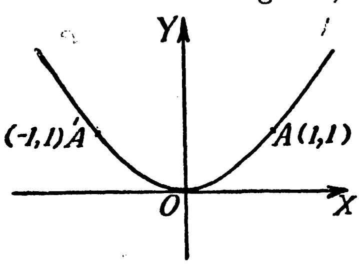
(*i*) *when n is a* **positive even** *integer ;* (*ii*) *when n is a* **positive odd** *integer ;* (*iii*) *when n is a* **negative even** *integer* (*iv*) *when n is a* **negative odd** *integer.*

The functions belonging to the same sub-class will be seen to have graphs similar in general outlines and differing only in details.

Each of these four cases will now be taken up one by one.

2·11. Let *n be a* **positive even** *integer.*

The following are, obviously, the properties of the graph of $y=x^n$ whatever positive even integral value, n may have.



$y=x^n$,
(n, a positive even integer)
Fig. 12.

(*i*) $y=0$, when $x=0$;
$y=1$, when $x=1$;
$y=1$, when $x=-1$.

The graph, therefore, passes through the points,

O (0, 0), A (1, 1), A' (−1, 1).

(*ii*) y is positive when x is positive or negative. Thus no point on the graph lies in the third or the fourth quadrant.

(*iii*) We have

$$y=x^n=(-x)^n,$$

so that the same value of y corresponds to two equal and opposite values of x. The graph is, therefore, symmetrical about the y-axis.

(*iv*) The variable y gets larger and larger, as x gets larger and larger numerically. Moreover, y can be made as large as we like by taking x sufficiently large numerically. The graph is, therefore, not closed.

The graph of $y=x^n$, for *positive even integral* value of n is, in general outlines, as given in Fig. 12. page. 18.

2·12. *Let n be a* **positive odd** *integer.*

The following are the properties of $y=x^n$, whatever positive odd integral value n may have.

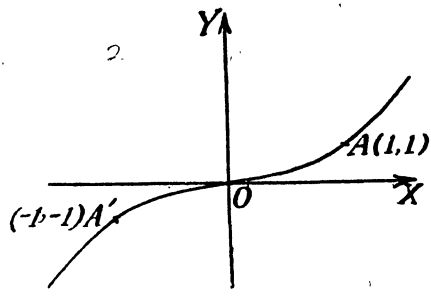
(*i*) $y=0$, when $x=0$;
$y=1$, when $x=1$;
$y=-1$, when $x=-1$

The graph, therefore, passes through the points

$O(0, 0)$, A $(1, 1)$, A' $(-1,-1)$.

(*ii*) y is positive or negative according as x is positive or negative.

Thus, no point on the graph lies in the second or the fourth quadrant.

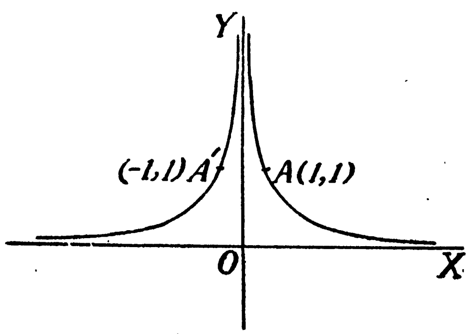


$y=x^n$,
(n, a positive odd integer)
Fig 13.

(*iii*) The numerical value of y increases with an increase in the numerical value of x.

Also, the numerical value of y can be made as large as we like by taking x sufficiently large numerically.

2·13. *Let n be a* **negative even** *integer, say,* $-m$, $(m>0)$.



$y=x^n$, (n, a negative even integer)
Fig. 14

Here,

$$y=x^n=\frac{1}{x^{-n}}=\frac{1}{x^m}.$$

The following are the properties of $y=x^n$ whatever negative even integral value, n, may have

(*i*) Determination of y for $x=0$ involves the meaningless operation of division by 0 and so y is not defined for $x=0$.

(*ii*) $y=1$ when $x=1$ or -1, so that the graph passes through the two points $A\ (1, 1)$ and $A'\ (-1, 1)$.

(*iii*) y is positive whether x be positive or negative. Thus no point on the graph lies in the third or in the fourth quadrant.

(*iv*) We have

$$y=x^n=(-x)^n,$$

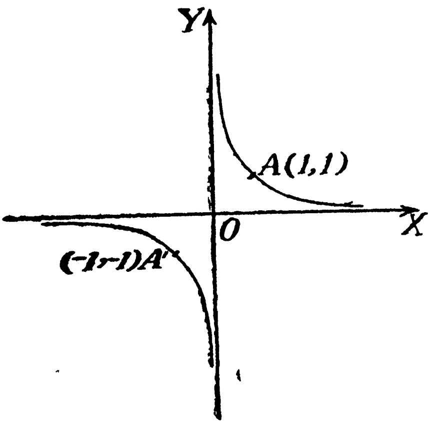
so that the same value of y corresponds to the two equal and opposite values of x. The graph is, therefore, symmetrical about y-axis.

(*v*) As x, starting from 1, increases, x^m also increases so that y decreases. Also y can be made as small as we like (*i.e.*, as near zero as we like) by taking x sufficiently large.

Again, as x, starting from 1, approaches zero, x^m decreases so that y increases. Also y can be made as large as we like by taking x sufficiently near 0.

The variation of y for negative values of x may be, now, put down by symmetry.

2·14. *Let n be a* **negative odd** *integer.*



$y=x^n$
(n, a negative odd integer)
Fig. 15.

The statement of the various properties for this case is left to the reader. The graph only is shown here. (Fig. 15)

2·2. Before considering the graphs of other functions, we allow ourselves some digression in this article to facilitate some of the later considerations.

2·21. Monotonic functions. Increasing or decreasing functions.

Def. 1. *A function $y=f(x)$ is called a* **monotonically increasing** *function in an interval if it is such that throughout the interval a larger value of x gives a large value of y, i.e.*, an increase in the value of x always causes an increase in the value of y.

Def. 2. *A function is called a* **monotonically decreasing** *function, if it is such that an increase in the value of x always causes a decrease in the value of the function.*

Thus, if x_1 and x_2 are *any* two numbers in an interval such that

$$x_2 > x_1,$$

then for a monotonically increasing function in the interval

$$f(x_2) > f(x_1)$$

and for a monotonically decreasing function in the interval

$$f(x_2) < f(x_1)$$

A function which is either monotonically increasing or decreasing is called a *monotonic* function.

Ex. Show that for a monotonic function $f(x)$ in an interval (a, b), the fraction

$$[f(x_2)-f(x_1)]/(x_2-x_1)$$

keeps the same sign for any pair of different numbers, x_1, x_2 of (a, b).

Illustration. Looking at the graphs in § 2·1, we notice that x^n is

(*i*) monotonically decreasing in the interval $(-\infty, 0)$ and monotonically increasing in the interval $[0, \infty]$ when n is a positive even integer ;

(*ii*) monotonically increasing in the intervals $[-\infty, 0]$, $[0, \infty]$ when n is a positive odd integer.

(*iii*) monotonically increasing in the interval $(-\infty, 0)$ and monotonically decreasing in the interval $(0, \infty)$ when n is a negative even integer.

(*iv*) monotonically decreasing in the intervals $(-\infty, 0)$, $(0, \infty)$ when n is a negative odd integer.

2·22. Inverse functions.

Let

$$y=f(x) \qquad \text{...(1)}$$

be a given function of x and suppose that we can solve this equation for x in terms of y so that we may write x as a function of y, say

$$x=\varphi(y). \qquad \text{...(2)}$$

Then $\varphi(y)$ is called the *inverse* of $f(x)$.

This process of defining and determining the inverse of a given function may be accompanied with difficulties and complications as illustrated below :—

(*i*) The functional equation (1) may not always be solvable for x as a function of y as, for instance, the case

$$y=x^2+2x^3.$$

(*ii*) The functional equation (1) may not always determine a *unique* value of y in terms of x as, for instance, in the case of the functional equation

$$y=1+x^2, \qquad \text{...(3)}$$

which, on solving for x, gives

$$x=\pm\sqrt{(y-1)}, \qquad \text{...(4)}$$

so that to each value of y there correspond two values of x. This situation is explained by the fact that in (3) two values of x which are equal in absolute value but opposite in sign give rise to the same value of y and so, conversely, to a given value of y there must naturally correspond two values of x which gave rise to it.

We cannot, therefore, look upon, x, as a function of, y, in the usual sense, for, according to our notion of a function there must correspond *just one* value of, x, to a value of, y.

In view of these difficulties, it is worthwhile to have a simple test which may enable us to determine whether a given function $y=f(x)$ admits of an inverse function or not.

If a function $y=f(x)$ is such that it *increases monotonically in an interval* $[a, b]$ *and takes up every value between its smallest value* $f(a)$ *and its greatest value* $f(b)$, then it is clear that to each value of y between $f(a)$ and $f(b)$, there corresponds one and only one value of x which gave rise to it, so that x is a function of y defined in the interval $[f(a), f(b)]$.

Similar conclusion is easily reached if y decreases monotonically.

Illustrations of this general rule will appears in the following articles.

2·3. *To draw the graph of*

$$y=x^{\frac{1}{n}};$$

n being any positive or negative integer.

From an examination of the graph of $y=x^n$ drawn in § 2·1, or even otherwise, we see that $y=x^n$

(*i*) is monotonic and positive in the interval $(0, \infty)$ whatever integral value n may have ;

(*ii*) takes up every positive value.

Thus, from § 2·2 or otherwise, we see that to every *positive* value of y there corresponds one and only one *positive* value of x such that

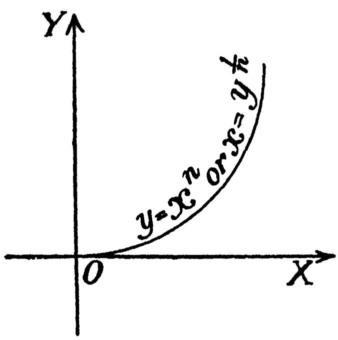
$$y=x^n \text{ or } x=y^{\frac{1}{n}}.$$

Thus

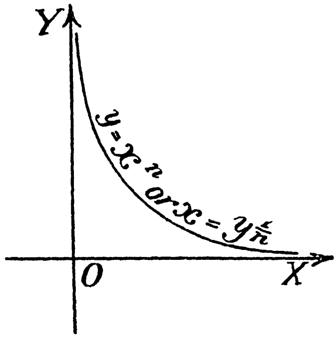
$$x=y^{\frac{1}{n}}$$

determines x as a function of y in the interval $(0, \infty)$; x being also always positive.

The part of the graph of $y=x^n$, lying on the first quadrant, is also the graph of $x=y^{\frac{1}{n}}$.



when n is a positive integer.
Fig. 16.



when n is a negative integer.
Fig. 17.

To draw the graph of $y=x^{\frac{1}{n}}$ when x is independent and y dependent, we have to change the point (h, k) in

$$x=y^{\frac{1}{n}}$$

to the point (k, h) *i.e.*, find the reflection of the curve

$$x=y^{\frac{1}{n}}$$

in the line $y=x$. This is illustrated in the figure given below.

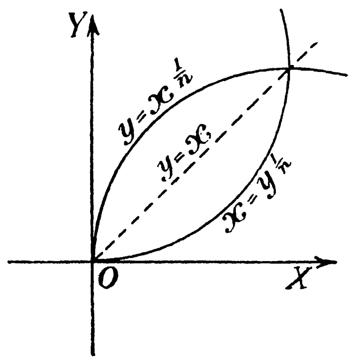
Cor. $x^{p/q}$ means $(x^p)^{1/q}$ where the root is to be taken positively and x is also positive.

Note. It may be particularly noted that in order that $x^{1/n}$ may have *one and only one* value, we have to restrict x to positive values only and $x^{1/n}$ is then to be taken to mean the positive nth root of x.

It is not enough to say that $x^{1/n}$ is an nth *root of* x; we must say the positive nth *root of* the positive number x.

For, if n be even and x be positive, then x has two nth roots, one positive and the other negative, so that the nth root is not unique and if n be even and x be negative, then x has no nth root so that $x^{1/n}$ does not exist.

Thus x and $x^{1/n}$ are both positive.



when n is a positive integer.
Fig. 18.

2·4. The exponential function a^x. The meaning of a^x when a is positive and x is any rational number, is already known to the student. (It has also been considered in § 2·3 above). To give the rigorous meaning of a^x when the index, x is *irrational* is beyond the scope of this book. However, to obtain some idea of the meaning of a^x in this case, we proceed as follows :

We find points on the graph of $y=a^x$ corresponding to the rational values of x. We then join them so as to determine a continuous curve. The ordinate of the point of this curve corresponding to any irrational value of x as its abscissa, is then taken to be the value of a^x for that particular value.

Note. The definition of a^x when x is irrational, as given here, is incomplete in as much as it assumes that what we have done is actually possible and that also uniquely. Moreover it gives no precise analytical way of representing a^x as a decimal expression.

Graph of $y=a^x$. In order to draw the graph of

$$y=a^x,$$

we note its following properties :—

2·41. *Let* $a>1$.

(*i*) The function a^x is always positive whether x be positive or negative.

(*ii*) It increases monotonically as x increases and can be made as large as we like by taking x sufficiently large.

(*iii*) $a^0=1$ so that the point (0, 1) lies on the graph.

(*iv*) Since $a^x=1/a^{-x}$, we see that a^x is positively very small when x is negatively very large and can be made as near zero as we like provided we give to x a negative value which is sufficiently large numerically.

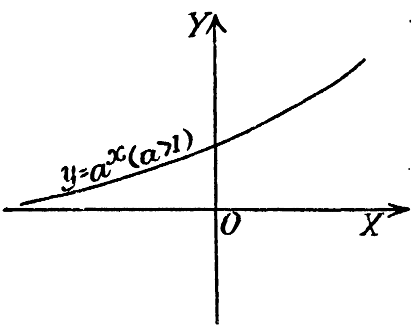


Fig. 19.

With the help of these facts, we can draw the graph which is as shown in (Fig. 19).

2·42. *Let* $0<a<1$.

(*i*) The function a^x is always positive whether x be positive or negative.

(*ii*) It decreases monotonically as x increases and can be made as near zero as we like by taking x sufficiently large.

(*iii*) $a^0=1$. Therefore the point (0, 1) lies on the graph.

(*iv*) Since $a^x=1/a^{-x}$, we see that a^x is positively very large when x is negatively very large. Also it can be made as large as we like if we give to x a negative value which is sufficiently large numerically.

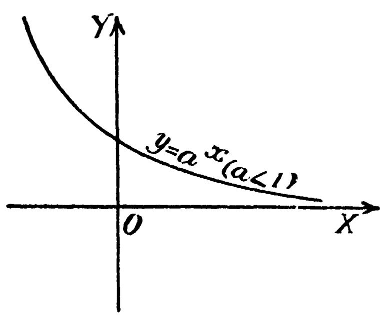


Fig. 20.

Thus we have the graph as drawn in (Fig. 20).

Note 1. In the case of the function a^x it is the exponent x which is a variable and the base is a constant. This is the justification for the name "*Exponential function*" given to it. For the function x^n, the base is a variable and the exponent is a constant.

Note 2. a^x never vanishes and is always positive :

Ex. Draw the graphs of the functions :

(*i*) $2^{1/x}$ (*ii*) $2^{1/x^2}$ (*iii*) $2^{-1/x}$ (*iv*) $2^{-1/x^2}$.

2·5. Graph of the logarithmic function

$$\mathbf{y=\log_a x}\ ;$$

a, x being any positive numbers.

We know that

$$y=a^x$$

can be written as

$$x=\log_a y.$$

We have seen in § 2·4 that $y=a^x$ is monotonic and takes up every positive value as x increases taking all values from $-\infty$ to $+\infty$.

Thus to any positive value of y there corresponds one and only one value of x.

In the figure of § 2·4, we take y as independent variable and suppose that it continuously varies from 0 to ∞. Thus we see that

(*i*) x monotonically increases from $-\infty$ to ∞, if $a>1$;

(*ii*) x monotonically decreases from ∞ to $-\infty$, if $a<1$.

Also, $x=0$ for $y=1$.

Considering the functional equation

$$y=\log_a x,$$

we see that as the independent variable x varies from 0 to ∞ the dependent variable y monotonically increases from $-\infty$ to ∞ if $a>1$ and monotonically decreases from ∞ to $-\infty$ if $a<1$.

Also, $y=0$ for $x=1$.

We thus have the graphs as drawn.

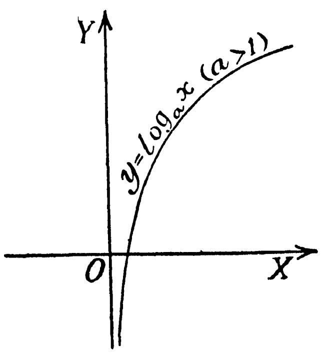


Fig. 21.

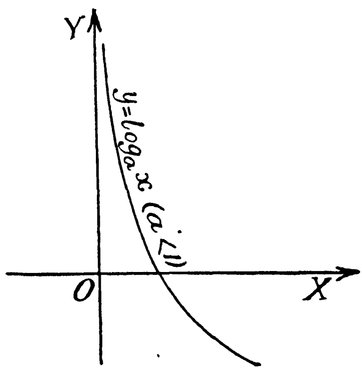


Fig. 22.

Note. The graph of $y=\log_a x$ is the reflection of $y=a^x$ in the line $y=x$.

Some important properties of $\log_a x$.

We may, now, note the following important properties of

$$y=\log_a x.$$

(*i*) $\log_a x$ is defined for positive values of x only and the base a is also positive.

(*ii*) If $a>1$, $\log_a x$ can be made as large as we like by taking x sufficiently large and can be made as small as we like by taking x sufficiently near 0.

For $a<1$, we have similar statements with obvious alterations.

(*iii*) $\log_a 1=0$.

When $a>1$, we have, $\log_a x>0$, if $x>1$, and $\log_a x<0$, if $x<1$;

When $a<1$, we have, $\log_a x<0$, if $x>1$, and $\log_a x>0$, if $x<1$.

2·6. Trigonometric functions.

It will be assumed that the student is already familiar with the definitions, properties and the nature of variations of the fundamental trigonometrical functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\text{cosec } x$. The most important property of these functions is their *periodic character*, the period for $\sin x$, $\cos x$, $\sec x$, $\text{cosec } x$ being 2π and that for $\tan x$, $\cot x$ being π.

We will only produce their graphs and restate some of their important and well known properties here.

2·61. y=sin x.

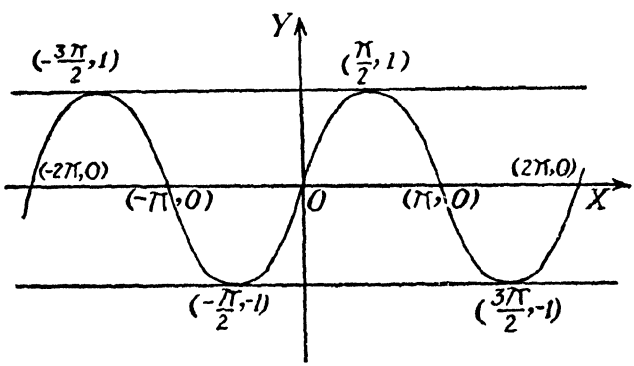


Fig. 23.

(*i*) It is defined for every value of x.

(*ii*) It increases monotonically from -1 to 1 as x increases from $-\pi/2$ to $\pi/2$; it decreases monotonically from 1 to -1 as x increases from $\pi/2$ to $3\pi 2$, and so on.

(*iii*) $-1 \leqslant \sin x \leqslant 1$, *i.e.*, $|\sin x| \leqslant 1$, whatever value x may have.

2·62. y=cos x.

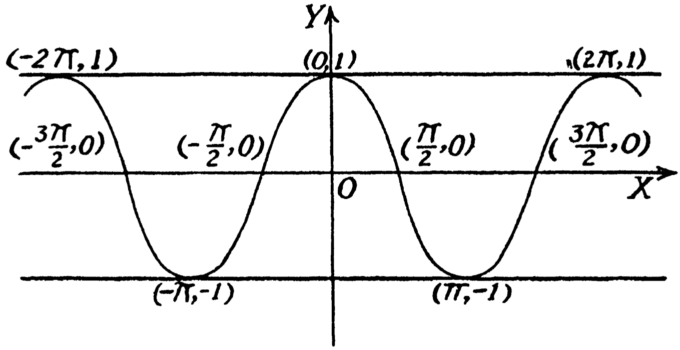


Fig. 24.

(*i*) It is defined for every value of x.

(*ii*) It decreases monotonically from 1 to -1 as x increases from 0 to π ; and so on.

(*iii*) $|\cos x| \leqslant 1$ whatever value x may have.

2·63. y=tan x.

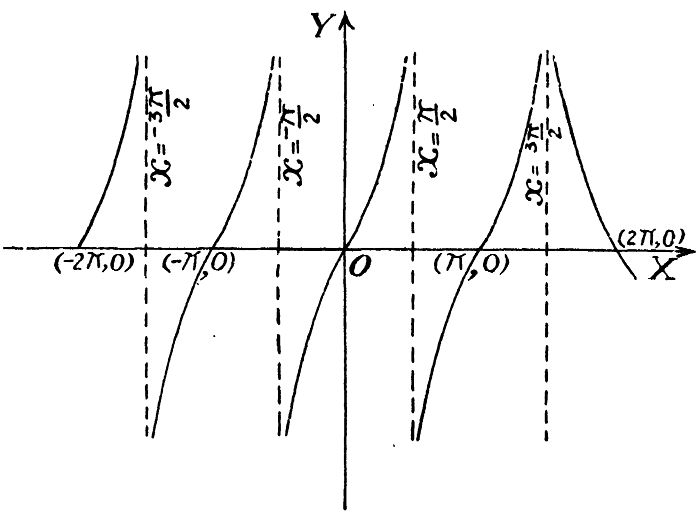


Fig. 25.

(*i*) It is defined for all values of x excepting

$$\ldots\ldots -\tfrac{7}{2}\pi,\ -\tfrac{5}{2}\pi,\ -\tfrac{3}{2}\pi,\ -\tfrac{1}{2}\pi,\ \tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi,\ \tfrac{5}{2}\pi,\ \tfrac{7}{2}\pi,\ldots\ldots$$

i.e., $(2n+1)\pi/2$, where n is any integer, positive or negative. It fact it will be recalled that for each of these angles the base of the right-angled triangle which defines tan x becomes 0 and so the definition of tan x, as being the ratio of height to base, involves division by 0 which is a meaningless operation.

(*ii*) It increases monotonically from $-\infty$ to $+\infty$ as x increases from $-\pi/2$ to $\pi/2$; it increases monotonically from $-\infty$ to $+\infty$ as x increases from $\pi/2$ to $3\pi/2$, and so on.

(*iii*) It can be made positively as large as well like provided we take

$$x<\pi/2 \text{ and sufficiently near to it ;}$$

and can be made negatively as large as we like provided we take

$$x>-\pi/2 \text{ and sufficiently near to it.}$$

2·64. y=cot x.

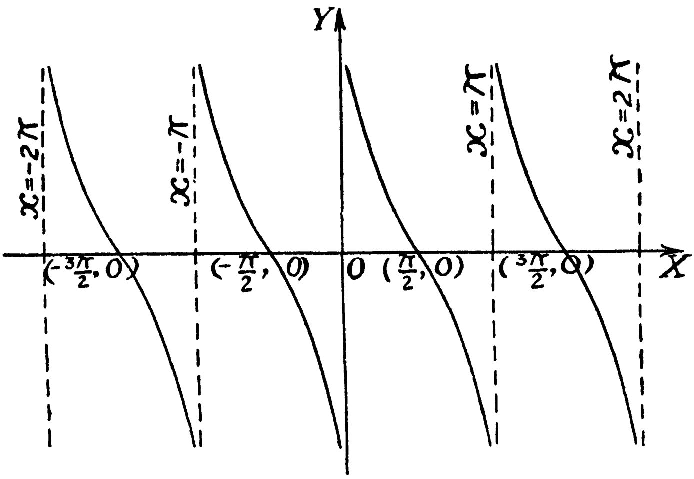


Fig. 26.

(*i*) It is defined for all values of x excepting

$$\ldots\ldots -4\pi,\ -3\pi,\ -2\pi,\ -\pi,\ 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ldots\ldots$$

i.e., $n\pi$, where n has any integral value, since for each of these angles the height of the right-angled triangle which defines cot x is zero and the definition of cot x being the ratio of base to height involves division by zero.

(*ii*) It decreases monotonically from ∞ to $-\infty$ as x increases from 0 to π ; it decreases monotonically from ∞ to $-\infty$ as x increases from π to 2π ; and so on.

(*iii*) It can be made positively as large as we like provided we take

$$x>0 \text{ and sufficiently near to it ;}$$

and can be made negatively as large as we like provided we take

$$x<0 \text{ and sufficiently near to it.}$$

2·65. y = sec x.

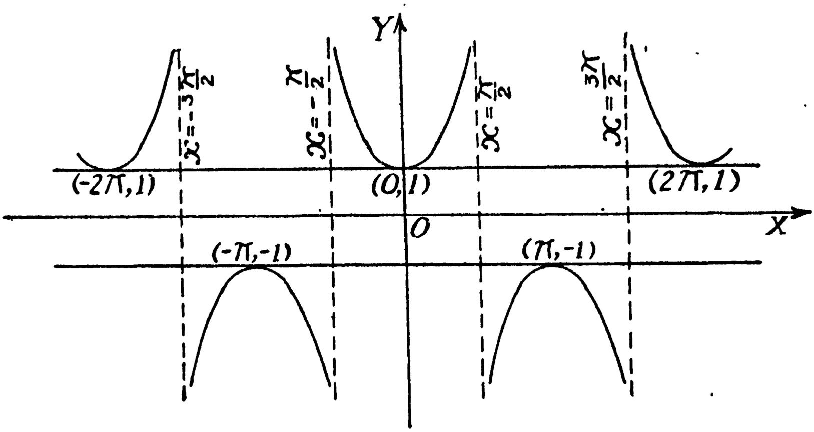


Fig.

It is defined for all values of x excepting

$$\ldots\ldots -\tfrac{5}{2}\pi,\ -\tfrac{3}{2}\pi,\ -\tfrac{1}{2}\pi,\ \tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi,\ \tfrac{5}{2}\pi,\ldots\ldots$$

i.e., $(2n+1)\,\pi/2$, where n has any integral value, since for each of these angles, the base of the right angled triangle which defines sec x is zero and as such the definition of sec x involves division by zero.

(*ii*) $|\sec x| \geqslant 1$ whatever value x may have.

(*iii*) It increases monotonically from 1 to ∞ when x increases from 0 to $\pi/2$; it increases monotonically from $-\infty$ to -1 when x increases from $\pi/2$ to π, and so on.

(*iv*) It can be made positively as large as we like by taking

$x<\pi/2$ and sufficiently near to it ;

and can be made negatively as large as we like by taking

$x>\pi/2$ and sufficiently near to it.

2·66. y = cosec x.

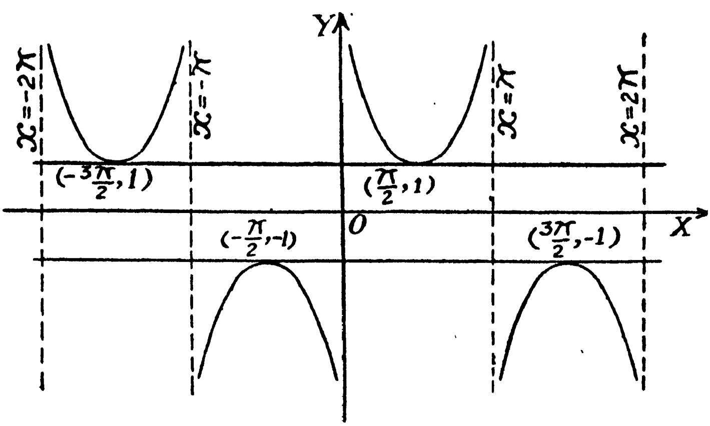


Fig. 28.

(*i*) It is defined for all values of x excepting

$$\ldots\ldots -3\pi,\ -2\pi,\ -\pi,\ 0,\ \pi,\ 2\pi,\ 3\pi, \ldots\ldots$$

i.e., $n\pi$, when n has any integral value.

(*ii*) $|\operatorname{cosec} x| \geqslant 1$ whatever value x may have.

(*iii*) It decreases monotonically from -1 to $-\infty$ as x increases from $-\pi/2$ to 0; it decreases monotonically from ∞ to 1 as x increases from 0 to $\pi/2$.

(*iv*) It can be made positively as large as we like by taking

$x>0$ and sufficiently near to it;

and can be made negatively as large as we like by taking

$x<0$ and sufficiently near to it.

2·7. Inverse trigonometrical functions. The inverse trigonometrical functions

$$\sin^{-1}x,\ \cos^{-1}x,\ \tan^{-1}x,\ \cot^{-1}x,\ \sec^{-1}x,\ \operatorname{cosec}^{-1}x$$

are generally defined as the inverse of the corresponding trigonometrical functions. For instance, $\sin^{-1}x$ is defined as the angle whose sine is x. The definition, as it stands, is *incomplete* and ambiguous as will now be seen.

We consider the functional equation

$$x=\sin y \qquad \ldots (i)$$

where y is the independent and x the dependent variable.

Now, to each value of y, there corresponds just one value of x in (*i*). On the other hand, the same value of x corresponds to an unlimited number of values of the angle so that to any given value of x between -1 and 1 there corresponds an unlimited number of values of the angle y whose sine is x. Thus $\sin^{-1}x$, as defined above, is not *unique*.

The same remark applies to the remaining trigonometric functions also.

We now proceed to *modify* the definitions of the inverse trigonometrical functions so as to remove the ambiguity referred to here.

2·71. $y=\sin^{-1}x$.

Consider the functional equation

$$x=\sin y.$$

We know that as y increases from $-\pi/2$ to $\pi/2$, then x increases monotonically, taking up every value between -1 and 1, so that to each value of x between -1 and 1 there corresponds *one and only one* value of y lying between $-\pi/2$ and $\pi/2$. Thus there is one and only one angle lying between $-\pi/2$ and $\pi/2$ with a given sine.

Accordingly we define $\sin^{-1}x$ as follows :—

$\sin^{-1}x$ is the angle **lying between** $-\pi/2$ **and** $\pi/2$, *whose sine is x.*

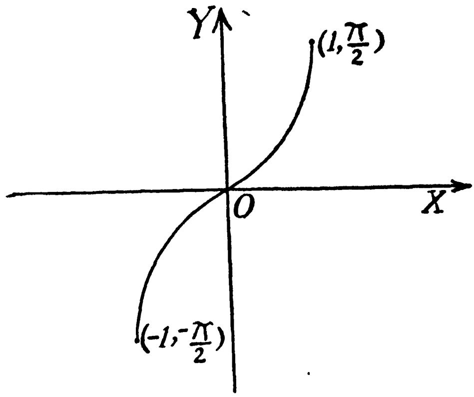


Fig. 29.

To draw the graph of

$y = \sin^{-1}x$,

we note that

(*i*) y increases monotonically from $-\pi/2$ to $\pi/2$ as x increases from -1 to 1.

(*ii*) $\sin^{-1} 0 = 0$, $\sin^{-1} 1 = \pi/2$, $\sin^{-1}(-1) = -\pi/2$.

(*iii*) $\sin^{-1}x$ is defined in the interval $[-1, 1]$ only.

We, thus, have the graph as drawn.

Note 1. We know that if $x = \sin y$, then x varies monotonically taking up every value between -1 and 1 as y increases from $\pi/2$ to $3\pi/2$, $3\pi/2$ to $5\pi/2$, and so on. Thus the definition could also have been equally suitably modified by restricting $\sin^{-1}x$ to any of the intervals $(\pi/2, 3\pi/2)$, $(3\pi/2, 5\pi/2)$ instead of $(-\pi/2, \pi/2)$. What we have done here is, however, more usual.

Note 2. The graph of $y = \sin^{-1}x$ is the reflection of $y = \sin x$ in the line $y = x$.

2·72. $y = \cos^{-1}x$.

Consider the functional equation

$$x = \cos y.$$

We know that as y increases from 0 to π, then x decreases monotonically taking up every value between 1 and -1. Thus, there is *one and only one* angle, lying between 0 and π, with a given cosine. Accordingly we define $\cos^{-1}x$ as follows :—

$\cos^{-1}x$ is the angle **lying between 0 and** π, *whose cosine is x.*

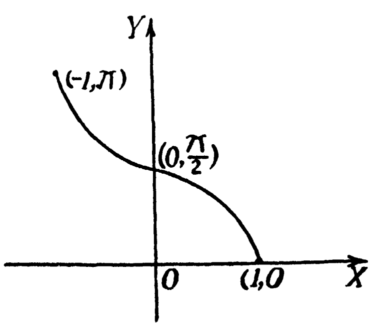


Fig. 30,

To draw the graph of

$y = \cos^{-1}x$,

we note that

(*i*) y decreases monotonically from π to 0 as x increases from -1 to 1.

(*ii*) $\cos^{-1}(-1) = \pi$, $\cos^{-1} 0 = \pi/2$, $\cos^{-1}(1) = 0$.

(*iii*) $\cos^{-1}x$ is defined in the interval $[-1, 1]$ only.

Note. We know that $x = \cos y$ varies monotonically, taking up every value between -1 and 1 as y increases from π to 2π, 2π to 3π and so on. Thus the definition could also have been equally suitably modified by restricting $\cos^{-1} x$ to any of the intervals, $[\pi, 2\pi]$, $[2\pi, 3\pi]$ etc., instead of (0π).

2·73. $y = \tan^{-1} x$.

We consider the functional equation

$$x = \tan y.$$

We know that as y increases from $-\pi/2$ to $\pi/2$, then x increases monotonically from $-\infty$ to ∞ taking up every value. Thus there is *one and only one* angle between $-\pi/2$ and $\pi/2$ with a given tangent. Accordingly we have the following definition of $\tan^{-1} x$:

$\tan^{-1}x$ is the angle, **lying between $-\pi/2$ and $\pi/2$,** *whose tangent is x.*

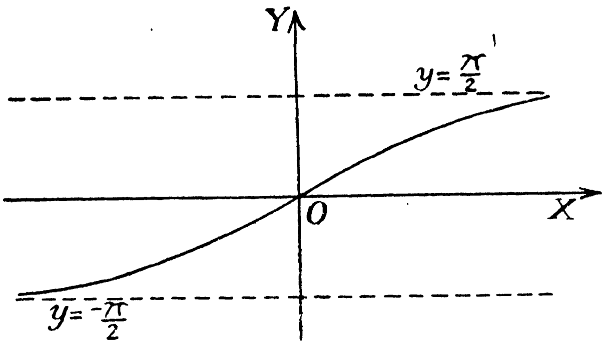


Fig. 31

To draw the graph (Fig. 31) *of*

$$y = \tan^{-1} x,$$

note that y increases monotonically from $-\pi/2$ to $\pi/2$ as x increases from $-\infty$ to ∞ and then $\tan^{-1} 0 = 0$.

2·74. $y = \cot^{-1} x$.

Consider the functional equation

$$x = \cot y.$$

We know that as y increases from 0 to π, then x decreases monotonically from $+\infty$ to $-\infty$ taking up every real value. Thus, there is one and only one angle between 0 and π with given cotangent.

In view of this, we define $\cot^{-1} x$ as follows :

$\cot^{-1} x$ is the angle, **lying between 0 and π,** *whose cotangent is x.*

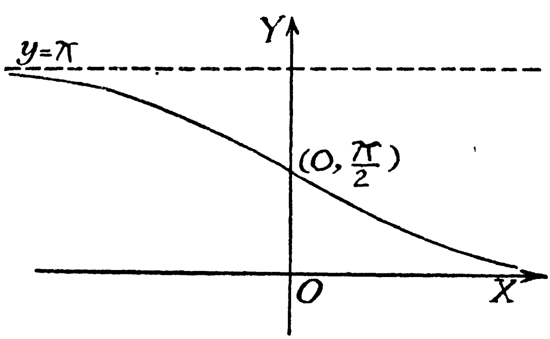


Fig. 32

To draw the graph (Fig. 32) *of*

$$y = \cot^{-1} x,$$

we note that y decreases monotonically from π to 0 as x increases from $-\infty$ to ∞ and $\cot^{-1} 0 = \pi/2$.

2·75. $y = \sec^{-1} x$.

Consider the functional equation

$$x = \sec y.$$

We know that as y increases from 0 to $\pi/2$, then x increases monotonically from 1 to ∞ ; also as y increases from $\pi/2$ to π, then x increases monotonically from $-\infty$ to -1.

Thus, there is *one and only one* value of the angle, lying between 0 and π, whose secant is any given number, not lying between -1 and 1. The following is, thus, the precise definition of $\sec^{-1} x$:—

$\sec^{-1} x$ is the angle, **lying between 0 and π**, *whose secant is x.*

To draw the graph of

$$y = \sec^{-1} x,$$

we note that y increases from 0 to $\pi/2$ as x increases from 1 to ∞ and y increases from $\pi/2$ to π as x increases from $-\infty$ to -1. Also

$$\sec^{-1} 1 = 0, \text{ and } \sec^{-1} (-1) = \pi.$$

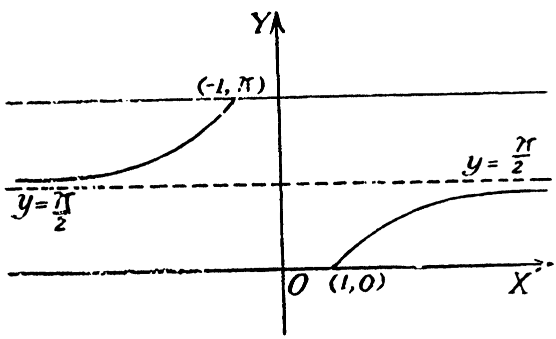


Fig. 33

2·76. $y = \text{cosec}^{-1} x$.

Consider the functional equation

$$x = \text{cosec } y.$$

We know that as y increases from $-\pi/2$ to 0, then x decreases monotonically from -1 to $-\infty$ and as y increases from 0 to $\pi/2$, it decreases monotonically from ∞ to 1. Thus there is *one and only one* value of the angle lying between $-\pi/2$ and $\pi/2$ having a given cosecant. We thus say that

$\text{cosec}^{-1} x$ is the angle, **lying between** $-\pi/2$ and $\pi/2$, whose *cosecant is x.*

To draw the graph of

$$y = \text{cosec}^{-1} x.$$

we note that y decreases from 0 to $-\pi/2$ as x increases from $-\infty$ to -1 any y decreases from $\pi/2$ to 0 as x increases from 1 to ∞.

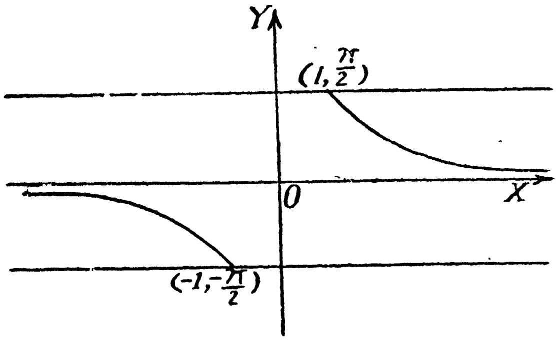


Fig. 34

Ex. What other definitions of $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$ would have been equally suitable.

Note. The graphs of $y=\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$ are the reflections of $y=\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ respectively in the line $y=x$.

2·8. Function of a Function. The notion of a *function of function* will be introduced by means of examples.

Ex. 1. Consider the two equations

$$u=x^2 \quad \ldots(1)$$

$$y=\sin u. \quad \ldots(2)$$

To any given value of x, corresponds a value of u as determined from (1) and to this value of u, corresponds a value of y as determined from (2) so that we see that y is a function of u which is again a function of x, *i.e.*, y is a function of a function of x.

From a slightly different point of view, we notice that since the two equations (1) and (2) associate a value of y to any given value of x, they determine *y as a function of x* defined for the entire aggregate of real numbers.

Combining the two equations, we get

$$y=\sin x^2$$

which defines y directly as a function of x and not through the intermediate variable u.

Ex. 2. Consider the two equations

$$u=\sin x, \quad \ldots(1)$$

$$y=\log u. \quad \ldots(2)$$

The equation (1) determines a value of u corresponding to every given value of x. The equation (2) then associates a value of y to this value of u in case it is positive. But we know that u is positive if and only if x lies in the open intervals

$$\ldots\ldots\ldots\ldots(-4\pi, -3\pi), (-2\pi, -\pi), (0, \pi), (2\pi, 3\pi) \quad \ldots(3)$$

Thus these two equations define y as a function of x where the domain of variation of x is the set of intervals (3).

General consideration.

Let

$$u=f(x),$$

and

$$y=\phi(u),$$

be two functional equations such that $f(x)$ is defined in the interval $[a, b]$ and $\phi(u)$ in $[c, d]$.

Let, further, each value of $f(x)$ lie in $[c, d]$.

Clearly, then, these two equations determine y as a function of x defined for the interval $[a, b]$.

2·9. Classification of functions. Algebraic and Transcendental. Any given function is either

(*i*) *Algebraic* or (*ii*) *transcendental.*

A function is said to be algebraic if it arises by performing upon the variable x and any number of constants a **finite** *number of operations of addition, subtraction, multiplication, division and root extraction. Thus,*

$y=x^2+7x+3$, $y=(7x^2+2)/(3x^4+5x^2)$, $y=\sqrt{(x+3\sqrt{x^3}+\sqrt{x})}$ *are algebraic functions.*

A function is said to be transcendental if it is not algebraic.

The exponential, logarithmic, trigonometric and inverse trigonometric functions are all transcendental functions.

Two particular cases of algebraic functions. There are two specially important types of algebraic functions, namely,

(*i*) *Polynomials* ;

(*ii*) *Rational functions.*

A function of the type

$$a_0x^m+a_1x^{m-1}+\ldots\ldots\ldots a_{m-1}x+a_m$$

where a_0, $a_1, \ldots\ldots\ldots, a_m$ are constants and m is a positive integer, is called a *polynomial in x.*

A function which appears as a quotient of two polynomials such as

$$\frac{a_0x^m+a_1x^{m-1}+\ldots+a_{m-1}x+a_m}{b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n}$$

is called a *rational function* of x.

A rational function of x is defined for every value of x excluding those for which the denominator vanishes.

Exercises

1. Write down the values of

(*i*) $\sin^{-1} \frac{1}{2}$, (*ii*) $\tan^{-1}(-1)$, (*iii*) $\sec^{-1} 2$, (*iv*) $\cos^{-1}(-\frac{1}{2})$.

2. *For what domains of values of the independent variable are the following functions defined :—*

(*i*) $\sin^{-1}\sqrt{x}$. (*ii*) $\log_2 \sin x$.

[**Sol.** (*i*) $\sin^{-1}\sqrt{x}$ is defined only for those values of x for which

$$-\leqslant \sqrt{x} \leqslant 1$$

and this will be the case if, and only if,

$$0 \leqslant x \leqslant 1,$$

so that $\sin^{-1}\sqrt{x}$ is defined in the interval [0, 1].

(*ii*) log sin x is defined only for the values of x for which

$$\sin x > 0$$

i.e., in the open intervals

$$(0, \pi), (2\pi, 3\pi), \ldots\ldots$$

or generally in $(2n\pi, \overline{2n+1}\ \pi)$; n being any integer.

(*iii*) $\log \sin^{-1} x$. (*iv*) $\log [(1+x)/(1-x)]$.

(*v*) $\log [(x+1)/(x-1)]$ (*vi*) $\log (\tan^{-'} x$

(*vii*) $(\sin x)^x$. (*viii*) $(\log_5 x)^x$.

(*ix*) $\log [x+\sqrt{(x^2-1)}]$. (*x*) $x \tan^{-1} x$.

(*xi*) $\dfrac{1}{\sqrt{[(x+1)(x+2)]}}$ (*xii*) $\dfrac{1}{\sin^3\theta - \cos^3\theta}$

(*xiii*) $\dfrac{1}{\sqrt{(1-x^2)}}$. (*xiv*) $\dfrac{a^x+a^{-x}}{a^x-a^{-x}}$ $(a \geqslant 0)$.

3. Find out the intervals in which the following functions are monotonically increasing or decreasing :—

(*i*) 2^x. (*ii*) $(\frac{1}{2})^x$. (*iii*) $3^{1/x}$. (*iv*) $4^{1/x^2}$. (*v*) $2 \sin x$.

4. Distinguish between the two functions

$$x, \frac{1}{\left(\frac{1}{x}\right)}.$$

CHAPTER III

CONTINUITY AND LIMIT

Introduction. The statement that a function of x is defined in a certain interval means that to each value of x belonging to the interval there corresponds a value of the function. *The value of the function for any particular value of x may be quite independent of the value of the function for another value of x* and no relationship in the various values of a function is implied in the definition of a function as such.

Thus, if x_1, x_2 are any two values of the independent variable so that $f(x_1)$, $f(x_2)$ are the corresponding values of the function $f(x)$, then $|f(x_2)-f(x_1)|$ may be large even though $|x_2-x_1|$ is small. It is because the value $f(x_2)$ assigned to the function for $x=x_2$ is quite independent of the value $f(x_1)$ of the function for $x=x_1$.

We now propose to study the change $|f(x_2)-f(x_1)|$ relative to the change $|x_2-x_1|$ by introducing the notion of continuity and discontinuity of a function.

In the next article, we first analyse the intuitive notion of *continuity* that we already possess and then state its precise analytical meaning in the form of a definition.

3·1. Continuity of a function at a point. Intuitively, *continuous* variation of a variable implies absence of *sudden changes* while it varies so that in order to arrive at a suitable definition of continuity, we have to examine the precise meaning of this implication in its relation to a function $f(x)$ which is defined in any interval $[a, b]$. Let, c, be any point of this interval.

The function $f(x)$ will vary continuously at $x=c$ if, as x changes from c to either side of it, the change in the value of the function is not sudden, *i.e.*, the change in the value of the function is *small* if only the change in the value of x is also *small.*

We consider a value $c+h$ of x belonging to the interval $[a, b]$. Here, h, which is the change n the independent variable x, may be positive or negative.

Then, $f(c+h)-f(c)$, is the corresponding change in the dependent variable $f(x)$, which, again, may be positive or negative.

For continuity, we require that $c+h)-f(c)$ should be numerically small, if h is numerically small. This means that $|f(c+h)-f(c)|$ can be made as small as we like by taking $|h|$ sufficiently small.

The precise analytical definition of continuity should not involve the use of the word *small*, whose meaning is definite, as there exists no definite and absolute standard of smallness. Such a definition would now be given.

To be more precise, we finally say that

A function $f(x)$ is continuous at $x=c$ if, corresponding to any positive number, ε, arbitrarily assigned, there exists a positive number δ such that

$$|f(c+h)-f(c)|<\varepsilon$$

for all values of, h, such that

$$|h| < \delta.$$

This means that $f(c+h)$ lies between $f(c)-\varepsilon$ and $f(c)+\varepsilon$ for all values of h lying between $-\delta$ and δ.

Alternatively, replacing $c+h$ by x, we can say that

$f(x)$ is continuous at $x=c$, if there exists an interval $(c-\delta, c+\delta)$ around c such that, for all values of x in this interval, we have

$$f(c)-\varepsilon < f(x) < f(c)+\varepsilon;$$

ε being any positive number arbitrarily assigned.

Meaning of continuity explained graphically. Draw the graph of the function $f(x)$ and consider the point $P[c, f(c)]$ on it.

Draw the lines

$$y=f(c)-\varepsilon,\ y=f(c)+\varepsilon \qquad \ldots(1)$$

which lie on different sides of the point P and are parallel to x-axis.

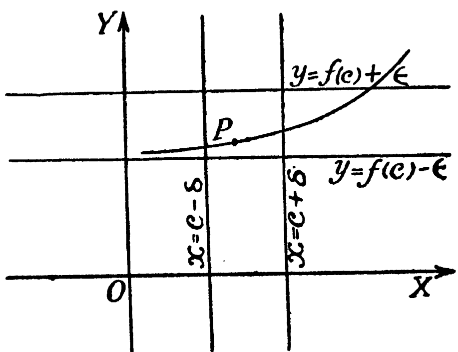


Fig. 35.

Here, ε is any arbitrarily assigned positive number and measures the degree of closeness of the lines (1) from each other.

The continuity of $f(x)$ for $x=c$, then, requires that we should be able to draw two lines

$$x=c-\delta,\ x=c+\delta, \quad \ldots(2)$$

which are parallel to y-axis and which lie around the line $x=c$, such that every point of the graph between the two lines (2) lies also between the two lines (1).

3·11. Continuity of a function in an interval. In the last articles, we dealt with the definition of continuity of a function $f(x)$ *at a point* of its interval of definition. We now extend this definition to *continuity in an interval* and say that

A function $f(x)$ is continuous in an interval $[a, b]$, if it is continuous at every point thereof.

Discontinuity. *A function $f(x)$ which is not continuous for $x=c$ is said to be discontinuous for $x=c$.*

The notion of continuity and discontinuity will now be illustrated by means of some simple examples.

Examples

1. *Show that*

$$f(x)=3x+1$$

is continuous at $x=1$.

Here,

$$f(1)=,4$$

and

$$f(x)-f(1)=3x+1-4=3(x-1).$$

We will now attempt to make the numerical value of this difference smaller than any preassigned positive number, *say* ·001.

(*i*) Let $x>1$, so that $3(x-1)$ is positive and is, therefore, the numerical value of $f(x)-f(1)$.

Now

$$|f(x)-f(1)|=3(x-1)<\cdot 001,$$

if

$$x-1<\cdot 001/3$$

i e., if

$$x<1+\cdot 001/3=1\cdot 000\dot{3}. \qquad ...(i)$$

(*ii*) Let $x<1$, so that $3(x-1)$ is negative and the numerical value of $f(x)-f(1)$ is $3(1-x)$.

Now

$$|f(x)-f(1)|=3(1-x)<\cdot 001,$$

if

$$1-x<\cdot 001/3,$$

i.e., if

$$1-\cdot 001/3<x,$$

or

$$1-\cdot 000\dot{3}<x.$$

Combining (*i*) and (*ii*), we see that

$$|f(x)-f(1)|<\cdot 001,$$

for all values of x such that

$$1-\cdot 000\dot{3}<x<1+\cdot 000\dot{3}.$$

The test of continuity for $x=1$ is thus satisfied for the particular value ·001 of ε. We may similarly show that the test is true for the other particular values of ε also.

The *complete* argument is, however, as follows :—

Let ε be *any* positive number. We have

$$|f(x)-f(1)|=3|x-1|.$$

Now

$$3|x-1|<\varepsilon$$

if

$$|x-1|<\varepsilon/3,$$

i.e, if

$$1-\varepsilon/3<x<1+\varepsilon/3.$$

Thus, there exists an interval $(1-\varepsilon/3,\ 1+\varepsilon/3)$ around 1 such that for every value of x in this interval, the numerical value of the difference between $f(x)$ and $f(1)$ is less than a preassigned positive number ε. Here $\delta=\varepsilon/3$.

Hence $f(x)$ is continuous for $x=1$.

Note. It may be shown that, $3x+1$, is continuous for every value of x.

2. *Show that $f(x)=3x^2+2x-1$ is continuous for $x=2$.*

Let ε be any *given* positive number. We have

$$f(2)=15.$$

$$\therefore \quad |f(x)-f(2)| = |3x^2+2x-1-15|$$
$$= |x-2|\;|3x+8|.$$

We suppose that x lies between 1 and 3. For values of x between 1 and 3, $3x+8$ is positive and less than $(3 \cdot 3+8)=17$.

Thus when

$$1<x<3,$$

we have

$$|f(x)-f(2)|<17\,|x-2|.$$

Now

$$17\,|x-2|<\varepsilon,$$

if

$$|x-2|<\varepsilon/17.$$

Thus, we see that there exists a positive number $\varepsilon/17$ such that

$$|f(x)-f(2)|<\varepsilon$$

when

$$|x-2|<\varepsilon/17.$$

Therefore $f(x)$ is continuous for $x=2$.

Note. It may be shown that $3x^2+2x-1$, is continuous for every value of x.

3. *Prove that $\sin x$ is continuous for every value of x.*

It will be shown that $\sin x$ is continuous for any given value of x; *say c.*

Let ε be any *arbitrarily assigned positive number.* We have

$$|f(x)-f(c)| = |\sin x-\sin c|$$
$$= \left|2\cos\frac{x+c}{2}\sin\frac{x-c}{2}\right|$$
$$= \left|2\cos\frac{x+c}{2}\right|\;\left|\sin\frac{x-c}{2}\right|$$

Now $\left|\cos\dfrac{x+c}{2}\right|\leqslant 1$ for every value of x and c.

Also $$\left|\sin\frac{x-c}{2}\right|\leqslant\left|\frac{x-c}{2}\right|.$$

(From Elementary Trigonometry)

Thus we have

$$\left|\sin x-\sin c\right|=2\left|\cos\frac{x+c}{2}\right|\;\left|\sin\frac{x-c}{2}\right|$$
$$\leqslant 2.1.\left|\frac{x-c}{2}\right| = \left|x-c\right|.$$

Hence $|\sin x-\sin c|<\varepsilon$ when $|x-c|<\varepsilon$.

Thus, there exists an interval $(c-\varepsilon, c+\varepsilon)$ around c such that for every value of x in this interval

$$|\sin x - \sin c| < \varepsilon.$$

Hence, $\sin x$ is continuous for $x=c$ and, therefore, also for every value of x ; c being any number.

4. *Show that* $\sin^2 x$ *is continuous for every value of* x.

Let ε be any given positive number.

We have

$$\begin{aligned} |f(x)-f(c)| &= |\sin^2 x - \sin^2 c| \\ &= |\sin(x+c)|\ |\sin(x-c)| \\ &\leqslant |\sin(x-c)| \\ &\leqslant |x-c|. \end{aligned}$$

$\therefore$

$$|f(x)-f(c)| < \varepsilon,$$

when

$$|x-c| < \varepsilon.$$

Hence, $\sin^2 x$ is continuous for $x=c$ and, therefore, also for every value of x ; c being *any* number.

5. *Show that the function* $f(x)$, *as defined below, is discontinuous at* $x=\frac{1}{2}$.

$$f(x)=\begin{cases} x, & \text{when } 0 \leqslant x < \frac{1}{2}\ ; \\ 1, & \text{when } x=\frac{1}{2}\ ; \\ 1-x, & \text{when } \frac{1}{2} < x < 1. \end{cases}$$

The argument will be better grasped by considering the graph of this function.

The graph consists of the point P $(\frac{1}{2}, 1)$ and the lines OA, AC excluding the point A. Our intuition immediately suggests that there is a discontinuity at A ; there being a gap in the graph at A.

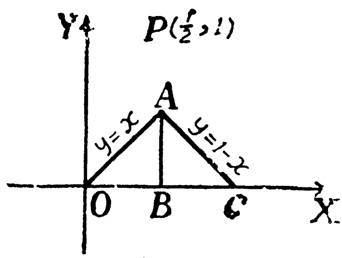


Fig. 36.

Also, analytically, we can see that for every value of x, round about $x=\frac{1}{2}$, $f(x)$ differs from $f(\frac{1}{2})$ which is equal to 1, by a number greater than $\frac{1}{2}$ so that there is no question of making the difference between $f(x)$ and $f(\frac{1}{2})$ less than *any* positive number arbitrarily assigned.

Therefore $f(x)$ is discontinuous at $x=\frac{1}{2}$.

3·2. Limit. The important concept of the limit of a function will now be introduced.

For the continuity of $f(x)$ for $x=c$, the values of the function for values of x near c lie near $f(c)$. In general, the two things, *viz.* ;

(*i*) The value $f(c)$ of the function for $x=c$, and

(*ii*) the values of $f(x)$ for values of x near c, are, independent. It may, in fact, sometimes happen that the values of the function for values of x near c lie near a number l which is not equal to $f(c)$ or that the values do not lie near any number at all. Thus, for example, we have seen in Ex. 5, above, that the values of the function for values of x near $\frac{1}{2}$ lie near $\frac{1}{2}$ which is different from the value, 1, of the function for $x=\frac{1}{2}$. In fact this inequality itself was the cause of discontinuity of this function for $x=\frac{1}{2}$.

The above remarks lead us to introduce the notion of limit as follows:

Def. $$\lim_{x \to c} f(x)=l.$$

A function $f(x)$ is said to tend to a limit, l, as x tends to c, if, corresponding to any positive number ε, arbitrarily assigned, there exists a positive number; δ, such that for every value of x in the interval $(c-\delta, c+\delta)$, other than c, $f(x)$ differs from l numerically by a number which is less than ε, i.e.,

$$|f(x)-l| - < \varepsilon,$$

for every value of x, other than c, such that

$$|x-c| < \delta.$$

Right handed and left handed limits.

$$\lim_{x \to (c+0)} f(x)=l. \qquad \lim_{x \to (c-0)} f(x)=l$$

A function $f(x)$ is said to tend to a limit, l, as x tends to, c, **from above,** *if, corresponding to any positive number, ε, arbitrarily assigned, there exists a positive number, δ, such that*

$$|f(x)-l| < \varepsilon$$

whenever

$$c < x < c+\delta.$$

In this case, we write

$$\lim_{x \to (c+0)} f(x)=l,$$

and say that, l is the right handed limit of $f(x)$.

A function $f(x)$ is said to tend to a limit, l, as x tends to c **from below,** *if corresponding to any positive number, ε, arbitrarily assigned, their exists a positive number, δ, such that*

$$|f(x)-l| < \varepsilon,$$

whenever

$$c-\delta < x < c.$$

In this case, we write

$$\lim_{x \to (c-0)} f(x)=l$$

and say that, l, is the left handed limit of $f(x)$.

From above it at once follows that

$$\lim_{x \to c} f(x)=l,$$

if, and only if,

$$\lim_{x \to (c+0)} f(x) = l = \lim_{x \to (c-0)} f(x)$$

It is important to remember that a limit may not always exist.
(Refer. Ex. 3, p. 46)

Remarks 1. In order that a function may tend to a limit it is necessary and sufficient that corresponding to *any* positive number, ε, a choice of δ is possible. The function will not either approach a limit or at any rate will not have the limit l, if for *some* ε, a corresponding δ does not exist.

2. The question of the limit of $f(x)$ as x approaches 'c' does *not* take any note of the value of $f(x)$ for $x=c$. The function may *not* even be defined for $x=c$.

3·21. Another form of the definition of continuity. Comparing the definitions of continuity and limit as given in § § 3·1, 3·2, we see that

f(x) is continuous for x = c if, and only if,

$$\lim_{x \to c} f(x) = f(c).$$

Thus the limit of a continuous function $f(x)$, as x tends to c, is equal to the value $f(c)$ of the function for $x=c$.

We see that a function $f(x)$ can fail to be continuous for $x=c$ in any one of the following three ways :—

(*i*) $f(x)$ is not defined for $x=c$;

(*ii*) $f(x)$ does not tend to a limit as x tends to c, *i.e.*,

$$\lim_{x \to c} f(x) \text{ does not exist.}$$

(*iii*) $\lim_{x \to c} f(x)$ exists and $f(c)$ is defined but

$$\lim f(x) \neq f(c).$$

Examples

1. *Examine the limit of the function*

$$\frac{x^2-1}{x-1},$$

as x tends to 1.

The function is defined for every value of x

$$y = \frac{x^2-1}{x-1} = x+1, \text{ when } x \neq 1.$$

Case 1. Firstly consider the behaviour of the values of y *for values of x greater than 1.*

Clearly, the variable y is greater than 2 when x is greater than 1.

If x, while remaining greater than 1, takes up values whose difference from 1 constantly diminishes, then y, while remaining greater than 2, takes up values whose difference from 2 constantly diminishes also.

In fact, difference between y and 2 can be made as small as we like by taking x sufficiently near 1.

For instance, consider the number ·001.

Then

$$| y-2 | = y-2 = x+1-2 < \cdot 001.$$

if $$x < 1\cdot 001.$$

Thus, for every value of x which is greater than 1 and less than 1·001, the absolute value of the difference between y and 2 is less than the number ·001 which we had arbitrarily selected.

Instead of the particular number ·001, we now consider any positive number ε.

Then

$$y-2 = x-1 < \varepsilon,$$

if

$$x < 1+\varepsilon.$$

Thus, there exists an interval $(1, 1+\varepsilon)$, *such that the value of* y, *for any value of* x *in this interval, differs from* 2 *numerically, by a number which is smaller than the positive number* ε, *selected arbitrarily.*

Thus the limit of y *as* x *approaches* 1 *through values greater than* 1, *is* 2 *and we have*

$$\lim_{x \to (1+0)} y = 2.$$

Case II. We now consider the behaviour of the values of y *for values of* x *less than* 1.

When x is less than 1, y is less than 2.

If, x, while remaining less than 1, takes up values whose difference from 1 constantly diminishes, then y, while remaining less than 2, takes up values whose difference from 2 constantly diminishes also.

Let, now, ε be any arbitrarily assigned positive number, however small.

We then have

$$| y-2 | = 2-y = 2-(x+1) = 1-x < \varepsilon,$$

if $$1-\varepsilon < x,$$

so that for every value of x less than 1 but $> 1-\varepsilon$, the absolute value of the difference between y and 2 is less than the number ε.

Thus there exists *an interval* $(1-\varepsilon, 1)$ *such that the value of y, for any x in the interval, differs from 2 numerically by a number which is smaller than the arbitrarily selected positive number* ε.

Thus the *limit of y, as x approaches* 1, *through values less than* 1, *is* 2 and we write

$$\lim_{x \to (1-0)} y=2.$$

Case III. Combining the conclusions arrived at in the last two cases, we see that corresponding to any arbitrarily assigned positive number ε, there exists an interval $(1-\varepsilon, 1+\varepsilon)$ around 1, such that for every value of x in this interval, *other than* 1, y differs from 2 numerically by a number which is less than ε, *i.e.*, we have

$$|y-2| < \varepsilon$$

or any x, other than 1, such that

$$|x-1| < \varepsilon.$$

Thus

$$\lim_{x \to 1} y=2, \text{ or, } y \to 2 \text{ as } x \to 1.$$

2. *Examine the limit of the function*

$$y=x \sin x,$$

as x approaches 0.

Case I. *Let* $x > 0$, so that y is also > 0, if we suppose

$$x < \pi/2.$$

For $0<x<\pi/2$, we have

$$\sin x<x,$$

so that

$$x \sin x<x^2, \text{ if } 0<x<\pi/2.$$

Let ε be any arbitrarily assigned positive number.

For values of x which are positive and less than $\sqrt{\varepsilon}$, we have

$$x^2 <\varepsilon$$

so that for such values of x, we have

$$x \sin x<\varepsilon.$$

Thus there exists an interval $(0, \sqrt{\varepsilon})$ such that for every value of x in this interval the numerical value of the difference between $x \sin x$ and 0 is less than the arbitrarily assigned positive number ε, so that we have a situation similar to that in case I of the Ex. 1 above. Thus

$$\lim_{x \to (0+0)} x \sin x=0.$$

Case II. *Let* $x<0$ so that $y>0$, if we suppose $x>-\pi/2$.

The values of the function for two values of x which are equal in magnitude but opposite in signs are equal. Hence, as in case I, we easily see that for any value of x in the interval $(-\sqrt{\varepsilon}, 0)$ the numerical value of the difference between $x \sin x$ and 0 is less than ε.

Hence
$$\lim_{x \to (0-0)} x \sin x = 0.$$

Case III. Combining the conclusions arrived at in the last two cases, we see that corresponding to any positive number ε arbitrarily assigned, there exists an interval $(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$ around 0, such that for any x belonging to this interval, the numerical value of the difference between $x \sin x$ and 0 is $<\varepsilon$,

i.e.,
$$| x \sin x - 0 | < \varepsilon.$$

Hence
$$\lim_{x \to 0} x \sin x = 0.$$

Note. It will be seen that the inequality $| x \sin x - 0 | < \varepsilon$ is satisfied even for $x=0$. But it should be carefully noted that no difference would arise as to the conclusion

$$\lim_{x \to 0} x \sin x = 0,$$

even if $x=0$ were an exception.

Again, we see that for $x=0$, the value of the function is

$$0 \sin 0 = 0$$

which is also its limit as x approaches 0. Thus in this case, the limit of the function is the same as its value. Thus this function is continuous for $x=0$.

The function $(x^2-1)/(x-1)$, as considered in Ex. 1, possesses a limit as x approaches 1, but does not have any value for $x=1$, so that it is discontinuous for $x=1$.

3. *Examine the limit of* $\sin (1/x)$ *as* x *approaches* 0.

Let

$$y = \sin (1/x)$$

so that y is a function defined for every value of x, other than 0

The graph of this function will enable the student to understand the argument better.

To draw the graph, we note the following points :—

(*i*) As x increases monotonically from $2/\pi$ to ∞, $1/x$ decreases from $\pi/2$ to 0 and therefore $\sin (1/x)$ decreases monotonically from 1 to 0 ;

(*ii*) as x increases monotonically from $2/3\pi$, to $2/\pi$, $1/x$ decreases monotonically from $3\pi/2$ to $\pi/2$ and therefore $\sin 1/x$ increases monotonically from -1 to 1 ;

(*iii*) as x increases from $2/5\pi$ to $2/3\pi$, $1/x$ decreases monotonically from $5\pi/2$ to $3\pi/2$ and therefore $\sin 1/x$ decreases from 1 to -1 ;

and so on.

Thus the positive values of x can be divided in an infinite number of intervals

$$\cdots\cdots\left[\frac{2}{7\pi},\ \frac{2}{5\pi}\right],\ \left[\frac{2}{5\pi},\ \frac{2}{3\pi}\right],\ \left[\frac{2}{3\pi},\ \frac{2}{\pi}\right],\ \left[\frac{2}{\pi},\ \infty\right];$$

such that the function decreases from 1 to 0 in the first interval on the right and then oscillates from -1 to 1 and from 1 to -1 alternatively in the others beginning from the second interval on the right.

It may similarly be seen that the negative values of x also divide themselves in an infinite set of intervals.

$$\left(-\infty,\ -\frac{2}{\pi}\right],\ \left[-\frac{2}{\pi},-\frac{2}{3\pi}\right],\ \left[-\frac{2}{3\pi},-\frac{2}{5\pi}\right]\cdots\cdots$$

such that the function decreases from 0 to -1 in the first interval on the left and then oscillates from -1 to 1 and from 1 to -1 alternately in the others beginning from the second interval on the left.

Hence, we have the graph (Fig. 37) as drawn.

The function oscillates between -1 and 1 more and more rapidly as x approaches nearer and nearer zero from either side. If we take any interval enclosing $x=0$, however small it may be then for an infinite number of points of this interval the function assumes the values 1 and -1.

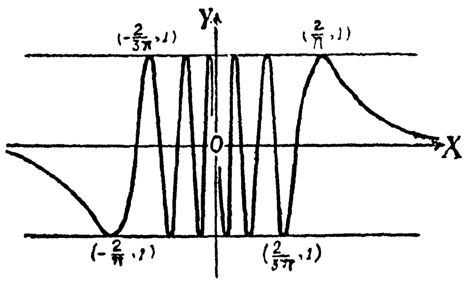


Fig. 37.

There can, therefore, exist no number which differs from the function by a number less than an arbitrarily assigned positive number for values of x near 0.

Hence

$$\lim_{x \to 0} (\sin 1/x) \text{ does not exist.}$$

This example illustrates an important fact that **a limit may not always exist.**

Note. The function considered above is not continuous for $x=0$. Neither is the function defined for $x=0$ nor does its limit exist when x tends to zero.

4. *Examine the limit of*

$$x \sin \frac{1}{x}$$

as x approaches 0.

Obviously the function is not defined for $x=0$. Now, for non-zero values of x

$$\left|x \sin \frac{1}{x}\right| = |x| \left|\sin \frac{1}{x}\right| \leqslant |x|$$

so that

$$\left|x \sin \frac{1}{x} - 0\right| = \left|x \sin \frac{1}{x}\right| \leqslant |x| < \varepsilon$$

when

$$|x-0| = |x| < \varepsilon.$$

Thus, we see that if, ε, be any positive number, then there exists an interval $(-\varepsilon, \varepsilon)$ around 0, such that for every value of x in this interval, with the sole exception of 0, $x \sin (1/x)$ differs from 0 by a number less than ε.

Hence

$$\lim_{x \to 0} \left(x \sin \frac{1}{x}\right) = 0.$$

In fact, as may be easily seen, the graph of

$$y = x \sin \frac{1}{x}$$

oscillates between

$$y = x \text{ and } y = -x$$

as x tends to zero.

This function is not continuous for $x=0$.

3·22. Infinite limits and variables tending to infinity. Meanings of

$$(i) \lim_{x \to c} f(x) = \infty \qquad (ii) \lim_{x \to c} f(x) = -\infty;$$

Def. *A function $f(x)$ is said to tend to ∞, as x tends to c, if, corresponding to any positive number G, however large, there exists a positive number δ such that for all values of x, other than c, lying in the interval* $[c-\delta, c+\delta]$,

$$f(x) > G.$$

Also, a function $f(x)$ is said to tend to $-\infty$, as x tends to c, if, corresponding to any positive number G, however large, there exists a positive number δ such that for all values of x in $[c-\delta, c+\delta]$ excluding c,

$$f(x) < -G.$$

Right handed and left handed limits can be defined as in § 3·2, p. 42.

We shall now consider some examples.

Examples

1. *Show that*

$$\lim_{x \to 0} \left(\frac{1}{x^2}\right) = \infty .$$

The function is not defined for $x=0$. We write $y=1/x^2$.

Case I. *Let* $x>0$. If x, while continuing to remain positive, diminishses, then $1/x^2$ increases.

Also, $1/x^2$ can be made *as large as we like*, if only we take x sufficiently near 0 and greater than it.

Consider any positive number, however large, say 10^6.

Then

$$1/x^2 > 10^6, \text{ if } x < 1/10^3,$$

so that for all positive values of $x<1/10^3$, we have

$$y > 10^6.$$

Instead of the particular number 10^6 we may consider any positive number G.

Then

$$1/x^2 > G, \text{ if } x < 1/\sqrt{G}.$$

Thus, there exists an interval $[0, 1/\sqrt{G}]$, such that for every value of x belonging to it, y is greater than the arbitrarily assigned positive number G. Hence

$$\lim_{x \to (0+0)} (1/x^2) = \infty .$$

Case II. *Let* $x<0$. Here also y is positive. If, x while continuing the remain negative, increases towards zero, then $1/x^2$ also increases.

Also if, G be any arbitrarily assigned positive number, then, for every value of x in the interval $(-1/\sqrt{G}, 0)$, we have

$$1/x^2 > G,$$

so that $1/x^2$ tends to ∞ as x tends to 0 through values less than it and we write in smybols

$$\lim_{x \to (0-0)} (1/x^2) = \infty .$$

Case III. Combining the conclusions arrived at in the last two cases we see that corresponding to *any* arbitrarily assigned positive number G, there exists an interval $[-1/\sqrt{G}, 1/\sqrt{G}]$, around 0, such that for every value of x (other than 0) belonging to this interval, we have

$$1/x^2 > G.$$

Thus

$$\lim_{x \to 0} (1/x^2) = \infty , \quad \text{or} \quad (1/x^2) \to \infty \text{ as } x \to 0.$$

2. *Show that*

$$\lim_{x\to(0+0)} \frac{1}{x}=\infty, \quad \lim_{x\to(0-0)} \frac{1}{x}=-\infty,$$

and

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist.}$$

The function is not defined for $x=0$. We write $y=1/x$.

Case I. *Let* $x>0$ so that y is positive. If x, while continuing to remain positive, diminishes, then $1/x$ increases.

Also, if G be any positive number taken arbitrarily, then

$$1/x>G, \text{ if } x<1/G.$$

Hence

$$\lim (1/x)=\infty \text{ when } x\to(0+0).$$

Case II. *Let* $x<0$ so that $y=1/x$ is negative. If x, while continuing to remain negative, increases towards zero, then $y=1/x$ decreases and numerically increases.

Also, if G be any positive number taken arbitrarily, then

$$1/x<-G, \text{ if } x>-1/G.$$

Hence

$$\lim (1/x)=-\infty \text{ when } x\to(0-0).$$

Case III. Clearly when $x\to 0$, $\lim (1/x)$ does not exist.

3·23. We will now give a number of definitions whose meanings the reader may easily follow.

(*i*) *A function $f(x)$ is said to tend to a limit l, as x tends to ∞ (or $-\infty$), if, corresponding to any arbitrarily assigned positive number ε, there exists a positive number G, such that*

$$|f(x)-l|<\varepsilon,$$

for every value of $x>G$, $(x<-G)$.

Symbolically, we write

$$\lim f(x)=l \text{ when } x\to\infty, \quad [\lim f(x)=l \text{ when } x\to-\infty].$$

(*ii*) *A function $f(x)$ is said to tend to ∞, as x tends to ∞ (or $-\infty$) if, corresponding to any positive number G, however large, there exists a positive number $\triangle$ such that*

$$f(x)>G, \textit{ for } x>\triangle, (<-\triangle).$$

Symbolically, we write

$$\lim f(x)=\infty \text{ when } x\to\infty, \ [\lim f(x)=\infty \text{ when } x\to-\infty)]$$

The reader may now similarly define

$$\lim f(x)=-\infty \text{ when } x\to\infty\ ; \lim f(x)=-\infty \text{ when } x\to-\infty$$

Note 1. Instead of ∞, we may write $+\infty$ to avoid confusion with $-\infty$.

Note 2. Rigorous solution of any problem on limits requires that the problem should be examined strictly according to the definitions given above in terms, of ε's, δ's, G's etc. But in practice this proves very difficult except in some very elementary cases. We may not, therefore, always insist on a rigorous solution of such probelms.

Exercises

1. Show that the following limits exist and have the values as given :

(*i*) $\lim_{x\to 1} (2x+3)=5$. (*ii*) $\lim_{x\to 2} (3x-4)=2$.

(*iii*) $\lim_{x\to 1} (x^2+3)=4$. (*iv*) $\lim_{x\to 3} (2x^2+x)=21$.

(*v*) $\lim_{x\to 3} (1/x)=\frac{1}{3}$. (*vi*) $\lim_{x\to 0} 1/(2x+3)=\frac{1}{3}$.

2. Show that

(*i*) $5x+4$ is continuous for $x=2$.

(*ii*) x^2+2 is continuous for $x=3$.

Also show that two functions are continuous for every value of x.

3. Examine the continuity for $x=0$, of the following functions :

(*i*) $$f(x)=\begin{cases} \dfrac{x^2-9}{x-3}, \text{ when } x\neq 3 \\ 6, \text{ when } x=3, \end{cases}$$

(*ii*) $$f(x)=\begin{cases} \sin\dfrac{1}{x}, \text{ when } x\neq 0 \\ 0, \quad \text{when } x=0, \end{cases}$$

4. Prove that $\cos x$ is continuous for every value of x.

5. Show that $\cos^2 x$ and $2+x+x^2$ are continuous for every value of x.

6. Draw the graph of the function $\varphi(x)$ which is equal to 0 when $x\leqslant 0$; to 1 when $0<x\leqslant 1$ and to 2 when $x>1$; and show that it has two points of discontinuity.

7. Investigate the points of continuity and discontinuity of the following function :—

$$f(x)=(x^2/a)-a \text{ for } x<a,\ f(a)=0,\ f(x)=a-(a^2/x), \text{ for } x>a.$$

8. Show that

$$f(x)=\cos\frac{1}{x} \text{ when } x\neq 0 \text{ and } f(0)=1.$$

is discontinuous for $x=0$.

9. Show that

$$\lim_{x\to(\frac{1}{2}\pi-0)} \tan x=\infty, \qquad \lim_{x\to(\frac{1}{2}\pi+0)} \tan x=-\infty$$

but $\lim_{x\to(\frac{1}{2}\pi)} \tan x$ does not exist.

[Refer § 2·63, p. 27]

10. Show that

$$\lim_{x\to(0+0)} \log x=-\infty.$$

[Refer § 2·5, p. 25].

11. *Examine*

$$\lim_{x\to 0} 2^{\frac{1}{x}}$$

[If x tends to 0 through positive values, then $1/x$ tends to ∞ and, therefore, $2^{1/x}$ tends to ∞.

If x tends to 0 through negative values, then $1/x$ tends to $-\infty$ and therefore, $2^{1/x}$ tends to 0.

Thus

$$\lim_{x\to(0+0)} 2^{\frac{1}{x}}=\infty, \qquad \lim_{x\to(0-0)} 2^{\frac{1}{x}}=0$$

but

$$\lim_{x\to 0} 2^{\frac{1}{x}}$$

does not exist.]

12. Show that

$$\lim_{x\to 0} 2^{-\frac{1}{x^2}}=0.$$

13. Show that if

$$f(x)=[x],$$

where $[x]$ denotes the greatest integer not greater than x, then

$$\lim_{x\to(1-0)} f(x)=0, \quad \lim_{x\to(1+0)} f(x)=1, \quad \lim_{x\to 1} f(x) \text{ does not exist.}$$

What is the value of the function for $x=1$? [Refer Fig. 11, p. 17]

14. Explain giving suitable examples, the distinction between the value of the function for $x=a$ and the limit of the same as x tends to a.

15. Give an example of a function which has a definite value at the origin but is, nevertheless, discountinuous there.

3·3. Theorems on limits.

Let $f(x)$ and $\varphi(x)$ be two functions of x such that

$$\lim_{x\to a} f(x)=l, \qquad \lim_{x\to a} \varphi(x)=m.$$

Then

(*i*) $$\lim_{x\to a} [f(x)+\varphi(x)]=\lim_{x\to a} f(x)+\lim_{x\to a} \varphi(x)=l+m,$$

i.e., the limit of the sum of two functions is equal to the sum of their limits ;

(*ii*) $$\lim_{x\to a} [f(x)-\varphi(x)]=\lim_{x\to a} f(x)-\lim_{x\to a} \varphi(x)=l-m,$$

i.e., the limit of the difference of two functions is equal to the difference of their limits ;

(*iii*) $$\lim_{x\to a} [f(x)-\varphi(x)]=\lim_{x\to a} f(x) \,.\, \lim_{x\to a} \varphi(x)=lm,$$

i.e., the limit of the product of two functions is equal to the product of their limits ;

(*iv*) $$\lim_{x\to a} [f(x)\div\varphi(x)]=\lim_{x\to a} f(x)\div\lim_{x\to a} \varphi(x)=l\div m, \ (m\neq 0)$$

i.e., the limit of the quotient of two functions is equal to the quotient of their limits provided the limit of the divisor is **not** *zero.*

These results are of fundamental importance but their formal proofs are beyond the scope of this book.

These results can easily be extended to the case of any *finite* number of functions.

Note. Inversion of operations. A beginner who notices the above statements for the first time may fail to see any meaning in them and he may not be able to appreciate the significance of the same. In order, therefore, to help him do so, some observations seem to be necessary here.

In mathematics we deal with different types of operations and sometimes the mathematical expressions are subjected to two or more operations and in such a case the *order* in which the operations are made must be taken into account. It cannot just be *assumed* that the order can be changed at will without altering the final outcome, *i.e.*, the final result may not be independent of the order of the operations.

The fact is that in each case the question as to whether the inversion of the order of operations is or is not valid has to be separately examined in relation to the nature of the operations.

We now examine the following statements :—

(*i*) $\log(m+n) \neq \log m + \log n$.

(*ii*) $\sin(A+B) \neq \sin A + \sin B$.

(*iii*) $a \times (b+c) = a \times b + a \times c$.

In (*i*) we find that the two operations involved, *viz.*, that of addition and taking of log are not invertible.

Similar is the case in (*ii*) where we have the operations of adding and taking of sine.

In (*iii*), however, we see that the two operations of addition and multiplication are invertible.

The theorems in this article amount to asserting the validity of the inversion of the operations of taking the limits with each of the four operations of addition, subtraction, multiplication and division.

The reader is advised to think of other similar cases also.

3·4. Continuity of the sum, difference, product and quotient of two continuous functions.

Let $f(x)$, $\varphi(x)$ be any two functions which are continuous for $x=a$, so that

$$\lim_{x\to a} f(x) = f(a), \qquad \lim_{x\to a} \varphi(x) = \varphi(a).$$

From the theorems in § 3·3, we see that

$$\begin{aligned}\lim_{x\to a} [f(x)+\varphi(x)] &= \lim_{x\to a} f(x) + \lim_{x\to a} \varphi(x)\\ &= f(a)+\varphi(a)\\ &= \text{value of } [f(x)+\varphi(x)], \text{ for } x=a,\end{aligned}$$

so that $f(x)+\varphi(x)$ is continuous at $x=a$.

The continuity of $f(x)-\varphi(x)$ and $f(x)\varphi(x)$ may similarly be proved.

For the quotient, we have

$$\lim_{x\to a} \frac{f(x)}{\varphi(x)} = \frac{\lim f(x)}{\lim \varphi(x)}, \text{ where } \lim_{x\to a} \varphi(x) \neq 0.$$

$$=\frac{f(a)}{\varphi(a)}=\text{value of }\frac{f(x)}{\varphi(x)},\text{ for } x=a,$$

so that $f(x)\div\varphi(x)$ is also continuous at $x=a$, provided

$$\lim \varphi(x)=\varphi(a)\neq 0, \text{ when } x \to a.$$

Thus, *the sum, the difference, the product and the quotient of two continuous functions are also continuous* (with one obvious exception in the case of a quotient).

3·41. Continuity of elementary functions.

We have seen in Ex. 3, p. 40 and Ex. 4, p. 51, that $\sin x$ and $\cos x$ are continuous for all values of x.

Hence from §3·4, above, we see that

$$\tan x=\frac{\sin x}{\cos x},\ \cot x=\frac{\cos x}{\sin x},\ \sec x=\frac{1}{\cos x},\ \operatorname{cosec} x=\frac{1}{\sin x}$$

are also continuous for all those values of x *for which they are defined*; points of discontinuity of these four functions arise when the denominators $\cos x$, $\sin x$ become zero and for such values of x, these functions themselves cease to be defined.

Also it may be easily seen that, x, and a constant are continuous functions of x. Thus by repeated applications of the results of §3·4 above, we see that every polynomial

$$a_0x^n+a_1x^{n-1}+\ldots+a_n$$

is a continuous function for every value of x and that every rational function

$$\frac{a_0x^n+a_1x^{n-1}+\ldots+a_n}{b_0x^m+b_1x^{m-1}+\ldots+b_m}$$

is a continuous function of x for every value of x except those for which the denominator becomes zero.

Finally the functions, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\operatorname{cosec}^{-1} x$, $\log_a x$, a^x are continuous for all those values of x for which they are defined. This is geometrically obvious from the graphs drawn in Chap. II. Analytical proofs are beyond the scope of this book.

3·5. Some important properties of continuous functions.

3·51. *Let $f(x)$ be continuous for $x=c$ and $f(c)\neq 0$. Then there exists an interval $[c-\delta, c+\delta]$ around c such that $f(x)$ has the sign of $f(c)$ for all values of x in this interval.*

Its truth is obvious if we remember that a continuous function does not undergo *sudden* changes so that if $f(x)$ is positive for any value c of x and also continuous thereat, it cannot suddenly become negative and must, therefore, remain positive for values of x in a certain neighbourhood of c.

In the *formal, precise* manner, it may be proved as follows ;

To *every* positive number ε, however small it may be, there will correspond a positive number δ such that

$$f(c)-\varepsilon<f(x)<f(c)+\varepsilon$$

for every value of x such that

$$c-\delta<x<c+\delta.$$

Let $f(c)>0$. In this case we *take* ε any positive number less than $f(c)$ so that $f(c)-\varepsilon$ and $f(c)+\varepsilon$ both are positive. Thus, $f(x)$ lies between two positive numbers and is, therefore, itself positive when x lies between $c-\delta$ and $c+\delta$.

Lt $f(c)<0$. Hence $f(c)-\varepsilon$ is already negative and $f(c)+\varepsilon$ will be negative if $\varepsilon<-f(c)$. Thus in this case if we *take* ε any positive number which is smaller than the positive number $-f(c)$, then we see that $f(x)$ lies between the two negative numbers $f(c)-\varepsilon$ and $f(c)+\varepsilon$ and is, therefore, itself negative when x lies in the interval $[c-\delta, c+\delta]$.

Hence the theorem.

Note. This simple but important property of continuous functions will be used in the chapters on Maxima, Minima and Points of inflexion.

3·52. *Let $f(x)$ be continuous in a closed interval $[a, b]$ and let $f(a)$, $f(b)$ have opposite signs. Then $f(x)$ is zero for at least one value of x, lying between a and b.*

Its truth is intuitively obvious, for, a continuous curve $y=f(x)$ going from a point on one side of x-axis to a point lying on the other cannot do so without crossing it.

Its formal, analytical proof is, however, beyond the scope of this book.

3·53. *Let $f(x)$ be continuous in a* **closed** *interval $[a, b]$. Then there exists points c and d in the interval $[a, b]$, where $f(x)$ assumes its greatest and least values M and m, i.e.,*

$$f(c)=M, f(d)=m.$$

The proof is beyond the scope of this book.

The theorem states that there is a value of a continuous function greater than every other of its values and as also a value smaller than every other value.

A discontinuous function may *not* possess greatest or least values as we now illustrate.

Consider the function defined as follows :—

$$f(x)=\begin{cases}1-x, & \text{for } 0<x\leqslant 1\\ \frac{1}{2}, & \text{for } x=0,\end{cases}$$

Its graph consists of the point $C(0, \frac{1}{2})$, and the line AB excluding the point B. The function possesses no greatest value; 1 not being a value of the function. If we consider any value less than 1, however near 1 ; it may be seen that there is a value of the function greater than that value.

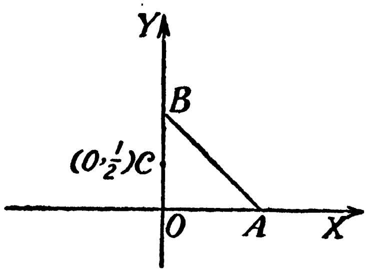


Fig. 38.

This is explained by the fact that the function is not continuous in the interval (0, 1) ; $x=0$ being a point of discontinuity.

Note. This property will be required to prove Rolle's Theorem in Chap. VI.

Example

Show that

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1},$$

when, n is any integer.

We *cannot* write

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \frac{\lim(x^n - a^n)}{\lim(x - a)}, \text{ for } \lim(x-a)=0.$$

In the present case the limit of the numerator is also zero.

Case I. Firstly suppose that n is a *positive integer*. By actual division,

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + \ldots\ldots + a^{n-1}$$

the equality being valid for every value of x other than a. As the limit does not depend upon the value for $x=a$, we can write

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} (x^{n-1} + x^{n-2}a + \ldots\ldots + a^{n-1}).$$

Being a polynomial, the function

$$x^{n-1} + x^{n-2}a + \ldots\ldots + a^{n-1}$$

is continuous for every value of x and, as such, its limit when $x \to a$ must be equal to its value for $x=a$. Thus

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = a^{n-1} + a^{n-2}a + \ldots\ldots + a^{n-1} = na^{n-1}$$

Case II. Now suppose that n is a *negative integer*, say $-m$, where m is a positive integer. We have

$$\frac{x^n - a^n}{x - a} = \frac{x^{-m} - a^{-m}}{x - a} = \frac{a^m - x^m}{x - a} \cdot \frac{1}{a^m x^m}$$

$$=-\frac{x^m-a^m}{x-a}\,\frac{1}{a^m x^m}.$$

$$\therefore\quad \lim_{x\to a}\frac{x^n-a^n}{x-a}=-\lim_{x\to a}\frac{x^m-a^m}{x-a}\;\lim_{x\to a}\frac{1}{a^m x^m}$$

$$=-ma^{m-1}.\frac{1}{a^m a^m}=-ma^{-m-1}=na^{n-1},$$

employing Case I and the fact that $1/a^m x^m$ is continuous for $x=a$. In the Case II, we must suppose that $a\neq 0$.

Exercises

1. Show that

(i) $\lim\limits_{x\to 0}\dfrac{x^2+7x}{x}=7.$ (ii) $\lim\limits_{x+1}\dfrac{x^3-3x+2}{x^3+x^2-5x+3}=\dfrac{3}{4}.$

(iii) $\lim\limits_{x\to 0}\dfrac{\sqrt{(1+x)}-1}{\sqrt{x}}=0.$ (iv) $\lim\limits_{x\to\infty}\dfrac{3x^3-2x^2+x}{4x^3+3x^2-2x+1}=\dfrac{3}{4}.$

2. If

$$f(x)=\frac{x^3+2x^2+x+2}{x^2+x-2},\text{ when } x\neq -2 \text{ and } f(-2)=k$$

find k so that $f(x)$ may be continuous for $x=-2$.

3. Show that

$$\lim_{h\to 0}\sin(x+h)=\sin x.$$

3·6. Some important and useful limits.

3·61. Limiting value of x^n *when n tends to infinity through positive integral values ; x being any given real number.*

(i) *Let* $x>1$.

We write $x=1+h$ so that h is positive.

By the Binomial theorem, we have

$$(1+h)^n=1+nh+\frac{n(n-1)}{2\,!}h^2+\ldots\ldots+\frac{n(n-1)\ldots .1}{n\,!}h^n,$$

where each term is positive.

Hence

$$x^n=(1+h)^n>1+nh.$$

As $(1+nh)$ tends to infinity with n, therefore x^n also tends to infinity.

To be more rigorous, we consider any pre-assigned positive number G. Then

$$1+nh>G,\text{ if } n>(G-1)/h.$$

If $\triangle$ be any positive integer greater than $(G-1)h$, then

$$x^n>G \text{ for } n>\triangle \text{ so that } x^n\to\infty \text{ when } n\to\infty.$$

(*ii*) *Let* $x=1$. Here $x^n=1$ for all values of n and therefore $x^n \to 1$ in this case.

(*iii*) *Let* $0<x<1$. Here, also, x^n is positive.

We write $x=1/(1+h)$ so that h is a positive number.

As before

$$(1+h)^n>1+nh.$$

$$\therefore \qquad 0<x^n=\frac{1}{(1+h)^n}<\frac{1}{1+nh}.$$

Now $1/(1+nh) \to 0$ as n tends to infinity. Therefore $x^n \to 0$ as $n \to \infty$.

To be more rigorous, we consider any positive number ε. Then

$$\frac{1}{1+nh}<\varepsilon, \text{ if } 1+nh>\frac{1}{\varepsilon} \text{ i.e., if } n>\left(\frac{1}{\varepsilon}-1\right)\Big/h.$$

If m be any integer which is $>\left(\frac{1}{\varepsilon}-1\right)\Big/h$, then

$$x^n<\varepsilon \text{ for } n \geqslant m.$$

Thus we see that x^n, which is always positive, lies between $-\varepsilon$ and ε $n \geqslant m$ so that for $0<x<1$

$$\lim x^n=0 \text{ when } n \to \infty.$$

(*iv*) *Let* $x=0$. Here $x^n=0$ for all n so that $x^n \to 0$.

(*v*) *Let* $-1<x<0$. Here x is negative so that x^n is positive for even values of n and negative for odd values. We write $x=-a$ so that a is a positive number less than 1.

Absolute value of x^n is a^n, *i.e.*,

$$|x^n|=a^n.$$

But $a^n \to 0$. Hence

$$x^n \to 0.$$

(*vi*) *Let* $x=-1$. Since x^n is alternately -1 and 1, therefore it neither tends to any finite limit nor to $\pm\infty$ in this case.

(*vii*) *Let* $x<-1$. Here, again, x^n is alternatively negative and positive. But in this case it takes values numerically greater than any assigned number. Hence x^n does not tend to a limit.

Thus, *we see that lim* x^n, *when* $n \to \infty$, *exists finitely if, and only if,*

$$-1<x \leqslant 1.$$

Also, it is 0 if $-1<x<1$, *i.e., if* $|x|<1$ *and is* 1 *for* $x=1$.

3·62. Limiting value of $x^n/n!$, *when n tends to infinity through positive integral values and x is any given real number.*

Let x be positive and let m, $m+1$ be the two consecutive integers between which x lies so that we have

$$m \leqslant x<m+1.$$

We write

$$\frac{x^n}{n!}=\frac{x}{1}\cdot\frac{x}{2}\cdot\frac{x}{3}\cdots\frac{x}{m}\cdot\frac{x}{m+1}\cdot\frac{x}{m+2}\cdots\frac{x}{n}.$$

Let $$p=\frac{x}{1}\cdot\frac{x}{2}\cdots\cdots\frac{x}{m},$$

so that p is a positive constant.

Also, each of $\frac{x}{m+2}, \frac{x}{m+3}, \ldots\ldots, \frac{x}{n}$ is $<\frac{x}{m+1}$.

$$\therefore \quad 0<\frac{x^n}{n!}<p\left(\frac{x}{m+1}\right)^{n-m}=\frac{p}{\left(\frac{x}{m+1}\right)^m}\left(\frac{x}{m+1}\right)^n$$

$$=k\left(\frac{x}{m+1}\right)^n, \text{ say}$$

where k is a positive constant independent of n.

Since, $x/(m+1)$ is positive and <1, therefore when $n\to\infty$,

$$\lim\left(\frac{x}{m+1}\right)^n=0 \qquad [\S\ 3{\cdot}61]$$

m being a constant.

Hence

$$\lim\frac{x^n}{n!}=0, \text{ when } n\to\infty.$$

Again, let x be any negative number, say $-\alpha$, so that α is positive. We have

$$\left|\frac{x^n}{n!}\right|=\left|\frac{(-1)^n\alpha^n}{n!}\right|=\frac{\alpha^n}{n!}.$$

Now, since $\frac{\alpha^n}{n!}\to 0$, we see that $\frac{x^n}{n!}$ also $\to 0$.

Hence $$\lim_{n\to\infty}\frac{x^n}{n!}=0,$$

whatever value x may have.

Here, of course, n tends to infinity through positive integral values only.

3·63. Limiting value of

$$\left(1+\frac{1}{n}\right)^n,$$

when n tends to infinity through positive integral values.

Step 1. By the Binomial Theorem for a positive integral index, we have

$$\left(1+\frac{1}{n}\right)^n=1+n.\frac{1}{n}+\frac{n(n-1)}{1.2}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1}{n^3}$$
$$+\ldots\ldots+\frac{n(n-1)(n-2)\ldots\ldots(n-n+1)}{1.2.3\ldots\ldots n}\cdot\frac{1}{n^n}$$
$$=1+1+\frac{1}{1.2}\left(1-\frac{1}{n}\right)+\frac{1}{1.2.3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$
$$+\ldots\ldots+\frac{1}{1.2\ldots n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{n-1}{n}\right).$$

The expression on the right is a sum of $(n+1)$ *positive* terms.

Changing n to $n+1$, we get

$$\left(1+\frac{1}{n+1}\right)^{n+1}=1+1+\frac{1}{1.2}\left(1-\frac{1}{n+1}\right)+\ldots\ldots\ldots$$
$$+\frac{1}{1.2\ldots\ldots(n+1)}\left(1-\frac{1}{(n+1)}\right)\left(1-\frac{2}{n+1}\right)\ldots\ldots\left(1-\frac{n}{n+1}\right)$$

Here, the right-hand side consists of the sum of $(n+2)$ *positive* terms. Also

$$1-\frac{1}{n}<1-\frac{1}{n+1},\quad 1-\frac{2}{n}<1-\frac{2}{n+1},\quad 1-\frac{3}{n}<1-\frac{3}{n+1},\ldots\ldots$$

so that each term in the expansion of

$$\left(1+\frac{1}{n+1}\right)^{n+1}$$

is greater than the corresponding term in the expansion of

$$\left(1+\frac{1}{n}\right)^n.$$

Thus, we conclude that

$$\left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^n$$

for all positive integral values of n, *i.e.*, $(1+1/n)^n$ increases *monotonically* as n increases.

Step II. We have

$$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1}{1.2}\left(1-\frac{1}{n}\right)+\frac{1}{1.2.3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$
$$+\ldots\ldots\ldots\frac{1}{1.2.3\ldots\ldots n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{n-1}{n}\right),\ \ldots(i)$$

$$<1+1+\frac{1}{2}+\frac{1}{1.2.3}+\cdots+\frac{1}{1.2.3\ldots n}$$

$$<1+1+\frac{1}{2}+\frac{1}{2.2}+\ldots+\frac{1}{2.2.2\ldots(n-1) \text{ factors}}$$

$$=1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}$$

$$=1+\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}=3-\frac{1}{2^{n-1}}<3, \text{ for all } n. \qquad \ldots(ii)$$

Thus we see that $\left(1+\frac{1}{n}\right)^n$ steadily increases as n takes up successively the series of positive integral values 1, 2, 3, etc., and remains less than 3 for all values of n.

Hence $\left(1+\frac{1}{x}\right)^n$ *approaches a finite limit as* $n \to \infty$.

Note 1. From (*i*) and (*ii*), we see that

$$2<\left(1+\frac{1}{n}\right)^n<3,$$

for every value of n so that *the limit is some number which lies between 2 and 3.*

The *mere existence* of the limit of $(1+1/n)^n$ has been shown here and at this stage nothing more can be said about the actual value of the limit which is *denoted by* e, except that it lies between 2 and 3.

The reader will be enabled to appreciate the argument better and also begin to feel a little more acquainted with, e, if he calculates the value of

$$a_n=\left(1+\frac{1}{n}\right)^n$$

for various successive positive integral values of n.

e.g.,

$$a_1=2,\ a_2=2\cdot 25,\ a_3=2\cdot 37,\ a_4=2\cdot 441.$$

In Chapter IX it will be shown how the value of e, correct to any number of places, can be determined without much inconvenience.

Note 2. The proof for the existence of the limit has been based on an intuitively obvious fact that a *monotonically increasing function which remains less than a fixed number tends to a limit.* The formal proof of this fact is beyond the scope of this book.

Cor. I. Lim $(1+1/x)^x=e$, *when* x *tends to infinity taking all the real numbers as its values.*

If x be any positive real number, then there exists a positive integer n such that

$$n \leqslant x < n+1.$$

$$\therefore \qquad \frac{1}{n} \geqslant \frac{1}{x} > \frac{1}{n+1}.$$

$$\text{or} \qquad 1+\frac{1}{n} \geqslant 1+\frac{1}{x} > 1+\frac{1}{n+1},$$

or $$\left(1+\frac{1}{n}\right)^{n+1} \geqslant \left(1+\frac{1}{x}\right)^{x} > \left(1+\frac{1}{n+1}\right)^{n}.$$

(In case the base is greater than 1, raising a greater number to a greater power, does not alter the direction of the inequality.)

We thus have

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{n} \geqslant \left(1+\frac{1}{n}\right)^{x} \geqslant \left(1+\frac{1}{n+1}\right)^{n+1} / \left(1+\frac{1}{n+1}\right)$$

Let $x \to \infty$. Then n and $(n+1) \to \infty$ through positive integral values. We know that

$$\lim_{n \to \infty} \left(1+\frac{1}{n}\right) = 1 = \lim_{n \to \infty} \left(1+\frac{1}{n+1}\right)$$

$$\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^{n} = e = \lim_{n \to \infty} \left(1+\frac{1}{n+1}\right)^{n+1}$$

Therefore

$$\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x} = e.$$

Cor. 2. $\displaystyle\lim_{x \to -\infty} \left(1-\frac{1}{x}\right)^{x} = e.$

Let $x = -y$ so that $y \to +\infty$ as $x \to -\infty$. We have

$$\left(1+\frac{1}{x}\right)^{x} = \left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^{y}$$

$$= \left(1+\frac{1}{y-1}\right)^{y} = \left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right).$$

$$\therefore \quad \lim_{x \to -\infty} \left(1+\frac{1}{x}\right)^{x} = \lim_{y \to \infty} \left[\left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right)\right]$$

$$= \lim_{y \to \infty} \left(1+\frac{1}{y-1}\right)^{y-1} \lim_{y \to \infty} \left(1+\frac{1}{y-1}\right)$$

$$= e \,.\, 1 = e.$$

Cor. 3. $\lim (1+z)^{1/z} = e$ *when* $z \to 0$.

Let $z = 1/x$ so that $x \to +\infty$ or $-\infty$ according as $z \to 0$ through positive or negative values.

Now

$$\lim_{z \to (0+0)} (1+z)^{1/z} = \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x} = e$$

and

$$\lim_{z \to (0-0)} (1+z)^{1/z} = \lim_{x \to -\infty} \left(1+\frac{1}{x}\right)^{x} = e.$$

$$\therefore \qquad \lim_{z\to 0} (1+z)^{1/z} = e.$$

Cor. 4. $$\lim_{x \to 0} \left(1+\frac{x}{a}\right)^{1/x} = \lim_{x \to 0} \left\{\left(1+\frac{x}{a}\right)^{a/x}\right\}^{1/a} = e^{1/a}.$$

Note. In higher Mathematics the number 'e' is taken as the base of logarithms which are then called *Natural logarithms.* The base e is generally not mentioned so that $\log x$ means $\log_e x$.

3·64. *To show that*

$$\lim_{x \to 0} \frac{a^x-1}{x} = \log_e a.$$

Let $a^x-1=y$ so that $y \to 0$ as $x \to 0$.

We have

$$a^x=1+y,$$

or

$$x \log a = \log (1+y), \text{ i.e., } = x \log (1+y)/\log a.$$

Hence

$$\frac{a^x-1}{x} = \frac{y}{\log (1+y)/\log a}$$

$$= \frac{1}{\frac{1}{y} \cdot \log (1+y) \frac{1}{\log a}}$$

$$= \log a . \frac{1}{\frac{1}{y} \log (1+y)} = \log a . \frac{1}{\log (1+y)^{1/y}}$$

$$\therefore \lim_{x \to 0} \frac{a^x-1}{x} = \lim_{y \to 0} \left[\log a \frac{1}{\log (1+y)^{1/y}}\right]$$

$$= \log a . \lim_{y \to 0} \frac{1}{\log (1+y)^{1/y}}$$

$$= \log a . \frac{1}{\lim_{y \to 0} \log (1+y)^{1/y}}$$

$$= \log a . \frac{1}{\log e} = \log a.$$

3·65. *To show that*

$$\lim_{x \to a} \frac{x^\lambda - a^\lambda}{x-a} = \lambda a^{\lambda-1} \qquad (a>0)$$

Let $x=a(1+y)$ so that $y \to 0$, as $x \to a$.

$$\therefore \frac{x^\lambda - a^\lambda}{x-a} = a^\lambda \frac{[(1+y)^\lambda - 1]}{ay} = a^{\lambda-1} \frac{(1+y)^\lambda - 1}{y}.$$

Again, we put

$$(1+y)^\lambda - 1 = z \text{ so that } z \to 0, \text{ as } y \to 0.$$

$$\therefore \quad (1+y)^\lambda = 1+z \text{ or } \lambda \log (1+y) = \log (1+z).$$

$$\therefore \quad \frac{x^\lambda - a^\lambda}{x-a} = a^{(\lambda-1)} \cdot \frac{z}{y}$$

$$= a^{(\lambda-1)} \cdot \frac{z}{\log (1+z)} \cdot \frac{\log (1+z)}{y}$$

$$= a^{(\lambda-1)} \cdot \frac{z}{\log (1+z)} \cdot \frac{\lambda \log (1+y)}{y}$$

$$= a^{(\lambda-1)} , \frac{1}{\log (1+z)^{1/z}} \cdot \lambda \log (1+y)^{1/y}$$

Hence

$$\lim_{x \to 0} \frac{x^\lambda - a^\lambda}{x-a} = \lambda a^{\lambda-1} \lim_{z \to 0} \frac{1}{\log (1+z)^{1/z}} \lim_{y \to 0} \log (1+y)^{1/y}$$

$$= \lambda a^{\lambda-1} \cdot \frac{1}{\log e} \cdot \log e = \lambda a^{\lambda-1}.$$

3·66. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, as proved in books on Elementary Trigonometry.

Examples

1. *Show that*

$$f(x) = (1+3x)^{1/x} \textit{ when } x \neq 0.$$
$$f(0) = e^3$$

is continuous for $x=0$.

Now

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left[(1+3x)^{1/3x} \right]^3$$

so that
$$\lim_{x \to 0} f(x) = e^3 = f(0).$$

Hence the result.

2. *Show that*

$$f(x)=(e^{\frac{1}{x}}-1)/(e^{\frac{1}{x}}+1) \qquad \textit{when } x\neq 0,$$
$$f(0)=0$$

is discontinuous at $x=0$.

When x tends to 0 through positive values, then $1/x \to +\infty$ and therefore $e^{\frac{1}{x}} \to +\infty$. Thus

$$\lim_{x\to(0+0)} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}=\lim_{x\to(0-0)} \frac{1-\dfrac{1}{e^{\frac{1}{x}}}}{1+\dfrac{1}{e^{\frac{1}{x}}}}=\frac{1-0}{1+0}=1.$$

When x tends to 0 through negative values, then $1/x\to -\infty$ and therefore $e^{\frac{1}{x}} \to 0$. Thus

$$\lim_{x\to(0+0)} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}=\frac{0-1}{0+1}=-1.$$

Hence we see that

$$\lim_{x\to(0+0)} f(x)\neq \lim_{x\to(0-0)} f(x)$$

so that

$$\lim_{x\to 0} f(x)$$

does not exist.

Hence the result.

3. *Show that*

$$f(x)=\begin{cases} \dfrac{x-1}{1+e^{1/(x-1)}} & \textit{when } x\neq 1 \\ 0 & \textit{when } x=1, \end{cases}$$

is continuous for $x=1$.

Now, as may easily be seen

$$\lim_{x\to(1+0)} e^{\frac{1}{x-1}}=\infty, \quad \lim_{x\to(1-0)} e^{\frac{1}{x-1}}=0.$$

Thus

$$\lim_{x\to(1+0)} f(x)=0, \qquad \lim_{x\to(1-0)} f(x)=0.$$

$$\lim_{x\to1} f(x)=0=f(0)$$

Hence the result.

4. *A sum of P rupees is given on interest at the rate of r% per annum. What will be the amount after t years, when the interest is being continuously added?*

Supposing that the interest is added after every nth part of a year, the amount after t years will be

$$P\left(1+\frac{r}{100n}\right)^{tn}$$

The required amount is its limit as $n\to\infty$.

We have

$$P\left(1+\frac{r}{100n}\right)^{tn}=P\left\{\left(1+\frac{r}{100n}\right)^{100n/r}\right\}^{tr/100}$$

which $\to Pe^{tr/100}$ as $n\to\infty$ and is thus the required amoun .

Exercises

1. Prove that

(i) $\lim_{x\to0}\dfrac{\sin ax}{\sin bx}=\dfrac{a}{b}$. (ii) $\lim_{x\to0}\dfrac{\tan x}{x}=1$.

(iii) $\lim_{x\to0}\dfrac{1-\cos x}{x^2}=\dfrac{1}{2}$.

2. Examine whether or not the following functions are continuous at $x=0$,

(i) $f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{when } x\neq0.\\ 1, & \text{when } x=0.\end{cases}$

(ii) $f(x)=\begin{cases}\dfrac{\sin 2x}{x}, & \text{when } x\neq0.\\ 1, & \text{when } x=0.\end{cases}$

(iii) $f(x)=\begin{cases}\dfrac{\tan 2x}{3x}, & \text{when } x\neq0.\\ \frac{2}{3}, & \text{when } x=0.\end{cases}$

(iv) $f(x)=\begin{cases}(1+x)^{1/x}, & \text{when } x\neq0.\\ 1, & \text{when } x=0.\end{cases}$

(v) $f(x)=\begin{cases}(1+2x)^{1/x}, & \text{when } x\neq 0.\\ e^2, & \text{when } x=0.\end{cases}$

(vi) $f(x)=\begin{cases}e^{-1/x^2}, & \text{when } x\neq 0,\\ 1, & \text{when } x=0.\end{cases}$

(vii) $f(x)=\begin{cases}\dfrac{e^{-1/x}}{1+e^{1/x}}, & \text{when } x\neq 0.\\ 1, & \text{when } x=0.\end{cases}$

(viii) $f(x)=\begin{cases}\dfrac{e^{1/x^2}}{e^{1/x^2}-1}, & \text{when } x\neq 0.\\ 1, & \text{when } x=0.\end{cases}$

3. Show that

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=\cos x.$$

Carefully state the results you employ.

4. Show that

(i) $(1/n^2+2/n^2+3/n^2+\ldots\ldots+n/n^2)\to\frac{1}{2}$.

and (ii) $(1^2/n^3+2^2/n^3+3^2/n^3+\ldots\ldots+n^2/n^3\to\frac{1}{3}$,

when $n\to\infty$ through positive integral values.

5. Draw the graphs of the functions

$$2^{-1/x^2},\ 2^{1/x^2}.\ 2^{1/x}.$$

3·7. Note on Hyperbolic Functions. In analogy with Trigonometric functions, the Hyperbolic functions are defined in the following manner :—

$$\sinh x=\frac{e^x-e^{-x}}{2};\qquad \cosh x=\frac{e^x+e^{-x}}{2};$$

$$\tanh x=\frac{\sinh x}{\cosh x}=\frac{e^x-e^{-x}}{e^x+e^{-x}};\qquad \coth x=\frac{\cosh x}{\sinh x}=\frac{e^x+e^{-x}}{e^x-e^{-x}};$$

$$\operatorname{sech} x=\frac{1}{\cosh x}=\frac{2}{e^x+e^{-x}};\qquad \operatorname{cosech} x=\frac{1}{\sinh x}=\frac{2}{e^x-e^{-x}};$$

3·71. Graph of sinh x.

The following properties of sinh x will enable us to draw the graphs :—

(i) sinh x is continuous for every value of x.

(ii) $\sinh 0=\dfrac{e^0-e^0}{2}=0.$

(*iii*) $\sinh x = \dfrac{e^x - e^{-x}}{2} = \dfrac{e^x - \dfrac{1}{e^x}}{2}$

and as x increases from 0, e^x monotonically increases and $1/e^x$ monotonically decreases. Thus as x increases from 0, sinh x monotonically increases.

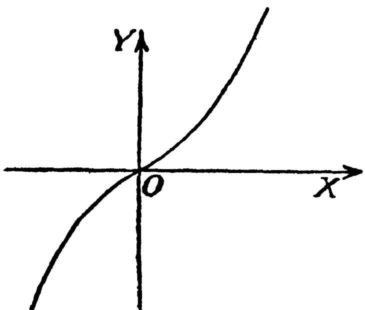


Fig. 39

(*iv*) $\lim\limits_{x \to \infty} (\sinh x) = \infty$.

(*v*) $\sinh (-x) = \dfrac{e^{-x} - e^x}{2}$

$$= -\frac{e^x - e^{-x}}{2} = -\sinh x.$$

Thus we have the graph as shown, in general outlines. (Fig. 39).

3·72. Graph of cosh x.

(*i*) cosh x is continuous for every value of x.

(*ii*) $\cosh 0 = \dfrac{e^0 + e^{-0}}{2} = 1.$

(*iii*) $\cosh x = \dfrac{e^x - e^{-x}}{2}$

$$= \sqrt{\left[\left(\frac{e^x - e^{-x}}{2}\right)^2 + 1\right]}$$

and as seen above $(e^x - e^{-x})/2$ monotonically increases, as x increases from 0. Thus cosh x monotonically increases as x increases from 0.

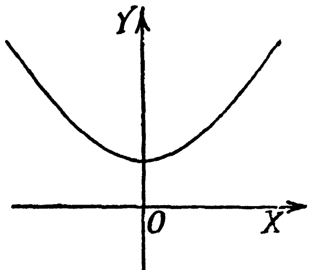


Fig. 40

(*iv*) $\lim\limits_{x \to \infty} \cosh x = \infty.$

(*v*) $\cosh (-x) = \cosh x.$

Hence the graph as shown.

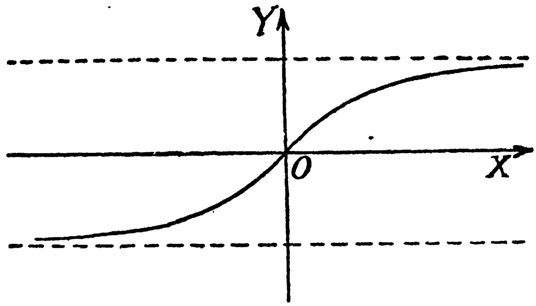
3·73. Graph of tanh x.

(*i*) tanh x is continuous for every value of x, the denominator $(e^x + e^{-x})$ not vanishing for any value of x.

(*ii*) $\tanh 0 = 0.$

(*iii*) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - \dfrac{2e^{-x}}{e^x + e^{-x}} = 1 - \dfrac{2}{e^x(e^x + e^{-x})}$

so that tanh x increases as x increases from 0 onward.



$y = \tanh x$

Fig. 41

(iv) $\lim\limits_{x\to\infty} \tanh x = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x\to\infty} \dfrac{1 - e^{-2x}}{1 + e^{-2x}} = 1.$

(v) $\tanh(-x) = -\tanh x.$

We may note that

$$|\tanh x| < 1,$$

for every value of x.

Note. The graphs of coth x, sech x and cosech x have not been given. The reader may, if he so desires, draw the same himself.

3·74. Some fundamental relations. The following fundamental relations can be at once deduced from the definitions :

(1) $\cosh^2 x - \sinh^2 x = 1.$

(2) $1 - \tanh^2 x = \text{sech}^2 x.$

(3) $\coth^2 x - 1 = \text{cosech}^2 x.$

(4) $\cosh^2 x + \sinh^2 x = \cosh 2x.$

(5) $2 \sinh x \cosh x = \sinh 2x.$

3·8. Inverse hyperbolic functions. In this section we obtain the logarithmic expressions for the inverse Hyperbolic Functions :

$$\sinh^{-1} x,\ \cosh^{-1} x,\ \tanh^{-1} x,\ \coth^{-1} x,\ \text{sech}^{-1} x,\ \text{cosech}^{-1} x,$$

which are defined as the inverses of the corresponding hyperbolic functions. This will also necessitate a slight modification of the definitions so as to make them *single-valued.*

A. Let

$$y = \sinh^{-1} x,$$

so that y is the number whose sinh is x.

$\therefore \quad x = \sinh y = \frac{1}{2}(e^y - e^{-y})$ *i.e.*, $e^{2y} - 2x.e^y - 1 = 0.$

$\therefore \quad e^y = x \pm \sqrt{(x^2+1)}$ or $y = \log[x \pm \sqrt{x^2+1})].$

It is easy to see that $x+\sqrt{(x^2+1)}$ is positive and $x-\sqrt{(x^2+1)}$ is negative for every value of x, positive or negative. Also we know that the logarithm of a negative number has no meaning in the field of real numbers so that $[x-\sqrt{(x^2+1)}]$ has to be rejected. Hence we have

$$\sinh^{-1}\mathbf{x}=\log[\mathbf{x}+\sqrt{(\mathbf{x}^2+1)}]$$

B. Let

$$y=\cosh^{-1}x.$$

We will see that there are always two values of y whose cosh is a given number.

Now

$$x=\cosh y=\tfrac{1}{2}(e^y+e^{-y}) \text{ i.e., } e^{2y}-2x.e^y+1=0.$$

$$\therefore \quad e^y=x\pm\sqrt{(x^2-1)} \text{ or } y=\log[x\pm\sqrt{(x^2-1)}].$$

Here we see that both $x+\sqrt{(x^2-1)}$, and $x-\sqrt{(x^2-1)}$ are positive and real when $x\geqslant 1$ so that in this case y has two values.

For $x\geqslant 1$, we have

$$x+\sqrt{(x^2-1)}>1$$

and

$$x-\sqrt{(x^2-1)}=\frac{1}{x+\sqrt{(x^2-1)}}<1,$$

so that

$$\log[x+\sqrt{(x^2-1)}]>0, \text{ and } \log[x-\sqrt{(x^2-1)}]<0.$$

To avoid this ambiguity, it is usual to modify the definition of $\cosh^{-1}x$ a little and say that $\cosh^{-1}x$ *is the positive number y whose cosh is x so that we have*

$$\cosh^{-1}\mathbf{x}=\log[\mathbf{x}+\sqrt{(\mathbf{x}^2-1)}].$$

Note. The ambiguity referred to here is a consequence of the fact that in $x=\cosh y$ two values of y give rise to the same value of x so that to a given value of x correspond two values of y which gave rise to it. (Refer § 3·72).

C. Let

$$y=\tanh^{-1}x, \text{ where } |x|<1.$$

$$\therefore \quad x=\tanh y=\frac{e^y-e^{-y}}{e^y+e^{-y}} \text{ so that } \frac{1+x}{1-x}=\frac{e^y}{e^{-y}}=e^{2y}.$$

Thus

$$\mathbf{y}=\tanh^{-1}\mathbf{x}=\frac{1}{2}\log\frac{1+\mathbf{x}}{1-\mathbf{x}}.$$

It is easy to see that

$$(1+x)/(1-x)>0, \text{ if } |x|<1.$$

We may similarly show that

D. $$\coth^{-1}x=\frac{1}{2}\log\frac{x+1}{x-1} \qquad |x|>1.$$

$[(x+1)/(x-1)]>0$, if $|x|>1$.

E. $$\operatorname{sech}^{-1}x=\log\frac{1+\sqrt{(1-x^2)}}{x}. \ [0<x<1].$$

F. $$\operatorname{cosech}^{-1}x=\log\frac{1\pm\sqrt{(1+x^2)}}{x}.$$

where the sign of the radical is positive or negative according as x is positive or negative.

Ex. 1. Show that $\sinh x$ tends to ∞ or $-\infty$ according as x tends to ∞ or $-\infty$.

2. Show that $\cosh x$ tends to ∞ whether x tends to ∞ or to $-\infty$.

3. Show that $\tanh x \to 1$ or $\to -1$ according as x tends to ∞ or to $-\infty$.

4. Show that

$$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$$
$$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y.$$

CHAPTER IV

DIFFERENTIATION

4·1. Introduction. Rate of Change. The subject of Differential Calculus which had its origin mainly in the geometrical problem of the determination of a tangent at a point of curve, has rendered possible the precise formulation of a large number of physical concepts such as *Velocity at a point*, *Acceleration at a point*, *Curvature at a point, Density at a point, Specific heat at any temperature*, etc. each of which appears as a **Local or instantaneous Rate of change** as against the **Average Rate of Change** which pertains to a finite interval of space or time and not to an instant of time and space.

The fundamental idea of **Local or instantaneous Rate of Change** pervading all these concepts underlies the analytical definition of differential co-efficient.

4·11. Derivability. Derivative. We consider a function $f(x)$ defined in any interval (a, b). Let c be any number of this interval so that $f(c)$ is the corresponding value of the function. We take $c+h$ any other number of this interval which lies to the right or left of c (*i.e.*, $c+h>$ or $<c$) according as h is positive or negative. The value of the function corresponding to it is $f(c+h)$.

Now, h, is the change in the independent variable x, and

$$f(c+h)-f(c)$$

is the corresponding change in the dependent variable $f(c)$.

The expression $[f(c+h)-f(c)]/h$, which is the ratio of these two changes, is a function of h and is not defined for $h=0$; c being a fixed number.

It is possible that the ratio tends to a limit as h tends to 0. This limit, if it exists, is called the derivative of $f(x)$ for $x=c$ and the function, then, is said to be derivable for this value.

Def. *$f(x)$ is said to be* **derivable** *at $x=c$ if*

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

exists and the limit is called the **Derivative** *or* **Differential co-efficient** *of the function $f(x)$ for $x=c$.*

The limit must be the same whether h tends to zero through positive or through negative values. The function will not be derivable if these limits are different.

The function $f(x)$ is said to be finitely derivable if its derivative is finite.

Ex. 1. *Show that x^2 is derivable for $x=1$ and obtain its derivative for $x=1$.*

Let

$$f(x)=x^2 \text{ so that } f(1)=1^2=1.$$

To find the derivative for $x=1$, we change x from 1 to $1+h$ so that the function changes from 1 to $(1+h)^2$.

Change in the function $=(1+h)^2-1=2h+h^2$.

$$\therefore \qquad \frac{f(1+h)-f(1)}{h}=\frac{2h+h^2}{h}=2+h, \text{ [as } h\neq 0]$$

which approaches 2 as h approaches 0.

Hence $f(x)$ is derivable and its derivative is 2 for $x=1$.

Ex. 2. *Show that $|\,x\,|$ is not derivable for $x=0$.*

Let

$$f(x)=|\,x\,| \text{ so that } f(0)=0.$$

It will be shown that the limit of $[f(0+h)-f(0)]/h$ does not exist, when h tends to 0.

Now

$$f(h)=\begin{cases} h, & \text{if } h>0\ ; \\ -h. & \text{if } h<0. \end{cases}$$

$$\therefore \qquad \frac{f(0+h)-f(0)}{h}=\frac{f(h)}{h}=1 \text{ or } -1$$

according as h is positive or negative.

Hence, $[f(0+h)-f(0)]/h \to 1$ as $h \to 0$ through positive values and $\to -1$ as $h \to 0$ through negative values.

Thus $[f(0+h)-f(0)]/h$ approaches different limits when h approaches 0 through positive or negative values so that it does not tend to a limit as h tends to 0.

Hence $|\,x\,|$ is *not* derivable for $x=0$.

Ex. 3. *If* $\qquad f(x)=x \sin \dfrac{1}{x}$ *when* $x\neq 0$

$$f(0)=0,$$

show that $f(x)$ is continuous for $x=0$ but has no differential co-efficient for $x=0$.

We have

$$f(x)-f(0)=x \sin \frac{1}{x}-0=x \sin \frac{1}{x}.$$

$$\therefore \qquad |\,f(x)-f(0)\,|=\left|\,x \sin \frac{1}{x}\right|$$

$$= |x| \; \left| \sin \frac{1}{x} \right| \leqslant |x|$$

Thus if ε be any pre-assigned number, we have

$$|f(x)-f(0)| < \varepsilon \text{ when } |x-0| < \varepsilon.$$

Hence

$$\lim_{x \to 0} f(x)=0=f(0)$$

so that $f(x)$ is continuous for $x=0$.

Again

$$\frac{f(x)-f(0)}{x-0}=\frac{x \sin \frac{1}{x}}{x}=\sin \frac{1}{x}$$

and, as seen in Ex. 3, p. 47, $\lim \sin (1/x)$ does not exist when $x \to 0$. Thus $f(x)$ has no differential co-efficient for $x=0$.

Ex. 4. Find the derivatives of

(*i*) $2x^2+3x-4$ for $x=5/2$. (*ii*) $1/x$ for $x=5$.

4·12. Derived function. In § 4·11, we have defined the derivative of a function $f(x)$ for a particular value, c, of the independent variable. Instead of considering a particular number c, we, now, consider any number x and determine

$$\lim \frac{f(x+h)-f(x)}{h} \text{ when } h \to 0$$

where x is kept constant in the process of taking the limit. We suppose here that the limit exists whatever value x may have provided it belongs to the interval of definition of the function.

This limit which is a function of x is called the *Derived function of Derivative of* $f(x)$ and is denoted by $f'(x)$.

Derivative of function is also called its **Differential co-efficient.**

The symbol $f'(c)$ then denotes the derivative of $f(x)$ for $x=c$.

Examples

1. *Find the derivative of* x^2.

Let

$$f(x)=x^2.$$

$$\therefore \quad f'(x)=\lim \frac{f(x+h)-f(x)}{h} \text{ when } h \to 0$$

$$= \lim \frac{(x+h)^2-x^2}{h} \text{ when } h \to 0$$

$$= \lim_{h \to 0} (2x+h) = 2x.$$

Thus $2x$ is the derivative or the derived function of x^2.

Puting $x=1$, we get

$$f'(1)=2,$$

which is the derivative of x^2 for $x=1$ and agrees with the result of **Ex. 1, § 4·11, p. 73.**

2. *Find the differential co-efficient of* $\sqrt{x}$.

Let

$$f(x)=\sqrt{x}.$$

$$f'(x)=\lim \frac{\sqrt{(x+h)}-\sqrt{x}}{h} \text{ when } h \to 0.$$

$$=\lim \left(\frac{\sqrt{(x+h)}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{(x+h)}+\sqrt{x}}{\sqrt{(x+h)}+\sqrt{x}}\right)$$

$$=\lim \frac{1}{\sqrt{(x+h)}+\sqrt{x}} = \frac{1}{2\sqrt{x}}, \text{ when } x \neq 0.$$

We start *afresh* to find derivative at $x=0$. We have

$$f'(0)=\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{\sqrt{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{h}} = \infty,$$

when $h \to 0$ through positive values ; $\sqrt{h}$ being not defined for negative values of h.

Ex. Find the derivatives of

(*i*) $1/(x^2+3)$. (*ii*) $1/\sqrt{x}$. (*iii*) x^3. (*iv*) ax^2+bx+c.

4·13. Another notation for the Derivative. In this notation the changes in the variables x and y are denoted by the *composite symbols* δx and δy respectively, so that

$$\delta y=f(x+\delta x)-f(x).$$

The derivative *i.e.*, $\lim (\delta y/\delta x)$, as $\delta x \to 0$, is then denoted by another *composite* symbol dy/dx.

Thus

$$\frac{dy}{dx} = \lim \frac{f(x+\delta x)-f(x)}{\delta x} \text{ when } \delta x \to 0.$$

Again, the value $f'(c)$ of the derivative of $y=f(x)$ for any particular value c, of x is denoted by $\left(\frac{dy}{dx}\right)_{x=c}$.

Note 1. dy/dx is a composite symbol denoting $\lim \delta y/\delta x$ and is not to be regarded as the quotient of dy by dx which have not so far been defined as separate symbols.

Note 2. The changes δx and δy in x, y are also known as *increments.*

Ex. 1. Find $\left(\frac{dy}{dx}\right)_{x=0}$ and $\frac{dy}{dx}$ when $y=\frac{2x}{x^2+1}$.

Ex. 2. If $y=\sqrt{(x^2+1)}$, find (dy/dx) when $x=-1$.

4·14. An important theorem. *Every finitely derivable function is continuous.*

Let $f(x)$ be derivable for $x=c$ so that the expression

$$[f(c+h)-f(c)]/h$$

tends to a finite limit as h tends to 0. We write

$$f(c+h)-f(c)=\frac{f(c+h)-f(c)}{h}\times h.$$

$$\therefore \lim_{h\to 0}[f(c+h)-f(c))=\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\times h\right)$$

$$=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\times\lim_{h\to 0}(h)$$

$$=f'(c)\times 0=0.$$

Hence

$$\lim_{h\to 0} f(c+h)=f(c), \textit{ i.e.,} \lim_{x\to c} f(x)=f(c).$$

Therefore $f(x)$ is continuous at $x=c$.

Cor. If $f(x)$ is derivable for every point of its interval of definition, then it is continuous in that interval.

Note. *The converse of this theorem is not necessarily true i.e.,* a function may be continuous for a value of x without being derivable for that value. For example, the function

$$y=|x|$$

is continuous but not derivable for $x=0$. [Ex. 2, 3 page 73]

Ex. 1. Show that the function $|x|+|x-1|$ is continuous for every value of x but is not derivable for $x=0$ and $x=1$.

Ex. 2. Construct a function which is continuous for every value of x but is not derivable for three values of x.

Ex. 3. Show that $f(x)=x^2 \sin(1/x)$ when $x\neq 0$, and $f(0)=0$, is continuous and derivable for $x=0$.

[For mor examples and exercises refer to the appendix to this chapter.]

4·15. Geometrical interpretation of a Derivative. *To show that $f'(c)$, i.e.,*

$$\left(\frac{dy}{dx}\right)_{x=c}$$

is the tangent of the angle which the tangent line to the curve $y=f(x)$ at the point $P[c, f(c)]$ makes with x-axis.

We take two points $P[c, f(c)]$ and $Q[(c+h), f(c+h)]$ on the curve $y=f(x)$.

Draw the ordinates PL, QM and draw $PN \perp MQ$. We have

$$PN=LM=h$$

and

$$NQ=MQ-LP$$
$$=f(c+h)-f(c)$$

$$\therefore \quad \tan\angle XRQ=\tan\angle NPQ$$
$$=\frac{NQ}{PN}$$
$$=\frac{f(c+h)-f(c)}{h} \quad \ldots (1)$$

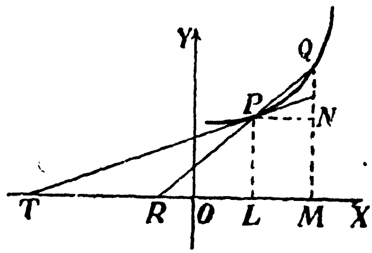


Fig. 42.

Here, $\angle XRQ$ is the angle which the chord PQ of the curve makes with the X-axis.

As h approaches 0, the point Q moving along the curve approaches the point P, the chord PQ approaches the tangent line TP as its limiting position, and $\angle XRQ$ approaches $\angle XTP$ which we denote by ψ.

On taking limits, the equation (1) gives

$$\tan \psi = f'(c).$$

Thus $f'(c)$ is the slope of the tangent to the curve $y=f(x)$ at $P[c, f(c)]$.

Cor. *The equation of the tangent at any point $P[c, f(c)]$ of the curve $y=f(x)$ is*

$$\mathbf{y-f(c)=f'(c)(x-c)}.$$

Note. The student should note that it is not necessary for *every curve to have a tangent line at every point* thereof. The existence of the tangent demands the existence of the derivative and we have seen in Ex. 2, and 3, § 4·11, p. 73 that every function is not derivable for every value of x.

For example, we know that $|x|$ is not derivable at $x=0$. The curve $y=|x|$ cannot therefore possess tangent at (0, 0).

This fact may be seen directly from the graph Fig. 8, p. 16 also.

Ex. 1. *Find the slope of the tangent to the parabola $y=x^2$ at the point* (2, 4).

Derivative of x^2 is $2x$ and its value for $x=2$ is 4. Hence the required slope of the tangent is 4.

Ex. 2 Show that the tangent to the hyperbola $y=1/x$ at (1, 1) makes an angle $3\pi/4$ with x-axis.

Ex. 3. Find the equations of the tangents to the parabola $y=x^2$ at the points $(-1, 1)$ and $(2, 4)$.

4·16. Expressions for velocity and acceleration of a particle moving in a straight line. Every thinking person is aware of the concepts of *Velocity* and *Acceleration* of a moving point. The difficulty arises in assigning *precise* measures to them. In practice, velocity at any instant is calculated by measuring the distance travelled in *some* short interval of time subsequent to the instant under consideration. This manner of calculating the velocity cannot clearly be precise, for different measuring agents may employ different intervals for the purpose. In fact this is only an approximate value of the actual velocity and some approximate value is all that we need in practice. The smaller the interval, the better is the approximation to the actual velocity.

In books not employing the method of Differential Calculus, velocity at any instant is defined as the distance travelled in an *infinitesimal* interval subsequent to the instant. Now there exists no such thing as an infinitesimal interval of time. We can take intervals of time as small as we like and in fact interval with duration smaller than any other is conceivable. The definition as it stands is thus meaningless. A meaning can, however, be attached to the above definition by supposing that the words 'velocity' and 'infinitesimal' in it really stand for approximation to the velocity' and 'some short interval of time', respectively.

The precise meaning to the velocity of a moving particle at any instant can only be given by employing the notion of Derivative.

4·161. Expression for velocity. The motion of a particle along a straight line is analytically represented by a functional equation

$$s=f(t),$$

where, s, represents the distance of the particle measured from some fixed point O on the line at time t.

Let P be the position of the particle at any given time t. Let, again, Q be its position after some short interval δt, and let $PQ=\delta s$.

The ratio $\delta s/\delta t$ is the average velocity over this interval and is an approximation to the actual velocity at P. We know intuitively that better approximations will be obtained by considering smaller values of δt.

We are thus led to *define the measure of the velocity at P* as being equal to

$$\lim_{\delta t\to 0} \frac{\delta s}{\delta t}, \text{ i.e., } \frac{ds}{dt}.$$

Hence, if v denotes the velocity, we have

$$v=\frac{ds}{dt}=\lim_{\delta t\to 0}\frac{f(t+\delta t)-f(t)}{\delta t}.$$

4·162. Expression for acceleration. Let v be the velocity at any given time t, and let $v+\delta v$ be the velocity after some short interval of time δt : δv is the change in velocity during time δt.

The ratio $\delta v/\delta t$ is the average acceleration during this interval δt and is an approximation to the actual acceleration at time t. The smaller values of δt will correspond to better approximations for the acceleration at time t. We are thus led to *define the measure of acceleration* as

$$\lim_{\delta t \to 0} \frac{\delta v}{\delta t}, \ i.e., \ \frac{dv}{dt}.$$

Ex. 1. Find the velocity and acceleration (*i*) at the end of 3 seconds, (*ii*) initially, in each of the following cases :—

(*a*) $s=t^2+2t+3$. (*b*) $s=1/(t+1)$. (*c*) $s=\sqrt{(t+1)}$.

Ex. 2. A particle moves along a straight line such that 's' is a quadratic function of t ; prove that its acceleration remains constant.

Ex. 3. If $s=t^3-2t^2+3t-4$, give the position, velocity and acceleration of the particle at the end of 0, 1, 2 seconds.

4·2. The remaining part of this chapter is devoted to determining the derivatives of functions. Some *general* theorems on differentiation which are required for the purpose will also be obtained in § 4·3. To provide for illustrations of these general theorems, we obtain, in this section, derivative of x^α where α is any real number.

4·21. Derivative of constant.

Let

$$y=c,$$

where, c, is a constant.

To every value of x corresponds the same value of y, so that the increment δy, corresponding to any increment δx, is zero.

$$\therefore \qquad \frac{\delta y}{\delta x}=\frac{0}{\delta x}=0.$$

$$\therefore \qquad \frac{dy}{dx}=\lim \frac{\delta y}{\delta x}=0, \ i.e., \ \frac{dc}{dx}=0.$$

Note. Looking at *derivative as the rate of change*, this result appears almost intuitive, as the rate of change of anything which does not change is necessarily zero.

The result may also be geometrically inferred from the fact that the slope of the tangent at any point of the curve $y=c$, which is a straight line parallel to x-axis, is 0.

4·22. Derivative of x^α *where α is any real* constant *number.*

Let

$$y=x^\alpha.$$

Let δy be the increment in y corresponding to the increment δx in x.

$$\therefore \qquad y+\delta y=(x+\delta x)^{\alpha},$$

$$\delta y=(y+\delta x)^{\alpha}-x^{\alpha},$$

$$\frac{\delta y}{\delta x}=\frac{(x+\delta a)^{\alpha}-x^{\alpha}}{\delta x}=\frac{(x+\delta x)^{\alpha}-x^{\alpha}}{(x+\delta x)-x},$$

$$\therefore \qquad \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{(x+\delta x)^{\alpha}-x^{\alpha}}{(x+\delta x)-x},$$

$$=\lim_{(x+\delta x)\to x}\frac{(x+\delta x)^{\alpha}-x^{\alpha}}{(x+\delta x)-x}=ax^{\alpha-1}.$$

(§ 3·65, p. 63)

Hence

$$\frac{\mathbf{d(x^{\alpha})}}{\mathbf{dx}}=\alpha\,\mathbf{x}^{\alpha-1},$$

where a is any real number, rational or irrational.

Another method. The derivative of x^{α} for rational values of a can also be obtained without employing the *general* limit theorem of § 3·65.

Case I. *Let α be any positive rational number, say, p/q.*

Here

$$\frac{\delta y}{\delta x}=\frac{(x+\delta x)^{p/q}-x^{p/q}}{\delta x}.$$

We write

$$z=x^{1/q} \text{ so that } z^{q}=x.$$

Then

$$z+\delta z=(x+\delta x)^{1/q} \text{ or } (z+\delta z)^{q}=x+\delta x.$$

$$\therefore \qquad \frac{\delta y}{\delta x}=\frac{(z+\delta z)^{p}-z^{p}}{(z+\delta z)^{q}-z^{q}}.$$

Let $\delta x\to 0$ so that δz also $\to 0$.

$$\therefore \qquad \frac{dy}{dx}=\lim \frac{(z+\delta z)^{p}-z^{p}}{(z+\delta z)^{q}-z^{q}} \text{ when } \delta z\to 0,$$

$$=\lim \frac{z^{p}+pz^{p-1}(\delta z)+\ldots+(\delta z)^{p}-z^{p}}{z^{q}+qz^{q-1}(\delta z)+\ldots+(\delta z)^{q}-z^{q}} \text{ when } \delta z\to 0.$$

$$=\frac{p.z^{p-1}}{q.z^{q-1}}=\frac{p}{q}.\ z^{p-q}=\frac{p}{q}\ .x^{(p-q)/q}=\frac{p}{q}\ .x^{p/q-1}$$

$$=\alpha x^{\alpha-1}.$$

Case II. *Let α be any negative rational number, say $-p/q$; p, q being both positive. We have*

$$\frac{\delta y}{\delta x}=\frac{(x+\delta x)^{-p/q}-x^{-p/q}}{\delta x}.$$

Writing $z=x^{1/q}$ and $z+\delta z=(x+\delta x)^{1/q}$, we obtain

$$\frac{\delta y}{\delta x}=\frac{(z+\delta z)^{-p}-z^{-p}}{(z+\delta z)^{q}-z^{p}}=\frac{z^{p}-(z+\delta z)^{p}}{z^{p}(z+\delta z)^{p}[(z+\delta z)^{q}-z^{q}]}.$$

Let $\delta x\to 0$ so that $\delta z\to 0$ also. As before, we get

$$\frac{dy}{dx}=\frac{1}{z^{p}.z^{p}}(-p)z^{p-1}.\ \frac{1}{q.z^{q-1}}$$

$$=-\frac{p}{q}.\ z^{-p-q}=-\frac{p}{q}.x^{(-p-q)/q}=-\frac{p}{q}\ .x^{-p/q-1}=\alpha x^{\alpha-1}$$

Ex. 1. (*i*) $\dfrac{d(x)^{\frac{1}{2}}}{dx}=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}x^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{x}}.$

(*ii*) $\dfrac{d(x^2)}{dx}=2x^{2-1}=2x.$

(*iii*) $\dfrac{d\left(\frac{1}{\sqrt{x}}\right)}{dx}=\dfrac{d\left(x^{-\frac{1}{2}}\right)}{dx}=-\frac{1}{2}x^{-\frac{1}{2}-1}=-\frac{1}{2}x^{-\frac{3}{2}}$

(*iv*) $\dfrac{d(x^e)}{dx}=ex^{e-1}.$

4·3. Some general theorems on differentiation.

4·31. Derivative of the sum or difference. Let u, v be two derivable functions of x.

Denoting their sum by y, we write

$$y=u+v. \qquad ...(i)$$

Let δu, δv, δy be the respective increments in u, v, y, corresponding to an increment δx in x so that x, u, v, y become $x+\delta x$, $u+\delta u$, $v+\delta v$, $y+\delta y$ respectively. We have

$$y+\delta y=u+\delta u+v+\delta v. \qquad ...(ii)$$

Subtracting (*i*) from (*ii*), and dividing by δx we get

$$\frac{\delta y}{\delta x}=\frac{\delta u}{\delta x}+\frac{\delta v}{\delta x}.$$

Let $\delta x \to 0$.

$$\therefore \qquad \lim \frac{\delta y}{\delta x} = \lim \left(\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x} \right) = \lim \frac{\delta u}{\delta x} + \lim \frac{\delta v}{\delta x},$$

or
$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{du}}{\mathbf{dx}} + \frac{\mathbf{dv}}{\mathbf{dx}}.$$

We may similarly prove that

$$\frac{d(u-v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

Generalisation. By a repeated application of the results obtained above, it can be proved that if $u_1, u_2, \ldots\ldots, u_n$ be any *finite* number of derivable functions and

$$y = u_1 \pm u_2 \pm u_3 \pm u_4 \pm \ldots\ldots \pm u_n,$$

then

$$\frac{dy}{dx} = \frac{du_1}{dx} \pm \frac{du_2}{dx} \pm \frac{du_3}{dx} \pm \ldots\ldots\ldots\ldots \pm \frac{du_n}{dx}.$$

We thus have the theorem : *Algebraic sum of any* **finite** *number of derivable functions is itself derivable and the derivative of the sum is equal to the sum of their derivatives.*

Ex. 1.

$$(i) \quad \frac{d(x^3+x^2)}{dx} = \frac{dx^3}{dx} + \frac{dx^2}{dx} = 3x^2 + 2x.$$

$$(ii) \quad \frac{d\left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)}{dx} = \frac{dx^{\frac{1}{2}}}{dx} - \frac{dx^{-\frac{1}{2}}}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}} - \left(-\frac{1}{2}\right)x^{-\frac{3}{4}}$$

$$= \frac{1}{2x^{\frac{1}{2}}} + \frac{1}{2x^{\frac{3}{2}}} = \frac{x+1}{2x^{\frac{3}{2}}}.$$

$$(iii) \quad \frac{d\left(\dfrac{x^2+1}{x}\right)}{dx} = \frac{d\left(\dfrac{x^2}{x} + \dfrac{1}{x}\right)}{dx}$$

$$= \frac{dx}{dx} + \frac{dx^{-1}}{dx}$$

$$= 1 + (-1)x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}.$$

Ex. 2. Find the derivatives of

(i) $\dfrac{1+x}{\sqrt{x}}$. (ii) $\dfrac{1+x^{\frac{1}{4}}}{x^{-\frac{1}{3}}}$. (iii) $\dfrac{x^2+x+1}{\sqrt{x}}$. (iv) $\dfrac{(1+x)\sqrt{x}}{\sqrt[3]{x}}$.

4·32. Derivative of a product.

Let

$$y=uv$$

where u, v, are two derivable functions of x.

Let δu, δv, δy, be the increments in u, v, y respectively corresponding to the increment δx in x. We have

$$y+\delta y=(u+\delta u)(v+\delta v)$$

$$=uv+u.\delta v+v.\delta u+\delta u.\delta v.$$

$$\therefore \qquad \delta y=u.\delta v+v.\delta u+\delta u.\delta v,$$

or

$$\frac{\delta y}{\delta u}=u.\frac{\delta v}{\delta x}+v.\frac{\delta u}{\delta x}+\delta u.\frac{\delta v}{\delta x}.$$

Let $\delta x \to 0$. Then δu also $\to 0$; for u, which is a derivable function, is continuous.

$$\therefore \qquad \frac{dy}{dx}=\lim\left[u.\frac{\delta v}{\delta x}+v.\frac{\delta u}{\delta x}+\delta u.\frac{\delta v}{\delta x}\right] \text{ when } \delta x \to 0$$

$$=\lim\left(u.\frac{\delta v}{\delta x}\right)+\lim\left(v.\frac{\delta u}{\delta x}\right)+\lim \delta u.\lim\frac{\delta v}{\delta x}$$

$$=u.\frac{dv}{dx}+v.\frac{du}{dx}+0.\frac{dv}{dx}=u.\frac{dv}{dx}+v.\frac{du}{dx}.$$

$$\therefore \qquad \frac{d(uv)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}. \qquad \ldots(i)$$

Thus we have the theorem :—*The product of two derivable functions is itself derivable and its derivative is the sum of the two products obtained by multiplying each function with derivative of the other.*

The derivative of the product of two functions=first function × derivative of the second+second function × derivative of the first.

The result (i) may also be re-written as

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}, \text{ if } u\neq 0,\ v\neq 0.$$

Note. It may be noted that the operations of differentiation and multiplication are not invertible.

i.e.,
$$\frac{d(u\,v)}{dx}\neq\frac{du}{dx}\frac{dv}{dx}.$$

Cor. 1. Generalisation. *Directive of the product of any finite number of derivable functions.*

We first take

$$y=u_1u_2u_3=(u_1u_2)u_3.$$

$$\therefore \qquad \frac{dy}{dx}=(u_1u_2)\frac{du_3}{dx}+u_3\frac{d(u_1u_2)}{dx}$$

$$=u_1u_2\frac{du_3}{dx}+u_3\left(u_1\frac{du_2}{dx}+u_2\frac{du_1}{dx}\right)$$

$$=u_1u_2\frac{du_3}{dx}+u_1u_3\frac{du_2}{dx}+u_3u_2\frac{du_1}{dx}.$$

On dividing by $y=u_1u_2u_3$, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u_1}\frac{du_1}{dx}+\frac{1}{u_2}\frac{du_2}{dx}+\frac{1}{u_3}\frac{du_3}{dx}.$$

By repeated application of this result, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u_1}\frac{du_1}{dx}+\frac{1}{u_2}\frac{du_2}{dx}+\ldots\ldots+\frac{1}{u_n}\frac{du_n}{dx}$$

$$=\sum_{r=1}^{r=n}\frac{1}{u_r}\frac{du_r}{dx}, \text{ where } y=u_1u_2u_3\ldots u_n=\prod_{r=1}^{r=n}u_r.$$

Cor. 2. Derivative of cu, where c is a constant and u any derivable function of x. Let

$$y=cu.$$

$$\therefore \qquad \frac{dy}{dx}=c\frac{du}{dx}+u\frac{dc}{dx}$$

$$=c\frac{du}{dx}+u.0=c\frac{du}{dx}.$$

Hence

$$\frac{d(cu)}{dx}=c\frac{du}{dx}.$$

Ex. 1. (*i*) $$\frac{d[(1+x)\sqrt{x}]}{dx}=(1+x)\frac{d\sqrt{x}}{dx}+\sqrt{x}\frac{d(1+x)}{dx}$$

$$=(1+x)\tfrac{1}{2}x^{-\frac{1}{2}}+\sqrt{x}.1$$

$$=\frac{1+x}{2\sqrt{x}}+\sqrt{x}=\frac{1+3x}{2\sqrt{x}}.$$

(*ii*) $$\frac{d(3x^3)}{dx}=\frac{3dx^3}{dx}\,3.3x^2=9x^2.$$

2. Find the derivatives of

(*i*) $(x+2)(3+x)$. (*ii*) $(x+2)^2(2x-3)$. (*iii*) $(2x+3)^2(2\sqrt{x}+3/\sqrt{x})$.

4·33. Derivative of a quotient.

Let

$$y=u/v, \qquad (v\neq 0)$$

where u, v are two derivable functions of x such that v is not zero for the value of x under consideration.

We have

$$y+\delta y=\frac{u+\delta u}{v+\delta v}. \qquad \ldots(i)$$

$$\therefore \qquad \delta y=\frac{u+\delta u}{v+\delta v}-\frac{u}{v}=\frac{v.\delta u-u.\delta v}{v(v+\delta v)}.$$

or

$$\frac{\delta y}{\delta x}=\frac{v.\dfrac{\delta u}{\delta x}-u.\dfrac{\delta v}{\delta x}}{v(v+\delta v)}.$$

Now, v, being a derivable function of x, is continuous.

Hence

$$\delta v \to 0 \text{ as } \delta x \to 0.$$

Thus

$$\frac{d\left(\dfrac{u}{v}\right)}{dx}=\frac{dy}{dx}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2},$$

so that **derivative of the quotient of two functions**

=[Derivate of Numer. (Denomr.)

—(Numer.) (Derivative of Denomr.)]÷Square of Denominator.

Ex. 1. (*i*) $$\frac{d\left(\dfrac{x}{1+x}\right)}{dx}=\frac{(1+x)\dfrac{dx}{dx}-x\dfrac{d(1+x)}{dx}}{(1+x)^2}$$

$$=\frac{(1+x).1-x.1}{(1+x)^2}=\frac{1}{(1+x)^2}.$$

(*ii*) $$\frac{d\left[\dfrac{x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}}\right]}{dx}=\frac{\left[1+x^{\frac{1}{2}}\right]\dfrac{dx^{\frac{3}{2}}}{dx}-x^{\frac{3}{2}}\dfrac{d(1+x^{\frac{1}{2}})}{dx}}{(1+x^{\frac{1}{2}})^2}$$

$$=\frac{(1+x^{\frac{1}{2}}).\frac{3}{2}x^{\frac{1}{2}}-x^{\frac{3}{2}}.\frac{1}{2}x^{-\frac{1}{2}}}{[1+x^{\frac{1}{2}}]^2}=\frac{\frac{3}{2}\sqrt{x}+x}{(1+\sqrt{x})^2}.$$

2. Obtain the derivatives of

(*i*) $\dfrac{3x+4}{4x+5}$. (*ii*) $\dfrac{x+1}{(x+2)^2}$. (*iii*) $\dfrac{x^{\frac{1}{2}}+x^{-\frac{1}{2}}}{x^{\frac{1}{3}}+x^{-\frac{1}{3}}}$.

Note 1. Being a derivable function, v is continuous. Also, we have supposed that $v \neq 0$ for the value of x under consideration. There is, therefore, an interval around x such that $v \neq 0$ for any point of the interval. (§3·51, p. 54.)

Thus if we suppose $x+\delta x$ to lie within this interval then, the corresponding value of v, *i.e.*, $v+\delta v \neq 0$. This fact justifies division by $v+\delta v$ in step (*i*).

Note 2. The importance of the results obtained above in § 4·3 lies in the fact that the derivative of any function which is an algebraic combination [*i.e.*, built up through the operations, of addition subtraction, multiplication and division of several others is itself expressible as an algebraic combination of the derivatives of the latter.

4·34. Derivative of a function of a function. *If $y=f(u)$ and $u=\phi(x)$, so y is a function of x, then*

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx},$$

f and ϕ being derivable functions of u and x respectively.

Let δx be any increment in x and δu the corresponding increment in u as determined from $u=\phi(x)$. Again, corresponding to the increment δu, in u, let δy be the increment in y as determined from $y=f(u)$. We write

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x}.$$

Let $\delta x \to 0$ so that $\delta u \to 0$.

$$\therefore \quad \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)=\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x}\right)$$

$$=\lim_{\delta u \to 0}\frac{\delta y}{\delta u}\cdot\lim_{\delta x \to 0}\frac{\delta u}{\delta x}.$$

Hence

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

The result is capable of immediate generalization. Thus if

$$y=f(u),\ u=\phi(v),\ v=\psi(x)$$

be three derivable functions so that y is a function of x, we have

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}.$$

Ex. *Find the derivatives of*

(*i*) $\sqrt{(1+x^2)}$, (*ii*) $\sqrt{[(1+x)/(1-x)]}$.

(*i*) We write $u=1+x^2,\ y=u^{\frac{1}{2}}$.

$$\therefore \quad \frac{du}{dx}=2x,\quad \frac{dy}{du}=\tfrac{1}{2}u^{-\frac{1}{2}}=\tfrac{1}{2}(1+x^2)^{-\frac{1}{2}}.$$

Hence

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\tfrac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}2x=\frac{x}{\sqrt{(1+x^2)}}.$$

or, without introducing u, we have

$$\frac{d\sqrt{(1+x^2)}}{dx}=\frac{d\sqrt{(1+x^2)}}{d(1+x^2)}\cdot\frac{d(1+x^2)}{dx}$$

$$=\tfrac{1}{2}(1+x^2)^{-\frac{1}{2}}2x=x/\sqrt{(1+x^2)}.$$

(*ii*) Let $u=\dfrac{1+x}{1-x}$, $y=u^{\frac{1}{2}}$.

$$\therefore\qquad \frac{du}{dx}=\frac{(1-x)\dfrac{d(1+x)}{dx}-(1+x)\dfrac{d(1-x)}{dx}}{(1-x)^2}$$

$$=\frac{(1-x)1-(1+x)(-1)}{(1-x)^2}=\frac{2}{(1-x)^2},$$

$$\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}.$$

Hence

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\cdot\frac{2}{(1-x)^2}=\frac{1}{(1+x)^{\frac{1}{2}}\,(1-x)^{\frac{3}{2}}}.$$

or, directly

$$\frac{d\sqrt{\left(\dfrac{1+x}{1-x}\right)}}{dx}=\frac{d\sqrt{\left(\dfrac{1+x}{1-x}\right)}}{d\left(\dfrac{1+x}{1-x}\right)}\cdot\frac{d\left(\dfrac{1+x}{1-x}\right)}{dx}$$

$$=\tfrac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\frac{2}{(1-x)^2}=\frac{1}{(1+x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}}.$$

Ex. Find the derivatives of

(*i*) $(ax+b)^n$. (*ii*) $1/(1+x^2)$. (*ii*) $\sqrt{(ax^2+2bx+c)}$.

(*iv*) $\dfrac{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}$. (*v*) $\sqrt{\left(\dfrac{a^2-x^2}{a^2+x^2}\right)}$. (*vi*) $\dfrac{1-a\sqrt[3]{x^2}}{1+a\sqrt[3]{x^2}}$.

(*vii*) $\dfrac{x\sqrt{(x^2-4)}}{\sqrt{(x^2-1)}}$.

4·35. Differentiation of inverse functions. Let $y=f(x)$ be any function derivable in its interval of definition. We suppose that it admits of an inverse function

$$x=\varphi(y),$$

as explained in §2·22, p. 21.

We have to find a relation between $f'(x)$ and $\varphi'(y)$.

Let δy be the increment in y corresponding to the increment δx in x, as determined from $y=f(x)$. The increment δx in x corresponds to the increment δy in y as determined from

$$x=\varphi(y).$$

We have

$$1=\frac{\delta y}{\delta x}\cdot\frac{\delta x}{\delta y} \text{ or } \frac{\delta x}{\delta y}=\frac{1}{\delta y/\delta x}.$$

Let $\delta x \to 0$.

$$\therefore \qquad \frac{dx}{dy}=\frac{1}{dy/dx} \text{ or } \frac{dy}{dx}\cdot\frac{dx}{dy}=1,$$

i.e.,

$$f'(x).\varphi'(y)=1.$$

Thus dy/dx and dx/dy *are reciprocal to each other.*

Ex. Verify the theorem for $y=x^2$ when $x=2$.

4·36. Differentiation of functions defined by means of a parameter. We consider two derivable functions

$$x=f(t),\ y=\varphi(t),$$

of 't'.

Assuming that $x=f(t)$ admits an inverse function $t=\psi(x)$ (§2·22) we obtain

$$y=\varphi[\psi(x)],$$

so that y is a function of x.

By the rule for the differentiation of a function of a function (§ 4·34), we have

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}.$$

Also

$$\frac{dt}{dx}=1\Bigg/\left[\frac{dx}{dt}\right].$$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{dt}\Bigg/\frac{dx}{dt}=\frac{\varphi'(t)}{f'(t)}.$$

Note. 't' is called a parameter.

Ex. 1. *Find dy/dx, when $x=at^2$, $y=2at$.*

We have

$$\frac{dx}{dt}=2at,\ \frac{dy}{dt}=2a.$$

$$\frac{dy}{dx}=\frac{dy}{dt}\Bigg/\frac{dx}{dt}=\frac{1}{t}.$$

2. Find dy/dx, when

(i) $x=a\dfrac{1-t^2}{1+t^2}$, $y=b\dfrac{2t}{1+t^2}$. (ii) $x=\dfrac{3at}{1+t^3}$, $y=\dfrac{3at^2}{1+t^3}$.

(iii) $x=\dfrac{2at^2}{1+t^2}$, $y=\dfrac{2at^3}{1+t^2}$.

(iv) $x=a\sqrt{\left(\dfrac{t^2-1}{t^2+1}\right)}$, $y=at\sqrt{\left(\dfrac{t^2-1}{t^2+1}\right)}$.

4·4. Derivatives of Trigonometrical functions. The symbols δx and δy stand for the increments in x and y and will always be used in this sense without any frequent mention of their meanings.

4·41. Derivative of sin x.

Let

$$y=\sin x.$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{\sin(x+\delta x)-\sin x}{\delta x}$$

$$=\frac{2\cos\frac{1}{2}(2x+\delta x)\sin\frac{1}{2}\delta x}{\delta x}$$

$$=\cos(x+\tfrac{1}{2}\delta x)\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}.$$

$$\therefore \quad \frac{dy}{dx}=\lim\cos(x+\tfrac{1}{2}\delta x)\,.\,\lim\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x},\text{ when }\delta x\to 0.$$

As $\cos x$ is a continuous function, we have, when $\delta x \to 0$,

$$\lim\cos(x+\tfrac{1}{2}\delta x)=\cos x.$$

Also when $\delta x \to 0$, $\lim\dfrac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}=1.$

$$\therefore \quad \frac{dy}{dx}=\cos x.$$

Thus

$$\mathbf{\frac{d(\sin x)}{dx}=\cos x.}$$

4·42. Derivative of cos x.

Let

$$y=\cos x$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{\cos(x+\delta x)-\cos x}{\delta x}$$

$$=\frac{-2\sin\frac{1}{2}(2x+\delta x)\sin\frac{1}{2}\delta x}{\delta x}$$

$$=-\sin(x+\tfrac{1}{2}\delta x)\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}.$$

As $\sin x$ is a continuous function, we have, when $\delta x \to 0$

$$\lim\sin(x+\tfrac{1}{2}\delta x)=\sin x.$$

$$\therefore \quad \frac{dy}{dx}=\lim_{\delta x\to 0}[-\sin(x+\tfrac{1}{2}\delta x)]\lim_{\delta x\to 0}\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}$$

$$=-\sin x.1=-\sin x.$$

Thus $$\frac{d(\cos x)}{dx}=-\sin x.$$

Ex. 1. *Find the derivatives of*

(*i*) $\sin 2x$. (*ii*) $\cos^3 x$. (*iii*) $\sqrt{(\sin\sqrt{x})}$.

(*i*) Let $y=\sin 2x$.

We write

$$u=2x, \text{ so that } y=\sin u.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\cos u \cdot 2=2\cos 2x.$$

or briefly

$$\frac{d(\sin 2x)}{dx}=\frac{d(\sin 2x)}{d(2x)}\cdot\frac{d(2x)}{dx}=\cos 2x \cdot 2=2\cos 2x$$

(*ii*) Let $y=\cos^3 x=(\cos x)^3$.

We write

$$u=\cos x \text{ so that } y=u^3.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=3u^2(-\sin x)=-3\cos^2 x \cdot \sin x.$$

or, briefly

$$\frac{d(\cos^3 x)}{dx}=\frac{d(\cos x)^3}{dx}=\frac{d(\cos x)^3}{d(\cos x)}\cdot\frac{d(\cos x)}{dx}$$

$$=3(\cos x)^2\times(-\sin x)=-3\cos^2 x \cdot \sin x$$

(*iii*) Let $y=\sqrt{(\sin\sqrt{x})}$.

We write $u=\sqrt{x}=x^{\frac{1}{2}}$ $\quad v=\sin u$,

so that $$y=\sqrt{v}=v^{\frac{1}{2}}.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dx}$$

$$=\tfrac{1}{2}v^{-\frac{1}{2}}\cdot\cos u\cdot\tfrac{1}{2}x^{-\frac{1}{2}}$$

$$=\tfrac{1}{4}\cdot\frac{\cos\sqrt{x}}{\sqrt{(\sin\sqrt{x})}}\cdot\frac{1}{\sqrt{x}}.$$

or, briefly

$$\frac{d\sqrt{(\sin\sqrt{x})}}{dx}=\frac{d\sqrt{(\sin\sqrt{x})}}{d\sin\sqrt{x}}\cdot\frac{d\sin\sqrt{x}}{d\sqrt{x}}\cdot\frac{d\sqrt{x}}{dx}$$

$$=\tfrac{1}{2}(\sin\sqrt{x})^{-\frac{1}{2}}\cdot\cos\sqrt{x}\cdot\tfrac{1}{2}x^{-\frac{1}{2}}$$

$$=\frac{1}{4}\cdot\frac{\cos\sqrt{x}}{\sqrt{x}\sqrt{(\sin\sqrt{x})}}.$$

2. *Find dy/dx for $t=\pi/2$, when*

$$x=2\cos t-\cos 2t,\ y=2\sin t-\sin 2t.$$

We have

$$\frac{dx}{dt}=-2\sin t-(-\sin 2t)(2)=-2\sin t+2\sin 2t,$$

$$\frac{dy}{dt}=2\cos t-(\cos 2t)2=2\cos t-2\cos 2t,$$

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{2(\cos t-\cos 2t)}{-2(\sin t-\sin 2t)}$$

Putting $t=\pi/2$, we obtain

$$\left(\frac{dy}{dx}\right)_{t=\pi/2}=-1\ .$$

3. Find the derivatives of

(*i*) $\sin^m x$, (*ii*) $\cos mx$, (*iii*) $\sin x^m$, (*iv*) $\cos^2 x$

(*v*) $\dfrac{\sin x}{x}$, (*vi*) $\dfrac{\sin^2 x}{1+\cos x}$, (*vii*) $\cos\sqrt{(ax^2+2bx+c)}$,

(*viii*) $\sin^m x.\cos^n x$.

4. Find dy/dx, when

(*i*) $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$.

(*ii*) $x=3\cos t-2\cos^3 t$, $y=3\sin t-2\sin^3 t$.

(*iii*) $x=a\cos^3 t$, $y=a\sin^3 t$. (*D.U. 1955*)

4·43. Derivative of tan x.

Let

$$y=\tan x$$

$$\therefore \qquad \frac{\delta y}{\delta x}=\frac{\tan(x+\delta x)-\tan x}{\delta x}$$

$$=\frac{\dfrac{\sin(x+\delta x)}{\cos(x+\delta x)}-\dfrac{\sin x}{\cos x}}{\delta x}$$

$$=\frac{\sin(x+\delta x)\cos x-\cos(x+\delta x)\sin x}{\delta x\cos(x+\delta x)\ .\cos x}$$

$$=\frac{\sin(x+\delta x-x)}{\delta x.\cos(x+\delta x).\cos x}$$

$$=\frac{1}{\cos(x+\delta x)}\cdot\frac{1}{\cos x}\cdot\frac{\sin\delta x}{\delta x}.$$

$$\therefore \qquad \frac{dy}{dx}=\frac{1}{\cos x}\cdot\frac{1}{\cos x}\cdot 1=\sec^2 x.$$

Thus $$\frac{\mathbf{d\,(tan\,x)}}{\mathbf{dx}}=\mathbf{sec^2\,x}.$$

Or, we write

$$y=\tan x=\frac{\sin x}{\cos x} \text{ so that}$$

$$\frac{dy}{dx}=\frac{\cos x.\dfrac{d(\sin x)}{dx}-\sin x.\dfrac{d(\cos x)}{dx}}{\cos^2 x}$$

$$=\frac{\cos x.\cos x+\sin x.\sin x}{\cos^2 x}$$

$$=\frac{1}{\cos^2 x}=\sec^2 x.$$

4·44. $$\frac{\mathbf{d(cot\,x)}}{\mathbf{dx}}=-\mathbf{cosec^2\,x}.$$

Its proof is left to the reader.

Ex 1. Find the derivatives of

(*i*) $\tan x+\cot x$. (*ii*) $\sin x.\tan 2x$. (*iii*) $x\tan x\cot 2x$.

(*iv*) $\dfrac{\tan x-\cot x}{\tan x+\cot x}$. (*v*) $\sqrt{\left(\dfrac{1-\tan x}{1+\tan x}\right)}$. (*vi*) $\sqrt{\left(\dfrac{1-\cos x}{1+\cos x}\right)}$.

4·45. Derivative of sec x.

Let

$$y=\sec x=\frac{1}{\cos x}.$$

$$\therefore \qquad \frac{\delta y}{\delta x}=\frac{\dfrac{1}{\cos(x+\delta x)}-\dfrac{1}{\cos x}}{\delta x}$$

$$=\frac{\cos x-\cos(x+\delta x)}{\delta x.\cos(x+\delta x).\cos x}$$

$$=\frac{2\sin\frac{1}{2}(2x+\delta x)\sin\frac{1}{2}\delta x}{\delta x.\cos(x+\delta x)\cos x}$$

$$=\sin(x+\tfrac{1}{2}\delta x)\frac{1}{\cos x}\cdot\frac{1}{\cos(x+\delta x)}\cdot\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}$$

$$\therefore \qquad \frac{dy}{dx}=\sin x.\frac{1}{\cos x}\cdot\frac{1}{\cos x}=\tan x\sec x.$$

Thus $\dfrac{d(\sec x)}{dx}=\tan x \sec x.$

Or, we write

$$y=\frac{1}{\cos x} \text{ so that}$$

$$\frac{dy}{dx}=\frac{\cos x\, 0-1\,(-\sin x)}{\cos^2 x}=\tan x \sec x.$$

4·46. $\dfrac{d(\operatorname{cosec} x)}{dx}=-\cot x \operatorname{cosec} x.$

Its proof is left to the reader.

Ex. Find the derivatives of

(*i*) $\operatorname{cosec}^3 3x$. (*ii*) $\sqrt{[\sec (ax+b)]}$.

(*iii*) $\sec\sqrt{(a+bx)}$. (*iv*) $\sec (\operatorname{cosec} x)$.

4·5. Derivatives of inverse trigonometrical functions. The precise definitions of inverse trigonometrical functions as given in § 2·7, p. 30 will have to be kept in mind to obtain their derivatives.

4·51. Derivative of $\sin^{-1} x$.

Let

$$y=\sin^{-1} x \text{ so that } x=\sin y.$$

$$\frac{dx}{dy}=\cos y.$$

or $$\frac{dy}{dx}=\frac{1}{\cos y}=\pm\frac{1}{\sqrt{(1-\sin^2 y)}}=\pm\frac{1}{\sqrt{(1-x^2)}}$$

where the sign of the radical is the same as that of $\cos y$.

By the def. of $\sin^{-1} x$, we have

$$-\pi/2 \leqslant \sin^{-1} x \leqslant \pi/2, \qquad i.e., \quad -\pi/2 \leqslant y \leqslant \pi/2$$

so that $\cos y$ is positive.

Hence $$\frac{d(\sin^{-1} x)}{dx}=\frac{1}{\sqrt{(1-x^2)}}.$$

4·52. Derivative of $\cos^{-1} x$.

Let

$$y=\cos^{-1} x \text{ so that } x=\cos y.$$

$\therefore$ $$\frac{dx}{dy}=-\sin y$$

or $$\frac{dy}{dx}=\frac{-1}{\sin y}=\pm\frac{-1}{\sqrt{(1-\cos^2 y)}}=\frac{-1}{\pm\sqrt{(1-x^2)}}$$

where the sign of the radical is the same as that of $\sin y$. By the def. of $\cos^{-1} x$, we have

$$0 \leqslant \cos^{-1}x \leqslant \pi \qquad i.e., \quad 0 \leqslant y \leqslant \pi.$$

Also if y lies between 0 and π, then $\sin y$ is necessarily positive.

Hence $$\frac{d(\cos^{-1} x)}{dx}=-\frac{1}{\sqrt{(1-x^2)}}.$$

4·53. Derivative of $\tan^{-1} x$.

Let

$$y=\tan^{-1} x \text{ so that } x=\tan y.$$

$\therefore$ $$\frac{dx}{dy}=\sec^2 y$$

or $$\frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}.$$

Thus $$\frac{d(\tan^{-1} x)}{dx}=\frac{1}{1+x^2}.$$

4·54. $$\frac{d(\cot^{-1} x)}{dx}=-\frac{1}{1+x^2}.$$

Its proof is left to the reader.

4·55. Derivative of $\sec^{-1} x$.

Let

$$y=\sec^{-1} x \text{ so that } x=\sec y.$$

$\therefore$ $$\frac{dx}{dy}=\sec y \tan y$$

or $$\frac{dy}{dx}=\frac{1}{\sec y \tan y}$$

$$=\pm\frac{1}{\sec y\sqrt{(\sec^2 y-1)}}=\pm\frac{1}{x\sqrt{(x^2-1)}}.$$

We take, $+$, sign before the radical and write

$$\frac{dy}{dx}=\frac{1}{x\sqrt{(x^2-1)}} \qquad \text{(Refer note below)}$$

Thus

$$\frac{d(\sec^{-1} x)}{dx}=\frac{1}{x\sqrt{(x^2-1)}}.$$

Note. This note is intended to show precisely what sign should be chosen before the radical. Now it is clear that the sign before the radical is the same as that of tan y. By the definition of $\sec^{-1} x$. [Refer § 2·75, p. 33], we have

$$0 \leqslant \sec^{-1} x < \pi/2 \;;\; \pi/2 < \sec^{-1} x \leqslant \pi$$

i.e.,
$$0 \leqslant y < \pi/2 \;;\; \pi/2 < y \leqslant \pi.$$

When x is positive so that it lies in the interval [1. ∞]. then y lies between 0 and $\pi/2$ and so *tan y* is positive ;

When x is negative so that it lies in the interval, [$-\infty$, -1], then y lies between $\pi/2$ and π and so *tan y* is negative.

Thus the sign of the radical is positive or negative aceording as x is positive or negative. Hence

$$\frac{dy}{dx} = \frac{1}{x\sqrt{(x^2-1)}} \text{ if } x>0 \text{ and } = -\frac{1}{x\sqrt{(x^2-1)}} \text{ if } x<0.$$

so that

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{(x^2-1)}}, \text{ for every admissible value of } x.$$

Thus
$$\frac{d(\sec^{-1} x)}{dx} = \frac{1}{|x|\sqrt{(x^2-1)}}.$$

4·56. Derivative of $\operatorname{cosec}^{-1}$ x.

Let

$$y = \operatorname{cosec}^{-1} x \text{ so that } x = \operatorname{cosec} y.$$

$$\therefore \quad \frac{dx}{dy} = -\cot y \operatorname{cosec} y$$

or
$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y}$$

$$= \pm \frac{-1}{\operatorname{cosec} y\sqrt{(\operatorname{cosec}^2 y - 1)}} = \pm \frac{-1}{x\sqrt{(x^2-1)}}$$

We take, $+$, sign before the radical and write

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{(x^2-1)}}.$$

Thus
$$\frac{d(\operatorname{cosec}^{-1} x)}{dx} = -\frac{1}{x\sqrt{(x^2-1)}}.$$

Note. The sign before the radical is the same as that of cot y.

By the definition of $\operatorname{cosec}^{-1} x$, we have

$$-\pi/2 \leqslant \operatorname{cosec}^{-1} x < 0 \;;\; 0 < \operatorname{cosec}^{-1} x \leqslant \pi/2,$$

i.e.,
$$-\pi/2 \leqslant y < 0 \quad ;\; 0 < y \leqslant \pi/2.$$

When x is positive so that it lies in the interval [1, ∞], then y lies between 0 and $\pi/2$ and so cot y is positive ;

when x is negative so that it lies in the interval [∞, -1], then y lies between $-\pi/2$ and 0 and so cot y is negative.

Thus the sign of the radical is positive or negative according as x is positive or negative. Hence we have

$$\frac{dy}{dx}=\frac{-1}{x\sqrt{(x^2-1)}} \text{ if } x>0 \text{ and } =\frac{-1}{-x\sqrt{x^2-1}} \text{ if } x<0,$$

so that we can write

$$\frac{dy}{dx}=-\frac{1}{|x|\sqrt{(x^2-1)}}, \text{ for every admissible value of } x.$$

Thus $$\frac{d(\operatorname{cosec}^{-1}x)}{dx}=-\frac{1}{|x|\sqrt{(x^2-1)}}.$$

Ex. 1. Find the derivatives of

(*i*) $\sin^{-1}\sqrt{x}$. (*ii*) $\sqrt{(\cot^{-1}\sqrt{x})}$.

(*iii*) $\tan^{-1}[(1+x)/(1-x)]$. (*iv*) $\tan^{-1}(\cos\sqrt{x})$.

(*v*) $\sec^{-1}x^2$. (*vi*) $\operatorname{cosec}^{-1}(x^{-\frac{1}{2}})$.

(*vii*) $\cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right)$. (*viii*) $\dfrac{x\sin^{-1}x}{\sqrt{(1-x^2)}}$.

(*ix*) $\tan^{-1}\dfrac{4\sqrt{x}}{1-4x}$. (*x*) $\cos^{-1}\left(\dfrac{a+b\cos x}{b+a\cos x}\right)$.

Ex. 2. Find dy/dx when

$$x=\sin^{-1}\sqrt{\left(\frac{t^2}{1+t^2}\right)},\quad y=\cos^{-1}\frac{1}{\sqrt{(1+t^2)}}.$$

4·61. Derivative of $\log_a x$. (*a, x are both positive*)

Let

$$y=\log_a x.$$

$\therefore$ $$\frac{\delta y}{\delta x}=\frac{\log_a(x+\delta x)-\log_a x}{\delta x}$$

$$=\frac{1}{\delta x}\log_a\left(\frac{x+\delta x}{x}\right)$$

$$=\frac{1}{x}\cdot\frac{x}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$=\frac{1}{x}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

Now $$\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}=e. \qquad (\S\ 3\cdot63,\ \text{cor. } 3)$$

Thus $$\frac{d(\log_a x)}{dx}=\frac{dy}{dx}=\frac{1}{x}\log_a e.$$

Cor. Let $a=e$ so that

$$y=\log_e x=\log x$$

$$\frac{dy}{dx}=\frac{1}{x}\cdot\log_e e=\frac{1}{x}.$$

Thus $$\frac{d(\log x)}{dx}=\frac{1}{x}.$$

4·62. Derivative of a^x.

Let $$y=a^x.$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{a^{x+\delta x}-a^x}{\delta x}=a^x.\frac{a^{\delta x}-1}{\delta x}.$$

$$\therefore \quad \frac{dy}{dx}=a^x.\lim_{\delta x\to 0}\frac{a^{\delta x}-1}{\delta x}$$

$$=a^x\log_e a. \qquad (\S 3{\cdot}64,\ p.\ 63)$$

Thus $$\frac{d(a^x)}{dx}=a^x\log_e a.$$

Cor. Let $a=e$ so that $y=e^x$.

$$\therefore \quad \frac{dy}{dx}=e^x\log_e e=e^x.$$

Thus $\dfrac{d(e^x)}{dx}=e^x.$

Ex. Find the derivatives of :—

(*i*) $\log\sin x$. (*ii*) $\cos(\log x)$. (*iii*) $e^{\sin x}$.

(*iv*) $\log[\sin(\log x)]$. (*v*) $\log\sqrt{(x^2+x+1)}$.

(*vi*) $\log\tan(\frac{1}{2}x+\frac{1}{4}\pi)$. (*vii*) $\log(\sec x+\tan x)$.

(*viii*) $\dfrac{e^{2x}}{\log x}$. (*ix*) $\sqrt{(a^{\sqrt{x}})}$.

(*x*) $\log_{10}(\sin^{-1}x^2)$. (*xi*) $\log[x+\sqrt{(x^2+a^2)}]$.

(*xii*) $\log(e^{mx}+e^{-mx})$. (*xiii*) $a^{2x}\sin^2 x$.

(*xiv*) $e^{\sqrt[3]{ax}}$. (*xv*) $\log\dfrac{a+b\tan x}{a-b\tan x}$.

Derivatives of hyperbolic functions.

4·71. Derivative of sinh x.

Let

$$y=\sinh x=\frac{e^x-e^{-x}}{2}.$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}[\tfrac{1}{2}(e^x-e^{-x})]=\tfrac{1}{2}(e^x+e^{-x})=\cosh x.$$

Thus $$\frac{\mathbf{d(\sinh x)}}{\mathbf{dx}}=\mathbf{\cosh x}.$$

4·72. Derivative of cosh x.

Let

$$y=\cosh x=\frac{e^x+e^{-x}}{2}.$$

$$\therefore \quad \frac{dy}{dx}=\frac{e^x-e^{-x}}{2}=\sinh x.$$

Thus $$\frac{\mathbf{d(\cosh x)}}{\mathbf{dx}}=\mathbf{\sinh x}.$$

4·73. Derivative of tanh x.

Let

$$y=\tanh x=\frac{\sinh x}{\cosh x}.$$

$$\therefore \quad \frac{dy}{dx}=\frac{\cosh x\,.\,\dfrac{d(\sinh x)}{dx}-\sinh x\,.\,\dfrac{d(\cosh x)}{dx}}{\cosh^2 x}$$

$$=\frac{\cosh x.\cosh x-\sinh x.\sinh x}{\cosh^2 x}$$

$$=\frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x}=\frac{1}{\cosh^2 x}=\text{sech}^2\, x.$$

Thus $$\frac{\mathbf{d(\tanh x)}}{\mathbf{dx}}=\mathbf{sech^2\, x}.$$

4·74. $$\frac{\mathbf{d(\coth x)}}{\mathbf{dx}}=-\mathbf{cosch^2\, x}.$$

Its proof is left to the reader.

4·75. Derivative of sech x.

Let

$$y=\text{sech}\, x=\frac{1}{\cosh x}.$$

$$\therefore \quad \frac{dy}{dx}=\frac{\cosh x.0-1.\sinh x}{\cosh^2 x}$$

$$=-\frac{\sinh x}{\cosh^2 x}=-\tanh x\ \text{sech}\ x.$$

Thus $$\frac{d(\text{sech } x)}{dx} = -\tanh x \text{ sech } x.$$

4·76. $$\frac{d(\text{cosech } x)}{dx} = -\coth x \text{ cosech } x.$$

Its proof is left to the reader.

Derivatives of inverse hyperbolic functions.

4·81. Derivative of $\sinh^{-1} x$.

Let

$$y = \sinh^{-1} x \text{ so that } x = \sinh y.$$

$$\therefore \quad \frac{dx}{dy} = \cosh y,$$

or $$\frac{dy}{dx} = \frac{1}{\cosh y} = \pm \frac{1}{\sqrt{(1+\sinh^2 y)}} = \pm \frac{1}{\sqrt{(1+x^2)}},$$

where the sign of the radical is the same as that of $\cosh y$ which we know, is always positive, (§ 3·72, page 68).

Hence $$\frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{(1+x^2)}}.$$

4·82. Derivative of $\cosh^{-1} x$.

Let

$$y = \cosh^{-1} x \text{ so that } x = \cosh y.$$

$$\therefore \quad \frac{dx}{dy} = \sinh y,$$

or $$\frac{dy}{dx} = \frac{1}{\sinh y} = \pm \frac{1}{\sqrt{(\cosh^2 y - 1)}} = \pm \frac{1}{\sqrt{(x^2-1)}},$$

where the sign of the radical is the same as that of $\sinh y$.

Now, $\cosh^{-1} x$ *i.e.*, y is always positive so that $\sinh y$ is positive (§ 3·71, page 68).

Hence $$\frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{(x^2-1)}}.$$

4·83. Derivative of $\tanh^{-1} x$. $[\,|x| < 1]$.

Let

$$y = \tanh^{-1} x \text{ so that } x = \tanh y.$$

$$\therefore \quad \frac{dx}{dy} = \text{sech}^2 y,$$

or $$\frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}.$$

Thus $$\frac{d(\tanh^{-1} x)}{dx} = \frac{1}{1-x^2}.$$

4·84. $$\frac{d(\coth^{-1} x)}{dx} = -\frac{1}{x^2-1}.\ [\,|x| > 1].$$

Its proof is left to the reader.

4·85. Derivative of $\text{sech}^{-1} x$.

Let

$$y = \text{sech}^{-1} x \text{ so that } x = \text{sech } y.$$

$$\therefore \qquad \frac{dx}{dy} = -\text{sech } y \,.\, \tanh y.$$

or
$$\frac{dy}{dx} = -\frac{1}{\text{sech } y \tanh y}$$

$$= \pm \frac{-1}{\text{sech } y \,.\, \sqrt{(1-\text{sech}^2 y)}} = \pm \frac{-1}{x\sqrt{(1-x^2)}},$$

where the sign of the radical is the same as that of $\tanh y$.

But we know that $\text{sech}^{-1} x$, *i.e.*, y is always positive, so that $\tanh y$ is always positive.

Hence
$$\frac{d(\text{sech}^{-1} x)}{dx} = -\frac{1}{x\sqrt{(1-x^2)}}.$$

4·86. Derivative of $\text{cosech}^{-1} x$.

Let

$$y = \text{cosech}^{-1} x \text{ so that } x = \text{cosech } y.$$

$$\therefore \qquad \frac{dx}{dy} = -\text{cosech } y \,.\, \coth y.$$

or
$$\frac{dy}{dx} = -\frac{1}{\text{cosech } y \,.\, \coth y}$$

$$= \pm \frac{-1}{\text{cosech } y \,.\, \sqrt{(\text{cosech}^2 y + 1)}} = \pm \frac{-1}{x\sqrt{(x^2+1)}},$$

when the sign of the radical is the same as that of $\coth y$.

Now, y, and therefore $\coth y$ is positive or negative according as x is positive or negative.

$$\therefore \qquad \frac{dy}{dx} = \frac{-1}{x\sqrt{(x^2+1)}} \text{ if } x > 0 \text{ and } = \frac{-1}{-x\sqrt{(x^2+1)}} \text{ if } x < 0.$$

Thus
$$\frac{d(\text{cosech}^{-1} x)}{dx} = \frac{-1}{|x| \sqrt{(x^2+1)}},$$

for all values of x.

Ex. Find the derivatives of

(*i*) $\log (\cosh x)$, (*ii*) $e^{\sinh^2 x}$, (*iii*) $\tan x \,.\, \tanh x$.

4·91. Logarithmic differentiation. In order to differentiate a function of the form u^v, where u, v are both variables, it is *necessary* to take its logarithm and then differentiate. This process which is known as *logarithmic differentiation* is also useful when the function to be differentiated is the product of a number of factors. The following examples illustrate this process.

Ex. 1. *Differentiate* $x^{\sin x}$.

Let $$y=x^{\sin x}$$

$$\therefore \qquad \log y=\log (x^{\sin x})=\sin x \,.\, \log x.$$

Differentiating, we get

$$\frac{1}{y} \cdot \frac{dy}{dx}=\cos x \,.\, \log x+\sin x \,.\, \frac{1}{x}.$$

Hence $$\frac{dy}{dx}=x^{\sin x}\left(\cos x \,.\, \log x+\frac{\sin x}{x}\right).$$

Ex. 2. *Differentiate* $[x^{\tan x}+(\sin x)^{\cos x}]$.

We write $$y=x^{\tan x}+(\sin x)^{\cos x}.$$

Let $$u=x^{\tan x}, \qquad \text{..(1)}$$

and $$v=(\sin x)^{\cos x}, \qquad \text{...(2)}$$

so that $$y=u+v$$

$$\therefore \qquad \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}.$$

From (1), we obtain, taking logarithm,

$$\log u=\tan x \,.\, \log x.$$

$$\therefore \qquad \frac{1}{u} \cdot \frac{du}{dx}=\sec^2 x \,.\, \log x+\tan x \,.\, \frac{1}{x},$$

i.e., $$\frac{du}{dx}=x^{\tan x}\left(\sec^2 x \,.\, \log x+\frac{\tan x}{x}\right). \qquad \text{...(3)}$$

From (2), we obtain, taking logarithm

$$\log v=\cos x \,.\, \log \sin x.$$

$$\therefore \qquad \frac{1}{v} \cdot \frac{dv}{dx}=-\sin x \,.\, \log \sin x+\cos x \,.\, \frac{1}{\sin x}\cos x$$

i.e., $$\frac{dv}{dx}=(\sin x)^{\cos x}\left(-\sin x \,.\, \log \sin x+\frac{\cos^2 x}{\sin x}\right). \qquad \text{...(4)}$$

Adding (3) and (4), we obtain dy/dx.

Ex. 3. *Differentiate*

$$\frac{x^{\frac{1}{2}}(1-2x)^{\frac{2}{3}}}{(2-3x)^{\frac{3}{4}}(3-4x)^{\frac{4}{5}}}.$$

Putting it equal to y, and taking logarithms, we obtain

$$\log y=\tfrac{1}{2}\log x+\tfrac{2}{3}\log(1-2x)-\tfrac{3}{4}\log(2-3x)-\tfrac{4}{5}\log(3-4x).$$

Differentiating, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{x}+\frac{2}{3}\cdot\frac{-2}{1-2x}-\frac{3}{4}\cdot\frac{-3}{2-3x}-\frac{4}{5}\cdot\frac{-4}{3-4x}$$

$$=\frac{1}{2x}-\frac{4}{3(1-2x)}+\frac{9}{4(2-3x)}+\frac{16}{5(3-4x)}$$

$$\therefore\quad \frac{dy}{dx}=y\left[\frac{1}{2x}-\frac{4}{3(1-2x)}+\frac{9}{4(2-3x)}+\frac{16}{5(3-4x)}\right].$$

Ex. 4. Find the differential co-efficients of :

(*i*) $(\cos x)^{\log x}$. (*ii*) $(1+x^{-1})^{x}$. (*iii*) e^{x^x}.

(*iv*) $(\tan x)^{\cot x}+(\cot x)^{\tan x}$, (*v*) $(\log x)^{x}+(\sin^{-1}x)^{\sin x}$.

(*vi*) $\dfrac{(1-x)^{1/2}(2-x^2)^{2/3}}{(3-x^3)^{3/4}(4-x^4)^{4/5}}$. (*vii*) $\dfrac{x\sqrt[3]{(x^2+4)}}{\sqrt{(x^2+3)}}$.

(*viii*) $\sin x\,.\,e^{x}\,.\log x\,.\,x^{x}$.

4·92. Preliminary transformation. In some cases, a preliminary transformation of the function to be differentiated facilitates the process of differentiation a good deal, as is illustrated by the following examples.

Ex. 1. *Differentiate*

$$\sin^{-1}\frac{2x}{1+x^2}.$$

Putting $x=\tan\theta$, we have

$$y=\sin^{-1}\frac{2x}{1+x^2}$$

$$=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}x.$$

$$\therefore\quad \frac{dy}{dx}=\frac{2}{1+x^2}.$$

Ex. 2. *Differentiate*

$$\tan^{-1}\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{\sqrt{(1+x)}+\sqrt{(1-x)}}.$$

Putting $x=\cos\theta$, we have

$$\sqrt{(1+x)}=\sqrt{(1+\cos\theta)}=\sqrt{\left(2\cos^2\frac{\theta}{2}\right)}=\sqrt{2}\,\cos\frac{\theta}{2}.$$

$$\sqrt{(1-x)}=\sqrt{(1-\cos\theta)}=\sqrt{\left(2\sin^2\frac{\theta}{2}\right)}=\sqrt{2}\ \sin\ \frac{\theta}{2}.$$

$$\therefore \qquad y=\tan^{-1}\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}$$

$$=\tan^{-1}\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

$$=\frac{\pi}{4}-\frac{\theta}{2}=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x.$$

$$\therefore \qquad \frac{dy}{dx}=\frac{1}{2\sqrt{(1-x^2)}}.$$

Ex. 3. *Find the differential co-efficient of*

$$\tan^{-1}\frac{2x}{1-x^2} \text{ with respect to } \sin^{-1}\frac{2x}{1+x^2}. \quad (P.U.\ 1954,\ 1956)$$

Let $\qquad y=\tan^{-1}\dfrac{2x}{1-x^2},\ z=\sin^{-1}\dfrac{2x}{1+x^2}.$

Putting $\quad x=\tan\theta$, we see that

$$y=\tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}=\tan^{-1}(\tan 2\theta)=2\theta=2\tan^{-1}x.$$

$$z=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}x.$$

$$\therefore \qquad \frac{dy}{dx}=\frac{2}{1+x^2},\ \frac{dz}{dx}=\frac{2}{1+x^2}.$$

$$\therefore \qquad \frac{dy}{dz}=\frac{dy}{dx}\cdot\frac{dx}{dz}=1.$$

Also otherwise, we have

$$y=z,$$

so that $\qquad \dfrac{dy}{dz}=1.$

Ex. 4. *If* $\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=a(x-y)$, *prove that*

$$\frac{dy}{dx}=\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}. \qquad (D.U.\ Hons.\ 1949\ ;\ P.U.\ 1952)$$

Putting $x=\sin\theta$ and $y=\sin\phi$, we have

$$\sqrt{(1-\sin^2\theta)}+\sqrt{(1-\sin^2\phi)}=a(\sin\theta-\sin\phi)$$

or $\qquad \cos\theta+\cos\phi=a(\sin\theta-\sin\phi)$

$$a=\frac{\cos\theta+\cos\phi}{\sin\theta-\sin\phi}=\frac{2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)}{2\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}$$

$\therefore \qquad \cot\frac{1}{2}(\theta-\phi)=a$

or $\qquad \frac{1}{2}(\theta-\phi)=\cot^{-1}a$

$$\theta-\phi=2\cot^{-1}a$$

$\therefore \qquad \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a.$

Differentiating, we get

$$\frac{1}{\sqrt{(1-x^2)}}-\frac{1}{\sqrt{(1-y^2)}}\frac{dy}{dx}=0,$$

or $$\frac{dy}{dx}=\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}.$$

Ex. 5. *Find the differential co-efficients of*

(*i*) $\tan^{-1}\dfrac{2x}{1-x^2}$. (*D.U. 1952*) $\qquad$ (*ii*) $\sin^{-1}(3x-4x^3)$.

(*iii*) $\tan^{-1}\dfrac{\sqrt{x}-x}{1+x^{3/2}}$. $\qquad$ (*iv*) $\tan^{-1}\left(\dfrac{1-\cos x}{1+\cos x}\right)^{1/2}$.

(*v*) $\cos^{-1}\dfrac{1-x^2}{1+x^2}$. $\qquad$ (*vi*) $\tan^{-1}\dfrac{\sqrt{x}+\sqrt{a}}{1-\sqrt{(ax)}}$.

(*vii*) $\sin^{-1}[x\sqrt{(1-x)}-\sqrt{\{x(1-x^2)\}}]$ (*D.U. Hons. 1954*)

[Show that this is equal to $\sin^{-1}x-\sin^{-1}\sqrt{x}$].

Ex. 6. Express in their simplest forms the differential co-efficients with respect to x of

(*i*) $\tan^{-1}\left(\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$. $\qquad$ (*ii*) $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$.

(*P.U. Supp. 1938*)

4·93. Differentiation "ab initio". To differentiate "ab initio" or, from first principles, means that the process of differentiation is to be performed without making any use of the theorems on the differentiation of sums, products, functions of functions etc., nor is any use to be made of the differential co-efficients of standard forms. We have already had numerous illustrations of it.

Ex. 1. *Differentiate $\sin^{-1}x$ 'ab initio'.* (*D.U. 1955*)

Let $\qquad y=\sin^{-1}x.$

$\therefore \qquad y+\delta y=\sin^{-1}(x+\delta x).$

We have $\qquad x=\sin y$

and $\qquad x+\delta x=\sin(y+\delta y)$

so that $\qquad \delta x=\sin(y+\delta y)-\sin y.$

Thus

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\sin (y+\delta y)-\sin y}=\frac{\delta y}{2\cos (y+\frac{1}{2}\delta y) \sin\frac{1}{2}\delta y}$$

$$=\frac{1}{\cos (y+\frac{1}{2}\delta y)}\cdot\left(\frac{\frac{1}{2}\delta y}{\sin \frac{1}{2} \delta y}\right)$$

Let $\delta x \to 0$ so that δy also $\to 0$.

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\cos y}\cdot 1=\frac{1}{\cos y}=\frac{1}{\sqrt{(1-x^2)}}.$$

Ex. 2. *Find, from first principles, the differential co-efficient of* $\sqrt{\sin x}$.

Let

$$y=\sqrt{\sin x}.$$

$$\therefore \quad y+\delta y=\sqrt{\sin (x+\delta x)},$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{\sqrt{\sin (x+\delta x)}-\sqrt{\sin x}}{\delta x}$$

$$=\frac{\sqrt{\sin (x+\delta x)}-\sqrt{\sin x}}{\delta x}\times\frac{\sqrt{\sin (x+\delta x)}+\sqrt{\sin x}}{\sqrt{\sin (x+\delta x)}+\sqrt{\sin x}}$$

$$=\frac{\sin (x+\delta x)-\sin x}{\delta x}\times\frac{1}{\sqrt{\sin (x+\delta x)}+\sqrt{\sin x}}$$

$$=\cos (x+\tfrac{1}{2}\delta x)\,\frac{\sin \frac{1}{2} \delta x}{\frac{1}{2}\delta x}\cdot\frac{1}{\sqrt{\sin (x+\delta x)}+\sqrt{\sin x}}$$

Let $\delta x \to 0$,

$$\therefore \quad \frac{dy}{dx}=\cos x \cdot 1\cdot\frac{1}{2\sqrt{\sin x}}$$

$$=\frac{1}{2}\,\frac{\cos x}{\sqrt{\sin x}}.$$

Ex. 3. Find, from first principles, the differential coefficients of :—

(*i*) $\sin x^2$. (*ii*) $\sin^2 x$. (*iii*) $\sqrt{x}$. (*D.U. 1953*)

(*iv*) $e^{\sin x}$. (*v*) $e^{\sqrt{x}}$. (*vi*) $\tan^{-1}x$.

Exercises

Find, from first principles, the differential coefficients of :—

1. $e^{(x^2)}$.
2. $\sqrt{(\tan x)}$.
3. $x \sin x$.
4. $\tan x^2$.
5. $\sec^{-1}x$.

Find the differential co-efficients of :—

6. $\dfrac{x\cos^{-1}x}{\sqrt{(1-x^2)}}$.

7. $\dfrac{(3x^2-1)\sqrt{(1+x^2)}}{x^3}$.

8. $\dfrac{x\sqrt{(x^2-4a^2)}}{\sqrt{(x^2-a^2)}}$.

9. $b\tan^{-1}\left(\dfrac{x}{a}\tan^{-1}\dfrac{x}{a}\right)$.

10. $e^{(x)^x}$.

11. $\tan^{-1}\dfrac{x^{\frac{1}{3}}+a^{\frac{1}{3}}}{1-a^{\frac{1}{3}}x^{\frac{1}{3}}}$.

12. $\tan^{-1}\dfrac{\cos x}{1+\sin x}$.

13. $\tan^{-1}\left(\dfrac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}$.

14. $\dfrac{(1-2x)^{\frac{2}{3}}(1+3x)^{-\frac{3}{4}}(1-4x)^{\frac{4}{5}}}{(1-6x)^{\frac{5}{6}}(1+7x)^{-\frac{6}{7}}(1-8x)^{\frac{7}{8}}}$.

15. $(x^x)^x$.

16. $x^{(x^x)}$.

17. $[1-x^2]^{\frac{3}{2}}\cdot\sin^{-1}x$.

18. $\log[\tanh(\frac{1}{2}x)]$.

19. $\tan^{-1}\dfrac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}$.

20. $\sqrt{\left(\dfrac{1+\tan x}{1-\tan x}\right)}$.

21. $\sin^{-1}[2ax\sqrt{(1-a^2x^2)}]$.

22. $10^{\log\sin x}$.

23. $x\log x\cdot\log(\log x)$.

24. $\tan^{-1}\dfrac{a+b\cos x}{b+a\cos x}$.

25. $(\sin x)^{\cos^{-1}x}$.

26. $e^{-ax^2}\cdot\sin(x\log x)$.

27. $\sin^{-1}\left(\dfrac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}\right)$.

28. $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x}$.

29. $\sin^{-1}\dfrac{2}{\sqrt{(x^4+a^4)}}$.

30. $\log\left[e^x\left(\dfrac{x-2}{x+2}\right)^{\frac{3}{4}}\right]$.

31. $\log\left(\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right)$.

32. $\tan^{-1}\left(\dfrac{x\sin a}{1-x\cos a}\right)$.

33. $9x^4\sin(3x-7)\log(1-5x)$.

34. $e^{ax}\cos(b\tan^{-1}x)$.

35. $\log[1-\sqrt{(1-e^{-n/\sin x})}]$.

36. $\cot x\coth x$.

37. $xa^x\sinh x$.

38. $\cos^{-1}\sqrt{\left(\dfrac{\cos 3x}{\cos^3 x}\right)}$.

39. $\left(1+\dfrac{1}{x}\right)^{x^2}$.

40. $\cos^{-1}\sqrt{\left[\dfrac{C(ax^2+c)}{c(Ax^2+C)}\right]}$.

41. Find the differential coefficient of :—

$$\tfrac{1}{3}\log[x+\sqrt[3]{(1-x^3)}]-\frac{1}{\sqrt{3}}\tan^{-1}\frac{3\sqrt[3]{(1-x^3)}-x}{\sqrt{(1-x)}}.$$

42. Differentiate

(i) $\dfrac{1}{4\sqrt{2}}\log\dfrac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{x\sqrt{2}}{1-x^2}$.

(ii) $\dfrac{1}{3}\log\dfrac{x-1}{\sqrt{(x^2+x+1)}}+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$.

43. Find dy/dx when

(i) $x=\dfrac{\sin^3 t}{\sqrt{(\cos 2t)}}$, $y=\dfrac{\cos^3 t}{\sqrt{(\cos 2t)}}$, at $t=\pi/6$.

(ii) $x=\sin t\sqrt{(\cos 2t)}$, $y=\cos t\sqrt{(\cos 2t)}$.

(iii) $x=a(\cos t+\log\tan\frac{1}{2}t)$, $y=a\sin t$.

44. If $x^y=e^{x-y}$, prove that $dy/dx=\log x/(1+\log x)^2$. (*P.U. 1955, 56*)

(Differentiate logarithmically)

45. Differentiate $\sin^2 x$ with respect to $(\log x)^2$. (*D.U. 1950*)

46. Differentiate $x^{\sin x}$ with respect to $(\sin x)^x$. (*P.U. 1959*)

47. Differentiate $\tan^{-1}[\{\sqrt{(1+x^2)}-1)\}/x]$ with respect to $\tan^{-1} x$. (*P.U. 1956*)

48. Find $\dfrac{dy}{dx}$ when $x=e^{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)}$.

49. Differentiate

$$\tan^{-1}\frac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}$$

with respect to $\cos^{-1} x^2$. (*D.U. Hons. 1949*)

50. Differentiate $(\log x)^{\tan x}$ with regard to $\sin(m\cos^{-1}x.)$ (*P. U. 1955, 56*)

51. *Differentiate the determinant.*

$$F(x)=\begin{vmatrix} f_1(x) & \varphi_1(x) & \psi_1(x) \\ f_2(x) & \varphi_2(x) & \psi_2(x) \\ f_3(x) & \varphi_3(x) & \psi_3(x) \end{vmatrix}$$

From first principles

we have

$$F(x+h)-F(x)=\begin{vmatrix} f_1(x+h) & \varphi_1(x+h) & \psi_1(x+h) \\ f_2(x+h) & \varphi_2(x+h) & \psi_2(x+h) \\ f_3(x+h) & \varphi_3(x+h) & \psi_3(x+h) \end{vmatrix}-\begin{vmatrix} f_1(x) & \varphi_1(x) & \psi_1(x) \\ f_2(x) & \varphi_2(x) & \psi_2(x) \\ f_3(x) & \varphi_3(x) & \psi_3(x) \end{vmatrix}$$

$$=\begin{vmatrix} f_1(x+h)-f_1(x) & \varphi_1(x+h)-\varphi_1(x) & \psi_1(x+h)-\psi_1(x) \\ f_2(x+h) & \varphi_2(x+h) & \psi_2(x+h) \\ f_1(x+h) & \varphi_3(x+h) & \psi_3(x+h) \end{vmatrix}$$

$$+\begin{vmatrix} f_1(x) & \varphi_1(x) & \psi_1(x) \\ f_2(x+h)-f_2(x) & \varphi_2(x+h)-\varphi_2(x) & \psi_2(x+h)-\psi_2(x) \\ f_3(x+h) & \varphi_3(x+h) & \psi_3(x+h) \end{vmatrix}$$

$$+\begin{vmatrix} f_1(x) & \varphi_1(x) & \psi_1(x) \\ f_2(x) & \varphi_2(x) & \psi_2(x) \\ f_3(x+h)-f_3(x) & \varphi_3(x+h)-\varphi_3(x) & \psi_3(x+h)-\psi_3(x) \end{vmatrix}$$

Dividing by h and making h tend to zero, we get

$$F'(x)=\begin{vmatrix} f_1' & \varphi_1' & \psi_1' \\ f_2 & \varphi_2 & \psi_2 \\ f_3 & \varphi_3 & \psi_3 \end{vmatrix}+\begin{vmatrix} f_1 & \varphi_1 & \psi_1 \\ f_2' & \varphi_2' & \psi_2' \\ f_3 & \varphi_3 & \psi_3 \end{vmatrix}+\begin{vmatrix} f_1 & \varphi_1 & \psi_1 \\ f_2 & \varphi_2 & \psi_2 \\ f_3' & \varphi_3' & \psi_3' \end{vmatrix}$$

The rule can be easily extended to the case of determinants of any order.

APPENDIX

EXAMPLES

1. *Show that*

$$f(x)=x^2 \sin (1/x) \text{ when } x\neq 0$$
$$f(0)=0$$

is derivable for every value of x but the derivative is not continuous for $x=0$. *(D.U. Hons. 1954)*

For $x\neq 0$, $f'(x)=2x \sin \dfrac{1}{x}+x^2 \cos\left(\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right)$

$$=2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

For $x=0$, we have

$$\frac{f(x)-f(0)}{x-0}=\frac{x^2 \sin \dfrac{1}{x}}{x}$$

$$=x \sin \frac{1}{x} \to 0 \text{ as } x \to 0.$$

$\therefore \qquad f'(0)=0.$

Thus the function possesses a derivative $f'(x)$ for every value of x and is given by

$$f'(x)=2x \sin \frac{1}{x} - \cos \frac{1}{x} \qquad \text{when } x\neq 0$$
$$f'(0)=0.$$

We have to show that $f'(x)$ is not continuous for $x=0$. We write

$$\cos \frac{1}{x}=2x \sin \frac{1}{x}-\left(2x \sin \frac{1}{x}-\cos \frac{1}{x}\right). \qquad ...(1)$$

Here

$$\lim_{x\to 0}\left(2x\sin\frac{1}{x}\right)=0.$$

In case

$$\lim_{x\to 0} f'(x),$$

i.e.,
$$\lim_{x\to 0}\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)$$

had existed, it would follow from (1) that $\lim_{x\to 0}(\cos 1/x)$ would also exist. But this is not the case.

Hence $\lim_{x\to 0} f'(x)$ does not exist. Thus $f'(x)$ is not continuous for $x=0$.

2. *Examine the continuity and derivability in the interval* $(-\infty, \infty)$ *for the following function*

$$f(x)=1 \text{ in } -\infty<x<0,$$
$$f(x)=1+\sin x \text{ in } 0\leqslant x<\tfrac{1}{2}\pi,$$
$$f(x)=2+(x-\tfrac{1}{2}\pi)^2 \text{ in } \tfrac{1}{2}\pi\leqslant x<\infty. \qquad (\textit{Mysore})$$

The function $f(x)$ is derivable for every value of x except perhaps for $x=0$ and $x=\pi/2$. Thus we shall now consider $x=0$ and $x=\pi/2$.

Firstly we consider $x=0$.

Now $$f(0)=1+\sin 0=1.$$

$$\lim_{x\to(0-0)} f(x)=1 \text{ and } \lim_{x\to(0+0)} f(x)=\lim_{x\to(0+0)}(1+\sin x)=1$$

$$\therefore \qquad \lim_{x\to 0} f(x)=1=f(0),$$

Hence $f(x)$ is continuous for $x=0$.

Again, for $x<0$,

$$\frac{f(x)-f(0)}{x-0}=\frac{1-1}{x}=0,$$

so that

$$\lim_{x\to(0-0)}\frac{f(x)-f(0)}{x-0}=0.$$

Also for $x>0$

$$\frac{f(x)-f(0)}{x-0}=\frac{1+\sin x-1}{x-0}=\frac{\sin x}{x}$$

so that

$$\lim_{x \to (0+0)} \frac{f(x)-f(0)}{x-0} = \lim_{x \to (0+0)} \frac{\sin x}{x} = 1.$$

Thus

$$\lim_{x \to (0+0)} \frac{f(x)-f(0)}{x-0} \neq \lim_{x \to (0-0)} \frac{f(x)-f(0)}{x-0}$$

Hence the function is not derivable for $x=0$.

Now we consider $x=\pi/2$.

We have

$$f\left(\frac{\pi}{2}\right) = 2+(\tfrac{1}{2}\pi-\tfrac{1}{2}\pi)^2 = 2,$$

$$\lim_{x \to (\frac{1}{2}\pi-0)} f(x) = \lim_{x \to (\frac{1}{2}\pi-0)} (1+\sin x) = 1+1 = 2,$$

$$\lim_{x \to (\frac{1}{2}\pi+0)} f(x) = \lim_{x \to (\frac{1}{2}\pi+0)} [2+(x-\tfrac{1}{2}\pi)^2] = 2.$$

$$\therefore \quad \lim_{x \to \pi/2} f(x) = 2 = f(\pi/2).$$

Hence $f(x)$ is continuous for $x=\pi/2$.

Again, for $0 \leqslant x < \frac{1}{2}\pi$,

$$\frac{f(x)-f(\frac{1}{2}\pi)}{x-\frac{1}{2}\pi} = \frac{(1+\sin x)-2}{x-\frac{1}{2}\pi} = \frac{1-\sin x}{\frac{1}{2}\pi-x},$$

Putting $\frac{1}{2}\pi - x = t$, we see that

$$\frac{1-\sin x}{\frac{1}{2}\pi-x} = \frac{1-\sin(\frac{1}{2}\pi-t)}{t}$$

$$= \frac{1-\cos t}{t} = \frac{2\sin^2\frac{1}{2}t}{t} = \sin\tfrac{1}{2}t\,\frac{\sin\frac{1}{2}t}{\frac{1}{2}t},$$

so that

$$\lim_{x \to (\frac{1}{2}\pi-0)} \frac{1-\sin x}{\frac{1}{2}\pi-x} = 0.$$

For $x > \frac{1}{2}\pi$,

$$\frac{f(x)-f(\frac{1}{2}\pi)}{x-\frac{1}{2}\pi} = \frac{2+(x-\frac{1}{2}\pi)^2-2}{x-\frac{1}{2}\pi}$$

$$= x-\tfrac{1}{2}\pi.$$

$$\therefore \quad \lim_{x \to (\frac{1}{2}\pi+0)} \frac{f(x)-f(\frac{1}{2}\pi)}{x-\frac{1}{2}\pi} = 0.$$

$\therefore$ $f'(\frac{1}{2}\pi)$ exists and is equal to 0.

Exercises

1. Discuss the existence of $f'(x)$ and $f''(x)$ at the origin for the function

$$f(x)=x^2 \sin \frac{1}{x} \text{ when } x\neq 0,$$
$$f(0)=0$$

(*B.U. 1953 ; D.U. Hons. 1952*)

2. Examine the differentiability of the function

$$f(x)= x^m \sin \frac{1}{x} \text{ when } x\neq 0, m>0$$
$$f(x)=0 \qquad \text{when } x=0$$

at the point $x=0$. Determine m when $f'(x)$ is continuous at the origin.
(*D.U. Hons. 1952*)

3. Determine whether $f(x)$ is continuous and has a derivative at the origin where

$$f(x)=\begin{cases} 2+x \text{ if } x\geqslant 0 \\ 2-x \text{ if } x<0. \end{cases}$$ (*B.U.*)

4. Show that

$$f(x)= |x| + |x-1|$$

is continuous but not derivable for $x=0$ and $x=1$.

5. Examine the function

$$f(x) =x\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, \quad x\neq 0$$
$$f(0)=0$$

as regards its continuity and the existence of its derivative at the origin.
(*D.U. Hons. 1951*)

6. Discuss the continuity and the differentiability of the function $f(x)$ where

$$f(x)=\begin{cases} 0, \text{ when } x \text{ is irrational or zero} \\ \dfrac{1}{q^3} \text{ when } x=\dfrac{p}{q}, \text{ a fraction in its lowest terms.} \end{cases}$$

(*D.U. 1953*)

7. Find from first principles the derivative of $f(x) =\dfrac{\sin (x^2)}{x}$ when $x\neq 0$ and $f(0)=0$ at a point $x=0$ and show that the derivative is continuous at $x=0$.
(*B.U. 1953*)

8. Show that the function

$$f(x)=x\{1+\tfrac{1}{3} \sin (\log x^2)\} \text{ ; for } x\neq 0, f(0)=0$$

is everywhere continuous but has no differential co-efficient at $x=0$.
(*B.U. 1952*)

9. If $f(x)=x \tan^{-1} \dfrac{1}{x}$ when $x\neq 0$ and $f(0)=0$, show that $f(x)$ is continuous but not derivable for $x=0$.

10. Is the function

$$f(x)=(x-a)\sin\frac{1}{x-a} \text{ for } x\neq 0$$

$$f(a)=0$$

continuous and differentiable at $x=a$? Give your answer with reasons.

(*P. U.*)

11. Discuss the continuity of $f(x)$ in the neighbourhood of the origin when $f(x)$ is defined as follows :

(*i*) $f(x)=x \log \sin x$ for $x\neq 0$, and $f(0)=0$.

(*ii*) $f(x)=e^{1/x}$ when $x\neq 0$ and $f(0)=0$. (*D.U. 1955*)

12. A function $f(x)$ is defined as being equal to $-x^2$, when $\leqslant 0$, to $5x-4$ when $0<x\leqslant 1$, to $4x^2-3x$ when $1<x<2$ and to $3x+4$ when $x=2$; discuss the continuity of $f(x)$ and the existence of $f'(x)$ for $x=0$, 1 and 2.

(*D.U. Hons. 1957*)

CHAPTER V

SUCCESSIVE DIFFERENTIATION

5·1. Notation. The derivative $f'(x)$ of a derivable function $f(x)$ is itself a function of x. We suppose that it also possesses a derivative, which we denote by $f''(x)$ and call the *second derivative* of $f(x)$. The *third derivative* of $f(x)$ which is the derivative of $f''(x)$ is denoted by $f'''(x)$ and so on.

Thus, the successive derivatives of $f(x)$ are represented by the symbols,

$$f'(x), f''(x), \ldots\ldots\ldots, f^n(x), \ldots\ldots$$

where each term is the derivative of the preceding one.

Alternatively, if $y=f(x)$, then d^ny/dx^n also denotes the nth derivative of y. Sometimes

$$y_1, y_2, y_3, \ldots\ldots\ldots, y_n \ldots\ldots$$

are used to denote the successive derivatives of y.

The symbols

$$f^n(a), \left[\frac{d^ny}{dx^n}\right]_{x=a}, \text{ or } y_n(a)$$

denote the value of the nth derivative of $y=f(x)$ for $x=a$.

Examples

1. *If* $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$, *find* d^2y/dx^2.

We have

$$\frac{dx}{d\theta}=a(-\sin\theta+\sin\theta+\theta\cos\theta)=a\,\theta\cos\theta,$$

$$\frac{dy}{d\theta}=a(\cos\theta-\cos\theta+\theta\sin\theta)=a\,\theta\sin\theta.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{dy}{d\theta}\bigg/\frac{dx}{d\theta}=\tan\theta.$$

$$\therefore \quad \frac{d^2y}{dx^2}=\sec^2\theta\,\frac{d\theta}{dx} \qquad \text{[Note this step]}$$

$$=\sec^2\theta.\frac{1}{a\,\theta\cos\theta}=\frac{\sec^3\theta}{a\,\theta}.$$

2. *If $y=\sin(\sin x)$, prove that*

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0. \qquad \textit{(D. U. 1953)}$$

We have

$$\frac{dy}{dx}=\cos(\sin x)\cos x,$$

$$\frac{d^2y}{dx^2}=-\sin(\sin x)\cos x\cos x-\cos(\sin x)\sin x$$

$$=-\sin(\sin x)\cos^2 x-\cos(\sin x)\sin x.$$

Making substitution, we see that

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0.$$

3. *Change the independent variable to θ in the equation*

$$\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{y}{(1+x^2)^2}=0,$$

by means of the transformation

$$x=\tan\theta. \qquad \textit{(P. U. 1932)}$$

We have

$$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=\frac{dy}{d\theta}\Big/\sec^2\theta=\cos^2\theta.\frac{dy}{d\theta},$$

$$\frac{d^2y}{dx^2}=-2\cos\theta\sin\theta\cdot\frac{d\theta}{dx}\cdot\frac{dy}{d\theta}+\cos^2\theta\cdot\frac{d^2y}{d\theta^2}\cdot\frac{d\theta}{dx}$$

$$=-2\cos\theta\sin\theta\cos^2\theta\cdot\frac{dy}{d\theta}+\cos^2\theta\cdot\frac{d^2y}{d\theta^2}\cdot\cos^2\theta$$

$$=-2\sin\theta\cos^3\theta.\frac{dy}{d\theta}+\cos^4\theta\frac{d^2y}{d\theta^2}.$$

Substituting the values of x, dy/dx and d^2y/dx^2 in the given differential equation, we see that it becomes

$$-2\sin\theta\cos^3\theta\frac{dy}{d\theta}+\cos^4\theta\frac{d^2y}{d\theta^2}+\frac{2\tan\theta}{1+\tan^2\theta}.\cos^2\theta\frac{dy}{d\theta}$$
$$+\frac{y}{(1+\tan^2\theta)^2}=0,$$

or $$-2\sin\theta\cos^3\theta\frac{dy}{d\theta}+\cos^4\theta.\frac{d^2y}{d\theta^2}+2\sin\theta.\cos^3\theta\frac{dy}{d\theta}+\cos^4\theta\cdot y=0$$

or $$\frac{d^2y}{d\theta^2}+y=0.$$

4. *If* $ax^2+2hxy+by^2+2gx+2fy+c=0$, *show that*

$$\frac{d^2y}{dx^2}=\frac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^3}. \qquad (P.U)$$

Solving the given equation as a quadratic in y, we get

$$by+hx+f=\pm[(h^2-ab)x^2+2(hf-bg)x+(f^2-bc)]^{\frac{1}{2}}$$

$$\therefore \quad b\frac{dy}{dx}+h=\pm\tfrac{1}{2}[(h^2-ab)x^2+2(hf-bg)x+(f^2-bc)]^{-\frac{1}{2}} \times [2(h^2-ab)x+2(hf-bg)]$$

$$=\frac{(h^2-ab)x+(hf-bg)}{hx+by+f}. \qquad ...(1)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2}=\frac{(h^2-ab)(hx+by+f)-\left(h+b\dfrac{dy}{dx}\right)[(h^2-ab)x+(hf-bg)]}{(hx+by+f)^2}.$$

Substituting for $b\dfrac{dy}{dx}+h$ from (1), we get the required result.

Exercises

1. $y=\dfrac{\log x}{x}$, show that $\dfrac{d^2y}{dx^2}=\dfrac{2\log x-3}{x^3}$.

2. $y=\log(\sin x)$, show that $y_3=\dfrac{2\cos x}{\sin^3 x}$.

3. Show that $y=x+\tan x$ satisfies the differential equation

$$\cos^2 x\frac{d^2y}{dx^2}-2y+2x=0.$$

4. If $y=[(a+bx)/(c+dx)]$, then $2y_1y_3=3y_2^2$.

5. If $x=2\cos t-\cos 2t$ and $y=2\sin t-\sin 2t$, find the value of d^2y/dx^2 when $t=\frac{1}{2}\pi$. (P.U.)

6. If $x=a\sin 2\theta\,(1+\cos 2\theta)$, $y=a\cos 2\theta\,(1-\cos 2\theta)$, prove that

$$\frac{[1+(dy/dx)^2]^{\frac{3}{2}}}{d^2y/dx^2}=4a\cos 3\theta.$$

7. If $y=(1/x)^x$, show that $y_2(1)=0$.

8. If $p^2=a^2\cos^2\theta+b^2\sin^2\theta$, prove that

$$p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}. \qquad (Rajputana)$$

9. If $y=x \log [(ax)^{-1}+a^{-1}]$, prove that

$$x(x+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=y-1.$$

10. If $x=\sin t$, $y=\sin pt$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0. \qquad (P.U.)$$

11. Change the independent variable to z in the equation

$$\frac{d^2y}{dx^2}+\cot x\frac{dy}{dx}+4y \operatorname{cosec}^2 x=0$$

by means of the transformation

$$z=\log \tan \tfrac{1}{2}x.$$

12. If y is a function of x and $x=1/z$, show that

$$\frac{dy}{dx}=-z^2\frac{dy}{dz}\;;\;\frac{d^2y}{dx^2}=2z^3\frac{dy}{dz}+z^4\frac{d^2y}{dz^2}.$$

13. Show that

$$\frac{dx}{dy}=\frac{1}{dy/dx}\;;\;\frac{d^2x}{dy^2}=-\frac{d^2y/dx^2}{(dy/dx)^3},$$

and find the value of d^3x/dy^3 in terms of dy/dx, d^2y/dx^2, d^3y/dx^3.

Also show that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=-\frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}.$$

5·2. Calculation of the nth derivative. Some standard results.

5·21. Let $\quad y=(ax+b)^m.$

$$\therefore \quad y_1=ma(ax+b)^{m-1},$$

$$y_2=m(m-1)a^2(ax+b)^{m-2},$$

$$y_3=m(m-1)(m-2)a^3(ax+b)^{m-3},$$

so that in general

$$y_n=m(m-1)(m-2)\ldots\ldots(m-n+1)a^n(ax+b)^{m-n}.$$

In case, m is a positive integer, y_n can be written as

$$\frac{m!}{(m-n)!}a^n(ax+b)^{m-n},$$

so that, the mth derivative of $(ax+b)^m$ is a constant *viz.*, $m!\,a^m$ and the $(m+1)$th derivative along with the other higher successive derivatives are all zero.

Cor. 1. Putting, $m=-1$, we get

$$y_n=(-1)(-2)\ldots\ldots(-n)a^n(ax+b)^{-1-n}.$$

$$\therefore \qquad \frac{d^n\left(\dfrac{1}{ax+b}\right)}{dx^n}=\frac{(-1)^n(n\,!)a^n}{(ax+b)^{n+1}}.$$

Cor. 2. Let $y=\log(ax+b)$.

$$y_1=\frac{a}{ax+b}.$$

Hence
$$y_n=\frac{(-1)^{n-1}(n-1)\,!}{(ax+b)^n}a^n.$$

$$\therefore \qquad \frac{d^n\{\log(ax+b)\}}{dx^n}=\frac{(-1)^{n-1}(n-1)!\ a^n}{(ax+b)^n}.$$

5·22. Let

$$y=a^{mx}.$$

$$\therefore \qquad y_1=ma^{mx}\log\ a,$$

$$y_2=m^2a^{mx}\ (\log\ a)^2,$$

so that, in general

$$y_n=m^na^{mx}(\log\ a)^n.$$

Cor. Putting e for a, we get

$$\frac{d^n(e^{mx})}{dx^n}=m^ne^{mx}.$$

5·23. Let

$$y=\sin\ (ax+b).$$

$$\therefore \qquad y_1=a\ \cos\ (ax+b)=a\ \sin\ (ax+b+\tfrac{1}{2}\pi),$$

$$y_2=a^2\cos\ (ax+b+\tfrac{1}{2}\pi)=a^2\ \sin\ (ax+b+\tfrac{2}{2}\pi),$$

$$y_3=a^3\cos\ (ax+b+\tfrac{2}{2}\pi)=a^3\ \sin\ (ax+b+\tfrac{3}{2}\pi),$$

so that, in general

$$\frac{d^n\sin\ (ax+b)}{dx^n}=a^n\sin\left[\ ax+b+\frac{n\pi}{2}\right].$$

5·24. Similarly

$$\frac{d^n\cos\ (ax+b)}{dx^n}=a^n\cos\left[\ ax+b+\frac{n\pi}{2}\right].$$

5·25. Let

$$y=e^{ax}\sin\ (bx+c).$$

$$\therefore \qquad y_1=ae^{ax}\sin\ (bx+c)+e^{ax}b\cos\ (bx+c)$$

$$=e^{ax}[a\sin\ (bx+c)+b\cos\ (bx+c)].$$

In order to put y in a form which will enable us to make the required generalisation, we determine two constant numbers r and φ such that

$$a=r\cos\varphi,\ b=r\sin\varphi$$

$$\therefore \qquad r=\sqrt{(a^2+b^2)},\ \varphi=\tan^{-1}(b/a).$$

Hence, we have

$$y_1=re^{ax}\sin(bx+c+\varphi).$$

Thus y_1 arises from y on multiplying by the constant r and increasing the angle by the constant ϕ.

Thus similarly

$$y_2=r^2e^{ax}\sin(bx+c+2\varphi).$$

Hence, in general

$$\frac{d^n[e^{ax}\sin(bx+c)]}{dx^n}=r^n e^{ax}\sin(bx+c+n\phi)$$

where

$$r=\sqrt{(a^2+b^2)},\ \varphi=\tan^{-1}(b/a).$$

5·26. Similarly

$$\frac{d^n[e^{ax}\cos(bx+c)]}{dx^n}=(a^2+b^2)^{\frac{1}{2}n}e^{ax}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right).$$

5·3. Determination of nth derivative of Algebraic rational function. Partial Fractions. In order to determine the nth derivative of any algebraic rational function, we have to decompose it into partial fractions.

Sometimes it also becomes necessary to apply Demoivre's theorem which states that

$$(\cos\theta\pm i\sin\theta)^n=\cos n\theta\pm i\sin n\theta,$$

where n is any integer, positive or negative, and $i=\sqrt{(-1)}$.

Examples

1. *Find the nth derivative of*

$$\frac{x^2}{(x+2)(2x+3)}$$

Throwing it into partial fractions, we obtain

$$\frac{x^2}{(x+2)(2x+3)}=\frac{1}{2}\left[1-\frac{8}{x+2}+\frac{9}{2x+3}\right].$$

$$\therefore \frac{d^n}{dx^n}\left[\frac{x^2}{(x+2)(2x+3)}\right]=-\frac{(-1)^n n!\,.\,8}{2(x+2)^{n+1}}+\frac{(-1)^n n!\,.\,2^n\,.\,9}{2(2x+3)^{n+1}}$$

$$=\frac{(-1)^n n!}{2}\left[\frac{9.2^n}{(2x+3)^{n+1}}-\frac{8}{(x+2)^{n+1}}\right].$$

2. *Find the differential co-efficient of*

$$\frac{x}{x^2+a^2}.$$

We have

$$\frac{x}{x^2+a^2}=\frac{x}{(x+a\,i)(x-a\,i)}$$

$$=\frac{1}{2}\left[\frac{1}{x-a\,i}+\frac{1}{x+a\,i}\right]$$

$$\therefore \qquad \frac{d^n}{dx_n}\left(\frac{x}{x^2+a^2}\right)=\frac{(-1)^n n\,!}{2}\left[\frac{1}{(x-a\,i)^{n+1}}+\frac{1}{(x+a\,i)^{n+1}}\right]$$

To render the result free from 'i' and express the same in real form, we determine two numbers r and θ such that

$$x=r\cos\theta,\ a=r\sin\theta.$$

$$\therefore \qquad r=\sqrt{(x^2+a^2)},\ \theta=\tan^{-1}(a/x).$$

$$\therefore \qquad \frac{1}{(x-a\,i)^{n+1}}=\frac{1}{r^{n+1}}\cdot(\cos\theta-i\sin\theta)^{-(n+1)}$$

$$=\frac{1}{r^{n+1}}[\cos(n+1)\theta+i\sin(n+1)\theta],$$

and

$$\frac{1}{(x+a\,i)^{n+1}}=\frac{1}{r^{n+1}}(\cos\theta+i\sin\theta)^{-(n+1)}$$

$$=\frac{1}{r^{n+1}}[\cos(n+1)\theta-i\sin(n+1)\theta].$$

Hence

$$\frac{d^n\left(\dfrac{x}{x^2+a^2}\right)}{dx^n}=\frac{(-1)^n n\,!\,.\cos(n+1)\theta}{r^{n+1}}$$

where

$$r=\sqrt{(x^2+a^2)},\ \theta=\tan^{-1}(a/x).$$

Exercises

1. Find the nth differential co-efficients of

(*i*) $\dfrac{x^4}{(x-1)(x-2)}$. (*D.U. Hons., 1954*) (*ii*) $\dfrac{x+1}{x^2-4}$.

(*iii*) $\dfrac{4x}{(x-1)^2(x+1)}$.

2. Find the tenth and the nth differential co-efficients of

$$\frac{x^2+4x+1}{x^3+2x^2-x-2}.$$ (*P.U.*)

3. Find the nth differential co-efficients of

(i) $\dfrac{x}{1+3x+2x^2}$. (ii) $\dfrac{1}{x^4-a^4}$.

(iii) $\dfrac{1}{x^2+x+1}$. (iv) $\dfrac{x}{x^2+x+1}$.

4. Show that the nth differential co-efficient of

$$\frac{1}{1+x+x^2+x^3}$$

is

$$\tfrac{1}{2}(-1)^n(n\,!)\sin^{n+1}\theta[\sin(n+1)\theta-\cos(n+1)\theta$$
$$+(\sin\theta+\cos\theta)^{-n-1}], \text{ where } \theta=\cot^{-1}x.$$

5. Prove that the value of the nth differential co-efficient of $x^3/(x^2-1)$ for $x=0$ is zero, if n is even, and $-(n\,!)$ if n is odd and greater that 1. (*P.U.*)

6. Show that the nth derivative of $y=\tan^{-1}x$ is

$$(-1)^{n-1}(n-1)\,!\sin n(\tfrac{1}{2}\pi-y)\sin^n(\tfrac{1}{2}\pi+y). \qquad (P.U.\ 1935)$$

7. Find the nth differential co-efficients of

(i) $\tan^{-1}\dfrac{1+x}{1-x}$. (ii) $\tan^{-1}\dfrac{x\sin\alpha}{1-x\cos\alpha}$.

8. If $y=\tan^{-1}x$, show that

$$\frac{d^ny}{dx^n}=(n-1)\,!\cos\,[ny+(n-1)\frac{\pi}{y}]\cos^n y. \qquad (P.U.\ 1953)$$

9. If $y=x(x+1)\log(x+1)^3$, prove that

$$\frac{d^ny}{dx^n}=\frac{3(-1)^{n-1}(n-3)\,!\,(2x+n)}{(x+1)^{n-1}}$$

provided that $n\not<3$.

10. If $y=x\log\dfrac{x-1}{x+1}$, prove that

$$\frac{d^ny}{dx^n}=(-1)^n(n-2)\,!\,.\left[\frac{x-n}{(x-1)^n}-\frac{x+n}{(x+1)^n}\right].$$

5·4. The nth derivative of the product of the powers of sines and cosines. In order to find out the nth derivative of such a product, we have to express it as the sum of the sines and cosines of multiples of the independent variable. Example 1 below, which has been solved, will illustrate the process.

Ex. . *Find the nth differential co-efficients of*

(i) $\cos^4 x$. (ii) $e^{ax}\cos^2 x\sin x$. (*D.U. 1951*)

i) We know that

$$\cos^2 x=\frac{1+\cos 2x}{2}$$

$$\therefore \quad \cos^4 x=\left(\frac{1+\cos 2x}{2}\right)^2$$

$$=\frac{1}{4}+\frac{\cos 2x}{2}+\frac{\cos^2 2x}{4}$$

$$=\tfrac{1}{4}+\tfrac{1}{2}\cos 2x+\tfrac{1}{8}(1+\cos 4x)$$

$$=\tfrac{3}{8}+\tfrac{1}{2}\cos 2x+\tfrac{1}{8}\cos 4x.$$

Hence $\dfrac{d^n(\cos^4 x)}{dx^n}=\tfrac{1}{2}.2^n\cos\left(2x+\dfrac{n\pi}{2}\right)+\tfrac{1}{8}.4^n\cos\left(4x+\dfrac{n\pi}{2}\right).$

(*ii*) $\cos^2 x\sin x=\tfrac{1}{2}(1+\cos 2x)\sin x$

$$=\tfrac{1}{2}\sin x+\tfrac{1}{4}.2\sin x\cos 2x$$

$$=\tfrac{1}{2}\sin x+\tfrac{1}{4}(\sin 3x-\sin x)$$

$$=\tfrac{1}{4}\sin x+\tfrac{1}{4}\sin 3x.$$

Hence

$$\frac{d^n}{dx^n}(e^{ax}\cos^2 x\sin x)=\tfrac{1}{4}\frac{d^n}{dx^n}(e^{ax}\sin x)+\tfrac{1}{4}\frac{d^n}{dx^n}(e^{ax}\sin 3x$$

$$=\tfrac{1}{4}(a^2+1)^{\frac{1}{2}n}e^{ax}\sin\left(x+n\tan^{-1}\frac{1}{a}\right)$$

$$+\tfrac{1}{4}(a^2+9)^{\frac{1}{2}n}e^{ax}\sin\left(3x+n\tan^{-1}\frac{3}{a}\right).$$

2. Find the nth differential co-efficients of

(*i*) $\sin^3 x$.

(*ii*) $\cos x\cos 2x\cos 3x$.

(*ii*) $\sin^2 x\cos^3 x$.

(*iv*) $e^x\sin^4 x$.

(*v*) $e^{2x}\cos x\sin^2 2x$.

5·5. Leibnitz's Theorem. The nth derivative of the product of two functions. If *u, v be two functions possessing derivatives of the nth order, then*

$$(uv)_n=u_n v+{}^nC_1 u_{n-1}v_1+{}^nC_2 u_{n-2}v_2+\ldots\ldots$$
$$+\ldots\ldots+{}^nC_r u_{n-r}v_r+\ldots\ldots+{}^nC_n uv_n.$$

This theorem will be proved by *Mathematical induction.*

Step I. By direct differentiation, we have

$$(uv)_1=u_1 v+uv_1,$$

and

$$(uv)_2=u_2 v+u_1 v_1+u_1 v_1+uv_2$$
$$=u_2 v+{}^2C_1 u_1 v_1+{}^2C_2 uv_2.$$

Thus the theorem is true for $n=1, 2,$

Step II. We *assume* that the theorem is true for a particular value of n, say m, so that we have

$$(uv)_m=u_m v+{}^mC_1 u_{m-1}v_1+{}^mC_2 u_{m-2}v_2+\ldots\ldots$$
$$+{}^mC_{r-1}u_{m-r+1}v_{r-1}+{}^mC_r u_{m-r}v_r+\ldots\ldots+{}^mC_m uv_m.$$

Differentiating both sides, we get

$$(uv)_{m+1}=u_{m+1}\,v+u_m\,v_1+{}^mC_1\,u_m\,v_1+{}^mC_1\,u_{m-1}\,v_2$$
$$+{}^mC_2\,u_{m-1}\,v_2+{}^mC_2\,u_{m-2}v_3+\ldots+{}^mC_{r-1}\,u_{m-r+2}v_{r-1}$$
$$+{}^mC_{r-1}\,u_{m-r+1}\,v_r+{}^mC_r\,u_{m-r+1}\,v_r+{}^mC_r\,u_{m-r}\,v_{r+1}+\ldots\ldots+$$
$$\ldots\ldots{}^mC_m\,uv_{m+1}$$
$$=u_{m+1}\,v+(1+{}^mC_1)u_m\,v_1+({}^mC_1+{}^mC_2)\,u_{m-1}\,v_2+\ldots\ldots+$$
$$({}^mC_{r-1}+{}^mC_r)u_{m-r+1}\,v_r+\ldots+{}^mC_m\,uv_{m+1}.$$

But we know that

$${}^mC_{r-1}+{}^mC_r={}^{m+1}C_r,$$
$$1+{}^mC_1=1+m={}^{m+1}C_1,$$
$${}^mC_m=1={}^{m+1}C_{m+1}.$$

$$\therefore\quad (uv)_{m+1}=u_{m+1}v+{}^{m+1}C_1\,u_mv_1+{}^{m+1}C_2u_{m-1}\,v_2+\ldots\ldots$$
$$+{}^{m+1}C_ru_{m-r+1}\,v_r+\ldots+{}^{m+1}C_{m+1}\,uv_{m+1},$$

from which we see that if the theorem is true for any value m of n, then it is also true for the next higher value $m+1$ of n.

Conclusion. In step I, we have seen that the theorem is true for $n=2$. Therefore it must be true for $n=2+1$, *i.e.*, 3 and so for $n=3+1$, *i.e.*, 4, and so for every value of n.

Examples

1. *Find the nth derivative of* $x^2\,e^x\cos x$. *(P.U.)*

To find the nth derivative of $x^2\,e^x\cos x$, we look upon $e^x\cos x$ as the first factor and x^2 as the second.

$$\therefore\quad (x^2\,e^x\cos x)_n=(e^x\cos x)_nx^2+{}^nC_1(e^x\cos x)_{n-1}.2x$$
$$+{}^nC_2(e^x\cos x)_{n-2}.2$$
$$=2^{\frac{1}{2}n}.e^x\cos(x+n\tan^{-1}1).x^2$$
$$+n2^{\frac{1}{2}(n-1)}\,e^x\cos(x+n-1).\tan^{-1}1)2x$$
$$+\frac{n(n-1)}{}\,.2^{\frac{1}{2}\,n-2}\quad e^x\cos(x+\overline{n-2}\tan^{-1}1).2$$
$$=2^{\frac{1}{2}(n-2)}\,e^x\Big[\,2x^2\cos\left(x+n\,\frac{\pi}{4}\right)$$
$$+2^{\frac{3}{2}}.nx\,.\cos\left(\,x+\overline{n-1}\,\frac{\pi}{4}\right)+n(n-1)\cos\left(\,x+\overline{n-2}\,\frac{\pi}{4}\right)\Big].$$

2. *If*

$$y=a\cos(\log x)+b\sin(\log x),$$

show that

$$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0.$$

(D.U. 1952)

Differentiating, we get

$$y_1=\frac{-a\sin(\log x)}{x}+\frac{b\cos(\log x)}{x}$$

or $$xy_1=-a\sin(\log x)+b\cos(\log x).$$

Differentiating again, we get

$$xy_2+y_1=-\frac{a\cos(\log x)}{x}-\frac{b\sin(\log x)}{x}$$

or $$x(xy_2+y_1)=-[a\cos(\log x)+b\sin(\log x)]=-y$$

or $$x^2y_2+xy_1+y=0.$$

Differentiating n times by Leibnitz's theorem, we obtain

$$x^2y_{n+2}+{}^nC_1.2x.y_{n+1}+{}^nC_2.2.y_n+xy_{n+1}+{}^nC_1.1.y_n+y_n=0$$

or $$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$$

Exercises

1. Find the nth derivative of

(*i*) x^ne^x. (*ii*) $x^3\cos x$.

(*iii*) $e^{ax}[a^2x^2-2nax+n(n+1)]$. (*iv*) $e^x\log x$.

(P.U. 1954, 56)

2. If $y=x^2\sin x$, prove that

$$\frac{d^ny}{dx^n}=(x^2-n^2+n)\sin\left(x+\frac{n\pi}{2}\right)-2nx\cos\left(x+\frac{n\pi}{2}\right).$$

3. If $f(x)=\tan x$, prove that

$$f^n(0)-{}^nC_2f^{n-2}(0)+{}^nC_4f^{n-4}(0)\ldots=\sin\frac{n\pi}{2}.$$

[P.U. Supp. 1936)

[Write $f(x)\cos x=\sin x$ and apply Leibnitz theorem.]

4. Differentiate the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+a^2y=0,$$

n times with respect to x. *(D.U. 1950)*

5. Find the nth derivative of the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(a^2-m^2)y=0.$$

5·6. Determination of the value of the nth derivative of a function for x = 0. Sometimes it is possible to obtain the value of the nth derivative of a function for $x=0$ directly without finding the general expression for the nth derivative which cannot, in general, be obtained in a convenient form. Examples 1 and 2 below which have been solved will make the procedure clear. As will be seen in Ch. IX, the values of the derivatives for $x=0$ are required, to expand a function by Maclaurin's theorem.

Examples

1. *Find the value of the nth derivative of* $e^{m \sin^{-1} x}$ *for* $x=0$,

Let $$y=e^{m \sin^{-1} x} \qquad \dots (1)$$

$$y_1=e^{m \sin^{-1} x} \cdot \frac{m}{\sqrt{(1-x^2)}} \qquad \dots (2)$$

or $$(1-x^2)y_1^2=m^2y^2.$$

Differentiating, we get
$$(1-x^2)\, 2y_1y_2-2xy_1^2=2m^2yy_1.$$

Dividing by $2y_1$, we obtaain
$$(1-x^2)y_2-xy_1=m^2y. \qquad \dots (3)$$

Differentiating n times by Leibnitz's theorem, we get
$$(1-x^2)y_{n+2}-2nxy_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n=m^2y_n$$
$$(1-x^2)y_{n+2}-(2n+1)\, xy_{n+1}-(n^2+m^2)y_n=0.$$

Putting $x=0$, we get
$$y_{n+2}(0)=(n^2+m^2)y_n(0). \qquad \dots (4)$$

From (1), (2) and (3), we obtain
$$y(0)=1,\ y_1(0)=m,\ y_2(0)=m^2.$$

Putting $n=1, 2, 3, 4$, etc. in (4), we get
$$y_3(0)=(1^2+m^2)\, y_1(0)=m(1^2+m^2)\ ;$$
$$y_4(0)=(2^2+m^2)\, y_2(0)=m^2(2^2+m^2)\ ;$$
$$y_5(0)=(3^2+m^2)\, y_3(0)=m(1^2+m^2)(3^2+m^2)\ ;$$
$$y_6(0)=(4^2+m^2)\, y_4(0)=m^2(2^2+m^2)(4^2+m^2).$$

In general
$$y_n(0)=\begin{cases} m^2(2^2+m^2)(4^2+m^2)\dots[(n-2)^2+m^2], & \text{when } n \text{ is even,} \\ m(1^2+m^2)\ (3^2+m^3)\dots[(n-2)^2+m^2], & \text{when } n \text{ is odd.} \end{cases}$$

(P.U. 1955)

2. If $y=(\sin^{-1} x)^2$, *prove that*

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0 \qquad \dots (1)$$

Differentiate the above equation n times with respect to x. Also find the value of all the derivatives of y for $x=0$.

(P.U. 1955)

Differentiating $y=(\sin^{-1} x)^2$, we get

$$y_1=\frac{2 \sin^{-1} x}{\sqrt{(1-x^2)}}. \qquad \dots (2)$$

$\therefore \quad (1-x^2)\, y_1^2=(2 \sin^{-1} x)^2=4y.$

Differentiating again, we get

$$2(1-x^2)\, y_1 y_2-2xy_1^2=4y_1. \qquad \ldots(3)$$

Dividing by $2y_1$, we get

$$(1-x^2)\, y_2-xy_1-2=0 \qquad \ldots(4)$$

which is (1).

Differentiating this n times by Leibnitz's theorem, we obtain

$$(1-x^2)y_{n+2}+ny_{n+1}\,(-2x)+\frac{n(n-1)}{2}\,y^n(-2)-xy_{n+1}-ny_n\,.\,1=0.$$

or
$$(1-x^2)\, y_{n+2}-(2n+1)\, xy_{n+1}-n^2y_n=0. \qquad \ldots(5)$$

Putting $x=0$, we obtain

$$y_{n+2}(0)=n^2y_n(0). \qquad \ldots(6)$$

From (2), $\quad y_1(0)=0.$

From (4), $\quad y_2(0)=2. \qquad (7)$

Putting $n=1, 3, 5, 7$ successively in (6), we see that

$$0=y_1=y_3=y_5=y_7=\ldots\ldots$$

Again, putting $n=2, 4, 6\ldots\ldots$in (6), we see that

$$y_4(0)=2^2y_2(0)=2.2^2\ ;$$
$$y_6(0)=4^2y_4(0)=2.2^2.4^2\ ;$$
$$y_8(0)=6^2y_6(0)=2\cdot2^2.4^2.6^2,$$

In general, if n is even, we obtain

$$y_n(0)=2.2^2\cdot4^2.6^2\ldots(n-2)^2, \text{ when } n\neq2.$$

Note. The result (5), obtained on dividing (4) by $2y_1$, is not a legitimate conclusion when $y_1=0$, which is the case when $x=0$. Thus, it is not valid to derive any conclusion from (5) and (6) for $x=0$.

But these results may be *obtained by proceeding to the* limit as $x\to0$ instead of putting $x=0$. This may be shown as follows :

We can easily convince ourselves that the derivative of every order of y as calculated from (2) will contain some power of $(1-x^2)$ in its denominator and will therefore be always continuous except for $x=\pm1$, so that lim $y_2=y_2(0)$ and lim $y_n=y_n(0)$, as $x\to0$.

Exercises

1. If $u=\tan^{-1} x$, prove that

$$(1+x^2)\frac{d^2u}{dx^2}+2x\,\frac{du}{dx}=0$$

and hence determine the values of the derivatives of u when $x=0$ *(M.T.)*

2. If

$$y=\sin\,(m \sin^{-1} x), \text{ show that}$$
$$(1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-m^2)y_n$$

and find y_n (0). *(P.U. 1958)*

3. Find y_n (0) when $y=\log\,[x+\surd(1+x^2)]$.

4. If $y=[x+\sqrt{(1+x^2)}]^m$, find $y_n(0)$.

5. If

$$y=e^{m\cos^{-1}x}$$

show that

$$(1-x^2)y_{n+2}-(2n+1)x\,y_{n+1}-(n^2+m^2)y_n=0$$

and find $y_n(0)$. (*D.U. Hons. 1953*)

6. If $y=(\sinh^{-1}x)^2$, prove that

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}+(2n+1)\,x\frac{d^{n+1}y}{dx^{n+1}}+n^2\frac{d^ny}{dx^n}=0.$$

Hence find at $x=0$ the value of d^ny/dx^n. (*B.U.1952*)

Exercises

1. Show that if

$$x(1-x)y_2-(4-12x)y_1-36y=0$$

then

$$x(1-x)y_{n+2}-[4-n-(12-2n)x]y_{n+1}-(4-n)(9-n)y_n=0.$$

(*B.U.*)

2. If y_n denotes the nth differential co-efficient of $e^{ax}\sin bx$ and $\theta=\tan^{-1}(b/a)$, prove that

$$y_n=(a\sec\theta)^n e^{ax}\sin(bx+n\theta).$$

Also show that

$$y_{n+1}-2ay_n+(a^2+b^2)y_{n-1}=0,$$

3. If $y=(A+Bx)\cos Kx+(C+Dx)\sin Kx$, prove that

$$\frac{d^4y}{dx^4}+2K^2\frac{d^2y}{dx^2}+K^4y=0.$$ (*M.T.*)

4. Find the third differential co-efficient of

$$\tan^{-1}\frac{x}{\sqrt{(1-x^2)}}.$$ (*P.U.*)

5. Prove that

$$\frac{d^4}{dx^4}\sqrt{(1+x^2)}=-\frac{12x^2-3}{\sqrt{(1+x^2)^7}}.$$ (*B.U.*)

6. Show that if $u=\sin nx+\cos nx$, then

$$u_r=n^r[1+(-1)^r\sin 2nx]^{\frac{1}{2}}$$

when u_r denotes the rth differential co-efficient of u with respect to x.

(*Lucknow*)

7. Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)=(-1)^n\frac{n\,!}{x^{n+1}}\left(\log x-1-\frac{1}{2}-\frac{1}{3}\ldots\ldots-\frac{1}{n}\right).$$

(*P.U.*)

8. Prove that

$$\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)=\left[\mathrm{P}\sin\left(x+\frac{n\pi}{2}\right)+\mathrm{Q}\left(\cos x+\frac{\pi}{2}\right)\right]\div x^{n+1}$$

where P and Q stand for

$$x^n-n(n-1)x^{n-2}+n(n-1)(n-2)(n-3)x^{n-4}+\ldots\ldots$$

and

$$nx^{n-1}-n(n-1)(n-2)x^{n-3}+\ldots\ldots$$

respectively.

9. Find the value of nth derivative of

$$\frac{x^3-x}{(x^2-4)^2}$$

for $x=0$. (*Trinity College*

10. Prove that if $x=\cot\theta$, $(0<\theta<\pi)$, then

$$\frac{d^n\theta}{dx^n}=(-1)^{n-1}(n-1)!\sin n\theta\sin^n\theta\ ;$$

where n is any positive integer.

11. If

$$I_n=\frac{d^n}{dx^n}(x^n\log x).$$

Prove that

$$I_n=nI_{n-1}+(n-1)!$$

and hence show that

$$_n=n!\left(\log x+1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right).$$ (*D.U. Hons. 1949*)

$$\left[\ I_n=\frac{d^{n-1}}{dx^{n-1}}\ \frac{d}{dx}(x^n\ \text{og}\ x)=\frac{d^{n-1}}{dx^{n-1}}(nx^{n-1}\log x+x^{n-1})\right.$$

$$=n\ \frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x)+\frac{d^{n-1}}{dx^{n-1}}(x^{n-1})=nI_{n-1}+(n-1)!\Big]$$

Rewriting this relation as

$$\frac{I_n}{n!}=\frac{I_{n-1}}{(n-1)!}+\frac{1}{n}$$

and replacing n by $n, n-1, \ldots.., 3, 2$, we get the required result.]

12. If $y=\dfrac{d^n}{dx^n}(x^2-1)^n$, show that

$$(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0.$$ (*P.U. 1957*)

Hence show that y satisfies the *Legendre's* equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0.$$

13. If U_n denotes the nth derivative of $(Lx+M)/(x^2-2Bx+C)$., prove that

$$\frac{x^2-2Bx+C}{(n+1)(n+2)}U_{n+2}+\frac{2(x-B)}{n+1}U_{n+1}+U_n=0.$$ (*M.U.*)

14. If $y=x^2e^x$, then

$$\frac{d^ny}{dx^n}=\frac{1}{2}n(n-1)\frac{d^2y}{dx^2}-n(n-2)\ \frac{dy}{dx}+\frac{1}{2}(n-1)(n-2)y.$$

(*Agra, P.U., D.U., 1955*)

15. If

$$\cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^n$$

prove that

$$x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0.$$

(*D.U. Hons., and Pass, 1949 ;*
P.U. 1958 Sept.)

16. If

$$y=(\tan^{-1} x)^2,$$

then

$$(x^2+1)^2\frac{d^2y}{dx^2}+2x(x^2+1)\frac{dy}{dx}=2.$$

Deduce that

$$(x^2+1)^2\frac{d^{n+2}y}{dx^{n+2}}+(4n+2)x(x^2+1)\frac{d^{n+1}y}{dx^{n+1}}+2n^2(3x^2+1)\frac{d^ny}{dx^n}$$

$$+2n(n-1)(2n-1)x\frac{d^{n-1}y}{dx^{n-1}}+n(n-1)^2(n-2)\frac{d^{n-2}y}{dx^{n-2}}=0.$$

(Birmingham)

17. If

$$x=\tan(\log y),$$

prove that

$$(1+x^2)\frac{d^{n+1}y}{dx^{n+1}}+(2nx-1)\frac{d^ny}{dx^n}+n(n-1)\frac{d^{n-1}y}{dx^{n-1}}=0. \qquad \textit{(B.U.)}$$

18. I $y^{\frac{1}{m}}+y^{-\frac{1}{m}}=2x$, prove that

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0,$$

where y_n denotes the nth derivative of y. *(D.U. Hons., 1947 : P.U. 1959)*

19. If $y=e^{\frac{1}{2}x^2}\cos x$, show that

$$y_{2n+2}(0)-4ny_{2n}(0)+(2n-1)2n\,y_{2n-2}(0)=0.$$

20. If $y=(1+x^2)^{\frac{1}{2}m}\sin(m\tan^{-1}x)$, show that

$$y_{2n}(0)=0 \text{ and } y_{2n+1}(0)=(-1)^n\,m(m-1)(m-2)\ldots.(m-2n).$$

21. If $y=\sin(m\cos^{-1}\sqrt{x})$ then

$$\lim_{x\to 0}\frac{y_{n+1}}{y_n}=\frac{4n^2-m^2}{4n+2}. \qquad \textit{(B.U., D.U., 1955)}$$

22. If $x+y=1$, prove that

$$\frac{d^n}{dx^n}(x^ny^n)=n!\,[y^n-({}^nC_1)^2y^{n-1}x+({}^nC_2)^2y^{n-2}x^2\ldots+(-1)^nx^n]$$

(D.U. Hons. 1950 ; M.U.)

23. By forming in two different ways the nth derivative of x^{2n}, prove that

$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2\,.\,2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2\,.\,2^2\,.\,3^2}+\ldots\ldots=\frac{2n!}{(n!)^2}$$

[Equate the nth derivative of x^{2n} with the nth derivative of the product of x^n and x^n and put $x=1$]. *(Lucknow)*

24. Prove that

$$\left(\sin^2\theta\frac{d}{d\theta}\right)^n\sin^2\theta=n!\,.\sin(n+1)\theta\,.\sin^{n+1}\theta.$$

25. If $y=\dfrac{1}{x^3-a^3}$, prove that

$$y_n=\frac{(-1)^n n!}{3a^2}\cdot\frac{1}{(x-a)^{n+1}}+\frac{(-1)^n n!}{3a^2}\cdot\frac{2\cos[(n+1)\varphi+\frac{2}{3}\pi]}{(x^2+ax+a^2)^{\frac{1}{2}(n+1)}}$$

where
$$\varphi=\tan^{-1}\frac{a\sqrt{3}}{2x+a}.$$

26. If $u=\dfrac{x^2-x+1}{x^2+x+1}$ and $\tan\theta=\dfrac{\sqrt{3}}{2x+1}$

show that

$$\frac{d^n u}{dx^n}=\frac{4}{\sqrt{3}}(-1)^n n!\,\frac{\sin[(n+1)\theta-\frac{1}{3}\pi]}{(x^2+x+1)^{\frac{1}{2}(n+1)}}.$$

(D. U. Hons. 1946)

CHAPTER VI

GENERAL THEOREMS

MEAN VALUE THEOREMS

Introduction. By now, the student must have learnt to distinguish between theorems applicable to a class of functions and those concerning some particular functions like $\sin x$, $\log x$ etc. The theorems applicable to a class of functions are known as *general theorems*.

Some general theorems which will play a very important part in the following chapters will be obtained in this chapter. Of these *Rolle's theorem* is the most fundamental.

6·1. Rolle's Theorem. *If a function $f(x)$ is derivable in an interval $[a, b]$, and also $f(a)=f(b)$, then there exists at least one value 'c' of x lying within $[a, b]$ such that $f'(c)=0$.*

The function $f(x)$ being derivable in the interval $[a, b]$ is continuous, (§ 4·14, p. 76). By virtue of continuity, it has a greatest value M and a least value m in the interval, (§3·53, p. 55) so that there are two numbers c and d such that

$$f(c)=M, f(d)=m.$$

Now either $$M=m, \qquad \ldots (i)$$

or $$M \neq m. \qquad \ldots (ii)$$

When the greatest value coincides with the least value as in case (i), the function reduces to a constant so that the derivative $f'(x)$ is equal to 0 for every value of x and therefore the theorem is true in this case.

When M and m are unequal, as in case (ii), at least one of them must be different from the equal values $f(a)$, $f(b)$. Let $M=f(c)$ be different from them. The number 'c' being different from a and b, lies *within* the interval $[a, b]$.

The function $f(x)$ which is derivable in the interval $[a, b]$ is, in particular, derivable for $x=c$, so that

$$\lim \frac{f(c+h)-f(c)}{h} \text{ when } h \to 0$$

exists and is the same when $h \to 0$ through positive or negative values.

As $f(c)$ is the greatest value of the function, we have

$$f(c+h) \leqslant f(c)$$

whatever positive or negative value h has.

Thus

$$\frac{f(c+h)-f(c)}{h} \leqslant 0 \text{ for } h>0, \qquad \ldots(1)$$

and

$$\frac{f(c+h)-f(c)}{h} \geqslant 0 \text{ for } h<0. \qquad ...(2)$$

Let $h \to 0$ through positive values. From (1), we get

$$f'(c) \leqslant 0. \qquad ...(3)$$

Let $h \to 0$ through negative values. From (2), we get

$$f'(c) \geqslant 0. \qquad ...(4)$$

The relations (3) and (4) will both be true if, and only if

$$f'(c)=0.$$

The same conclusion would be similarly reached if it is the least value m which differs from $f(a)$ and $f(b)$.

Hence the theorem is proved.

Geometrical Statement of the theorem.

If a curve has a tangent at every point thereof, and the ordinates of its extremities A, B are equal, then there exists at least one point P of the curve other than A, B, the tangent at which is parallel to x-axis.

In this geometrical form the theorem is quite self-evident, as the student may himself realise by drawing some curves satisfying the conditions of the statement.

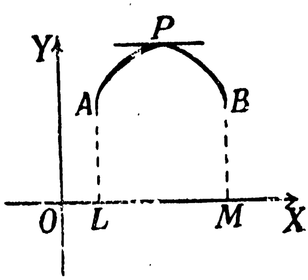


Fig. 43.

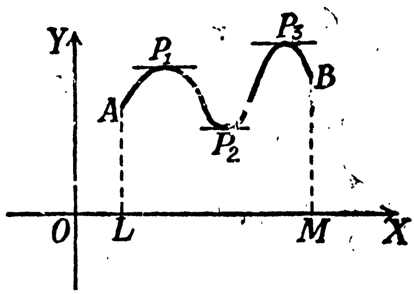


Fig. 44.

This intuitive consideration is also sometimes regarded to be a proof of the theorem.

Note 1. It will be seen that the point 'c' where the derivative has been shown to vanish lies strictly *within* the interval $[a, b]$, so that it coincides neither with a nor with b. In view of this fact, the theorem is generally stated as follows :—

If a function $f(x)$ is such that

1. *It is continuous in the closed interval* $[a, b]$,
2. *It is derivable in the open interval* (a, b),
3. $f(a)=f(b)$.

then, there exists at least one point 'c' of the open interval (a, b), *such that* $f'(c)=0$

Note 2. The conclusion of Rolle's theorem may not hold good for a function which does not satisfy any of its conditions.

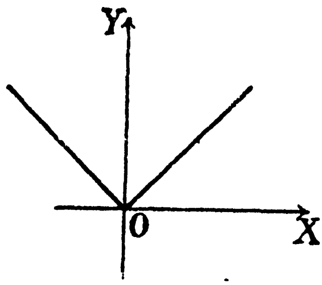


Fig. 45.

To illustrate this remark, we consider the function $y=f(x)=|x|$ in the interval $[-1, 1]$.

It is continuous in $[-1, 1]$ and $f(1)$, $f(-1)$ are both equal to 1.

Its derivative $f'(x)$ is 1 for $0<x\leqslant 1$ and -1, for $-1\leqslant x<0$, and does not exist for $x=0$, so that $f'(x)$ nowhere vanishes.

(Ex. 2. p. 73)

The failure is explained by the fact that $|x|$ is not derivable in $[-1, 1]$, in as much as the derivative does not exist for $x=0$, which is a point of the interval.

Ex. 1. *Verify Rolle's theorem for*

(*i*) x^2 *in* $[-1, 1]$. (*ii*) $x(x+3)e^{-\frac{1}{2}x}$ *in* $[-3, 0]$.

(*i*) Let

$$f(x)=x^2 \text{ so that } f(1)=1=f(-1).$$

Also, x^2 is derivable in $[-1, 1]$.

The conditions of the theorem being satisfied, the derivative $f'(x)$ must vanish for at least one value of x lying within $[-1, 1]$.

Also directly we see that the derivative $2x$ vanishes for $x=0$ which value lies within $[1, -1]$. Hence the verification.

(*ii*) Let

$$f(x)=x(x+3)e^{-\frac{1}{2}x}.$$

We have

$$f(-3)=0=f(0),$$

and $f(x)$ is derivable in the interval $[-3, 0]$. We have

$$f'(x)=(2x+3)e^{-\frac{1}{2}x}+x(x+3)e^{-\frac{1}{2}x}(-\tfrac{1}{2})$$
$$=\frac{(-x^2+x+6)}{2}e^{-\frac{1}{2}x}.$$

Putting $f'(x)=0$, we get

$$-x^2+x+6=0.$$

This equation is satisfied by $x=-2, 3$. Of these two values of x, for which $f'(x)$ is zero, -2, belongs to the interval $[-3, 0]$ under consideration.

Hence the verification.

Ex. 2. Verify Rolle's theorem in the interval $[a, b]$ for the functions

(*i*) $\log [(x^2+ab)/(a+b)x]$.

(*ii*) $(x-a)^m(x-b)^n$; m, n being positive integers.

Ex. 3. Verify Rolle's theorem for the functions

(*i*) $\sin x/e^x$ in $[0, \pi]$; (*ii*) $e^x (\sin x - \cos x)$ in $[\pi/4, 5\pi/4]$.

Ex. 4. By considering the function $(x-2) \log x$, show that the equation $x \log x = 2 - x$ is satisfied by at least one value of x lying between 1 and 2.

6·2. Lagrange's mean value theorem. *If a function $f(x)$ is derivable in the interval $[a, b]$, then there exists at least one value 'c' of x lying within $[a, b]$ such that*

$$\frac{\mathbf{f(b)-f(a)}}{\mathbf{b-a}} = \mathbf{f'(c)}.$$

To prove the theorem, we define a new function $\varphi(x)$ involving $f(x)$ and designed so as to satisfy the conditions of Rolle's theorem.

Let

$$\varphi(x) = f(x) + Ax,$$

where A is a constant to be determined such that

$$\varphi(a) = \varphi(b).$$

Thus

$$f(a) + Aa = f(b) + Ab.$$

$$\therefore \qquad A = -\frac{f(b)-f(a)}{b-a}.$$

Now, $f(x)$ is derivable in $[a, b]$. Also x is derivable and, A, is a constant. Therefore, $\varphi(x)$, is derivable in $[a, b]$ and its derivative is

$$f'(x) + A.$$

Thus, $\varphi(x)$ satisfies all the conditions of Rolle's theorem. There is, therefore, at least one value 'c' of x, lying within $[a, b]$ such that

$$\varphi'(c) = 0.$$

$$\therefore \qquad 0 = \varphi'(c) = f'(c) + A, \text{ i.e., } A = -f'(c),$$

or
$$\frac{\mathbf{f(b)-f(a)}}{\mathbf{b-a}} = \mathbf{f'(c)}. \qquad \ldots(i)$$

Another form of the statement of Lagrange's mean value theorem. *If a function $f(x)$ is derivable in an interval $[a, a+h]$, there exists at least one number 'θ' lying between 0 and 1 such that*

$$\mathbf{f(a+h) = f(a) + hf'(a+\theta h)}.$$

We write $b - a = h$ so that h denotes the length of the interval $[a, b]$ which may now be written as $[a, a+h]$.

The number, c, which lies between a and $a+h$ is greater than a by some fraction of h, so that we may write

$$c = a + \theta h,$$

where θ is some number between 0 and 1. Thus the equation (*i*) becomes

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h),$$

or
$$\mathbf{f(a+h) = f(a) + hf'(a+\theta h)}. \qquad [0 < \theta < 1]$$

Geometrical statement of the theorem. *If a curve has a tangent at each of its points, then there exists at least one point P on the curve such that the tangent at P is parallel to the chord AB joining its extremities.*

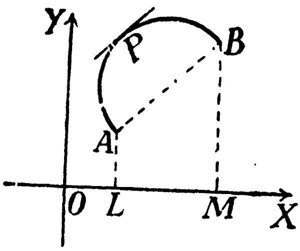


Fig. 46.

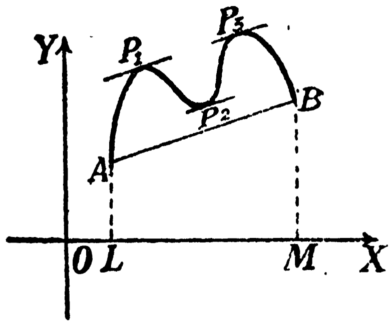


Fig. 47.

We will now see the equivalence of the analytical and geometrical statements.

If $f(x)$ is derivable in $[a, b]$, then the curve $y=f(x)$ has a tangent at each point of the curve lying between the extremities $A\ [a, f(a)]$ and $B\ [b, f(b)]$.

The slope of the chord $AB=\dfrac{f(b)-f(a)}{b-a}$.

Let P be the point $(c, f(c))$ on the curve ; c, being such that

$$\frac{f(b)-f(a)}{b-a}=f'(c). \qquad \ldots(ii)$$

The slope of the tangent at $P=f'(c)$.

From (ii), we see that the slopes of the tangent at P and the chord AB are equal.

Thus there exists a point P on the curve the tangent at which is parallel to the chord AB.

Note 1. Now $[f(b)-f(a)]$ is the change in the function $f(x)$ as x changes from a to b so that $[f(b)-f(a)]/(a-b)$ is the *average* rate of change of the function $f(x)$ over the interval $[a, b]$. Also $f'(c)$ is the *actual* rate of change of the function for $x=c$. Thus the theorem states that the *average rate of change* of a function over an interval is also the *actual rate of change* of the function at some point of the interval. In particular, for instance, the average velocity over any interval of time is equal to the actual velocity at some instant belonging to the interval ; velocity being rate of change of distance *w.r.* to time. This interpretation of the theorem justifies the name 'Mean Value' for the theorem.

Note 2. If we draw some curves satisfying the conditions of the theorem, we will realise that the theorem, as stated in the geometrical form, is almost self-evident.

Ex. 1. *If*

$$f(x)=(x-1)(x-2)(x-3)\ ;\ a=0,\ b=4,$$

find the value of c. (*D.U. Hons. 1954*)

We have

$$f(b)=f(4)=3.2.1=6,$$

$$f(a)=f(0)=-6,$$

$$\therefore \qquad \frac{f(b)-f(a)}{b-a}=\frac{12}{4}=3.$$

Also

$$f'(x)=(x-2)(x-3)+(x-3)(x-1)+(x-1)(x-2)$$
$$=3x^2-12x+11.$$

Consider now the equation

$$\frac{f(b)-f(a)}{b-a}=f'(c),$$

i.e., $$3=3c^2-12c+11.$$

or $$3c^2-12c+8=0$$

$$\therefore \qquad c=\frac{6+2\sqrt{3}}{3},\ \frac{6-2\sqrt{3}}{3}.$$

Taking $\sqrt{3}=1{\cdot}732...$, we may see that both these values of c' belong to the interval $[0, 4]$.

Ex. 2. Verify the mean value theorem for

(*i*) $\log x$ in $[1, e]$. (*ii*) x^3 in $[a, b]$.

(*iii*) lx^2+mx+n in $[a, b]$.

Ex. 3. Find 'c' of the mean value theorem, if

$$f(x)=x(x-1)(x-2)\ ;\ a=0,\ b=\tfrac{1}{2}.$$

(*D.U. Hons. 1951*)

Ex. 4. Find 'c' so that $f'(c)=[f(b)-f(a)]/(b-a)$ in the following cases :—

(*i*) $f(x)=x^2-3x-1\ ;\ a=-\frac{11}{7},\ b=\frac{13}{7}$.

(*ii*) $f(x)=\sqrt{(x^2-4)}\ ;\ a=2,\quad b=3.$

(*iii*) $f(x)=e^x\ ;\ a=0,\ b=1.$

Ex. 5. Applying Lagrange's mean value theorem, in turn to the functions $\log x$ and e^x, determine the corresponding values of θ in terms of a and h.

Deduce that

(*i*) $0<[\log(1+x)]^{-1}-x^{-1}<1$; (*ii*) $0<\dfrac{1}{x}\log\dfrac{e^x-1}{x}<1$.

Ex. 6. Explain the failure of the theorem in the interval $[-1, 1]$ when

$$f(x)=1/x,\ (x\neq 0)\ ;\ f(0)=0.$$

6·3. Some important deductions from the mean Value theorem. Meaning of the sign of Derivative. We consider a function $f(x)$ derivable in an interval $[a, b]$. Let x_1, x_2 be *any* two points belonging to the interval such that $x_2 > x_1$. Applying the mean value theorem to the interval $[x_1, x_2]$, we see that there exists a number ξ between x_1 and x_2 such that

$$\mathbf{f(x_2)-f(x_1)=(x_2-x_1)f'(\xi).} \qquad ...(i)$$

6·31. *Let* $f'(x)=0$ *throughout the interval* $[a, b]$.

From (i) we get

$$f(x_2)=f(x_1),$$

where x_1, x_2 are *any* two values of x. Thus we see that every two values of the function are equal. Hence $f(x)$ is a constant. We thus prove :—

"If the derivative of a function vanishes for all values of x in an interval, then the function must be a constant.

This is the converse of the theorem, "Derivative of a constant is zero."

Cor. *If two functions* $f(x)$ *and* $F(x)$ *have the same derivative for every value of x in* $[a, b]$ *then they differ only by a constant.*

We write

$$\phi(x)=f(x)-F(x).$$

$$\therefore \qquad \phi'(x)=f'(x)-F'(x)=0.$$

Hence, $\phi(x)$, *i.e.*, $f(x)-F(x)$ is a constant.

6·32. *Let* $f'(x)>0$ *for every value of* x in $[a, b]$.

From (i), we get

$$f(x_2)-f(x_1)>0 \text{ i.e., } f(x_2)>f(x_1)\ ;$$

for, x_2-x_1 and $f'(\xi)$ are both positive.

Hence $f(x)$ is an increasing function of x. We have thus proved :—

"A function whose derivative is positive for every value of x in an interval is a monotonically increasing function of x in that interval."

6·33. *Let* $f'(x)<0$ *for every value of x in* $[a, b]$.

From (i), we get

$$f(x_2)-f(x_1)<0 \text{ i.e., } f(x_2)<f(x_1)\ ;$$

for, x_2-x_1 is positive and $f'(\xi)$ negative.

Hence $f(x)$ is a decreasing function of x. We have thus proved :—

"A function whose derivative is negative for every value of x in an interval is a monotonically decreasing function of x in that interval."

Note. The above conclusions remain valid even if $f'(x)$ vanishes at the end points a, b of the interval, for the 'c' of the mean value theorem never coincides with a, or b.

6·4. Cauchy's mean value theorem. *If two functions $f(x)$ and $F(x)$ are derivable in an interval $[a, b]$ and $F'(x) \neq 0$ for any value of x in $[a, b]$, then there exists at least one value 'c' of x lying within $[a, b]$, such that*

$$\frac{f(b)-f(a)}{F(b)-F(a)} = \frac{f'(c)}{F'(c)}.$$

Firstly, we note that $[F(b)-F(a)] \neq 0$: for if it were 0, then $F(x)$ would satisfy the conditions of the Rolle's theorem and its derivative would therefore vanish for at least one value of x and the hypothesis that $F'(x)$ is never 0 would be contradicted.

Now, we define a new function $\varphi(x)$ involving $f(x)$ and $F(x)$ and designed so as to satisfy the conditions of Rolle's theorem.

Let

$$\varphi(x) = f(x) + AF(x),$$

where A is a constant to be determined such that

$$\varphi(a) = \varphi(b).$$

Thus

$$f(a) + AF(a) = f(b) + AF(b).$$

$$\therefore \qquad A = -\frac{f(b)-f(a)}{F(b)-F(a)}, \text{ for } [F(b)-F(a)] \neq 0.$$

Now, $f(x)$ and $F(x)$ are derivable in $[a, b]$. Also A is a constant. Therefore, $\varphi(x)$ is derivable in $[a, b]$ and its derivative is

$$f'(x) + AF'(x).$$

Thus, $\varphi(x)$ satisfies the conditions of Rolle's theorem. There is, therefore, at least one value, c, of x lying within $[a, b]$ such that

$$\phi'(c) = 0.$$

$$\therefore \qquad 0 = \phi'(c) = f'(c) + AF'(c),$$

or

$$f'(c) = -AF'(c),$$

Dividing by $F'(c)$ which $\neq 0$, we get

$$\frac{\mathbf{f'(c)}}{\mathbf{F'(c)}} = \frac{\mathbf{f(b)-f(a)}}{\mathbf{F(b)-F(a)}}.$$

Hence the theorem.

Another form of the statement of Cauchy's mean value theorem. *If two functions $f(x)$ and $F(x)$ are derivable in an interval $[a, a+h]$ and $F'(x) \neq 0$, then there exists at least one number θ between 0 and 1, such that*

$$\frac{f(a+h)-f(a)}{F(a+h)-F(a)} = \frac{f'(a+\theta h)}{F'(a+\theta h)} \qquad (0<\theta<1)$$

The equivalence of the two statements can be easily seen as in the case of Lagrange's mean value theorem.

Note. Taking $F(x)=x$, we may easily see that Lagrange's theorem is only a particular case of Cauchy's.

Ex. 1. Verify the theorem for the functions x^2 and x^4 in the interval $[a, b]$; a, b being positive.

Ex. 2. If, in the Cauchy's mean value theorem, we write for $f(x)$, $F(x)$; (*i*) x^2, x ; (*ii*) $\sin x$, $\cos x$; (*iii*) e^x, e^{-x} ; show that in each case 'c' is the arithmetic mean between a and b.

Ex. 3. If, in the Cauchy's mean value theorem, we write for $f(x)$, and $F(x)$, $\sqrt{x}$ and $1/\sqrt{x}$ respectively then, c, is the geometric mean between a and b, and if we write $1/x^2$ and $1/x$ then, c, is the harmonic mean between a and b.

6·5. Examples

1. *Show that*

$$x^3-3x^2+2x+2$$

is monotonically increasing in every interval.

Let

$$f(x)=x^3-3x^2+3x+2.$$

$$\therefore \quad f'(x)=3x^2-6x+3=3(x-1)^2.$$

Thus $f'(x)>0$ for every value of x except 1 where it vanishes. Hence $f(x)$ is monotonically increasing in every interval.

2. *Separate the intervals in which the polynomial*

$$2x^3-15x^2+36x+1$$

is increasing or decreasing.

Also draw a graph of the function.

Let

$$y=f(x)=2x^3-15x^2+36x+1.$$

$$\therefore \quad f'(x)=6x^2-30x+36=6(x-2)(x-3),$$

so that

$$f'(x)>0 \text{ for } x<2 \;;$$
$$f'(x)<0 \text{ for } 2<x<3 \;;$$
$$f'(x)>0 \text{ for } x>3 \;;$$
$$f'(x)=0 \text{ for } x=2 \text{ and } 3.$$

Thus, $f'(x)$ is positive in the interval $(-\infty, 2)$ and $(3, \infty)$ and negative in the interval $(2, 3)$.

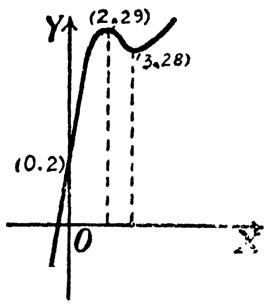


Fig. 48.

Hence $f(x)$ is monotonically increasing in the intervals $(-\infty, 2]$ $[3, \infty)$ and monotonically decreasing in the interval $[2, 3]$.

To draw the graph of the function, we note the following additional points :—

(*i*) $f(2)=29$; (*ii*) $f(3)=28$;

(*iii*) $f(0)=2$,

(*iv*) $f(x)\to-\infty$ as $x\to-\infty$.

(*v*) $f(x)\to+\infty$ as $x\to+\infty$.

3. *Show that*

$$x/(1+x) < \log(1+x) < x \text{ for } x > 0.$$

We write

$$f(x)=\log(1+x)-\frac{x}{1+x}.$$

$$\therefore \quad f'(x)=\frac{1}{1+x}-\frac{1}{(1+x)^2}=\frac{x}{(1+x)^2}.$$

Thus $\quad f'(x) > 0$ for $x > 0$ and $=0$ for $x=0$.

Hence $f(x)$ is monotonically increasing in the interval $[0, \infty]$.

Also $\quad f(0)=0.$

$\therefore \quad f(x) > f(0)=0$ for $x > 0$.

Hence $f(x)$ is positive for every positive value of x, so that

$$\log(1+x) > x/(1+x) \text{ for } x > 0.$$

Again, we write

$$F(x)=x-\log(1+x),$$

so that $$F'(x)=1-1/(1+x)=x/(1+x)$$

Thus, $\quad F'(x) > 0$ for $x > 2$ and is 0 for $x=0$.

Therefore $F(x)$ is monotonically increasing in the interval $[0, \infty]$. Also $F(0)=0$.

$$F(x) > F(0)=0 \text{ for } x > 0.$$

Hence $F(x)$ is positive for positive values of x, so that

$$x > \log(1+x) \text{ for } x > 0.$$

Exercises

1. Show that

(*i*) $x/\sin x$ increases steadily from $x=0$ to $x=\pi/2$. *(P.U.)*

(*ii*) $x/\tan x$ decreases monotonically from $x=0$ to $x=\pi/2$.

(*iii*) the equation $\tan x-x=6$ has one and only one root in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. *(B.U. 1952)*

(*iv*) $\tan^{-} x>\dfrac{x}{1+\frac{1}{3}x^2}$ if $0 < \tan^{-1} x < \dfrac{\pi}{2}$.

2. Show that $x-\sin x$ is an increasing function throughout any interval of values of x. Determine for what values of a, $ax-\sin x$ is a steadily increasing function. *(M.U.)*

3. Determine the intervals in which the function

$$(x^4+6x^3+17x^2+32x+32)e^{-x}$$

is increasing or decreasing.

4. Separate the intervals in which the function

$$(x^2+x+1)/(x^2-x+1)$$

is increasing or decreasing.

5. Determine the intervals in which the function $(4-x^2)^2$ is increasing or decreasing. Also draw its graph.

6. Find the greatest and least values of the function

$$x^3-9x^2+24x \text{ in } [0, 6].$$

7. Show that $x^{-1}\log(1+x)$ decreases as x increases from 0 to ∞.

8. Show that if $x > 0$,

(i) $x-\frac{x^2}{2} < \log(1+x) < x-\frac{x^2}{2(1+x)}$.

(ii) $x-\frac{x^2}{2}+\frac{x^3}{3(1+x)} < \log(1+x) < x-\frac{x^2}{2}+\frac{x^3}{3}$. (*B.U. 1953*)

9. Show that

$$x < -\log(1-x) < x(1-x)^{-1} \text{ for } 0 < x < 1.$$

10. Prove that e^{-x} lies between

$$1-x \text{ and } 1-x+\tfrac{1}{2}x^2.$$

11. Show that $\sin x$ lies between

$$x-\frac{x^3}{6} \text{ and } x-\frac{x^3}{6}+\frac{x^5}{120}.$$

12. If $0 < x < 1$, show that

$$2x < \log\frac{1+x}{1-x} < 2x\left(1+\tfrac{1}{3}\cdot\frac{x^2}{1-x^2}\right).$$

Hence taking $x=\frac{1}{2n+1}$, $n > 0$, deduce that

$$e<\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e\,.\,e^{\frac{1}{12n(n+1)}}$$

(*D.U. Hons. 1951*)

13. Show that

$$\frac{\tan x}{x} > \frac{x}{\sin x} \text{ if } 0 < x < \frac{\pi}{2}.$$

14. Show that

$$x-1 > \log x > (x-1)x^{-1},$$

and $$x^2-1 > 2x\log x > 4(x-1)-2\log x$$

for $$x > 1.$$ (*M.T.*)

15. If $f(x)$ is derivable in the interval $[a-h, a+h]$, prove that

(i) $f(a+h)-f(a-h)=h[f'(a+\theta_1 h)+f'(a-\theta_1 h)]$,
where $0 < \theta_1 < 1$;

(ii) $f(a+h)-2f(a)+f(a-h)=h[f'(a+\theta_2 h)-f'(a-\theta_2 h)]$
where $0 < \theta_2 < 1$.

16. The derivative of a function $f(x)$ is positive for every value of x in an interval $[c-h, c]$, and negative for every value of x in $[c, c+h]$; show that $f(c)$ is the greatest value of the function in the interval $[c-h, c+h]$.

6·6. Higher mean value theorem or Taylor's development of function in a finite form. *If a function $f(x)$ possesses the derivatives $f'(x), f''(x), f'''(x), \ldots\ldots f^n(x)$, up to a certain order n for every value of x in the interval, $[a, a+h]$, then there exists at least one number θ, between 0 and 1, such that*

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\frac{h^3}{3\,!}f'''(a)+\ldots\ldots$$
$$+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(a)+\frac{h^n}{n\,!}f^n(a+\theta h).$$

We define a new function $\varphi(x)$ involving $f(x)$ and its drivatives $f'(x), f''(x), \ldots\ldots, f^n(x)$ designed so as to satisfy the conditions for Rolle's theorem. Let

$$\varphi(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{2\,!}f''(x)+\ldots\ldots$$
$$+\ldots\ldots+\frac{(a+h-x)^{n-1}}{(n-1)\,!}f^{n-1}(x)+\frac{(a+h-x)^n}{n!}A$$

where A is a constant to be determined such that

$$\varphi(a)=\varphi(a+h).$$

Thus we get A from the equation

$$f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(a)+\frac{h^n}{n\,!}A$$
$$=f(a+h) \qquad \ldots(i)$$

Now, it is given that $f(x), f'(x), f''(x), \ldots\ldots, f^{n-1}(x)$ are derivable in the interval $[a, a+h]$.

Also, $a+h-x$, $(a+h-x)^2/2\,!$, ..., $(a+h-x)^n/n\,!$ are derivable in $[a, a+h]$. Therefore $\varphi(x)$ is derivable in $[a, a+h]$. Also

$$\varphi'(x)=f'(x)-f'(x)+(a+h-x)f''(x)-(a+h-x)f''(x)$$
$$+\frac{(a+h-x)^2}{2\,!}f'''(x)-\frac{(a+h+x)^2}{2\,!}f'''(x)+\ldots\ldots$$
$$-\frac{(a+h-x)^{n-2}}{(n-2)\,!}f^{n-1}(x)+\frac{(a+h-x)^{n-1}}{(n-1)\,!}f^n(x)-\frac{(a+h-x)^{n-1}}{(n-1)\,!}A$$
$$=\frac{(a+h-x)^{n-1}}{(n-1)\,!}[f^n(x)-A],$$

other terms cancelling in pairs.

Thus $\varphi(x)$ satisfies all the conditions for Rolle's theorem. There exists, therefore, at least one number θ between 0 and 1 such that

$$\varphi'(a+\theta h)=0$$

$$\therefore \quad 0=\varphi'(a+\theta h)=\frac{(h-\theta h)^{n-1}}{(n-1)\,!}[f^n(a+\theta h)-A]$$
$$=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)\,!}[f^n(a+\theta h)-A]$$

$\therefore \quad f^n(a+\theta h)=A$; for $(1-\theta)\neq 0$.

Substituting this value of A in (i), we get

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(a)$$

$$+\frac{h^n}{n\,!}f^n(a+\theta h). \qquad [0<\theta<1] \qquad \ldots(ii)$$

The $(n+1)$th term

$$\frac{h^n}{n\,!}f^n(a+\theta h)$$

is called **Lagrange's form of remainder** after n terms in the Taylor's expansion of $f(a+h)$ in ascending integral powers of h.

Note. Taking $n=1$ we see that Lagrange's mean value theorem is only a particular case of the Taylor's development obtained here.

Cor. Maclaurin's development. Instead of considering the interval $[a, a+h]$, we now consider the interval $[0, x]$ so that we change a to 0 and h to x in (ii). We get

$$f(x)=f(0)=xf'(0)+\frac{x^2}{2\,!}f''(0)+\ldots\ldots+\frac{x^{n-1}}{(n-1)\,!}f^{n-1}(0)+\frac{x^n}{n\,!}f^n(\theta x) \qquad \ldots(iii)$$

which holds when *the function $f(x)$ possesses the derivatives*

$$f'(x), f''(x), \ldots\ldots\ldots, f^n(x)$$

in the interval $[0, x]$.

The formula (iii) is known as *Maclaurin's development* of $f(x)$ in $[0, x]$ in the finite form with Lagrange's form of remainder.

6·7. Taylor's development of a function with Cauchy's form of remainder. *If a function $f(x)$ possesses the derivatives $f'(x), f''(x) \ldots\ldots, f^n(x)$ up to a certain order n in the interval $[a, a+h]$, then there exists at least one number θ between 0 and 1 such that*

$$f(a+h)=f(a)+f'(a)+\frac{h^2}{2\,!}f''(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(a)$$

$$+\frac{h^n}{(n-1)\,!}(1-\theta)^{n-1}f^n(a+\theta h).$$

We define a new function $\varphi(x)$ as follows :—

$$\varphi(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{2\,!}f''(x)+\ldots\ldots$$

$$+\frac{(a+h-x)^{n-1}}{(n-1)\,!}f^{n-1}(x)+(a+h-x)A,$$

where A is a constant to be determined such that

$$\varphi(a+h)=\varphi(a).$$

Thus, we get A from the equation

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+hA \quad \dots(i)$$

It is easy to see that $\varphi(x)$ is derivable in $[a, a+h]$.

Hence $\varphi(x)$ satisfies the conditions for Rolle's theorem so that there exists at least one number θ between 0 and 1 such that

$$\varphi'(a+\theta h)=0$$

Now, $$\varphi'(x)=\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(a+\theta h)-A\,;$$

other terms cancelling in pairs.

$\therefore$ $$0=\varphi'(a+\theta h)=\frac{(h-\theta h)^{n-1}}{(n-1)!}f^n(a+\theta h)-A$$

i.e., $$A=\frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h).$$

Substituting this value of A in (i), we get

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+\frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h) \quad \dots(ii)$$

The $(n+1)$th term

$$\mathbf{\frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h)}$$

is called **Cauchy's form of remainder** after n terms in the expansion of $f(a+h)$ in ascending integral powers of h.

Cor. Maclaurin's development. Changing a to 0 and h to x in (ii), we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2}f''(0)+\dots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)$$
$$+\frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^n(\theta x)$$

which is known as *Maclaurin's development of* $f(x)$ in the interval $[0, x]$ with Cauchy's form of remainder after n terms. It holds when $f(x)$ possesses the derivatives $f'(x), f''(x), \dots\dots, f^n(x)$ in $[0, x]$.

Ex. 1. *Show that*

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\dots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}e^{\theta x}.$$

Here $f(x)=e^x$. Therefore $f^n(x)=e^x$.

$\therefore$ $f(0)=f'(0)=\dots=f^{n-1}(0)=1, \quad f^n(\theta x)=e^{\theta x}.$

Substituting these values in (*iii*) (§ 6·7, p. 142), we obtain the result.

Ex. 2. Show that, for every value of x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots\dots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} +$$
$$(-1)^n \frac{x^{2n}}{2n!} \sin \theta x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\dots\dots + (-1)^n \frac{x^{2n}}{2n!} +$$
$$(-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} \sin \theta x.$$

Ex. 3. Show that

$$\log (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots\dots + (-1)^{n-2}.\frac{x^{n-1}}{(n-1)} +$$
$$(-1)^{n-1}.\frac{x^n}{n(1+\theta x)^n}.$$

Ex. 4. Find, by Maclaurin's theorem, the first four terms and the remainder after n terms of the expression of $e^{ax} \cos bx$ in terms of the ascending powers of x.

APPENDIX

Examples

1. *Show that the number θ which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches the limit $1/(n+1)$ as h approaches zero provided that $f^{n+1}(x)$ is continuous and different from zero at $x=a$.* (*D.U. Hons. 1950*)

Applying Taylor's Theorem with remainders after n terms and $(n+1)$ terms successively, we obtain

$$f(a+h)=f(a)+hf'(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^n(a+\theta h),$$

$$f(a+h)=f(a)+hf'(a)+\ldots\ldots+\frac{h^n}{n!}f^n(a)+\frac{h^{n+1}}{(n+1)!}f^{n+1}(a+\theta' h)$$

These give

$$\frac{h^n}{n!}f^n(a+\theta h)=\frac{h^n}{n!}f^n(a)+\frac{h^{n+1}}{(n+1)!}f^{n+1}(a+\theta' h),$$

or $$f^n(a+\theta h)-f^n(a)=\frac{h}{n+1}f^{n+1}(a+\theta' h).$$

Applying Lagrange's Mean Value Theorem to the left-hand side, we have

$$\theta h f^{n+1}(a+\theta''\theta h)=\frac{h}{n+1}f^{n+1}(a+\theta' h).$$

or $$\theta=\frac{1}{n+1}\cdot\frac{f^{n+1}(a+\theta' h)}{f^{n+1}(a+\theta''\theta h)}.$$

Let $h \to 0$.

$$\therefore \qquad \lim \theta=\frac{1}{n+1}.$$

2. *Show that*

$$x^2>(1+x)\,[\log(1+x)]^2 \textit{ for } x>0.$$

We put $\quad f(x)=x^2-(1+x)\,[\log(1+x)]^2.$

$$\therefore \quad f'(x)=2x-1.\,[\log(1+x)]^2-(1+x).\,2\log(1+x).[1/(1+x)]$$
$$=2x-[\log(1+x)]^2-2\log(1+x).$$

The form of $f'(x)$ is such that we cannot immediately decide as to its sign. We, therefore, proceed to determine the second derivative.

$$f''(x)=2-2\log(1+x).1/(1+x)-2/(1+x)$$
$$=2\left[\frac{x-\log(1+x)}{1+x}\right]$$

which is >0 for $x>0$. (See Ex. 3, p. 139)

$\therefore$ $f'(x)$ is monotonically increasing in the interval $[0, \infty)$.

Also

$$f'(0)=0.$$

Therefore

$$f'(x)>0 \text{ for } x>0.$$

Hence $f(x)$ is monotonically increasing in the interval $[0, \infty)$. Also

$$f(0)=0$$

Therefore

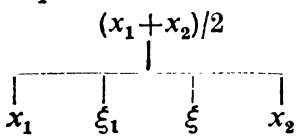
$$f(x)>0 \text{ for } x>0.$$

Hence

$$x^2>(1+x)[\log(1+x)]^2 \text{ for } x>0.$$

3. *If $\varphi''(x)>0$ for every value of x, then*
$$\varphi[\tfrac{1}{2}(x_1+x_2)]\leqslant\tfrac{1}{2}[\varphi(x_1)+\varphi(x_2)],$$
for every pair of values of x_1 and x_2.

Suppose that $x_2>x_1$.



We write

$$\varphi(x_1)+\varphi(x_2)+2\varphi[\tfrac{1}{2}(x_1+x_2)]$$
$$=(\varphi(x_2)-\varphi\tfrac{1}{2}\{(x_1+x_2)\}]-[\varphi\tfrac{1}{2}\{x_1+x_2)\}-\varphi(x_1)]. \quad ...(1)$$

Applying Lagrange's mean value theorem to the function $\varphi(x)$ for the intervals $[x_1, (x_1+x_2)/2]$ and $[(x_1+x_2)/2, x_2]$ we see that there exist numbers ξ_1, ξ_2 belonging to the two intervals respectively, such that

$$\varphi(x_2)-\varphi[\tfrac{1}{2}(x_1+x_2)]=[x_2-\tfrac{1}{2}(x_1+x_2)]\varphi'(\xi_2)=\tfrac{1}{2}(x_2-x_1)\varphi'(\xi_2) \quad ...(2)$$

and

$$[\varphi\tfrac{1}{2}(x_1+x_2)-\varphi(x_1)]=[\tfrac{1}{2}(x_1+x_2)-x_1]\varphi'(\xi_1)=\tfrac{1}{2}(x_2-x_1)\varphi'(\xi_1) \quad ...(3)$$

Thus from (1), (2), (3), we obtain

$$\varphi(x_1)+\varphi(x_2)-2\varphi[\tfrac{1}{2}(x_1+x_2)]=\tfrac{1}{2}(x_2-x_1)[\varphi'(\xi_2)-\varphi'(\xi_1]. \quad ...(4)$$

Applying the mean value theorem to the function $\varphi'(x)$ for the interval $[\xi_1, \xi_2]$, we see that there exists a numer η such that

$$\varphi'(\xi_2)-\varphi'(\xi_1)=(\xi_2-\xi_1)\varphi''(\eta). \quad ...(5)$$

From (4) and (5), we obtain

$$[\varphi(x_1)+\varphi(x_2)]-\varphi[\tfrac{1}{2}(x_1+x_2)]=\tfrac{1}{2}(x_2-x_1)(\xi_2-\xi_1)\varphi''(\eta).$$

Since x_2-x_1, $\xi_2-\xi_1$ and $\varphi''(\eta)$ are all positive, we obtain the required result.

Exercises

1. Show that 'θ' (which occurs in the Lagrange's mean value theorem) approaches the limit $\frac{1}{2}$ as 'h' approaches 0, provided that $f''(a)$ is not zero.

(P.U. 1949 ; D.U. Hons. 1949)

[It should also be assumed that $f''(x)$ is not continuous.]

2. Show that

$$f(a+h)=f(a)+hf'(a)+\tfrac{1}{2}h^2f''(a+\theta h),$$

where θ lies between 0 and 1 and prove that

$$\lim_{h \to 0} \theta=\tfrac{1}{3},$$

specifying the necessary conditions. *(C.U. Hons. 1953)*

3. Assuming $f''(x)$ continuous in $[a, b]$, show that

$$f(c)-f(a)\,\frac{b-c}{b-a}-f(b)\,\frac{c-a}{b-a}=\tfrac{1}{2}(c-a)(c-b)f''(\xi),$$

where c and ξ both lie in $[a, b]$. *(P.U.)*

(Take $\varphi(x)=f(x)+Ax+Bx^2$ and determine A, B such that

$$\varphi(a)=\varphi(b)=\varphi(c).$$

Now apply Rolle's Theorem to the intervals $[a, c]$ and $[c, b]$.]

4. The second derivative $f''(x)$ of a function $f(x)$ is continuous for $a\leqslant x\leqslant b$ and at each point x, the signs of $f(x)$ and $f''(x)$ are the same. Prove that if $f(x)$ vanishes at points c and d, where $a\leqslant c<d\leqslant b$, then it vanishes everywhere between c and d. *(B.U. 1952)*

5. $f(x)$, $\varphi(x)$ and $\psi(x)$ are three functions derivable in an interval (a, b) ; show that there exists a point ξ such that

$$\begin{vmatrix} f'(\xi) & \varphi'(\xi) & \psi'(\xi) \\ f(a) & \varphi(a) & \psi(a) \\ f(b) & \varphi(b) & \psi(b) \end{vmatrix}=0 \qquad (a<\xi<b).$$

Deduce Lagrange's and Cauchy's mean value theorems.

6. Prove that

$$\lim_{h \to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}=f''(x),$$

if $f''(x)$ exists.

7. Discuss the applicability of Rolle's Theorem when

(i) $f(x)=\tan x$ and $a=0$, $b=\pi$.

(ii) $f(x)=2+(x-1)^{\frac{2}{3}}$ and $a=0$, $b=2$.

(iii) $f(x)=(x-c)^{\frac{2}{3}}-c^{\frac{2}{3}}$ when $a=0$ and $b=2c$

(iv) $f(x)=\dfrac{1}{1+x}+\dfrac{1}{1-x}$ and $a=-1$, $b=1$

CHAPTER VII

MAXIMA AND MINIMA

GREATEST AND LEAST VALUES

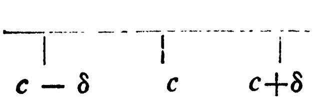
7·1. In this chapter we shall be concerned with the application of Calculus for determining the values of a function which are greatest or least in their *immediate neighbourhood* technically known as *Maximum and Minimum* values. A knowledge of these values of a function is of great help in drawing its graph and in determining its greatest and least values in any given *finite* interval.

It will be assumed that $f(x)$ possesses continuous derivatives of every order that come in question.

Maximum value of a function. Let, c, be any interior point of the interval of definition of a function $f(x)$. Then we say that $f(c)$ is a maximum value of $f(x)$, if it is the greatest of all its values for values of x lying in some neighbourhood of c. To be more definite and to avoid the vague words 'Some neighbourhood', we say that $f(c)$ is a *maximum value of the function, if there exists some interval* $(c-\delta,\ c+\delta)$ *around c such that*

$$f(c) > f(x)$$

for all values of x, other than c, lying in this interval.



This, again, is equivalent to saying that $f(c)$ is a maximum value of $f(x)$, if

$$f(c) > f(c+h),\ i.e.,\ f(c+h)-f(c) < 0$$

for values of h lying between $-\delta$ and δ, *i.e.*, for values of h sufficiently small in numerical value.

Minimum value of a function. $f(c)$ is said to be a minimum value of $f(x)$, if it is the least of all its values for values of x lying in some neighbourhood of c.

This is equivalent to saying that $f(c)$ *is a minimum value of* $f(x)$ *if there exists a positive* δ *such that*

$$f(c) < f(c+h),\ i.e.,\ f(c+h)-f(c) > 0$$

for values of h lying between $-\delta$ and δ, *i.e.*, for values of h sufficiently small in numerical value.

Note 1. The term extreme value is used both for a maximum as well as for a minimum value, so that $f(c)$ *is an extreme value if* $f(c+h)-f(c)$ *keeps an invariable sign for values of h sufficiently small numerically.*

Note 2. While ascertaining whether any value $f(c)$ is an extreme value or not, we compare $f(c)$ with the values of the function for values of x in any immediate neighbourhood of c, so that the values of the function outside the neighbourhood do not come into question at all.

Thus, a maximum value may not be the greatest and a minimum value may not be the least of all the values of the function in any finite interval. In fact a function can have several maximum and minimum values and a minimum value can even be greater than a maximum value.

A glance at the adjoining graph of $f(x)$ shows that the ordinates of points P_1, P_3, P_5 are the maximum and the ordinates of the points P_2, P_4 are the minimum values of $f(x)$ and that the ordinates of P_4 which is minimum is greater than the ordinate of P_1 which is a maximum.

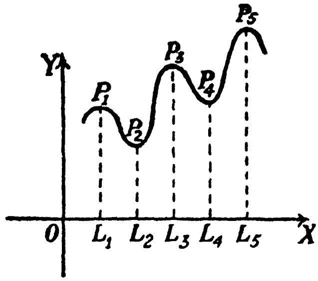


Fig. 49.

7·2. A necessary condition for extreme values. *To prove that a necessary condition for $f(c)$ to be an extreme value of $f(x)$ is that*

$$f'(c)=0.$$

Let $f(c)$ be a maximum value of $f(x)$.

There exists an interval $(c-\delta, c+\delta)$, around c, such that, if, $c+h$ is any number, belonging to this interval, we have

$$f(c+h)<f(c),$$

Here, h may be positive or negative. Thus

$$\frac{f(c+h)-f(c)}{h}<0 \text{ if } h>0, \qquad \dots (i)$$

$$\frac{f(c+h)-f(c)}{h}>0 \text{ if } h<0. \qquad \dots (ii)$$

If h tends to 0 through positive values, we obtain from (i),

$$f'(c)\leqslant 0. \qquad \dots (iii)$$

If h tends to 0 through negative values, we obtain from (ii),

$$f'(c)\geqslant 0. \qquad \dots (iv)$$

The relations (iii) and (iv) will simultaneously be true, if and only if

$$f'(c)=0.$$

It can similarly be shown that $f'(c)=0$, if $f(c)$ is minimum value of $f(x)$.

Cor. Greatest and least values of a function in any interval. *The greatest and least values of $f(x)$ in any interval $[a, b]$ are either $f(a)$ and $f(b)$, or are given by the values of x for which $f'(x)=0$.*

The greatest and least value of a function are also its extreme values in case they are attained at a point strictly *within* the interval so that the derivative must be zero at the corresponding point.

The theorem now easily follows.

Note 1. Geometrically interpreted, the necessary condition for extreme values obtained above states : "*Tangent to a curve at a point P where the ordinate is a maximum or minimum is parallel to x-axis.*"

When stated in this geometrical form, the theorem appears almost self-evident. [Refer Fig. 48 ; p. 149.]

Note 2. *The vanishing of f (c) is only a necessary but not sufficient condition for $f(x)$ to be an extreme value.* To see this, we consider the function $f(x)=x^3$ for $x=0$.

For value of x greater than 0, $f(x)$ is positive and is, therefore, greater than $f(0)$ which is 0 ; and for values of x less than 0, $f(x)$ is negative and is, therefore, less than $f(0)$.

Thus $f(0)$ is not an extreme value even though $f'(0)=0$.

Note 3. Stationary value. *A function $f(x)$ is said to be stationary for $x=c$ if the derivative $f'(x)$ vanishes for $x=c$, i e.*, if $f'(c)=0$; also then $f(c)$ is said to be a *stationary* or a *turning* value of $f(x)$. The term stationary arises from the fact that the *rate of change $f'(x)$ of the function $f(x)$ with respect to x is zero* for a value of x for which $f(x)$ is stationary.

It will be noted that a maximum or a minimum value is also a stationary value but a stationary value may neither be a maximum nor a minimum value.

Ex. 1. *Find the greatest and least values of*

$$3x^4-2x^3-6x^2+6x+1$$

in the interval [0, 2].

Let

$$f(x)=3x^4-2x^3-6x^2+6x+1.$$

$$\therefore \quad f'(x)=12x^3-6x^2-12x+6$$
$$=6(x-1)(x+1)(2x-1).$$

Thus

$$f'(x)=0 \text{ for } x=1,\ -1,\ \tfrac{1}{2}.$$

The value $x=-1$ does not belong to the interval [0, 2] and is not, therefore, to be considered.

Now

$$f(1)=2, f(\tfrac{1}{2})=39/16.$$

Also

$$f(0)=1, f(2)=21.$$

Thus the least value is 1 and the greatest value is 21.

Ex. 2. Find the greatest and least values of

$$x^4-4x^3-2x^2+12x+1$$

in the interval [−2, 5].

Def Change of Sign. A function is said to change sign from positive to negative as x passes through a number c, if there exists some left-handed neighbourhood $(c-h, c)$ of c for every point of which the function is positive, and also there exists some right-fhanded neighbourhood $(c, c+h)$ of c for every point of which the unction is negative.

A similar meaning with obvious alterations can be assigned to the statement "A function changes sign from negative to positive, as x passes through c".

It is clear that if a continuous function $f(x)$ changes sign as x passes through c, then we must have $f(c)=0$.

Ex. 1. Show that the function

$$\varphi(x)=(x+2)(x-1)^2(2x-1)(x-3)$$

changes sign from positive to negative as x passes through $\frac{1}{2}$ and from negative to positive as x passes through -2 or 3; also show that it does not change sign as x passes through 1.

Ex. 2. Show that the function

$$\varphi(x)=(2x+3)(x+4)(x-2)(x-1)^3$$

changes sign from positive to negative as x passes through -4 and 1 and from negative to positive as x passes through $-\frac{3}{2}$ and 2.

7·2. Sufficient criteria for extreme values. *To prove that $f(c)$ is an extreme value of $f(x)$ if and only if $f'(x)$ changes sign as x passes through c, and to show that $f(c)$ is a maximum value if the sign changes from positive to negative and a minimum value in the contrary case.*

Case I. *Let $f'(x)$ change sign from positive to negative as x passes through c.*

In some left-handed neighbourhood of c, $f'(x)$ is positive and so $f(x)$ is monotonically increasing in his neighbourhood. (§6·32, p. 136). Therefore $f(c)$ is the greatest of all the values of $f(x)$ in this left-handed neighbourbood.

In some right-handed neighbourhood of c, $f'(x)$ is negative and so $f(x)$ is monotonically increasing in this neighbourhood (§ 6·33, p. 136). Therefore $f(c)$ is the greatest of all the values of $f(x)$ in this right-handed neighbourhood.

Hence $f(c)$ is the greatest of all the values of $f(x)$ in a certain complete neighbourhood of c and so, by def., $f(c)$ is a maximum value of $f(x)$.

Case II. *Let $f'(x)$ change sign from negative to positive as x passes through c.*

It can similarly be shown that in this case $f(c)$ is the least of all the values of $f(x)$, in a certain complete neighbourhood of c and so, by def., $f(c)$ is a minimum value of $f(x)$.

Case III. If $f'(x)$ *does not change sign*, *i.e.*, has the same sign in a certain complete neighbourhood of c, then $f(x)$ is either monotonically increasing or monotonically decreasing throughout this neighbourhood so that $f(c)$ is not an extreme value of $f(x)$.

Note. Geometrically interpreted, the theorem states that the tangent to a curve at every point in a certain left-handed neighbourhood of the point P whose ordinate is a maximum (minimum) makes an acute angle (obtuse angle) and the tangent at any point in a certain right-handed neighbourhood of P makes an obtuse angle (acute angle) with x-axis. In case, the tangent on *either* side of P makes an acute angle (or obtuse angle), the ordinate of P is neither a maximum nor a minimum.

Ex. 1. *Examine the polynomial*

$$10x^6-24x^5+15x^4-40x^3+108$$

for maximum and minimum values.

Let

$$f(x)=10x^6-24x^5+15x^4-40x^3+108.$$

$$\therefore \quad f'(x)=60x^5-120x^4+60x^3-120x^2$$
$$=60x^2(x^3-2x^2+x-2)$$
$$=60x^2(x^2+1)(x-2)$$

Thus, $f'(x)=0$ for $x=0$ and $x=2$ so that we expect extreme values of $f(x)$ for $x=0$ and 2 only.

Now,

for $x<0$, $\quad f'(x)<0$;

for $0<x<2$ $\quad f'(x)<0$;

for $x>2$, $\quad f'(x)>0$.

Here, $f'(x)$ does not change sign as x passes through 0 so that $f(0)$ is neither a maximum nor a minimum value.

Also, since $f'(x)$ changes sign from negative to positive as x passes through 2, therefore $f(2)=-100$ is a minimum value.

Ex. 2. *Find the extreme values of*

$$x^4-8x^3+22x^2-24x+1$$

and distinguish between them.

Let

$$y=f(x)=x^4-8x^3+22x^2-24x+1.$$

$$\therefore \quad f'(x)=4x^3-24x^2+44x-24$$
$$=4(x^3-6x^2+11x-6)$$
$$=4(x-1)(x-2)(x-3)$$

so that $f'(x)=0$ for $x=1, 2, 3$.

Therefore, $f(x)$ can have extreme values for $x=1, 2, 3$ only.

Now, for $x<1$, $\quad f'(x)<0$;

for $1<x<2$, $\quad f'(x)>0$;

for $2<x<3$, $\quad f'(x)<0$;

for $x>3$, $\quad f'(x)>0$.

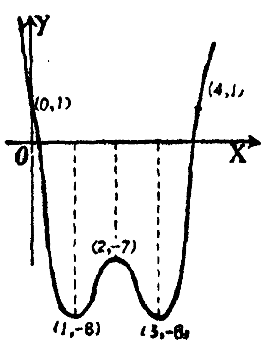


Fig. 50

Since $f'(x)$ changes sign from negative to positive as x passes through 1 and 3, therefore $f(1)=-8$ and $f(3)=-8$ are the two minimum values.

Again since, $f'(x)$ changes sign from positive to negative as x passes through 2, therefore, $f(2)=-7$ is a maximum value.

Ex. 3. Find the extreme values of

$$5x^6+18x^5+15x^4-10.$$

Ex. 4. Show that the maximum and minimum values of

$$(x+1)(x+4)/(x-1)(x-4)$$

are -9 and $-1/9$ respectively.

Ex. 5. Show that

$$9x^5+30x^4+35x^3+15x^2+1$$

is maximum when $x=-2/3$ and minimum when $x=0$.

Also, find its greatest and least values in the intervals $[-2/3, 0]$, and $[-2, 2]$.

Ex. 6. Show that

$$x^5-5x^4+5x^3-1$$

has a maximum value when $x=1$, a minimum value when $x=3$ and neither when $x=0$. *(D.U. 1948)*

7·4. Use of derivatives of second and higher orders. The derivatives of the first order only have so far been employed for determining and distinguishing between the extreme values of a function. As shown in the present article, the same thing can sometimes be done more conveniently by employing derivatives of the second and higher orders.

All along this discussion it will be assumed that $f(x)$ possesses continuous derivatives of every order, that come in question, in the neighbourhood of the point c.

7·41. Theorem 1. *$f(c)$ is a minimum value of $f(x)$, if*

$$f'(c)=0 \text{ and } f''(c)>0.$$

Applying Taylor's theorem with remainder after two terms, we get

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2\,!}f''(c+\theta_2 h)\;; \qquad (0<\theta_2<1)$$

$$\therefore \quad f(c+h)-f(c)=\frac{h^2}{2\,!}f''(c+\theta_2 h). \qquad \because f'(c)=0$$

As the value $f''(c)$ of $f''(x)$ is positive for $x=c$, there exists an interval around, c, for every point of which the second derivative is positive. (§ 3·51, p. 54)

Let $c+h$ be any point of this interval. Then, $c+\theta_2 h$, is also a point of this interval and accordingly $f''(c+\theta_2 h)$ is positive. Also $h^2/2\,!$ is positive.

Thus we see that there exists an interval around c for every point $c+h$, of which, $f(c+h)-f(c)$ is positive, *i.e.*, $f(c)<f(c+h)$.

Hence $f(c)$ is a minimum value of $f(x)$.

7·42. Theorem 2. *$f(c)$ is a maximum value of $f(x)$, if*

$$f'(c)=0 \text{ and } f''(c)<0.$$

As in theorem 1, we have by Taylor's theorem,

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2\,!}f''(c+\theta_2 h),$$

i.e.,
$$f(c+h)-f(c)=\frac{h^2}{2\,!}f''(c+\theta_2 h).$$

As $f''(c)$ is negative, there exists an interval around c for every point of which the second derivative is negative. Thus as in the preceding case, there exists an interval around c for every point, $c+h$ of which $f(c+h)-f(c)$, is negative, *i.e.*, $f(c)>f(c+h)$;

Hence $f(c)$ is a maximum value of $f(x)$.

7·43. General Criteria. *Let*

$$f'(c)=f''(c)=\ldots=f^{n-1}(c)=0, \text{ but } f^n(c)\neq 0,$$

Then $f(c)$ is

(*i*) *a minimum value of $f(x)$, if $f^n(c)>0$ and n is even* ;

(*ii*) *a maximum value of (x), if $f^n(c)<0$ and n is even* ;

(*iii*) *neither a maximum nor a minimum value if n is odd.*

Applying Taylor's theorem with remainder after n terms, we

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2\,!}f''(c)+\ldots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(c)$$
$$+\frac{h^n}{2\,!}f^n(c+\theta_n h)\ ;\ 0<\theta_n<1\ ;$$

so that because of the given conditions,

$$f(c+h)-f(c)=\frac{h^n}{n\,!}f^n(c+\theta_n h).$$

As $f^n(c)\neq 0$, there exists an interval around c for every point x of which the nth derivative $f^n(x)$ has the same sign, *viz.*, that of $f^n(c)$.

Thus for every point, $c+h$, of this interval, $f^n(c+\theta_n h)$ has the sign of $f^n(c)$.

Also when n is even, $h^n/n\,!$; is positive, whether, h, be positive or negative and when n is odd, $h^n/n\,!$, changes sign with the change in the sign of h.

Hence, as in the preceding cases we have the criteria as stated.

Note. The result proved in § 7·41 and § 7·42 is only a particular case of the general criteria established in ω 7·43 above.

Examples

1. *Find the maximum and minimum values of the polynomial*

$$8x^5-15x^4+10x^2.$$

Let $\qquad f(x)=8x^5-15x^4+10x^2.$

$\therefore \qquad f'(x)=40x^4-60x^3+20x$

$$=20x(2x^3-3x^2+1)$$

$$=20x(x-1)^2(2x+1).$$

Hence $\qquad f'(x)=0$ for $x=0, 1, -\tfrac{1}{2}.$

Again $\qquad f''(x)=160x^3-180x^2+20$

$$=20(8x^3-9x^2+1).$$

Now, $f''(-\tfrac{1}{2})=-45$ which is negative so that $f(-\tfrac{1}{2})=\tfrac{21}{16}$ is a maximum value.

Again, $f''(0)=20$ which is positive so that $f(0)=0$ is a minimum value.

As $f''(1)=0$, we have to examine $f'''(1)$.

Now $\qquad f'''(x)=480x^2-360x.$

$\therefore \qquad f'''(1)=120$ which is not zero.

Hence $f(1)$ is neither a maximum nor a minimum value.

2. *Investigate for maximum and minimum values the function*

$$\sin x+\cos 2x.$$

Let

$$y=\sin x+\cos 2x. \qquad \ldots(i)$$

$\therefore \qquad dy/dx=\cos x-2\sin 2x$

$$=\cos x-4\sin x\cos x.$$

Putting $\qquad dy/dx=0$, we get

$$\cos x=0 \text{ or } \sin x=\tfrac{1}{4}.$$

We consider values of x between 0 and 2π only, for the given function is periodic with period 2π.

Now $\qquad \cos x=0$ gives $\quad x=\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$

and $\qquad \sin x=\tfrac{1}{4}$ gives $x=\sin^{-1}\tfrac{1}{4}$ and $\pi-\sin^{-1}\tfrac{1}{4}$,

$\sin^{-1}\tfrac{1}{4}$ lying between 0 and $\pi/2$.

Now $\qquad d^2y/dx^2=-\sin x-4\cos 2x.$

For $x=\pi/2$, $d^2y/dx^2=3$ which is positive ;

For $x=3\pi/2$, $d^2y/dx^2=5$ which is positive ;.

For $x=\sin^{-1}\tfrac{1}{4}$ and $\pi-\sin^{-1}\tfrac{1}{4}$,

$$d^2y/dx^2=-\sin x-4(1-2\sin^2 x)=-15/4$$

which is negative.

Therefore y is a maximum for $x=\sin^{-1}\frac{1}{4}, \pi-\sin^{-1}\frac{1}{4}$ and is a minimum for $x=\pi/2, 3\pi/2$.

Putting these values of x in (i), we see that $\frac{9}{8}, \frac{9}{8}$ are the two maximum values and $0, -2$ are the two minimum values.

Exercises

1. Investigate the maximum and minimum values of

(i) $2x^3-15x^2+36x+10$. (*P.U. 1945*)

(ii) $3x^4-4x^3+5$.

2. Find the values of x for which $x^6-6ax^5+9a^2x^4+a^6$ has minimum values. $(a>0)$.

3. Determine the values of x for which the function

$$12x^5-45x^4+40x^3+6$$

attains (1) a maximum value, (2) a minimum value. (*P.U. 1941*)

4. Find the maximum value of $(x-1)(x-2)(x-3)$. (*P,U. 1939*)

5. Find the maxima and minima as well as the greatest and the least values of the function $y=x^3-12x^2+45x$ in the interval $[0, 7]$. (*P.U. 1942*)

6. Find for what values of x the following expression is a maximum or minimum :

$$2x^3-21x^2+36x-20.$$

7. Find the extreme values of $x^3/(x^4+1)$.

8. Show that $(x+1)^2/(x+3)^3$ has a maximum value $2/27$ and a minimum value 0.

9. Show that x^x is minimum for $x=e^{-1}$.

10. Show that the maximum value of $(1/x)^x$ is $e^{1/e}$.

11. Find the maximum value of $(\log x)/x$ in $<x<\infty$. (*P.U. 1955*)

12. Find the extreme value of $a^{x+1}-a^x-x$. $(a>1)$.

13. Find the maximum and minimum values of $x+\sin 2x$ in $0\leq x\leq 2\pi$.

14. Find the values of x for which $\sin x-x\cos x$ is a maximum or a minimum.

15. Find the maximum and minimum values of

$$x-\sin 2x+\tfrac{1}{3}\sin 3x,$$

in $(-\pi\leq x\leq\pi)$.

16. Show that $\sin x(1+\cos x)$ is a maximum when $x=\frac{1}{3}\pi$.

17. Discuss the maxima and minima in the interval $[0, \pi]$ of the sums

(i) $\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x$.

(ii) $\cos x+\frac{1}{2}\cos 2x+\frac{1}{3}\cos 3x$.

Also, obtain their greatest and the least values in the given interval.

18. Find the minimum and maximum values of

(i) $\sin x\cos 2x$. (ii) $a\sec x+b\,\mathrm{cosec}\, x$, $(0<a<b)$.

(iii) $\sin x\cos^2 x$. (iv) $e^x\cos(x-a)$.

19. Show that $(3-x)e^{2x}-4xe^x-x$ has no maximum or minimum value for $x=0$.

20. Find the maxima and minima of the radii vectors of the curve

$$\frac{c^4}{r^2}=\frac{a^2}{\sin^2\theta}+\frac{b^2}{\cos^2\theta}.$$ *(Delhi, Aligarh)*

21. Find the maximum and minimum values of x^2+y^2 where

$$ax^2+2hxy+by^2=1.$$

[Taking $x=r\cos\theta$, $y=r\sin\theta$, the question reduces to finding the extreme value of r^2, where

$$\frac{1}{r^2}=a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta]$$

7·5. In the following, we shall apply the theory of maxima and minima to solve problems involving the use of the same. It will be seen that, in general, we shall not need to find the second derivative and complete decision would be made at the stage of the first derivative only when we have obtained the stationary values. In this connection, it will be found useful to determine the limits between which the independent variable lies. Suppose that these limits are a, b.

If y is 0 for $x=a$ and $x=b$ and positive otherwise and has only *one* stationary value, then the stationary value is necessarily the maximum and the greatest.

If $y\to\infty$ as $x\to a$ and as $x\to b$ and has only *one* stationary value, then the stationary value is necessarily the minimum and the least.

In connection with the problems concerning spheres, cones and cylinders the following results would be often needed :

1. *Sphere of Radius r.*

   Volume $=\frac{4}{3}\pi r^3$. Surface $=4\pi r^2$.

2. *Cylinder of height, h, and radius of circular base, r.*

   Volume $=\pi r^2h$. Curved surface $=2\pi rh$.

   The area of each plane face $=\pi r^2$.

3. *Cone of height, h, and radius of circular base, r.*

   Semi-vertical angle $=\tan^{-1}(r/h)$,

   Slant height $=\sqrt{(r^2+h^2)}$,

   Volume $=\frac{1}{3}\pi r^2h$,

   Curved surface $=\pi r\sqrt{(r^2+h^2)}$.

Here cones and cylinders are always supposed to be right circular.

7·6. Application to Problems.

Examples

1. *Show that the height of an open cylinder of given surface and greatest volume is equal to the radius of its base.*

Let r be the radius of the circular base ; h, the height ; S, the surface and V, the volume of the open cylinder. Therefore,

$$S=\pi r^2+2\pi rh, \qquad ...(i)$$

$$V=\pi r^2 h. \qquad \text{...}(ii)$$

Here, as given, S is a constant and V is a variable. Also, h, r are variables. Substituting the value of h, as obtained from (i), in (ii), we get

$$V=\pi r^2\left(\frac{S-\pi r^2}{2\pi r}\right)=\frac{Sr-\pi r^3}{2}, \qquad \text{...}(iii)$$

which gives V in terms of *one* variable r.

$$\frac{dV}{dr}=\frac{S-3\pi r^2}{2},$$

so that $dV/dr=0$ only when $r=\sqrt{(S/3\pi)}$: negative value of r being inadmissible. Thus V has only *one* stationary value.

As V must be positive, we have

$$Sr-\pi r^3>0, \textit{i.e.}, Sr>\pi r^3 \text{ or } r>\sqrt{(S/\pi)}.$$

Thus r varies in the interval $(0, \sqrt{(S/\pi)})$.

Now $V=0$ for the points $r=0$ and $\sqrt{(S/\pi)}$, and is positive for every other admissible value of x. Hence V is greatest for $r=\sqrt{(S/3\pi)}$.

Substituting this value of r in (i), we get

$$h=\frac{S-\pi r^2}{2\pi r}=\frac{S-\pi\dfrac{S}{3\pi}}{2\pi\sqrt{\dfrac{S}{3\pi}}}.$$

$$=\frac{2S}{3}\cdot\frac{1}{2\pi}\cdot\sqrt{\frac{3\pi}{S}}=\sqrt{\frac{S}{3\pi}}.$$

Hence $h=r$ for a cylinder of greatest volume and given surface.

2. *Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.*

Let r be the radius OA of the base and h, the height OV of the given cone.

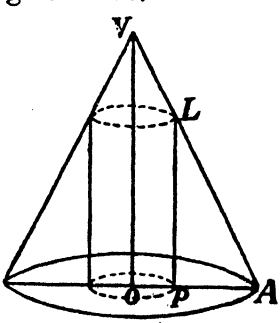


Fig. 51.

We inscribe in it a cylinder, the radius of whose base is $OP=x$, as shown in the figure. We note that x may take up any value between 0 and r.

To determine the height PL, of this cylinder, we have

$$\frac{PL}{OV}=\frac{PA}{OA}, \textit{ i.e.}, \frac{PL}{h}=\frac{r-x}{r}.$$

$$PL=\frac{h(r-x)}{r}$$

If S be the curved surface of the cylinder, we have

$$S=2\pi.\,OP.PL=\frac{2\pi xh(r-x)}{r}=\frac{2\pi h}{r}(rx-x^2),$$

$$\therefore \quad \frac{dS}{dx}=\frac{2\pi h}{r}(r-2x)=0 \text{ for } x=r/2.$$

Thus S has only one stationary value.

Now S is 0 for $x=0$ and $x=r$ and is positive for values of x lying between 0 and r.

Therefore S is greatest for $x=r/2$.

3. *Find the surface of the right circular cylinder of greatest surface which can be inscribed in a sphere of radius r.*

We construct a cylinder as shown in the figure. OA is the radius of the base and CB is the height of this cylinder.

Let $\angle AOB=\theta$, so that θ lies between 0 and $\pi/2$.

$$\because \frac{OA}{OB}=\cos\theta.$$

$$\therefore OA=OB\cos\theta=r\cos\theta.$$

$$\because \frac{AB}{OB}=\sin\theta,$$

$$\therefore AB=OB\sin\theta=r\sin\theta.$$

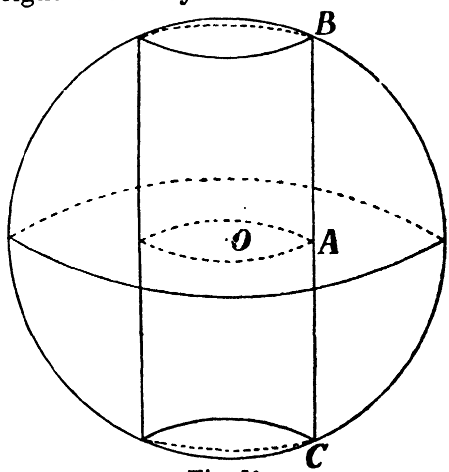


Fig. 52.

If S be the surface, we have

$$S=2\pi.\,OA^2+2\pi.OA.BC.$$

$$=2\pi r^2(\cos^2\theta+\sin 2\theta)\ldots(1)$$

$$\therefore \frac{dS}{d\theta}=2\pi r^2(-2\cos\theta\sin\theta+2\cos 2\theta)$$

$$=2\pi r^2(2\cos 2\theta-\sin 2\theta).$$

$\therefore dS/d\theta=0$ gives

$$2\cos 2\theta-\sin 2\theta=0, \text{ i.e., } \tan 2\theta=2. \quad \ldots(2)$$

Let, θ_1, be a root of $\tan 2\theta=2$.

As $\tan 2\theta_1=2$, $\therefore \sin 2\theta_1=2/\sqrt{5}$ and $\cos 2\theta_1=1/\sqrt{5}$.

From (1), we see that

when $\theta=0$, $S=2\pi r^2$; when $\theta=\pi/2$, $S=0$,

$$\text{when } \theta=\theta_1,\ S=2\pi r^2\left(\frac{1+\cos 2\theta_1}{2}+\sin 2\theta_1\right)$$

$$=\frac{\pi r^2(5+5\sqrt{5})}{5}$$

which is greater than $3\pi r^2$.

Hence $\dfrac{\pi r^2(5+\sqrt{5})}{5}$ is the required greatest surface. (Cor. p. 149).

4. *Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is* $6r\sqrt{3}$. *(P. U. 1934)*

We take one vertex A of the triangle at a distance x from the centre O. Let AO meet the circle at P. The two tangents from A and the tangent at P determine an isosceles triangle ABC circumscribing the given circle. We have $OL=r$.

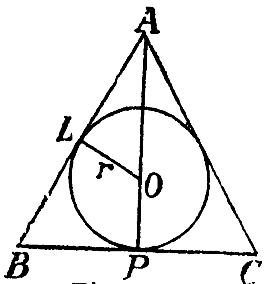


Fig. 53.

$$\therefore \quad AL=\sqrt{(OA^2-OL^2)}=\sqrt{(x^2-r^2)}.$$

Also $BP=AP \tan \angle BAP$

$$=AP \tan \angle LAO$$

$$=AP.\frac{OL}{AL}=(r+x).\frac{r}{\sqrt{(x^2-r^2)}}.$$

If, p, denote the perimeter of the triangle, we have

$$p=AB+AC+BC$$

$$=2AB+2BP$$

$$=2(AL+LB)+2BP$$

$$=2AL+4BP, \text{ (for, } BL=BP)$$

$$=2\sqrt{(x^2-r^2)}+\frac{4(r+x)r}{\sqrt{(x^2-r^2)}}=2\frac{(x+r)^2}{\sqrt{(x^2-r^2)}}. \qquad \text{... } i)$$

$$\therefore \frac{dp}{dx}=2\frac{2(x+r).\sqrt{(x^2-r^2)}-x(x+r)^2(x^2-r^2)^{-\frac{1}{2}}}{x^2-r^2}$$

$$=2\frac{2(x+r)(x^2-r^2)-x(x+r)^2}{(x^2-r^2)^{\frac{3}{2}}}$$

$$=2\frac{(x+r)^2(x-2r)}{(x^2-r^2)^{\frac{3}{2}}}.$$

so that $dp/dx=0$ for $x=2r$; negative value, $-r$, of x being inadmissible.

Now, x may take up any value $>r$ only, so that it varies in the interval (r, ∞).

From (i), we see that $p \to \infty$ as $x \to r$.

Again, dividing the numerator and denominator by x^2, we get

$$p=\frac{2x(1+r/x)^2}{\sqrt{(1-r^2/x^2)}},$$

so that $p \to \infty$ also as $x \to \infty$.

Hence p is least for $x=2r$. Putting this value of x in (i), we see that the least value of p is $6r\sqrt{3}$.

5. *A cone is circumscribed to a sphere of radius r ; show that when the volume of the cone is a minimum, its altitude is 4r and its semi-vertical angle* $\sin^{-1}\frac{1}{3}$. (*Madras 1953 ; P.U. 1930*)

We take the vertex A of the cone at a distance x from the centre O of he sphere. (See Fig. 53. p. 160).

By drawing tangent lines from A, as shown in the figure, we construct the cone circumscribing the sphere.

Let the semi-vertical angle BAP of the cone be θ.

Now, if v be the volume of the cone, we have

$$v=\tfrac{1}{3}\pi \,.\, BP^2 \,.\, AP,$$

which will be now expressed in terms of x. We have

$$AP=AO+OP=r+x.$$

Since $\sin\theta=\dfrac{OL}{OA}=\dfrac{r}{x}$, $\quad\therefore\ \tan\theta=\dfrac{r}{\sqrt{(x^2-r^2)}}$.

Again, since $\dfrac{BP}{AP}=\tan\theta$,

$$\therefore \qquad BP=AP\tan\theta=(r+x)\,.\,\frac{r}{\sqrt{(x^2-r^2)}}.$$

Thus $\qquad v=\pi\dfrac{(r+x)^2r^2}{3(x^2-r^2)}(x+r)=\dfrac{\pi r^2(x+r)^2}{3(x-r)}$.

$$\therefore \qquad \frac{dv}{dx}=\frac{\pi r^2(x+r)(x-3r)}{3(x-r)^2}.$$

Thus dv/dx is 0 for $x=3r$.

Here x can take up any value$\geqslant r$ and $v\to\infty$ when $x\to r$ and when $x\to\infty$. Thus v is minimum and least for $x=3r$.

Hence, for minimum volume, the altitude of the cone

$$=AP=r+3r=4r,$$

and the semi-vertical angle θ

$$=\sin^{-1}\frac{r}{x}=\sin^{-1}\frac{r}{3r}=\sin^{-1}\tfrac{1}{3}.$$

6. *Normal is drawn at a variable point P of an ellipse*

$$x^2/a^2+y^2/b^2=1\ ;$$

find the maximum distance of the normal from the centre of the ellipse. (*P.U. 1935*)

We take any point $P(a\cos\theta, b\sin\theta)$ on the ellipse ; θ being the eccentric angle of the point.

The equation of the tangent at P is $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$.

Therefore, slope of the tangent at $P=-\dfrac{b\cos\theta}{a\sin\theta}$.

Hence, slope of the normal at $P=\dfrac{a\sin\theta}{b\cos\theta}$.

Therefore equation of the normal at P is

$$y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta}(x-a\cos\theta),$$

or $$ax\sin\theta-by\cos\theta=(a^2-b^2)\sin\theta\cos\theta.$$

If, p, be its perpendicular distance from the centre (0, 0), we obtain

$$p=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}}. \qquad ...(i)$$

$$\therefore \qquad \frac{dp}{d\theta}=(a^2-b^2)\frac{b^2\cos^4\theta-a^2\sin^4\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}.$$

Putting $dp/d\theta=0$, we get

$$\tan^4\theta=\frac{b^2}{a^2}, \ i.e., \tan\theta=\pm\sqrt{\left(\frac{b}{a}\right)}$$

Because of the symmetry of the ellipse about the two co-ordinate axes, it is enough to consider only those values of θ which lie between 0 and $\pi/2$ so that we reject the negative value of $\tan\theta$.

Now, $p=0$ when $\theta=0$ or $\pi/2$ and p is positive when θ lies between 0 and $\pi/2$. Therefore p is maximum when $\tan\theta=\sqrt{(b/a)}$. Substituting this value in (i), we see that the maximum value of p is $a-b$.

7. *Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is $\frac{3}{2}$ c miles per hour.*

Let v miles per hour be the velocity of the boat so that $(v-c)$ miles per hour is its velocity relative to water when going against the current.

Therefore the time required to cover a distance of d miles

$$=\frac{d}{v-c} \text{ hours.}$$

The petrol burnt per hour$=kv^3$, where k is a constant. Thus the total amount, y, of petrol burnt is given by

$$y=k\frac{v^3d}{v-c}=kd\frac{v^3}{v-c}.$$

$$\frac{dy}{dv}=kd\frac{3v^2(v-c)-1.v^3}{(v-c)^2}.$$

Putting $dy/dv=0$, we get $v=0$ and $\frac{3}{2}c$. Of these $v=0$ is inadmissible.

Also $y\to\infty$ when $v\to c$ and when $v\to\infty$.

Thus $v=\frac{3}{2}c$ gives the least value of y.

Exercises

1. Divide a number 15 into two parts such that the square of one multiplied with the cube of the other is a maximum.

2. Show that of all rectangles of given area, the square has the smallest perimeter.

3. Find the rectangle of greatest perimeter which can be inscribed in a circle of radius a.

4. If 40 square feet of sheet metal are to be used in the construction of an open tank with a square base, find the dimensions so that the capacity is greatest possible. (*P.U.*)

5. A figure consists of a semi-circle with a rectangle on its diameter. Given that perimeter of the figure is 20 feet, find its dimensions in order that its area may be a maximum. (*Patna, Allahabad*)

6. A, B are fixed points with co-ordinates $(0, a)$ and $(0, b)$ and P is a variable point $(x, 0)$ referred to rectangular axes ; prove that $x^2=ab$ when the angle APB is a maximum. (*P.U. 1935*)

7. A given quantity of metal is to be cast into a half-cylinder, *i.e.*, with a rectangular base and semi-circular ends. Show that in order the total surface area may be minimum the ratio of the length of the cylinder to the diameter of its circular ends is $\pi/(\pi+2)$. (*Aligarh 1949*)

8. The sum of the surfaces of a cube and a sphere is given ; show that when the sum of their volumes is least, the diameter of the sphere is equal to the edge of the cube.

9. The strength of a beam varies as the product of its breadth and the square of its depth. Find the dimensions of the strongest beam that can be cut from a circular log of wood of radius a units. (*D.U. 1953*)

10. The amount of fuel consumed per hour by a certain steamer varies as the cube of its speed. When the speed is 15 miles per hour, the fuel consumed is $4\frac{1}{2}$ tons of coal per hour at Rs. 4 per ton. The other expenses total Rs. 100 per hour. Find the most economical speed and the cost of a voyage of 1980 miles. (*P.U. 1949*)

11. Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$. (*D.U. 1952*)

12. Show that the right circular cylinder of the given surface and maximum volume is such that its height is equal to the diameter of its base.

13. Show that the height of a closed cylinder of given volume and least surface is equal to its diameter.

14. Given the total surface of the right circular cone, show that when the volume of the cone is maximum, then the semi-vertical angle will be $\sin^{-1}\frac{1}{3}$.

15. Show that the right cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of its base.

16. Show that the curved surface of a right circular cylinder of greatest curved surface which can be inscribed in a sphere is one-half of that of the sphere.

17. A cone is inscribed in a sphere of radius r ; prove that its volume as well as its curved surface is greatest when its altitude is $4r/3$.

18. Find the volume of the greatest cylinder that can be inscribed in a cone of height h and semi-vertical angle α. (*D.U. 1955*)

19. A thin closed rectangular box is to have one edge n times the length of another edge and the volume of the box is given to be v. Prove that the least surface s is given by

$$ns^3=54(n+1)^2v^2. \qquad \text{(P.U.)}$$

20. Prove that the area of the triangle formed by the tangent at any point of the ellipse $x^2/a^2+y^2/b^2=1$ and its aves is aminimuu for the point

$$(a/\sqrt{2}, b/\sqrt{2}).$$

21. Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse, the triangle having its vertex coincident with one extremity of the major axis. *(Allahabad 1939)*

22. A perpendicular is let fall from the centre to a tangent to an ellipse. Find the greatest value of the intercept between the point of contact and the foot of the perpendicular.

23. A tangent to an ellipse meets the axes in P and Q ; show that the least value of PQ is equal to the sum of the semi-axes of the ellipse and also that PQ is divided at the point of contact in the ratio of its semi-axes.

24. N is the foot of the perpendicular drawn from the centre O on to the tangent at a variable point P on the ellipse $x^2/a^2+y^2/b^2=1$ $(a>b)$. Prove that the maximum area of the triangle OPN is $(a^2-b^2)/4$.

25. One corner of the rectangular sheet of the paper, width one foot, is folded over so as to just reach the opposite edge of the sheet ; find the minimum length of the crease.

26. If $f'(x)$ exists throughout an interval $a \leqslant x \leqslant b$, prove that the greatest and least value of $f(x)$ in the interval are either $f(a)$ and $f(b)$ or are given by the values of x for which $f'(x)=0$.

A grocer requires cylindrical vessels of thin metal with lids, each to contain exactly a given volume V. Show that if he wishes to be as economical as possible in metal, the radius r of the base is given by $2\pi r^3=V$.

If, for other reasons, it is impracticable to use vessels in which the diameter exceeds three-fourths of the height, what should be the radius of the base of each vessel ? *(P.U.)*

27. A tree trunk, l, feet long is in the shape of a frustum of a cone the radii of its ends being a and b feet $(a>b)$. It is required to cut from it a beam of uniform square section. Prove that the beam of the greatest volume that can be cut is $al/3(a-b)$ feet long. *(Agra ; P.U.)*

28. Find the volume of the greatest right circular cone that can be described by the revolution about a side of a right-angled triangle of hypotenuse 1 foot. *(P.U. 1940)*

29. A rectangular sheet of metal has four equal square portions removed at the corners, and the sides are then turned up so as to form an open rectangular box. Show that when volume contained in the box is a maximum, the depth will be

$$\tfrac{1}{6}\{(a+b)-(a^2-ab+b^2)^{\frac{1}{2}}\},$$

where a, b are the sides of the original rectangle. *(Banaras 1953)*

30. The parcel post regulations restrict parcels to be such that the length plus the girth must not exceed 6 feet and the length must not exceed $3\frac{1}{2}$ feet Determine the parcels of greatest volume that can be sent up by post if the from of the parcel be a right circular cylinder. Will the result be affected if the greatest length permitted were only $1\frac{3}{4}$ feet. *(Patna)*

31. Show that the maximum rectangle inscribable in a circle is a square. *(P.U. Suppl. 1944)*

CHAPTER VIII

EVALUATION OF LIMITS

INDETERMINATE FORMS

8·1. We know that $x \to a$,

$$\lim \frac{f(x)}{F(x)} = \frac{\lim f(x)}{\lim F(x)} \text{ if } \lim F(x) \neq 0,$$

so that this theorem on limits fails to give any information regarding the limit of a fraction whose denominator tends to zero as its limit.

Now, suppose, that the denominator $F(x) \to 0$ as $x \to a$.

The numerator $f(x)$ may or may not tend to zero. If it does not tend to zero, then $f(x)/F(x)$ *cannot* tend to any *finite* limit. For, if possible, let it tend to finite limit, say l. We write

$$f(x) = \frac{f(x)}{F(x)} F(x),$$

so that, in this case, we have

$$\lim f(x) = \lim \left[\frac{f(x)}{F(x)} \cdot F(x) \right] = \lim \frac{f(x)}{F(x)} \cdot \lim F(x) = l.0 = 0.$$

Thus we have a contradiction.

Three types of behaviour are possible in this case. The fraction may tend to $+\infty$, or $-\infty$, or its limit may not exist. For example, when $x \to 0$, (so that the limit of the denominator in each of the following cases is 0), we see that

(*i*) $\lim (1/x^2) = +\infty$; (*ii*) $\lim (1/-x^2) = -\infty$;

(*iii*) $\lim (1/x)$ does not exist.

The case where the limits of the numerator of a fraction is also zero is more important and interesting. A general method of determining the limit of such a fraction will be given in this chapter. For the sake of brevity, we say that a *fraction whose numerator and denominator both tend to zero as x tends to a, assumes the* **indeterminate form 0/0 for x=a.**

It may be of interest to notice that the determination of the differential co efficient dy/dx is itself equivalent to finding the limit of a fraction $\delta y/\delta x$ which assumes an indeterminate form $0/0$.

Other cases of limits which are reducible to this form will also be considered in this chapter.

In what follows it will always be assumed that $f(x)$, $F(x)$ and $\phi(x)$ possess continuous derivatives of every order that come in question in a certain interval enclosing $x = a$.

8·2. The Indeterminate form $\frac{0}{0}$. *To determine*

$$\lim_{x \to a} [f(x)/F(x)]$$

when

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} F(x).$$

As $f(x)$, $F(x)$ are assumed to be continuous for $x=a$, we have

$$f(a) = \lim_{x \to a} f(x) = 0 \; ; \; F(a) = \lim_{x \to a} F(x) = 0.$$

By Taylor's theorem, with remainder after one term, we have

$$f(a+h) = f(a) + hf'(a+\theta_1 h) \quad = hf'(a+\theta_1 h),$$
$$F(a+h) = F(a) + hF'(a+\theta'_1 h) = hF'(a+\theta'_1 h).$$

Hence

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{h \to 0} \frac{f(a+h)}{F(a+h)}$$
$$= \lim_{h \to 0} \frac{f'(a+\theta_1 h)}{F'(a+\theta'_1 h)} = \frac{f'(a)}{F'(a)}, \text{ if } F'(a) \neq 0.$$

This argument fails if $F'(a)=0$. The case when $F'(a)=0$ but $f'(a) \neq 0$ has already been discussed in § 8·1.

Now, let $f'(a) = F'(a) = 0$.

Again, by Taylor's theorem with remainder after two terms, we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta_2 h) = \frac{h^2}{2!} f''(a+\theta_2 h),$$

$$F(a+h) = F(a) + hF'(a) + \frac{h^2}{2!} F''(a+\theta'_2 h) = \frac{h^2}{2!} F''(a+\theta'_2 h).$$

Hence

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{h \to a} \frac{f(a+h)}{F(a+h)}$$
$$= \lim_{h \to 0} \frac{f''(a+\theta_2 h)}{F''(a+\theta'_2 h)} = \frac{f''(a)}{F''(a)}, \text{ if } F''(a) \neq 0.$$

The case of failure which arises when $F''(a)=0$ can be examined as before. In general, let

$$f(a) = f'(a) = f''(a) = \ldots\ldots = f^{n-1}(a) = 0$$
$$F(a) = F'(a) = F''(a) = \ldots\ldots = F^{n-1}(a) = 0 \; ;$$

and

$$F^n(a) \neq 0.$$

By Taylor's theorem, with remainder after n terms, we get

$$f(a+h)=f(a)+hf'(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)\frac{h^n}{n!}\,f^n(a+\theta_n h)$$

$$=\frac{h^n}{n!}f^n(a+\theta_n h).$$

$$F(a+h)=F(a)+hF'(a)+\ldots+\frac{h^{n-1}}{(n-1)!}F^{n-1}(a)+\frac{h^n}{n!}\,F^n(a+\theta'_n h)$$

$$=\frac{h^n}{n!}\,F^n(a+\theta'_n h)$$

Hence

$$\lim_{x\to a}\frac{f(a)}{F(x)}=\lim_{h\to 0}\frac{f(a+h)}{F(a+h)}$$

$$=\lim_{h\to 0}\frac{f^n(a+\theta_n h)}{F^n(a+\theta'_n h)}=\frac{f^n(a)}{F^n(a)}\text{, for } F^n(a)\neq 0.$$

Ex. 1. *Determine* $\lim \dfrac{e^x-e^{-x}-2\log(1+x)}{x\sin x}$, *where* $x\to 0$.

$$\begin{cases} f(x)=e^x-e^{-x}-2\log(1+x), & \therefore\quad f(0)=0,\\ F(x)=x\sin x\,; & \therefore\quad F(0)=0.\end{cases}$$

$$\begin{cases} f'(x)=e^x+e^{-x}-\dfrac{2}{1+x}, & \therefore\quad f'(0)=0,\\ F'(x)=x\cos x+\sin x. & \therefore\quad F'(0)=0.\end{cases}$$

$$\text{Again}\ \begin{cases} f''(x)=e^x-e^{-x}+\dfrac{2}{(1+x)^2}, & \therefore\quad f''(0)=2,\\ F''(x)=-x\sin x+2\cos x, & \therefore\quad F''(0)=2.\end{cases}$$

$$\therefore\quad \lim_{x\to 0}\frac{e^x-e^{-x}-2\log(1+x)}{x\sin x}=\frac{2}{2}=1.$$

The process may be conveniently exhibited as follows :—

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2\log(1+x)}{x\sin x}$$

$$=\lim_{x\to 0}\frac{e^x+e^{-x}-2/(1+x)}{x\cos x+\sin x}$$

$$=\lim_{x\to 0}\frac{e^x-e^{-x}+2/(1+x)^2}{-x\sin x+2\cos x}=\frac{2}{2}=1.$$

Ex. 2. *Find the values of a and b in order that*

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}$$

ay be equal to 1. *(D.U. 1944, 1959)*

The function is of the form (0/0) for all values of a and b.

$$\therefore \quad \lim_{x \to 0} \frac{x(1+a\cos x)-b\sin x}{x^3}$$

$$= \lim_{x \to 0} \frac{1+a\cos x-ax\sin x-b\cos x}{3x^2}.$$

The denominator being 0 for $x=0$, the fraction will tend to a finite limit if and only if the numerator is also zero for $x=0$. This requires

$$1+a-b=0. \qquad \ldots(1)$$

Again supposing this relation satisfied, we have

$$\lim_{x \to 0} \frac{1+a\cos x-ax\sin x-b\cos x}{3x^2}$$

$$= \lim_{x \to a} \frac{-2a\sin x-ax\cos x+b\sin x}{6x}$$

$$= \lim_{x \to 0} \frac{-3a\cos x+ax\sin x+b\cos x}{6}$$

$$= \frac{b-3a}{6}.$$

As given, $\dfrac{b-3a}{6}=1$, *i.e.*, $b-3a=6$. $\qquad \ldots(2)$

From (1) and (2), we have

$$a=-\frac{5}{2}, \quad b=-\frac{3}{2}.$$

Ex. 3. Determine the limits of the following :

(*i*) $\dfrac{1+\log x-x}{1-2x+x^2}$, $(x \to 1)$. (*ii*) $\dfrac{\cos^2 \pi x}{e^{2x}-2ex}$, $(x \to \frac{1}{2})$. (*P.U.*)

(*iii*) $\dfrac{\sinh x-x}{\sin x-x\cos x}$, $(x \to 0)$. (*iv*) $\dfrac{xe^x-\log(x+1)}{\cosh x-\cos x}$, $(x \to 0)$.

(*v*) $\dfrac{\log(1-x^2)}{\log\cos x}$. $(x \to 0)$. (*vi*) $\dfrac{e^{ax}-e^{-ax}}{\log(1+bx)}$, $(x \to 0)$.

(*D.U. Hons. 1952*)

Ex. 4. Evaluate the following :

(*i*) $\displaystyle\lim_{x \to 0} \frac{xe^x-\log(1+x)}{x^2}$. (*D.U. 1952*) (*ii*) $\displaystyle\lim_{x \to 0} \frac{x\cos x-\log(1+x)}{x^2}$.

(*D. U. Hons. 1951, P.U. 1957*)

(*iii*) $\displaystyle\lim_{x \to 0} \frac{e^x\sin x-x-x^2}{x^2+x\log(1-x)}$. (*D.U. 1953*) (*iv*) $\displaystyle\lim_{x \to 0}\left\{\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right\}$.

(*D.U. 1955*)

Ex. 5. If the limit of

$$\frac{\sin 2x + a\sin x}{x^3},$$

as x tends to zero, be finite, find the value of a and the limit. (*P.U.*)

8·3. Preliminary transformation. Sometimes a preliminary transformation involving the use of known results on limits, such as

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{\tan x}{x} = 1$$

simplifies the process a good deal. These limits may also be used to shorten the process at an intermediate stage.

Ex. 1. *Find lim* $\dfrac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$, $(x \to 0)$.

The inconvenience of continuously differentiating the denominator, which involves $\tan^2 x$ as a factor, may be partially avoided as follows. We write

$$\frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$$

$$= \frac{1+\sin x-\cos x+\log(1-x)}{x^3} \cdot \left(\frac{x}{\tan x}\right)^2$$

$$\therefore \quad \lim_{x\to 0} \frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$$

$$= \lim_{x\to 0} \frac{1+\sin x-\cos x+\log(1-x)}{x^3} \cdot \lim_{x\to 0}\left(\frac{x}{\tan x}\right)^2$$

$$= \lim_{x\to 0} \frac{1+\sin x-\cos x+\log(1-x)}{x^3} \cdot 1$$

$$= \lim_{x\to 0} \frac{1+\sin x-\cos x+\log(1-x)}{x^3}.$$

To evaluate the limit on the R.H.S., we notice that the numerator and denominator both become 0 for $x=0$.

$$\therefore \quad \lim_{x\to 0} \frac{1+\sin x-\cos x+\log(1-x)}{x^3}$$

$$= \lim_{x\to 0} \frac{\cos x+\sin x-\dfrac{1}{1-x}}{3x^2}, \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-\sin x+\cos x-\dfrac{1}{(1-x)^2}}{6x} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-\cos x-\sin x-\dfrac{2}{(1-x)^3}}{6} = -\frac{3}{6} = -\frac{1}{2}.$$

Ex. 2. *Evaluate the following* :—

$$\lim_{x\to 0} \frac{\cosh x - \cos x}{x \sin x}.$$

We have

$$\frac{\cosh x-\cos x}{x \sin x}=\frac{\cosh x-\cos x}{x^2}\cdot\frac{x}{\sin x}.$$

$$\therefore \quad \lim_{x\to 0}\frac{\cosh x-\cos x}{x\sin x}=\lim_{x\to 0}\frac{\cosh x-\cos x}{x^2}\cdot\lim_{x\to 0}\frac{x}{\sin x}$$

$$=\lim_{x\to 0}\frac{\cosh x-\cos x}{x^2}.$$

Since the numerator and denominator are both zero for $x=0$, therefore

$$\lim_{x\to 0}\frac{\cosh x-\cos x}{x^2}=\lim_{x\to 0}\frac{\sinh x+\sin x}{2x}$$

$$=\lim_{x\to 0}\frac{\cosh x+\cos x}{2}$$

$$=1.$$

Ex. 3. Determine the limits of the following functions :—

(*i*) $\dfrac{x^2+2\cos x-2}{x\sin^3 x}$, $(x\to 0)$.

(*ii*) $\dfrac{\sin x-\log(e^x\cos x)}{x\sin x}$, $(x\to 0)$.

(*iii*) $\dfrac{x-\log(1+x)}{1-\cos x}$, $(x\to 0)$. (*iv*) $\log(1-x)\cot \pi x$, $(x\to 0)$.

(*v*) $\dfrac{\sin 2x+2\sin^2 x-2\sin x}{\cos x-\cos^2 x}$, $(x\to 0)$.

(*vi*) $\dfrac{\tan x-x}{x^2\tan x}$, $(x\to 0)$. (*vii*) $\dfrac{\tan^2 x-x^2}{x^2\tan^2 x}$, $(x\to 0)$.

8·4. The Indeterminate form $\dfrac{\infty}{\infty}$. *To determine*

$$\lim_{x\to a}\,[f(x)/F(x)],$$

when

$$\lim_{x\to a} f(x)=\infty=\lim_{x\to a} F(x).$$

*Let $f'(x)/F'(x)\to l$ as $x\to a$. *It will be shown that $f(x)/F(x)$ also $\to l$.*

Suppose, $x>a$. As $f'(x)/F'(x)\to l$, when $x\to a$, we make it arbitrarily near l by taking x sufficiently near a, *say between a, and $a+\delta$*.

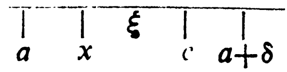
[*Another proof is given in the following sub-section.]

We now take any two numbers c and x which lie between a and $a+\delta$ and apply Cauchy's mean value theorem $f(x)$ and $F(x)$ for the interval (c, x). We thus have

$$\frac{f(x)-f(c)}{F(x)-F(c)}=\frac{f'(\xi)}{F'(\xi)}, \qquad \dots (i)$$

where ξ lies between c and x and, therefore, between a and $a+\delta$. We re-write (i) as

$$\frac{f(x)}{F(x)}\left[\frac{1-f(c)/f(x)}{1-F(c)/F(x)}\right]=\frac{f'(\xi)}{F'(\xi)}$$



or

$$\frac{f(x)}{F(x)}=\frac{1-F(c)/F(x)}{1-f(c)/f(x)}\cdot\frac{f'(\xi)}{F'(\xi)}.$$

Keeping c fixed, we make $x \to a$. Then $f(c)$ and $F(c)$ are fixed and, by our hypothesis, $f(x)$ and $F(x)$ tend to infinity.

Therefore, $\dfrac{1-F(c)/F(x)}{1-f(c)/f(x)} \to 1$ as $x \to \infty$. Thus $f(x)/F(x)$ can be made arbitrarily near l by taking x sufficiently near a so that it $\to l$ as $x \to a$ through values greater than a.

Similarly, it may be shown that $f(x)/F(x) \to l$ as $x \to a$ through values less than a.

Hence

$$\lim_{x\to a}\frac{f(x)}{F(x)}=\lim_{x\to a}\frac{f'(x)}{F'(x)},$$

when

$$\lim_{x\to a} f(x)=\infty=\lim_{x\to a} F(x).$$

Note. The above investigation rests on the hypothesis that $f'(x)/F'(x)$ tends to a limit as $x \to a$. This part of the hypothesis necessarily implies that $f'(x)/F'(x)$ has a meaning for values of x near a so that $f'(x)$, $F'(x)$ both exist and $F'(x)\neq 0$ for such values. This justifies the use of Cauchy's mean value theorem in the above investigation.

8·41. *A proof of the above result is also often given as follows :—*

$\because \quad \lim f(x)=\infty$ and $\lim F(x)=\infty$,

$\therefore \quad \lim \dfrac{1}{f(x)}=0$ and $\lim \dfrac{1}{F(x)}=0.$

Now
$$\lim\frac{f(x)}{F(x)}=\lim\frac{\dfrac{1}{F(x)}}{\dfrac{1}{f(x)}} \qquad \left(\frac{0}{0}\right)$$

$$=\lim\frac{-\dfrac{F'(x)}{[F(x)]^2}}{-\dfrac{f'(x)}{[f(x)]^2}}$$

$$=\lim\left(\left[\frac{f(x)}{F(x)}\right]^2\frac{F'(x)}{f(x)}\right)=\lim\left[\frac{f(x)}{F(x)}\right]^2\lim\frac{F'(x)}{f'(x)}. \quad ...(1)$$

Let

$$\lim\,[f(x)/F(x)]=l \text{ where } l\neq 0 \text{ and } l\neq\infty.$$

$\therefore$ from (1), we get

$$l=l^2\lim\frac{F'(x)}{f'(x)},$$

or
$$\lim\frac{f'(x)}{F'(x)}=l=\lim\frac{f(x)}{F(x)}. \quad ...(2)$$

Supposing now $l=0$.

$$\therefore\quad l+1=\lim\frac{f(x)}{F(x)}+1=\lim\frac{f(x)+F(x)}{F(x)}.$$

Applying the above result,

$$l+1=\lim\frac{f(x)+F(x)}{F(x)}=\lim\frac{f'(x)+F'(x)}{F'(x)}$$

$$=\lim\frac{f'(x)}{F'(x)}+1,$$

or
$$l=\lim\frac{f'(x)}{F'(x)}. \quad ...(3)$$

Finally let $l=\infty$ so that

$$\lim\frac{F(x)}{f(x)}=0.$$

By the preceding result,

$$0=\lim\frac{F(x)}{f(x)}=\lim\frac{F'(x)}{f'(x)}.$$

$$\therefore\quad \lim\frac{f'(x)}{F'(x)}=\infty. \quad ...(4)$$

Hence we see that always

$$\lim\frac{f(x)}{F(x)}=\lim\frac{f'(x)}{F'(x)}, \quad ...(5)$$

when $\lim f(x)=\infty=\lim F(x)$.

Note 1. The above second proof is incomplete in the sense that it assumes that $\lim\,[f(x)/F(x)]$ exists. The first proof did not assume this existence.

Note 2. While evaluating

$$\lim\frac{f(x)}{F(x)} \text{ when it is of the form } \frac{\infty}{\infty},$$

we must try to change over to the form 0/0 as soon as it may be conveniently possible, for, otherwise we may go on indefinitely without ever arriving at the end of the process.

Ex. 1. *Determine* $\lim \dfrac{\log (x-a)}{\log (e^x-e^a)}$, as $x \to a$.

Here, the numerator and the denominator both tend to ∞ as x tends to a.

$$\therefore \quad \lim_{x \to a} \frac{\log (x-a)}{\log (e^x-e^a)} = \lim_{x \to a} \frac{\dfrac{1}{x-a}}{\dfrac{e^x}{e^x-e^a}}$$

$$= \lim_{x \to a} \frac{e^x-e^a}{e^x(x-a)} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{e^x}{e^x(x-a)+e^x} = \frac{e^a}{e^a} = 1.$$

Ex. 2. Determine the limits of the following functions :

(*i*) $\dfrac{\log \sin x}{\cot x}$; $(x \to 0)$. (*ii*) $\dfrac{\tan x}{\tan 3x}$, $\left(x \to \dfrac{\pi}{2}\right)$.

(*iii*) $\dfrac{\log \tan x}{\log x}$, $(x \to 0)$. (*iv*) $\dfrac{\log \tan 2x}{\log \tan x}$, $(x \to 0)$.

(*v*) $\log_{\tan x} \tan 2x$, $(x \to 0)$.

$\left[\textbf{Hint.} \ \log_{\tan x} \tan 2x = \dfrac{\log \tan 2x}{\log \tan x}.\right]$

(*vi*) $\log (1-x) \cot (x\pi/2)$, $(x \to 1)$.

8·6. The indeterminate form $0 \cdot \infty$. *To determine*

$$\lim_{x \to a} [f(x). F(x)],$$

when

$$\lim_{x \to a} f(x)=0, \quad \lim_{x \to a} F(x)=\infty.$$

To determine this limit, we write

$$f(x).F(x)=\frac{f(x)}{1/F(x)}=\frac{F(x)}{1/f(x)},$$

so that these new forms are of the type $0/0$ and ∞/∞ respectively and the limit can, therefore, be obtained by § 8·2 or by § 8·4.

In this case, we say that $f(x).F(x)$ assumes the indeterminate form $0 \cdot \infty$ at $x=a$.

Ex. 1. *Determine* $\lim (x \log x)$, *as* $x \to 0$.

We write

$$x \log x = \frac{\log x}{1/x}.$$

$$\therefore \quad \lim_{x \to 0} (x \log x) = \lim_{x \to 0} \frac{\log x}{1/x}, \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\dfrac{1}{x}}{\dfrac{-1}{x^2}} = \lim_{x \to 0} (-x) = 0.$$

The reader may see that writing

$$x \log x = \frac{x}{(1/\log x)}$$

which is of the form (0/0) and employing the corresponding result of § 8·2 would not be of any avail.

Note. We know that $1/x$ does not tend to a limit as $x \to 0$. In fact we have

$$\lim_{x\to(0+0)} \frac{1}{x} = \infty, \qquad \lim_{x\to(0-0)} \frac{1}{x} = - \quad .$$

Again, $\log x$ is defined for positive values of x only so that there is no question of making $x \to 0$ through negative values while determining

$$\lim_{x\to 0} (x \log x).$$

Thus, here $x \to 0$ really means $x \to (0+0)$ so that $1/x$ does tend to a limit.

Ex. 2. Determine the limits of the following functions :

(*i*) $x \log \tan x$, $(x \to 0)$. (*ii*) $x \tan (\pi/2 - x)$, $(x \to 0)$.

(*iii*) $(a-x) \tan (\pi x/2a)$, $(x \to 0)$.

8·6. The Indeterminate for $\infty - \infty$. *To determine*

$$\lim_{x\to a} [f(x) - F(x)],$$

when

$$\lim_{x\to a} f(x) = \infty = \lim_{x\to a} F(x)$$

We write

$$f(x) - F(x) = \left[\frac{1}{F(x)} - \frac{1}{f(x)}\right] \div \frac{1}{f(x)\,F(x)},$$

so that the numerator and denominator both tend to 0 as x tends to a. The limit may now be determined with the help of § 8·2.

In this case, we say that $[f(x) - F(x)]$ assumes the indeterminate form $\infty - \infty$ for $x = a$.

Note. In order to evaluate the limit of a function which assumes the form, $\infty - \infty$, it is necessary to express the same as a function which assumes the form 0/0 or ∞/∞.

Ex. 1. *Determine*

$$\lim \left\{\frac{1}{x-2} - \frac{1}{\log (x-1)}\right\} \text{ as } x \to 2.$$

We write

$$\frac{1}{x-2} - \frac{1}{\log (x-1)} = \frac{\log (x-1) - (x-2)}{(x-2) \log (x-1)},$$

and see that the new form is of the type 0/0 when $x \to 2$.

$$\lim_{x\to 2} \left[\frac{1}{x-2} - \frac{1}{\log (x-1)}\right] = \lim_{x\to 2} \frac{\log (x-1) - (x-2)}{(x-2) \log (x-1)}$$

The numerator and the denominator are both 0 for $x=2$. On using the method of § 8·2, we may show that the required limit is $-\frac{1}{2}$.

Ex. 2. Determine the limits of the following functions :

(*i*) $\left(\frac{1}{x}-\frac{1}{x^x-1}\right)$, $(x \to 0)$.

(*ii*) $\left(\frac{a}{x}-\cot\frac{x}{a}\right)$. $(x \to 0)$. $\left[\text{Write } \cot\frac{x}{a}=\frac{\cos(x/a)}{\sin(x/a)}\right]$

(*iii*) $\left(\frac{1}{x^2}-\cot^2 x\right)$, $(x \to 0)$.

(*iv*) $\frac{\pi x-1}{2x^3}+\frac{\pi}{(e^{2\pi x}-1)x}$, $(x \to 0)$.

8·7. The Indeterminate forms, 0^0, 1^∞, ∞^0. *To determine*

$$\lim\left[f(x)^{F(x)}\right] \text{ as } x \to a,$$

when

(*i*) $\lim f(x)=0$; $\lim F(x)=0$.

(*ii*) $\lim f(x)=0$; $\lim F(x)=\infty$.

(*iii*) $\lim f(x)=\infty$; $\lim F(x)=0$.

We write

$$y=[f(x)]^{F(x)},$$

so that

$$\log y=F(x).\log f(x).$$

In each of the three cases, we see that the right hand side assumes the indeterminate form $0.\infty$ and its limit may, therefore, be determined by the method given in § 8·5.

Let $\lim\limits_{x \to a} [F(x).\log f(x)]=l$.

$\therefore$ $\lim \log y=l$,

or $\log \lim y=l$ or $\lim y=e^l$.

Hence $\lim\left[f(x)^{F(x)}\right]=e^l$

or brevity, we say that $\left[f(x)^{F(x)}\right]$ assumes the indeterminate forms 0^0, 1^∞, ∞^0 respectively for $x=a$.

Ex. 1. *Determine*

$$\lim (x-a)^{x-a} \text{ as } x \to a. \qquad (0^0 \text{ form}).$$

Let $\qquad y=(x-a)^{x-a}.$

$$\therefore \qquad \log y=(x-a)\log(x-a)=\frac{\log (x-a)}{1/(x-a)} \cdot \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x \to a} \log y = \lim_{x \to a} \frac{\log (x-a)}{1/(x-a)}$$

$$= \lim_{x \to a} \frac{\dfrac{1}{x-a}}{\dfrac{-1}{(x-a)^2}} = \lim_{x \to a} -(x-a)=0.$$

Hence $\log \lim y=0$, *i.e.*, $\lim y=e^0=1$

Thus $\qquad \lim (x-a)^{x-a}=1$ when $x \to a$.

Note. Here it is understood that x tends to a through values greater than a, for otherwise the base $(x-a)$ would be negative and $(x-a)^{x-a}$ would have no meaning.

Ex. 2. *Determine*

$$\lim (\cos x)^{1/x^2} \text{ as } x \to 0.$$

Let $\qquad y=(\cos x)^{1/x^2}.$

$$\therefore \qquad \log y=\frac{\log \cos x}{x^2}. \qquad \left(\frac{0}{0}\right)$$

$$\therefore \qquad \lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log \cos x}{x^2}$$

$$= \lim_{x \to 0} \frac{-\tan x}{2x} = -\frac{1}{2}.$$

$$\therefore \qquad \log (\lim_{x \to 0} y) = -\tfrac{1}{2} \text{ or } \lim_{x \to 0} y=e^{-\frac{1}{2}}$$

Hence
$$\lim_{x \to 0} (\cos x)^{1/x^2} = e^{-\frac{1}{2}}.$$

Ex. 3. Determine the limits of the following functions :—

(*i*) $x^x, (x \to 0)$. (*ii*) $x^{(1-x)^{-1}}, (x \to 1)$. (*P.U. 1923*)

(*iii*) $(\cot x)^{\sin 2x}, (x \to 0)$. (*iv*) $(\sin x)^{\tan x}, (x \to \pi/2)$.

(v) $\left(\dfrac{2x+1}{x+1}\right)^{x^{-1}}$, $(x \to 0)$. (vi) $(1+\sin x)^{\cot x}$, $(x \to 0)$.

(vii) $\left(\dfrac{\sinh x}{x}\right)^{x^{-2}}$, $(x \to 0)$. (*D.U. 1949*)

(viii) $x^{\frac{1}{x-1}}$, $(x \to 1)$. (*P.U. 1951*)

Exercises

Determine the limits of the following functions :

1. $\dfrac{e^x - e^{-x} - x}{x^2 \sin x}$, $(x \to 0)$.

2. $\dfrac{\log x}{x^3}$, $(x \to \infty)$.

3. $\dfrac{1 + x\cos x - \cosh x - \log(1+x)}{\tan x - x}$, $(x \to 0)$.

4. $\dfrac{\log(1+x)\log(1-x) - \log(1-x^2)}{x^4}$, $(x \to 0)$.

5. $\left(\dfrac{x-1}{2x^2} + \dfrac{e^{-x}}{2x \sinh x}\right)$, $(x \to 0)$.

6. $\dfrac{1 - x + \log x}{1 - \sqrt{(2x - x^2)}}$, $(x \to 1)$.

7. $(2x \tan x - \pi \sec x)$, $(x \to \pi/2)$.

8. $(\cot x)^{\frac{1}{\log x}}$, $(x \to 0)$.

9. $\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$, $(x \to 0)$.

10. $\dfrac{(x - 4x^4)^{\frac{1}{2}} - (x/4)^{\frac{1}{3}}}{1 - (8x^3)^{\frac{1}{4}}}$, $(x \to \frac{1}{2})$.

11. $\dfrac{\log(1 + x\sin x)}{\cos x - 1}$, $(x \to 0)$.

12. $(\sin x)^{\tan^2 x}$, $(x \to \pi/2)$.

13. $(\cos ax)^{b/x^2}$, $(x \to 0)$.

14. $(1-x^2)^{1/\log(1-x)}$, $(x \to 1)$.

15. $\left[\dfrac{2(\cosh x - 1)}{x^2}\right]^{1/x^2}$, $(x \to 0)$.

16. $\left(2 - \dfrac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$, $(x \to a)$. (*B.U. 1953*)

17. $\left(\sin^2 \dfrac{\pi}{2 - ax}\right)^{\sec^2 \frac{\pi}{2 - bx}}$, $(x \to 0)$.

18. $\dfrac{a^{\sin x} - a}{\log \sin x}$, $(x \to \pi/2)$.

19. $\dfrac{x^x - x}{x - 1 - \log x}$, $(x \to 1)$.

20. $(\sec x)^{\cot x}$, $(x \to \pi/2)$.

21. $(2-x)^{\tan \pi x/2}$, $(x \to 1)$.

22. $\dfrac{1-4\sin^2(\pi x/6)}{1-x^2}$, $(x \to 1)$.

23. $\dfrac{a^b-b^a}{a^a-b^b}$, $(a \to b)$.

24. $\left(\dfrac{a_1^x+a_2^x+\ldots+a_n^x}{n}\right)^{1/x}$, $(x \to 0)$.

25. $\dfrac{(1+x)^{1/x}-e}{x}$, $(x \to 0)$.

26. $\dfrac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}$, $(x \to 0)$.

(*D.U. Hons. 1959*)

[For solutions of Ex. 25 and Ex. 26 by Infinite series refer § 9·5, p. 185]

27. $\dfrac{\log_{\sec \frac{1}{2}x} \cos x}{\log_{\sec x} \cos \frac{1}{2}x}$, $(x \to 0)$.

28. $x\left[\left(1+\dfrac{a}{x}\right)^x-e^a\right]$, $(x \to \infty)$.

29. $x\left[\left(1-\dfrac{a}{x}\right)^{-x}-\left(1+\dfrac{a}{x}\right)^x\right]$, $(x \to \infty)$.

30. $\left[\dfrac{f'(x)}{f(x)-f(a)}-\dfrac{1}{x-a}\right]$, $(x \to a)$.

31. Evaluate

$$\lim_{x \to 0} \frac{\sin x \sin^{-1}x-x^2}{x^6}.$$

32. If

$$f(x)=e^{-1/x^2}, x \neq 0$$
$$f(0)=0,$$

show that the derivative of every order of $f(x)$ vanishes for $x=0$, *i.e.*,

$$f^n(0)=0, \text{ for all } n.$$

33. Discuss the continuity of $f(x)$ at the origin when

$$f(x)=x \log \sin x \text{ for } x \neq 0 \text{ and } f(0)=0.$$

(*D.U. Hons. 1955*)

CHAPTER IX

TAYLOR'S INFINITE SERIES

EXPANSIONS OF FUNCTIONS

9·1. Infinite Series. Its convergence and sum.

Let

$$u_1, u_2, u_3, \ldots\ldots\ldots, u_n \ldots\ldots\ldots$$

be an infinite set of numbers given according to some law. Then a symbol of the form

$$u_1 + u_2 + u_3 + \ldots\ldots + u_n + \ldots\ldots\ldots$$

is called an **infinite series**. Here each term is followed by another so that there is no last term.

If we add the first two terms of this infinite series, and then add the sum so obtained to the third, and thus go on adding each term to the sum of the previous term, we see that, as there is no last term of the series, we will never arrive at the end of this process. In the case of a *finite series*, however, this process of addition will be completed at some stage, howsoever large a number of terms the series may consist of.

Thus, in the ordinary sense, the expression '*Sum of an infinite series*', has no meaning. A meaning is assigned to this expression by employing the notion of limit in the manner we now describe.

Let S_n denote the sum of the first n terms of the series so that S_n is a function of the positive integral variable n. If S_n *tends to a finite limit* S, *as* n *tends to infinity, then the series is said to be convergent and* S *is said to be its sum.*

In case, S_n does not tend to a finite limit, we say that the series does not converge.

The question of the sum of a non-convergent series does not arise.

We may find an approximate value of the sum of a convergent infinite series by adding a sufficiently large number of its terms.

Illustrations. Consider the infinite geometric series

$$1 + r + r^2 + r^3 + \ldots\ldots + r^n + \ldots\ldots$$

We know that

$$S_n = \frac{1-r^n}{1-r}, \qquad \text{when } r \neq 1;$$

$$S_n = n \qquad \text{when } r = 1.$$

We have now to examine $\lim S_n$ when $n \to \infty$.
[Refer § 3·61, p. 57].

For $r=1$, $S_n=n$ which tends to ∞.

For $|r| < 1$, $\lim r^n=0$, so that $\lim S_n=1/(1-r)$.

For $r>1$, $\lim r^n=\infty$ and, therefore, $\lim S_n=\infty$.

For $r \leqslant 1$, $\lim r^n$ and, therefore, $\lim S_n$ does not exist.

Hence, *we see that the infinite geometric series converges if and only if* $|r| < 1$, *and the sum of the infinite series then, is* $1/(1-r)$.

9·2. Taylor's infinite series. We suppose that a given function $f(x)$ possesses derivatives of every order in an interval $[a, a+h]$.

Then, however large a positive integer n may be, there exists a number θ, lying between 0 and 1, such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+R_n$$

where
$$R_n=\frac{h^n}{n!}f^n(a+\theta h).$$

(Taylor's development with Lagrange's form of remainder.)

We write

$$S_n=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$

so that

$$f(a+h)=S_n+R_n.$$

Suppose that $R_n \to 0$, as $n \to \infty$. It is then clear that

$$\lim_{n\to\infty} S_n=f(a+h),$$

so that we see that the series

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\ldots\ldots$$

converges and that its sum is equal to $f(a+h)$.

Thus we have proved that

(*i*) *if $f(x)$ possesses derivatives of every order in the interval $[a, a+h]$ and*

(*ii*) *the remainder*

$$\frac{h^n}{n!}f^n(a+\theta h),$$

tends to 0 as n tends to infinity, then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots.+\frac{h^n}{n!}f^n(a)+\ldots\ldots \quad \ldots(1)$$

This is known as Taylor's theorem for the development of $f(a+h)$ in an infinite series of ascending integral powers of h, *i.e.*, power series in h.

The series (1) is known as *Taylor's series.*

9·3. Maclaurin's infinite series. Putting 0 for a and x for h in the Taylor's infinite series, we see that if

(*i*) *$f(x)$ possesses derivatives of every order in the interval $[0, x]$ and*

(*ii*) *the remainder*

$$\frac{x^n}{n!} f^n(\theta x)$$

tends to 0 as n tends to infinity, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!} f''(0)+\ldots\ldots+\frac{x^n}{n!} f^n(0)+\ldots \qquad \ldots(2)$$

This is known as Maclaurin's theorem for the development or expansion of $f(x)$ in an infinite series of ascending integral powers of x, *i.e.*, power series in x.

The series (2) is known as *Maclaurin's series.*

Note. It may be seen that instead of considering Lagrange's form of remainder, we may as well consider Cauchy's form.

9·4. Formal expansion of functions. We have seen that in order to find out if any given function can be expanded as an infinite Taylor and Maclaurin series it is necessary to examine the behaviour of R_n as n tends to infinity. To put down R_n, we require to obtain the general expression for the nth derivative of the function, so that we fail to apply Taylor's or Maclaurin's theorem to expand in a power series a function for which a general expression for the nth derivative cannot be determined. Other more powerful methods have, accordingly, been discovered to obtain such expansions whenever they are possible. But to deal with these methods is not within the scope of this book.

Formal expansion of a function as a power series may, however, be obtained by *assuming* that it can be so expanded, *i.e.*, R_n does tend to 0 as n tends to infinity. Thus we have the result :

If $f(x)$ can be expanded as an infinite Maclaurin's series, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!} f''(0)+\ldots\ldots \qquad \ldots(1)$$

Such an investigation will not give any information as to the range of values of x for which the expansion is valid.

To obtain the expansion of a function, on the assumption that it is possible, we have only to calculate the values of its derivatives for $x=0$ and substitute them in (1).

In the Appendix, we shall obtain the expansions of

$$e^x, \sin x, \cos x, \log(1+x). (1+x)^m$$

without assuming the possibility of expansion by actually examining the behaviour of R_n for the functions.

In the following, however, we obtain these and other expansions by *assuming the possibility of expansion.*

9·41. Expansion of e^x.

Let

$$f(x)=e^x.$$

$$\therefore \qquad f^n(x)=e^x.$$

Thus

$$f^n(0)=e^0=1.$$

Substituting in the Maclaurin's series, we obtain

$$e^x=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\dots\dots+\frac{x^n}{n\,!}+\dots \qquad \dots (1)$$

which is known as **Exponential Series.**

Cor. 1. Changing x into $x \log a$ in (1), we get

$$a^x=e^{x \log a}=1+(x \log a)+\frac{x^2}{2\,!}(\log a)^2+\dots\dots$$

This result may also be obtained directly by employing Maclaurin's series.

Cor 2. Putting, 1 for x, we obtain

$$e=1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\dots$$

from which we may obtain the values of, e, upto any number of decimal places.

9·42. Expansion of sin x.

Let

$$f(x)=\sin x.$$

$$\therefore \qquad f^n(x)=\sin\left(x+\frac{n\pi}{2}\right).$$

Thus

$$f^n(0)=\sin\frac{n\pi}{2}.$$

$$\therefore \qquad f'(0)=1, f''(0)=0, f'''(0)=-1, f''''(0)=0, \text{ etc.},$$

so that we see that the values of $f^n(0)$ for different values of n form a successively appearing periodic cycle of four values

$$1, 0, -1, 0.$$

Making these substitutions in the Maclaurin's series, we obtain the **sine series.**

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots\ldots$$

9·43. We may similarly obtain the **cosine series :**

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots\ldots$$

9·44. Expansion of log (1+x).

Let

$$f(x) = \log (1+x).$$

$$\therefore \qquad f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

Thus

$$f^n(0) = (-1)^{n-1}(n-1)!$$

$\therefore f'(0)=1, f''(0)=-1, f'''(0)=2!, f''''(0)=-3!$ and so on.

Making substitutions in the Maclaurin's series, we obtain the **Logarithmic Series :**

$$\log (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots\ldots$$

9·45. Expansion of $(1+x)^m$.

Let

$$f(x) = (1+x)^m.$$

$$\therefore \qquad f^n(x) = m(m-1)\ldots\ldots(m-n+1)(1+x)^{m-n}.$$

Thus

$$f^n(0) = m(m-1)\ldots\ldots(m-n+1).$$

$\therefore f'(0)=m,\ f''(0)=m(m-1),\ f'''(0)=m(m-1)(m-2)$, etc.

Making substitutions, we obtain the **Binomial Series :**

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \ldots\ldots + \frac{m(m-1)\ldots(m-n+1)}{n!} x^n + \ldots$$

In case m is a positive integer, we obtain a finite series on the right.

Note 1. If we examine the behaviour of R_n, as is done in the Appendix, we can show that (*i*) *the Exponential, sine and cosine series are valid for every value x,* (*ii*) *the Logarithmic Series is valid for* $-1<x<1$, *and* (*iii*) *the Binomial Series is valid for* $-1<x<1$.

Note 2. In the following we shall consider some more cases of expansions of functions, in each case assuming the possibility of expansion. It will be seen that in some cases we may also obtain the expansion by using any of the series obtained above.

Examples

1. *Assuming the possibility of expansion, expand, tan x, as far as the term in x^5.*

Let

$$f(x)=\tan x.$$

$$\therefore \quad f'(x)=\sec^2 x=1+\tan^2 x.$$

$$f''(x)=2\tan x\sec^2 x$$
$$=2\tan x(1+\tan^2 x)$$
$$=2\tan x+2\tan^3 x.$$

$$f'''(x)=2\sec^2 x+6\tan^2 x\sec^2 x$$
$$=2(1+\tan^2 x)+6\tan^2 x(1+\tan^2 x)$$
$$=2+8\tan^2 x+6\tan^4 x.$$

$$f^{iv}(x)=16\tan x\sec^2 x+24\tan^3 x\sec^2 x$$
$$=16\tan x(1+\tan^2 x)+24\tan^3 x(1+\tan^2 x)$$
$$=16\tan x+40\tan^3 x+24\tan^5 x.$$

$$f^{v}(x)=16\sec^2 x+120\tan^2 x\sec^2 x+120\tan^4 x\sec^2 x.$$

Thus

$$f(0)=0, f'(0)=1, f''(0)=0, f'''(0)=2, f^{iv}(0)=0, f^{v}(0)=16.$$

Substituting these values in the Maclaurin's series § 9·4, we obtain

$$\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\ldots.$$

2. *Assuming the possibility of expansion, expand $f(x)=e^{m\sin^{-1}x}$ in ascending integral powers of x.*

In Ex. 1, § 5·6, p. 124 we proved that

$$f_n(0)=\begin{cases} m^2(2^2+m^2)(4^2+m^2)\ldots\ldots[(n-2)^2+m^2]\ ; n \text{ even} \\ m(1^2+m^2)(3^2+m^2)\ldots\ldots[(n-2)^2+m^2]\ ; n \text{ odd.}\end{cases}$$

Substituting these values in the Maclaurin's series, we get

$$e^{m\sin^{-1}x}=1+mx+\frac{m^2}{2!}x^2+\frac{m(1^2+m^2)}{3!}x^3+\frac{m^2(2^2+m^2)}{4!}x^4+\ldots$$

3. Use of known series. *By Maclaurin's theorem or otherwise find the expansion of*

$$\sin(e^x-1),$$

upto and including the term in x^4.

First four derivatives are needed to expand the given function by Maclaurin's theorem upto the term required.

The required expansion can also be obtained by employing the Exponential and the sine series and thus avoiding the process of calculating the derivatives which is often very inconvenient.

$$\because \qquad e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots$$

$$\therefore \qquad e^x - 1 = x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots\ldots$$

$$= z, \text{ say.}$$

Now, $\sin(e^x - 1) = \sin z$

$$= z - \frac{z^3}{3\,!} + \frac{z^5}{5\,!} - \ldots$$

$$= \left(x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots\ldots \right)$$

$$- \frac{1}{3\,!}\left(x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \ldots \right)^3 + \ldots$$

$$= x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots - \frac{1}{6}\left(x^3 + \frac{3}{2} x^4 + \ldots \right)$$

$$= x + \frac{x^2}{2} - \frac{5x^4}{24} \ldots\ldots$$

9·5. Use of infinite series for evaluating the limits of indeterminate forms. The following examples will illustrate the procedure.

Examples

1. *Find* $\lim\limits_{x \to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$. (*D.U. 1953*)

Using the infinite series for e^x, $\sin x$ and $\log(1-x)$, we obtain

$$\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{1}{2\,!} x^2 + \frac{1}{3\,!} x^3 \ldots \right)\left(x - \frac{x^3}{3\,!} + \ldots \right) - x - x^2}{x^2 - x\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \right)}$$

$$= \lim \frac{\frac{1}{3}x^3 - 0.x^4 \ldots}{-\frac{1}{2}x^3 - \frac{1}{3}x^4 \ldots}$$

$$= \lim \frac{\frac{1}{3} + 0.x \ldots\ldots}{-\frac{1}{2} - \frac{1}{3}x \ldots\ldots} = -\frac{2}{3}.$$

2. *Find*

(*i*) $\lim\limits_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x}$

(*ii*) $\lim\limits_{x \to 0} \dfrac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$. (*D.U. Hons. 1954*)

(*Cf.* *Ex. 25*, *Ex. 26*, p. 178)

Let $$y=(1+x)^{1/x}$$

$$\log y=\frac{1}{x}\log(1+x)$$

$$=\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\ldots\right)$$

$$=1-\frac{x}{2}+\frac{x^2}{3}-\ldots\ldots$$

or $$y=e^{1-\frac{x}{2}+\frac{x^2}{3}-\ldots\ldots}$$

$$=e.e^{-\frac{x}{2}+\frac{x^2}{3}-\ldots\ldots}$$

$$=e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\ldots\right)+\frac{1}{2\,!}\left(-\frac{x}{2}+\frac{x^2}{3}\ldots\ldots\right)^2+\ldots\ldots\right]$$

$$=e\left[1-\frac{x}{2}+\frac{11x^2}{24}-\ldots\ldots\right]$$

(*i*) $$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=\lim_{x\to 0}\frac{e\left(1-\frac{x}{2}+\frac{11x^2}{24}\ldots\right)-e}{x}$$

$$=-\frac{e}{2}.$$

(*ii*) $$\lim\frac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}=\lim\frac{e\left(1-\frac{x}{2}+\frac{11x^2}{24}\ldots\ldots\right)-e+\frac{1}{2}ex}{x^2}$$

$$=\frac{11e}{24}.$$

Exercises

(*In the following, the possibility of expansion may also be assumed*)

1. Prove that

$$e^{ax}\cos bx=1+ax+\frac{a^2-b^2}{2}x^2+\frac{a(a^2-3b^2)}{3\,!}x^3+\ldots$$
$$+\frac{(a^2+b^2)^{\frac{1}{2}n}}{n\,!}x^n\cos\left(n\tan^{-1}\frac{b}{a}\right)+\ldots$$

2. Prove that

$$e^{ax}\sin bx = bx + abx^2 + \frac{3a^2b-b^3}{3\,!}x^3 + \ldots .$$

$$+\frac{(a^2+b^2)^{\frac{1}{2}n}}{n\,!}x^n \sin\left(n\tan^{-1}\frac{b}{a}\right)+\ldots$$

3. Obtain the nth term in the expansion of $\tan^{-1} x$ in ascending powers of x. (*P.U. 1951*)

4. Show that $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 \ldots\ldots$

5. Prove that $e^x \sin^2 x = x^2 + x^3 + \frac{1}{6}x^4 \ldots .$

6. If $y = \log[x + \sqrt{(1+x^2)}]$, prove that

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

Differentiate this n times and deduce the expansion of y in ascending powers of x in the form

$$y = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \ldots .$$

(*P.U. 1937*)

7. If $y = \sin\log(x^2+2x+1)$, prove that

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0.$$

Hence or otherwise expand y in ascending powers of x as far as x^6.

8. Prove that

$$(\sin^{-1}x)^2 = 2.\frac{x^2}{2\,!} + \frac{2\cdot 2^2}{4\,!}x^4 + \frac{2\cdot 2^2\cdot 4^2}{6\,!}x^6 + \ldots\ldots$$

$$+\frac{2\cdot 2^2\cdot 4^2\ldots(2n-2)^2}{(2n)\,!}x^{2n} + \ldots$$

9. Show that

$$e^{m\tan^{-1}x} = 1 + mx + \frac{m^2}{2\,!}x^2 + \frac{m(m^2-2)}{3\,!}x^3 + \frac{m^2(m^2-8)}{4\,!}x^4 + \ldots .$$

10. Obtain the following expressions :—

(*i*) $\log\tan(\frac{1}{4}\pi + x) = 2x + \frac{4}{3}x^3 + \frac{4}{3}x^5 + \ldots .$

(*ii*) $\log(1+\sin x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dfrac{x^4}{12} + \ldots\ldots$

(*iii*) $\dfrac{x}{2}\cdot\dfrac{e^x+1}{e^x-1} = 1 + \dfrac{1}{6}\cdot\dfrac{x^2}{2\,!} - \dfrac{1}{30}\cdot\dfrac{x^4}{4\,!} + \dfrac{1}{42}\cdot\dfrac{x^6}{6\,!} - \ldots .$

(*iv*) $\log\dfrac{\tan x}{x} = \dfrac{x^2}{3} + \dfrac{7}{90}x^4 + \ldots$

(*v*) $\dfrac{x}{e^x-1} = 1 - \dfrac{x}{2} + \dfrac{1}{6}\cdot\dfrac{x^2}{2\,!} - \dfrac{1}{30}\cdot\dfrac{x^4}{4\,!} + \ldots .$

(*vi*) $\sin^{-1}x = x + \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{x^5}{5} + \ldots .$

(*vii*) $\log\sec x = \dfrac{1}{2}x^2 + \dfrac{1}{12}x^4 + \dfrac{1}{45}x^6 + \ldots .$ (*D.U. 1953*)

(*viii*) $\log \sin (x+h) = \log \sin x + h \cot x - \frac{h^2}{2!} \operatorname{cosec}^2 x$
$+\frac{h^3}{3!} . 2 \cot x \operatorname{cosec}^2 x - \dots$

(*ix*) $\tan^{-1}(x+h) = \tan^{-1}x + h \sin z \frac{\sin z}{1} - (h \sin z)^2 \frac{\sin 2z}{2}$
$+(h \sin z)^3 \frac{\sin 3z}{3} - \dots$

when $z = \cot^{-1} x$.

11. Obtain the expansion of $e^{\sin x}$ in powers of x as far as x^4. (*D.U. 1949*)

12. Obtain the first three terms of the expansion of $\log(1+\tan x)$ in powers of x. (*P.U. 1955*)

13. Apply Maclaurin's theorem to find the expansion of $e^x/(e^x+1)$ as far as the term in x^3. (*D.U. 1955*)

14. Expand $\log \sin x$ in powers of $(x-3)$.

[Write $\log \sin x = \log \sin (3+x-3)$ and replace x by 3 and h by $x-3$ in the expansion of $\log \sin (x+h)$ in powers of h.]

15. Expand $3x^3-2x^2+x-4$ in powers of $x-2$.

16. Show that

(*i*) $\lim\limits_{x \to 0} \frac{x-\sin x}{x^3} = \frac{1}{6}$. (*ii*) $\lim\limits_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = 1$.

(*iii*) $\lim\limits_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex + \frac{11}{24}ex^2}{x^3} = -\frac{7e}{16}$.

17. Obtain by Maclaurin's theorem, the first four terms of the expansion of $e^{x \cos x}$ in ascending powers of x. Hence or otherwise show that

$$\lim_{x \to 0} \frac{e - e^{x \cos x}}{x - \sin x} = 3.$$

Appendix

We shall now obtain the expansions of

$$e^x, \sin x, \cos x, \log(1+x), (1+x)^m$$

without making any assumption as to the possibility of expansion.

Expansion of e^x.

Let $f(x) = e^x$. Therefore, $f^n(x) = e^x$, so that $f(x)$ possesses derivatives of every order for every value of x.

Taking Lagrange's form of remainder, we have

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} \cdot e^x.$$

We know that when $n \to \infty$,

$$x^n/n! \to 0 \quad \text{(See § 3·62, p. 58)}$$

whatever value x may have.

$\therefore \qquad R_n \to 0$ as $n \to \infty$.

Conditions for Maclaurin's infinite expansion are thus satisfied.

Also $\qquad f^n(0) = e^0 = 1$

Making substitutions in the Maclaurin's series, we get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

which is valid for every value of x.

Expansion of sin x.

Let $f(x) = \sin x$. $\qquad \therefore \; f^n(x) = \sin(x + \frac{1}{2}n\pi)$,

so that $f(x)$ possesses derivatives of every order for every value of x. We have

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right).$$

Since

$$|R_n| = \left|\frac{x^n}{n!}\right| \cdot \left|\sin\left(\theta x + \frac{n\pi}{2}\right)\right| \leqslant \left|\frac{x^n}{n!}\right|,$$

we see that $R_n \to 0$ as $n \to \infty$ for every value of x.

Thus the conditions for Maclaurin's infinite expansion are satisfied. Now

$$f^n(0) = \sin\frac{n\pi}{2}.$$

Making these substitutions in the Maclaurin's series, we get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

which is valid for every value of x.

Expansion of cos x. As above, it may easily be shown that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

for every value of x.

Expansion of log (1+x).

Let

$$f(x) = \log(1+x).$$

We know that $\log(1+x)$ possesses derivatives of every order for $(1+x) > 0$, *i.e.*, for $x > -1$.

Moreover

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \qquad (x > -1).$$

If R_n, denotes Lagrange's form of remainder, we have

$$R_n=\frac{x^n}{n\,!}\,f^n(\theta x)=(-1)^{n-1}.\,\left(\frac{x}{1+\theta x}\right)^n.$$

(*i*) Let $0 \leqslant x \leqslant 1$, so that $x/(1+\theta x)$ and, therefore, also, $[x/(1+\theta x)]^n$ is positive and <1, whatever value n may have. Since, also, $1/n \to 0$, as $n \to \infty$, we see that $R_n \to 0$.

Thus

$$R_n \to 0 \text{ as } n \to \infty \text{ when } 0\leqslant x\leqslant 1.$$

(*ii*) Let $-1<x<0$.

In this case $x/(1+\theta x)$ may not be numerically less than unity so that we fail to draw any definite conclusion from Lagrange's form of R_n.

Taking Cauchy's form of remainder, we have

$$R_n=\frac{x^n}{(n-1)\,!}\,(1-\theta)^{n-1}f^n(\theta x)$$

$$=(-1)^{n-1}\,.\,x^n\,.\,\frac{1}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}.$$

Here $(1-\theta)/(1+\theta x)$ is positive and less than 1, and

$$\frac{1}{1+\theta x}<\frac{1}{1-|\,x\,|}.$$

Also

$$x^n \to 0 \text{ as } n \to \infty. \qquad (\S\ 3{\cdot}61,\ \text{p. } 57)$$

$$\therefore \qquad R_n \to 0 \text{ as } n \to \infty.$$

Hence, the conditions for Maclaurin's theorem are satisfied for $-1<x\leqslant 1$.

Also

$$f^n(0)=(-1)^{n-1}(n-1)\,!.$$

Making these substitutions in the Maclaurin's series, we get

$$\log\,(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

for $-1<x\leqslant 1$.

Expansion of $(1+x)^m$; (*m* is any real number).

Let

$$f(x)=(1+x)^m.$$

When m is any real number, $(1+x)^m$ possesses continuous derivatives of every order only when $1+x>0$, *i.e.*, when $x>-1$.

Now,

$$f^n(x)=m(m-1)(m-2)\ldots\ldots(m-n+1)(1+x)^{m-n}.$$

We notice that if m is any positive integer, then the derivatives of $f(x)$ of order higher than mth vanish identically and thus, for $n > m$, R_n identically vanishes, so that $(1+x)^m$ is expanded in a finite series consisting of $(m+1)$ terms.

If m be not a positive integer, then no derivative vanishes identically so that we have to examine this case still further.

If R_n denotes Cauchy's form of remainder, we get

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1} m(m-1)...(m-n+1)(1+\theta x)^{m-n}$$

$$= x^n . \frac{m(m-1)(m-n+1)}{(n-1)!}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}(1+\theta x)^{m-1}.$$

Let

$$-1 < x < 1, \text{ i.e., } |x| < 1.$$

Also

$$0 < \theta < 1.$$

$\therefore$

$$0 < 1-\theta < 1+\theta x,$$

or

$$0 < \left(\frac{1-\theta}{1+\theta x}\right) < 1,$$

or

$$0 < \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} < 1. \qquad ...(i)$$

Let $(m-1)$ *be positive.*

Now

$$0 < 1+\theta x < 1+1=2,$$

$\therefore$

$$0 < (1+\theta x)^{m-1} < 2^{m-1} \qquad ...(ii)$$

Let, now, $m-1$ *be negative.*

Now, since

$$\theta x \geqslant -|x|,$$

$\therefore$

$$1+\theta x \geqslant 1-|x|,$$

or

$$(1+\theta x)^{m-1} \leqslant (1-|x|)^{m-1}.$$

*Also we know that

$$\frac{m(m-1)...(m-n+1)}{(n-1)!}x^n \to 0 \text{ when } |x| < 1.$$

Thus, $R_n \to 0$ as $n \to \infty$ when $|x| < 1$.

The conditions for Maclaurin's expansion are, therefore, satisfied. Now

$$f^n(0) = m(m-1)......(m-n+1).$$

*Refer to the foot note on the next page.

Making substitutions, we get

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\ldots$$

when $-1 < x < 1$.

Ex. 1. Justify the Maclaurin's expansions of

$$e^{ax}\sin bx,\ e^{ax}\cos bx,\ \tan^{-1} x.$$

Ex. 2. Show that $\log x$ and $\cot x$ cannot be expanded as Maclaurin's series.

Ex. 3. Show that the Maclaurin's infinite expansion is not valid for $f(x)$ where

$$f(x)=e^{-1/x^2} \text{ when } x\neq 0 \text{ and } f(0)=0.$$

[Refer Ex. 32. p. 178.]

**To prove that when* $|x| < 1$,

$$u_n=\frac{m(m-1)\ldots(m-n+1)}{(n-1)!}x^n \to 0 \text{ as } n \to \infty.$$

Changing n to $n+1$, we get

$$u_{n+1}=\frac{m(m-1)\ldots(m-n)}{n!}x^{n+1}$$

$$\therefore \quad \frac{u_{n+1}}{u_n}=\frac{m-n}{n}x=-\left(1-\frac{m}{n}\right)x.$$

or $$\left|\frac{u_{n+1}}{u_n}\right|=\left|1-\frac{m}{n}\right||x| \to |x| \text{ as } n\to\infty.$$

As $|x| < 1$, we can find a positive number $k < 1$, and a positive integer p such that $\left|\frac{u_{n+1}}{u_n}\right| < k$, for $n \geqslant p$. Thus

$$|u_{p+1}| < k\,|u_p|$$
$$|u_{p+2}| < k\,|u_{p+1}|$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$|u_n| < k\,|u_{n-1}|.$$

Multiplying we get

$$|u_n| < k^{n-p}\,|u_p| = k^n\,|u_p|\,k^p.$$

Now $|u_p|/k^p$ is a constant and $k^n \to 0$ as $n \to \infty$.

$\therefore$ $|u_n|$ and so also $u_n \to 0$ as $n \to \infty$.

CHAPTER X

FUNCTIONS OF TWO VARIABLES

PARTIAL DIFFERENTIATION

10·1. The notions of continuity, limit and differentiation in relation to functions of two variables, will be briefly explained in this chapter. A few theorems of elementary character will also be proved.

The subject of functions of two variables is capable of extension to functions of n variables, but the treatment of the subject in this generalised form is not within the scope of this book. Only a few examples dealing with functions of three or more variables may be given.

10·2. Functions of two variables. As in the case of functions of a single variable, we introduce the notion of functions of two variables by considering some examples.

(*i*) The relation

$$z=\sqrt{(1-x^2-y^2)}, \qquad \ldots(1)$$

between x, y, z, determines a value of z corresponding to every pair of numbers x, y, which are such that $x^2+y^2 \leqslant 1$.

Denoting a pair of numbers x, y, geometrically by a point on a plane as explained in § 1·7, p. 14, we see that the points (x, y) for which $x^2+y^2 \leqslant 1$ lie on or within the circle whose centre is at the origin and radius is 1.

The region determined by the point (x, y) is called the *domain* of the point (x, y).

Now, we say that the relation (1) determines z as a function of two variables, x, y defined for the domain bounded by the unit circle $x^2+y^2=1$.

(*i*) Consider the relation

$$z=\sqrt{[(a-x)(x-b)]}+\sqrt{[(c-y)(y-d)]}, \qquad \ldots(2)$$

where $a<b$; $c<d$.

Now $(a-x)(x-b)$ is non-negative if $a \leqslant x \leqslant b$ and $(c-y)(y-d)$ is non-negative if $c \leqslant y \leqslant d$.

The points (x, y) for which $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ determine a rectangular domain bounded by the lines

$$x=a, \ x=b \ ; \ y=c, \ y=d.$$

The relation (2) determines a value of z corresponding to every point (x, y) of this rectangular domain. Thus z is a function of x and y defined for the domain.

(*iii*) The relation

$$z = e^{-x^2 - y^2},$$

determines z as a function of x, y defined for the whole plane.

In general, we say that z *is a function of two variables defined for a certain domain, if it has a value corresponding to every point* (x, y) *of the domain.*

The relation of functionality is expressed by the symbols f, φ etc., as in the case of functions of a single variable, so that we may write $z = f(x, y)$, $\phi(x, y)$, etc.

Ex. Determine the domains of definition of

(*i*) $z = 1/[\log x + \log y]$. (*ii*) $z = x^y + y^x$.

10·3. Neighbourhood of a point (a, b). Let δ be any positive number. The points (x, y) such that

$$a - \delta \leqslant x \leqslant a + \delta, \quad b - \delta \leqslant y \leqslant b + \delta$$

determine a square bounded by the lines

$$x = a - \delta,\ x = a + \delta\ ;\ y = b - \delta,\ y = b + \delta.$$

Its centre is at the point (a, b). This square is called a neighbourhood of the point (a, b). For every value of, δ, we will get a neighbourhood.

10·4. Continuity of a function of two variables.

Let (a, b) be any point of the domain of definition of

$$z = f(x, y).$$

As in the case of functions of one variable, we say that $f(x, y)$ is continuous at (a, b), if for points (x, y) near (a, b), the value $f(x, y)$ of the function is near $f(a, b)$ *i.e.*, $f(x, y)$ can be made as near $f(a, b)$ as we like by taking the points (x, y) sufficiently near (a, b).

Formally, we say that $f(x, y)$ *is continuous at* (a, b), *if, corresponding to any pre-assigned positive number* ε, *there exists a positive number* δ *such that*

$$| f(x, y) - f(a, b) | < \varepsilon$$

for all points (x, y) *in the square*

$$a - \delta \leqslant x \leqslant a + \delta,\ b - \delta \leqslant y \leqslant b + \delta.$$

Thus for continuity at (a, b), there exists a square bounded by the lines

$$x = a - \delta,\ x = a + \delta\ ;\ y = b - \delta,\ y = b + \delta$$

such that, for any point (x, y) of this square, $f(x, y)$ lies between

$$f(a, b) - \varepsilon, \text{ and } f(a, b) + \varepsilon$$

where ε is any positive number, however small.

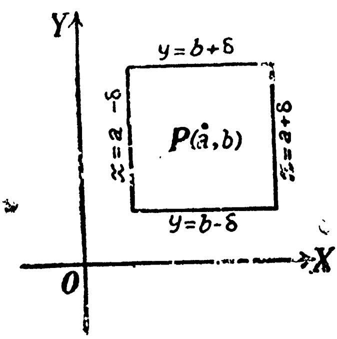


Fig. 54

10·41. Continuity in a domain. *A function $f(x, y)$ is said to be continuous in a domain if it is continuous at every point of the same.*

10·42. Special Case. Let $f(x, y)$ be continuous at (a, b) and let ε be any positive number, however small. Then there exists a square bounded by the lines

$$x=a-\delta,\ x=a+\delta\ ;\ y=b-\delta,\ y=b+\delta$$

such that for points (x, y) of the square, the numerical value of the difference between $f(x, y)$ and $f(a, b)$ is less than ε.

In particular, if we consider points of this square lying on the line $y=b$, we see that for values of x lying between $a-\delta$ and $a+\delta$, the numerical value of the difference between $f(x, b)$ and $f(a, b)$ is less than ε. Also, such a choice of δ is possible for every positive number ε. This is equivalent to saying that $f(x, b)$ is a continuous function of a single variable x for $x=a$.

It may be similarly shown that $f(a, y)$ is a continuous function of y for $y=b$.

Thus we have shown that a *continuous function of two variables is also a continuous function of each variable separately.*

10·5. Limit of a function of two variables.

$$\lim f(x, y)=l, \text{ as } (x, y) \to (a, b).$$

A function $f(x, y)$ is said to tend to the limit l, as x tends to a and y tends to b, i.e., as (x, y) tends to (a, b), if, corresponding to any pre-assigned positive number ε, there exists a positive number δ such that

$$|\ f(x, y)-l\ | < \varepsilon$$

for all points (x, y), other than (a, b), lying within the square,

$$a-\delta \leqslant x \leqslant a+\delta\ ;\ b-\delta \leqslant y \leqslant b+\delta.$$

This means that corresponding to every positive number ε, there exists a neighbourhood such that for every point (x, y) of this neighbourhood, other than $f(a, b)$, $f(x, y)$ lies between $l-\varepsilon$ and $l+\varepsilon$.

10·51. Limit of a continuous function. Comparing the definitions of limit and continuity as given in § 10·4, and § 10·5, we see that *$f(x, y)$ is continuous at (a, b), if and only if*

$$\lim f(x, y)=f(a, b) \text{ as } (x, y) \to (a, b),$$

i.e.,

the limit of the function = actual value of the function.

The same thing may also be expressed by saying that for continuity at (a, b), we have

$$\lim f(a+h, b+k)=f(a, b)$$

as $(h, k) \to (0, 0)$.

10·6. Partial derivatives.

Let

$$z=f(x, y).$$

Then

$$\lim \frac{f(a+h, b)-f(a, b)}{h}, \text{ as } h \to 0,$$

if it exists, is said to be the partial derivatives of $f(x, y)$ w. r. t. x at (a, b) and is denoted by

$$\left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\right)_{(\mathbf{a}, \mathbf{b})} \text{ or } \mathbf{f}_x(\mathbf{a}, \mathbf{b}).$$

It will be seen that to find the partial derivative of $f(x, y)$ *w.r.t.* x at (a, b), we put y equal to b and consider the change in the function as x changes from a to $a+h$ so that

$$\left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\right)_{(a, b)}$$

is the ordinary derivative of $f(x, b)$ w.r. to x for $x=a$.

Again,

$$\lim \frac{f(a, b+k)-f(a, b)}{k} \text{ as } k \to 0.$$

if it exists, is called the partial derivative of $f(x, y)$ w.r. to y at (a, b), and is denoted by

$$\left(\frac{\partial \mathbf{z}}{\partial \mathbf{y}}\right)_{(\mathbf{a}, \mathbf{b})} \text{ or } \mathbf{f}_y\ (\mathbf{a}, \mathbf{b})$$

so that

$$\left(\frac{\partial z}{\partial y}\right)_{(a, b)}$$

is the ordinary derivative of $f(a, y)$ w.r. to y for $y=b$.

If $f(x, y)$ possesses a partial derivative *w.r.* to x at every point of its domain of definition, then the values of these partial derivatives themselves define a function of two variables for the same domain. This function is called the partial derivative of the function *w.r.* to x and is denoted by $\partial z/\partial x$ or $\mathbf{f}_x(\mathbf{x}, \mathbf{y})$ or simply $\mathbf{f}_x$.

Thus

$$f_x(x, y)=\lim \frac{f(x+h, y)-f(x, y)}{h} \text{ as } h \to 0,$$

where y is kept constant in the process of taking the limit. We can similarly define the partial derivative of f, *w.r.* to y which is denoted by

$$\partial \mathbf{z}/\partial y,\ \mathbf{f}_y(\mathbf{x}, \mathbf{y}) \text{ or } \mathbf{f}_y.$$

10·61. Partial derivatives of higher orders. We can form partial derivatives of $\partial z/\partial x$ and $\partial z/\partial y$ just as we formed those of z.

Thus we have

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

which are called the second order partial derivatives of z and are denoted by

$$\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{x^2}, f_{yx}$$

respectively.

Similarly the second order partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

are respectively denoted by

$$\frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2} \text{ or } f_{xy}, f_{y^2}.$$

Thus, there are *four* second order partial derivatives of z at any point (a, b).

The partial derivatives $\partial^2 z/\partial y\, \partial x$ and $\partial^2 z/\partial x\, \partial y$ are distinguished by the *order* in which z is *successively* differentiated *w.r.* to x and y. But, it will be seen that, *in general*, they are equal. The proof is given in the Appendix.

10·7. Geometrical representation of function of two variables. We take a pair of perpendicular lines OX and OY. Through the point O, we draw a line $Z'OZ$ perpendicular to the XY plane and call it Z-axis.

Any one of the two sides of Z-axis may be assigned a positive sense. Lengths, z, will be measured parallel to this axis.

The three co-ordinate axes, taken in pairs determine three planes, *viz*, XY, YZ and ZX which are taken as the *co-ordinate planes*.

Let $z = f(x, y)$ be a function defined in any domain lying in the XY plane.

To each point (x, y) of this domain, there corresponds a value of z. Through this point, we draw the line perpendicular to XY plane equal in length to z, so that we arrive at another point P denoted as (x, y, z), lying on one or the other side of the plane according as z in positive or negative.

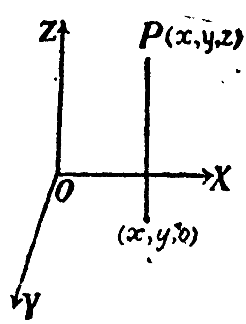


Fig. 55

Thus to each point of the domain in the XY plane there corresponds a point P. The aggregate of the points P determines a surface which is said to represent the function geometrically.

10·71. Geometrical interpretation of partial derivatives of the first order.

Let

$$z=f(x, y). \qquad ...(i)$$

We have seen that the functional equation (*i*) represents a *surface* geometrically. We now seek the geometrical interpretation of the partial derivatives

$$\left[\frac{\partial z}{\partial x}\right]_{a, b} \text{ and } \left[\frac{\partial z}{\partial y}\right]_{a, b}.$$

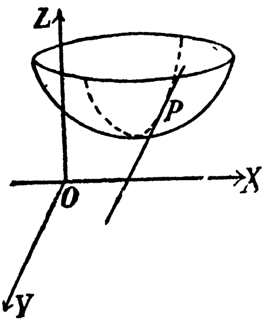


Fig. 56

The point $P[a, b, f(a, b)]$ on the surface corresponds to the values a, b of the independent variables x, y.

If a variable point, starting from P, changes its position on the surface such that y remains constantly equal to b, then it is clear that the locus of the point is the curve of intersection of the surface and the plane $y=b$.

On this curve x and z vary according to the relation

$$z=f(x, b).$$

Also, $\left(\frac{\partial z}{\partial x}\right)_{(a, b)}$ is the ordinary derivative of $f(x, b)$ *w.r.* to x for $x=a$.

Hence, we see that $\left(\frac{\partial z}{\partial x}\right)_{(a, b)}$ *is the tangent of the angle which the tangent to the curve, in which the plane $y=b$ parallel to the ZX plane cuts the surface at $P[a, b, f(a, b)]$, makes with X-axis.*

Similarly, it may be seen that $\left(\frac{\partial z}{\partial y}\right)_{(a, b)}$ *is the tangent of the angle which the tangent to the curve of intersection of the surface and the plane $x=a$ makes with Y-axis.*

10·8. Homogeneous Functions. Ordinarily, $f(x, y)$ is said to be a homogeneous function of order n, if the degree of each of its terms in x and y is equal to n. Thus

$$a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\ldots\ldots+a_{n-1}xy^{n-1}+a_ny^n \qquad ...(i)$$

is a homogeneous function of order n.

This definition of *homogeneity* applies to polynomial functions only. To enlarge the concept of homogeneity so as to bring even transcendental functions within its scope, we say that z *is a homogeneous function of order or degree n, if it is expressible as*

$$x^nf(y/x).$$

The polynomial function (i) which can be written as

$$x^n\left[a_0+a_1\frac{y}{x}+a_2\left(\frac{y}{x}\right)^2+\ldots\ldots+a_n\left(\frac{y}{x}\right)^n\right]$$

is a homogeneous function of order n according to the new definition also. The functions

$$x^n \sin(y/x),\ (\sqrt{y}+\sqrt{x})/(y+x)$$

are homogeneous according to the second definition only. Here the degree of $x^n \sin(y/x)$ is n. Also

$$\frac{\sqrt{y}+\sqrt{x}}{y+x}=\frac{\sqrt{x}\left[1+\dfrac{\sqrt{y}}{\sqrt{x}}\right]}{x\left[1+\dfrac{y}{x}\right]}=x^{-\frac{1}{2}}\frac{1+\sqrt{\dfrac{y}{x}}}{1+\dfrac{y}{x}}$$

so that it is of degree $-\frac{1}{2}$.

10 81. Euler's theorem on Homogeneous Functions.

If z be a homogeneous function of x, y of order n, then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$$

We have

$$z=x^n f\left(\frac{y}{x}\right).$$

$$\therefore \quad \frac{\partial z}{\partial x}=nx^{n-1}f\left(\frac{y}{x}\right)+x^n f'\left(\frac{y}{x}\right)\times-\frac{y}{x^2}$$

$$=nx^{n-1}f\left(\frac{y}{x}\right)-yx^{n-2}f'\left(\frac{y}{x}\right),$$

and

$$\frac{\partial z}{\partial y}=x^n f'\left(\frac{y}{x}\right).\frac{1}{x}=x^{n-1}f'\left(\frac{y}{x}\right).$$

Thus

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nx^n f\left(\frac{y}{x}\right)=nz.$$

Cor. *If z is a homogeneous function of x, y of degree n, then*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z.$$

Differentiating

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz, \qquad \ldots(1)$$

partially with respect to x and y separately, we obtain

$$\frac{\partial z}{\partial x}+x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}, \qquad \ldots(2)$$

$$x\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial y^4}=n\frac{\partial z}{\partial y}. \qquad \ldots(3)$$

Multiplying (2), (3) by x, y respectively and adding, we obtain

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z,$$

where we have employed (1) and assumed the equality of $\partial^2 z/\partial x\partial y$ and $\partial^2 z/\partial y\partial x$.

Examples

1. *If*

$$u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y},$$

prove that

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}. \qquad (D.U.\ 1953;\ P.U.)$$

We have

$$\frac{\partial u}{\partial y}=x^2\frac{1}{1+\dfrac{y^2}{x^2}}\cdot\frac{1}{x}-2y\tan^{-1}\frac{x}{y}+y^2\frac{1}{1+\dfrac{x^2}{y^2}}\cdot\frac{x}{y^2}$$

$$=\frac{x^3}{x^2+y^2}+\frac{xy^2}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}$$

$$=x-2y\tan^{-1}\frac{x}{y}.$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$=1-2y\frac{1}{1+\dfrac{x^2}{y^2}}\cdot\frac{1}{y}$$

$$=1-\frac{2y^2}{y^2+x^2}=\frac{x^2-y^2}{x^2+y^2}.$$

2. *If*

$$u=\frac{1}{\sqrt{(x^2+y^2+z^2)}}$$

show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0. \qquad (D.U.\ 1952\ ;\ P.U.\ 1949)$$

We have

$$\frac{\partial u}{\partial x}=-\tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\,2x=-x(x^2+y^2+z^2)^{-\frac{3}{2}},$$

$$\frac{\partial^2 u}{\partial x^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}+3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}.$$

Similarly or by symmetry

$$\frac{\partial^2 u}{\partial y^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}+3y^2(x^2+y^2+z^2)^{-\frac{5}{2}},$$

$$\frac{\partial^2 u}{z^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}+3z^2(x^2+y^2+z^2)^{-\frac{5}{2}}.$$

Adding, we obtain the result as given.

3. *If*

$$u=\tan^{-1}\frac{x^3+y^3}{x-y},$$

prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$$

Here u is not a homogeneous function. We, however, write

$$z=\tan u=\frac{x^3+y^3}{x-y}=x^2\,\frac{1+(y/x)^3}{1-(y/x)},$$

so that z is a homogeneous function of x, y of order 2.

$$\therefore \qquad x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z. \qquad \ldots(1)$$

But

$$\frac{\partial z}{\partial x}=\sec^2 u\,\frac{\partial u}{\partial x}\ ;\ \frac{\partial z}{\partial y}=\sec^2 u\,\frac{\partial u}{\partial y}. \qquad \ldots(2)$$

Substituting in (1), we obtain

$$\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}\right)=2z=2\tan u,$$

or

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\sin u}{\cos u}\cos^2 u=\sin 2u. \qquad \ldots(3)$$

Exercises

1. Find the first order partial derivatives of

   (*i*) $\tan^{-1}(x+y)$. (*ii*) $e^{ax} \sin by$.

   (*iii*) $\log(x^2+y^2)$.

2. Find the second order partial derivatives of

   (*i*) e^{x-y}. (*ii*) e^{x^y}.

   (*iii*) $\tan(\tan^{-1}x+\tan^{-1}y)$.

3. Verify that

$$\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x};$$

when u is

   (*i*) $\sin^{-1}\dfrac{x}{y}$. (*ii*) $\dfrac{xy}{\sqrt{(1+x^2+y^2)}}$.

   (*iii*) $\log(y \sin x+x \sin y)$.

4. Find the value of

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2}+\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2},$$

when

$$a^2x^2+b^2y^2-c^2z^2=0.$$

5. Verify Euler's theorem for

   (*i*) $z=ax^2+2hxy+by^2$. (*ii*) $z=(x^2+xy+y^2)^{-1}$.

   (*iii*) $z=\sin^{-1}\dfrac{x}{y}+\tan^{-1}\dfrac{y}{x}$. (*iv*) $z=x^n \log\dfrac{y}{x}$.

   (*v*) $z=\dfrac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}$.

6. If $u=f(y/x)$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$ **(*D.U. 1950 ; P.U. 1954*)**

7. If $z=xy\,f(x/y)$, show that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z.$$ **(*P.U.*)**

8. If $z=\tan(y+ax)+(y-ax)^{\frac{3}{2}}$, find the value of

$$\frac{\partial^2 z}{\partial x^2}-a^2\frac{\partial^2 z}{\partial y^2}.$$ **(*P.U. 1945*)**

9. If $z=\tan^{-1}(y/x)$, verify that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0.$$ **(*P.U.*)**

10. If $z(x+y)=x^2+y^2$, show that

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right).$$ (*Allahabad*)

11. If $z=3xy-y^3+(y^2-2x)^{\frac{3}{2}}$, verify that

$$\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial^2 z}{\partial y \partial x} \text{ and } \frac{\partial^2 z}{\partial x^2}\,\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2.$$ (*M.T.*)

12. If $u=\log \dfrac{x^2+y^2}{x+y}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1$.

13. (*a*) If $u=\sin^{-1}\dfrac{x^2+y^2}{x+y}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\tan u$. (*P.U. 1954*)

(*b*) If $u=\sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$ (*P.U. 1955*)

14. If $z=f(x+ay)+\varphi(x-ay)$, prove that

$$\frac{\partial^2 z}{\partial y^2}=a^2\frac{\partial^2 z}{\partial x^2}.$$

15. If $z=(x+y)+(x+y)\varphi(y/x)$, prove that

$$x\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y \partial x}\right)=y\left(\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial x \partial y}\right).$$

16. If $u=f(ax^2+2hxy+by^2)$, $v=\varphi(ax^2+2hxy+by^2)$, prove that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$ (*M.T.*)

17. If $\theta=t^n e^{-r^2/4t}$, find the value of n which will make

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right)=\frac{\partial \theta}{\partial t}.$$ (*M.T.*)

18. If $u=f(r)$ where $r=\sqrt{(x^2+y^2)}$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

19. If $u=\log(x^2+y^2+z^2)$, prove that

$$x\frac{\partial^2 u}{\partial y \partial z}=\frac{\partial^2 u}{\partial z \partial x}=z\frac{\partial^2 u}{\partial x \partial y}.$$

20. If $V=r^m$ where $r^2=x^2+y^2+z^2$, show that

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=m(m+1)r^{m-2}.$$

21. If $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$, find

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}.$$ (*D.U. Hons. 1953*)

[Refer Ex. 3, p. 201]

Differentiating (2), *w.r.* to x and y, we get

$$\frac{\partial^2 z}{\partial x^2}=2\sec^2 u\tan u\left(\frac{\partial u}{\partial x}\right)^2+\sec^2 u\frac{\partial^2 u}{\partial x^2},$$

$$\frac{\partial^2 z}{\partial y\partial x}=2\sec^2 u\tan u\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+\sec^2 u\frac{\partial^2 u}{\partial y\partial x},$$

$$\frac{\partial^2 z}{\partial y^2}=2\sec^2 u\tan u\left(\frac{\partial u}{\partial x}\right)^2+\sec^2 u\frac{\partial^2 u}{\partial y^2}.$$

As z is a homogeneous function of x, y of order 2, we have, by Cor. § 10·81 P. 199.

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=2(2-1)z=2z.$$

Making substitutious, we obtain

$$\sec^2 u\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right]$$
$$+2\sec^2 u\tan u\left[x^2\left(\frac{\partial u}{\partial x}\right)^2+2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial v}+y^2\left(\frac{\partial u}{\partial y}\right)^2\right]=2x$$

Using (3) p. 201, we obtain

$$\sec^2 u\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right]=2\tan u-2\sec^2 u\tan u\,.\,\sin^2 2u.$$

$$\therefore\qquad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(1-4\sin^2 u)\sin 2u.$$

22. If

$$u=\sin^{-1}\left\{\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right\}^{\frac{1}{2}},$$

show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}\left(13+\tan^2 u\right).\quad (D.U.\ Hons.\ 1949)$$

10·82. Choice of independent variables. A new Notation. To introduce the new notation, we consider a particular case. Suppose that

$$x=r\cos\theta,\ y=r\sin\theta\ ;\qquad ..(i)$$

and we are required to find $\partial x/\partial r$.

Before beginning partial differentiation, we have to ask ourselves a question. "*What are the independent variables ?*" Hitherto the notation has always been such as to suggest readily what the independent variables are and no ambiguity was possible.

In the present case we have four variables x, y, r, θ connected by two relations so that any one of them may be expressed in terms of two of the remaining three. Thus, x may be expressed in terms of

(a) r, θ ; or (b) r, y ; or (c) θ, y.

In case (c), $\partial x/\partial r$ has no meaning. In cases (a) and (b), where $\partial x/\partial r$ has a meaning there is no reason to suppose that the two values of $(\partial x/\partial r)$ as determined from them, where we regard θ and y constants respectively, are equal. Some modification of the notation is therefore necessary to distinguish between these two values.

For the sake of distinction, these two values are respectively denoted as

$$\left[\frac{\partial x}{\partial r}\right]_\theta, \left[\frac{\partial x}{\partial r}\right]_y.$$

Thus $\left[\frac{\partial x}{\partial r}\right]_\theta$ *means the partial derivative of* x *w. r. to* r *when* r, θ *are the independent variables.*

From $x=r\cos\theta$, we have

$$\left[\frac{\partial x}{\partial r}\right]_\theta=\cos\theta.$$

To find $\left[\frac{\partial x}{\partial r}\right]_y$, we have to express x in terms of r and y.

From (i), we obtain

$$r^2=x^2+y^2 \text{ so that } x=\sqrt{(r^2-y^2)}.$$

$$\therefore \qquad \left[\frac{\partial x}{\partial r}\right]=\frac{r}{\sqrt{(r^2-y^2)}}=\frac{r}{a}=\sec\theta.$$

Thus
$$\left[\frac{\partial x}{\partial r}\right]_\theta\neq\left[\frac{\partial x}{\partial r}\right]_y.$$

Ex. 1. If $x=r\cos\theta$, $y=r\sin\theta$, find the values of

$$\left[\frac{\partial x}{\partial\theta}\right]_r, \left[\frac{\partial x}{\partial r}\right]_\theta, \left[\frac{\partial y}{\partial\theta}\right]_r, \left[\frac{\partial x}{\partial r}\right]_\theta, \left[\frac{\partial x}{\partial\theta}\right]_y \text{ and } \left[\frac{\partial y}{\partial r}\right]_x.$$

(P.U. 1938)

Ex. 2. v is the volume, s the curved surface, h, the height and r the radius of the circular base of a right circular cylinder, show that

$$(i)\ \left[\frac{\partial v}{\partial s}\right]_h\left[\frac{\partial s}{\partial v}\right]_r=2. \qquad (ii)\ \left[\frac{\partial h}{\partial r}\right]_v=2\left[\frac{\partial h}{\partial r}\right]_s.$$

Ex. 3. Explain the meanings of the partial differential co-efficients $\partial x/\partial r$ and $\partial r/\partial x$ where x, y are the rectangular cartesian co-ordinates of a point and r, θ are its polar co-ordinates. Prove that

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]. \qquad (P.U.\ 1936)$$

Ex. 4. Prove that

$$\frac{\partial^2\theta}{\partial x\partial y}=\frac{\partial^2(\log r)}{\partial x^2}=-\frac{\partial^2(\log r)}{\partial y^2}=-\frac{1}{r^2}\cos 2\theta,$$

where

$$x=r\cos\theta,\ y=r\sin\theta.$$

10·9. For the following developments, it will be assumed that $f(x, y)$ possesses continuous partial derivative *w. r.* to x and y in the domain of definition of the function.

10·91. Theorem on Total Differentials. We consider a function

$$z=f(x, y). \qquad ..(i)$$

Let (x, y), $(x+\delta x, y+\delta y)$ be any two points so that $\delta x, \delta y$ are the changes in the independent variables x and y. Let δz be the consequent change in z.

We have

$$z+\delta z=f(x+\delta x, y+\delta y). \qquad ...(ii)$$

From (i) and (ii), we get

$$\begin{aligned}\delta z&=f(x+\delta x, y+\delta y)-f(x, y)\\&=[f(x+\delta x, y+\delta y)-f(x+\delta x, y)]\\&\qquad+[f(x+\delta x, y)-f(x, y)] \qquad ...(iii)\end{aligned}$$

so that we have subtracted and added

$$f(x+\delta x, y).$$

Here the change δz has been expressed as the sum of two differences, to each of which we shall apply Lagrange's mean value theorem.

We regard $f(x+\delta x, y)$ as a function of y only ; $x+\delta x$ being supposed constant, so that by the mean value theorem,

$$f(x+\delta x, y+\delta y)-f(x+\delta x, y)=\delta y f_y(x+\delta x, y+\theta_1\delta y).$$

We write

$$f_y(x+\delta x, y+\theta_1\delta y)-f_y(x, y)=\varepsilon_2 \qquad ...(iv)$$

so that ε_2 depends on δx, δy, and because of the assumed continuity of $f_y(x, y)$ tends to zero as δx and δy both tend to 0.

Again we regard $f(x, y)$ as a function of x only, y being supposed constant, so that by the mean value theorem, we have

$$f(x+\delta x, y)-f(x, y)=\delta x f_x(x+\theta_2\delta x, y).$$

We write

$$f_x(x+\theta_2\delta x, y)-f_x(x, y)=\varepsilon_1 \qquad ...(v)$$

so that ε_1 depends upon δx and, because of the assumed continuity of $f_x(x, y)$, tends to 0 as δx tends to 0.

From (*iii*), (*iv*), (*v*), we get

$$\delta z=\delta x f_x(x, y)+\delta y f_y(x, y)+\varepsilon_1\delta x+\varepsilon_2\delta y$$
$$=\underbrace{\frac{\partial z}{\partial x}\cdot\delta x+\frac{\partial z}{\partial y}\cdot\delta y}+\underbrace{\varepsilon_1\delta x+\varepsilon_2\delta y}.$$

Thus the change δz in z consists of two points as marked. Of these the first is called the **differential** of z and is denoted by dz. Thus

$$dz=\frac{\partial z}{\partial x}\,\delta x+\frac{\partial z}{\partial y}\delta y. \qquad ...(vi)$$

Let

$$z=x$$

so that

$$dx=dz=1\,.\,\delta x=\delta x.$$

Similarly, by taking $z=y$, we show that

$$\delta y=dy.$$

Thus (*vi*) takes the form

$$\mathbf{dz}=\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\mathbf{dx}+\frac{\partial \mathbf{z}}{\partial \mathbf{y}}\mathbf{dy}.$$

It should be carefully noted that the *differentials dx and dy of the independent variables x and y are the* **actual** *changes δx and δy, **but** the differential dz of the dependent variable z is not the same as the change δz; it being the* **principal part** *of the increment δz.*

Cor. Approximate Calculations. From above we see *that the approximate change dz in z corresponding to the small changes δx and δy in x, y is*

$$\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy,$$

which has been denoted by dz.

Examples

1. *Find the percentage error in the area of an ellipse when an error of $+1$ per cent, is made in measuring the major and minor axes.*

If a, b, A denote semi-major axis, semi-minor axis and area of an ellipse respectively, we have the relation

$$A=\pi ab.$$

Here $$\frac{\partial A}{\partial a}=\pi b,\ \frac{\partial A}{\partial b}=\pi a.$$

$$\therefore \qquad dA=\pi b da+\pi a db.$$

Since we are given that

$$da=\frac{a}{100},\ \ db=\frac{b}{100},$$

$$\therefore \qquad dA=\frac{\pi ab}{100}+\frac{\pi ab}{100}=\frac{2A}{100}.$$

Therefore the percentage of error in $A=2$.

2. *The sides of an acute-angled triangle are measured. Prove that the increment in A due to small increments in a, b, c is given by the equation*

$$bc \,.\, \sin A \,.\, \delta A=-a(\cos C\delta b+\cos B\delta c-\delta a).$$

Supposing that the limits of error in the length of any side are $\pm\mu$ *per cent, where* μ *is small, prove that the limits of error in A are approximately*

$$\pm 1{\cdot}15(\mu a^2/bc \sin A) \text{ degrees.} \qquad (M.T.)$$

From elementary Trigonometry, we know that

$$2 \cos A=\frac{b^2+c^2-a^2}{bc}.$$

Here A is a function of three variables, a, b, c.

$$\therefore \quad -2 \sin A \,.\, dA=\frac{2b^2c-c(b^2+c^2-a^2)}{b^2c^2}\,db$$

$$+\frac{2c^2b-b(b^2+c^2-a^2)}{b^2c^2}\,dc-\frac{2a}{bc}\,da,$$

or $$-2 \sin A \,.\, dA=\frac{a^2+b^2-c^2}{b^2c}\,db+\frac{c^2+a^2-b^2}{c^2b}\,dc-\frac{2a}{bc}\,da.$$

$$=\frac{2ab\cos C}{b^2c}\,db+\frac{2\,ca\cos B}{c^2b}\,dc-\frac{2a}{bc}\,da.$$

$$\therefore \quad bc \sin A \,.\, dA=-a(\cos C \,.\, db+\cos B \,.\, dc-da).$$

The limits of the errors db, dc, da are

$$\pm b\mu/100, \qquad \pm c\mu/100, \qquad \pm a\mu/100.$$

As the triangle is acute-angled, therefore $\cos B$, $\cos C$ are positive. Therefore the limits of the error dA are

$$\frac{-a(b\cos C+c\cos B+a)}{bc \sin A}\cdot\frac{\mu}{100}, \quad \frac{-a(-b\cos C-c\cos B-a)}{bc \sin A}\cdot\frac{\mu}{100}$$

i.e., $$\pm\frac{2a^2}{bc \sin A}\cdot\frac{\mu}{100} \text{ radians}$$

or $$\pm\frac{2a^2}{bc \sin A}\cdot\frac{\mu}{100}\cdot\frac{180}{\pi} \text{ degrees}$$

or $$\pm(1{\cdot}15)\frac{\mu a^2}{bc \sin A} \text{ degrees approximately.}$$

Exercises

1. Find the percentage error in calculating the area of a rectangle when an error of 2 per cent is made in measuring its sides.

2. Show that the error in calculating the time period of a pendulum at any place is zero if an error of $+\mu$ per cent be made in measuring its length and gravity at the place.

3. In a triangle ABC, measurements are taken of the side c and angles A, B and length a is calculated from these measurements. If Δc, ΔA, ΔB are the small errors in these measurements, show that the error Δa in a is given by

$$\Delta a = \frac{c \sin B}{\sin^2 (A+B)} \Delta A + \frac{\sin A}{\sin (A+B)} \Delta c - a \cot (A+B) \Delta B.$$

4. ABC is an acute-angled triangle with fixed base BC. If δb, δc, δA and δB are small increments in b, c, A and B respectively, the vertex A is given a small displacement δx parallel to BC, prove that

(*i*) $c\delta b + b\delta c + bc \cot A \, \delta A = 0$. (*ii*) $c\delta B + \sin B \delta x = 0$. (*M.T.*)

5. The area of a triangle whose sides are a, b, c is $\triangle$. Prove that the error corresponding to errors δa, δb, δc in the sides is approximately given by

$$2\triangle \delta \Delta = s^2 \delta p - s\delta q - abc\delta s,$$

where $2s = a+b+c$, $2p = a^2+b^2+c^2$, $3q = a^3+b^3+c^3$. (*M.T.*)

6. The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to

$$s^2 D^{\frac{2}{3}} t^2.$$

Find approximately the percentage increase of work necessary when the displacement is increased by 1%, the time diminished by 1%, and the distance diminished by 3%.

7. The height h and the semi-vertical angle α of a cone are measured, and from them A, the total area of the cone including the base, is calculated. If h and α are in error by small quantities δh and $\delta \alpha$ respectively, find the corresponding error in the area. Show further that if $\alpha = \pi/6$, an error of $+1$ per cent in h will be approximately compensated by an error of $-0{\cdot}33$ degree in α. (*M.T.*)

10·92. Composite functions.

Let

$$z = f(x, y) \; ; \qquad \ldots(i)$$

and let

$$x = \varphi(t) \qquad \ldots(ii)$$

$$y = \psi(t), \qquad \ldots(iii)$$

so that x, y are themselves functions of a third variable t.

The functional equations, (*i*), (*ii*), (*iii*) are said to define z as a *composite function* of t.

Again, let

$$x = \phi(u, v,), \qquad \ldots(iv)$$

$$y = \psi(u, v) \; ; \qquad \ldots(v)$$

so that x, y are functions of the variables u, v. Here the functional equations (*i*), (*iv*), (*v*) define z as a function of u, v, which is called *a composite function* of u and v.

10·93. Differentiation of composite functions.

Let

$$z=f(x, y),$$

possess continuous partial derivatives and

let $$x=\varphi(t),$$
$$y=\psi(t),$$

possess continuous derivatives.

Then

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

Let t, $t+\delta t$ be any two values. Let δx, δy, δz be the changes in x, y, z consequent to the change δt in t. We have

$$x+\delta x=\varphi(t+\delta t),\ y+\delta y=\psi(t+\delta t)$$
$$z+\delta z=f(x+\delta x,\ y+\delta y)$$
$$\therefore \qquad \delta z=f(x+\delta x,\ y+\delta y)-f(x, y)$$
$$=[f(x+\delta x,\ y+\delta y)-f(x, y+\delta y)]$$
$$+[f(x, y+\delta y)-f(x, y)].$$

As in § 10·91, p. 207, we apply Lagrange's mean value theorem to the two differences on the right, and obtain

$$\delta z=\delta x f_x(x+\theta_1\delta x,\ y+\delta y)+\delta y f_y(x,\ y+\theta_2\delta y),$$
$$(0<\theta_1,\ \theta_2<1).$$

$$\therefore \qquad \frac{\delta z}{\delta t}=\frac{\delta x}{\delta t}\cdot f_x(x+\theta_1\delta x,\ y+\delta y)+\frac{\delta y}{\delta t}f_y(x,\ y+\theta_2\delta y). \quad ...(i)$$

Let $\delta t\to 0$ so that δx and $\delta y\to 0$.

Because of the continuity of partial derivatives, we have

$$\lim_{(\delta x,\ \delta y)\to(0,\ 0)} f_x(x+\theta_1\ \delta x,\ y+\delta y)=f_x(x,\ y)=\frac{\partial z}{\partial y},$$

$$\lim_{\delta y\to 0} f_y(x,\ y+\theta_2\delta y)=f_y(x,\ y)=\frac{\partial z}{\partial y}.$$

Hence, in the limit, (i) becomes

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}. \quad ..(ii)$$

Cor 1. Let

$$z=f(x,\ y),$$

possess continuous first order partial derivatives *w.r.* to x, y.

Let

$$x=\varphi(u,\ v,),$$
$$y=\psi(u,\ v),$$

possess continuous first order partial derivatives.

To obtain $\partial z/\partial u$, we regard v as a constant so that x and y may be supposed to be functions of u only. Then, by the above theorem, we have

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

It may similarly be shown that

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

Examples

1. *Find dz/dt when*

$$z=xy^2+x^2y,\ x=at^2,\ y=2at.$$

Verify by direct substitution.

Now

$$\frac{\partial z}{\partial x}=y^2+2xy,\ \frac{\partial z}{\partial y}=2xy+x^2,$$

$$\frac{dx}{dt}=2at,\ \frac{dy}{dt}=2a.$$

Substituting these values in (*ii*), § 10·93, p. 210, we get

$$\frac{dz}{dt}=(y^2+2xy)\,2at+(2xy+x^2)2a$$
$$=(4a^2t^2+4a^2t^3)\,2at+(4a^2t^3+a^2t^4)2a$$
$$=a^3(16t^3+10t^4).$$

Again

$$z=x^2y+xy^2=2a^3t^5+4a^3t^4.$$

$$\therefore\quad \frac{dz}{dt}=10a^3t^4+16a^3t^3=a^3(16t^3+10t^4).$$

Hence the verification.

2. *z is a function of x and y. Prove that if*

$$x=e^u+e^{-v},\ y=e^{-u}-e^v,$$

then

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}\,; \qquad (D.U. 1955;\ L.U.)$$

We look upon z as a composite function of u, v.

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$
$$=\frac{\partial z}{\partial x}\cdot e^u-\frac{\partial z}{\partial y}\cdot e^{-u}\,;$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= -\frac{\partial z}{\partial x} \cdot e^{-v} - \frac{\partial z}{\partial y} \cdot e^{v}.$$

Subtracting, we get

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u}+e^{-v}) - \frac{\partial z}{\partial y}(e^{-u}-e^{v})$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

3. *If $H=f(y-z, z-x, x-y)$; prove that,*

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} - \frac{\partial H}{\partial z} = 0.$$

Let

$$u=y-z,\ v=z-x,\ w=x-y,$$

so that

$$H=f(u, v, w).$$

We have expressed H as a composite function of x, y, z.

$$\therefore \quad \frac{\partial H}{\partial x} = \frac{\partial H}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial H}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial H}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$= \frac{\partial H}{\partial u} \cdot 0 + \frac{\partial H}{\partial v}(-1) + \frac{\partial H}{\partial w} \cdot 1$$

$$= -\frac{\partial H}{\partial v} + \frac{\partial H}{\partial w}. \qquad \ldots(i)$$

Similarly

$$\frac{\partial H}{\partial y} = -\frac{\partial H}{\partial w} + \frac{\partial H}{\partial u}$$

$$\frac{\partial H}{\partial z} = -\frac{\partial H}{\partial u} + \frac{\partial H}{\partial v}.$$

Adding, we get the result.

4. *H is a homogeneous function of x, y, z of order n ; prove that*

$$x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} + z\frac{\partial H}{\partial z} = nz.$$

[This is Euler's theorem for a homogeneous function of three independent variables.] We have

$$H = x^{n}f\left[\frac{y}{x}, \frac{z}{x}\right] = xf(u, v) \text{ where } y/x=u,\ z/x=v.$$

$$\therefore \quad \frac{\partial H}{\partial x} = nx^{n-1}f(u, v) + x^{n}\left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}\right).$$

But

$$\frac{\partial u}{\partial x}=-\frac{y}{x^2}, \frac{\partial v}{\partial x}=-\frac{z}{x^2}.$$

Hence

$$\frac{\partial H}{\partial x}=nx^{n-1}f(u, v)-x^{n-2}\left(y\frac{\partial f}{\partial u}+z\frac{\partial f}{\partial v}\right).$$

Again

$$\frac{\partial H}{\partial y}=x^n\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$$

$$=x^{n-1}\frac{\partial f}{\partial u}, \text{ for } \frac{\partial u}{\partial y}=\frac{1}{x}\cdot\frac{\partial v}{\partial y}=0.$$

Similarly

$$\frac{\partial H}{\partial z}=x^{n-1}\frac{\partial f}{\partial v}.$$

$$\therefore \qquad x\frac{\partial H}{\partial x}+y\frac{\partial H}{\partial y}+z\frac{\partial H}{\partial z}=nx^nf(u, v)=nz.$$

10·94. Implicit functions. Let $f(x, y)$ be any function of two variables. Ordinarily, we say that, since on solving the equation

$$f(x, y)=0 \qquad ...(i)$$

we can obtain y as a function of x, the equation (i) defines y as an *implicit* function of x.

There arises a theoretical difficulty here. Without investigation we cannot say that corresponding to each value of x, the equation (i) must determine one and only one value of y so that the equation (i) always determines, y, as a function of x and in fact, this is not the case, in general. The investigation of the conditions under which the equation (i) does define y as a function of x is not, however, within the scope of this book.

Assuming that the conditions under which the equation (i) defines y as a derivable function of x are satisfied, we shall *now obtain the values of dy/dx and d^2y/dx^2 in terms of the partial derivatives* $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y^2$ of 'f' *w.r.* to x and y.

Now, $f(x, y)$ is a function of two variables x, y and y is again a function of x so that we may regard $f(x, y)$ as a composite function of x. Its derivative with respect to x is

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} \; i.e., \; \frac{\partial f}{\partial x}+\frac{\partial f}{\partial x}\cdot\frac{dy}{dx}.$$

Also $f(x, y)$ considered as a function of x alone, is identically equal to 0. Therefore its derivative *w.r.* to x is 0.

Hence $$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial x}\cdot\frac{dy}{dx}=0.$$

i.e.,
$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y}, \text{ if } f_y \neq 0.$$

Differentiating again *w.r.* to x, regarding $\partial f/\partial x$ and $\partial f/\partial y$ as composite functions of x, we get

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{dy}{dx}\right)\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{dy}{dx}\right)}{\left(\frac{\partial f}{\partial y}\right)^3}$$

$$= -\frac{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial f}{\partial y}\right)^3}.$$

Hence

$$\mathbf{\frac{dy}{dx} = -\frac{f_x}{f_y}}, \qquad \ldots(ii)$$

and

$$\mathbf{\frac{d^2y}{dx^2} = -\frac{f_x{}^2(f_y)^2 - 2f_{yx}f_xf_y + f_y{}^2(f_x)^2}{(f_y)^3}}. \qquad \ldots(iii)$$

Without making use of § 10·93, we may obtain dy/dx also as follows :

Now $$f(x, y) = 0.$$

Let δx be the increment in x and δy the consequent increment in y, so that

$$f(x+\delta x, y+\delta y) = 0.$$

$\therefore$
$$f(x+\delta x, y+\delta y) - f(x, y) = 0,$$

or
$$f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y) = 0.$$

$\therefore$
$$\delta x f_x(x+\theta_1 \delta x, y+\delta y) + \delta y f_y(x, y+\theta_2 \delta y) = 0,$$

or
$$\frac{\delta y}{\delta x} = -\frac{f_x(x+\theta_1\delta x, y+\delta y)}{f_y(x, y+\theta_2 y)}.$$

Let $\delta x \to 0$.

$\therefore$ $$\mathbf{\frac{dy}{dx} = -\frac{f_x}{f_y}}, \text{ if } \mathbf{f_y \neq 0}.$$

Example

Prove that if $y^3 - 3ax^2 + x^3 = 0$, then

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Let

$$f(x, y) \equiv y^3 - 3ax^2 + x^3 = 0.$$

$$\therefore \quad f_x = -6ax + 3x^2, f_y = 3y^2 ;$$

$$f_{x^2} = -6a + 6x, f_{xy} = 0, f_{y^2} = 6y.$$

Substituting these values in (*iii*), on p. 214, we get

$$\frac{d^2y}{dx^2} = -\frac{6(x-a)9y^4 + (3x^2 - 6ax)^2 6y}{27y^6}$$

$$= -2\frac{(x-a)(3ax^2 - x^3) + (x^2 - 2ax)^2}{y^5}$$

$$= -2\frac{a^2x^2}{y^5}.$$

Thus
$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Or, directly, differentiating the given relation *w.r.* to x,

$$3y^2\frac{dy}{dx} = 6ax - 3x^2,$$

or
$$\frac{dy}{dx} = \frac{2ax - x^2}{y^2}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{(2a-2x)y^2 - 2y(2ax - x^2)\, dy/dx}{y^4}$$

$$= \frac{(2a-2x)y^2 - 2y(2ax - x^2)(2ax - x^2)/y^2}{y^4}$$

$$= \frac{2(a-x)y^3 - 2(2ax - x^2)^2}{y^5} = -\frac{2a^2x^2}{y^5},$$

as before.

Exercises

1. If $u = x - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$, find $\partial u/\partial r$.
2. If $z = (\cos y)/x$ and $x = u^2 - v$, $y = e^v$. find $\partial z/\partial v$.
3. If $z = \dfrac{\sin u}{\cos v}$, $u = \dfrac{\cos y}{\sin x}$, $v = \dfrac{\cos x}{\sin y}$, find $\partial z/\partial x$.
4. If $u = (x+y)/(1-xy)$; $x = \tan(2r - s^2)$, $y = \cot(r^2 s)$, find $\partial u/\partial s$.
5. Find dy/dx in the following cases :—

   (*i*) $x \sin(x-y) - (x+y) = 0$. (*ii*) $y^{x^y} = \sin x$.

   (*iii*) $(\cos x)^y - (\sin y)^x = 0$. (*iv*) $x^y = y^x$.

   (*v*) $(\tan x)^y + y^{\cot x} = a$. *(P.U. 1955)*
6. If $F(x, y, z) = 0$ find $\partial z/\partial x$, $\partial z/\partial y$. *(P.U. 1938)*
7. If $z = xyf(y/x)$ and z is a constant, show that

$$\frac{f'(y/x)}{f(y/x)} = \frac{x[y + x(dy/dx)]}{y[y - x(dy/dx)]}.$$ *(P.U. 1934)*

8. If $f(x, y)=0$, $\varphi(y, z)=0$, show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \varphi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \varphi}{\partial y}.$$ *(P.U.)*

9. If $x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}=a$, show that

$$\frac{d^2y}{dx^2} = \frac{a}{(1-x^2)^{\frac{3}{2}}}.$$ *(P.U. 1935)*

10. If u and v are functions of x and y defined by

$$x=u+e^{-v} \sin u, \; y=v+e^{-v} \cos u,$$

prove that

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$ *(P.U. 1936)*

11. If A, B, C are the angles of a triangle such that

$$\sin^2 A+\sin^2 B+\sin^2 C=\text{constant},$$

prove that

$$\frac{dA}{dB} = \frac{\tan C-\tan B}{\tan A-\tan C}.$$ *(P.U. 1937)*

12. If $ax^2+2hxy+by^2+2gx+2fy+c=0$, prove that

$$\frac{d^2y}{dx^2} = \frac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^3}.$$

13. Show that at the point of the surface

$$x^x y^y z^z = c$$

where $x=y=z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -\left[x \log ex \right]^{-1}.$$

14. Find d^2y/dx^2 in the following cases :—

(*i*) $x^3+y^3=3axy$. (*P.U.*) (*ii*) $x^4+y^4=4a^2xy$.

(*iii*) $x^5+y^5=5a^3xy$. (*iv*) $x^5+y^5=5a^3x^2$.

15. If $f(x, y)=0$ and $f_x \neq 0$, prove that

$$\frac{dx}{dy} = -\frac{f_y}{f_x}, \quad \frac{d^2x}{dy^2} = -\frac{f_{x}^2 (f_y)^2 - 2f_{xy} f_x f_y + f_y^2 (f_x)^2}{(f_x)^3}.$$

16. Given that

$$f(x, y) \equiv x^3+y^3-3axy=0, \text{ show that}$$

$$\frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2} = \frac{4a^6}{xy(xy-2a^2)^3}.$$

17. If u is a homogeneous function of the nth degree in (x, y, z) and if

$$u=f(X, Y, Z),$$

where X, Y, Z are the first differential co-efficients of, u, with respect to x, y, z respectively, prove that

$$X \frac{\partial f}{\partial X} + Y \frac{\partial f}{\partial Y} + Z \frac{\partial f}{\partial Z} = \frac{n}{n-1} u.$$

(Delhi Hons. 1950)

APPENDIX

EQUALITY OF REPEATED DERIVATIVES

A. 1. Equality of f_{xy} and f_{yx}. It has been seen that the two repeated second order partial derivatives are generally equal. They are not, however, *always* equal as is shown below by considering two examples. It is easy to see *a priori* also why $f_{yx}(a, b)$ may be different from $f_{xy}(a, b)$.

We have

$$f_{xy}(a, b) = \lim_{h \to 0} \frac{f_y(a+h, b) - f_y(a, b)}{h}.$$

Also $f_y(a+h, b) = \lim_{k \to 0} \dfrac{f(a+h, b+k) - f(a+h, b)}{k}$,

and $\quad f_y(a, b) = \lim_{k \to 0} \dfrac{f(a, b+k) - f(a, b)}{k}.$

$$\therefore f_{xy}(a, b) = \lim_{h \to 0} \lim_{k \to 0} \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{\phi(h, k)}{hk}, \text{ say.}$$

It may similarly be shown that

$$f_{yx}(a, b) = \lim_{k \to 0} \lim_{h \to 0} \frac{\phi(h, k)}{hk}.$$

Thus we see that $f_{xy}(a, b)$ and $f_{yx}(a, b)$ are *repeated* limits of the same expression taken in *different* orders. Also the two repeated limits may not be equal, as, for example

$$\lim_{h \to 0} \lim_{k \to 0} \frac{h-k}{h+k} = \lim_{h \to 0} \frac{h}{h} = 1,$$

and

$$\lim_{k \to 0} \lim_{h \to 0} \frac{h-k}{h+k} = \lim_{k \to 0} \frac{-k}{k} = -1.$$

Examples

1. *Prove that $f_{xy} \neq f_{yx}$ at the origin for the function*

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2},$$

when x, y are not simultaneously zero and $f(0, 0) = 0$. (D.U. Hons. 1954)

We have $f_{xy}(0, 0) = \lim_{h \to 0} \dfrac{f_y(0+h, 0) - f_y(0, 0)}{h}$ ...(1)

Also $$f_y(0, 0)=\lim_{k\to 0}\frac{f(0, 0+k)-f(0, 0)}{k} \qquad \ldots(2)$$

$$=\lim_{k\to 0}\frac{f(0, k)-f(0, 0)}{k}=\lim_{k\to 0}\frac{0}{k}=0.$$

and $$f_y(h, 0)=\lim_{k\to 0}\frac{f(h, 0+k)-f(h, 0)}{k}$$

$$=\lim_{h\to 0}\frac{hk(h^2-k^2)}{k(h^2+k^2)}=h. \qquad \ldots(3)$$

Thus from (1), (2) and (3)

$$f_{xy}(0, 0)=\lim_{h\to 0}\frac{h-0}{h}=1.$$

Again $$f_{yx}(0, 0)=\lim_{k\to 0}\frac{f_x(0, 0+k)-f_x(0, 0)}{k} \qquad \ldots(4)$$

Also $$f_x(0, 0)=\lim_{h\to 0}\frac{f(0+h, 0)-f(0, 0)}{h}=\lim_{h\to 0}\frac{0}{h}=0, \qquad \ldots(5)$$

and $$f_x(0, k)=\lim_{h\to 0}\frac{f(0+h, k)-f(0, k)}{h}$$

$$=\lim_{h\to 0}\frac{hk(h^2-k^2)}{h(h^2+k^2)}=-k. \qquad \ldots(6)$$

From (4), (5) and (6),

$$f_{yx}(0, 0)=\lim_{k\to 0}\frac{-k-0}{k}=-1.$$

Thus $f_{xy}(0, 0)=1\neq -1=f_{yx}(0, 0)$,

2. *Show that*

$$f_{xy}(0, 0)\neq f_{yx}(0, 0),$$

where $$f(x, y)=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$$

if $xy\neq 0$ and is zero elsewhere. (*B.U. 1953*)

We have

$$f_{xy}(0, 0)=\lim_{h\to 0}\frac{f_y(h, 0)-f_y(0, 0)}{h},$$

$$f_y(h, 0)=\lim_{k\to 0}\frac{f(h, k)-f(h, 0)}{k}$$

$$=\lim_{k\to 0}\frac{1}{k}\left(h^2\tan^{-1}\frac{k}{h}-k^2\tan^{-1}\frac{h}{k}\right)$$

$$= \lim_{k \to 0} \left[h \left(\frac{\tan^{-1} k/h}{k/h} \right) - k \tan^{-1} \frac{h}{k} \right]$$

$$= h \,.\, 1 - 0 = h, \text{ for } [\tan^{-1} t/t] \to 1 \text{ as } t \to 0.$$

$$f_y(0,\ 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = 0.$$

$$\therefore \qquad f_{xy}(0,\ 0) = \lim_{h \to 0} \frac{h - 0}{h} = 1.$$

We may similarly show that

$$f_{yx}(0, 0) = -1.$$

Hence the result.

A. 2. Equality of f_{xy} and f_{xy}. Theorem. *If $z = f(x, y)$ possesses continuous second order partial derivatives $\partial^2 z/\partial x \partial y$ and $\partial^2 z/\partial y \partial x$, then*

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Consider the expression

$$\varphi(h, k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y).$$

For the sake of brevity, we write

$$\psi(x) = f(x, y+k) - f(x, y), \qquad \ldots(1)$$

so that

$$\varphi(h, k) = \psi(x+h) - \psi(x). \qquad \ldots(2)$$

By Lagrange's mean value theorem,

$$\psi(x+h) - \psi(x) = h\psi'(x + \theta_1 h). \qquad 0 < \theta_1 < 1. \quad \ldots(3)$$

Also

$$\psi'(x) = f_x(x, y+k) - f_x(x, y). \qquad \ldots(4)$$

$\therefore$

$$\varphi(h, k) = h[f_x(x + \theta_1 h, y+k) - f_x(x + \theta_1 h, y)]. \qquad \ldots(5)$$

Again applying the mean value theorem to the right side of (5), we obtain

$$\varphi(h, k) = hk\, f_{yx}(x + \theta_1 h, y + \theta_2 k). \qquad 0 < \theta_2 < 1.$$

Thus

$$\frac{\varphi(h, k)}{hk} = f_{yx}(x + \theta_1 h, y + \theta_2 k). \qquad \ldots(6)$$

Again considering $F(y) = f(x+h, y) - f(x, y)$ instead of $\psi(x)$ and proceeding as before, we may prove that

$$\frac{\varphi(h, k)}{hk} = f_{xy}(x + \theta_3 h, y + \theta_4 k). \qquad \ldots(7)$$

$$\therefore \qquad f_{xy}(x + \theta_1 h, y + \theta_2 k) = f_{xy}(x + \theta_3 h, y + \theta_4 k).$$

Let $h \to 0$ and $k \to 0$. Then, because of the assumed continuity of the Partial Derivatives, we obtain

$$f_{yx}(x, y) = f_{xy}(x, y).$$

A. 3. Taylor's theorem for a function of two variables.

If $f(x, y)$ possesses continuous partial derivatives of the nth order in any neighbourhood of a point (a, b) and if $(a+h, b+k)$ be any point of this neighbourhood, then there exists a positive number θ which is less than 1, such that

$$f(a+h, b+k) = f(a, b) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(a, b)$$

$$+ \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(a, b) + \ldots\ldots\ldots\ldots$$

$$+ \ldots\ldots\ldots + \frac{1}{(n-1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n-1} f(a, b)$$

$$+ \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(a+\theta h, b+\theta k). \quad (0<\theta<1).$$

Lemma.

We write

$$z = f(x, y) ;$$

and

$$x = a+ht, \; y = b+kt,$$

so that z is a function of t.

Now, we have, by § 10·93, p. 210,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{\partial z}{\partial x} h + \frac{\partial z}{\partial y} k = h \frac{\partial z}{\partial x} + k \frac{\partial z}{\partial y}.$$

Now, we agree to write

$$h \frac{\partial z}{\partial x} + k \frac{\partial z}{\partial y} \equiv \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) z,$$

in the form of the **operator**

$$h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}$$

operating on the **operand** z.

$$\therefore \qquad \frac{dz}{dt} = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) z$$

This equality shows that the operators

$$\frac{d}{dt} \text{ and } h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}$$

are equivalent.

Employing the equivalence of these two operators, we obtain

$$\frac{d^2z}{dt^2}=\frac{d}{dt}\left(\frac{dz}{dt}\right)$$
$$=\frac{d}{dt}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)z$$
$$=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)z.$$

If, now, we agree to write

$$\left(h\frac{\partial}{dx}+k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)\equiv\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2,$$

we have

$$\frac{d^2z}{dt^2}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2z.$$

Continuing in this manner, we see that

$$\frac{d^nz}{dt^n}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^nz,$$

where the operator

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n$$

implies the repeated application of the operator

$$\left(h\frac{\partial}{dx}+k\frac{\partial}{\partial y}\right)$$

n times. Thus we have arrived at the following result :

If $z=f(x, y)$ and $x=a+ht$, $y=b+kt$, where a, b, h, k, are constants, then

$$\frac{d^nz}{dt^n}=\left(h\frac{\partial}{dx}+k\frac{\partial}{\partial y}\right)^nz.$$

Proof of Taylor's Theorem.

We write

$$f(x, y)=f(a+ht, b+kt)=g(t),$$

and apply Maclaurin's theorem to the function $g(t)$ of the single variable t.

There exists a positive number θ between 0 and 1 such that

$$g(t)=g(0)+tg'(0)+\frac{t^2}{2\,!}g''(0)+\dots\dots\dots\dots$$
$$+\dots\dots+\frac{t^{n-1}}{(n-1)\,!}g^{n-1}(0)+\frac{t^n}{n\,!}g^n(\theta t).$$

For $t=1$, this becomes $(0<\theta<1)$ ..(*i*)

$$g(1)=g(0)+g'(0)+\frac{1}{2\,!}g''(0)+\dots\dots$$
$$+\dots+\frac{1}{(n-1)\,!}g^{n-1}(0)+\frac{1}{n\,!}g^n(\theta).$$

Now

$$g(1)=f(a+h,\ b+k).$$
$$g(0)=f(a,\ b).$$

Also since

$$g^n(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x,\ y),$$

$$\therefore \quad g'(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,\ b),$$

$$g''(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,\ b),$$

$$\cdot\cdot \qquad \cdot\cdot \qquad \cdot\cdot \qquad \cdot\cdot$$

$$g^{n-1}(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,\ b),$$

$$g^n(\theta)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h,\ b+\theta k).$$

Substituting these values in (i), we have the Taylor's theorem's as stated.

Exercises

1. Show that

(i) $$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 z=h^2\frac{\partial^2 z}{\partial x^2}+2hk\frac{\partial^2 z}{\partial x\partial y}+k^2\frac{\partial^2 z}{\partial y^2}.$$

(ii) $$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 z=h^3\frac{\partial^3 z}{\partial x^3}+3h^2k\frac{\partial^3 z}{\partial x^2\partial y}$$
$$+3hk^2\frac{\partial^3 z}{\partial x\partial y^2}+k^3\frac{d^3 z}{\partial y^3}.$$

2. By mathematical indication or otherwise, show that

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n z=h^n\frac{\partial^n z}{\partial x^n}+{}^nc_1h^{n-1}k\frac{\partial^n z}{\partial x^{n-1}\partial y}+\ldots\ldots$$
$$+\ldots+{}^nc_rh^{n-r}k^r\frac{\partial^n z}{\partial x^{n-r}\partial y^r}+\ldots\ldots$$
$$+\ldots\ldots\ldots\ldots\ldots+k^n\frac{\partial^n z}{\partial y^n}.$$

3. Show that

(i) $\sin x\sin y=xy-\frac{1}{6}[(x^3+3xy^2)\cos\theta x\sin\theta y$
$+(y^3+3x^2y)\sin\theta x\cos\theta y]\ ;\ 0<\theta<1.$

(ii) $e^{ax}\sin by=by+abxy$
$+\frac{1}{6}e^u[(a^3x^3-3ab^2xy^2)\sin v+(3a^2bx^2y-b^3y^3)\cos v],$

where $u=a\theta x$, $v=b\theta y$.

MAXIMA AND MINIMA

A. 4. Maxima and Minima of a function of two variables.

Def. Maximum value. *$f(a, b)$ is a* **maximum** *value of the function $f(x, y)$, if there exists some neighbourhood of the point (a, b) such that for every point $(a+h, b+k)$ of this neighbourhood,*

$$\mathbf{f(a, b)>f(a+h, b+k).}$$

Minimum value. *$f(a, b)$ is* **minimum** *value of the function $f(x, y)$, if there exists some neighbourhood of the point (a, b) such that for every point $(a+h, b+k)$ of this neighbourhood*

$$\mathbf{f(a, b)<f(a+h, b+k).}$$

Extreme value. *$f(a, b)$ is said to be an extreme value of $f(x, y)$, if it is a maximum or a minimum value.*

A. 41. *The* **necessary** *conditions for $f(a, b)$ to be an extreme value of $f(x, y)$ are that*

$$f_x(a, b)=0, f_y(a, b)=0.$$

If $f(a, b)$ is an extreme value of the function $f(x, y)$ of two variables x and y then, clearly, it is also an extreme value of the function $f(x, b)$ of one variable x for $x=a$ and as such its derivative $f_x(a, b)$ for $x=a$ must necessarily be zero. Similarly we may show that

$$f_y(a, b)=0.$$

Note 1. As in the case of single variable, the conditions obtained above are necessary and not sufficient. For example, if $f(x, y)=0$ when $x=0$ or $y=0$ and $f(x, y)=1$ elsewhere, then

$$f_x(0, 0)=0, f_y(0, 0)=0,$$

but $f(0, 0)$ is not an extreme value.

Note 2. Stationary value. A function $f(x, y)$ is said to be stationary for $x=a$, $y=b$ or $f(a, b)$ is said to be a stationary value of $f(x, y)$ if

$$f_x(a, b)=0, f_y(a, b)=0.$$

Thus every extreme value is a stationary value but the converse may not be true.

A. 42. Sufficient conditions. *To show that*

$$f_x(a, b)=0, f_y(a, b)=0,$$

and

$$f_{x^2}(a, b)=A, f_{xy}(a, b)=B, f_{y^2}(a, b)=C$$

then

(*i*) *$f(a, b)$ is a max. value if $AC-B^2>0$ and $A<0$,*

(*ii*) *$f(a, b)$ is a min. value if $AC-B^2>0$ and $A>0$,*

(*iii*) *$f(a, b)$ is not an extreme value if*

$$AC-B^2<0,$$

(*iv*) *the case is doubtful and needs further consideration, if*

$$AC-B^2=0.$$

It may be noticed that $A\neq 0$ *if*

$$AC-B^2>0.$$

By Taylor's Theorem with remainder after three terms, we obtain

$$f(a+h, b+k)=f(a, b)+\left(h \frac{\partial}{\partial x}+k \frac{\partial}{\partial y}\right)f(a, b)$$
$$+\frac{1}{2\,!}\left(h \frac{\partial}{\partial x}+k \frac{\partial}{\partial y}\right)^2 f(a, b)$$
$$+\frac{1}{3\,!}\left(h \frac{\partial}{\partial x}+k \frac{\partial}{\partial y}\right)^3 f(a+\theta h, b+\theta k),$$

or $$f(a+h, b+k)-f(a, b)=hf_x(a, b)+kf_y(a, b)$$
$$+\tfrac{1}{2}[h^2f_{x^2}(a, b)+2hkf_{xy}(a, b)+k^2f_{y^2}(a, b)]$$
$$+\tfrac{1}{6}[h^3f_{x^3}(u, v)+3h^2kf_{x^2y}(u, v)+3hk^2f_{xy^2}(u, v)+k^3f_{y^3}(u, v)],$$

where $u=a+\theta h$, $v=b+\theta k$.

Now

$$f_x(a, b)=0, f_y(a, b)=0.$$

Also, we have written

$$f_{x^2}(a, b)=A, f_{xy}(a, b)=B, f_{y^2}(a, b)=C$$

$$\therefore \quad f(a+h, b+k)-f(a, b)=\tfrac{1}{2}(Ah^2+2Bhk+Ck^2)+\rho,$$

where ρ is of the third degree in h and k.

We assume that for sufficiently small values of h, k, the sign of

$$\tfrac{1}{2}(Ah^2+2Bhk+Ck^2)+\rho$$

is the same as that of

$$Ah^2+2Bhk+Ck^2.$$

Case 1.

Let $AC-B^2>0$. In this case neither A nor C can be zero. We write

$$Ah^2+2Bhk+Ck^2=\frac{1}{A}[(Ah+Bk)^2+(AC-B^2)k^2].$$

Since $AC-B^2$ is positive, we see that

$$(Ah+Bk)^2+(AC-B^2)k^2$$

is always positive except when

$$Ah+Bk=0, k=0.$$

i.e., when $h=0$, $k=0$, when it is zero.

Thus we see that $Ah^2+2Bhk+Ck^2$ always retains the same sign which is that of A.

Thus $f(a, b)$ is an extreme value in this case and will be a maximum or minimum according as A is negative or positive.

Case II.

Let $AC-B^2<0$.

Firstly, we suppose that $A\neq 0$. We write

$$Ah^2+2Bhk+Ck^2=[(Ah+Bk)^2+(AC-B^2)k^2]/A.$$

Since $AC-B^2$ is negative, we see that this expression takes up values with different signs when $k=0$ and when $Ah+Bk=0$.

Thus in this case $f(a, b)$ is not an extreme value.

The proof is similar when $C\neq 0$.

In case $A=0$ as well as $C=0$, we have

$$Ah^2+2Bhk+Ck^2=2Bhk,$$

so that the expression does assume values with different signs and accordingly $f(a, b)$ is not an extreme value.

Case III.

Let $AC-B^2=0$.

Suppose that $A\neq 0$. We have

$$Ah^2+2Bhk+Ck^2=[(Ah+Bk)^2+(AC-B^2)k^2]/A$$
$$=(Ah+Bk)^2/A.$$

Here the expression becomes zero, when

$$Ah+Bk=0,$$

so that the nature of the sign of

$$f(a+h, b+k)-f(a, b)$$

depends upon the consideration of ρ. The case is, therefore, doubtful.

If, now, $A=0$ then, because of the condition $AC=B^2$, we must have $B=0$.

$$\therefore \qquad Ah^2+2Bhk+Ck^2=Ck^2,$$

so that the expression is zero when $k=0$ whatever h may be. The case is again doubtful.

Examples

1. *Find the extreme values of*

$$xy(a-x-y).$$

We write

$$f(x, y)=xy(a-x-y)=axy-x^2y-xy^2.$$

$$\therefore \qquad f_x(x, y)=ay-2xy-y^2,$$
$$f_y(x, y)=ax-x^2-2xy.$$
$$A=f_{x^2}(x, y)=-2y,$$
$$B=f_{xy}(x, y)=a-2x-2y$$
$$C=f_{y^2}(x, y)=-2x.$$

We now solve the equations $f_x=0$ and $f_y=0$.

Thus we have

$$\left.\begin{aligned} ay-2xy-y^2=0, \\ ax-x^2-2xy=0. \end{aligned}\right\}$$

These are equivalent to

$$y(a-2x-y)=0,\ x(a-x-2y)=0,$$

so that we have to consider the four pairs of equations, *viz.*,

$$y=0,\ x=0\ ;$$
$$a-2x-y=0,\ x=0\ ;$$
$$y=0,\ a-x-2y=0\ ;$$
$$a-2x-y=0,\ a-x-2y=0.$$

Solving these, we obtain the following pairs of values of x and y which make the function stationary :

$$(0, 0), (0, a), (a, 0), (\tfrac{1}{3}a, \tfrac{1}{3}a).$$

For $(0, 0)$,

$A=0$, $B=a$, $C=0$ so that $AC-B^2$ is negative.

Thus $f(0, 0)$ is not an extreme value of $f(x, y)$.

For $(0, a)$,

$A=-2a$, $B=-a$, $C=0$ so that $AC-B^2$ is negative.

Thus $f(0, a)$ is also not an extreme value of $f(x, y)$.

We may similarly show that $f(a, 0)$ is also not an extreme value of the function.

For $(\frac{1}{3}a, \frac{1}{3}a)$,

$A=-\frac{2}{3}a$, $B=-\frac{1}{3}a$, $C=-\frac{2}{3}a$ so that $AC-B^2$ is positive.

Thus $f(\frac{1}{3}a, \frac{1}{3}a)$ is an extreme value and will be a maximum or a minimum according as, A, is negative or positive, *i.e.*, according as, a, is positive or negative.

The extreme value $f(\frac{1}{3}a, \frac{1}{3}a)=\frac{1}{27}a^3$.

2. *Find the extreme value of*

$$2(x-y)^2-x^4-y^4. \qquad \textit{(D.U. Hons.)}$$

We write $\quad f(x, y)=2(x-y)^2-x^4-y^4.$

$$\therefore \quad f_x(x, y)=4(x-y)-4x^3.$$
$$f_y(x, y)=-4(x-y)-4y^3.$$
$$f_{x^2}(x, y)=4-12x^2.$$
$$f_{xy}(x, y)=-4.$$
$$f_{y^2}(x, y)=4-12y^2.$$

We now solve the equations $f_x=0, f_y=0$, which are

$$\left.\begin{aligned} 4(x-y)-4x^3=0, \\ -4(x-y)-4y^3=0. \end{aligned}\right\} \qquad \begin{aligned} &\ldots(1) \\ &\ldots(2) \end{aligned}$$

Adding these, we obtain

$$-4(x^3+y^3)=0$$

or $$(x+y)(x^2-xy+y^2)=0,$$

which shows that

either $$x+y=0$$

or $$x^2-xy+y^2=0.$$

We have, thus, to consider the two pairs of equations, *viz.*,

$$\left.\begin{aligned} x-y-x^3=0 \\ x+y=0 \end{aligned}\right\} \qquad \ldots(3)$$

and $$\left.\begin{aligned} x-y-x^3=0 \\ x^2-xy+y^2=0 \end{aligned}\right\} \qquad \ldots(4)$$

The equations (3) give the following pairs of solutions :

$$(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}).$$

The equations (4) give only (0, 0) as the real solution.

For (0, 0),

$$A=4, B=-4, C=4 \text{ so that } AC-B^2=0$$

and accordingly this case needs *further* examination.

For $(\sqrt{2}, -\sqrt{2})$

$$A=-20, B=-4, C=-20 \text{ so that } AC-B^2 \text{ is positive.}$$

$\therefore f(\sqrt{2}, -\sqrt{2})$ is an extreme value and is, in fact, a maximum value as, A, is negative.

We may similarly see that $f(-\sqrt{2}, \sqrt{2})$ is also a maximum value.

Note. The case which arises when $x=0$, as well as $y=0$ can be disposed of by an elementary consideration as follows:

Now $f(0, 0)=0$.

For points, $(x, 0)$ along x-axis, where $y=0$, the value of the function

$$=2x^2-x^4=x^2(2-x^2)$$

which is positive for points in the neighbourhood of the origin.

Again for points along the line, $y=x$ the value of the function $=-2x^4$ which is negative.

Thus in every neighbourhood of the point (0, 0) there are points where function assumes positive values *i.e.*, $>f(0, 0)$ and there are points where the function assumes negative values *i.e.*, $<f(0, 0)$.

Hence $f(0, 0)$ is not an extreme value.

3. *Find the minimum value of*

$$x^2+y^2+z^2$$

when

$$ax+by+cz=p.$$

We write

$$u=x^2+y^2+z^2,$$

so that we have to find the minimum value of a function of three variables x, y, z which are connected by a single relation, *viz.*,

$$ax+by+cz=p.$$

We re-write this relation in the form

$$z=(p-ax-by)/c,$$

so that z has been expressed as a function of x and y. We now obtain

$$u=x^2+y^2+\frac{(p-ax-by)^2}{c^2},$$

where, u, has been expressed as a function of *two* independent variables x and y.

We have

$$u_x(x, y)=2x-\frac{a}{c^2}(p-ax-by),$$

$$u_y(x, y)=2y-\frac{2b}{c^2}(p-ax-by).$$

Equating to zero these two first order partial derivatives, we obtain

$$x=ap/(a^2+b^2+c^2),$$
$$y=bp/(a^2+b^2+c^2).$$

Again, we have

$$A=u_{x^2}(x, y)=2+\frac{2a^2}{c^2},$$

$$B=u_{xy}(x, y)=\frac{2ab}{c^2},$$

$$C=u_{y^2}(x, y)=2+\frac{2b^2}{c^2}.$$

$$\therefore \qquad AC-B^2=4\left(1+\frac{a^2}{c^2}\right)\left(1+\frac{b^2}{c^2}\right)-\frac{4a^2b^2}{c^4}$$

$$=4\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right).$$

Since $AC-B^2$ is positive and, A, is also positive, therefore, u, is minimum for the values of x and y in question. The minimum value of, u, therefore, is

$$p^2/(a^2+b^2+c^2).$$

Exercises

1. Examine the following functions for extreme values:—

(i) $y^2+4xy+3x^2+x^3$. (ii) $y^2+x^2y+ax^4$.

(iii) $x^2+xy+y^2+ax+by$. (iv) $x^3y^2(12-3x-4y)$.

(v) $x^2y(x+2y-4)$. (vi) $(x^2+y^2)e^{6x+2x^2}$.

(vii) $3x^2-y^2+x^3$. (viii) $2x^2y+x^2-y^2+2y$.

(ix) $2\sin(x+2y)+3\cos(2x-y)$. (x) $x^3y^2(1-x-y)$.

(*B.U. 1952*)

2. Find all the stationary points of the function

$$x^3+3xy^2-15x^2-15y^2+72x,$$

examining whether they are maxima or minima. (*D.U. Hons. 1952*)

3. Find the shortest distance between the lines

$$(x-x_1)/l_1=(y-y_1)/m_1=(z-z_1)/n_1$$
$$(x-x_2)/l_2=(y-y_2)/m_2=(z-z_2)/n_2.$$

A. 5. Stationary values under subsidiary conditions. *To find the stationary values of*

$$u=f(x, y, z, w). \qquad ...(1)$$

where the four variables x, y, z, w are subjected to the two subsidiary conditions

$$\varphi(x, y, z, w)=0 \qquad ...(2)$$
$$\psi(x, y, z, w)=0 \qquad ...(3)$$

Now, we can look upon the equations (2) and (3) as determining any two of the four variables x, y, z, w in terms of the remaining two. We may suppose, for the sake of definiteness, that (2) and (3) determine z and w as functions of x, y. Thus u is a function of x, y, z, w where z and w are functions of x and y so that u is essentially a function of two independent variables x and y. Therefore for stationary values of, u, the two partial derivatives of, u, with respect to x and y, obtained after z and w have been replaced by their values in terms of x and y, are respectively zero.

Equating to zero the partial derivatives of, u, with respect to x and y, we obtain

$$f_x+f_z\frac{\partial z}{\partial x}+f_w\frac{\partial w}{\partial x}=0, \qquad ..(4)$$

$$f_y+f_z\frac{\partial z}{\partial y}+f_w\frac{\partial w}{\partial y}=0. \qquad ...(5)$$

Again, differentiating (2) and (3) partially with respect to x and y, we obtain

$$\varphi_x+\varphi_z\frac{\partial z}{\partial x}+\varphi_w\frac{\partial w}{\partial x}=0 \qquad ...(6)$$

$$\varphi_y+\varphi_z\frac{\partial z}{\partial y}+\varphi_w\frac{\partial w}{\partial y}=0. \qquad ...(7)$$

$$\psi_x+\psi_z\frac{\partial z}{\partial x}+\psi_w\frac{\partial w}{\partial x}=0. \qquad ...(8)$$

$$\psi_y+\psi_z\frac{\partial z}{\partial y}+\psi_w\frac{\partial w}{\partial y}=0. \qquad ...(9)$$

If, now, $\partial z/\partial x$, $\partial w/\partial x$, $\partial z/\partial y$, $\partial w/\partial y$ be eliminated out of the six equations (4)—(9), we shall obtain two eliminants, say,

$$F_1(x, y, z, w)=0, \qquad ...(10)$$
$$F_2(x, y, z, w)=0. \qquad ...(11)$$

Then the four equations (2), (3), (10) and (11) determine the values of the unknowns x, y, z and w for which u is stationary.

Note. This method for determining stationary values of a function under subsidiary conditions outlined above is *unsymmetrical* in character in regard to its treatment of the variables involved. This defect has been remedied by Lagrange who has given a symmetrical method based on the introduction of certain *undetermined multipliers*. This method is explained in the following section.

A. 51. Lagrange's method of undetermined multipliers. We multiply the equations (6) and (8) of the preceding section by λ_1 and λ_2 respectively and add to (4), so that we obtain

$$(f_x+\lambda_1\varphi_x+\lambda_2\psi_x)+(f_z+\lambda_1\varphi_z+\lambda_2\psi_z)\frac{\partial z}{\partial x}$$
$$+(f_w+\lambda_1\varphi_w+\lambda_2\psi_w)\frac{\partial w}{\partial x}=0. \quad ...(12)$$

Again, we multiply the equations (7) and (9) of the preceding section by λ_1 and λ_2 and add to (5), so that we obtain

$$(f_y+\lambda_1\varphi_y+\lambda_2\psi_y)+(f_z+\lambda_1\varphi_z+\lambda_2\psi_z)\frac{\partial z}{\partial y}$$
$$+(f_w+\lambda_1\varphi_w+\lambda_2\psi_w)\frac{\partial w}{\partial y}=0. \quad ...(13)$$

We, now, suppose that λ_1 and λ_2 are determined so as to make

$$f_z+\lambda_1\varphi_z+\lambda_2\varphi_z=0, \quad ...(14)$$
$$f_w+\lambda_1\varphi_w+\lambda_2\psi_w=0. \quad ...(15)$$

With this choice of the values of λ_1 and λ_2, the equations (12) and (13) give

$$f_x+\lambda_1\varphi_x+\lambda_2\psi_x=0, \quad ...(16)$$
$$f_y+\lambda_1\varphi_y+\lambda_2\psi_y=0. \quad ...(17)$$

The equations (2) and (3) of the preceding section along with the equations (14), (15), (16), (17) determine values of λ_1, λ_2, and of x, y, z and w, which render u stationary.

If, *now, we write*

$$g=f+\lambda_1\varphi+\lambda_2\psi,$$

we see that the equations (14) *to* (17) *are obtained by equating to zero the four partial derivatives of g, with respect to x, y, z and w so that all the four variables are being treated on a uniform basis.* In practice, therefore, the necessary equations are to be put down by setting up the *auxiliary* function g.

Note. The method outlined above is applicable to a function of any number of variables which are subjected to any set of subsidiary conditions.

Examples

1. *Find the lengths of the axes of the section of the ellipsoid* $x^2/a^2+y^2/b^2+z^2/c^2=1$, *by the plane* $lx+my+nz=0$.

Let (x, y, z) be any point of the section. We write

$$r^2=x^2+y^2+z^2. \quad ...(i)$$

We have to find the stationary values of r^2, where x, y, z are subjected to the two subsidiary conditions

$$\Sigma x^2/a^2=1,\ \Sigma lx=0. \qquad \text{... (ii)}$$

We write

$$g(x, y, z)=(x^2+y^2+z^2)+\lambda_1\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+\lambda_2(lx+my+nz).$$

Equating to zero the partial derivatives of $g(x, y, z)$ *w.r.t.* x, y and z, we obtain

$$g_x=2x+\frac{2\lambda_1 x}{a^2}+l\lambda_2=0, \qquad \text{.. (iii)}$$

$$g_y=2y+\frac{2\lambda_1 y}{b^2}+m\lambda_2=0, \qquad \text{... (v)}$$

$$g_z=2z+\frac{2\lambda_1 z}{c^2}+n\lambda_2=0. \qquad \text{... (v)}$$

The equations (*ii*), (*iii*), (*iv*) and (*v*) will determine the values of $\lambda_1, \lambda_2, x, y, z$. These values of x, y, z, when substituted in (*i*), will determine the stationary values of r^2. These operations amount to eliminating $\lambda_1, \lambda_2, x, y$ and z from (*i*), (*ii*), (*iii*), (*iv*) and (*v*).

We multiply (*iii*), (*iv*) and (*v*) by x, y, z respectively and add so that, on making use of (*i*) and (*ii*), we obtain

$$2r^2+2\lambda_1=0 \text{ or } \lambda_1=-r^2.$$

With this value of λ_1, the equations (*iii*), (*iv*) and (*v*) can be re-written as

$$\frac{l}{1-\dfrac{r^2}{a^2}}=-\frac{2x}{\lambda_2},$$

$$\frac{m}{1-\dfrac{r^2}{b^2}}=-\frac{2y}{\lambda_2},$$

$$\frac{n}{1-\dfrac{r^2}{c^2}}=-\frac{2z}{\lambda_2}.$$

Multiplying these by l, m, n respectively and adding, we obtain

$$\frac{l^2}{1-\dfrac{r^2}{a^2}}+\frac{m^2}{1-\dfrac{r^2}{b^2}}+\frac{n^2}{1-\dfrac{r^2}{c^2}}=0,$$

which is a quadratic in r^2 and determines, as its roots, the two stationary values of r^2.

The geometrical considerations, now show that these two stationary values of r^2, are the squares of the semi-axes of the section, in question.

2. *If*
$$u=a^3x^2+b^3y^2+c^3z^2$$
where
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1,$$
show that the stationary value of u, is given by
$$x=\Sigma a/a,\ y=\Sigma a/b,\ z=\Sigma a/c.$$

We write

$$g(x, y, z)=a^3x^2+b^3y^2+c^3z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right).$$

Equating to zero the partial derivatives of $g(x, y, z)$ *w r.* to x, y and z, we obtain

$$2a^3x-\frac{\lambda}{x^2}=0,\quad 2b^3y-\frac{\lambda}{y^2}=0,\ 2c^3z-\frac{\lambda}{z^2}=0.$$

These give
$$\tfrac{1}{2}\lambda=a^3x^3=b^3y^3=c^3z^3,$$
or
$$ax=by=cz. \qquad \ldots (i)$$

The equation (i) along with the given subsidiary condition $\Sigma 1/x=1$, determine x, y and z.

Exercises

1. Find the minimum value of $x^2+y^2+z^2$ when
   (*i*) $x+y+z=3a$.
   (*ii*) $xy+yz+zx=3a^2$.
   (*iii*) $xyz=a^3$.

2. Find the extreme value of xy when
$$x^2+xy+y^2=a^2.$$

3. Find the greatest value of $ax+by$ when
$$x^2+xy+y^2=3k^2. \qquad \textit{(D.U. Hons. 1953)}$$

4. Find the perpendicular distance of the point (a, b, c) from the plane
$$lx+my+nz=0,$$
by the Lagrange's method of undetermined multipliers.

5. Which point of the sphere $\Sigma x^2=1$ is at the maximum distance from the point (2, 1, 3) ?

6. Find the lengths of the axes of the conic
$$ax^2+2hxy+by^2=1.$$

7. In a plane triangle, find the maximum value of
$$\cos A \cos B \cos C.$$

8. Find the extreme values of
$$x^2+y^2+z^2$$

subject to

$$x^2+\frac{y^2}{2}+\frac{z^2}{3}=2.$$
$$3x+2y+z=0.$$

9. Show that the maximum and minimum values of, r^2, where
$$r^2=a^2x^2+b^2y^2+c^2z^2,\ x^2+y^2+z^2=1 \text{ and } lx+my+nz=0$$
are given by the equation
$$\frac{l^2}{a^2-r^2}+\frac{m^2}{b^2-r^2}+\frac{n^2}{c^2-r^2}=0. \qquad \text{[D.U. 1955]}$$

10. If two variables x and y are connected by the relation
$$ax^2+by^2=ab,$$
show that the maximum and the minimum values of the function
$$x^2+xy+y^2$$
will be the values of θ given by the equation
$$4(\theta-a)(\theta-b)=ab. \qquad \text{(D.U. Hons. 1957)}$$

Miscellaneous Exercises

1. Find the points of continuity and discontinuity of the following functions :—
$$f(x)=\frac{1}{1-e^{\frac{1}{x}}} \text{ for } x\neq 0,\ f(0)=0.$$

2. Determine the points of continuity and discontinuity of the function $f(x)$ defined by
$$f(n+t)=\begin{cases} t, \text{ if } 0\leqslant t<\frac{1}{2} \\ 0, \text{ if } t=\frac{1}{2} \\ t-1, \text{ if } \frac{1}{2}<t\leqslant 1, \end{cases}$$
where n is any integer. (B.U. 1952)

3. Determine the points of discontinuity of
$$\tan\frac{\pi x}{x+1}. \qquad \text{(C.U. Hons. 1954)}$$

4. Draw the graph of
$$y=\lim_{n\to\infty}\frac{x^n}{x^n+1}$$
and find the points of discontinuity.

5. Determine the points of discontinuity of

(i) $[x]+[-x]$. (ii) $|x|+[x]$.

(iii) $\lim\limits_{n\to\infty}\left(\frac{2}{\pi}\tan^{-1}nx\right)$.

6. Show that the function $\varphi(x)$ which is equal to 0 when $x=0$; to $\frac{1}{2}-x$ when $0<x<\frac{1}{2}$: to $\frac{1}{2}$ when $x=\frac{1}{2}$; to $\frac{3}{2}-x$ when $\frac{1}{2}<x<1$; and to 1 when $x=1$, has three points of discontinuity which you are required to find. (Patna)

7. Examine the continuity of the function
$$f(x)=\frac{e^{\frac{1}{x}}\sin(1/x)}{1+e^{\frac{1}{x}}},\ x\neq 0,$$
$$f(0)=0. \qquad \text{(D.U. Hons. 1947)}$$

8. Find dB/dA where A, B, C are the angles of a triangle and satisfy the relation

$$\sin B \sin C+\sin C \sin A+\sin A \sin B=h,$$

where, h, is constant.

9. Find

$$\frac{1}{r}+\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right),$$

in terms of r, when

$$r^2=a^2 \cos 2\theta.$$

10. If $ax^2+2hxy+by^2=1$, prove that

$$x\left(\frac{d^2x}{dy^2}\right)^{-\frac{1}{3}}+y\left(\frac{d^2y}{dx^2}\right)^{-\frac{1}{3}}+(ab-h^2)^{-\frac{1}{3}}=0.$$

11. Find

$$\frac{d^n}{dx^n}\left(\frac{1}{x^2+2ax\cos a+a^2}\right). \qquad (B.U.)$$

12. If

$$y=e^x\frac{d^n}{dx^n}(x^n e^{-x}),$$

show that

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny=0. \qquad (M.T.)$$

13. If $y=\cos(m\sin^{-1}x)$, prove that

(i) $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+m^2y=0,$

(ii) $(1-x^2)\dfrac{d^{n+1}y}{dx^{n+1}}-(2n-1)x\dfrac{d^ny}{dx^n}+[m^2-(n-1)^2]\dfrac{d^{n-1}y}{dx^{n-1}}=0.$

If y can be expanded in a series of ascending powers of x, prove that when $m=6$,

$$y=1-18x^2+48x^4-32x^6\ldots. \qquad (M.T.)$$

14. If

$$y=\frac{\log[x+\sqrt{(1+x^2)}]}{\sqrt{(1+x^2)}}$$

prove that

$$(1+x^2)\frac{dy}{dx}+xy=1.$$

Assuming that y can be expanded as a series

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$$

prove that

$$a_0=0,\ a_1=1 \text{ and for } m>1$$

$$a_{2m}=0,\quad a_{2m+1}=(-1)^m\frac{2.4.6\ldots(2m)}{3.5.7\ldots(2m+1)}.$$

15. If $e^{x\cos\alpha}\cos(x\sin\alpha)$ can be expanded in ascending powers of x, show that the co-efficient of x^n is $\cos n\alpha/n!$ (B.U.)

16. (a) Give the first three non-vanishing terms in the expansion of

$$\sin^{-1}(\tfrac{1}{2}\sin x).$$

(b) Show that the expansion upto x^4 of $x/\sinh x$ is

$$1-\frac{x^2}{3!}+\frac{14x^4}{6!}. \qquad (B.U.)$$

17. Use Taylor's theorem to prove that the only function $f(x)$ satisfying

$$\frac{d^2y}{dx^2}=-f(x) \text{ for all } x,\ f(0)=-1,\ f'(0)=0,$$

is given by

$$f(x)=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}\ldots \quad \ldots \quad \ldots \text{ ad inf.}$$

(*D.U. Hons. 1952*)

18. Prove that whatever the functions f and g,

(*i*) $$z=xf(x+y)+yg(x+y)$$

satisfies the relation

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2}=0.$$

(*ii*) $$z=xf(y/x)+g(y/x),$$

satisfies the relation

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2}=0.$$

19. Given that z is a function of u and v, while

$$u=x^2-y^2-2xy,\ v=y,$$

prove that the equation

$$(x+y)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=0.$$

is equivalent to $\partial z/\partial y=0$.

20. If $u=(1-2xy+y^2)^{-\frac{1}{2}}$, prove that

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right)=0.$$

(*B.U. 1955*)

21. If $u=\sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u \cos 2u}{4\cos^3 u}.$$ (*P.U. Hons.*)

22. Prove that if u is a homogeneous function of the nth degree in x, y, z, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

Prove also that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(r^2u)=2(2n+3)\,u+r^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right).$$

where

$$r^2=x^2+y^2+z^2.$$

(*P.U. 1935*)

13. If $f(r)=r^{-\frac{1}{2}}\ (a+\log r)$ where $r=\sqrt{(x^2+y^2+z^2)}$, show that

$$\frac{\partial^2 f(r)}{\partial x^2}+\frac{\partial^2 f(r)}{\partial y^2}+\frac{\partial^2 f(r)}{\partial x^2}+\frac{f(r)}{4r^2}=0.$$ (*B.U.*)

24. If

$$\frac{\partial^2 v}{\partial x^2}=rk\frac{\partial v}{\partial t},$$

has a solution of the form

$$Ae^{-ax}\sin(wt-bx).$$

prove that

$$a=b=\sqrt{(\tfrac{1}{2}rwk)}.$$

25. $u=\log(x^3+y^3-x^2y-xy^2)$, prove that

(*i*) $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=2(x+y)^{-1}$.

(*ii*) $\dfrac{\partial^2 u}{\partial x^2}+2\dfrac{\partial u^2}{\partial x\partial y}+\dfrac{\partial^2 u}{\partial y^2}=-4(x+y)^{-2}$. (*B.U. 1952*)

26. Show that if $a=0$ and $f(x)=\log(1+x)$ then 'θ' of the Lagrange's mean value theorem is a continuous function of, h, which decreases steadily from 1 to 0 as, h, increases from -1 to ∞.

27. Show that

(*i*) $x<\log[1/(1-x)]<x/(1-x)$ where $0<x<1$.

(*ii*) $1+\dfrac{x}{2\sqrt{(1+x)}}<\sqrt{(1+x)}<1+\dfrac{x}{2}$, where $-1<x<0$.

28. A slight error $\delta\alpha$ is made in measuring the semi-vertical angle of a cone which circumscribes a sphere of radius R. Find the approximate error in the calculated volume of the cone and show that, for a given $\delta\alpha$, the least value of the error is

$$\frac{64\pi R^3}{75}\cdot\sqrt{5}.\delta\alpha.$$

29. If the three sides a, b, c of a triangle are measured, the error in the angle A, due to given small error in the sides, is

$$dA=\frac{\sin A}{\sin B\sin C}\cdot\frac{da}{a}-\cot C\frac{db}{b}-\cot B\frac{dc}{c}$$

30. Find $\lim\dfrac{1-ae^{-x}-be^{-2x}-ce^{-3x}}{1-ae^{x}-be^{2x}-ce^{3x}}$, $(x\to 0)$.

when

(*i*) $a=3, b=-5, c=4$; (*ii*) $a=3, b=-4, c=2$;

(*ii*) $a=3, b=-3, c=1$.

31. Find $\lim\limits_{x\to 0}\dfrac{3x\log(\sin x/x)^2+x^3}{(x-\sin x)(1-\cos x)}$. (*B.U.*)

32. Find $\lim\limits_{x\to 0}\dfrac{(1+x)e^{-x}-(1-x)e^{x}}{x(e^{x}-e^{-x})-2x^2e^{-x}}$. (*B.U.*)

33. Obtain

(*i*) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{\tan x}$. (*ii*) $\lim\limits_{x\to 0}\dfrac{e^{ax}-e^{-ax}}{\log(1+bx)}$. (*iii*) $\lim\limits_{x\to 0}\dfrac{x^2\sin 1/x}{\tan x}$.

34. Find $\lim\limits_{x\to 0}\dfrac{\tan(\sin x)-\sin(\tan x)}{\sin x-x\cos x-\frac{1}{2}\sin^2 x}$. (*D.U. Hons. 1955*)

35. Examine where the following functions are continuous for $x=0$.

(*i*) $f(x)=\dfrac{\tan x-x}{x^2\tan x}$, when$(x\neq 0)$, $f(0)=\frac{1}{3}$.

(*ii*) $f(x) = \dfrac{\log(1-x^2)}{\log\cos x}$, $(x \neq 0)$, $f(0) = 1$.

(*iii*) $f(x) = (\cos x)^{\cot^2 x}$, $(x \neq 0)$, $f(0) = 1/\sqrt{e}$.

36. Having given that the fraction

$$(\tan x - xe^{\sin x} + x^2 + \tfrac{1}{6}x^3)/x^n$$

maintains a finite non-zero limit as x tends to zero, prove that n must be equal to 5. (*Indian Police 1932*)

37. Find the maximum values of the function

$$(x^2 - 3x + 2)/(x^2 + 3x + 2). \qquad \text{(P.U. 1938)}$$

38. Sketch the curve

$$y = \sin x + \tfrac{1}{2}\sin 2x. \qquad \text{(P.U. Hons. 1943)}$$

from $x=0$ to $x=2\pi$, indicating correctly the positions of maxima and minima, if any, of the function.

39. Show that the function

$$4(x-1)^{-1} - 9(x+1)^{-1}$$

has a single maximum and a single minimum value and that the function does not lie between its maximum and minimum. Draw the graph of the function.

40. Find the maximum and the minimum values of $(1-x)^2 e^x$.

Show that $e^x - (1+x)/(1-x)$ steadily decreases as x increases from $-\infty$ to 1 and that it has one and only one minimum for values of x between $x=1$ and $x=+\infty$. (*P.U. 1938*)

41. Prove that $\frac{1}{8}(35\sin^4 x - 40\sin^2 x + 8)$ ranges in value between unity and $-\frac{4}{7}$ and has also $\frac{3}{8}$ as a maximum value.

42. Show that $\psi(x) = \frac{1}{2}\sin x \tan x - \log \sec x$ is positive and increasing in the interval $0 < x < \pi/2$. (*M.T.*)

43. Find the maxima and the minima of, y, where

$$(ay+b)(cy+d)^{-1} = \sin^2 x + 2\cos x + 1 \text{ and } (ad-bc) \neq 0. \qquad \text{(B.U.)}$$

44. If $x^4 - axy^2 - a^3y = 0$, prove that y is maximum where $3xy + 4a^2 = 0$ and a minimum where $x=0$. (*B.U.*)

45. Draw the curve $r = a(2\cos\theta + \cos 3\theta)$. Show that the extreme values of the radius vector are $3a$ and $a/3\sqrt{3}$.

46. The sum of the surfaces of a cube and a sphere is given. Show that when the sum of their volumes is least, the diameter of the sphere is equal to the edge of the cube.

47. Given the volume of a right cone; required its dimensions when the surface is least possible.

48. Prove that the volume of a right circular cylinder of greatest volume which can be inscribed in a sphere, is $\sqrt{3}/3$ times that of the sphere.

49. An ellipse is inscribed in an isosceles triangle of height h and base $2k$ and having one axis lying along the perpendicular from the vertex of the triangle to the base. Show that the maximum area of the ellipse is $\sqrt{3}\pi hk/9$.

50. A sector has to be cut from a circular sheet of metal so that the remainder can be formed into a conical shaped vessel of maximum capacity. Find the angle of the sector. (*M.U.*)

51. Find the greatest rectangle which can be described so as to have two of its corners on the latus rectum and the other two on the portion of the curve cut off by the latus rectum of the parabola.

52. Find the greatest and least values of $(\sin x)^{\sin x}$ (*B.U.*)

CHAPTER XI

SOME IMPORTANT CURVES

11·1. The following chapters will be devoted to a discussion of such types of properties of curves as are best studied with the help of Differential Calculus. It will, therefore, be useful if we acquaint ourselves at this stage with some of the important curves which will frequently occur.

11·2. Explicit Cartesian equations of Curves. A few curves whose equations are of this form have already been traced in Chapter II. We now trace another very important curve

$$y=c \cosh \frac{x}{c},$$

which is known as **Catenary.**

The following particulars about the curve will enable us to trace it :—

(*i*) Since

$$\cosh \frac{x}{c}=\frac{1}{2}\left[e^{x/c}+e^{-x/c}\right],$$

on changing x to $-x$, we see that

$$y=c \cosh \frac{x}{c}=c \cosh \left(-\frac{x}{c}\right),$$

so that the two values of x which are equal in magnitude but opposite in sign give rise to the same value of y.

Hence *the curve is symmetrical about y-axis.*

(*ii*) *When* $x=0$, $y=c$ *so that* A $(0, c)$ is a point on the curve.

(*iii*) $\dfrac{dy}{dx}=\sinh \dfrac{x}{c}$, which is positive when x is positive.

Thus, y is monotonically increasing in $[0, \infty)$.

Also, $\left[\dfrac{dy}{dx}\right]_{x=0}=0$, *i.e.*, the slope of the tangent is, 0, for $x=0$ so that the tangent is parallel to x-axis at A $(0, c)$.

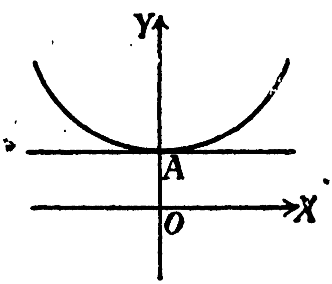


Fig. 57

Hence if the point $P(x, y)$ on the curve starts moving from the position $A(0, c)$ such that its abscissa increases, then its ordinate also increases. Also since $y \to \infty$ when $x \to \infty$, the curve is not closed.

The curve being symmetrical about y-axis, we have its shape as shown in the adjoining figure.

Note. In books on Statics it is shown that the curve in which a uniformally heavy fine chain hangs freely under gravity is a Catenary.

11·3. Parametric Cartesian Equations of Curves.

Let $f(t)$, $F(t)$ be two functions of t defined for some interval. We write

$$x=f(t) \qquad \ldots(i)$$
$$y=F(t) \qquad \ldots(ii)$$

To each value of t, there corresponds a pair of numbers x, y as determined from the equations (i), (ii). To this pair of numbers x, y there corresponds a point in a plane on which a pair of rectangular co-ordinate axes has been marked. Thus the two equations associate to each value of, t, a point in the plane. The two equations (i) and (ii) then constitute the *parametric equations* of the curve determined by the points (x, y) which arise for different values of the *parameter* t.

The point P on the curve corresponding to any particular value, t, of the parameter is denoted as the point $P(t)$ or simply 't'.

We assume that the reader is familiar with the standard parametric equations of a parabola and the Ellipse so that we may only restate them here. Afterwards the parametric equations of a Cycloid, Epicycloid and Hypocycloid will be obtained from their geometrical definitions.

11·31. Parabola. The parametric equations

$$x=at^2, \; y=2at$$

represent the parabola having its axis along x-axis and the tangent at the vertex along y-axis, and latus rectum equal to $4a$.

The point $P(t)$ describes the part ABO of the parabola in the direction of the arrow-head as the parameter, t, continuously increases from $-\infty$ to 0.

For the vertex O, $t=0$. Again, the point describes the part OCD of the parabola in the direction of the arrow-head as the parameter t continuously increases from 0 to ∞.

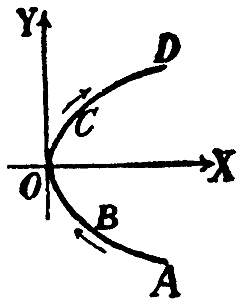


Fig. 58

11·32. The Ellipse. The parametric equations

$$x=a\cos\theta, \; y=b\sin\theta$$

represent the ellipse whose axes lies along x-axis and y-axis and are of lengths $2a$, $2b$, respectively.

The point $P(\theta)$ describes the parts AB, BA', $A'B$, $B'A$ of the ellipse in the direction of the arrow-head as the parameter, θ, continuously increases in the intervals $(0, \frac{1}{2}\pi)$, $(\frac{1}{2}\pi, \pi)$ $(\pi, \frac{3}{2}\pi)$, $(\frac{3}{2}\pi, 2\pi)$ respectively.

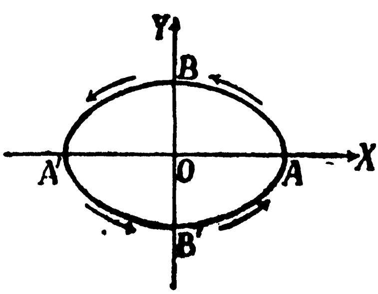


Fig. 59.

11·33. Cycloid *The Cycloid is the curve traced out by a point marked on the circumference of a circle as it rolls without sliding along a fixed straight line.*

Let the rolling circle start from the position in which the generating point P coincides with some point O of the fixed line $X'X$.

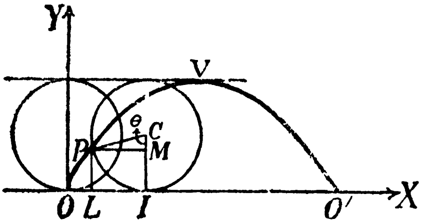


Fig. 60

Take the point O as origin and the fixed line as x-axis. When the rolling circle has *rolled* on to the position shown in the figure, the generating point has moved from O to $P(x, y)$ so that

$$OI = \text{arc } PI.$$

Let, a be the radius of the circle. Let, θ, be the angle between CP and CI so that it is the angle through which the radius drawn to the general point has rotated while the circle rolls from the initial to its present position. Thus arc $PI = a\theta$. We have

$$x = OL = OI - LI = OI - PM = a\theta - a \sin \theta = a(\theta - \sin \theta) \; ;$$

$$y = LP = IM = IC - MC = a - a \cos \theta = a(1 - \cos \theta).$$

Thus $$\left.\begin{aligned} \mathbf{x} &= \mathbf{a(\theta - \sin \theta)} \\ \mathbf{y} &= \mathbf{a(1 - \cos \theta)} \end{aligned}\right\}$$

are the parametric equations of the Cycloid ; θ being the parameter.

Note. While the circle makes one complete revolution, the point P describes one complete arch OVO of the Cycloid, so that θ increases from 0 to 2π as the point P moves from O to O'.

The position V of the generating point which is reached after the circle revolved through two right angles is shown as the *vertex* of the Cycloid.

The Cycloid evidently consists of an endless succession of exactly congruent portions each of which represents one complete revolution of the rolling circle.

11·34. Epicycloid and Hypocycloid. *The curve traced out by a point marked on the circumference of a circle as it rolls without sliding along a fixed circle, is called an* **Epicycloid** *or* **Hypocycloid** *according as the rolling circle is outside or inside the fixed circle.*

We shall first consider the case of an Epicycloid.

Let O be the centre and, a, the radius of the fixed circle. Let the rolling circle start from the position in which the generating point P coincides with some point O of the fixed circle. We take the point O as origin and OA as X-a is.

The generating point has moved on from A to $P(x, y)$ when the rolling circle has rolled on to the position shown in the figure so that

arc AI = arc PI.

Let $\angle AOC = \theta$, $\angle ICP = \phi$

$\therefore$ $a\theta$ = arc AI = arc $PI = b\phi$

i.e., $\phi = a\theta/b$.

Here, θ, is the angle through which the line joining the centres of the two circles rotates while the rolling circle rolls from its initial to its present position.

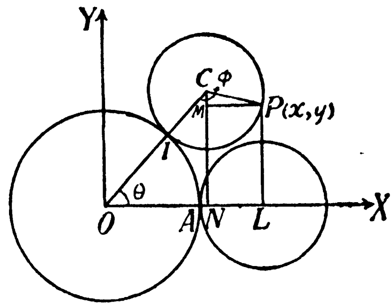


Fig. 61.

$$\angle NCP = \angle OCP - \angle OCN$$
$$= \phi - (\tfrac{1}{2}\pi - \theta) = \theta + \phi - \tfrac{1}{2}\pi.$$

Therefore

$$x = OL$$
$$= ON + NL$$
$$= ON + MP$$
$$= OC \cos\theta + CP \sin(\theta + \phi - \tfrac{1}{2}\pi)$$
$$= (a+b)\cos\theta - b\cos(\theta+\phi)$$
$$= (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta\ ;$$

$$y = LP$$
$$= NC - MC$$
$$= (a+b)\sin\theta - b\cos(\theta+\phi-\tfrac{1}{2}\pi)$$
$$= (a+b)\sin\theta - b\sin(\theta+\phi)$$
$$= (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta.$$

Thus

$$\mathbf{x = (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta,}$$

$$\mathbf{y = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta,}$$

are the parametric equations of the Epicycloid ; θ being tbe parameter.

The tracing point would describe an Hypocycloid, if the rolling circle were within the fixed circle. Its equation can easily be obtained by changing b to $-b$ in the equations of the Epicycloid.

Thus

$$\mathbf{x = (a-b)\cos\theta + b\cos\frac{b-a}{b}\theta,}$$

$$\mathbf{y = (a-b)\sin\theta + b\sin\frac{b-a}{b}\theta,}$$

are the parametric equations of the Hypocycloid.

In making one complete revolution the rolling circle will describe $2b\pi$ of the length of the circumference of the fixed circle.

Let the ratio a/b of the radii of the circles be a *rational* number p/q in its lowest terms so that

$$\frac{a}{b}=\frac{p}{q}, \text{ i.e., } aq=bp \text{ or } 2\pi a.q.=2\pi b.p.$$

This relation shows that in making, p, complete revolutions, the rolling circle describes the circumference of the fixed circle q, times and then the generating point returns to its original position. Therefore the path consists of the repetition of the same, p, identical portions.

In case a/b is irrational, the tracing point will never return to its original position and so the path will consist of an *endless* series of exactly congruent portions.

Some Particular cases of Epicycloid and Hypocycloid.

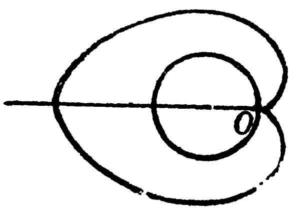


Fig. 62.

(*i*) For $a=b$, the equations of the epicycloid become

$$x=2a\cos\theta-a\cos 2\theta,$$
$$y=2a\sin\theta-a\sin 2\theta.$$

In this case the generating point will return to its original position after the rolling circle has made one complete revolution. The shape of the curve is shown in the adjoined figure.

(*ii*) **Four cusped hypocycloid.** If $a=4b$, the equations of hypocycloid become

$$x=\tfrac{3}{4}a\cos\theta+\tfrac{1}{4}a\cos 3\theta,$$
$$y=\tfrac{3}{4}a\sin\theta-\tfrac{1}{4}a\sin 3\theta.$$

Now,

$$\cos 3\theta=4\cos^3\theta-3\cos\theta,\ \sin 3\theta=3\sin\theta-4\sin^3\theta.$$

$$\therefore \qquad \mathbf{x=a\cos^3\theta,\ y=a\sin^3\theta},$$

are the parametric equations of a curve known as **four cusped hypocycloid or astroid.**

Here $a/b=4$ so that the path consists of the repetitions of four portions. The thickly drawn curve $ABA'B'$ gives the path of the point.

For the points A, B, A', B' the values of θ are $0, \pi/2, \pi, 3\pi/2$ respectively.

Eliminating θ between the parametric equations of this curve, we obtain

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},$$

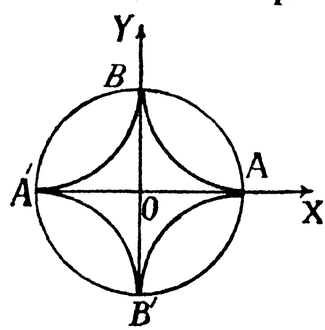


Fig. 63.

which is also the form in which the equation of a four-cusped hypocycloid is sometimes given.

Ex. Show that hypocycloid becomes a straight line for $a=2b$.

11·4. Implicit cartesian equations of curves. If $f(x, y)$ be a function of the two variables x and y, then

$$f(x, y)=0.$$

is the *implicit* equation of the curve determined by the points whose co-ordinates satisfy it.

Curves whose equations are of the form $f(x, y)=0$ possess many points of interest not offered by curves with explicit equations of the form $y=F(x)$.

In the following, we consider only rational algebraic functions $f(x, y)$ so that the form of the equation, when arranged according to ascending powers of x and y, is

$$\begin{array}{c} a_0 \\ +b_0x+b_1y \\ +c_0x^2+c_1xy+c_2y^2 \\ +d_0x^3+d_1x^2y+d_2xy^2+d_3y^3 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ +l_0x^n+l_1x^{n-1}y+l_2x^{n-2}y^2+\dots+l_ny^n=0. \end{array}$$

The same equation may, in a concise form, be written as

$$u_0+u_1+u_2+u_3+\dots\dots+u_{n-1}+u_n=0,$$

where, u_k, represents the general homogeneous polynomial of the kth degree in x and y.

The degree of any term means the sum of the indices of x and y in that term and the degree of the curve means the highest of the degrees of each term.

Branches of a curve. If, in the rational algebraic equation

$$f(x, y)=0$$

of degree n, we replace x by any fixed value, then there will result an equation in, y, whose degree will be less than or equal to n. On solving, this equation will give as many values of, y, as is its degree. Thus with the given value of, x, as abscissa there will be as many points on the curve as are the different real roots of the equation. As, x, goes on taking different values, each of these points will separately describe what is known as a **branch** of the curve.

We now trace a few important curves. In each case we have to examine how, y, will vary as x, starting from some fixed value, increases or decreases.

11·41. $(x^2+y^2)x-ay^2=0$. $(a>0)$. Cissoid of Diocles.

We write the equation as

$$y^2(a-x)=x^3,$$

i.e., $$y=\pm x\sqrt{\left[\frac{x}{a-x}\right]},$$

so that, we see, that to a value of x correspond two values of y which are equal in magnitude but opposite in sign. The two values of y determine the two branches of the Cissoid which are symmetrically situated about x-axis.

We consider one branch,

$$y=x\sqrt{\left[\frac{x}{a-x}\right]},$$

and the form of the other branch can be seen by symmetry.

We have

$$\frac{dy}{dx}=\frac{(\frac{3}{2}a-x)\sqrt{x}}{(a-x)\sqrt{(a-x)}}.$$

The following particulars about the curve will enable us to trace it :—

(*i*) The expression $x/(a-x)$ under the radical is negative when x is negative or when x is greater than a and is positive when x lies between 0 and a. Thus for y to be real, x must lie between 0 and a.

Hence the curve is entirely situated between the lines $x=0$ and $x=a$.

(*ii*) When $x=0$, $y=0$ so that the branch passes through the origin.

(*iii*) $dy/dx=0$, when $x=0$ or $\frac{3}{2}a$. The values $\frac{3}{2}a$ of x is outside the interval $[0, a]$ of the admissible values of x.

Thus the slope of the tangent at the origin to the branch is 0 so that the branch touches the x-axis at the origin.

(*iv*) For values of x between 0 and a, the value of dy/dx is positive so that the ordinate y monotonically increases.

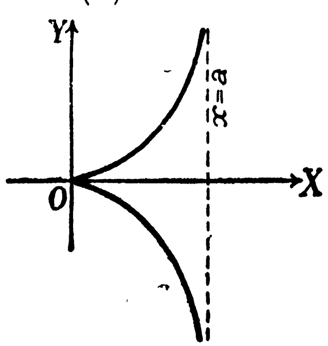


Fig. 64.

Also, as $x \to a$, $y \to \infty$.

Hence we have the shape of the curve as shown, the two symmetrical branches lying in the first and the fourth quadrants.

Note. It is important to notice that origin is a point common to the two branches and the two branches have a common tangent there.

11·42. $(x^2+y^2)x-a(x^2-y^2)=0$, $(a>0)$. **Strophoid.**

We write the given equation as

$$y^2(a+x)=x^2(a-x),$$

i.e.,
$$y=\pm x\sqrt{\left[\frac{a-x}{a+x}\right]},$$

so that we see that to a value of x there correspond two values of y (giving rise to two branches) which are equal in magnitude but opposite in sign. The two branches of the curve are, therefore, symmetrically situated about x-axis.

We have

$$\frac{dy}{dx}=\pm\frac{a^2-ax-x^2}{(a+x)\sqrt{(a^2-x^2)}}.$$

Some particulars which enable us to trace the curve will now be obtained.

(*i*) The expression $(a-x)/(a+x)$ under the radical is positive if and only if x lies between $-a$ and a. Thus the curve entirely lies between the lines $x=-a$ and $x=a$.

(*ii*) When $x=0$ or a, both the values of y are 0 so that the points (0, 0) and (a, 0) lie on both the branches.

(*iii*) When $x=0$, $dy/dx=\pm 1$ so that the slopes of the two tangents at the origin to the two branches are ± 1. Hence $y=x$ and $y=-x$ are two distinct tangents to the two branches at the origin.

For both the branches dy/dx tends to infinity as $x\to a$ so that at $(a, 0)$ the tangent to either branch is parallel to y-axis.

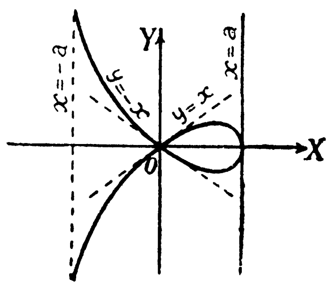


Fig. 65.

(*iv*) $dy/dx=0$ for values of x given by

$$a^2-ax-x^2=0,$$

i.e., for $\qquad x=-a(1\pm\sqrt{5})/2.$

The value $-a(1+\sqrt{5})/2$ does not belong to the interval $[-a, a]$ of the admissible values of x.

Thus for both the branches,

$$dy/dx=0$$

for
$$x=-a(1-\sqrt{5})/2$$
$$=a(\sqrt{5}-1)/2$$

only.

(*v*) When $x\to -a$, $y\to+\infty$ for one branch and $-\infty$ for the other.

Hence we have the shape of the curve as shown.

Note. Both the branches of the curve pass through the origin and have distinct tangents there.

11·43. $ay^2-x(x+a)^2=0$, $(a > 0)$.

Clearly,

$$\sqrt{a}y=\sqrt{x}(x+a),\ \sqrt{a}y=-\sqrt{x}(x+a),$$

are the two branches of this curve.

Also $$\frac{dy}{dx}=\pm\frac{1}{\sqrt{a}}\left[\frac{2}{3}\sqrt{x}+\frac{a}{2\sqrt{x}}\right]=\pm\frac{2x+a}{2\sqrt{(ax)}}.$$

(*i*) The point $(-a, 0)$ lies on both the branches. To no other negative value of x corresponds a real value of y.

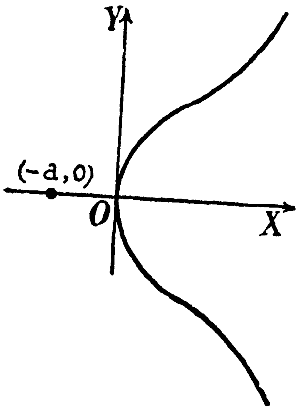


Fig. 66.

The value of dy/dx is not real for $(-a, 0)$.

(*ii*) $(0, 0)$ lies on both the branches. Also, dy/dx tends to infinity for either branch as $x \to 0$. Thus y-axis is a tangent to the two branches at the origin.

(*iii*) As x takes up positive values only one value of dy/dx is always positive and the other always negative. Thus the ordinate y for one branch monotonically increases and for the other monotonically decreases.

(*iv*) When $x \to \infty$,

$y \to \infty$ for one branch and $\to -\infty$ for the other.

Also when $x \to \infty$,

$$\frac{dy}{dx}=\pm\frac{1}{\sqrt{a}}\left[\frac{3\sqrt{x}}{2}+\frac{a}{2\sqrt{x}}\right] \text{ tends to infinity.}$$

This shows that as we proceed to infinity along the curve, the tangent tends to become parallel to y-axis. This is possible if and only if at some point, the curve changes its concavity from downwards to upwards.

We have the shape of the curve as in Fig. 66.

Note. The peculiar nature of the point $(-a, 0)$ on the curve may be carefully noted. This point lies on either branch, but no point in its immediate neighbourhood lies on the curve.

11·44. $x^3-ay^2=0$, semi-cubical parabola.

The discussion of this equation which is very simple is left to the student.

Its shape is shown in the adjoined Fig. 67.

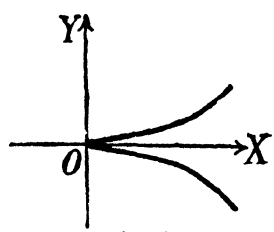


Fig. 67.

11·45. $x^3+y^3-3axy=0$. Folium of Descartes.

It will be traced in Chapter XVIII.

Important Note Putting $y=tx$ in the equation $y^2(a-x)=x^3$ of the Cissoid, we obtain

$$x=\frac{at^2}{1+t^2},\quad y=\frac{at^3}{1+t^2},$$

which are its parametric equations ; t being the parameter.

We may similarly show that

$$x=a\frac{1-t^2}{1+t^2},\quad y=a\frac{t(1-t^2)}{1+t^2},$$

$$x=at^2,\qquad y=at^3;$$

$$x=\frac{3at}{1+t^3},\quad y=\frac{3at^2}{1+t^3},$$

are the parametric equations of the Strophoid, Semi-cubical Parabola and Folium respectively.

This method of determining parametric equations is not general and applies only to such curves as we have here considered.

11·5. Polar co-ordinates. Besides the cartesian, there are other systems also for representing points and curves analytically. Polar system, which is one of them, will be described here.

In this system we start with a fixed line OX, called the *initial line* and a fixed point on it, called the *pole*.

If P be any given point, the distance $OP=r$ is called the *radius vector* and $\angle XOP=\theta$, the *vectorial angle*. The two together are referred to as the *polar co-ordinates* of P.

11·51. Unrestricted variation of polar co-ordinates. If we were concerned with assigning polar co-ordinates to only *individual* points in the plane, then it would clearly be enough to consider the radius vector to have positive values only and the vectorial angle θ to lie between 0 and 2π. But, while considering points whose co-ordinates satisfy a given relation between r and θ, it becomes necessary to remove this restriction and consider both r and θ to be capable of varying in the interval $(-\infty, \infty)$. The necessary conventions for this will be introduced now.

The angle, θ, will be regarded as the measure of rotation of a line which starting from OX revolves round it ; the measure being positive or negative according as the rotation is counter-clock-wise or clock-wise.

To find the point (r, θ) where r is negative and, θ, has any value, we proceed as follows :

Let the revolving line starting from OX revolve though θ. We produce this final position of the revolving line backwards through O. The point P on this produced line such that $OP=|r|$ is the required point (r, θ).

The positions of the points $(1, 9\pi/4)$, $(-1, 9\pi/4)$, $(1, -9\pi/4)$, $(-1, -9\pi/4)$ have been marked in the diagrams below.

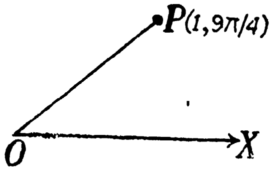


Fig. 68.

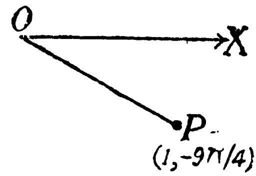


Fig. 69.

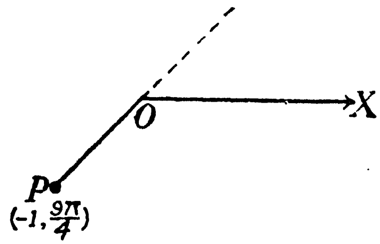


Fig. 70.

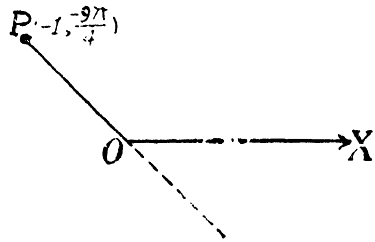


Fig. 71.

It will be seen that according to the conventions introduced here a point can be represented in an infinite number of ways. For example, the points $(-1, \pi/4)$, $(1, 5\pi/4)$, $(1, 2n\pi+5\pi/4)$, (n is any integer), are identical.

11 52. Transformation of co-ordinates. Take the initial line OX of the polar system as the positive direction of X-axis and the pole O as origin for the Cartesian system The positive direction of Y-axis is to be such that the line OX after revolving through $\pi/2$ in counter-clock-wise direction comes to coincide with it.

Let (x, y) and (r, θ) be the cartesian and polar co-ordinates respectively of any point P in the plane.

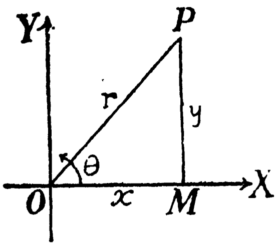


Fig. 72

From the $\triangle\ OMP$, we get

$OM/OP = \cos\theta$, *i.e.*, $x = r\cos\theta$ .. (*i*)

$MP/OP = \sin\theta$, *i.e.*, $y = r\sin\theta$ .. (*ii*)

The equations (*i*) and (*ii*) determine the cartesian co-ordinates (x, y) of the point P in terms of its polar co-ordinates (r, θ) and *vice versa*

11 6. Polar Equations of Curves. Any explicit or implicit relation between, r and θ will give a curve determined by the points whose co-ordinates satisfy that relation.

Thus the equations

$$r = f(\theta) \text{ or } F(r, \theta) = 0$$

determine curves.

The co-ordinates of two points symmetrically situated about the initial line or of the form (r, θ) and $(r, -\theta)$ so that their vectorial angles differ in sign only.

Hence *a curve will be symmetrical about the initial line if, on changing θ to $-\theta$, its equation does not change.* For instance, the curve $r=a(1+\cos\theta)$ is symmetrical about the initial line, for,

$$r=a(1+\cos\theta)=a[1+\cos(-\theta)].$$

It may be noted that

$r=a$ represents a circle with its centre at the pole and radius a ; and

$\theta=b$ represents a line through the pole obtained by revolving the initial line through the angle b.

A few important curves will now be treated. To trace polar curves, we generally consider the variation in r as θ varies.

11·61. $r=a(1-\cos\theta)$. Cardioide.

(*i*) The curve is symmetrical about the initial line.

(*ii*) When $\theta=0$, $r=0$.

(*iii*) When θ increases from 0 to $\pi/2$, $\cos\theta$ decreases from 1 to 0 and, therefore, r increases continuously from 0 to a. When $\theta=\pi/2$, $r=a$.

(*iv*) When θ increases from $\pi/2$ to π, $\cos\theta$ decreases from 0 to -1 and, therefore, r increases from a to $2a$. When $\theta=\pi$, $r=2a$.

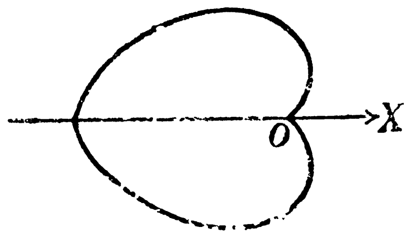


Fig. 73

The variation of θ from π to 2π need not be considered because of symmetry about the initial line.

Hence the curve is as shown.

Ex. Trace the curve $r=a(1+\cos\theta)$.

11·62. $r^2=a^2\cos 2\theta$. Lemniscate of Bernouilli.

It is symmetrical about the initial line and so we need consider the variation in r as θ varies from 0 to π only.

We consider positive values of r only.

(*i*) When $\theta=0$, $r=a$.

(*ii*) When θ increase from 0 to $\pi/4$, 2θ increases from 0 to $\pi/2$ so that $\cos 2\theta$ decreases from 1 to 0 and, therefore, r decreases from a to 0.

When $\theta=\pi/4$, $r=0$.

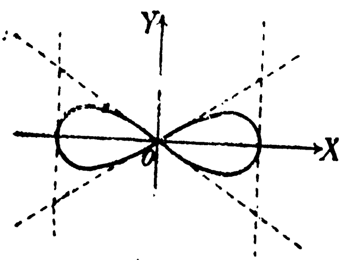


Fig. 74

(*iii*) When θ increases from $\pi/4$ to $\pi/2$ and from $\pi/2$ to $3\pi/4$, $\cos 2\theta$ remains negative and so r is not real. Thus no point on the curve corresponds to these values of θ.

When $\theta=3\pi/4$, $r=0$.

(*iv*) When θ increases from $3\pi/4$ to π, 2θ increases from $3\pi/2$ to 2π to that

$\cos 2\theta$ increases from 0 to 1 and, therefore, r increases from 0 to a.

Hence we have the curve as shown.

It is easy to see that the point P will describe exactly the same curve even if we take, r, to be negative.

The curve consists of two loops situated between the lines

$$\theta=\pi/4,\ \theta=3\pi/4.$$

To obtain the cartesian equation of the lemniscate, we re-write the polar equation as

$$r^2=a^2(\cos^2\theta-\sin^2\theta)$$

or

$$r^4=a^2(r^2\cos^2\theta-r^2\sin^2\theta).$$

Therefore, we obtain

$$(x^2+y^2)^2=a^2(x^2-y^2)$$

which is a well-known form of the equation of the lemniscate and is of the fourth degree.

11·63. $\mathbf{r^m=a^m\cos m\theta}$,

for $m=1,\ -1,\ 2,\ -2,\ \frac{1}{2},\ -\frac{1}{2}$.

(1) *For* $m=1$, we have

$$r=a\cos\theta$$

or

$$r^2=ar\cos\theta$$

i.e.,

$$x^2+y^2=ax,$$

which is a circle (Fig. 75) with its centre at $(a/2, 0)$ and radius $(a/2)$.

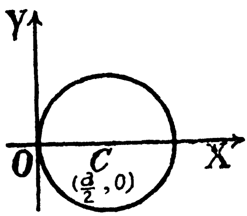


Fig. 75

(2) *For* $m=-1$, we have

$$r^{-1}=a^{-1}\cos(-\theta),$$

or

$$a=r\cos\theta$$

i.e.,

$$a=x$$

which is a straight line perpendicular (Fig. 76) to the initial line and at a distance, a, from it.

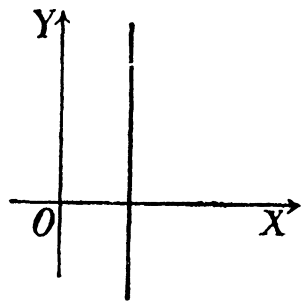


Fig. 76.

(3) *For* $m=2$, we have

$$r^2=a^2\cos 2\theta,$$

which is lemniscate.

(4) *For* $m=-2$, we have

$$r^{-2}=a^{-2}\cos(-2\theta)$$

or

$$a^2=r^2\cos 2\theta$$

$$=r^2(\cos^2\theta-\sin^2\theta)$$

or

$$a^2=x^2-y^2,$$

which is known to be a rectangular hyperbola (Fig. 77). To trace this curve, write its equation in the form

$$r^2=\frac{a^2}{\cos 2\theta}$$

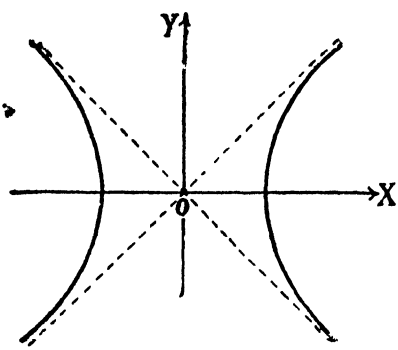


Fig. 77

and note the following points about it :—

(*i*) It is symmetrical about the initial line.

(*ii*) When $\theta=0$, $r=a$. When θ increases from 0 to $\pi/4$, r increases from a to ∞.

(*iii*) When θ varies from $\pi/4$ to $3\pi/4$, r remains imaginary.

(*iv*) When θ varies from $3\pi/4$, to π, r decreases from ∞ to a.

(5) *For* $m=\frac{1}{2}$, we have

$$r^{\frac{1}{2}}=a^{\frac{1}{2}}\cos\tfrac{1}{2}\theta$$

or $$r=a\cos^2\tfrac{1}{2}\theta,$$

or $$2r=a(1+\cos\theta),$$

which is a Cardioide.

(6) *For* $m=-\frac{1}{2}$, we have

$$r^{-\frac{1}{2}}=a^{-\frac{1}{2}}\cos(-\tfrac{1}{2}\theta)$$

i.e., $$a^{\frac{1}{2}}=r^{\frac{1}{2}}(\cos\tfrac{1}{2}\theta)$$

or $$a=r\cos^2\tfrac{1}{2}\theta$$
$$=r(1+\cos\theta)/2$$

or $$\frac{2a}{r}=1+\cos\theta,$$

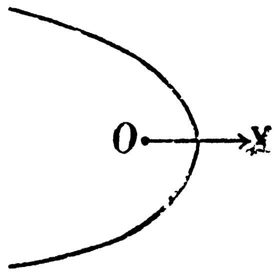


Fig. 78.

which is known to be a *parabola* (Fig. 78).

To trace this curve (Fig. 78), we re-write its equation in the form

$$r=\frac{2a}{1+\cos\theta}$$

and note the following points about it :—

(*i*) It is symmetrical about the initial line.

(*ii*) When $\theta=0$, $r=a$. When θ increases from 0 to π, $1+\cos\theta$ decreases from 2 to 0 and, therefore, r increase from a to ∞.

11·64. r = aθ, (a > 0). Spiral of Archimedes.

(*i*) When $\theta=0$, $r=0$, so that the curve goes through the pole.

(*ii*) When θ increases, r increases.

Also,

$$\text{when } \theta\to+\infty,\ r\to+\infty,$$
$$\text{when } \theta\to-\infty,\ r\to-\infty.$$

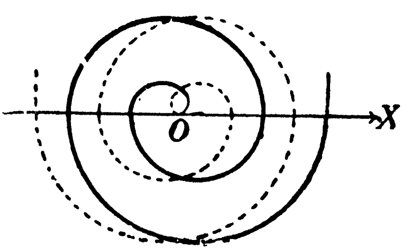


Fig. 79.

Thus the curve starting from the pole goes round it both ways an infinite number of times. The continuously drawn line corresponds to positive values of θ and the dotted one to negative values of θ.

11·65. $r\theta=a$, $(a>0)$. Hyperbolical Spiral.

Here

$$r=a/\theta.$$

(*i*) r is positive or negative according as θ is positive or negative.

(*ii*) When $|\theta|$ increases, $|r|$ decreases.

Also

$$\text{when } |\theta| \to 0, \ |r| \to \infty.$$
$$\text{when } |\theta| \to \infty, \ |r| \to 0.$$

(*iii*) Now $\qquad r=a/\theta$

or

$$r \sin \theta=(a \sin \theta)/\theta$$

or

$$y=(a \sin \theta)/\theta \text{ which} \to a \text{ as } \theta \to 0.$$

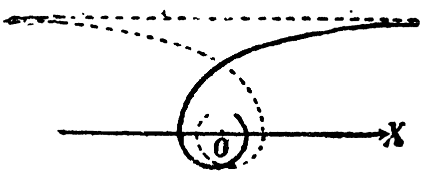


Fig. 80.

The ordinate of every point on the curve approaches a as θ approaches 0. We thus have the curve as drawn.

The continuously drawn line corresponds to positive values of θ, while the dotted one corresponds to negative value of θ.

11·66. $r=ae^{b\theta}$, $(a, b>0)$. (Equiangular Spiral)

(*i*) When $\theta=0$, $r=a$.

(*ii*) When θ increases, r also increases.

Also when $\theta \to \infty$, $r \to \infty$; when $\theta \to -\infty$, $r \to 0$.

(*iii*) r is always positive,

We thus have the curve as drawn.

The justification of the adjective 'Equiangular' will appear in Chapter XII, Ex. 1. p. 269.

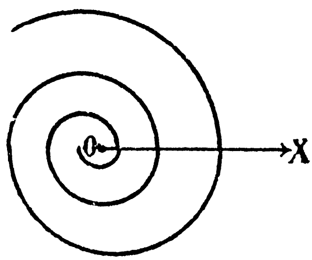


Fig. 81

11·67. $r=a \sin 3\theta$, $(a>0)$. *Three leaved rose.* (*D.U. 1949*)

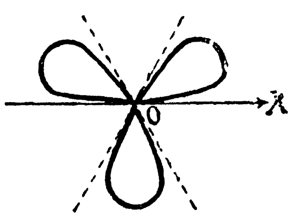


Fig. 82

(*i*) When $\theta=0$, $r=0$. As θ increases from 0 to $\pi/6$, 3θ increases from 0 to $\pi/2$ and, therefore, r increases from 0 to a.

(*ii*) As θ increases from $\pi/6$ to $\pi/3$, r decreases from a to 0.

(*iii*) As θ increases from $\pi/3$ to $\pi/2$, r remains negative and numerically increases from 0 to a.

(*iv*) As θ increases from $\pi/2$ to $2\pi/3$, r remains negative and numerically decreases from a to 0.

It may similarly be shown that as θ increases from $2\pi/3$ to $5\pi/6$ and from $5\pi/6$ to π, the point $P(r, \theta)$ describes the second loop above the initial line.

If θ increases beyond π, the same loops of the curve are repeated and we do not get any new point.

11·68. $\mathbf{r=a \sin 2\theta}$. *Four leaved rose.*

The discussion of this equation is left to the reader. Its shape only is shown in figure 83.

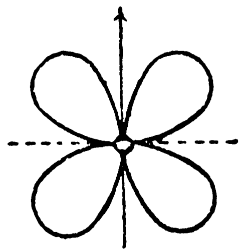


Fig. 83.

CHAPTER XII

TANGENTS AND NORMALS

[**Introduction.** This and the following chapters of the book will be devoted to the applications of Differential Calculus to Geometry. The part of Geometry thus treated is known as Differential Geometry of plane curves.]

Section I

Cartesian Co-ordinates

12·1. EQUATIONS OF TANGENT AND NORMAL.

12·11. Explicit Cartesian Equations. It was shown in Ch. IV § 4·15, p. 77 that if ψ be the angle which the tangent at any point (x, y) on the curve $y=f(x)$ makes with x-axis, then

$$\tan\psi=\frac{dy}{dx}=f'(x).$$

Therefore, *the equation of the tangent at any point* (x, y) *on the curve* $y=f(x)$ *is*

$$\mathbf{Y-y=f'(x)(X-x)}, \qquad \ldots(i)$$

where X, Y are the current co-ordinates of any point on the tangent.

The normal to the curve $y=f(x)$ at any point (x, y) is the straight line which passes through that point and is perpendicular to the tangent to the curve at that point so that its slope is, $-1/f'(x)$. Hence *the equation of the normal at* (x, y) *to the curve* $y=f(x)$ *is*

$$\mathbf{(X-x)+f'(x)(Y-y)=0}.$$

12·12. Implicit Cartesian Equations. For any point (x, y) on the curve $f(x, y)=0$ where $\partial f/\partial y\neq 0$, we have

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Hence the equations of the tangent and the normal at any point (x, y) *on the curve* $f(x, y)=0$, are

$$\mathbf{(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0} \textbf{ and } \mathbf{(X-x)\frac{\partial f}{\partial y}-(Y-y)\frac{\partial f}{\partial x}=0}.$$

respectively.

Cor. *A symmetrical form of the equation of the tangent to a rational Algebraic curve.* Let $f(x, y)$ be a rational algebraic function of x, y of degree n. We make $f(x, y)$ a homogeneous function of three variables x, y, z by multiplying each of its terms with a suitable power of z. Then by Euler's theorem, (§ 10·81, p. 199) we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x, y, z)=0$$

i.e.,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=-z\frac{\partial f}{\partial z},$$

so that the equation

$$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0$$

of the tangent takes the form

$$\mathbf{X}\frac{\partial f}{\partial x}+\mathbf{Y}\frac{\partial f}{\partial y}=x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=-z\frac{\partial f}{\partial z},$$

or

$$\mathbf{X}\frac{\partial f}{\partial x}+\mathbf{Y}\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=0,$$

or

$$\mathbf{X}\frac{\partial \mathbf{f}}{\partial \mathbf{x}}+\mathbf{Y}\frac{\partial \mathbf{f}}{\partial \mathbf{y}}+\mathbf{Z}\frac{\partial \mathbf{f}}{\partial \mathbf{z}}=0,$$

where, for the sake of *symmetry*, the co-efficient z of $\partial f/\partial z$ has been replaced by Z.

The symbols Z and z are to be both put equal to unity after differentiation.

This elegant form of the equation of the tangent to a rational algebraic curve proves very convenient in practice.

12·13. Parametric Cartesian Equations. At any point 't' of the curve $x=f(t)$, $y=F(t)$, where $f'(t)\neq 0$, we have

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{F'(t)}{f'(t)}.$$

Hence *the equations of the tangent and the normal at any point 't' of the curve $x=f(t)$, $y=F(t)$ are*

$$[\mathbf{X}-\mathbf{f(t)}]\mathbf{F'(t)}-[\mathbf{Y}-\mathbf{F(t)}]\mathbf{f'(t)}=\mathbf{0},$$

and

$$[\mathbf{X}-\mathbf{f(t)}]\mathbf{f'(t)}+[\mathbf{Y}-\mathbf{F(t)}]\mathbf{F'(t)}=\mathbf{0},$$

respectively.

Examples

1. *Find the equations of the tangent and the normal at any point (x, y) of the curves*

(*i*) $y=c\cosh(x/c)$. (*Catenary*) (*ii*) $x^m/a^m+y^m/b^m=1$.

(*i*) For $y=c\cosh(x/c)$, we have

$$\frac{dy}{dx}=\sinh\frac{x}{c}.$$

Hence the equations of the tangent and the normal at any point (x, y), *i.e.*, $(x, c\cosh x/c)$ are

$$Y-c\cosh\frac{x}{c}=(X-x)\sinh\frac{x}{c}$$

and

$$\left(Y-c\cosh\frac{x}{c}\right)\sinh\frac{x}{c}+X-x=0,$$

respectively ; X, Y being the current co-ordinates.

(*ii*) Let $$f(x, y)\equiv\frac{x^m}{a^m}+\frac{y^m}{b^m}-1=0.$$

$$\therefore \quad \frac{\partial f}{\partial x}=\frac{mx^{m-1}}{a^m}; \quad \frac{\partial f}{\partial y}=\frac{my^{m-1}}{b^m}.$$

Hence

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{b^m x^{m-1}}{a^m y^{m-1}}.$$

Therefore, the equation of the tangent at (x, y)

$$Y-y=-\frac{b^m x^{m-1}}{a^m y^{m-1}}(X-x),$$

or

$$\frac{Xx^{m-1}}{a^m}+\frac{Yy^{m-1}}{b^m}=\frac{x^m}{a^m}+\frac{y^m}{b^m}=1,$$

for (x, y) lies on the given curve.

Also, the equation of the normal at (x, y) is

$$Y-y=\frac{a^m y^{m-1}}{b^m x^{m-1}}(X-x),$$

or

$$\frac{X-x}{b^m x^{m-1}}=\frac{Y-y}{a^m y^{m-1}}.$$

Another method. The given equation is of degree m. Making it homogeneous, we get

$$f(x, y, z)=\frac{x^m}{a^m}+\frac{y^m}{b^m}-z^m,$$

so that

$$\frac{\partial f}{\partial x}=\frac{mx^{m-1}}{a^m}, \quad \frac{\partial f}{\partial y}=\frac{my^{m-1}}{b^m}, \quad \frac{\partial f}{\partial y}=-mz^{m-1}.$$

Hence the equation of the tangent is

$$\frac{X.mx^{m-1}}{a^m}+\frac{Y.my^{m-1}}{b^m}-Z.mz^{m-1}=0. \qquad \text{(cor. § 12·12, p. 254)}$$

Putting $Z=z=1$, we obtain

$$\frac{Xx^{m-1}}{a^m}+\frac{Yy^{m-1}}{b^m}=1,$$

as the required equation of the tangent.

2. *Find the equations of the tangent and normal at $\theta=\pi/2$ to the Cycloid*

$$x=a(\theta-\sin\theta),\ y=a(1-\cos\theta).$$

We have

$$\frac{dx}{d\theta}=a(1-\cos\theta)=2a\sin^2\frac{\theta}{2},$$

$$\frac{dy}{d\theta}=a\sin\theta=2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{dy}{d\theta}\bigg/\frac{dx}{d\theta}=\cot\frac{\theta}{2}.$$

Thus

$$\frac{dy}{dx}=1, \text{ for } \theta=\frac{\pi}{2}.$$

Also, for $\theta=\pi/2$, we have $x=a(\pi/2-1)$ and $y=a$. Hence the equation of the tangent at $\theta=\pi/2$, *i.e.*, at the point $[a(\pi/2-1),\ a]$ is

$$Y-a=1\ [X-a(\tfrac{1}{2}\pi-1)] \text{ or } X-Y=\tfrac{1}{2}a\pi-2a,$$

and the equation of the normal is

$$Y-a=-1[X-a(\tfrac{1}{2}\pi-1)] \text{ or } X+Y=\tfrac{1}{2}a\pi.$$

3. *Prove that the equation of the normal to the Astroid*

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},$$

may be written in the form

$$x\sin\phi-y\cos\phi+a\cos 2\phi=0.$$

Differentiating

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}, \qquad \ldots(i)$$

we get

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0$$

or

$$\frac{dy}{dx}=-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}};$$

Therefore the slope of the normal at any point (x, y)

$$=x^{\frac{1}{3}}/y^{\frac{1}{3}}.$$

But the slope of the given line $=\tan\phi$.

We write

$$x^{\frac{1}{3}}/y^{\frac{1}{3}}=\tan\phi. \qquad ...(ii)$$

Equations (i) and (ii) have now to be solved to find x and y. They give

$$y^{\frac{2}{3}}\tan^2\phi+y^{\frac{2}{3}}=a^{\frac{2}{3}},$$

or

$$y^{\frac{2}{3}}=a^{\frac{2}{3}}\cos^2\phi, \text{ i.e., } y=a\cos^3\phi.$$

Substituting this value of y in (ii), we get

$$x^{\frac{1}{3}}=a^{\frac{1}{3}}\sin\phi, \text{ i.e., } x=a\sin^3\phi.$$

Thus $\tan\phi$ is the slope of the normal at $(a\sin^3\phi, a\cos^3\phi)$. The equation of the normal at the point is

$$y-a\cos^3\phi=\frac{\sin\phi}{\cos\phi}(x-a\sin^3\phi),$$

or

$$y\cos\phi-a\cos^4\phi=x\sin\phi-a\sin^4\phi,$$

i.e.,

$$x\sin\phi-y\cos\phi+a(\cos^2\phi-\sin^2\phi)(\cos^2\phi+\sin^2\phi)=0,$$

i.e.,

$$x\sin\phi-y\cos\phi+a\cos 2\phi=0,$$

which is the given equation.

4. *Find the condition for the line*

$$x\cos\theta+y\sin\theta=p, \qquad ...(i)$$

to touch the curve

$$x^m/a^m+y^m/b^m=1. \qquad ...(ii)$$

In Ex. 1, (ii) page 255, it was shown that the equation of the tangent at any point (x, y) of the curve (ii) is

$$\frac{Xx^{m-1}}{a^m}+\frac{Yy^{m-1}}{b^m}=1; \qquad ...(iii)$$

where X, Y are the current co-ordinates.

We re-write the equation (i) in the form

$$X\cos\theta+Y\sin\theta-p=0, \qquad ...(iv)$$

taking X, Y as current co-ordinates instead of x, y.

The equations (iii) and (iv) represent the same line.

$$\therefore \qquad \frac{x^{m-1}}{a^m \cos\theta} = \frac{y^{m-1}}{b^m \sin\theta} = \frac{1}{p},$$

or

$$x = \left(\frac{a^m \cos\theta}{p}\right)^{1/(m-1)}, \; y = \left(\frac{b^m \sin\theta}{p}\right)^{1/(m-1)}.$$

The point (x, y) lies on the given curve. Therefore

$$\frac{1}{a^m}\left(\frac{a^m \cos\theta}{p}\right)^{m/(m-1)} + \frac{1}{b^m}\left(\frac{b^m \sin\theta}{p}\right)^{m/(m-1)} = 1,$$

or

$$(a\cos\theta)^{m/(m-1)} + (b\sin\theta)^{m/(m-1)} = p^{m/(m-1)},$$

which is the required condition.

5. *Show that the length of the perpendicular from the foot of the ordinate on any tangent to the Catenary*

$$y = c \cosh (x/c),$$

is constant.

Equation of the tangent at *any* point (x, y) of the Catenary is $X \sinh x/c - Y + (c \cosh x/c - x \sinh x/c) = 0$ (See Ex. 1. p. 255). The foot of the ordinate of the point (x, y) is the point $(x, 0)$. The length of the perpendicular from $(x, 0)$ to the tangent

$$= \frac{x \sinh x/c - 0 + (c \cosh x/c - x \sinh x/c)}{\sqrt{(\sinh^2 x/c + 1)}}$$

$$= \frac{c \cosh x/c}{\cosh x/c} = c,$$

which is free of x, y and is, therefore, constant.

6. *Show that the length of the portion of the tangent to the astroid*

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

intercepted between the co-ordinate axes is constant.

Differentiating the given equation, we get

$$\frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-\frac{1}{3}} \frac{dy}{dx} = 0,$$

i.e., $$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

Therefore the equation of the tangent at *any* point (x, y) is

$$Y - y = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(X - x),$$

or

$$Xy^{\frac{1}{3}}+Yx^{\frac{1}{3}}=x^{\frac{1}{3}}y^{\frac{1}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})=x^{\frac{1}{3}}y^{\frac{1}{3}}a^{\frac{2}{3}}. \quad ..(i)$$

The tangent (*i*) cuts X-axis where $Y=0$. Hence its intersection with X-axis is

$$A(x^{\frac{1}{3}}a^{\frac{2}{3}}, 0).$$

Similarly, on putting $X=0$ in (*i*), we find that its intersection with Y-axis is

$$B(0, y^{\frac{1}{3}}a^{\frac{2}{3}}).$$

Therefore $AB=\sqrt{[x^{\frac{2}{3}}a^{\frac{4}{3}}+y^{\frac{2}{3}}a^{\frac{4}{3}}]}=\sqrt{[a^{\frac{4}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})]}$

$$=\sqrt{(a^{\frac{4}{3}}a^{\frac{2}{3}})}=a,$$

which is free of x and y and is, therefore, a constant.

Exercises

1. Find the tangent and normal to each of the following curves :—

(*i*) $y^2=4ax$ at $(a, -2a)$. (*ii*) $x^2/a^2+y^2/b^2=1$ at (x', y').

(*iii*) $xy=c^2$ at $(cp, c/p)$. (*iv*) $(x^2+y^2)x-ay^2=0$ at $x=a/2$.

(*v*) $(x^2+y^2)x-a(x^2-y^2)=0$ for $x=-3a/5$.

(*vi*) $x^2(x-y)+a^2(x+y)=0$ at $(0, 0)$.

(*vii*) $x=2a\cos\theta-a\cos 2\theta$, $y=2a\sin\theta-a\sin 2\theta$ at $\theta=\pi/2$.

(*viii*) $c^2(x^2+y^2)=x^2y^2$ at $(c/\cos\theta, c/\sin\theta)$.

(*ix*) $x=\dfrac{2at^2}{1+t^2}$, $y=\dfrac{2at^3}{1+t^2}$ at $t=\frac{1}{2}$.

2. Find the equation of the tangent to the curve $c^2(x^2+y^2)=x^2y^2$ in the form $x\cos^3\theta+y\sin^3\theta=c$.

3. Show that the tangent to the curve $3xy^2-2x^2y=1$ at $(1, 1)$ meets the curve again at $(-16/5, -1/20)$.

4. Prove that the equation of the tangent at any point $(4m^2, 8m^3)$ of the semicubical parabola $x^3=y^2$ is $y=3mx-4m^3$ and show that it meets the curve again at $(m^2, -m^3)$, where it is a normal if $9m^2=2$. (*D.U. Hons.*, *1957*)

5. Find where the tangent is parallel to X-axis and where it is parallel to Y-axis for the following curves :

(*i*) $x^3+y^3=a^3$. (*ii*) $x^3+y^3=3axy$. (*Folium*)

(*iii*) $25x^2+12xy+4y^2=1$.

6. Find the equations of the tangent and the normal to the curve

$$y(x-2)(x-3)-x+7=0$$

at the point where it cuts X-axis. (*Delhi 1948*)

7. Show that the line

$$x\cos\theta+y\sin\theta-p=0,$$

will touch the curve

$$x^my^n-a^{m+n}=0,$$

if

$$p^{m+n}m^mn^n=(m+n)^{m+n}a^{m+n}\cos^m\theta\sin^n\theta.$$

(*P.U. 1951*)

8. Tangent at any point of the curve $(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1$ meets the co-ordinate axes in A and B. Show that the locus of the point with (OA, OB) as co-ordinates is

$$x^2/a^2+y^2/b^2=1$$

9. Show that the tangent at any point (x, y) on the curve

$$y^m=ax^{m-1}+x^m,$$

makes intercepts

$$\frac{ax}{(m-1)a+mx} \text{ and } \frac{ay}{m(a+x)}$$

on the co-ordinate axes.

10. Prove that the sum of the intercepts on the co-ordinate axes of any tangent to

$$\sqrt{x}+\sqrt{y}=\sqrt{a}$$

is constant.

11. Prove that the portion of the tangent to the curve

$$\frac{x+\sqrt{(a^2-y^2)}}{a}=\log\frac{a+\sqrt{(a^2-y^2)}}{y},$$

intercepted between the point of contact and X-axes is constant.

12. Show that the normal at any point of the curve

$$x=a\cos\theta+a\theta\sin\theta,\ y=a\sin\theta-a\theta\cos\theta$$

is at a constant distance from the origin. (*B.U.*)

13. Show that the length of the portion of the normal to the curve

$$x=a(4\cos^3\theta-3\cos\theta),\ y=a(4\sin^3\theta-9\sin\theta)$$

intercepted between the co-ordinate axes is constant.

14. Show that the tangent and the normal at every point of the curve

$$x=ae^{\theta}(\sin\theta-\cos\theta),\ y=ae^{\theta}(\sin\theta+\cos\theta)$$

are equidistant from the origin.

15. Show that the distance from the origin of the normal at any point of the curve

$$x=ae^{\theta}(\sin\tfrac{1}{2}\theta+2\cos\tfrac{1}{2}\theta),\ y=ae^{\theta}(\cos\tfrac{1}{2}\theta-2\sin\tfrac{1}{2}\theta)$$

is twice the distance of the tangent.

16. Show that the tangent at any point of the curve

$$x=a(t+\sin t\cos t),\ y=a(1+\sin t)^2,$$

makes angle $\frac{1}{2}(\pi+2t)$ with X-axis.

17. Tangents are drawn from the origin to the curve $y=\sin x$. Prove that their points of contact lie on $x^2y^2=x^2-y^2$. (*D.U 1953*)

18. If

$$p=x\cos\theta+y\sin\theta,$$

touch the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}}+\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=1,$$

prove that

$$p^n=(a\cos\theta)^n+(b\sin\theta)^n.$$

(*P.U. 1941 ; D.U. 1955*)

19. The tangent at any point on the curve $x^3+y^3=2a^3$ cuts off lengths p and q on the co-ordinate axis ; show that

$$p^{-\frac{3}{2}}+q^{-\frac{3}{2}}=2^{-\frac{1}{2}}a^{-\frac{3}{2}}. \qquad (P.U.)$$

20. The tangent at any point $P(x_1, y_1)$ of the curve

$$y=x^2-x^3$$

meets it again at Q. Find the co-ordinates of the mid-point of PQ and show that its locus is

$$y=1-9x+28x^2-28x^3. \qquad (B.U.)$$

21. Prove that the distance of the point of contact of any tangent to

$$2a^m xy^{m-1}=y^{2m}-a^{2m},$$

from the origin as the segment of the x-axis between the origin and the tangent.

22. Show that the normal to the curve

$$5x^5-10x^3+x+2y+6=0$$

at $(0,-3)$ is tangent at the two points where it meets the curve again.

23. Show that the tangent to the curve

$$25x^5+5x^4-45x^3-5x^2+2x+6y-24=0.$$

at $(-1, 1)$ is also a normal at two points of the curve.

12·2. Angle of intersection of two curves.

Def. *The angle of intersection of two curves at a point of intersection is the angle between the tangents to the two curves at that point.*

Examples

1. *Find the angle of intersection of the parabolas*

$$y^2=4ax, \qquad ..(i)$$

and

$$x^2=4by, \qquad ...(ii)$$

at the point other than the origin.

We have

$$x^4=16b^2y^2=16b^2\,;\ 4ax=64ab^2x,$$

or

$$x(x^3-64ab^2)=0.$$

$$\therefore \qquad x=0 \text{ and } 4a^{\frac{1}{3}}b^{\frac{2}{3}}.$$

Substituting these values of x in (ii), we get

$$y=0 \text{ for } x=0 \text{ and } y=4a^{\frac{2}{3}}b^{\frac{1}{3}} \text{ for } x=4a^{\frac{1}{3}}b^{\frac{2}{3}}.$$

Therefore $(0, 0)$ and $(4a^{\frac{1}{3}}b^{\frac{2}{3}},\ 4a^{\frac{2}{3}}b^{\frac{1}{3}})$ are the two points of intersection.

Differentiating (i), we get

$$2y\,dy/dx=4a \text{ or } dy/dx=2a/y.$$

Therefore, for the curve (i),

$$(dy/dx) \text{ at } 4(a^{\frac{1}{3}}b^{\frac{2}{3}},\ 4a^{\frac{2}{3}}b^{\frac{1}{3}})=a^{\frac{1}{3}}/2b^{\frac{1}{3}}.$$

Differentiating (ii), we get

$$2x=4b\frac{dy}{dx} \text{ or } \frac{dy}{dx}=\frac{x}{2b}.$$

Therefore, for the curve (ii),

$$(dy/dx) \text{ at } (4a^{\frac{1}{3}}b^{\frac{2}{3}},\ 4a^{\frac{2}{3}}b^{\frac{1}{3}})=2a^{\frac{1}{3}}/b^{\frac{1}{3}}.$$

Thus, if m, m' be the slopes of the tangents to the two curves, we have

$$m=a^{\frac{1}{3}}/2b^{\frac{1}{3}},\ m'=2a^{\frac{1}{3}}/b^{\frac{1}{3}}.$$

The required angle

$$=\tan^{-1}\frac{m-m'}{1+mm'}=\tan^{-1}\frac{\dfrac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}}-\dfrac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}}}{1+\dfrac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}}\cdot\dfrac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}}}=\frac{a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}}+b^{\frac{2}{3}})}.$$

2. *Find the condition that the curves*

$$ax^2+by^2=1,\ a_1x^2+b_1y^2=1,$$

should intersect orthogonally.

Let (x', y') be a point of intersection so that we have

$$ax'^2+by'^2=1,$$

$$a_1x'^2+b_1y'^2=1.$$

$$\therefore \qquad \frac{x'^2}{-b+b_1}=\frac{y'^2}{-a_1+a}=\frac{1}{ab_1-a_1b},$$

or

$$x'^2=(b_1-b)/(ab_1-a_1b),\ y'^2=(a-a_1)/(ab_1-a_1b).$$

Differentiating the first equation, we get

$$2ax+2by\frac{dy}{dx}=0 \text{ or } \frac{dy}{dx}=-\frac{ax}{by}.$$

$$\therefore \qquad \left[\frac{dy}{dx}\right]_{(x',\ y')}=-\frac{ax'}{by'}.$$

Similarly for the second curve

$$\left[\frac{dy}{dx}\right]_{(x',\ y')}=-\frac{a_1x'}{b_1y'}.$$

For orthogonal intersection, we have

$$\frac{-ax'}{by'}\times\frac{-a_1x'}{b_1y'}=-1,\ i.e.,\ aa_1x'^2+bb_1y'^2=0. \qquad ...(i)$$

Substituting the values of x', y' in (i), we obtain

$$\frac{aa_1(b_1-b)}{ab_1-a_1b}+\frac{bb_1(a-a_1)}{ab_1-a_1b}=0,$$

or

$$\frac{b_1-b}{bb_1}+\frac{a-a_1}{aa_1}=0,$$

i.e.,

$$\frac{1}{b}-\frac{1}{b_1}=\frac{1}{a}-\frac{1}{a_1},$$

as the required condition.

Exercises

1. Find the angle between
   (i) $x^2-y^2=a^2$ and $x^2+y^2=a\sqrt{2}$. *(P.U. 1955)*
   (ii) $y^2=ax$ and $x^3+y^3=3axy$.
2. Prove that the curves
$$x^3+2xy^2-10a^2x+12a^2y+3a^3=0,$$
$$y^3+2xy^2-5a^2x-a^3=0,$$
interesect at $\tan^{-1}(88/73)$ at $(3a, -2a)$.
3. Find the condition for $y=mx$ to cut at right angles the conic
$$ax^2+2hxy+by^2=1.$$
Hence find the directions of the axes of the conic.

12·3. Lengths of the tangent, normal, sub-tangent and sub-normal at any point of a curve.

Let the tangent and the normal at any point (x, y) of the curve meet the X-axis at T and G respectively. Draw the ordinate PM.

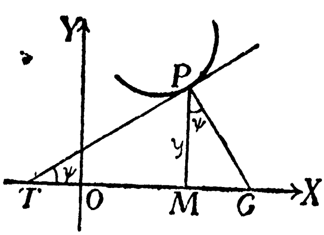


Fig. 84

Then *the lines TM, MG are called the sub-tangent and sub-normal respectively.*

The lengths PT, PG are sometimes referred to as the lengths of the tangent and the normal respectively.

Clearly

$$\angle MPG=\psi.$$

Also

$$\tan\psi=\frac{dy}{dx}.$$

From the figure, we have

(*i*) **Tangent** $=TP=MP$ cosec $\psi=y\sqrt{(1+\cot^2\psi)}$

$$=y\sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}.$$

(*ii*) **Sub-tangent** $=TM=MP\cot\psi=y\,\dfrac{dx}{dy}$.

(*iii*) **Normal** $=GP=MP\sec\psi=y\sqrt{(1+\tan^2\psi)}$

$$=y\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}.$$

(*iv*) **Sub-normal** $=MG=MP\tan\psi=y\,\dfrac{dy}{dx}$.

Exercises

1. Prove that the sub-normal at any point of the parabola

$$y^2=4ax$$

is constant.

2. Show that the sub-normal at any point of the curve

$$y^2x^2=a^2(x^2-a^2)$$

varies inversely as the cube of its abscissa.

3. Find the length of the tangent, length of the normal, sub-tangent and sub-normal at the point θ, on the four cusped hypocycloid

$$x=a\cos^3\theta,\ y=a\sin^3\theta.$$

4. Show that the sub-tangent at any point of the curve

$$x^m y^n=a^{m+n}$$

varies as the abscissa. (*Banaras*)

5. Show that for the curve

$$x=a+b\log[b+\sqrt{(b^2-y^2)}]-\sqrt{(b^2-y^2)},$$

sum of the sub-normal and sub-tangent is constant. (*P.U. 1938*)

6. Show that for the curve

$$y=hn^{-\frac{1}{2}}e^{-h^2x^2},$$

the product of the abscissa and the sub-tangent is constant.

7. Show that the sub-tangent at any point of the exponential curve

$$y=ae^{x/b}$$

is constant.

8. For the catenary

$$y=c\cosh(x/c),$$

prove that the length of the normal is y^2/c. (*P.U. 1941*)

9. Prove that the sum of the tangent and sub-tangent at any point of the curve

$$e^{y/a}=x^2-a^2.$$

varies as the product of the corresponding co-ordinates.

10. Show that in the curve

$$y=a\log(x^2-a^2),$$

the sum of the tangent and the sub-tangent varies as the product of the co-ordinates of the point. [Compare Ex. 9 above]. (*D.U. 1952*)

12·4. Pedal equations or p, r equations.

Def. *A relation between the distance, r, of any point on the curve from the origin (or pole), and the length of the perpendicular, p, from the origin (or pole) to the tangent at the point is called pedal equation of the curve.*

12·41. *To determine the pedal equation of a curve whose Cartesian equation is given.*

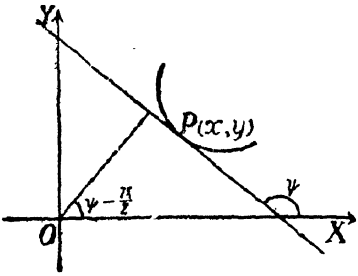


Fig. 85.

Let the equation of the curve be

$$y=f(x). \qquad ...(i)$$

Equation of the tangent at any point (x, y) is

$$Y-y=f'(x)(X-x)$$

or

$$Xf'(x)-Y+[y-xf'(x)]=0.$$

If, p, be the length of the perpendicular from (0, 0) to this tangent, we have

$$p=\frac{y-xf'(x)}{\sqrt{[1+f'^2(x)]}}.$$

Also

$$r^2=x^2+y^2. \qquad ...(iii)$$

Eliminating x, y between (i), (ii) and (iii), we obtain the required pedal equation of the curve (i).

Example

Find the pedal equation of the parabola

$$y^2=4a(x+a).$$

Tangent at (x, y) is

$$Y-y=\frac{2a}{y}\cdot(X-x)$$

or

$$2aX-yY+(y^2-2ax)=0. \qquad ...(i)$$

The length, p, of the perpendicular from the origin to the tangent (i) is given by

$$p=\frac{y^2-2ax}{\sqrt{(4a^2+y^2)}}$$

$$=\frac{4a(x+a)-2ax}{\sqrt{[4a^2+4a(x+a)]}}$$

$$=\frac{2a(x+2a)}{\sqrt{[4a(x+2a)]}}=\sqrt{[a(x+2a)]}. \qquad ...(ii)$$

Also

$$r^2=x^2+y^2=x^2+4a(x+a)=(x+2a)^2. \qquad ...(iii)$$

From (*ii*) and (*iii*), we obtain

$$p^2=ar,$$

which is the required pedal equation.

Exercises

1. Show that the pedal equation of ellipse

$$x^2/a^2+y^2/b^2=1,$$

is

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}. \qquad (P.U.\ 1955)$$

2. Show that the pedal equation of the astroid

$$x=a\cos^3\theta,\ y=a\sin^3\theta,$$

is

$$r^2=a^2-3p^2.$$

3. Show that the pedal equation of the curve

$$x=2a\cos\theta-a\cos 2\theta,\ y=2a\sin\theta-a\sin 2\theta,$$

is

$$9(r^2-a^2)=8p^2.$$

4. Show that the pedal equation of the curve

$$x=ae^{\theta}(\sin\theta-\cos\theta),\ y=ae^{\theta}(\sin\theta+\cos\theta),$$

is

$$r=\sqrt{2}p.$$

5. Show that the pedal equation of the curve

$$x=a(3\cos\theta-\cos^3\theta),\ y=a(3\sin\theta-\sin^3\theta),$$

is

$$3p^2(7a^2-r^2)=(10a^2-r^2)^2.$$

6. Show that the pedal equation of the curve

$$c^2(x^2+y^2)=x^2y^2.$$

is

$$1/p^2+3/r^2=1/c^2.$$

Section II

Polar Co-ordinates

12·5. Angle between radius vector and tangent.

Let $P(r, \theta)$ be any given point on the curve.

Take any other point $Q(r+\delta r, \theta+\delta\theta)$ on the curve. Produce OP to L.

Let PT be the tangent at P and let $\angle LPT=\phi$.

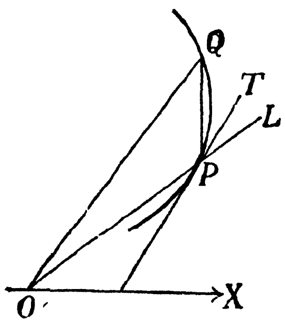


Fig. 86

Let $\angle LPQ=\alpha$ so that $\angle LPT=\phi$ is the limit of α as $Q\to P$. We have

$$\angle OPQ=\pi-\alpha.$$

Also

$$\angle OQP=\angle LPQ-\angle POQ=\alpha-\delta\theta.$$

Applying the sine formula to the $\triangle OPQ$, we get

$$\frac{OQ}{OP}=\frac{\sin\angle OPQ}{\sin\angle OQP}$$

or

$$\frac{r+\delta r}{r}=\frac{\sin(\pi-\alpha)}{\sin(\alpha-\delta\theta)}=\frac{\sin\alpha}{\sin(\alpha-\delta\theta)}.$$

$$\therefore\quad \frac{\delta r}{r}=\frac{\sin\alpha}{\sin(\alpha-\delta\theta)}-1$$

$$=\frac{\sin\alpha-\sin(\alpha-\delta\theta)}{\sin(\alpha-\delta\theta)}$$

$$=2\cos\frac{2\alpha-\delta\theta}{2}\sin\frac{\delta\theta}{2}\,\frac{1}{\sin(\alpha-\delta\theta)}$$

or

$$\frac{1}{r}\cdot\frac{\delta r}{\delta\theta}=\frac{\cos(\alpha-\frac{1}{2}\delta\theta)}{\sin(\alpha-\delta\theta)}\cdot\frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}.$$

Let $Q\to P$ so that $\delta\theta\to 0$, $\delta r\to 0$, and $\alpha\to\phi$.

Therefore

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{\cos\phi}{\sin\phi}\cdot 1$$

or

$$\mathbf{\tan\phi=r\frac{d\theta}{dr}}.$$

Cor. Angle of intersection of two curves. If ϕ_1, ϕ_2 be angles between the common radius vector and the tangents to the two curves at a point of intersection, then their angle of intersection is

$$|\phi_1-\phi_2|.$$

Note. Precise meaning of φ. For a point P of the curve,

$$r=f(\theta),$$

φ is precisely defined to be the angle through which the positive direction of the radius vector (*i.e.*, the direction of the radius vector produced) has to rotate to coincide with the direction of the tangent in which θ increases. The direction of the tangent in which, θ, increases is taken as the positive direction of the tangent.

Examples

1. *Show that the radius vector is inclined, at a constant angle to the tangent at any point on the equiangular spiral*

$$r = ae^{b\theta}. \qquad \ldots (i)$$

Differentiating (i) *w.r.* to θ, we get

$$\frac{dr}{d\theta}=b \,.\, ae^{b\theta}=br, \text{ i.e., } r\frac{d\theta}{dr}=\frac{1}{b}$$

$\therefore$ $$\tan\phi=\frac{1}{b},$$

or $$\phi=\tan^{-1}\frac{1}{b},$$

which is a constant. This property of equiangular spiral justifies the adjective '*Equiangular*'. [Refer § 11·66, p. 252]

2. *Find the angle of intersection of the Cardioides*

$$r=a(1+\cos\theta),\ r=b(1-\cos\theta).$$

Let $P(r_1, \theta_1)$ be a point of intersection. Let ϕ_1, ϕ_2 be the angles which OP makes with the tangents to the two curves.

For the curve $r=a(1+\cos\theta)$ we have

$$\frac{dr}{d\theta}=-a\sin\theta,$$

or

$$r\frac{d\theta}{dr}=-\frac{a(1+\cos\theta)}{a\sin\theta}=-\frac{2\cos^2\theta/2}{2\sin\theta/2.\cos\theta/2}=-\cot\frac{\theta}{2}.$$

$\therefore$ $$\tan\phi=\tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right),$$

or

$$\phi=\frac{\pi}{2}+\frac{\theta}{2}.$$

Hence

$$\phi_1=\frac{\pi}{2}+\frac{\theta_1}{2}.$$

For the curve $r=b(1-\cos\theta)$, we have

$$\frac{dr}{d\theta}=b\sin\theta,$$

or

$$\tan\phi=r\frac{d\theta}{dr}=\frac{b(1-\cos\theta)}{b\sin\theta}=\tan\frac{\theta}{2}.$$

$\therefore$ $$\phi=\frac{\theta}{2}.$$

Hence

$$\phi_2=\frac{\theta_2}{2}.$$

Therefore, $\phi_1 - \phi_2 = \pi/2$ and hence the curves cut each other at right angles.

Exercises

1. Find for the curves

(i) $r = a(1 - \cos\theta)$. (Cardioide). (ii) $r^m = a^m \cos m\theta$.

(iii) $2a/r = 1 + \cos\theta$. (Parabola) (iv) $r^m = a^m (\cos m\theta + \sin m\theta)$.

2. Show that the two curves

$$r^2 = a^2 \cos 2\theta \text{ and } r = a(1 + \cos\theta)$$

intersect at an angle $3\sin^{-1}(3/4)^{\frac{1}{4}}$.

3. Show that the curves $r^m = a^m \cos m\theta$, $r^m = a^m \sin m\theta$ cut each other orthogonally.

4. Find the angles between the curves

(i) $r = a\theta$, $r\theta = a$; (ii) $r = a\theta/(1+\theta)$, $r = a/(1+\theta^2)$;

(iii) $r = a \operatorname{cosec}^2(\theta/2)$, $r = b\sec^2(\theta/2)$;

(iv) $r = a\log\theta$, $r = a/\log\theta$; (v) $r^2 \sin 2\theta = 4$, $r^2 = 16\sin 2\theta$;

(vi) $r = ae^{\theta}$, $re^{\theta} = b$.

5. Show that in the case of the curve $r = a(\sec\theta \pm \tan\theta)$, if a radius OPP' be drawn cutting the curve in P and P' and if the tangents at P, P' meet in T, then $TP = TP'$. *(M.U.)*

6. Show that the tangents to the cardioide

$$r = a(1 + \cos\theta)$$

at the points whose vectorial angles are $\pi/3$ and $2\pi/3$ are respectively parallel and perpendicular to the initial line.

7. The tangents at two points P, Q lying on the same side of the initial line of the cardioide $r = a(1 + \cos\theta)$, are perpendicular to each other ; show that the line PQ subtends an angle $\pi/3$ at the pole.

8. Show that the tangents drawn at the extremities of any chord of the cardioide $r = a(1 + \cos\theta)$ which passes through the pole are perpendicular to each other.

9. If two tangents to the cardioide $r = a(1 + \cos\theta)$ are parallel, show that the line joining their points of contact subtends an angle $2\pi/3$ at the pole.

10. Show that

$$\tan = \varphi \left(x\frac{dy}{dx} - y \right) \Big/ \left(y\frac{dy}{dx} + x \right).$$

(Agra 1952)

126·. Length of the perpendicular from pole to the tangent. From the pole O draw OY perpendicular to the tangent at any point $P(r, \theta)$ on the curve $r = f(\theta)$; Y being its foot.

Let $OY = p$

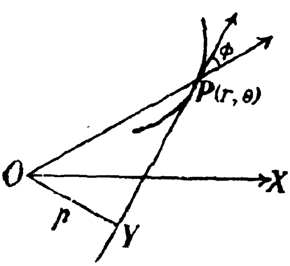


Fig. 87

From the $\triangle OPY$, we get

$$\frac{p}{r} = \sin\phi$$

or

$$\mathbf{p = r\sin\phi},$$

which is a very useful expression for p.

We now express, p, in terms of the co-ordinates r, θ and the derivative of, r, w.r. to θ. We have

$$\tan\phi = r\frac{d\theta}{dr}$$

$$\therefore \qquad \sin\phi = \pm\frac{r}{\sqrt{[r^2+(dr/d\theta)^2]}}.$$

$$\therefore \qquad p = \pm\frac{r^2}{\sqrt{[r^2+(dr/d\theta)^2]}},$$

which we can write as

$$\frac{1}{p^2} = \frac{r^2+(dr/d\theta)^2}{r^4} = \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \qquad \ldots(ii)$$

Yet, another form of p will be obtained, if we write

$$r = 1/u,$$

so that

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}.$$

Substituting these values in (*ii*), we get

$$\frac{1}{p^2} = u^2+\left(\frac{du}{d\theta}\right)^2. \qquad \ldots(iii)$$

12·7. Lengths of polar sub-tangent and polar sub-normal. Let the line through the pole O, drawn perpendicular to the radius vector OP, meet the tangent and normal at P in points T and G respectively.

Then, *OT, OG are respectively called the polar sub-tangent and polar sub-normal at P.*

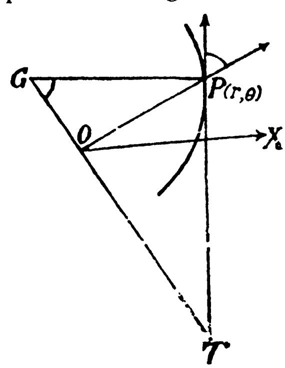


Fig. 88

From the $\triangle OPT$, we get

$$\frac{OT}{OP} = \tan\phi,$$

or

$$OT = OP\tan\phi = r^2\frac{d\theta}{dr}.$$

Hence polar sub-tangent

$$= r^2\frac{d\theta}{dr} = -\frac{d\theta}{du} \text{ where } u = \frac{1}{r}.$$

From the $\triangle OPG$, we get

$$\frac{OG}{OP} = \cot\phi,$$

or

$$OG = OP\cot\phi$$

$$= r\cdot\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{dr}{d\theta}.$$

Hence, **polar sub-normal** $= \dfrac{dr}{d\theta} = -\dfrac{1}{u^2}\cdot\dfrac{du}{d\theta}$ where $u = \dfrac{1}{r}$.

Note. The lengths PT and PG are sometimes called the lengths of the polar tangent and polar normal at P. We have

$$PT=OP\sec\phi=r\sqrt{[1+\tan^2\phi]}=r\sqrt{\left[1+r^2\left(\frac{d\theta}{dr}\right)^2\right]}$$

$$PG=OP\operatorname{cosec}\phi=r\sqrt{[1+\cot^2\phi]}=\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}.$$

12·8. Pedal equations. *To obtain the pedal equation of a curve whose polar equation is given.*

Let

$$r=f(\theta) \text{ be the given curve.} \qquad \ldots(i)$$

We have

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \qquad \ldots(ii)$$

Eliminating θ between (i) and (ii), we obtain the required pedal equation of the curve.

The pedal equation is sometimes more conveniently obtained by eliminating θ and ϕ between (i) and

$$\tan\phi=r\frac{d\theta}{dr},$$

$$p=r\sin\phi.$$

Exercises

1. Show that for the curve

$$\theta=\cos^{-1}\frac{r}{k}-\sqrt{\left[\frac{k^2-r^2}{r^2}\right]},$$

the length of polar tangent is constant.

2. Prove that for the curve

$$r^{-1}=A\theta+B,$$

the polar sub-tangent is constant.

3. Show that for the spiral $r=a\theta$, the polar sub-normal is constant.

4. Find the polar sub-tangent of

(i) $r=a(1+\cos\theta)$. (Cardioide) (ii) $2a/r=1+e\cos\theta$. (Conic)
(iii) $r=a\theta^2/(\theta-1)$.

5. Show that for the curve : $r=ae^{b\theta^2}$.

$$\frac{\text{polar sub-normal}}{\text{polar sub-tangent}} \text{ varies as } \theta^2,$$

6. Find the polar sub-normal of

(i) $r=a/\theta$; (ii) $r=a+b\cos\theta$.

7. Find p for the curve

$$r=ae^{\theta}/(1+\theta)^2.$$

8. *Find the pedal equation of $r^m = a^m \cos m\theta$.*

Logarithmically differentiating, we get

$$\frac{m}{r} \cdot \frac{dr}{d\theta} = -m \tan m\theta$$

$\therefore$ $$\tan \phi = r \frac{d\theta}{dr} = -\cot m\theta = \tan\left(\tfrac{1}{2}\pi + m\theta\right).$$

or $$\phi = \tfrac{1}{2}\pi + m\theta.$$

Thus $$p = r \sin \phi = r\left(\sin \tfrac{1}{2}\pi + m\theta\right) = r \cos m\theta.$$

$\therefore$ $$r^m = a^m \cdot \frac{p}{r} \text{ or } pa^m = r^{m+1}.$$

which is the required pedal equation.

9. Find the pedal equations of

(*i*) $r = a/\theta$. (*ii*) $r = ae^{\theta \cot \alpha}$.

(*iii*) $l/r = 1 + e \cos \theta$. (*iv*) $r = a\theta$.

(*v*) $r = a(1 + \cos \theta)$. (*vi*) $r = a \sin m\theta$.

(*vii*) $r(1 - \sin \tfrac{1}{2}\theta)^2 = a$. (*viii*) $r = a + b \cos \theta$.

(*ix*) $r^m = a^m \sin m\theta + b^m \cos m\theta$.

10. Prove that the locus of the extremity of the polar sub-normal of the curve $r = f(\theta)$ is the curve

$$r = f'(\theta - \tfrac{1}{2}\pi).$$

Hence show that the locus of the extremity of the polar sub-normal of the equiangular spiral $r = ae^{m\theta}$ is another equiangular spiral. (*P.U. 1940*)

11. Prove that the locus of the extremity of the polar sub-tangent of the curve

$$\frac{1}{r} + f(\theta) = 0$$

is

$$u = f'(\tfrac{1}{2}\pi + \theta).$$

Hence show that the locus of the extremity of the polar sub-tangent of the curve

$$r = (1 + \tan \tfrac{1}{2}\theta)/(m + n \tan \tfrac{1}{2}\theta)$$

is a cardioide. (*Allahabad 1943*)

12. Show that the pedal equation of the spiral $r = a \operatorname{sech} n\theta$ is of the form

$$\frac{1}{p^2} = \frac{A}{r^2} + B. \qquad (\textit{Delhi 1949})$$

*

CHAPTER XIII

DERIVATIVE OF ARCS

13·1. On the meaning of the lengths of arcs. The intuitive notion of the length of an arc of a curve is based upon the *assumption* that it is possible for an inextensible fine string to take the form of the given curve so that we may then stretch it along the number axis and find out the number which measures its length.

This assumption, which is not analytical, cannot be the basis of analytical treatment of the subject of lengths of arcs of curves. Also to define lengths of arcs analytically is not within the scope of this book.

Hence we have to base our treatment on an *axiom* which concerns the numerical measure of lengths of arcs and we now proceed to give it.

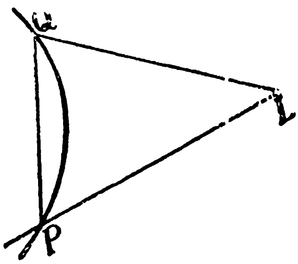


Fig. 89

Axiom. *If P, Q, be any two points on a curve such that the arc PQ is throughout its length concave to the chord PQ, then*

$$\text{Chord } PQ < \text{arc } PQ < PL + QL,$$

where PL and QL are any two lines enclosing the curve.

An important deduction from the axiom. *If P, Q be any two points on a curve, then*

$$\lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ} = 1.$$

To prove it, we take Q so near P that the arc PQ is everywhere concave to the chord PQ.

Let PL be the tangent at P. Draw

$$QL \perp PL.$$

Let $\angle LPQ = \alpha$; we have

$$\text{chord } PQ < \text{arc } PQ < PL + LQ,$$

or

$$1 < \frac{\text{arc } PQ}{\text{chord } PQ} < \frac{PL}{PQ} + \frac{LQ}{PQ},$$

or

$$1 < \frac{\text{arc } PQ}{\text{chord } PQ} < \cos\alpha + \sin\alpha. \qquad \ldots(i)$$

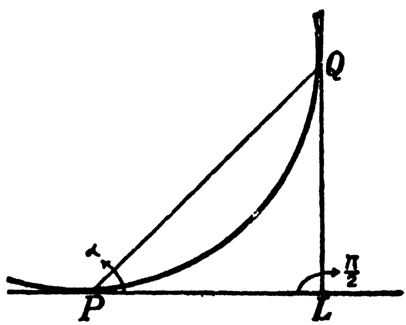


Fig. 90

Let $Q \to P$ so that chord PQ tends to the tangent PL as its limiting position and $\alpha \to 0$.

From (*i*), we get

$$\lim_{Q \to P} \frac{\text{arc PQ}}{\text{chord PQ}} = 1,$$

for

$$(\cos\alpha + \sin\alpha) \to 1 \text{ as } \alpha \to 0.$$

13·2. Length of arc as a function. Let $y = f(x)$ be the equation of a curve on which we take a fixed point A.

To any given value of x corresponds a value of y, *viz.*, $f(x)$; to this pair of number x and $f(x)$ corresponds a point P on the curve, and this point P has some arcual lengths 's' from A. Thus 's' is a function of x for the curve $y = f(x)$.

Similarly, we can see that 's' is a function of the parameter 't' for the curve

$$x = f(t),\ y = F(t), \qquad \textit{(Parametric Equation)}$$

and is a function of θ for the curve

$$r = f(\theta). \qquad \textit{(Polar Equation)}$$

13·3. Cartesian Equations. *To prove that*

$$\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]},$$

for the curve

$$y = f(x).$$

Let 's' denote arcual distance of any point $P(x, y)$ from some fixed point A on the curve.

We take another point $Q(x + \delta x,\ y + \delta y)$ on the curve near P.

Let an arc $AQ = s + \delta s$ so that

arc $PQ = \delta s$.

From the rt.-angled $\triangle PQN$, we have

$$PQ^2 = PN^2 + NQ^2$$
$$= (\delta x)^2 + (\delta y)^2,$$

or

$$\left(\frac{PQ}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2,$$

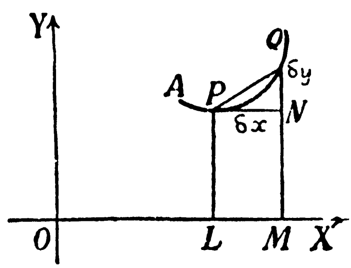


Fig. 91

or

$$\left[\frac{\text{chord } PQ}{\text{arc } PQ}\right]^2 \cdot \left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2.$$

Let $Q \to P$ so that in the limit

$$1.\ \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

or

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

We make a convention that for the curve $y=f(x)$, 's' is measured positively in the direction of, x, increasing so that, s, increases with x. Hence ds/dx is positive.

Thus we have

$$\frac{ds}{dx}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]},$$

taking positive sign before the radical.

Cor. 1. If $x=f(y)$ be the equation of the curve, then 's' is a function of y and, as above, it can be shown that

$$\frac{ds}{dy}=\sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}.$$

Cor. 2. From the right-angled $\triangle\, PNQ$, we have

$$\cos \angle NPQ=\frac{PN}{PQ}=\frac{\delta x}{\delta s}\cdot\frac{\text{arc } PQ}{\text{chord } PQ}.$$

Let $Q \to P$ so that $\angle NPQ \to \psi$, where ψ is the angle which the positive direction of the tangent at P makes with X-axis.

$$\therefore \qquad \cos \psi=\frac{dx}{ds}\cdot 1=\frac{dx}{ds}.$$

Hence

$$\cos \psi=\frac{dx}{ds}.$$

Similarly, we can show that

$$\sin \psi=\frac{dy}{ds}.$$

13·4. Parametric Equations. *To prove that*

$$\frac{ds}{dt}=\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]},$$

for the curve

$$x=f(t),\ y=F(t).$$

Let 's' denote the actual distance of any point $P(t)$ on the curve from a fixed point A of the same.

We take another point $Q(t+\delta t)$ on the curve. Let $x+\delta x$, $y+\delta y$ be its co-ordinates.

Let

$$\text{arc } PQ=\delta s.$$

From the $\triangle\, PNQ$, (Fig. 91, p. 275), we get

$$PQ^2=PN^2+NQ^2=(\delta x)^2+(\delta y)^2,$$

or

$$\left(\frac{PQ}{\delta t}\right)^2=\left(\frac{\delta x}{\delta t}\right)^2+\left(\frac{\delta y}{\delta t}\right)^2,$$

or

$$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2\left(\frac{\delta s}{\delta t}\right)^2=\left(\frac{\delta x}{\delta t}\right)^2+\left(\frac{\delta y}{\delta t}\right)^2.$$

Let $Q \to P$ so that in the limit

$$\left(\frac{ds}{dt}\right)^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2.$$

We measure 's' positively in the direction of, t, increasing so that ds/dt is positive. Thus

$$\frac{ds}{dt}=\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}.$$

13·5. Polar Equations. *To prove that*

$$\frac{ds}{d\theta}=\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]},$$

for the curve

$$r=f(\theta).$$

Let 's' denote the arcual length of any point $P(r, \theta)$ from some fixed point A on the curve.

We take another point $Q(r+\delta r, \theta+\delta\theta)$ on the curve near P.

Let arc $AQ=s+\delta s$ so that arc $PQ=\delta s$.

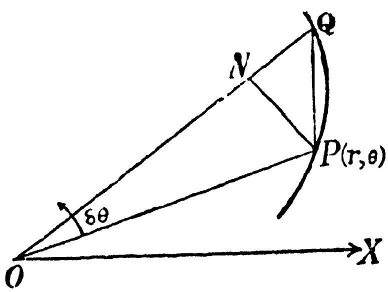


Fig. 92

Draw $PN \perp OQ$.

Now

$$PN/OP=\sin\delta\theta \text{ so that } PN=r\sin\delta\theta.$$

Again

$$ON/OP=\cos\delta\theta \text{ so that } ON=r\cos\delta\theta.$$

Hence
$$\begin{aligned}NQ&=OQ-ON\\&=r+\delta r-r\cos\delta\theta\\&=r(1-\cos\delta\theta)+\delta r\\&=2r\sin^2\tfrac{1}{2}\delta\theta+\delta r.\end{aligned}$$

From the $\triangle PNQ$, we get

$$PQ^2 = PN^2 + NQ^2$$

$$= r^2 \sin^2 \delta\theta + (2r \sin^2 \tfrac{1}{2}\delta\theta + \delta r)^2.$$

Dividing by $(\delta\theta)^2$, we get

$$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \left(\frac{\delta s}{\delta\theta}\right)^2 = r^2 \left(\frac{\sin \delta\theta}{\delta\theta}\right)^2$$

$$+ \left(r \,.\, \sin \tfrac{1}{2}\delta\theta.\ \frac{\sin \frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta} + \frac{\delta r}{\delta\theta}\right)^2.$$

Let $Q \to P$. Therefore

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 \,.\, 1 + \left(r.0 \times 1 + \frac{dr}{d\theta}\right)^2$$

$$= r^2 + \left(\frac{dr}{d\theta}\right)^2.$$

We measure 's' positively in the direction of θ increasing so that $ds/d\theta$ is positive. Thus

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}.$$

Cor. 1. If $\theta = f(r)$ be the equation of the curve, then 's' is a function of r and, as above, it can be shown that

$$\frac{ds}{dr} = \sqrt{\left[1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right]}.$$

Cor. 2. $\sin \angle PQN = \dfrac{PN}{PQ} = \dfrac{r \sin \delta\theta}{PQ}$

$$= r \frac{\sin \delta\theta}{\delta\theta} \cdot \frac{\delta\theta}{\delta s} \cdot \frac{\text{arc } PQ}{\text{chord } PQ}.$$

Let $Q \to P$ so that $\angle PQN \to \phi$ where, ϕ, is the angle between the positive directions of the tangent and the radius vector.

$$\therefore \quad \sin \phi = r.\, 1.\, \frac{d\theta}{ds} \,.\, 1$$

or

$$\sin \phi = r \frac{d\theta}{ds}.$$

Again, $\cos \angle PQN = \dfrac{NQ}{PQ}$

$$= \frac{2r \sin^2 \frac{1}{2}\delta\theta + \delta r}{PQ}$$

$$=\frac{2r\sin^2\frac{1}{2}\delta\theta+\delta r}{\delta\theta}\cdot\frac{\delta\theta}{\delta s}\cdot\frac{\text{arc } PQ}{\text{chord } PQ}$$

$$=\left(r\cdot\sin\tfrac{1}{2}\delta\theta\,\frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}+\frac{\delta r}{\delta\theta}\right)\frac{\delta\theta}{\delta s}\cdot\frac{\text{arc } PQ}{\text{chord } PQ}.$$

Let $Q\to P$

$$\therefore \qquad \cos\phi=\left(r.0.1+\frac{dr}{d\theta}\right)\frac{d\theta}{ds}.1$$

or

$$\cos\phi=\frac{dr}{ds}.$$

Exercises

1. Find ds/dx for the curves .—

(i) $y=c\cosh x/c$.
(ii) $y=a\log[a^2/(a^2-x^2)]$.
(iii) $3ay^2=x^2(a-x)$.
(iv) $x^3=ay^2$.
(v) $8a^2y^2=x^2(a^2-x^2)$.
(vi) $3ay^2=x(x-a)^2$.
(vii) $4ay+2a^2\log x=x^2$.

2. Find ds/dy for the curve

$$[4(x^2+y^2)-a^2]^3=27a^4y^2.$$

3. Show that $y\,ds/dy$ is constant for the curve

$$\frac{x+\sqrt{(a^2-y^2)}}{a}=\log\frac{a+\sqrt{(a^2-y^2)}}{y}.$$

4. Find $ds/d\theta$ for the following curves, θ being the parameter :—

(i) $x=a\cos\theta$, $y=b\sin\theta$ (Ellipse)
(ii) $x=a\cos^3\theta$, $y=\sin^3\theta$. (Astroid)
(iii) $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$. (Cycloid)
(iv) $x=ae^{\theta}\sin\theta$, $y=ae^{\theta}\cos\theta$.
(v) $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$.

5. Find $ds/d\theta$ for the following curves :—

(i) $r=a(1+\cos\theta)$. (Cardioide). (ii) $r^2=a^2\cos 2\theta$. (Lemniscate)
(iii) $r=ae^{\theta\cot\alpha}$. (Equiangular spiral)
(iv) $r=a\theta$.
(v) $r=a(\theta^2-1)$.
(vi) $r=a/(\theta^2-1)$.
(vii) $r^m=a^m\cos m\theta$. (Cosine spiral).
(viii) $r^m=a^m(\cos m\theta+\sin m\theta)$.

6. Show that for the hyperbolical spiral $r\theta=a$,

$$\frac{ds}{dr}=\frac{\sqrt{(r^2+a^2)}}{r}$$

7. Show that $r\,ds/dr$ is constant for the curve

$$\theta=\cos^{-1}\frac{r}{k}-\frac{\sqrt{(k^2-r^2)}}{r}.$$

8. Show that $\dfrac{1}{\sqrt{r}}\cdot\dfrac{ds}{dr}$ is constant for the curve :

$$\theta=2\left[\sqrt{\left(\frac{r-a}{a}\right)}-\tan^{-1}\sqrt{\left(\frac{r-a}{a}\right)}\right].$$

9. If p_1 and p_2 be the perpendiculars from the origin on the tangent and normal respectively at the point (x, y), and if $\tan\psi=dy/dx$, prove that

$$p_1=x\sin\psi-y\cos\psi, \text{ and } p_2=x\cos\psi+y\sin\psi\ ;$$

hence prove that

$$p_2=\frac{dp_1}{d\psi}.$$

10. Show that for any pedal curve $p=f(r)$,

$$\frac{ds}{dr}=\frac{r}{\sqrt{(r^2-p^2)}}.$$

CHAPTER XIV

CONCAVITY ; CONVEXITY

POINTS OF INFLEXION

14·1. Definitions. Consider a curve $y=f(x)$ and any point $P[c, f(c)]$ thereon. Draw the tangent at P.

We suppose that this tangent is not parallel Y-axis so that $f'(c)$ is some finite number.

Now there are three mutually exclusive possibilities to consider :—

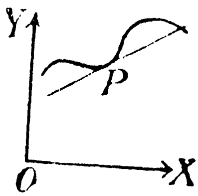


Fig. 93

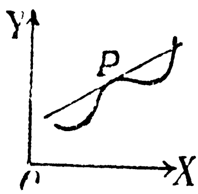


Fig. 94

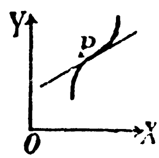


Fig. 95

(*i*) A portion of the curve on *both sides* of P, however small it may be, lies *above* the tangent P (*i.e.*, towards the positive direction of Y-axis).

In this case we say that the curve is *concave upwards* or *convex downwards at P*. (*See Fig 93*).

(*ii*) A portion of the curve on *both* sides of P, however small it may be, lies *below* the tangent at P (*i.e.*, towards the negative direction of Y-axis).

In this case we say that the curve is *concave downwards* or *convex upwards at P*. (*See Fig. 94*).

(*iii*) The two portions of the curve on the two sides of P lie on *different* sides of the tangent at P, *i e.*, the curve *crosses* the tangent at P. In this case we say that P is a point of *inflexion* on the curve. (*See Fig. 95*).

Thus, the curve in the adjoined figure 96 is concave upwards at every point between P_1 and P_2 and concave downwards at every point between P_2 and P_3.

P_2 and P_3 are the points of inflexion on the curve.

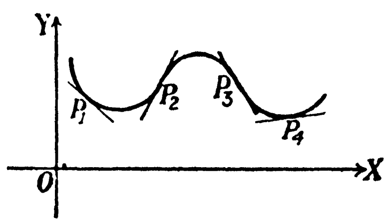


Fig. 96

From these figures, it is easy to see that the *curve changes the direction of its bending from concavity to convexity or vice versa as a point, moving along the curve, passing through a point of inflexion.* This property is also sometimes adopted as the definition of a point of inflexion.

Note. The property of concavity or convexity of a curve at any point is not an *inherent* property of the curve independent of the position of axes. Upward and downward directions, as also positive and negative directions of Y-axis, are fixed by convention and have no absolute meaning attached to them. But this is not the case for a point of inflexion. The point *where a curve crosses the tangent is an inflexional point* so that its existence in no way depends upon the choice of axes.

14·2. Criteria for concavity, convexity and inflexion. *To determine whether a curve $y=f(x)$ is concave upwards, concave downwards, or has a point of inflexion at $P\,[c, f(c)]$.*

We take a point $Q[c+h, f(c+h)]$ on the curve $y=f(x)$ lying near the point $P[c, f(c)]$.

The point Q lies to the right or left of the point P according as h is positive or negative.

Draw $QM \perp X$ axis meeting the tangent at P in R.

The equation of the tangent at P is

$$y-f(c)=f'(c)\,(x-c),$$

and therefore the ordinate MR of this tangent corresponding to the abscissa $c+h$ is

$$f(c)+hf'(c).$$

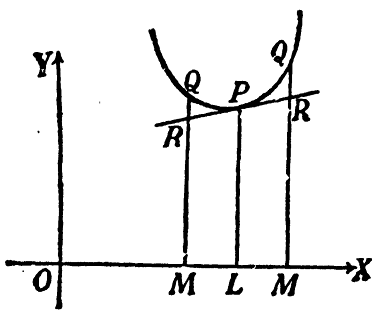


Fig. 97

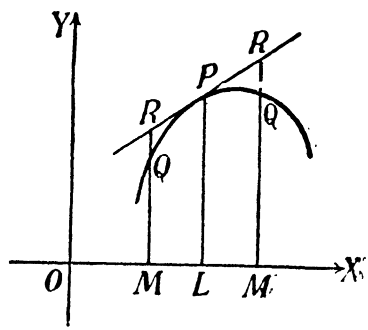


Fig. 98

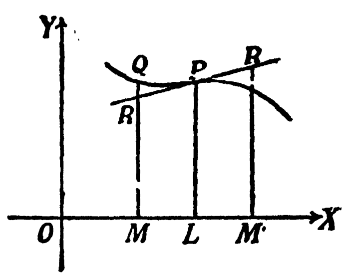


Fig. 99

Also, the ordinate MQ of the curve for the abscissa $c+h$ is $f(c+h)$.

$$\therefore \qquad RQ=MQ-MR=f(c+h)-f(c)-hf'(c).$$

For concavity upwards at P, (Fig. 97).

$MQ > MR$, *i.e.*, RQ, is positive when Q lies on *either* side of P.

For concavity downwards at P, (Fig. 98).

$MQ < MR$, *i.e.*, RQ is negative when Q lies on *either* side of P.

For inflexion at P, (Fig. 99).

RQ is positive when Q lies on one side of P and negative when Q lies on the other side of P.

Thus, we have to examine the behaviour of RQ, *i.e.*,

$$f(c+h)-f(c)-hf'(c)$$

for values of, h, which are sufficiently small numerically.

By Taylor's theorem, with remainder after two terms, we have

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2\,!}f''(c+\theta_2 h),$$

$$\therefore \qquad RQ=\frac{h^2}{2\,!}f''(c+\theta_2 h).$$

Case I. *Let* $\qquad f''(c)>0.$

As the value $f''(c)$ of $f''(x)$ is positive for $x=c$, there exists an interval around c for every point x of which the second derivative $f''(x)$ is positive. (§ 3·51, p. 54). Let $c+h$ be any point of this interval. Then $c+\theta_2 h$ is also a point of this interval and accordingly $f''(c+\theta_2 h)$ is positive.

Also

$$h^2/2\,!, \text{ is positive.}$$

$\therefore$

$$RQ>0,$$

for sufficiently small positive and negative values of h.

Hence the curve is concave upwards at P if $f''(c)>0$.

Case II. *Let* $\qquad f''(c)<0.$

In this case we may show, as above, that $RQ<0$ for sufficiently small positive and negative values of h.

Hence the curve is concave downwards at P if $f''(c)<0$.

Case III. *Let* $\qquad f''(c)=0$ *but* $f'''(c)\neq 0$.

By Taylor's theorem, with remainder after three terms, we have

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2!}f''(c)+\frac{h^3}{3!}f'''(c+\theta_3 h),$$

so that

$$RQ=\frac{h^3}{3!}f'''(c+\theta_3 h), \text{ for } f''(c)=0.$$

There exists an interval around c for every point x of which $f'''(x)$ has the sign of $f'''(c)$. Let $c+h$ be any point on this interval. Then $c+\theta_3 h$ is also a point of this interval and accordingly $f'''(c+\theta_3 h)$ has the sign of $f'''(c)$

But $h^3/3!$ changes sign with the change in the sign of h.

Thus RQ changes sign with the change in the sign h. *Hence the curve has inflexion at P if $f''(c)=0$ and $f'''(c)\neq 0$.*

Case IV. Generalisation.

Let

$f''(c)=f'''(0)=\ldots\ldots=f^{n-1}(c)=0$ *but* $f^n(c)\neq 0$.

By Taylor's theorem, with remainder after n terms, we have

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2!}f''(c)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(c)$$
$$+\frac{h^n}{n!}f^n(c+\theta_n h)$$

$$=f(c)+hf'(c)+\frac{h^n}{n!}f^n(c+\theta_n h),$$

so that

$$RQ=\frac{h^n}{n!}f^n(c+\theta_n h).$$

There exists an interval around, c, such that $f^n(c+\theta_n h)$ has the sign of $f^n(c)$ or every point $c+h$ of this interval.

Also $h^n/n!$ changes its sign or keeps the same sign while the sign of, h, changes, according as n is odd or even.

Thus RQ changes sign if n is odd and keeps the sign of $f^n(c)$, if n is even.

Hence if n is odd the curve has inflexion at P; if n is even, it is concave upwards or downwards according as $f^n(c)>0$, or <0.

14·3. Another Criterion for points of inflexion. We know that a curve has inflexion at P if it changes from concavity upwards to concavity downwards, or *vice versa*, as a point moving along the curve passes through P, *i.e.*, if there is a complete neighbourhood of P such that at every point on one side of P, lying in this neighbourhood, the curve is concave upwards and at every point on the other side of P

the curve is concave downwards. We thus see that *the curve, has inflexion at* $P[c, f(c)]$, *if* $f''(x)$ *changes* sign *as x passes through c.*

Note. We have already remarked that the position of the point of inflexion on a curve is independent of the choice of axes so that, in particular, the positions of x and y axes may be interchanged without affecting the positions of the points of inflexion on the curve. Thus the points of inflexion may also be determined by examining d^2x/dy^2 just as we examine d^2y/dx^2. For points where the tangent is parallel to X-axis, *i.e.*, where dy/dx is infinite, it becomes necessary to examine d^2x/d^2y instead of d^2y/dx^2.

14·4. Concavity and convexity with respect to a line. Let P be a given point on a curve and, l, a straight line not passing through P. Then the curve is said to be concave or convex at P with respect to l, according as a sufficiently small arc containing P and extending to both sides of it lies entirely within or without the acute angle formed by, l, and the tangent to the curve at P.

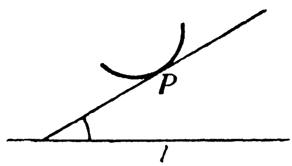


Fig. 100

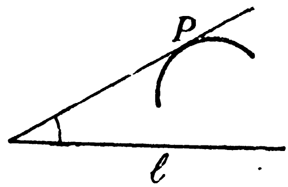


Fig. 101

Thus the curve is convex to l at P in Fig. 100 and concave in Fig. 101.

In the next section, we deduce a test for concavity and convexity with respect to the X-axis.

14·41. A test of concavity and convexity with respect to the X-axis.

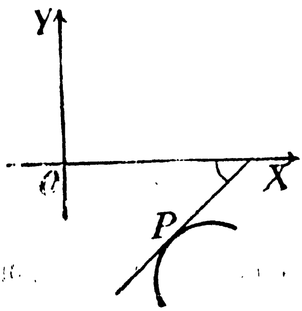


Fig. 102

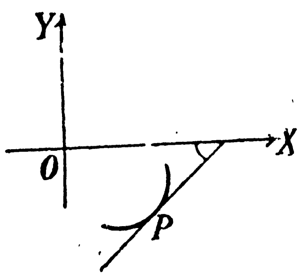


Fig. 103

From the examination of the figures 97, 98 and the figures 102, 103, we deduce the following :—

(*i*) A curve lying above the axis of x (so that the ordinate y is *positive*) *is convex or concave* with respect to the axis of x according as it is concave upwards or downwards, *i.e.*, according as d^2y/dx^2 is *positive or negative.*

(*ii*) A curve lying below the axis of x (so that the ordinate y is *negative) is convex or concave* with the respect to the axis of x according as it is concave downwards or upwards *i.e.*, according as d^2y/dx^2 *is negative or positive.*

Thus, in either case, we see that *a curve is convex or concave at P with respect to the axis of x according as*

$$y\frac{d^2y}{dx^2}$$

is positive or negative at P.

Examples

1. *Find if the curve*

$$y=\log x,$$

is concave or convex upwards throughout. (*P.U. 1932*)

Now

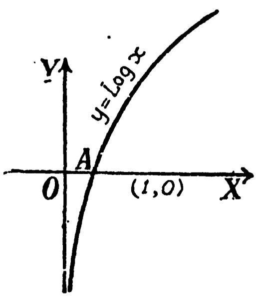


Fig. 104

$$\frac{dy}{dx}=\frac{1}{x}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{x^2},$$

which is always negative.

Hence the curve is concave downwards, *i.e.*, convex upwards throughout.

Note. We know that $y=\log x$ is negative or positive according as $0<x<1$ or $x>1$. Thus for $0<x<1$, yd^2y/dx^2 is positive and for $x>1$, yd^2y/dx^2 negative, so that the curve is convex *w.r.* to x-axis if $0<x<1$ and concave *w.r.* to x-axis if $x>1$.

2. *Show that $y=x^4$ is concave upwards at the origin.*

We have

$$\frac{dy}{dx}=4x^3,\ \frac{d^2y}{dx^2}=12x^2,\ \frac{d^3y}{dx^3}=24x,\ \frac{d^4y}{dx^4}=24,$$

so that

$$\frac{d^2y}{dx^2}=\frac{d^3y}{dx^3}=0 \text{ and } \frac{d^4y}{dx^4}\neq 0 \text{ but positive at } (0,\ 0).$$

Therefore the curve is concave upwards at the origin.

3. *Find the ranges of values of x for which*

$$y=x^4-6x^3+12x^2+5x+7$$

is concave upwards or downwards.

Also, determine its points of inflexion.

We have

$$\frac{dy}{dx}=4x^3-18x^2+24x+5,$$

$$\frac{d^2y}{dx^2}=12x^2-36x+24=12(x-1)(x-2).$$

Now,

$$\text{for } x<1, \quad \frac{d^2y}{dx^2}>0\ ; \text{ for } x=1, \frac{d^2y}{dx^2}=0\ ;$$

$$\text{for } 1<x<2, \frac{d^2y}{dx^2}<0\ ; \text{ for } x=2, \frac{d^2y}{dx^2}=0,$$

$$\text{for } x>2, \quad \frac{d^2y}{dx^2}>0.$$

Thus the curve is concave upwards in the intervals $[-\infty, 1)$, $(2, \infty]$ and is concave downwards in the interval $[1, 2]$.

It has inflexions for $x=1$ and $x=2$.

Therefore (1, 19) and (2, 33) are the two points of inflexion on the curve.

4. *Show that the curve* $(a^2+x^2)y=a^2x$ *has three points of inflexion.*

Here

$$\frac{dy}{dx}=\frac{a^2(a^2+x^2)-2a^2x^2}{(a^2+x^2)^2}=\frac{a^2(a^2-x^2)}{(a^2+x^2)^2},$$

$$\frac{d^2y}{dx^2}=a^2\frac{-2x(a^2+x^2)^2-4x(a^2+x^2)(a^2-x^2)}{(a^2+x^2)^4}$$

$$=a^2\frac{-2x(3a^2-x^2)}{(a^2+x^2)^3}=2a^2\frac{(x-\sqrt{3}a)x(x+\sqrt{3}a)}{(a^2+x^2)^3}.$$

$\therefore\ d^2y/dx^2=0$ for $x=\sqrt{3}a, 0, -\sqrt{3}a$.

It is easy to see that d^2y/dx^2 changes sign as, x, passes through each of these values. Hence the curve has inflexions at the corresponding points.

Thus

$$(\sqrt{3}a, \sqrt{3}a/4), (0, 0), (-\sqrt{3}a, -\sqrt{3}a/4)$$

are the three points of inflexion of the curve.

5. *Find the points of inflexion on the curve*

$$x=(\log y)^3.$$

Here, y, is the independent and, x, the dependent variable.

$$\frac{dx}{dy}=3(\log y)^2. \ \frac{1}{y},$$

$$\frac{d^2x}{dy^2}=6\log y\cdot\frac{1}{y^2}-\frac{3}{y^2}(\log y)^2$$

$$=\frac{3\log y}{y^2}(2-\log y).$$

$\therefore$ $\quad d^2x/dy^2=0$ if $\log y=0$ or $\log y=2$,

i.e., if

$$y=e^{\circ}=1 \text{ or } e^2.$$

Thus we expect points of inflexion on the curve for $y=1$ and e^2. Again

$$\frac{d^3x}{dy^3}=\frac{6}{y^3}-\frac{12\log y}{y^3}-\frac{6\log y}{y^3}+\frac{6}{y^3}(\log y)^2.$$

Therefore

$$\left(\frac{d^3x}{dy^3}\right)_{y=1}=6\neq 0 \text{ and} \left(\frac{d^3x}{dy^3}\right)_{y=e^2}=-\frac{6}{e^6}\neq 0.$$

Hence $y=1$ and e^2 give points of inflexion, so that (0, 1) and (8, e^2) are the two points of inflexion.

Exercises

1. Show that $y=e^x$ is everywhere concave upwards.

2. Examine the curve $y=\sin x$ for concavity and convexity in the interval $(0, 2\pi)$.

3. Find the ranges of values of x in which the curve

$$y=3x^5-40x^3+3x-20$$

is concave upwards or downwards. Also, find the points of inflexion.

4. Find the ranges of the values of x in which the curve

$$y=(x^2+4x+5)e^{-x}$$

is concave upwards or downwards. Also find its points of inflexion.

5. Find the intervals in which the curve

$$y=(\cos x+\sin x)e^x$$

is concave upwards or downwards; x varying in the interval $(0, 2\pi)$.

6. Find the points of inflexion the curves

(*i*) $y=ax^3+bx^2+cx+d$. (*ii*) $y=(x^3-x)/(3x^2+1)$.

(*iii*) $x=3y^4-4y^3+5$. (*iv*) $x=(y-1)(y-2)(y-3)$.

(*v*) $y=\dfrac{a^2(a-x)}{a^2+x^2}$. (*vi*) $y=\dfrac{x^3}{a^2+x^2}$.

(*vii*) $y=\dfrac{x}{x^2+2x+2}$. (*viii*) $y=be^{-(x/a)^2}$.

(*ix*) $xy=a^2\log(y/a)$. (*x*) $y=x^2\log(x^2/e^3)$.

(*xi*) $y^2=x(x+1)^2$. (*D.U. Hons. 1947*)

(*xii*) $a^2y^2=x^2(a^2-x^2)$. (*D.U. Hons. 1955*)

Also, obtain the equations of the inflexional tangents to the curves (*ii*), (*iii*) and (*xi*).

7. Show that in the curve

$$y=(1/h\sqrt{2\pi})e^{-x^2/2h^2},$$

the abscissae of the points of inflexion are $\pm h$.

8. Show that the line joining the two points of inflexion of the curve

$$y^2(x-a)=x^2(x+a)$$

subtends an angle $\pi/3$ at the origin.

9. Find the values of, x, for which the curve

$$54y=(x+5)^2(x^3-10),$$

has an inflexion and draw a rough sketch of the curve for $-6<x<3$, making the inflexions. (*M.T.*)

10. Show that the curve

$$ay^2=x(x-a)(x-b)$$

has two and only two points of inflexion.

11. Find the points of inflexion on the curves

(*i*) $x=a(2\theta-\sin\theta)$, $y=a(2-\cos\theta)$.

(*ii*) $x=a\tan t$, $y=a\sin t\cos t$.

(*iii*) $x=2a\cot\theta$, $y=2a\sin^2\theta$.

12. Show that the abscissae of the points of inflexion on the curve $y^2=f(x)$ satisfy the equation

$$[f'(x)]^2=2f(x)f''(x).$$ (*P.U.*)

13. Show that the points of inflexion of the curve

$$y^2=(x-a)^2(x-b)$$

lie on the line

$$3x+a=4b.$$ (*Lucknow*)

CHAPTER XV

CURVATURE

EVOLUTES

15·1. Introduction. Definition of Curvature. In our everyday language, we make statements which involve the comparison of bending or curvature of a road at two of its points. For instance, at times, we say, "The bend of the road is sharper at this place than at that." Here we depend upon intuitive means of comparing the curvature at two points provided the difference is fairly marked. But we are far from intuitively assigning any definite numerical measure to the curvature at any given point of a curve. In order to make '*Curvature*' a subject of Mathematical investigation, we have to assign, by some suitable definition, a numerical measure to it and this has to be done in a way which may be in agreement with our intuitive notion of curvature. This we proceed to do.

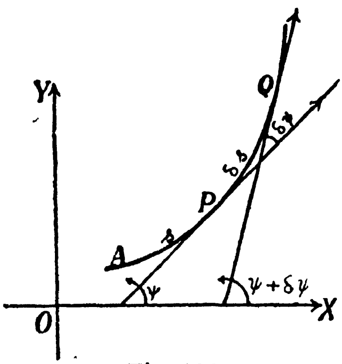


Fig. 105

We take a point Q on the curve lying near P.

Let A be any fixed point on the curve. Let

$$\text{arc } AP = s,$$
$$\text{arc } AQ = s + \delta s,$$

so that

$$\text{arc } PQ = \delta s,$$

Let ψ, $\psi+\delta\psi$, be the angles which the positive directions of the tangents at P and Q make with some fixed line.

The symbol, $\delta\psi$ denotes the angle through which the tangent turns as a point moves along the curve from P to Q through a distance δs. According to our intuitive feeling, $\delta\psi$ *will be large or small, as compared with* δs, *depending on the degree of sharpness of the bend.*

This suggests the following definitions :—

(*i*) *the* **total bending or total curvature** *of the arc PQ is defined to be the angle* $\delta\psi$;

(*ii*) *the* **average curvature** *of the arc PQ is defined to be the ratio* $\delta\psi/\delta s$;

and (*iii*) *the* **curvature** *of the curve at P is defined to be*

$$\lim_{Q\to P} \frac{\delta\psi}{\delta s}, \text{ i.e., } \frac{d\psi}{ds}.$$

Thus, by def., $d\psi/ds$ *is the curvature of the curve at any point* P

15·2. Curvature of a circle. *To prove that the curvature of a circle is constant.*

Intuitively, we feel that the curvature of a circle is uniform throughout its circumference and that the larger the radius of the circle, the smaller will be its curvature. It will now be shown that these intuitive conclusions are consequences of our formal definition of curvature.

Consider any circle with radius, r, and centre O.

Let P, Q be any points on the circle and let arc

$$PQ=\delta s.$$

Also let L be the point where the tangents PT, QT' at P and Q meet.

We have

$$\angle POQ=\angle TLT'=\delta\psi,$$

From Elementary Trigonometry

$$\frac{\text{arc } PQ}{OP}=\angle POQ, \quad i.e., \frac{\delta s}{r}=\delta\psi.$$

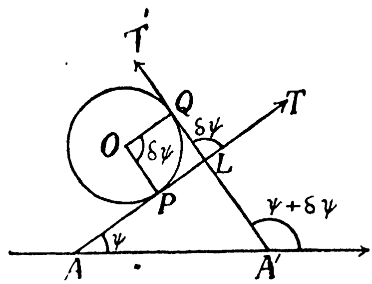


Fig. 106

$$\frac{\delta\psi}{\delta s}=\frac{1}{r}.$$

Let $Q \to P$ so that in the limit, we have

$$\therefore \qquad \frac{d\psi}{ds}=\frac{1}{r}.$$

Thus the curvature at *any* point of a circle is the reciprocal of its radius and is, thus, a constant.

Also, it is clear that the curvature, $1/r$, decreases as the radius r increases.

15·3 Radius of curvature. The *reciprocal* of the curvature of a curve at any point, in case it is $\neq 0$, is called its *radius o curvature* at the point and is generally denoted by, ρ, so that we have

$$\rho=\frac{ds}{d\psi}.$$

Thus the radius of curvature of a circle at any point is equal to its radius.

15·31. The expression $ds/d\psi$ for radius of curvature is suitable only for those curves whose equations are given by means of a relation between s and ψ. We must, therefore, transform it so that it may be applicable to other types of equations such as Cartesian, Polar, Pedal, Tangential Polar. etc.

It may be remarked that in the case of Cartesian and Polar equations, the fixed line will be taken as X-axis and initial line respectively. For the curve $y=f(x)$, the positive direction of the tangent is the one in which, x, increases, and for $r=f(\theta)$ the positive direction of the tangent is the one in which, θ, increases.

Ex. Find the radius of curvature at any point of the following :—

(*i*) $s=c \tan \psi$ (Catenary). (*ii*) $s=4a \sin \psi$ (Cycloid).

(*iii*) $s=4a \sin \frac{1}{3}\psi$ (Cardioide). (*iv*) $s=c \log \sec \psi$ (Tractrix).

(*v*) $s=a \log (\tan \psi+\sec \psi)+a \tan \psi \sec \psi$ (Parabola).

15·4. Radius of curvature for Cartesian Curves.

15·41. Explicit equations : y=f(x).

Now,

$$\tan \psi=\frac{dy}{dx}.$$

Differentiating, *w.r.* to s, we get

$$\sec^2\psi.\frac{d\psi}{ds}=\frac{d^2y}{dx^2}\cdot\frac{dx}{ds}.$$

$$\frac{ds}{d\psi}=\frac{(1+\tan^2\psi)\dfrac{ds}{dx}}{\dfrac{d^2y}{dx^2}}.$$

But, we have

$$\frac{ds}{dx}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}.$$

(13·3, p. 275)

where the positive sign is to be taken before the radical.

Hence

$$\rho=\frac{ds}{d\psi}=\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}=\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} \qquad \ldots(A)$$

Cor. The radius of curvature, ρ, is positive or negative according as d^2y/dx^2 is positive, or negative *i e.*, according as the curve is concave upwards or downwards. Also, the equation (A) shows that curvature is zero at a point of inflexion.

Note. Since the value of ρ is independent of the choice of X-axis and Y-axis, interchanging x and y we see that, ρ, is also given by

$$\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}\Big/\frac{d^2x}{dy^2}.$$

This formula is specially useful when the tangent is perpendicular to X-axis in which case $dx/dy=0$.

15·42. Implicit equations : f(x, y)=0.

At points where $\partial f/\partial y=f_y\neq 0$, we have, (§10·94, p. 213)

$$\frac{dy}{dx}=-\frac{f_x}{f_y},$$

$$\frac{d^2y}{dx^2} = -\frac{f_x^2(f_y)^2 - 2f_{xy}f_xf_y + f_y^2(f_x)^2}{(f_y)^3}$$

Substituting these values of dy/dx and d^2y/dx^2 in the formula A) above, we get

$$\rho = \pm \frac{[(f_x)^2 + (f_y)^2]^{\frac{3}{2}}}{f_x^2(f_y)^2 - 2f_{xy}f_xf_y + f_y^2(f_x)^2} \quad \ldots(B)$$

where the sign is $+$ or $-$ according as f_y is negative or positive.

Note. The form of the formula (B) shows that the expression for ρ will retain the same form at points where $f_y=0$ but $f_x \neq 0$. The exceptional case which arises for the points where f_x and f_y become simultaneously 0 will be considered in chapter XVII.

15·43. Parametric equations : $x=f(t)$, $y=F(t)$.

At points where $f'(t) \neq 0$, we have

$$\frac{dy}{dx} = \frac{F'(t)}{f'(t)}, \quad \ldots(i)$$

$$\frac{d^2y}{dx^2} = \frac{f'(t)F''(t) - F'(t)f''(t)}{[f'(t)]^2} \cdot \frac{1}{f'(t)}.$$

Substituting these values of dy/dx and d^2y/dx^2 in the formula A) above, we get

$$\rho = \pm \frac{[f'^2(t) + F'^2(t)]^{\frac{3}{2}}}{f'(t)F''(t) - F'(t)f''(t)}, \quad \ldots(C)$$

where the sign is $+$ or $-$ according as $f'(t)$ is positive or negative.

Note. The formula (C) shows that the expression for, ρ, will retain the same form for the points where $f'(t)=0$ but $F'(t) \neq 0$.

15·44. Newtonian Method. *If a curve passes through the origin and the axis of x is tangent at the origin, then*

$$\lim \frac{x^2}{2y}, \text{ as } x \to 0$$

gives the radius of curvature at the origin.

Here we obtain the values of y_1 and y_2 at the origin.

$$y_1(0) = \left(\frac{dy}{dx}\right)_{(0,\,0)} = 0.$$

Now, $x^2/2y$, assumes the indeterminate form 0/0 as $x \to 0$.

$$\lim_{x \to 0} \frac{x^2}{2y} = \lim_{x \to 0} \frac{2x}{2y_1} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{1}{y_2} = \frac{1}{y_2(0)}.$$

Also, from the formula (A) of § 15·41, we have at the origin

$$\rho=\frac{(1+0)^{\frac{3}{2}}}{y_2(0)}=\frac{1}{y_2(0)}.$$

Thus at the origin where x-axis is a tangent,

$$\rho=\lim_{x \to 0}\left(\frac{x^2}{2y}\right).$$

It can similarly be shown *that at the origin where Y-axis is a tangent,*

$$\rho=\lim_{x \to 0}\left(\frac{y^2}{2x}\right).$$

These two formulae are due to Newton.

15·45. Generalised Newtonian Formula. If a curve passes through the origin and X-axis is the tangent at the origin, we have

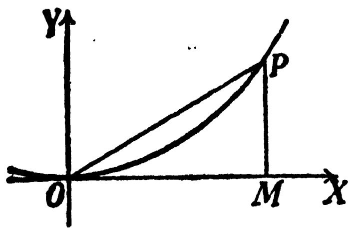


Fig. 107

$$\lim\frac{x^2+y^2}{2y}=\lim\left(\frac{x^2}{2y}+\frac{y}{2}\right)$$

$$=\lim\frac{x^2}{2y}$$

$=\rho$, at the orgin.

Here, $x^2+y^2=OP^2$, is the square of the distance of any point $P(x, y)$ on the curve from the origin O and, y, is the distance of the point P from the tangent X-axis at O.

Interpreted in general terms this conclusion can be stated as follows : (*see* Fig. 108.)

If OT be the tangent at any given point O of a curve, and PM, the length of the perpendicular drawn from any point P to the tangent at O, then the *radius of curvature at O*

$$=\lim\frac{OP^2}{2PM},$$

when the point P tends to O as its limit.

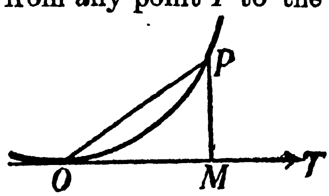


Fig. 108

Note. It will be proved later on in § 17·31, on p. 332 that the tangent at the origin lying on a curve is obtained by equating to zero the lowest degree terms in its equation.

Examples

1. *In the cycloid*

$$x=a(t+\sin t),\ y=a(1-\cos t),$$

prove that

$$\rho = 4a \cos \frac{t}{2}. \qquad (P.U.\ 1944)$$

We have

$$\frac{dx}{dt} = a(1+\cos t),\ \frac{dy}{dt} = a \sin t.$$

$$\therefore \quad \frac{dy}{dx} = \frac{a \sin t}{a(1+\cos t)} = \frac{2 \sin t/2 \cos t/2}{2 \cos^2 t/2} = \tan \frac{t}{2}.$$

$$\frac{d^2y}{dx^4} = \frac{1}{2} \sec^2 \frac{t}{2} \cdot \frac{dt}{dx}$$

$$= \frac{1}{2} \sec^2 \frac{t}{2} \cdot \frac{1}{2a \cos^2 \frac{t}{2}} = \frac{1}{4a} \cdot \frac{1}{\cos^4 \frac{t}{2}}$$

$$\therefore \quad \rho = \frac{[1+(dy/dx)^2]^{\frac{3}{2}}}{d^2y/dx^2} = \frac{(1+\tan^2 \frac{1}{2}t)^{\frac{3}{2}}}{\frac{1}{4a} \cdot \frac{1}{\cos^4 \frac{t}{2}}} = 4a \cos \frac{t}{2}.$$

2. *Show that the curvature of the point* $(3a/2, 3a/2)$ *on the Folium* $x^3+y^3=3axy$ *is* $-8\sqrt{2}/3a$.

Differentiating, we get

$$3x^2+3y^2 \frac{dy}{dx} = 3ay+3ax \frac{dy}{dx},$$

or

$$x^2+y^2 \frac{dy}{dx} = ay+ax \frac{dy}{dx}. \qquad \ldots (i)$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{(\frac{3}{2}a,\ \frac{3}{2}a)} = -1.$$

Again, differentiating (i), we get

$$2x+2y \left[\frac{dy}{dx}\right]^2 + y^2 \frac{d^2y}{dx^2} = a \frac{dy}{dx} + a \frac{dy}{dx} + ax \frac{d^2y}{dx^2}.$$

Substituting $3a/2$, $3a/2$, -1 for x, y, dy/dx respectively, we get

$$\left[\frac{d^2y}{dx^2}\right]_{(\frac{3}{2}a,\ \frac{3}{2}a)} = -\frac{32}{3a}.$$

Hence the curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$= \left[\frac{d^2y/dx^2}{[1+(dy/dx)^2]^{\frac{3}{2}}}\right] = -\frac{8\sqrt{2}}{3a}.$$

3. *Find the radius of curvature at the origin of the curve*

$$x^3-2x^2y+3xy^2-4y^3+5x^2-6xy+7y^2-8y=0.$$

It is easy to see that X-axis is the tangent at the origin.

[Refer note § 15·45, p. 294]

Dividing by y, we get

$$x \cdot \frac{x^2}{y} - 2x^2+3xy-4y^2+5\frac{x^2}{y}-6x+7y-8=0.$$

Let $x \to 0$ so that $\lim\limits_{x \to 0} (x^2/y)=2\rho$.

$$0.2\rho+5.2\rho-8=0,$$

or $$\rho=4/5.$$

Exercises

1. Find the radius of curvature at any point on the curves :—

(i) $y=c \cosh (x/c)$ (*Catenary*).

(ii) $x=a(\cos t+t \sin t)$, $y=a(\sin t-t \cos t)$.

(iii) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. (Astroid) (*D.U. 1953*)

(iv) $x=(a \cos t)/t$, $y=(a \sin t)/t$.

2. Find the radius of curvature at the origin for

(i) $x^4-y^4+x^3-y^3+x^2-y^2+y=0$.

(ii) $x^3y-xy^3+2x^2y-2xy^2+2y^2-3x^2+3xy-4x=0$.

(iii) $2x^4+3y^4+4x^2y+xy-y^2+2x=0$. (*P.U. Supp. 1939*)

3. Show that the radius of curvature of any point of the astroid

$$x=a \cos^3 \theta,\ y=a \sin^3 \theta$$

is equal to three times the length of the perpendicular from the origin to the tangent. (*Andhra 1951*)

4. Show that for the curve

$$x=a \cos \theta\,(1+\sin \theta),\ y=a \sin \theta\,(1+\cos \theta),$$

the radius of curvature is, a, at the point for which the value of the parameter is $-\pi/4$.

5. Show that the radius of the curvature at any point of the curve

$$x=t-c \sinh \frac{t}{c} \cosh \frac{t}{c},\ y=2c \cosh \frac{t}{c}.$$

is, $-2c \cosh^2 (t/c) \sinh (t/c)$, where t is the parameter.

6. Show that the radius of curvature at a point of the curve

$$x=ae^{\theta} (\sin \theta-\cos \theta),\ y=ae^{\theta} (\sin \theta+\cos \theta)$$

is twice the distance of the tangent at the point from the origin.

7. Prove that the radius of curvature at the point

$(-2a, 2a)$ on the curve $x^2y=a(x^2+y^2)$ is, $-2a$.

8. Show that the radius of curvature of the Lemniscate

$$(x^2+y^2)^2=a^2(x^2-y^2)$$

at the point where the tangent is parallel to x-axis is $\sqrt{2}a/3$.

9. Prove that the radius of curvature at the point $(2a, 0)$ on the curve

$$(x^2+y^2)^2-2ax(x^2+y^2)-a^3y=0,$$

is

$$(65)^{\frac{3}{2}}\, a/(536).$$

10. Find the radius of curvature for $\sqrt{(x/a)}-\sqrt{(y/b)}=1$ at the points where it touches the co-ordinate axes.

11. Show that the ratio of the radii of curvature at points on the two curves

$$xy=a^2,\ x^3=3a^2y,$$

which have the same abscissa varies as the square root of the ratio of the ordinates.

12. Show that the radius of curvature at each point of the curve

$$x=a(\cos t+\log\tan \tfrac{1}{2}t),\ y=a\sin t,$$

is inversely proportional to the length of the normal intercepted between the point on the curve and the X-axis.

13. Show that $3\sqrt{3}/2$ is the least value of $|\rho|$ for $y=\log x$.

14. Find the point on the curve $y=e^x$ at which the curvature is maximum, and show that the tangent at this point forms with the axes of co-ordinates a triangle whose sides are in the ratio $1 : \sqrt{2} : \sqrt{3}$. *(Rajputana 1951)*

15. Show that the radius of curvature of the curve given by

$$x^2y=a(x^2+a^2/\sqrt{5})$$

is least for the point $x=a$ and its value there is $9a/10$. *(B.U.)*

16. Prove that for the ellipse $x^2/a^2+y^2/b^2=1$,

$$\rho=a^2b^2/p^3\ ;$$

p, being the perpendicular from the centre upon the tangent at any point (x, y). *(P.U. 1944)*

17. Prove that for the ellipse

$$x^2/a^2+y^2/b^2=1,\ \rho=CD^3/ab$$

where CD is the semi-conjugate diameter to CP. *(Madras 1953)*

18. Employ generalised Newtonian Formula to show that the radius of curvature at any point of the ellipse $x^2/a^2+y^2/b^2=1$ is equal to

$$\frac{(\text{normal})^3}{(\text{semi-latus rectum})^2}.$$

19. The tangents at two points P, Q on the cycloid

$$x=a(\theta-\sin\theta),\ y=a(1-\cos\theta)$$

are at right angles ; show that if ρ_1, ρ_2 be the radii of curvatures at these points, then

$$\rho_1^2+\rho_2^2=16a^2.$$

20. Find the points on the parabola $y^2=8x$ at which the radius of curvature is $7\frac{13}{16}$.

21. (*a*) Prove that if, ρ be the radius of curvature at any point P on the parabola $y^2=4ax$ and S be its focus, then ρ^2 varies as $(SP)^3$. *(P.U. 1952)*

(*b*) ρ_1, ρ_2 are the radii of curvature at the extremities of a focal chord of a parabola whose semi-latus rectum is l ; prove that

$$(\rho_1)^{-\frac{2}{3}}+(\rho_2)^{-\frac{2}{3}}=(l)^{-\frac{2}{3}}$$

22. Show that in the curve

$$x=\tfrac{3}{2}a(\sinh u\cosh u+u),\ y=a\cosh^3 u,$$

if the normal at $P(x, y)$ meets the axis of x in G, the radius of curvature at P is equal to $3\,PG$. *(B.U. 1954)*

23. If x, y are given as functions of the arc s, show that

$$\rho=-\frac{dy/ds}{d^2x/ds^2}=\frac{dx/ds}{d^2y/ds^2};$$

also, show that

$$\frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2.$$

Find ρ for the catenary

$$x=c\log[s+\sqrt{(c^2+s^2)}],\ y=\sqrt{(c^2+s^2)}.$$

24. Show that for the curve $s=f(x)$,

$$\rho=\left[\left(\frac{ds}{dx}\right)^2\sqrt{\left\{\left(\frac{ds}{dx}\right)^2-1\right\}}\right]\Big/\frac{d^2s}{dx^2}.$$

25. Prove that for the curve $s=ce^{x/c}$,

$$c\rho=s(s^2-c^2)^{\frac{1}{2}}.$$

Find ρ for the curve $s^2=8ay$. *(Patna, 1952)*

15·46. Radius of curvature for polar curves : $r=f(\theta)$.

Here we are to express $ds/d\psi$ in terms of r and its derivatives with respect to θ.

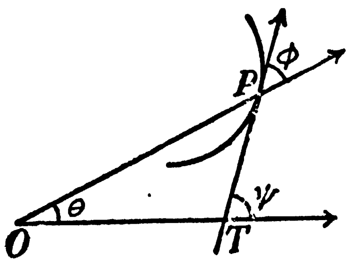


Fig. 109

From the figure, we see that

$$\psi=\theta+\phi.$$

$$\therefore\quad \frac{d\psi}{ds}=\frac{d\theta}{ds}+\frac{d\phi}{ds}$$

$$=\frac{d\theta}{ds}+\frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds}$$

$$=\frac{d\theta}{ds}\left(1+\frac{d\phi}{d\theta}\right) \qquad ...(1)$$

Now,
$$\tan\phi=\frac{r}{\dfrac{dr}{d\theta}}.$$

$$\therefore\quad \sec^2\phi\,\frac{d\phi}{d\theta}=\frac{\dfrac{dr}{d\theta}\cdot\dfrac{dr}{d\theta}-r\cdot\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2},$$

or
$$\frac{d\phi}{d\theta}=\frac{\left[\dfrac{dr}{d\theta}\right]^2-r\cdot\dfrac{d^2r}{d\theta^2}}{\left[1+\tan^2\phi\right]\left[\dfrac{dr}{d\theta}\right]^2}=\frac{\left[\dfrac{dr}{d\theta}\right]^2-r\cdot\dfrac{d^2r}{d\theta^2}}{\left[\dfrac{dr}{d\theta}\right]^2+r^2}. \qquad ...(2)$$

Also

$$\frac{ds}{d\theta}=\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}, \qquad (\S\ 13{\cdot}5,\ \text{p. } 277) \qquad ...(3)$$

where positive sign is to be taken before the radical.

From (1), (2) and (3), we obtain]

$$\frac{d\psi}{ds}=\frac{1}{\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}}\cdot\left\{1+\frac{\left(\frac{dr}{d\theta}\right)-r\frac{d^2r}{d\theta^2}}{r^2+\left(\frac{dr}{d\theta}\right)^2}\right\}$$

$$=\frac{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}$$

Therefore

$$\rho=\frac{(\mathbf{r}^2+\mathbf{r}_1{}^2)^{\frac{3}{2}}}{\mathbf{r}^2+2\mathbf{r}_1{}^2-\mathbf{r}\mathbf{r}_2},$$

where

$$r_1=dr/d\theta \text{ and } r_2=d^2r/d\theta^2.$$

Cor. Since curvature is zero at a point of inflexion, therefore at a point of inflexion on a polar curve,

$$r^2+2r_1{}^2-rr_2=0.$$

15·47. Radius of curvature for pedal curves.

We know that $\qquad \psi=\theta+\phi.$

$$\therefore \qquad \frac{d\psi}{ds}=\frac{d\theta}{ds}+\frac{d\phi}{ds}. \qquad \text{...}(i)$$

Differentiating *w.r.* to r, the relation

$$p=r\sin\phi,$$

we have

$$\frac{dp}{dr}=\sin\phi+r\cos\phi\frac{d\phi}{dr}$$

$$=\sin\phi+r\cos\phi\frac{d\phi}{ds}\cdot\frac{ds}{dr}.$$

Now $\sin\phi=r\dfrac{d\theta}{ds}$, $\cos\phi=\dfrac{dr}{ds}$. (§ 13·5, Cor. p. 278)

$$\therefore \qquad \frac{dp}{dr}=r\frac{d\theta}{ds}+r\frac{dr}{ds}\frac{ds}{dr}\frac{d\phi}{ds}$$

$$=r\left(\frac{d\theta}{ds}+\frac{d\phi}{ds}\right)=\frac{r}{\rho}, \text{ from } (i),$$

or $$\rho=r\frac{dr}{dp}.$$

Note. The relation $\psi=\theta+\phi$ is true whatever be the position of the point on a curve and the tangent at it relative to the initial line. We can satisfy ourselves on this point, if we examine a few different curves, keeping in view the following conventions :—

θ is the angle through which the positive direction of the initial line has to rotate to coincide with the positive direction of the radius vector ; ϕ is the angle through which the positive direction of the radius vector has to rotate to coincide with the positive direction of the tangent which is that of, θ, increasing; ψ is the angle through which the positive direction of the initial line has to rotate to coincide with the positive direction of the tangent.

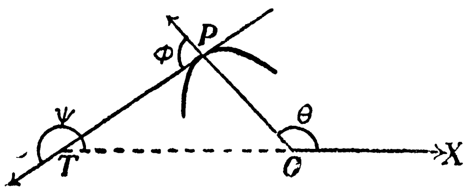


Fig. 110

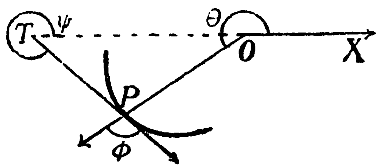


Fig. 111

For Fig. 110.

$$\angle POT=\pi-\theta,\ \angle OTP=\psi-\pi,\ \angle OPT=\pi-\phi.$$

$\therefore$ $$\pi-\theta+\psi-\pi+\pi-\phi=\pi,$$

or $$\psi=\theta+\phi.$$

For Fig. 111.

$$\angle POT=\theta-\pi,\ \angle OTP=2\pi-\psi,\ \angle OPT=\phi.$$

$\therefore$ $$\theta-\pi+2\pi-\psi+\phi=\pi.$$

or $$\psi=\theta+\phi.$$

15·48. Radius of curvature for tangential polar equations. A relation between p and ψ, holding for every point of a curve, is called *Tangential polar equation.*

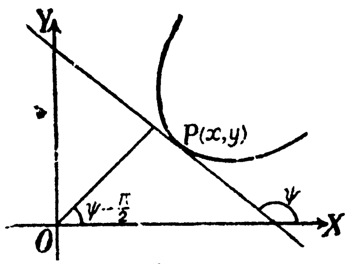


Fig. 112

The perpendicular drawn to the tangent at any point (x, y) from the origin makes angle, $\psi-\pi/2$, with x-axis.

Also p, is the length of this perpendicular.

Therefore the equation of the tangent at P is

$$p=X\cos(\psi-\tfrac{1}{2}\pi)+Y\sin(\psi-\tfrac{1}{2}\pi),$$

o

$$p=X\sin\psi-Y\cos\psi\ ;$$

X, Y being the current co-ordinates of any point on this tangent. Since the point $P(x, y)$ lies on this tangent, we have

$$p = x \sin \psi - y \cos \psi \qquad \ldots(i)$$

which is a relation between ψ, p, x, y for any point on the curve.

Differentiating (i), w r. to ψ, we get

$$\frac{dp}{d\psi} = x \cos \psi + y \sin \psi + \frac{dx}{d\psi} \sin\psi - \frac{dy}{d\psi} \cos \psi.$$

Now $\dfrac{dx}{d\psi} = \dfrac{dx}{ds} \cdot \dfrac{ds}{d\psi} = \rho \cos \psi$, $\dfrac{dy}{d\psi} = \dfrac{dy}{ds} \cdot \dfrac{ds}{d\psi} = \rho \sin \psi$

$$\therefore \quad \frac{dp}{d\psi} = x \cos \psi + y \sin \psi + \rho \cos \psi \sin \psi - \rho \sin \psi \cos \psi$$

$$= x \cos \psi + y \sin \psi. \qquad \ldots(ii)$$

Differentiating (ii) w.r. to ψ, we get

$$\frac{d^2p}{d\psi^2} = -x \sin \psi + y \cos \psi + \frac{dx}{d\psi} \cos \psi + \frac{dy}{d\psi} \sin \psi$$

$$= -x \sin \psi + y \cos \psi + \rho \cos^2\psi + \rho \sin^2\psi$$

$$= -x \sin \psi + y \cos \psi + \rho.$$

$$\therefore \quad \rho = x \sin \psi - y \cos \psi + \frac{d^2p}{d\psi^2},$$

or

$$\rho = p + \frac{d^2p}{d\psi^2}.$$

Example

For the curve $r^m = a^m \cos m\theta$, prove that

$$\rho = \frac{a^m}{(m+1)r^{m-1}}. \qquad (P.U.)$$

First Method. By logarithmic differentiation, we obtain

$$\frac{m}{r} \cdot \frac{dr}{d\theta} = -m \frac{\sin m\theta}{\cos m\theta}.$$

$$\therefore \quad r_1 = \frac{dr}{d\theta} = -r \tan m\theta.$$

$$\therefore \quad r_2 = \frac{d^2r}{d\theta^2}$$

$$= -rm \sec^2 m\theta - \tan m\theta. \frac{dr}{d\theta}$$

$$= -rm \sec^2 m\theta + r \tan^2 m\theta.$$

Hence $$\rho=\frac{(r^2+r^2\tan^2 m\theta)^{\frac{3}{2}}}{r^2+2r^2\tan^2 m\theta+r^2 m\sec^2 m\theta-r^2\tan^2 m\theta}$$

(See § 15·46, p. 298)

$$=\frac{r^3\sec^3 m\theta}{r^2\sec^2 m\theta+mr^2\sec^2 m\theta}$$

$$=\frac{r}{(m+1)(\cos m\theta)}=\frac{1}{m+1}\cdot\frac{a^m}{r^{m-1}}.$$

Second Method. Its pedal equation, as obtained in Ex. 8, page 273 is

$$pa^m=r^{m+1}$$

$$\therefore \qquad \frac{dp}{dr}=\frac{1}{a^m}(m+1)r^m.$$

Hence $$\rho=r\frac{dr}{dp}$$

$$=\frac{r.a^m}{(m+1)r^m}=\frac{1}{(m+1)}\cdot\frac{a^m}{r^{m-1}}.$$

Exercises

1. Find the radius of curvature of the curve $r=a\cos n\theta$ as a function of r. Also, show that at a point where $r=a$ its values is $a/(1+n^2)$. (*P.U.*)

2. Find the radius of curvature at the point (r, θ) on each of the following curves :—

(*i*) $r=a$. (*ii*) $r\theta=a$. (*iii*) $r^n=a^n\sin n\theta$

(*iv*) $\theta=\frac{\sqrt{(r^2-a^2)}}{a}-\cos^{-1}\frac{a}{r}$. (*vi*) $r(1+\cos\theta)=a$.

3. Find the radius of curvature of the curve $r=a\sin n\theta$ at the origin. (*Allahabad*)

4. Prove that for the cardioide $r=a(1+\cos\theta)$, ρ^2/r is constant. (*P.U. 1954 ; D.U. 1955*)

5. Find the radius of curvature at the point (p, r) on each of the following curves :—

(*i*) $p^2=ar$. (*ii*) $1/p^2=A/r^2+B$. (*iii*) $p^2(a^2+b^2-r^2)=a^2b^2$.

6. Find the radius of curvature for the ellipse

$$p^2=a^2\cos^2\psi+b^2\sin^2\psi.$$

7. Find the radii of curvature of the curves

(*i*) $p=a\sin\psi\cos\psi$.

(*ii*) $p^{m/(m+1)}=a^{m/(m+1)}\sin\frac{m}{m+1}\psi$.

8. Show that at the points in which the Archimedean spiral $r=a\theta$ intersects the hyperbolical spiral $r\theta=a$, their curvatures are in the ratio 3 : 1

9. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioide

$$r=a(1+\cos\theta),$$

which passes through the pole, then

$$\rho_1^2+\rho_2^2=16a^2/9.$$

10. Find the radius of curvature to the curve $r=a\,(1+\cos\theta)$ at the point where the tangent is parallel to the initial line.

11. A line is drawn through the origin meeting the cardioid $r=a\,(1-\cos\theta)$ in the points P, Q and the normals at P, Q meet in O; show that the radii of curvature at P and Q are proportional to PC and QC.

12. Find the points of inflexion on the curves

(*i*) $r=a\theta^2/(\theta^2-1)$. (*ii*) $r^2\theta=a^2$. (See Cor. § 15·46, p. 299)

13. Show that $(a, 0)$, in polar co-ordinates, is a point of inflexion on the curve $r=ae^{\theta}/(1+\theta)$.

14. Show that the curve $r=a\theta^n$ has points of inflexion if and only if, n lies between 0 and -1 and they are given by $\theta=\pm\sqrt{[-n(n+1)]}$.

15. Show that the curve $re^{\theta}=a\,(1+\theta)$ has no point of inflexion.

16. Prove that for any curve.

$$r/\rho=\sin\phi[1+d\phi/d\theta],$$

where ρ is the radius of curvature and $\tan\phi=rd\theta/dr$.

(*P. U. 1951 ; Delhi 1948*)

17. If the equation to a curve be given in polars $r=f(\theta)$ and if $u=\dfrac{1}{r}$

prove that the curvature is given by

$$\left(\frac{d^2u}{d\theta^2}+u\right)\sin^3\phi.$$

Deduce or otherwise prove that the curvature is given by

$$\frac{1}{r}\,\frac{dp}{dr}.$$

(*Bombay*)

18. The curve $r=ae^{\theta\cot\alpha}$ cuts any radius vector in the consecutive points $P_1, P_2, \ldots\ldots\ldots\ldots\ldots, P_n$. If ρ_n denotes the radius of curvature at P_n, prove that

$$\frac{1}{m-n}\log\frac{\rho_m}{\rho_n}$$

is constant for all integral values of m and n. (*Delhi Hons. 1947*)

19. Prove that in the curve

$$r^2=a^2\sin 2\theta.$$

the tangent turns three times as fast as the radius vector and that the curvature varies as the radius vector. (*Delhi, 1949*)

15·5. Centre of curvature *for any point P of a curve is the point which lies on the positive direction of the normal at P and is at a distance, ρ, from it.*

The distance, ρ, must be taken with a proper sign so that the normal of curvature lies on the positive or negative direction of the centre according as, ρ, is positive or negative.

The positive direction of the normal is obtained by rotating the positive direction of the tangent (the positive direction of the tangent to $y=f(x)$ is the one in which x increases) through $\pi/2$ in the anti-clockwise direction.

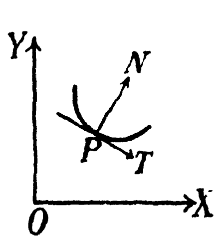


Fig. 113

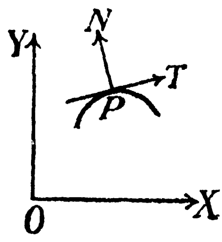


Fig. 114

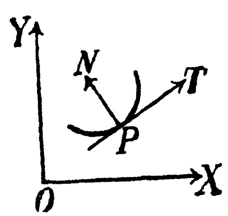


Fig. 115

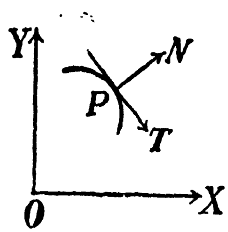


Fig. 116

From an examination of the figures above, we see that the *centre of curvature at any point of a curve lies on the side towards which the concavity is turned.*

15·51. *To find the co-ordinates of the centre of curvature for any point $P(x, y)$ of the curve $y=f(x)$.*

Let the positive direction of the tangent make angle, ψ, with X-axis so that the positive direction of the normal makes angle $\psi+\pi/2$ with X-axis.

The equation of the normal is

$$\frac{X-x}{\cos(\psi+\pi/2)}=\frac{Y-y}{\sin(\psi+\pi/2)}=r,$$

or

$$\frac{X-x}{-\sin\psi}=\frac{Y-y}{\cos\psi}=r,$$

where X, Y are the current co-ordinates of any point on the normal and, r is the variable distance of the variable point (X, Y) from (x, y).

Thus the co-ordinates (X, Y) of the point on the normal at a distance, r from (x, y) are

$$(x-r \sin \psi, y+r \cos \psi).$$

For the centre of curvature,

$$r=\rho.$$

Hence, if (X, Y) be the centre of curvature, we have

$$\left.\begin{aligned} X&=x-\rho \sin \psi, \\ Y&=y+\rho \cos \psi. \end{aligned}\right\} \quad \ldots(A)$$

But, we know that

$$\sin \psi = \frac{y_1}{\sqrt{[1+y_1^2]}} \; ; \cos \psi = \frac{1}{\sqrt{[1+y_1^2]}} \; ; \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$

$$\therefore \quad X = x - \frac{y_1(1+y_1^2)}{y_2}, \; Y = y + \frac{1+y_1^2}{y_2}.$$

Another method. If C be the centre of curvature for P, we have

$$PC=\rho.$$

Also, $\angle NCP = \pi/2 - \angle NPC$
$= \pi/2 - (\pi/2 - \angle XTP)$
$= \angle XTP = \psi.$

$\therefore$ $X = OM = OL - ML$
$= OL - NP$
$= OL - PC \sin \psi$
$= x - \rho \sin \psi.$

$Y = MC = NM + NC$
$= LP + NC$
$= LP + PC \cos \psi$
$= y + \rho \cos \psi.$

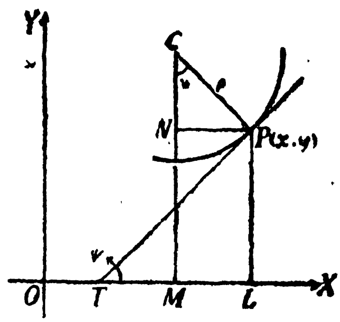


Fig. 117

Substituting the value of $\sin \psi$, $\cos \psi$ and ρ, we can obtain the values of X and Y.

15·52. Evolute. *The locus of the centres of curvature of a curve is called its evolute and a curve is said to be an involute of its evolute.*

15·53. The circle of curvature *of any point P of a curve is the circle whose centre is at the centre of curvature C and whose radiu is* $|\rho|$.

The circle of curvature will clearly touch the curve at P and its curvature will be the same as that of the curve.

15·54. Chord of curvature drawn in a given direction for any point of a curve is the chord of the circle of curvature through the point drawn in the said direction.

We will now determine expressions for the lengths of some important particular cases of the chords of curvature.

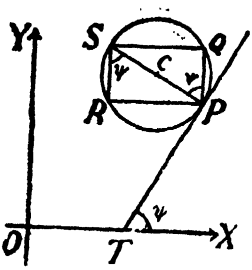


Fig. 118

(*i*) Let PT be the tangent and C the centre of curvature for any point P of a curve.

We draw the circle with C as centre and PC as radius.

PR and PQ are chords of this circle parallel to X-axis and Y-axis respectively.

Clearly, $\angle RSP = \psi = \angle SPQ$.

Now, $\quad PR = PS \sin \angle RPS = 2\rho \sin \psi$.

Thus, *the chord of curvature* $\parallel$ *X-axis*

$= 2\rho \sin \psi$.

Also, $PQ = PS \cos \angle SPQ = 2\rho \cos \psi$.

Thus *the chord of curvature* $\parallel$ *Y-axis* $= 2\rho \cos \psi$.

(*ii*) O is the pole. PT is the chord of the circle of curvature through the pole O.

PS is the chord of the circle of curvature perpendicular to the radius vector OP.

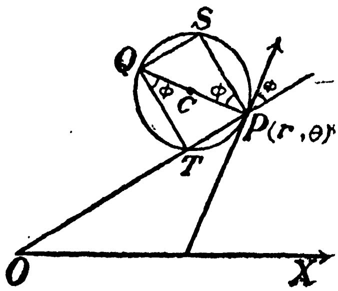


Fig. 119

Clearly $\angle TQP = \phi = \angle SPQ$.

$\therefore \quad PT = PQ \sin \angle TQP = 2\rho \sin \phi$.

Thus, *the chord of curvature through the pole* $= 2\rho \sin \phi$.

Also, $PS = PQ \cos \angle SPQ = 2\rho \cos \phi$,

Thus, *the chord of curvature* $\perp$ *radius vector* $= 2\rho \cos \psi$.

Examples

1. *Find the co-ordinates of the centres of curvature at any point (x, y) of the parabola $y^2 = 4ax$. Hence obtain its evolute.*

Differentiating, we get

$$2y \frac{dy}{dx} = 4a, \ i.e., \ \frac{dy}{dx} = \frac{2a}{y}.$$

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2} \cdot \frac{dy}{dx} = -\frac{4a^2}{y^3}.$$

If (X, Y) be the centre of curvature

$$X = x - \frac{\frac{2a}{y}\left(1 + \frac{4a^2}{y^2}\right)}{-\frac{4a^2}{y^3}}$$

$$=x+\frac{y^2+4a^2}{2a}=\frac{2ax+4ax+4a^2}{2a}=3x+2a. \quad \ldots (i)$$

$$Y=y+\frac{1+\dfrac{4a^2}{y^2}}{-\dfrac{4a^2}{y^3}}$$

$$=y-\frac{y(y^2+4a^2)}{4a^2}$$

$$=-\frac{y^3}{4a^2}=\mp\frac{(4ax)^{\frac{3}{2}}}{4a^2}=\mp\frac{2x^{\frac{3}{2}}}{a^{\frac{1}{2}}}. \quad \ldots (ii)$$

To find the evolute, we have to eliminate x from (i) and (ii). Thus,

$$Y^2=\frac{4x^3}{a}=\frac{4}{a}\left(\frac{X-2a}{3}\right)^3$$

or $$27aY^2=4(X-2a)^3,$$

is the required evolute.

2. *Find the evolute of the four cusped hypocycloid*

$$x=a\cos^3\theta,\ y=a\sin^3\theta,\ i.e.,\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

We can easily show that

$$\frac{dy}{dx}=-\tan\theta\ ;\ \frac{d^2y}{dx^2}=\frac{1}{3a}\sec^4\theta\ \text{cosec}\ \theta.$$

$$X=a\cos^3\theta+\frac{\tan\theta\,(1+\tan^2\theta)}{\sec^4\theta\ \text{cosec}\ \theta}.3a$$

$$=a\cos^3\theta+3a\sin^2\theta\cos\theta, \quad \ldots (i)$$

$$Y=a\sin^3\theta+\frac{1+\tan^2\theta}{\sec^4\theta\ \text{cosec}\ \theta}.3a$$

$$=a\sin^3\theta+3a\cos^2\theta\sin\theta. \quad \ldots (ii)$$

To eliminate θ, we separately add and subtract (i), (ii). Therefore

$$X+Y=a(\cos\theta+\sin\theta)^3 \text{ or } (X+Y)^{\frac{1}{3}}=a^{\frac{1}{3}}(\cos\theta+\sin\theta).$$

$$X-Y=a(\cos\theta-\sin\theta)^3 \text{ or } (X-Y)^{\frac{1}{3}}=a^{\frac{1}{3}}(\cos\theta-\sin\theta)$$

On squaring and adding, we obtain

$$(X+Y)^{\frac{2}{3}}+(X-Y)^{\frac{2}{3}}=2a^{\frac{2}{3}},$$

as the required evolute.

3. *In the curve $y=a\log\sec(x/a)$, the chord of curvature parallel to Y-axis is of constant length.*

Here $\tan\psi=\dfrac{dy}{dx}=\tan\dfrac{x}{a}$. $\therefore\ \psi=\dfrac{x}{a}$.

Also, $\dfrac{d^2y}{dx^2}=\dfrac{1}{a}\sec^2\dfrac{x}{a}$.

$\therefore$ $\rho=a\dfrac{(1+\tan^2 x/a)^2}{\sec^2 x/a}=a\sec\dfrac{x}{a}$.

$\therefore$ chord of curvature parallel to Y-axis

$$=2\rho\cos\psi=2a\sec\frac{x}{a}\cos\frac{x}{a}=2a,$$

which is constant.

Exercises

1. Show that the evolute of the ellipse

$$x=a\cos\theta,\ y=b\sin\theta$$

is

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$$

2. (*a*) Show that for the hyperbola $x^2/a^2-y^2/b^2=1$,

$$a^4X=(a^2+b^2)x^3,\ b^4Y=-(a^2+b^2)y^3,$$

and the equation of the evolute is

$$(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}=(a^2+b^2)^{\frac{2}{3}}. \qquad (P.U.)$$

(*b*) Prove that the evolute of the hyperbola

$$2xy=a^2,$$

is

$$(x+y)^{\frac{2}{3}}-(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}. \qquad (P.U.\ 1955)$$

3. Show that the evolute of the tractrix

$$x=a\,(\cos t+\log\tan\tfrac{1}{2}t),\ y=a\sin t$$

is the catenary

$$y=a\cosh(x/a).$$

4. Prove that the centres of curvature at points of a cycloid lie on an equal cycloid. (*P.U. Supp. 1944*)

5. Show that $(21a/16, 21a/16)$ is the centre of curvature for the point $(3a/2, 3a/2)$ of the Folium $x^3+y^3=3axy$.

6. Show that the centre of curvature of the point $P(a, a)$ of the curve $x^4+y^4=2a^2xy$ divides the line OP in the ratio $6:1$; O being the origin of co-ordinates.

7. Show that the parabolas

$$y=-x^2+x+1,\ x=-y^2+y+1$$

have the same circle of curvature at the point $(1, 1)$.

8. Show that

$$(x-\tfrac{3}{4}a)^2+(y-\tfrac{3}{4}a)^2=\tfrac{1}{2}a^2,$$

is the circle of curvature of the curve

$$\sqrt{x}+\sqrt{y}=\sqrt{a},$$

at the point $(a/4, a/4)$.

9. Find the circle of curvature at the origin for the curve

$$x+y=ax^2+by^2+cx^3. \quad \text{(Delhi Hons. 1951)}$$

10. Show that the circle of curvature at the origin of the parabola

$$y=mx+\frac{x^2}{a},$$

is

$$x^2+y^2=a(1+m^2)(y-mx). \quad \text{(D.U. 1955)}$$

11. (a) Show that the circle of curvature, at the point $(am^2, 2am)$ of the parabola $y^2=4ax$, has for its equation

$$x^2+y^2-6am^2x-4ax+4am^3y=3a^2m^4. \quad \text{(D.U. Hons. 1957)}$$

(b) Find the equation of the circle of curvature at the point $(0, b)$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. *(D.U. Hons. 1959)*

12. Show that the radii of curvature of the curve

$$x=ae^{\theta}(\sin\theta-\cos\theta),\ y=ae^{\theta}(\sin\theta+\cos\theta),$$

and its evolute at corresponding points are equal.

13. Find the radius of curvature at any point P of $y=c\cosh(x/c)$ and show that $PC=PG$ where C is the centre of curvature at P and G the point of intersection of the normal at P with x-axis. *(Allahabad)*

14. Show that the chord of curvature through the pole of the equiangular spiral $r=ae^{m\theta}$ is $2r$.

15. Show that the chord of curvature through the pole of the equiangular spiral $r=ae^{b\theta}$ is bisected at the pole.

16. If c_x and c_y be the chords of curvature parallel to the axes at any point of the curve $y=ae^{x/a}$, prove that

$$\frac{1}{(c_x)^2}+\frac{1}{(c_y)^2}=\frac{1}{2ac_x}. \quad \text{(P.U. 1948)}$$

17. If c_x and c_y be the chords of curvature parallel to the axes at any point of the catenary $y=c\cosh(x/c)$, prove that

$$4c^2(c_x^2+c_y^2)=c_y^4.$$

18. Show that the chord of curvature through the pole of the cardioide

$$r=a(1-\cos\theta).$$

is $\frac{4}{3}r$.

19. If c_r and c_θ be the chords of curvature of the cardioide $r=a(1+\cos\theta)$ through the pole and perpendicular to the radius vector, then

$$3(c_r^2+c_\theta^2)=8a.c_r.$$

20. Show that the chord of curvature through the pole of the curve

$$r^m=a^m\cos m\theta,$$

is

$$2r/(m+1). \quad \text{(Gujrat 1952)}$$

21. Show that the chord of curvature through the pole for the curve

$$p=f(r),$$

is

$$2f(r)/f'(r). \quad \text{(Lucknow)}$$

22. Show that for the curve $p=ae^{br}$, the chord of curvature through the pole is of constant length.

23. For the lemniscate $r^2=a^2\cos 2\theta$, show that the length of the tangent from the origin to the circle of curvature at any point is $r\sqrt{3}/3$. *(B.U.)*

24. If P is any point on the curve $r^2=a^2 \cos 2\theta$ and Q is the intersection of the normal at P with the line through O at right angles to the radius vector OP, prove that the centre of curvature corresponding to P is a point of trisection of PQ. (*L.U.*)

25. If P is any point on the curve $r=a\,(1+\cos\theta)$ and Q is the intersection of the normal at P with the line through the pole O perpendicular to OP, prove that the centre of curvature at P is a point of trisection of PQ remote from P.

26. The circle of curvature at any point P of the Lemniscate $r^2=a^2 \cos 2\theta$ meets the radius vector OP at A, show that

$$OP : AP=1:2 ;$$

O being the pole.

27. ρ_1, ρ_2 are the radii of curvature at the corresponding points of a cycloid and its evolute ; prove that $\rho_1^2+\rho_2^2$ is a constant.

28. Show that the chord of curvature through the focus of a parabola is four times the focal distance of the point and the chord of curvature parallel to the axis has the same length. (*Rajputana 1952*)

29. Prove that the distance between the pole and the centre of curvature corresponding to any point on the curve $r^n=a^n \cos n\theta$ is

$$\frac{[a^{2n}+(n^2-1)r^{2n}]^{\frac{1}{2}}}{(n+1)r^{n-1}}.$$

15·55. Two important properties of the evolute.

If (X, Y) be the centre of curvature for any point $P(x, y)$ on the given curve, we have

$$X=x-\rho \sin\psi,\ Y=y+\rho\cos\psi.$$

Differentiating *w.r.* to x, we obtain

$$\frac{dX}{dx}=1-\rho\cos\psi\frac{d\psi}{dx}-\sin\psi\frac{d\rho}{dx}$$

$$=1-\frac{ds}{d\psi}\frac{dx}{ds}\frac{d\psi}{dx}-\sin\psi\frac{d\rho}{dx}$$

$$=-\sin\psi\frac{d\rho}{d} ; \qquad \dots (i)$$

$$\frac{dY}{dx}=\frac{dy}{dx}-\rho\sin\psi\frac{d\psi}{dx}+\cos\psi\frac{d\rho}{dx}$$

$$=\frac{dy}{dx}-\frac{ds}{d\psi}\frac{dy}{ds}\frac{d\psi}{dx}+\cos\psi\frac{d\rho}{dx}.$$

$$=\cos\psi\frac{d\rho}{dx}. \qquad \dots (ii)$$

From (*i*) and (*ii*),

$$\frac{dY}{dX}=-\cot\psi. \qquad \dots (iii)$$

Now, dY/dX is the slope of the tangent to the evolute at P' and, $-\cot\psi$, is the slope of the normal PP' in the original curve at P. By (*ii*) the slopes of two lines, which have a point P' in common, are equal, and therefore they coincide.

Thus, *the normals to a curve are the tangents to its evolute.*

Again, we square (*i*) and (*ii*) and add.

$$\therefore \qquad \left(\frac{dX}{dx}\right)^2+\left(\frac{dY}{dx}\right)^2=\left(\frac{d\rho}{dx}\right)^2.$$

Let, S, be the length of the arc of the evolute measured from one fixed point on it upto (X, Y) so that

$$\left(\frac{dS}{dx}\right)^2=\left(\frac{dX}{dx}\right)^2+\left(\frac{dY}{dx}\right)^2$$

Here, x is a parameter for the evolute.

$$\therefore \qquad \left(\frac{dS}{dx}\right)^2=\left(\frac{d\rho}{dx}\right)^2, \qquad \ldots(iv)$$

$$\frac{dS}{dx}=\frac{d\rho}{dx}, \qquad \ldots(v)$$

or

$$S=\rho+c,$$

where, c, is constant.

Let M_1, M_2 be the two points on the evolute corresponding to the points L_1, L_2 on the original curve. Let ρ_1, ρ_2 be the values of, ρ, for L_1, L_2 and S_1, S_2 be the values of S for M_1, M_2.

$$\therefore \qquad S_1=\rho_1+c,$$
$$S_2=\rho_2+c.$$

Thus

$$S_2-S_1=\rho_2-\rho_1,$$

i.e., arc M_1M_2=difference between the radii of curvatures at L_1, L_2.

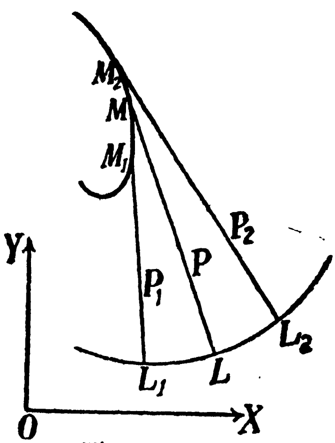


Fig. 120

Thus, we have shown that *difference between the radii of curvatures at two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.*

Note. We suppose that S is measured positively in the direction of x, increasing so that dS/dx is positive. Also, $d\rho/dx$ is positive or negative according as, ρ, increases or decreases as x increases.

Thus

$$\frac{dS}{dx}=\frac{d\rho}{dx} \text{ or } -\frac{d\rho}{dx}$$

according as, ρ, increases or decreases for the values of, x, under consideration.

It is easy to see that the conclusion arrived at in this section remains the same if we consider $dS/dx=-d\rho/dx$ instead of $dS/dx=d\rho/dx$. It should, however, be noted that the conclusion holds good only for that part of the curve for which ρ, constantly increases or decreases so that $d\rho/dx$ keeps the same sign.

Ex. *Find the length of the arc of the evolute of the parabola*

$$y^2=4ax$$

which is intercepted between the parabola.

The evolute is

$$27ay^2=4(x-2a)^3. \qquad \text{(Ex. 1, p. 30)}$$

Let L, M be the points of intersection of the evolute LAM and the parabola.

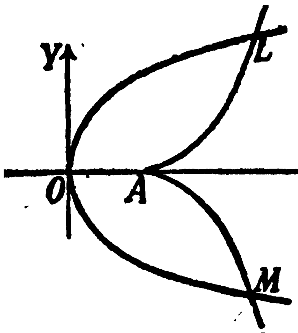


Fig. 121

To find the co-ordinates of the points of intersection L, M, we solve the two equations simultaneously.

We get

$$27a \;.\; 4ax=4(x-2a)^3,$$

or

$$x^3-6ax^2-15a^2x-8a^3=0.$$

Now, $8a$, $-a$, $-a$ are the roots of this cubic equation of which $x=8a$ is the only admissible value ; $-a$ being negative.

$\therefore$ $(8a, 4\sqrt{2}a)$, $(8a, -4\sqrt{2}a)$ are the co-ordinates of L, M.

If (X, Y) be the centre of curvature for any point (x, y) on the parabola, we have

$$X=3x+2a,\ Y=-y^3/4a^2. \qquad \text{(Ex. 1, p. 306)}$$

Thus $A(2a, 0)$ is the centre of curvature for $O(0, 0)$ and $L(8a, 4\sqrt{2}a)$ is the centre of curvature for $P(2a, -2\sqrt{2}a)$.

The radius of curvature at $O=OA=2a$.

The radius of curvature at $P=PL=6\sqrt{3}a$.

$\therefore$ arc $AL=PL-OA=2a(3\sqrt{3}-1)$.

Hence the required length $MAL=4a(3\sqrt{3}-1)$.

Ex. 2. Show that the whole length of the evolute of the ellipse

$$x^2/a^2+y^2/b^2=1$$

is $4(a^2/b-b^2/a)$.

Ex. 3. Show that the whole length of the evolute of the astroid

$$x=a\cos^3\theta,\ y=a\sin^3\theta$$

is $12a$.

CHAPTER XVI

ASYMPTOTES

16·1. Definition. *A straight line is said to be an* **Asymptote** *of an infinite branch of a curve, if, as the point P recedes to infinity along the branch, the perpendicular distance of P from the straight line tends to 0.*

Illustration. The line $x=a$ is an asymptote of the *Cissoid*

$$y^2(a-x)=x^3. \qquad \text{(See § 11·41)}$$

It is easy to see that as $P(x, y)$ moves to infinity, its distance from the line $x=a$ tends to zero.

Ex. What are the asymptotes of the curves

$y=\tan x$; $y=\cot x$; $y=\sec x$ and $y=\operatorname{cosec} x$.

16·2. Determination of Asymptotes. We know that the equation of a line which is *not parallel to X-axis* is of the form

$$y=mx+c. \qquad ...(i)$$

The abscissa, x, must tend to infinity as the point $P(x, y)$ recedes to infinity along this line.

We shall now determine, m, and, c, so that the line (i) may be an asymptote of the given curve.

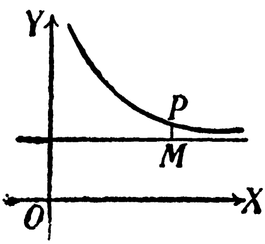


Fig. 122

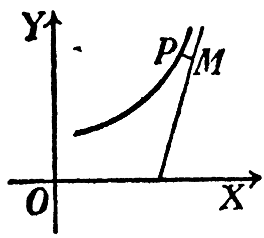


Fig. 123

If $p=MP$ be the perpendicular distance of any point $P(x, y)$ on the infinite branch of a given curve from the line (i), we have

$$p=\frac{y-mx-c}{\sqrt{(1+m^2)}}.$$

Now

$$p \to 0 \text{ as } x \to \infty.$$

$$\therefore \qquad \lim(y-mx-c)=0,$$

which means that, when $x \to \infty$.

$$\lim(y-mx)=c.$$

Also,

$$y/x-m=(y-mx).\ (1/x).$$

$\therefore$ $\lim\ (y/x-m)=\lim\ (y-mx).\ \lim\ 1/x=c.\ 0=0$

or

$$\lim\ (y/x)=m,$$

when $x \to \infty$.

Hence

$$\mathbf{m}=\lim_{\mathbf{x} \to \infty} (\mathbf{y}/\mathbf{x}),\ \mathbf{c}=\lim_{\mathbf{x} \to \infty} (\mathbf{y}-\mathbf{mx}).$$

We have thus the following method to determine asymptotes which are not parallel to y-axis :—

(*i*) *Find* $\lim\limits_{x \to \infty} (y/x)$; *let* $\lim\limits_{x \to \infty} (y/x)=m.$

(*ii*) *Find* $\lim\limits_{x \to \infty} (y-mx)$; *let* $\lim\limits_{x \to \infty} (y-mx)=c.$

Then $y=mx+c$ *is an asymptote*

The values of y will be different according to the different branches along which P recedes to infinity, and so we expect several values of lim (y/x) corresponding to the several values of y and also several corresponding values of lim $(y-mx)$. Thus a curve may have more than one asymptote.

This method will determine all the asymptotes except those which are parallel to Y-axis. To determine such asymptotes, we start with the equation $x=my+d$ which can represent every straight line not parallel to X-axis and show, that when $y \to \infty$

$$m=\lim\ (x/y) \text{ and } d=\lim\ (x-my).$$

The asymptotes not parallel to any axis can be obtained either way.

Ex. 1. *Examine the Folium*

$$x^3+y^3-3axy=0, \qquad \ldots(i)$$

for asymptotes.

The given equation is of the third degree.

To find lim (y/x), divide the equation (i) by x^3, so that

$$1+\left(\frac{y}{x}\right)^3-3a\ \frac{y}{x}\ .\ \frac{1}{x}=0.$$

Let $x \to \infty$. We then get

$$1+m^3=0 \text{ or } (m+1)(m^2-m+1)=0.$$

$\therefore$ $m=-1.$

The roots of $m^2-m+1=0$ are not real.

To find lim $(y-mx)$ when $m=-1$, *i.e.*, to find lim $(y+x)$, we put $y+x=p$ so that, p, is a variable which $\to c$ when $x \to \infty$.

Putting $p-x$ for y in the equation (*i*), we get

$$x^3+(p-x)^3-3ax(p-x)=0,$$

or

$$3(p+a)x^2-3(p^2+ap)x+p^3=0,$$

which is of the *second* degree in x.

Dividing by x^2, we get

$$3(p+a)-3(p^2+ap).\ \frac{1}{x}+p^3.\ \frac{1}{x^2}=0.$$

Let $x \to \infty$. We then have

$$3(c+a)=0 \text{ or } c=-a.$$

Hence

$$y=-x-a \text{ or } x+y+a=0,$$

is the only asymptote of the given curve.

If we start with $x=my+d$, we get no new asymptotes. Thus

$$x+y+a=0,$$

is the only asymptote of the given curve.

Ex. 2. Find the asymptotes of the following curves :

(*i*) $x^2(x-y)+ay^2=0$. (*ii*) $x^3+y^3=3ax^2$.

(*iii*) $y^3=x^3+ax^2$.

16·3. Working rules for determining asymptotes. Shorter Methods. In practice the rules obtained below for determining asymptotes are found more convenient than the method which involves direct determination of

$$\lim (y/x) \text{ and } \lim (y-mx).$$

Firstly we shall consider the case of asymptotes parallel to the co-ordinate axes and then that of oblique asymptotes.

16·31. Determination of the asymptotes parallel to the co-ordinate axes.

Asymptotes parallel to Y-axis.

Let

$$x=k \qquad \ldots(i)$$

be an asymptote of the curve, so that we have to determine k.

Here, y, alone tends to infinity as a point $P(x, y)$ recedes to infinity along the curve.

The distance PM of any point $P(x, y)$ on the curve from the line (i) is equal to $x-k$.

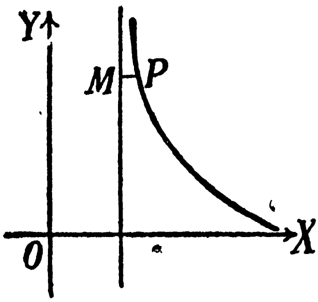


Fig. 124

$\therefore \quad \lim (x-k)=0$ when $y \to \infty$,

or

$\lim x=k$ when $y \to \infty$,

which gives k.

Thus to find the asymptotes parallel to Y-axis, we find the definite value or values k_1, k_2, etc., to which x tends as y tends to ∞. Then $x=k_1$, $x=k_2$, etc. are the required asymptotes.

We will now obtain a simple rule to obtain the asymptotes of a rational *Algebraic* curve which are parallel to Y-axis.

We arrange the equation of the curve in descending powers of y, so that it takes the form

$$y^m\phi(x)+y^{m-1}\phi_1(x)+y^{m-2}\phi_2(x)+\ldots\ldots\ldots=0 \ ; \qquad \ldots(i)$$

where

$$\phi(x),\ \phi_1(x),\ \phi_2(x), \text{ etc.},$$

are polynomials in x.

Dividing the equation (i) by y^m, we get

$$\phi(x)+(1/y).\ \phi_1(x)+(1/y^2).\ \phi_2(x)+\ldots\ldots\ldots=0. \qquad \ldots(ii)$$

Let $y \to \infty$. We write

$$\lim x=k.$$

The equation (ii) gives

$$\phi(k)=0,$$

so that, k, is a root of the equation $\phi(x)=0$.

Let k_1, k_2 etc., be the roots of $\phi(x)=0$. Then the asymptotes parallel to Y-axis are

$$x=k_1,\ x=k_2, \text{ etc.}$$

From algebra, we know that $(x-k_1)$, $(x-k_2)$, etc., are the factors of $\phi(x)$ which is the co-efficient of the highest power y^m of y in the given equation.

Hence we have the rule :—*The asymptotes parallel to Y-axis are obtained by equating to zero the real linear factors in the co-efficient of the highest power of, y, in the equation of the curve.*

The curve will have no asymptote parallel to Y-axis, if the co-efficient of the highest power of, y, is a constant or if its linear factors are all imaginary.

Asymptotes parallel to X-axis. As above it can be shown that *the asymptotes, which are parallel to X-axis, are obtained by equating to zero the real linear factors in the co-efficient of the highest power of, x, in the equation of the curve.*

16·32. *To determine the asymptotes of the* **general rational algebraic** *equation*

$$U_n+U_{n-1}+U_{n-2}+\ldots\ldots+U_2+U_1+U_0=0, \qquad \ldots(i)$$

where, U_r, is a homogeneous expression of degree, r, in x, y.

We write U_r in the form

$$U_r \equiv x^r \phi_r(y/x)$$

where $\phi_r(y/x)$ is a polynomial in y/x of degree, r, at most.

So we write (i) in the form

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots\ldots$$

$$+x\phi_1\left(\frac{y}{x}\right)+\phi_0\left(\frac{y}{x}\right)=0. \qquad \ldots(ii)$$

Dividing by x^n, we get

$$\phi_n\left(\frac{y}{x}\right)+\frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right)+\frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots\ldots$$

$$+\frac{1}{x^{n-1}}\phi_1\left(\frac{y}{x}\right)+\frac{1}{x^n}\phi_0\left(\frac{y}{x}\right)=0. \quad \ldots(iii)$$

On taking limits, as $x \to \infty$, we obtain the equation

$$\phi_n(\mathbf{m})=0, \qquad \ldots(iv)$$

which determines the slopes of the asymptotes.

Let m_1 be one of the roots of this equation so that $\phi_n(m_1)=0$. We write

$$y-m_1x=p_1, \text{ i.e., } \frac{y}{x}=m_1+\frac{p_1}{x}.$$

Substituting this value of y/x in (ii), we get

$$x^n\phi_n\left(m_1+\frac{p_1}{x}\right)+x^{n-1}\phi_{n-1}\left(m_1+\frac{p_1}{x}\right)+x^{n-2}\phi_{n-2}\left(m_1+\frac{p_1}{x}\right)+\ldots$$

$$\ldots\ldots+x\phi_1\left(m_1+\frac{p_1}{x}\right)+\phi_0\left(m_1+\frac{p_1}{x}\right)=0.$$

Expanding each term by Taylor's theorem, we get

$$x^n\left[\phi_n(m_1)+\frac{p_1}{x}\phi'_n(m_1)+\frac{p_1^2}{2x^2}\phi n''(m_1)+\ldots\ldots\right]$$

$$+x^{n-1}\left[\phi_{n-1}(m_1)+\frac{p_1}{x}\phi'_{n-1}(m_1)+\ldots\ldots\right]$$

$$+x^{n-2}\left[\phi_{n-2}(m_1)+\frac{p_1}{x}\phi'_{n-2}(m_1)+\ldots\right]+\ldots=0.$$

Arranging terms according to descending powers of x, we get

$$x^n\phi_n(m_1)+x^{n-1}[p_1\phi_n'(m_1)+\phi_{n-1}(m_1)]$$
$$+x^{n-2}\left[\frac{p_1^2}{2}\phi''_n(m_1)+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right]+\ldots=0$$

Putting $\phi_n(m_1)=0$ and then dividing by x^{n-1}, we get

$$[p_1\phi'_n(m_1)+\phi_{n-1}(m_1)]$$
$$+\left[\frac{p_1^2}{2}\phi''_n(m_1)+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right]\frac{1}{x}+\ldots.=0 \quad \ldots(v)$$

Let $x \to \infty$. We write $\lim p_1=c_1$. Therefore

$$\mathbf{c_1\phi'_n(m_1)+\phi_{n-1}(m_1)=0}, \quad \ldots(vi)$$

or

$$c_1=-\frac{\phi_{n-1}(m_1)}{\phi'_n(m_1)}, \text{ if } \phi'_n(m_1)\neq 0.$$

Therefore

$$y=m_1x-\frac{\phi_{n-1}(m_1)}{\phi'_n(m_1)},$$

is the asymptote corresponding to the slope m_1, if $\phi'_n(m_1)\neq 0$.

Similarly

$$y=m_2x-\frac{\phi_{n-1}(m_2)}{\phi'_n(m_2)} \;;\; y=m_3x-\frac{\phi_{n-1}(m_3)}{\phi'_n(m_3)}, \text{ etc.},$$

are the asymptotes of the curve corresponding to the slopes m_2, m_3 etc., which are the roots of $\phi_n(m)=0$, if $\phi'_n(m_2)$, $\phi'_n(m_3)$, etc., are not 0.

Exceptional case. *Let* $\phi'_n(m_1)=0$.

If $\phi'_n(m_1)=0$ but $\phi_{n-1}(m_1)\neq 0$, then the equation (vi) does not determine any value of c_1 and, therefore, there is no asymptote corresponding to the slope m_1.

Now suppose

$$\phi'_n(m_1)=0=\phi_{n-1}(m_1).$$

In this case, (vi) becomes an identity and we have to re-examine the equation (v) which now becomes

$$\left[\frac{p_1^2}{2}\phi''_n(m_1)+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right]+\left[\ldots\right]\frac{1}{x}$$
$$+\ldots.=0.$$

On taking limits, as $x \to \infty$, we see that c_1 is a root of the equation

$$\mathbf{(c_1^2/2)\phi''_n(m_1)+c_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)=0},$$

which determines two values of c_1, say c_1', c_1'', provided that

$$\phi''_n(m_1)\neq 0.$$

Thus

$$y=m_1x+c_1', \ y=m_1x+c_1'',$$

are *the two* asymptotes corresponding to the slope m_1. These are clearly parallel.

This is known as the *case of parallel asymptotes.*

Important Note. *The polynomial $\phi_n(m)$ is obtained by putting $x=1$ and $y=m$ in the highest degree terms $x^n\phi_n(y/x)$, and $\phi_{n-1}(m)$, $\phi_{n-2}(m)$, etc., are obtained from $x^{n-1}\phi_{n-1}(y/x)$, $x^{n-2}\phi_{n-2}(y/x)$, etc., in a similar way.*

Examples

1. *Find the asymptotes parallel to co-ordinate axes, of the curves:*

(*i*) $(x^2+y^2)x-ay^2=0$. (*ii*) $x^2y^2-a^2(x^2+y^2)=0$.

(*i*) The co-efficient of the highest power y^2 of y is $x-a$. Hence the asymptote parallel to y axis is $x-a=0$.

The co-efficient of the highest power x^3 of x is 1 which is a constant. Hence there is no asymptote parallel to x-axis.

(*ii*) The co-efficient of the highest power y^2 of y is x^2-a^2.

Also,

$$x^2-a^2=(x-a)(x+a).$$

Hence

$$x-a=0, \ x+a=0$$

are the two asymptotes parallel to y-axis.

It may similarly be shown that $y-a=0$, $y+a=0$ are the two asymptotes parallel to x-axis.

2. *Find the asymptotes of the cubic curve*

$$2x^3-x^2y+2xy^2+y^3-4x^2+8xy-4x+1=0.$$

Putting $x=1$, $y=m$ in the third degree and second degree terms separately, we get

$$\phi_3(m)=2-m-2m^2+m^3, \ \phi_2(m)=-4+8m.$$

The slopes of the asymptotes are given by

$$\phi_3(m)=m^3-2m^2-m+2=0,$$

or

$$(m+1)(m-1)(m-2)=2.$$

$\therefore$

$$m=-1, 1, 2$$

Again, c is given by

$$c\phi'_3(m)+\phi_2(m)=0,$$

i.e.,

$$c(-1-4m+3m^2)+(-4+8m)=0.$$

Putting $m=-1, 1, 2$, we get $c=2, 2, -4$ respectively.

Therefore the asymptotes are

$$y=-x+2, y=x+2, y=2x-4.$$

3. *Find the asymptotes of*

$$x^3-x^2y-xy^2+y^3+2x^2-4y^2+2xy+x+y+1=0.$$

Here $\phi_3(m)=1-m-m^2+m^3=(1-m)-m^2(1-m)$

$$=(1-m^2)(1-m)=(1-m)^2(1+m).$$

$$\phi_2(m)=2+2m-4m^2.$$

The slopes of the asymptotes, given by $\phi_3(m)=0$, are 1, 1, -1.

To determine c, we have

$$c\phi'_3(m)+\phi_2(m)=0;$$

i.e.,

$$c(-1-2m+3m^2)+(2+2m-4m^2)=0. \quad ...(i)$$

For $m=-1$, this gives $c=1$ and therefore, $y=-x+1$ is the corresponding asymptote.

For $m=1$, the equation (i) becomes 0. $c+0=0$ which is identically true. In this exceptional case, c, is determined from the equation

$$(c^2/2).\ \phi_3''(m)+c\phi'_2(m)+\phi_1(m)=0,$$

i.e.,

$$(c^2/2)(-2+6m)+c(2-8m)+1(1+m)=0.$$

For $m=1$, this becomes

$$2c^2-6c+2=0, \text{ i.e., } c=(3\pm\sqrt{5})/2.$$

Hence $y=x+(3\pm\sqrt{5})/2$ are the two parallel asymptotes corresponding to the slope 1.

We have thus obtained all the asymptotes of the curve.

Exercises

Find the asymptotes parallel to co-ordinate axes of the following curves :—

1. $y^2x-a^2(x-a)=0.$
2. $x^2y-3x^2-5xy+6y+2=0.$
3. $y=x/(x^2-1).$
4. $a^2/x^2+b^2/y^2=1.$

Find the asymptotes of the following curves :—

5. $x(y-x)^2=x(y-x)+2.$
6. $x^2(x-y)^2+a^2(x^2-y^2)=a^2xy.$ *(D.U. 1952)*
7. $(x-y)^2(x^2+y^2)-10(x-y)x^2+12y^2+2x+y=0.$ *(P.U.)*
8. $x^2y+xy^2+xy+y^2+3x=0.$ *(P.U.)*
9. $(x-y+1)(x-y-2)(x+y)=8x-1.$ *(P.U.)*
10. $y^3-x^2y+2xy^2-y+1=0.$ *(P.U.)*
11. $y(x-y)^2=x+y.$ *(B.U.)*
12. $x^2y^2(x^2-y^2)^2=(x^2+y^2)^3.$ *(B.U.)*

13. $y(y-1)^2=x^2-1$. (*L.U.*)

14. $xy^2=(x+y)^2$. (*L.U.*)

15. $(x+y)(x-y)(2x-y)-4x(x-2y)+4x=0$. (*L.U.*)

16. $xy^2-x^2y-3x^2-2xy+y^2+x-2y+1=0$. (*L.U.*)

17. $2x(y-3)^2=3y(x-1)^2$. (*Agra*)

18. $(y-a)^2(x^2-a^2)=x^4+a^4$. (*Lucknow*)

19. $(x+y)^2(x^2+xy+y^2)=a^2x^2+a^3(y-x)$.

20. $y^3+3y^2x-x^2y-3x^3+y^2-2xy+3x^2+4y+5=0$.

21. $(x-y)^2(x-2y)(x-3y)-2a(x^3-y^3)-2a^2(x-2y)(x+y)=0$, (*Dehi Hons. 1950*)

22. $y^2=x^4/(a^2-x^2)$. (*P.U. 1955 Supp.*)

Show that the following curves have no asymptotes :—

23. $x^4+y^4=a^2(x^2-y^2)$.

24. $y^2=x(x+1)^2$.

25. $a^4y^2=x^5(2a-x)$.

26. $x^2(y^2+x^2)=a^2(x^2-y^2)$.

27. Find the equation of the tangent to the curve $x^3+y^3=3ax^2$ which is parallel to its asymptote.

28. An asymptote is sometimes defined as a straight line which cuts the curve in two points at infinity. Criticise this definition and replace it by a correct definition. (*P.U. 1955 Supp.*)

16·33. Some deductions from § 16·32.

(*i*) *The number of asymptotes of an algebraic curve of the nth degree cannot exceed n.*

The slopes of the asymptotes which are not parallel to Y-axis are given as the roots of the equation $\phi_n(m)=0$ which is of degree n at the most.

In case the curve possesses one or more asymptotes parallel to Y-axis, then it is easy to see that the degree of $\phi_n(m)=0$ will be smaller than n by at least the same number.

Hence the result.

(*ii*) *The asymptotes of an algebraic curve are parallel to the lines obtained by equating to zero the factors of the highest degree terms in its equation.*

Let, m, be a root of the equation $\phi_n(m)=0$, so that the line $y-m_1x=0$ is parallel to an asymptote.

By elementary algebra, $(y/x-m_1)$ is a factor of $\phi_n(y/x)$ and hence, $y-m_1x$, is a factor of $x^n\phi_n(y/x)$, *i.e.*, U_n. Also conversely, we see that if, $y-m_1x$, is a factor of U_n then m_1 is a root of $\phi_n(m)=0$.

In case the highest degree terms contain, x, as a factor, then a little consideration will show that the curve will possess asymptotes parallel to $x=0$, *i.e.*, to y-axis.

(*iii*) Case of parallel asymptotes.

In this case, m_1, satisfies the three equations

$$\phi_n(m)=0,\ \phi'_n(m)=0,\ \phi_{n-1}(m)=0.$$

Since $\phi_n(m)$ and its derivative $\phi'_n(m)$ vanish for $m=m_1$, therefore, by elementary algebra, m_1 is a double root of $\phi_n(m)=0$ and therefore, $(y-m_1x)^2$ is a factor of the highest degree terms U_n.

Also, since m_1 is a root of $\phi_{n-1}(m)=0$, $y-m_1x$ is a factor of the $(n-1)$th degree terms U_{n-1}.

Thus, we see that in the exceptional case of § 16·32, a twice repeated linear factor of U_n is also a non-repeated factor of U_{n-1}.

There will be *no* asymptote with slope m_1, if m_1 is a root of $\phi_n(m)=0$, $\phi'_n(m)=0$ but *not* of $\phi_{n-1}(m)=0$, *i.e.*, if $(y-m_1x)^2$ is a factor of U_n and $y-m_1x$ is not a factor of U_{n-1}.

Note. The results obtained in the paragraphs (*ii*) and (*iii*) above enable us to shorten the process of determining the asymptotes as shown in the following examples.

The first step will always consist in factorising the expression formed of the highest degree terms in the given equation.

Examples

1. *Find the asymptotes of the Folium*

$$x^3+y^3-3axy=0.$$

The curve has no asymptotes parallel to co-ordinate axes.

Factorizing the highest degree terms, we get

$$(x+y)(x^2-xy+y^2)-3axy=0,$$

so that, $y+x$, is the only real linear factor of the highest degree terms.

Hence the curve has only one real asymptote which is parallel to the line $y+x=0$ whose slope is -1.

We have, now to find, lim $(y+x)$, when $x\to\infty$ and $y/x\to-1$.

We have

$$y+x=\frac{3ayx}{x^2-xy+y^2}=\frac{3a\,.\,(y/x)}{1-(y/x)+(y/x)^2}.$$

In the limit, we have

$$\lim\,(y+x)=\frac{-3a}{1+1+1}=-a \qquad \ldots(i)$$

$\therefore$

$$y=-x-a,\ i.e.,\ y+x+a=0,$$

is the only real asymptote of the folium.

It is easy to see that we could have eliminated the step (*i*), and simplified the process by saying that the required asymptote is

$$y+x=\lim \frac{3a(y/x)}{1-(y/x)+(y/x)^2},$$

when $x \to \infty$ and $y/x \to -1$.

2. *Find the asymptotes of*

$$x^3+4x^2y+4xy^2+5x^2+15xy+10y^2-2y+1=0.$$

Equating to zero the co-efficient of the highest power y^2 of y, we see that

$$4x+10=0, \text{ i.e., } 2x+5=0,$$

is one asymptote.

Factorising the highest degree terms, we get

$$x(2y+x)^2+5x^2+15xy+10y^2-2y+1=0.$$

Here $2y+x$ is a repeated linear factor of highest degree terms, *i.e.*, 3rd degree. There will, therefore, be no asymptote parallel to $2y+x=0$ if $(2y+x)$ is not a factor of the 2nd degree terms also. But this is not the case. In fact, the equation is

$$x(2y+x)^2+5(x+y)(x+2y)-2y+1=0.$$

Therefore, the curve has two asymptotes parallel to

$$2y+x=0.$$

We have now to find $\lim (y+\frac{1}{2}x)$ when $x \to \infty$ and $y/x \to -\frac{1}{2}$. Let $\lim (y+\frac{1}{2}x)=c$ so that $\lim (2y+x)=2c$.

Dividing by x, the equation becomes

$$(2y+x)^2+5(2y+x)(1+y/x)-2y/x+1/x=0.$$

In the limit,

$$4c^2+5.2c(1-\tfrac{1}{2})-2(-\tfrac{1}{2})+0=0, \qquad \ldots(i)$$

or $$4c^2+5c+1=0$$

$\therefore$ $$c=-\tfrac{1}{4}, -1.$$

Hence $$y=-\tfrac{1}{2}x-\tfrac{1}{4} \text{ and } y=-\tfrac{1}{2}x-1,$$

i.e., $$4y+2x+1=0 \text{ and } 2y+x+2=0,$$

are the two more asymptotes.

It is easy to see that we could have eliminated the step (*i*) and simplified the process by saying that the asymptotes are

$$(2y+x)^2+5(2y+x).\lim (1+y/x)+\lim (-2y/x+1/x)=0,$$

i.e.,

$$(2y+x)^2+5(2y+x).\tfrac{1}{2}+1=0 \text{ or } 2(2y+x)^2+5(2y+x)+2=0,$$

which gives

$$2y+x+2=0 \text{ and } 4y+2x+1=0.$$

3. *Find the asymptotes of*

$$(x-y)^2(x^2+y^2)-10(x-y)x^2+12y^2+2x+y=0. \qquad (P.U.)$$

The asymptotes parallel to the two imaginary lines $x^2+y^2=0$ are imaginary. To obtain the two asymptotes parallel to the lines $x-y=0$, we re-write the equation, on dividing it by (x^2+y^2), as

$$(x-y)^2-10\,(x-y)\,\frac{1}{1+(y/x)^2}+\frac{12(y/x)^2+2/x+y/x\;1/x}{1+(y/x)^2}=0.$$

We take the limits when $x \to \infty$ and $y/x \to 1$. Therefore the asymptotes are

$$(x-y)^2-10(x-y)\lim\frac{1}{1+(y/x)^2}+\lim\frac{12(y/x)^2+2/x+y/x.1/x}{1+(y/x)^2}=0.$$

i.e., $\qquad (x-y)^2-5(x-y)+6=0,$

i.e., $\qquad x-y-2=0,\ x-y-3=0.$

4. *Find the asymptotes of*

$$(x-y+2)(2x-3y+4)(4x-5y+6)+5x-6y+7=0.$$

The asymptote parallel to $x-y+2=0$ is

$$x-y+2+\lim\frac{5x-6y+7}{(2x-3y+4)(4x-5y+6)}=0,$$

when $x \to \infty$ and $y/x \to 1$,

i.e., $$x-y+2+\lim\left[\frac{5-6y/x+7/x}{(2-3y/x+4/x)(4-5y/x+6/x)}\cdot\frac{1}{x}\right]=0.$$

or $$x-y+2=0,$$

as the limit is zero because of the factor $1/x$.

Similarly, we can show that

$$2x-3y+4=0,\ 4x-5y+6=0$$

are also the asymptotes of the curve.

16·4. Asymptotes by Inspection. *If the equation of a curve of the nth degree can be put in the form*

$$F_n+F_{n-2}=0,$$

where F_{n-2} is of degree $(n-2)$ at the most, then every linear factor of F_n, when equated to zero will give an asymptote, provided that no straight line obtained by equating to zero any other linear factor of F_n is parallel to it or coincident with it.

Let $ax+by+c=0$ be a non-repeated factor of F_n. We write

$$F_n=(ax+by+c)F_{n-1},$$

where F_{n-1} is of degree $(n-1)$. The asymptote parallel to $ax+by+c=0$ is

$$ax+by+c+\lim \frac{F_{n-2}}{F_{n-1}}=0,$$

when $x \to \infty$ and $y/x \to -a/b$.

To determine the limit (F_{n-2}/F_{n-1}), we divide the numerator as well as the denominator by x^{n-1} and see that $1/x$ appears as a factor so that $F_{n-2}/F_{n-1} \to 0$ as $x \to \infty$.

Thus

$$ax+by+c=0,$$

is an asymptote.

Exercises

Find the asymptotes of the following curves :—

1. $xy(x+y)=a(x^2-a^2)$.
2. $(x-1)(x-2)(x+y)+x^2+x+1=0$.
3. $y^3-x^3+y^2+x^2+y-x+1=0$.
4. $x(y^2-3by+2b^2)=y^3-3bx^2+b^3$.
5. $x^3+6x^2y+11xy^2+6y^3+3x^2+12xy+11y^2+2x+3y+5=0$. (P.U.)
6. $x^2(3y+x)^2+(3y+x)(x^2+y^2)+9y^2+6xy+9y-6x+9=0$.
7. $(y^2+xy-2x^2)^2+(y^2+xy-2x^2)(2y+x)-7y^2-19xy-28x^2+x+2y+3=0$.
8. $x(y-3)^3=4y(x-1)^3$.
9. $(a+x)^2(b^2+x^2)=x^2y^2$.
10. $y^3-5xy^2+8x^2y-4x^3-3y^2+9xy-6x^2+2y-2x+1=0$.

16·5. Intersection of a curve and its asymptotes.

Any asymptote of a curve of the nth degree cuts the curve in $(n-2)$ points.

Let $y=mx+c$ be an asymptote of the curve

$$x^n\phi_n(y/x)+x^{n-1}\phi_{n-1}(y/x)+x^{n-2}\phi_{n-2}(y/x)+\ldots\ldots=0.$$

To find the points of intersection, we have to solve the two equations simultaneously.

The abscissae of the points of intersection are the roots of the equation

$$x^n\phi_n(m+c/x)+x^{n-1}\phi_{n-1}(m+c/x)+x^{n-2}\phi_{n-2}(m+c/x)+\ldots\ldots=0. \quad \ldots(i)$$

Expanding each term by Taylor's theorem and arranging according to descending powers of x, we get

$$x^n\phi_n(m)+[c\phi'_n(m)+\phi_{n-1}(m)]x^{n-1}+$$
$$[\tfrac{1}{2}c^2\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)]x^{n-2}+\ldots\ldots=0. \quad \ldots(ii)$$

As $y=mx+c$ is an asymptote, the co-efficients of x^n and x^{n-1} are both zero.

Thus the equation (*ii*) reduces to that of $(n-2)$th degree and, therefore, determines $(n-2)$ values of x. Hence the result.

Cor. 1. The, n, asymptotes of a curve of the nth degree cut it in $n(n-2)$ points.

Cor. 2. *If the equation of a curve of the nth degree can be put in the form $F_n+F_{n-2}=0$ where F_{n-2} is of degree $(n-2)$ at the most and F_n consists of, n, non-repeated linear factors, then the $n(n-2)$ points of intersection of the curve and its asymptotes lie on the curve*

$$F_{n-2}=0.$$

The result follows at once from the fact that $F_n=0$ is the joint equation of the, n, asymptotes. At the points of intersection of the curve and its asymptotes, the two equations $F_n=0$ and $F_n+F_{n-2}=0$ hold simultaneously and therefore at such points we have $F_{n-2}=0$.

Particular cases

(*i*) *For a cubic*, $n=3$, and therefore the asymptotes cut the curve in $3(3-2)=3$ points which lie on a curve of degree $3-2=1$, *i.e.*, on a straight line.

(*ii*) *For a quartic*, $n=4$, and, therefore the asymptotes cut the curve in $4(4-2)=8$ points which lie on a curve of degree $4-2=2$ *i.e.*, on a conic.

Examples

1. *Find the asymptotes of the curve*

$$x^2y-xy^2+xy+y^2+x-y=0 \qquad (P.U.\ 1955)$$

and show that they cut the curve again in three points which lie on the line

$$x+y=0. \qquad (P.U.\ 1940)$$

The asymptotes of the given curve, as may be easily shown are

$$y=0,\ x=1,\ x-y+2=0.$$

The joint equation of the asymptotes is

$$y(x-1)(x-y+2)=0,$$

i.e.,
$$x^2y-xy^2+xy+y^2-2y=0.$$

The equation of the curve can be written as

$$x^2y-xy^2+xy+y^2-2y+(x+y)=0.$$

Here

$$F_3=x^2y-xy^2+xy+y^2-2y,\ F_1=x+y.$$

Hence the points of intersection lie on the line

$$F_1\equiv x+y=0.$$

2. *Show that the asymptotes of the quartic*

$$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3+xy+7y^2-1=0,$$

cut the curve in the eight points which lie on a circle.

The asymptotes of the curve are

$$x+2y=0,\ x-2y+1=0,\ x-3y=0,\ x+3y-1=0,$$

so that their joint equations is

$$(x+2y)(x-2y+1)(x-3y)(x+3y-1)=0,$$

i.e., $$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3-x^2+xy+6y^2=0.$$

The equation of the curve can be written as

$$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3-x^2+xy+6y^2+(x^2+y^2-1)=0.$$

Hence the points of intersection lie on the circle

$$x^2+y^2-1=0.$$

3, *Find the equation of the cubic which has the same asymptotes as the curve*

$$x^3-6x^2y+11xy^2-6y^3+x+y+1=0.$$

and which passes through the points (0, 0), (1, 0) and (0, 1).
(Delhi Hons. 1948)

We write

$$F_3 \equiv x^3-6x^2y+11xy^2-6y^3$$
$$=(x-y)(x-2y)(x-3y).$$
$$F_1 \equiv x+y+1.$$

The equation of the curve can be written in the form $F_3+F_1=0$ where F_3 has non-repeated linear factors. Thus $F_3=0$ is the joint equation of the asymptotes of the cubic.

The general equation of the cubic is of the form

$$F_3+ax+by+c=0,$$

or

$$x^3-6x^2y+11xy^2-6y^3+ax+by+c=0,$$

where $ax+by+c$ is the general linear expression.

In order that it may pass through the points (0, 0), (1, 0) and (0, 1), we must have

$$c=0,$$
$$1+a=0 \text{ or } a=-1,$$
$$-6+b=0 \text{ or } b=6.$$

Thus the required cubic is

$$x^3-6x^2y+11xy^2-6y^3-x+6y=0.$$

Exercises

1. Show that the asymptotes of the cubic

$$x^3-xy^2-2xy+2x-y=0,$$

cut the curve again in points which lie on the line

$$3x-y=0.$$

2. If a right line is drawn through the point $(a, 0)$ parallel to the asymptote of the cubic $(x-a)^3-x^2y=0$, prove that the portion of the line intercepted by the axes is bisected by the curve. (C.U.)

3. Through any point P on the hyperbola $x^2-y^2=2ax$, a straight line is drawn parallel to the only asymptote of the curve $x^3+y^3=3ax^2$ meeting the curve in A and B; show that P is the mid-point of AB.

4. Show that $y=x+a$ is the only asymptote of the curve

$$x^2(x-y)+ay^2=0.$$

A straight line parallel to the asymptote meets the curve in P, Q; show that the mid-point of PQ lies on the hyperbola

$$x(x-y)+ay=0.$$

5. Find the equation of the straight line on which lie three points of intersection of the cubic

$$x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0$$

and its asymptotes.

6. Find the asymptotes of the curve

$$4x^4-13x^2y^2+9y^4+32x^2y-42y^3-20x^2+74y^2-56y+4x+16=0,$$

and show that they pass through the intersection of the curve with $y^2+4x=0$. (*D.U. Hons. 1953*)

7. Find all the asymptotes of the curve

$$3x^3+2x^2y-7xy^2+2y^3-14xy+7y^2+4x+5y=0.$$

Show that the asymptotes meet the curve again in three points which lie on a straight line, and find the equation of this line. (*D.U. Hons. 1952*)

8. Find the equation of the cubic which has the same asymptotes as the curve

$$x^3-6x^2y+11xy^2-6y^3+x+y+1=0,$$

and which touches the axis of y at the origin and passes through the point $(3, 2)$. (*Delhi Hons. 1949, 1955*)

9. Find the asymptotes of the curve

$$4(x^4+y^4)-17x^2y^2-4x(4y^2-x^2)+2(x^2-2)=0,$$

and show that they pass through the points of intersection of the curve with the ellipse $x^2+4y^2=4$. (*Delhi Hons. 1951, 1959*)

10. Find the asymptotes of the curve

$$(2x-3y+1)^2(x+y)-8x+2y-9=0$$

and show that they intersect the curve again in three points, which lie on a straight line. Obtain the equation of the line. (*D.U. Hons. 1957*)

16·6. Asymptotes by expansion. *To show that*

$$y=mx+c$$

is an asympotate of the curve

$$y=mx+c+A/x+B/x^2+C/x^3+\ldots\ldots \quad \ldots(1)$$

Dividing by x, we have

$$y/x = m + c/x + A/x^2 + B/x^3 + C/x^4 + \ldots$$

so that when $x \to \infty$,

$$\lim (y/x) = m. \qquad \text{.. (2)}$$

Also from (1) we have

$$(y - mx) = c + A/x + B/x^2 + C/x^3 + \ldots\ldots$$

so that when $x \to \infty$,

$$\lim (y - mx) = c. \qquad \text{...(3)}$$

From (2) and (3), we deduce that

$$y = mx + c$$

is an asymptote of the curve (1).

16·7. Position of a curve with respect to an asymptote. *To find the position of the curve*

$$y = mx + c + A/x + B/x^2 + C/x^3 + \ldots$$

with respect to its asymptote

$$y = mx + c.$$

Let $A \neq 0$. Let y_1 and y_2 denote the ordinates of the curve and the asymptote corresponding to the same abscissa x. We have

$$\begin{aligned} y_1 - y_2 &= A/x + B/x^2 + C/x^3 + \ldots. \\ &= (1/x)(A + B/x + C/x^2 + \ldots.) \end{aligned} \qquad \text{..(1)}$$

By taking x sufficiently large, we can make

$$B/x + C/x^2 + \ldots.$$

as small as we like. We suppose that x is so large numerically that this expression is numerically less than A. Thus, for sufficiently large values of x, the expression

$$A + B/x + C/x^2 + D/x^3 + \ldots. \qquad \text{..(2)}$$

has the sign of A.

Thus if A be positive, the expression (1) is positive for sufficiently large values of x so that, from (1) we deduce that when x is *positively* sufficiently large, then $y_1 - y_2$ is positive, *i.e.*, the curve lies above the asymptote and when x is *negative* but numerically large, $y_1 - y_2$ is negative, *i.e.*, the curve lies below the asymptote.

Similarly, we may deduce that if A be negative, then the curve lies below the asymptote when, x, is positive but sufficiently large and lies above the asymptote when, x, is negative but sufficiently large numerically.

Let $A = 0$, $B \neq 0$; we have

$$y_1 - y_2 = (1/x^2)(B + C/x + D/x^2 + \ldots.).$$

As above we can show that for numerically sufficiently large values of x, the expression

$$B + C/x + D/x^2 + \ldots.$$

has the sign of B.

In this case, the curve lies on the same side of the asympotate both for positive and negative values of x ; it will be above or below the asymptote according as B is positive or negative.

If $B=0$ and $C\neq 0$, we will have a situation similar to that of case (1).

Ex. *Find the asympotates of the curve*

$$y^2=x(x-a)(x-2a)/(x+3a),$$

and determine on which side of the asymptotes the curve lies.

We have

$$y=\pm\sqrt{\left[\frac{x(x-a)(x-2a)}{x+3a}\right]}$$

$$=\pm x\left(1-\frac{a}{x}\right)^{\frac{1}{2}}\left(1-\frac{2a}{x}\right)^{\frac{1}{2}}\left(1-\frac{3a}{x}\right)^{-\frac{1}{2}}$$

$$=\pm\left(1-\frac{a}{2x}-\frac{a^2}{8x^2}\cdots\right)\left(1-\frac{a}{x}-\frac{a^2}{2x^2}\cdots\right)\left(1-\frac{3a}{2x}+\frac{27a^2}{8x^2}\cdots\right)$$

$$=\pm x\left(1-\frac{3a}{x}+\frac{11a^2}{2x^2}\cdots\right)$$

Thus we have two values of y, *viz.*,

$$y=x-3a+\tfrac{11}{2}a^2x\ldots\ldots$$

$$y=-x+3a-\tfrac{11}{2}a^2x\ldots.$$

Therefore

$$y=x-3a,\ y=-x+3a$$

are two asymptotes.

The difference between the ordinate of the curve and that of the asymptote $y=x-3a$ being

$$\tfrac{11}{2}a^2x-\ldots\ldots,$$

we see that the curve lies above the asymptote when x is positive and below it when x is negative.

It may similarly be seen that the curve lies below the second asymptote when x is positive and above it when x is negative.

It is easy to see that

$$x=-3a$$

is also an asymptote of the curve. To find the position of the curve relative to this asymptote, we suppose that

$$x=-3a+A/y+B/y^2+C/y^3+\ldots\ldots$$

Substituting this value of x in the equation of the curve, we have

$$y^2\left[\frac{A}{y}+\frac{B}{y^2}+\frac{C}{y^3}+\ldots\ldots\ldots\right]$$

$$=\left[-3a+\frac{A}{y}+\ldots\right]\left[-4a+\frac{A}{y}+\ldots\right]\left[-5a+\frac{A}{y}+\ldots\right]$$

Equating the co-efficients of like powers of y, we have

$$A=0.$$

$$B=-60a^3, \text{ etc.}$$

$$\therefore \qquad x=-3a-\frac{60a^3}{y^2}+\ldots\ldots$$

The difference between the abscissae of the curve and the asymptote $x=-3a$, for the same value of y, being

$$-60a^3/y^2+\ldots\ldots,$$

which is negative whether y be positive or negative, we see that the curve lies towards the negative side of x-axis.

Ex. 2. Find the asymptotes and their position with regard to the following curves :—

(*i*) $x^3+y^3=3ax^2$. (*ii*) $x^3+y^3=3axy$. (*iii*) $x^2(x-y)+y^2=0$.

16·8. Asymptotes in polar co-ordinates.

Lamma. The Polar Equation of a Line. *The polar equation of any line is*

$$p=r\cos(\theta-\alpha),$$

where, p, is the length of the perpendicular from the pole to the line and α, is the angle which this perpendicular makes with the initial line.

Let OY be the perpendicular on the given line ; Y being its foot.

We are given that

$$OY=p\,;\ \angle XOY=\alpha.$$

If $P(r, \theta)$ be any point on the line, we have

$$\angle YOP=\theta-\alpha.$$

Now, $\dfrac{OY}{OP}=\cos\angle YOP.$

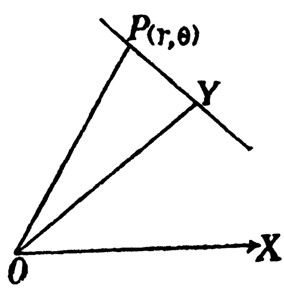


Fig. 125.

$\therefore \qquad p/r=\cos(\theta-\alpha)$, *i.e.*, $p=r\cos(\theta-\alpha)$,

which is the required equation of the line.

To determine the asymptotes of the curve

$$r=f(\theta), \qquad ..(i)$$

we have to obtain the constants, p, and, α, so that any line

$$p=r\cos(\theta-\alpha), \qquad ..(ii)$$

is the asymptote of the given curve.

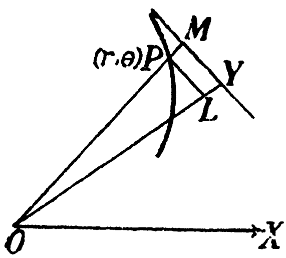


Fig. 126.

Let $P(r, \theta)$ be any point on the curve (*i*).

Draw $OY \perp$ the line (*ii*).
Draw $PL \perp OY$ and $PM \perp$ the line (*ii*).

Now,

$$\begin{aligned} PM &= LY \\ &= OY - OL \\ &= p - OP \cos(\theta - \alpha) \\ &= p - r\cos(\theta - \alpha). \quad \ldots(iii) \end{aligned}$$

Now $r \to \infty$ as the point recedes to infinity along the curve. Let $\theta \to \theta_1$ when $r \to \infty$.

We have

$$\frac{PM}{r} = \frac{p}{r} - \cos(\theta - \alpha).$$

Now when $r \to \infty$, $PM \to 0$ so that

$$\frac{PM}{r} = PM \cdot \frac{1}{r} \to 0 \text{ and } \frac{p}{r} \to 0.$$

$\therefore$ $\lim \cos(\theta - \alpha) = 0$

or $\lim (\theta - \alpha) = \pi/2$,

or $\theta_1 - \alpha = \pi/2$, *i.e.*, $\alpha = \theta_1 - \pi/2$.

This gives α.

Again, $p = OY$ *is the polar sub-tangent of the point at which the asymptote touches the curve, i.e., the point at infinity on the curve.* This may be seen as follows :—

Join the pole O to the point at infinity on the curve *i.e.*, draw through O a line parallel to the asymptote. This line is the radius vector of the point at ∞.

Draw through O a line perpendicular to the asymptote meeting it at Y. Then, by def. OY is the polar sub-tangent of the point at infinity on the curve. (§ 12·7, p. 271).

Thus

$$p = -\left[\frac{d\theta}{du}\right]_{\theta = \theta_1}, \text{ where } u = \frac{1}{r}.$$

Note Without employing the notion of the polar sub-tangent and the point at infinity, the value of p, may also be obtained as follows :—

From (*iii*) we have, when $r \to \infty$,

$$\begin{aligned} p &= \lim [r \cos(\theta - \alpha)] \\ &= \lim [r \cos(\theta - \theta_1 + \pi/2)] \\ &= \lim [r \sin(\theta - \theta_1)] = \lim_{\theta_1 \to \theta} \frac{\sin(\theta_1 - \theta)}{1/r} \end{aligned}$$

which is of the form (0/0).

$$\therefore \qquad p = \lim_{\theta \to \theta_1} \frac{\cos(\theta_1 - \theta)}{\dfrac{1}{r^2}\dfrac{dr}{d\theta}}$$

$$= \lim_{\theta \to \theta_1} \left[r^2 \frac{d\theta}{dr} \right] = \lim \left[-\frac{d\theta}{du} \right] \text{ where } u = \frac{1}{r}$$

Hence the asymptote is

$$\lim\left(-\frac{d\theta}{du} \right) = r \cos(\theta - \alpha)$$

$$= r \cos\left(\theta - \theta_1 + \frac{\pi}{2} \right)$$

$$= r \sin(\theta_1 - \theta).$$

where θ_1 is the limit of θ as $r \to \infty$ *i.e.*, as $u \to 0$.

Working rule for obtaining asymptotes to polar curves.

Change r to $1/u$ in the given equation and find out the limit of θ as $u \to 0$.

Let θ_1, be any one of the several possible limits of θ.

Determine $(-d\theta/du)$ and its limit as $u \to \theta$ and $\theta \to \theta_1$.

Let this limit be p.

Then

$$p = r \sin(\theta_1 - \theta),$$

is the corresponding asymptote.

To draw the asymptote.

Through the pole O draw a line making angle $(\theta_1 - \frac{1}{2}\pi)$ with the initial line ; on this line take a point Y such that

$$OY = \lim(-d\theta/du).$$

The line drawn through Y perpendicular to OY is the required asymptote.

Examples

1. *Find the asymptote of the hyperbolic spiral $r\theta = a$.*

Here

$$\theta = a/r = au \text{ so that } \theta \to 0 \text{ as } u \to 0.$$

Here

$$\theta_1 = \lim \theta = 0.$$

Since $\qquad u = \theta/a,$

we have

$$du/d\theta = 1/a \text{ or } d\theta/du = a.$$

Therefore

$$-a = r \sin(0 - \theta) = -r \sin \theta,$$

i.e., $\qquad r \sin \theta = a,$

is the asymptote.

2. *Find the asymptote of the curve*

$$r=\frac{a}{\frac{1}{2}-\cos\theta}.$$

Here

$$u=\frac{1}{r}=\frac{1}{a}\left(\tfrac{1}{2}-\cos\theta\right).$$

When $u\to 0$, $(\frac{1}{2}-\cos\theta)\to 0$ so that $\cos\theta\to\frac{1}{2}$.

$$\therefore\qquad \theta_1=\pm\pi/3.$$

Now,

$$\frac{du}{d\theta}=\frac{1}{a}\sin\theta \text{ or } -\frac{d\theta}{du}=-\frac{a}{\sin\theta}.$$

$$\therefore\qquad -\frac{d\theta}{du}\to-\frac{2a}{\sqrt{3}} \text{ as } \theta\to\frac{\pi}{3},$$

and

$$\frac{-d\theta}{du}\to\frac{2a}{\sqrt{3}} \text{ as } \theta\to-\frac{\pi}{3}.$$

$$\therefore\quad -\frac{2a}{\sqrt{3}}=r\sin\left(\frac{\pi}{3}-\theta\right),\ i.e.,\ 4a=r(\sqrt{3}\sin\theta-3\cos\theta),$$

and $$\frac{2a}{\sqrt{3}}=r\sin\left(-\frac{\pi}{3}-\theta\right),\ i.e.,\ -4a=r(\sqrt{3}\sin\theta+3\cos\theta),$$

are the two asymptotes.

Exercises

Find the asymptotes of the following curves :—

1. $r=\dfrac{a\theta}{\theta-1}$.
2. $r=\dfrac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}$.
3. $r=a\sec\theta+b\tan\theta$. (*P.U.*)
4. $r^2=a^2(\sec^2\theta+\operatorname{cosec}^2\theta)$.
5. $r\sin 2\theta=a\cos 3\theta$.
6. $2r^2=\tan 2\theta$.
7. $r\theta\cos\theta=a\cos 2\theta$. (*P.U.*)
8. $r\sin n\theta=a$. (*P.U.*)
9. $r^n\sin n\theta=a^n$.
10. $r=a\tan\theta$.
11. $r=a\log\theta$.
12. $r\log\theta=a$.
13. $r(1-e^{\theta})=a$.
14. $r(\theta^2-\pi^2)=2a\theta$.
15. $r\sin\theta=ae^{\theta}$.
16. $r(\pi+\theta)=ae^{\theta}$.

17. Find the equation of the asymptotes of the curve given by the equation

$$r^nf_n(\theta)+r^{n-1}f_{n-1}(\theta)+\ldots\ldots+f_0(\theta)=0$$

(*P.U. Hons., 1938*)

18. Show that all the asymptotes of the curve

$$r\tan n\theta=a,$$

touch the circle

$$r=a/n.$$

19. Find the asymptotes of the curve

$$r\cos 2\theta=a\sin 3\theta.$$

(*Delhi Hons., 1948*)

CHAPTER XVII

SINGULAR POINTS

MULTIPLE POINTS, DOUBLE POINTS

17·1. Introduction. Cusps, Nodes and Conjugate points. The cases of curves considered in § 11·4, p. 243 show that curves with implicit equations of the form $f(x, y)=0$ exhibit some peculiarities which are not possessed by the curves with explicit equations of the form $y=F(x)$. These peculiarities arise from the fact that the equation $f(x, y)=0$ may not define, y, as a single valued function of x. In fact, to each value of x corresponds as many values of y as is the degree of the equation in y, and these different values of y give rise to different branches of the curve.

We recall to ourselves the following three curves considered in § 11·4.

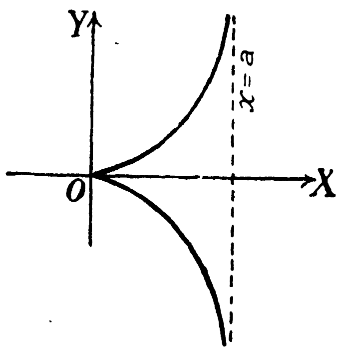


Fig. 127.

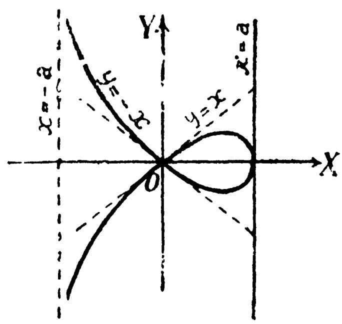


Fig. 128.

(*i*) Origin is a point common to the two branches of the Cissoid (Fig. 127)

$$y^2(a-x)=x^3, \qquad (\S\ 11{\cdot}41)$$

and the two branches have a *common* tangent there.

Such a point on a curve is called a **cusp**.

(*ii*) Origin is a point common to the two branches of the Strophoid (Fig. 128)

$$(x^2+y^2)x-a(x^2-y^2)=0, \qquad (\S\ 11{\cdot}42)$$

and the two branches have *different* tangents there.

Such a point on a curve is called a **node**.

(*iii*) $(-a, 0)$ is a point common to the two branches of the curve (Fig. 129)

$$ay^2-x(x+a)^2=0 \qquad (\S\ 11{\cdot}43)$$

and the two branches have *imaginary* tangents there. There is no point in the immediate neighbourhood of the point $(-a, 0)$ which lies on the curve. Here, a, is positive.

Such a point on a curve is called an **isolated** or **conjugate** point.

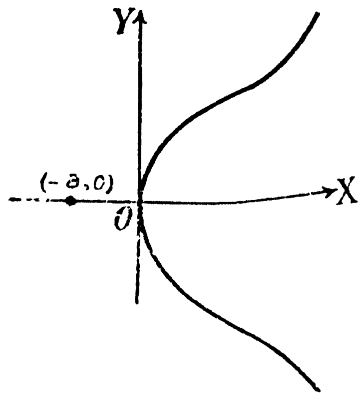


Fig. 129.

17·2. Definitions.

Double points. Cusp. Node. Conjugate point. *A point through which there pass* **two** *branches of a curve is called a* **double point.**

A curve has two tangents at a double point, one for each branch.

The double point will be a **node**, *a* **cusp** *or an* **isolated** *point according as the two tangents are different and real, coincident or imaginary.*

Multiple point. *A point through which there pass, r, branches of a curve is called a* **multiple point of the rth order** so that a curve has, r, tangents at a multiple point of the rth order.

Thus a double point is a multiple point of the second order. A multiple point of the third order is also called a *triple* point. A multiple point is also, sometimes, called a **singular point.**

17·3. A simple rule for writing down the *tangent or tangents at the origin* to rational algebraic curves is obtained in the following article.

17·31. Tangents at the origin. The general equation of *rational algebraic* curve of the nth degree which passes through the origin O, when arranged according to ascending powers of x and y, is of the form

$$(b_1x+b_2y)+(c_1x^2+c_2xy+c_3y^2)+(d_1x^3+d_2x^2y+..)+...=0, \quad ...(i)$$

where the constant term is absent.

Let $P(x, y)$ be any point on the curve. The slope of the chord OP is y/x. Limiting position of the chord OP, when $P \to O$, is the tangent at O so that when $x \to 0$ and $y \to 0$,

$$\lim (y/x)=m,$$

is the slope of this tangent.

From (i), we have, after dividing by x,

$$\left(b_1+b_2\frac{y}{x}\right)+\left(c_1x+c_2y+c_3y.\frac{y}{x}\right)+(d_1x^2+d_2xy+..)+...= .$$

On taking limits, when $x \to 0$, we get

$$b_1+b_2m=0 \text{ so that } m=-b_1/b_2, \text{ if } b_2\neq 0.$$

Hence

$$y/x=-b_1/b_2,$$

i.e.,

$$b_1x+b_2y=0, \quad ...(ii)$$

is the tangent at the origin. This may be written down by equating to zero the lowest degree (first degree) terms in the equation (i).

If $b_2=0$ but $b_1\neq 0$, then, considering the slope of OP with reference to Y-axis, it can be shown that the tangent retains the same form.

Let $b_1=b_2=0$ so that the equation takes the form

$$(c_1x^2+c_2xy+c_3y^2)+(d_1x^3+d_2x^2y+......)+...=0. \quad ...(iii)$$

Dividing by x^2 and then taking limits as $x \to 0$, we get

$$c_1+c_2m+c_3m^2=0, \quad ...(iv)$$

which is a quadratic equation in, m, and determines as its *two* roots the slopes of the *two tangents* so that the origin is a double point in this case.

The equation of either tangent at the origin is

$$y=mx, \quad ..(v)$$

when m is a root of (iv). Eliminating m between (iv) and (v), we obtain

$$c_1x^2+c_2xy+c_3y^2=0, \quad ...(vi)$$

as the joint equation of the two tangents at the origin. This can be written down by equating to zero the lowest degree terms in (iii).

The equation (vi) becomes an identity if $c_1=c_2=c_3=0$. In this case the second degree terms, also, do not appear in the equation of the curve. It can now be similarly shown that the equation of the tangents can still be written down by equating to zero the terms of the lowest degree which is third in this case.

In general, we see *that the equation of the tangent or tangents at the origin is obtained by equating to zero the terms of the lowest degree in the equation of the curve.*

The origin will be a multiple point on a curve whose equation does not, at least, contain the constant and the first degree terms.

Illustrations.

(*i*) The origin is a node on the curve

$$x^3+y^3-3axy=0,$$

and $x=0$, $y=0$ are the two tangents thereat.

(*ii*) The origin is a cusp on the curve

$$(x^2+y^2)x-2ay^2=0,$$

and $y=0$ is the cuspidal tangent.

(*iii*) The origin is an isolated point on the curve

$$a^2x^2+b^2y^2=(x^2+y^2)^2,$$

and $ax \pm iby=0$ are the two imaginary tangents thereat.

(*iv*) The origin is a triple point on the curve

$$2y^5+5x^5-3x(x^2-y^2)=0,$$

and $x=0$, $x=y$, $x=-y$ are the three tangents thereat.

Exercises

Find the tangents at the origin to the following curves :—

1. $(x^2+y^2)^2=4a^2xy$.
2. $y^2(a^2-x^2)=x^2(b-x)^2$.
3. $(x^2+y^2)(2a-x)=b^2x$.
4. $a^2(x^2-y^2)=x^2y^2$.
5. $(x^2+y^2)^3=a^2(x^2-y^2)^2$.

Example

Find the equation of the tangent at $(-1, -2)$ *to the curve*

$$x^2+2x^2+2xy-y^2+5x-2y=0,$$

and show that this point is a cusp.

We will shift the origin to the point $(-1, -2)$. To do so we have to write

$$x=X-1, y=Y-2,$$

where X, Y are the current co-ordinates of a point on the curve with reference to the new-axes. The transformed eqation is

$$(X-1)^3+2(X-1)^2+2(X-1)(Y-2)-(Y-2)^2+5(X-1)-2(Y-2)=0,$$

or

$$X^3-X^2+2XY-Y^2=0.$$

Equating to zero the lowest degree terms, we get

$$-X^2+2XY-Y^2=0, \text{ i.e., } (Y-X)^2=0,$$

which are two coincident lines, and, therefore, the point is a cusp and the cuspidal tangent, *i.e.*, the tangent at the cusp with reference to the new axes is

$$Y-X=0.$$

To find the equation of the cuspidal tangent with reference to the given system of axes, we write

$$X=x+1, \quad Y=y+2.$$

Hence the tangent at $(-1, -2)$ is

$$(y+2)-(x+1)=0, \text{ i.e., } y=x-1.$$

Exercises

Find the equations of the tangents to the following curves :—

1. $y^2(a^2+x^2)=x^2(a^2-x^2)$ at $(\pm a, 0)$.
2. $(x-2)^2=y(y-1)^2$ at $(2, 1)$.
3. $x^4-4ax^3-2ay^3+4a^2x^2+3a^2y^2-a^4=0$ at $(a, 0)$ and $(2a, a)$.
4. Show that the origin is a node ; a cusp or a conjugate point on the curve

$$y^2=ax^2+ax^3,$$

according as, a, is positive, zero or negative. *(Delhi Hons. 1950)*

17·4. Conditions for any point (x, y) to be a multiple point of the curve

$$f(x, y)=0.$$

In § 10·94, p. 213, we have seen that at a point (x, y) of the curve

$$f(x, y)=0,$$

the slope of the tangent, dy/dx is given by the equation

$$f_x+f_y\frac{dy}{dx}=0. \quad \text{..(1)}$$

At a multiple point of a curve, the curve has at least two tangents and accordingly *dy/dx must have at least two values at a multiple point.*

The equation (1), being of the first degree in dy/dx, can be satisfied by more than one value of dy/dx, if, and only if,

$$f_x=0, f_y=0.$$

Thus we see that *the necessary and sufficient conditions for any point (x, y) on $f(x, y)=0$ to be a multiple point are that*

$$f_x(x, y)=0, f_y(x, y)=0.$$

To find multiple points (x, y), we have therefore to find the values of (x, y) which simultaneously satisfy the three equations

$$f_x(x, y)=0, f_y(x, y)=0, f(x, y)=0.$$

17·41. To find the slopes of the tangents at a double point.

Differentiating (1) *w.r.* to x, we have

$$f_x{}^2+f_{xy}\frac{dy}{dx}+\left(f_{yx}+f_y{}^2\frac{dy}{dx}\right)\frac{dy}{dx}+f_y\frac{d^2y}{dx^2}=0,$$

so that at the multiple point, where $f_y=0, f_x=0$, the values of dy/dx are the roots of the quadratic equation

$$f_y{}^2\left(\frac{dy}{dx}\right)^2+2f_{xy}\frac{dy}{dx}+f_x{}^2=0. \qquad \ldots(2)$$

In case $f_x{}^2, f_{xy}, f_y{}^2$, are not all zero and $f_x=0=f_y$, the point (x, y) will be a double point and will be a node, cusp or conjugate according as the values of dy/dx are real and distinct, equal or imaginary i.e., according as

$$(f_{xy})^2-f_x{}^2f_y{}^2>0, =0, <0.$$

If $f_x{}^2=f_{xy}=f_y{}^2=0$; the point (x, y) will be multiple point of order higher than the second.

Example

Find the multiple points on the curve

$$x^4-2ay^3-3a^2y^2-2a^2x^2+a^4=0.$$

Also, find the tangents at the multiple point.

Let $\quad f(x, y)=x^4-2ay^3-3a^2y^2-2a^2x^2+a^4.$

$\therefore \quad f_x(x, y)=4x^3-4a^2x.$

$f_y(x, y)=-6ay^2-6a^2y.$

$f_x(x, y)=0$ gives $x=0, a, -a.$

$f_y(x, y)=0$ gives $y=0, -a.$

Hence the two partial derivatives vanish for the points

$$(0, 0)\ (0,-a), (a, 0), (a,-a), (-a, 0), (-a,-a).$$

Of these the only points on the curve are

$$(a, 0), (-a, 0), (0,-a).$$

Hence these are the only three multiple points, on the curve.

To find the tangents at the multiple points, we proceed as follows :—

First method.

We have

$$f_x{}^2=12x^2-4a^2, f_{xy}=0, f_y{}^2=-12ay-6a^2.$$

Since at $(a, 0)$.

$$f_x{}^2=8a^2, f_{xy}=0, f_y{}^2=-6a^2,$$

therefore, by the equation (2), the values of dy/dx at $(a, 0)$ are given by

$$-6a^2(dy/dx)^2+8a^2=0,$$

i.e., $$dy/dx = \pm 2/\sqrt{3}.$$

The two values being both real, the point $(a, 0)$ is a node. The tangents at $(a, 0)$ are

$$y = \pm(2/\sqrt{3})(x-a).$$

It may similarly be shown that the tangents at $(-a, 0)$ and $(0, -a)$ are

$$y = \pm\sqrt{\tfrac{4}{3}}(x+a),\ y+a = \pm\sqrt{\tfrac{2}{3}}x.$$

Second method. Differentiating the given equation *w.r.* to x, we get,

$$4x^3 - 6ay^2y_1 - 6a^2yy_1 - 4a^2x = 0,$$

which identically vanishes for the multiple points.

Differentiating again, we get

$$12x^2 - 12ayy_1^2 - 6ay^2y_2 - 6a^2y^2_1 - 6a^2yy_2 - 4a^2 = 0.$$

From this we see that

(*i*) for $(a, 0)$, $y_1^2 = \frac{3}{4}$, *i.e.*, $y_1 = \pm\sqrt{\frac{3}{4}}$.

(*ii*) for $(-a, 0)$, $y_1^2 = \frac{3}{4}$, *i.e.*, $y_1 = \pm\sqrt{\frac{3}{4}}$.

(*iii*) for $(0, -a)$, $y_1^2 = \frac{2}{3}$, *i.e.*, $y_1 = \pm\sqrt{\frac{2}{3}}$.

Knowing the slopes of the tangents, we can now put down their equations.

Third method. To find the tangents at $(a, 0)$, we shift the origin to this point. The transformed equation is

$$(x = X+a,\ y = Y+0)$$

$$X+a)^4 - 2aY^3 - 3a^2Y^2 - 2a^2(X+a)^2 + a^4 = 0,$$

or

$$X^4 + 4X^3a - 2aY^3 + 4a^2X^2 - 3a^2Y^2 = 0.$$

The tangents at the new origin are

$$4a^2X^2 - 3a^2Y^2 = 0,$$

or

$$Y = \pm\sqrt{(4/3)}X.$$

The tangents at the multiple point $(a, 0)$, therefore, are

$$y = \pm\sqrt{(4/3)}(x-a).$$

It may similarly be shown that

$$y = \pm\sqrt{(4/3)}(x+a) \text{ and } y+a = \pm\sqrt{(2/3)}\ x,$$

are the tangents at the multiple points $(-a, 0)$ and $(0, -a)$ respectively.

The three mutiple points on the curve are nodes.

Exercises

Find the position and nature of the multiple points on the following curves :—

1. $x^2(x-y)+y^2=0.$ *(D. U. 1951)*
2. $y^3=x^3+ax^2.$ *(D. U. 1950 ; P. U.)*
3. $x^4+y^3-2x^3+3y^2=0.$ *(P. U.)*

4. $xy^2-ax^2+2a^2x-a^3=0.$

5. $y^2=(x-1)(x-2)^2.$

6. $ay^2=(x-a)^2(x-b)^2.$

7. $x^4-4ax^3+2ay^3+4a^2x^2-3a^2y^2-a^4=0.$

8. $x^3+y^3-12x-27y+70=0.$

9. $x^4+4ax^3+4a^2x^2-b^2y^2-2b^3y-a^4-b^4=0.$

10. $x^4+y(y+4a)^3+2x^2(y-5a)^2=5a^2x^2.$ (*B. U.*)

11. $x^3+2x^2+2xy-y^2+5x-2y=0.$ (*P. U.*)

12. $(2y+x+1)^2-4(1-x)^5=0.$ (*P. U.*)

13. $(x+y)^3-\sqrt{2}(y-x+2)^2=0.$ (*P. U.*)

14. $(y^2-a^2)^3+x^4(2x+3a)^2=0.$ (*P. U.*)

15. $x^2y^2=(a+y)^2(b^2-y^2)$; distinguishing between the cases $b \lessgtr a.$

(*D. U. Hons. 1953*)

Find the equations of the tangents at the multiple points of the following curves :—

16. $x^4-4ax^3-2ay^3+4a^2x^2+3a^2y^2-a^4=0.$

17. $x^4-8x^3+12x^2y+16x^2+48xy+4y^2-64y=0.$

18. $(y-2)^2=x(x-1)^2.$

19. Show that each of the curves

$$(x\cos\alpha-y\sin\alpha-b)^3=c(x\sin\alpha+y\cos\alpha)^2,$$

for all different values of α, has a cusp : show also that all the cusps lie on a circle.

17·5. Types of cusps. We know that two branches of a curve have a common tangent at a cusp. There are *five* different ways in which the two branches stand in relation to the common tangent and the common normal as illustrated by the following figures :—

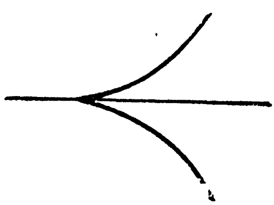


Fig. 130
Single cusp of 1st species

Fig. 131
Single cusp of 2nd species

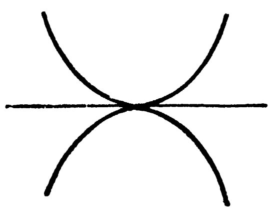


Fig. 132
Double cusp of 1st species

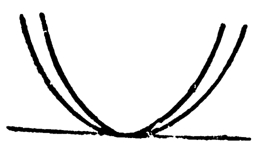


Fig. 133
Double cusp of 2nd species

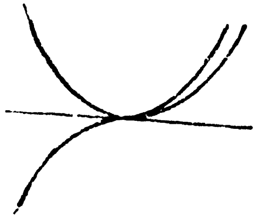


Fig.
Point of oscu-inflexion

In Fig. 130, the two branches lie on *the same side* of the common normal and on the *different sides* of the tangent.

In Fig. 131, the two branches lie on *the same side* of the normal and on *the same side* of the tangent.

In Fig. 132, the two branches lie on *the different sides* of the normal and on the *different sides* of the tangent.

In Fig. 133, the two branches lie on the *different sides* of the normal and on the *same side* of the tangent.

In Fig. 134, the two branches lie on the *different sides* of the normal but on one side they lie on the same, and on the other on opposite sides of the common tangent. One branch has inflexion at the point.

It will thus be seen that *the cusp is single or double according as the two branches lie on the same or different sides of the common normal. Also it is of the first or second species according as the branches lie on the different or the same side of the common tangent.*

Examples

1. *Find the nature of the cusps on the following curves :—*

(*i*) $y^2=x^3$. (*ii*) $y^2-x^4=0$. (*iii*) $(y-4x^2)^2=x^7$.

(*i*) $y=0$ is the cuspidal tangent. Since x cannot be negative, the two branches lie only on the same side of the common normal so that the cusp is single. See Fig. 130.

Again, $y=\pm x^{\frac{3}{2}}$ so that to each positive value of x correspond two values of y which are of opposite signs and hence the two branches lie on different sides of the common tangent and the cusp is of first species.

(*ii*) Two branches of $y^2-x^4=0$, are the two parabolas $y-x^2=0$ and $y+x^2=0$ which lie on different sides of the common tangent $y=0$ and extend to both sides of the common normal $x=0$. Thus the origin is a double cusp of first species. See Fig. 132.

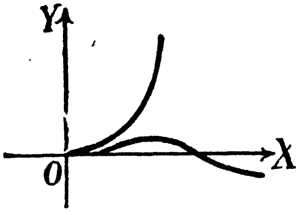


Fig. 135

(*iii*) Here

$$y=4x^2 \pm x^{\frac{7}{2}},$$

where the two signs correspond to the two branches ; $y=0$ is the cuspidal tangent.

Since x cannot take up negative values, the two branches lie only on the same side of the common normal and the cusp, therefore, is single.

Now, one value of y is always positive and, therefore, the corresponding branch lies above X-axis. Again

$$4x^2 > x^{\frac{7}{2}}, \text{ if } 4 > x^{\frac{3}{2}} \text{ i.e., if } 4^{\frac{2}{3}} > x.$$

Thus, for the values of x lying between 0 and $4^{\frac{2}{3}}$, the second value of y is also positive and, therefore, the corresponding branch lies above Y-axis in the vicinity of the origin. Thus the cusp is of the second species.

2. *Show that the curve*

$$y^2=2x^2y+x^3y+x^3,$$

has a single cusp of the first species at the origin. (*Delhi Hons. 1949*)

Equating to zero the lowest degree terms, we see that the origin is a cusp and $y=0$ is the cuspidal tangent.

We re-write the given equation in the form, quadratic in y,

$$y^2-yx^2(2+x)-x^3=0,$$

and solve it for y, so that

$$y=\frac{x^2(2+x) \pm \sqrt{[x^4(2+x)^2+4x^3]}}{2}$$

For positive values of x, we have

$$x^4(2+x)^2+4x^3 > x^4(2+x)^2,$$

or

$$\sqrt{[x^4(2+x)^2+4x^3]} > x^2(2+x),$$

so that to positive values of x correspond two values of y with opposite signs. Thus the two branches lie on opposite sides of x-axis when x is positive.

Again, we have

$$x^4(2+x)^2+4x^3=x^3(4+4x+4x^2+x^3).$$

For values of x which are sufficiently small in numerical value,

$$4+4x+4x^2+x^3$$

is positive, for the same $\rightarrow$ 4 when $x \rightarrow 0$.

Thus for negative values of x which are sufficiently small in numerical value,

$$x^3(4+4x+4x^2+x^3),$$

is negative so that the values of y are imaginary. Thus x cannot take up negative values.

Hence the curve has a single cusp of first species at the origin.

Exercises

Find the nature of the cusps on the following curves :—

1. $x^2(x-y)+y^2=0$.
2. $x^2(x+y)-y^2=0$.
3. $x^3+y^3-2ay^2=0$.
4. $a^4y^2=x^5(2a-x)$.
5. $(y-x)^2+x^6=0$.
6. $x^6-ayx^4-a^3x^2y+a^4y^2=0$.
7. $x^5-ax^3y-a^2x^2y+a^2y^2=0$.
8. Examine the curve

$$x^5+16x^2y-64y^2=0$$

for singularities. *(Delhi Hons. 1948)*

9. Prove that the curve $x^3+y^3=ax^2$ has a cusp of the first species at the origin and a point of inflexion where $x=a$. *(Lucknow Hons. 1950)*

10. Show that the curve

$$y^3=(x-a)^2(2x-a)$$

has a single cusp at $(a, 0)$. *(D.U. 1955)*

17·6. Radii of curvature at multiple points. The formula for the radius of curvature at any point (x, y) on the curve $f(x, y)=0$, as obtained in § 15·42, page 292, becomes meaningless at a multiple point where $f_x=f_y=0$. At a multiple point we expect as many values of, ρ, as its order. Of course, these values of ρ may not be all distinct.

The following examples will illustrate the method of determining the values of ρ at such points.

Examples

1. *Find the radii of curvature at the origin of the branches of the curve*

$$y^4+2axy^2=ax^3+x^4.$$

Here, $2xy^2=x^3$, *i.e.*, $x=0$, $y=\pm(1/\sqrt{2})x$ are the three tangents at the origin so that it is a *triple* point.

To find, ρ, for the branch which touches $x=0$, we find $\lim(y^2/2x)$. To do this, we write

$$y^2/2x=\rho_1, \text{ i.e., } x=y^2/2\rho_1,$$

and substitute this value of x in the given equation. Lim $\rho_1=\rho$ is the radius of curvature of the corresponding branch at the origin. We get

$$y^4+\frac{2ay^4}{2\rho_1}=a\frac{y^6}{8\rho_1^3}+\frac{y^8}{16\rho_1^4}$$

or

$$1+\frac{a}{\rho_1}=a\frac{y^2}{8\rho_1^3}+\frac{y^4}{16\rho_1^4}.$$

Let $y \to 0$ so that we have $1 + a/\rho = 0$.

Thus, ρ, for this branch $= -a$.

To find, ρ, for the other branches we proceed as follows.

Suppose that the equation of either branch is given by

$$y = f(0) + xf'(0) + \frac{x^2}{2\,!} f''(0) + \ldots.$$

We have, ere

$$f(0) = 0.$$

Also we write $f'(0) = p, f''(0) = q$. Thus we have

$$y = px + \tfrac{1}{2}qx^2 + \ldots.$$

Making substitution in the given equation, we get

$$(px + \tfrac{1}{2}qx^2 + \ldots)^4 + 2ax(px + \tfrac{1}{2}qx^2 + \ldots)^2 = ax^3 + x^4$$

Equating co-efficients of x^3 and x^4, we get

$$2ap^2 = a,\ p^4 + 2apq = 1.$$

These give

$$p = \frac{1}{\sqrt{2}}, \quad q = \frac{3\sqrt{2}}{8a};$$

$$p = -\frac{1}{\sqrt{2}}, \quad q = -\frac{3\sqrt{2}}{8a}.$$

$$\therefore \qquad \rho = \frac{(1+p^2)}{p} = \pm 2\sqrt{3}a,$$

for the two branches.

2. *Show that the pole is a triple point on the curve*

$$r = a(2 \cos \theta + \cos 3\theta),$$

and that the radii of curvature of the three branches are

$$\sqrt{3}a/2,\ a/2,\ \sqrt{3}a/2.$$

The radius vector, r, vanishes for the values of, θ, given by

$$2 \cos \theta + \cos 3\theta = 0,$$

i.e.,

$$2 \cos \theta + 4 \cos^3 \theta - 3 \cos \theta = 0,$$

or

$$\cos \theta\ (4 \cos^2 \theta - 1) = 0,$$

or

$$\cos \theta = 0,\ \cos \theta = \tfrac{1}{2},\ \cos \theta = -\tfrac{1}{2}.$$

Thus, $r = 0$ when $\theta = \pi/3,\ \pi/2,\ 2\pi/3$ so that the pole is a triple point.

We now proceed to find ρ. We have

$$r_1=a(-2\sin\theta-3\sin 3\theta),\ r_2=a(-2\cos\theta-9\cos 3\theta).$$

For $\theta=\pi/3$,

$$r_1=-\sqrt{3}a, \qquad r_2=8a\ ;$$

for $\theta=\pi/2$,

$$r_1=a \qquad r_2=0\ ;$$

or $\theta=2\pi/3$,

$$r_1=-\sqrt{3}a, \qquad r_2=-8a.$$

Also, $r=0$ for each of these branches.

Putting these values in the formula

$$\rho=\frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2},$$

we get the required result.

Exercises

1. Show that the radius of curvature at the origin for both the branches of the curve

$$y^2(a-x)=x^2(a+x)$$

is $\sqrt{2}a$.

2. Find the radius of curvature at the point (1, 2) for the curve

$$(y-2)^2=x(x-1)^2.$$

3. Find the radii of curvature at the origin of the two branches of the curve given by the equations

$$y=t-t^3,\ x=1-t^2. \qquad (P.U.)$$

(For the origin $t=\pm 1$, so that the two branches correspond to these two values of t).

Find the radii of curvature at the origin of the following curves :

4. $x^3+y^3=3axy$. (*Folium*).

5. $x^2-3xy-4y^2+y^3+y^4x+x^5=0$.

6. $x^5+ax^2y^2-ax^3y-2a^2xy^2+a^2y^3=0$

7. Show that $(a, 0)$, in polar co-ordinates, is a triple point on the curve

$$r=a(1+2\sin\tfrac{1}{2}\theta),$$

and find the radii of curvature at the point.

CHAPTER XVIII

CURVE TRACING

18·1. We have already traced some curves in Chapters II and XII. The general problem of curve tracing, in its elementary aspects, will be taken up in this chapter.

It will be seen that the equations of curves which we shall trace are generally solvable for y, x or r. Some equations which are not solvable for y or x may be rendered solvable for r, on transformation from Cartesian to Polar system.

18·2. Procedure for tracing Cartesian Equations.

I. *Find out if the curve is symmetrical about any line.* In this connection, the following rules, whose truth is evident are helpful :—

(*i*) A curve is symmetrical about X-axis if the powers of y which occur in its equation are all even ;

(*ii*) A curve is symmetrical about Y-axis if the powers of x which occur in its equation are all even ;

(*iii*) A curve is symmetrical about the line $y=x$ if, on interchanging x and y, its equation does not change.

II. *Find out if the origin lies on the curve.* If it does, write down the tangent or tangents thereat. In case the origin is a multiple point, find out its nature.

III. *Find out the points common to the curve and the co-ordinate axes if there be any.* Also obtain the tangents at such points.

IV. *Find out dy/dx and the points where the tangent is parallel to the co-ordinate axes.* At such point, the ordinate or abscissa generally changes its character from increasing to decreasing or *vice versa.*

V. *Find out such points on the curve whose presence can be easily detected.*

VI. *Find out points of inflexion.* (It may not always be necessary).

VII. *Find out multiple points,* if any, and their nature.

VIII. *Find out the asymyptotes* and the points in which each asymptote meets the curve.

IX. *Find out if there is any region of the plane such that no part of the curve lies in it.*

Such a region is generally obtained on solving the equation for one variable in terms of the other, and find out the set of values of one variable which make the other imaginary.

X. If possible, solve the equation for one variable in terms of the other and find out, by examining the sign of the derivative, or otherwise, the ranges of the values of the independent variable for which the dependent variable monotonically increases.

Important Note. The student must remember that a mere knowledge of symmetry, asymptotes, double points, etc., will not enable him to trace a curve, this knowledge being only piece-meal; asymptotes indicate the nature of the curve in parts of the plane sufficiently far removed from the origin and the tangent at any point gives an idea of the curve in the neighbourhood of the point. To draw a curve completely, it may be necessary to solve the given equation for one variable, say, y, in terms of another and then to examine how y varies as x varies continuously. For this purpose the examination of dy/dx proves very helpful.

Examples

18·3. Equations of the form

$$y^2=f(x).$$

1. *Trace the curve*

$$y^2(a^2+x^2)=x^2(a^2-x^2).$$

We note the following particulars about this curve :—

(*i*) It is symmetrical about both the axes.

(*ii*) It passes through the origin and $y=\pm x$ are the two tangents thereat. Thus the origin is a node.

(*iii*) It meets X-axis at $(a, 0)$, $(0, 0)$ and $(-a, 0)$ and meets Y-axis at $(0, 0)$ only. The tangents at $(a, 0)$ and $(-a, 0)$ are $x=a$ and $x=-a$ respectively.

(*iv*) $$\frac{dy}{dx}=\pm\frac{a^4-2a^2x^2-x^4}{(a^2+x^2)^{\frac{3}{2}}(a^2-x^2)^{\frac{1}{2}}},$$

which becomes zero, when

$$a^4-2a^2x^2-x^4=0, \text{ i.e., when } x^2=(-1\pm\sqrt{2})a^2.$$

Thus the only real values of x for which dy/dx vanishes are

$$\pm\sqrt{(-1+\sqrt{2})}a.$$

(*v*) It has no asymptotes.

(*vi*) Solving the equation for y, we re-write it as

$$y^2=\frac{x^2(a^2-x)}{a^2+x^2} \text{ or } y=\pm x\sqrt{\left[\frac{a^2-x^2}{a^2+x^2}\right]}.$$

We see that, for y to be real, a^2-x^2 must be non-negative and therefore, x must lies between $-a$ and a. Hence the entire curve lies between the lines $x=a$ and $x=-a$.

Conclusion. In order to trace the curve we have to connect the above isolated facts. We proceed to do this now. The curve being symmetrical about *both* the axes, we firstly consider the part of the curve lying in first the quadrant where x, y are positive and

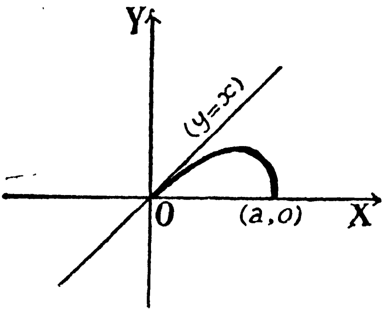


Fig. 136.

$$y=x\sqrt{\left[\frac{a^2-x^2}{a^2+x^2}\right]}.$$

When $x=0$, then $y=0$. When x increases, starting from 0, then y, which is positive, also increases and goes on increasing till $x=\sqrt{(-1+\sqrt{2})}a$ where $dy/dx=0$, *i.e.*, where the tangent is parallel to X-axis. Since $y=0$ when $x=a$, we see that when x increases from $\sqrt{(-1+\sqrt{2})}\, a$ to a, y decreases.

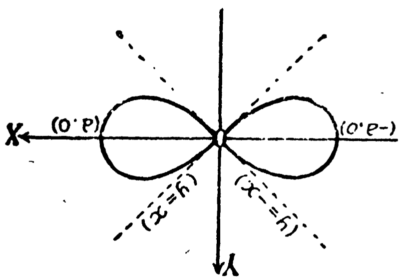


Fig. 137

The variable x cannot take up values greater than a, for, then, y is not real. Thus the part of the curve in the first quadrant is as shown in Fig. 136. The curve being symmetrical about the two axes, its complete shape is as shown in Fig. 137.

2. *Trace the curve*

$$y^2(x-a)=x^2(x+a).$$

(*i*) It is symmetrical about X-axis only.

(*ii*) It passes through the origin and $y^2+x^2=0$, *i.e.*, $y\pm ix=0$ are the two imaginary tangents thereat. Thus the origin is an isolated point.

(*iii*) It meets X-axis at $(-a, 0)$ and $(0, 0)$ and y-axis at the origin only ; $x=-a$ is the tangent at $(-a, 0)$.

(*iv*) $$\frac{dy}{dx}=\pm\frac{x^2-ax-a^2}{(x-a)^{\frac{3}{2}}(x+a)^{\frac{1}{2}}},$$

which vanishes when

Since $x^2-ax-a^2=0$, *i.e.*, when $x=\frac{1}{2}(1\pm\sqrt{5})a$.

For $x=(1-\sqrt{5})a/2$, which lies between $-a$ and a, y is not real. See (*vi*) below].

(*v*) $y=\pm(x+a)$ and $x=a$ are its three asymptotes.

(*vi*) Since $$y^2=x^2\frac{x+a}{x-a}=x^2\frac{x^2-a^2}{(x-a)^2},$$
we see that, for y to be real, x^2-a^2 must be non-negative, *i.e.*, must be $>a$ and $<-a$.

Hence no part of the curve lies between
$$x=a \text{ and } x=-a.$$

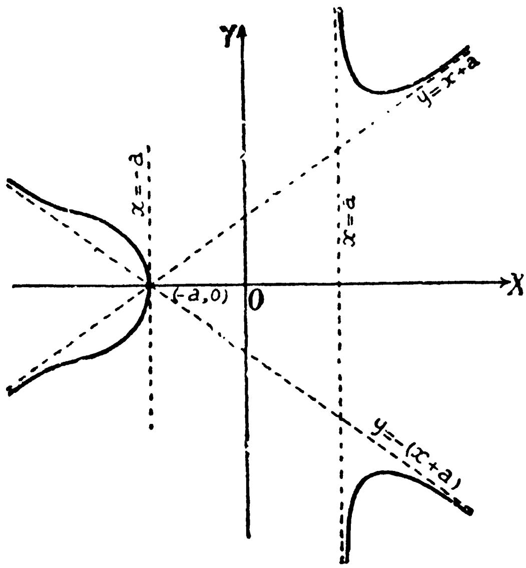


Fig. 138

Conclusion. The curve being symmetrical about the X axis, we consider the part of it lying in the first two quadrants where y is positive so that we have
$$y=\pm x\sqrt{\left(\frac{x+a}{x-a}\right)},$$
where we consider $+$ or $-$ sign according as x is positive or negative.

When $x=-a$, $y=0$. Also $x=-a$, is the tangent at the corresponding point $(-a, 0)$.

Let x vary continuously from $-a$ to $-\infty$. Then, since dy/dx is not 0 for any of these values of x, and $y=-(x+a)$ is an asymptote, we have the part of the curve in the second quadrant as shown in Fig. 138.

For values of x lying between $-a$ and a, y is not real.

Since $x=a$ is an asymptote and $dy/dx=0$ for $x=(1+\sqrt{5})\,a/2$, we see that y, starting from ∞, goes on decreasing as x increases from a to $(1+\sqrt{5})a/2$. As x increases from $(1+\sqrt{5})a/2$ to ∞, y again goes on increasing and, $y=x+a$ being another asymptote, we have the part of the curve in the first quadrant as shown.

The curve being symmetrical about the X-axis, its complete shape is as shown in Fig. 138.

We thus have the curve as drawn.

Note. The curve appears to have two points of inflexion, apparently necessitated by the fact that the curve could not be asymptotic to the lines $y=\pm(x+a)$ unless it changes the direction of its bending after leaving the point $(-a, 0)$ where it touches $x=-a$. In fact, by actual calculation, we see that $(-2a, 2a/\sqrt{3})$ and $(-2a, -2a/\sqrt{3})$ are its two points of inflexion.

3. *Trace the curve*

$$y^2x^2=x^2-a^2.$$

(*i*) It is symmetrical about both the axes.

(*ii*) It does not pass through the origin.

(*iii*) It meets X-axis at $(a, 0)$ and $(-a, 0)$ but it does not meet Y-axis at any point ; $x=a$ and $x=-a$ are the tangents at $(a, 0)$ and $(-a, 0)$ respectively.

(*iv*) $\dfrac{dy}{dx}=\pm\dfrac{a^2}{x^2\sqrt{(x^2-a^2)}}$,

which never becomes zero.

(*v*) $y=\pm 1$ are the two asymptotes.

From § 16·31, p. 319, it appears that $x=0$ is also an asymptote, but it will be seen below that it cannot be an asymptote.

(*vi*) Writing the equation in the forms

$$y^2=\frac{x^2-a^2}{x^2}, \quad x^2=\frac{a^2}{1-y^2},$$

we see that for x and y to be real, we have

$$x^2-a^2>0,\ i.e.,\ x<-a \text{ or } x>a$$

and

$$1-y^2>0,\ i.e.,\ -1<y<1.$$

As no part of the curve lies between the lines $x=-a$, and $x=a$ no branch of the curve can possibly be asymptotic to $x=0$.

Conclusion.

Consider

$$y=\sqrt{\left[\frac{x^2-a^2}{x^2}\right]}$$

which corresponds to the part of the curve in the first quadrant.

For the values of x lying between 0 and a, y is not real.

If $x=a$, $y=0$ so that $(a, 0)$ is a point on the curve and $x=a$, as seen above, is the tangent there.

As x increases from a onwards, y, which is positive must *constantly increase* ; dy/dx being never zero.

Since the line $y=1$ is an asymptote, the part of the curve in the first quadrant can now be easily shown.

The curve being symmetrical about both the axes, its complete shape is as shown in the Fig. 139.

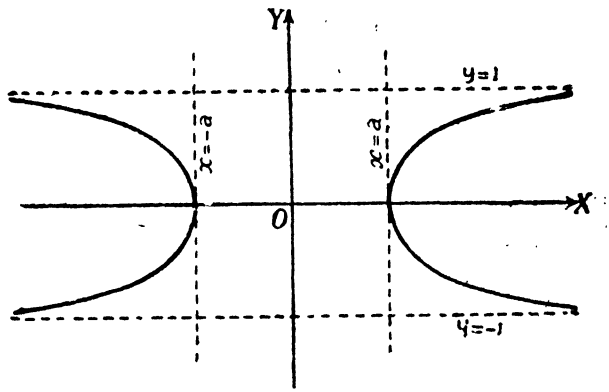


Fig. 139

4. *Trace the curve*

$$y^2=(x-a)(x-b)(x-c).$$

We suppose that a, b, c are positive and in ascending order of magnitude so that we have to consider the following four cases :—

(*i*) $a<b<c$. (*ii*) $a=b<c$. (*iii*) $a<b=c$. (*iv*) $a=b=c$.

Case I. $a<b<c$.

(*i*) It is symmetrical about X-axis.

(*ii*) It does not pass through the origin.

(*iii*) It meets X-axis at $(a, 0)$, $(b, 0)$ and $(c, 0)$, but it does not meet Y-axis ; $x=a$, $x=b$, $x=c$ are the tangents at $(a, 0)$, $(b, 0)$, $(c, 0)$, respectively.

(*iv*) It has no asymptotes.

(*v*) When $x<a$, $y^2<0$, *i e.*, y is not real ;

when $a<x<b$, $y^2>0$,

when $b<x<c$, $y^2<0$; *i.e.*, y is not real ;

when $x>c$, $y^2<0$.

Hence, no part of the curve lies to the left of the line $x=a$ and between the lines $x=b$, $x=c$.

(*vi*) As y^2 vanishes for $x=a$ and $x=b$, it must have a maximum for some value of x between a and b.

(*viii*) As x increases beyond c, y^2 also constantly increases, for each of the three factors then increases.

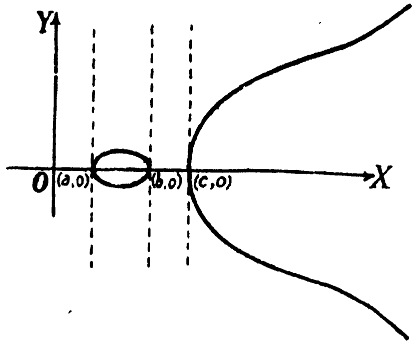


Fig. 140

We have

$$2y\,\frac{dy}{dx}=3x^2-2(a+b+c)x+(ab+bc+ca),$$

or
$$\frac{dy}{dx}=\frac{3x^2-2(a+b+c)x+(ab+bc+ca)}{2\sqrt{[(x-a)(x-b)(x-c)]}}$$
$$=x^{\frac{1}{2}}.\frac{3-2(a+b+c)(1/x)+(ab+bc+ca)(1/x^2)}{\sqrt{[(1-a/x)(1-b/x)(1-c/x)]}}$$

which $\to\infty$ as $x\to\infty$.

Thus the curve tends to become parallel to Y-axis as $x\to\infty$.

Since the curve, in departing from $(c, 0)$, where the tangent is parallel to Y-axis, must again tend to become parallel to Y-axis, we see that the curve must change its direction of bending at some point between $(c, 0)$ and ∞; and as such must have a point of inflexion.

Hence, we have the curve as shown in Fig. 140.

It consists of an oval and an infinite branch.

Case II. $a=b<c$. *The equation, now, is*

$$y^2=(x-a)^2(x-c).$$

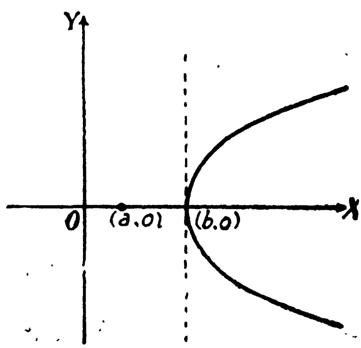


Fig. 141

It is easy to show that $(a, 0)$ is an isolated point, as if the oval of case I shrinks to the point $(a, 0)$.

The curve consists of an isolated point and an infinite branch and can be easily drawn. (Fig. 141 on the preceding page).

As in case I, it can be shown that the curve has two points of inflexion.

Case III. $a<b=c$. *The equation, now, is*

$$y^2=(x-a)(x-b)^2.$$

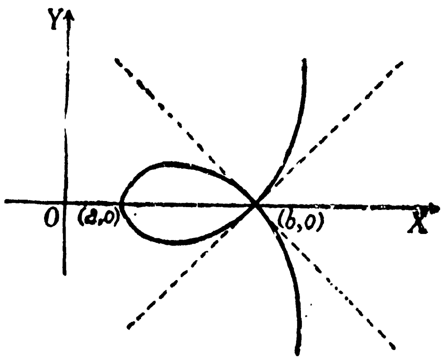


Fig. 142

It is easily shown that $(b, 0)$ is a node and

$$y=\pm\sqrt{(b-a)}(x-b)$$

are the nodal tangents.

The curve may now be easily drawn. (Fig. 142).

Case IV. $a=b=c$. *The equation, now, is*

$$y^2=(x-a)^3.$$

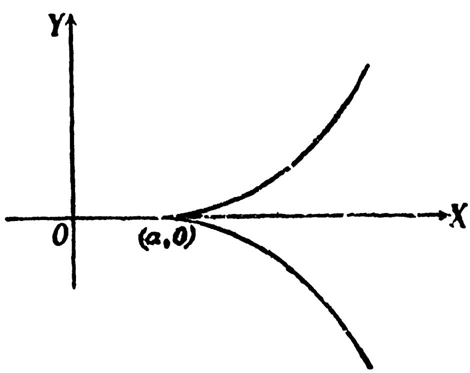


Fig. 143

It is easy to see that $(a, 0)$ is a cusp and $y=0$ is the cuspidal tangent.

The curve may be easily drawn. (Fig. 143).

5. *Trace the cissoid*

$$y^2(a-x)=x^3.$$

This curve has already been considered in § 11·41, p. 244 also. (See Fig. 64, p. 244).

We note the following particulars about the curve :—

(*i*) It is symmetrical about x-axis.

(*ii*) It passes through the origin and $y^2=0$, *i.e.*, $y=0$, $y=0$ are the two coincident tangents at the origin. Thus the origin is a cusp.

(*iii*) It meets the co-ordinate axes at the origin only.

(*iv*) $x=a$ is the only asymptote of the curve.

(*v*) Since

$$y^2=x^3/(a-x),$$

we see that y^2 is positive, *i.e.*, y is real only when x lies between and a.

(*vi*)
$$\frac{dy}{dx}=\frac{(\frac{3}{2}a-x)\sqrt{x}}{(a-x)^{\frac{3}{2}}},$$

which vanishes when $x=\frac{3}{2}a$ or 0.

But $x=\frac{3}{2}a$ is outside the range of admissible values of x. Thus dy/dx vanishes at no admissible value of x except $x=0$.

Combining all these facts, we may easily see that the shape of the curve is as shown in Fig. 64, p. 244.

Note. The student would do well to learn to trace the curves given in §§11·42, 11·43 also by the methods of this chapter.

6. *Trace the curve*

$$x^3+y^3=3ax^2,\ (a>0).$$

(*i*) It is neither symmetrical about the co-ordinate axes nor about the line $y=x$.

(*ii*) Origin is a cusp, and $x=0$ is the cuspidal tangent.

(*iii*) It meets X-axis at $(0, 0)$ and $(3a, 0)$ but meets Y-axis at the origin only ; $x=3a$ is the tangent at $(3a, 0)$.

(*iv*) $y+x=a$ is its only asymptote and the curve meets the asymptote at

$$(a/3,\ 2a/3).$$

(*v*) x and y cannot both be negative.

(*vi*) Now, $y^2dy/dx=x(2a-x)$ and, therefore,

$$dy/dx=0, \text{ for } x=2a.$$

Solving for y, we obtain

$$=[x^2(3a-x)]^{\frac{1}{3}}.$$

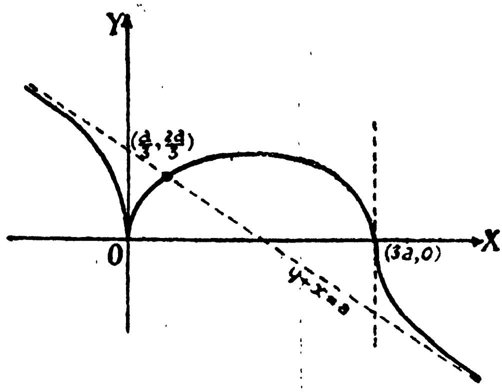


Fig. 144.

If $x=0$, then $y=0$ and Y-axis is the tangent there.

If $0<x<3a$, then $y>0$, and if x increases from 0, y also increases; y will go on increasing with x till $x=2a$ where $dy/dx=0$. When x increases beyond $2a$, y will constantly be decreasing ; $y=0$ for $x=3a$ and is negative for $x>3a$.

We now consider the negative values of x.

If x is negative, y is positive and constantly goes on increasing as x increases numerically, *i.e.*, as x varies from 0 to $-\infty$.

Also, $y+x=a$ is the only asymptote of the curve.

Taking all the above facts into consideration, we see that the complete curve is as shown in Fig. 144.

18·4. Equation of the form $y^2+yf(x)+F(x)=0$.

7. *Trace the curve*

$$x^2(x-y)+y^2=0$$

(*i*) It is symmetrical neither about the co-ordinate axes, nor about the line $y=x$.

(*ii*) Origin is a cusp and $y=0$ is the cuspidal tangent.

(*iii*) It meets the co-ordinate axes at the origin only.

(*iv*) $y=x+1$ is the only asymptote of the curve. It meets the curve at $(-\frac{1}{2}, \frac{1}{2})$.

(*v*) We re-write the equation as a quadratic in y and solve it. We have

$$y^2-yx^2+x^3=0,$$

or
$$y=\frac{x^2\pm\sqrt{[x^3(x-4)]}}{2},$$

so that y is imaginary if $x^3(x-4)$ is negative *i.e.*, if x lies between 0 and 4.

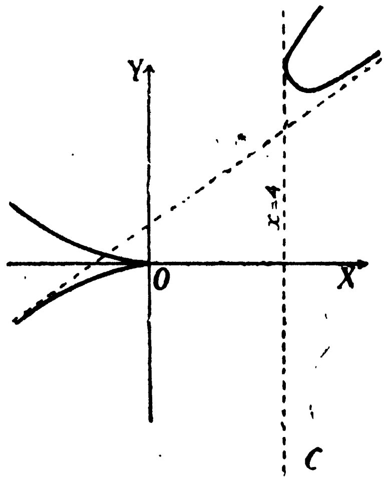


Fig. 145.

Thus, there is no part of the curve lying between the line $x=0$ and $x=4$.

(*vi*) $y=8$ when $x=4$ for both the branches, and $x=4$ is the tangent at the point.

The following additional remarks about the two branches of curve will facilitate the process of tracing the curve.

Now $$y=\frac{x^2+\sqrt{(x^4-4x^3)}}{2}=\frac{x^2+\sqrt{[x^3(x-4)]}}{2}, \quad \ldots.(i)$$

and $$y=\frac{x^2-\sqrt{(x^4-4x^3)}}{2}=\frac{x^3-\sqrt{[x^3(x-4)]}}{2}, \quad \ldots.(ii)$$

are the two branches. They meet where $x^4-4x^3=0$, *i.e.*, where $x=0$ or 4.

Thus the two branches meet at the points (0, 0) and (4, 8).

Consider the branch (*i*).

When x, starting from 4, increases, then y is positive and also constantly increases.

Also, when x starting from 0 decreases, and takes up negative values whose numerical value increases, then y is positive and constantly increases.

Now, consider the branch (*ii*).

When $x>4$, x^4-4x^3 is positive and $\sqrt{(x^4-4x^3)}<\sqrt{x^4}=x^2$ and, therefore, y is positive. Also, when x, starting from 4, increases, y decreases in the beginning.

When $x<0$, x^4-4x^3 is positive and $\sqrt{(x^4-4x^3)}>\sqrt{x^4}=x^2$ so that y is negative. Also, when x, starting from 0, decreases, then y numerically increases. Thus, we have the shape as shown in Fig. 145.

18·5. Polar curves.

8. *Trace the curve*

$$r=\frac{a\theta^2}{1+\theta^2}.$$

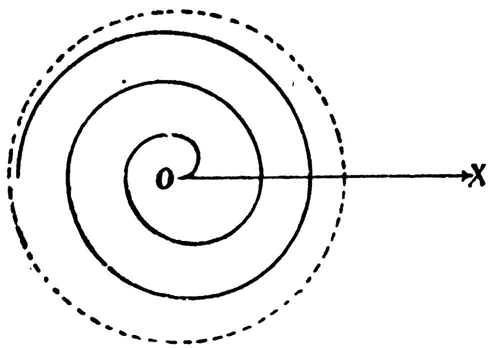


Fig. 146.

We first consider *positive* values θ only. We have

$$\frac{dr}{d\theta}=\frac{2a\theta}{(1+\theta^2)^2},$$

which is always positive so that r, constantly increases as, θ, increases.

Also,

$$r=0 \text{ when } \theta=0.$$

Again we have

$$\frac{a\theta^2}{1+\theta^2}=\frac{a}{\theta^{-2}+1} \text{ which} \to a \text{ as } \theta\to\infty.$$

Thus r, starting from its initial value 0, constantly increases as θ increases and approaches, a, as $\theta\to\infty$ so that the point (r, θ) approaches nearer and nearer the circle whose centre is at the pole and radius equal to a.

The circle is shown dotted in the Fig. 146.

The figure shows the part of the curve corresponding to the positive values of θ only, and the part of the curve for negative values is its reflection in the initial line.

9. *Trace the curve*

$$r=a(\sec\theta+\cos\theta).$$

Here

$$r=a\left(\frac{1}{\cos\theta}+\cos\theta\right)=a\,\frac{1+\cos^2\theta}{\cos\theta}.$$

(*i*) The curve is symmetrical about the initial line.

(*ii*) $r\cos\theta=a$, *i.e.*, $x=a$ is its asymptote.

(*iii*) $\frac{dr}{d\theta}=a\frac{\sin^3\theta}{\cos^2\theta}$ so that $\frac{dr}{d\theta}$ is positive when θ lies between θ and $\pi/2$.

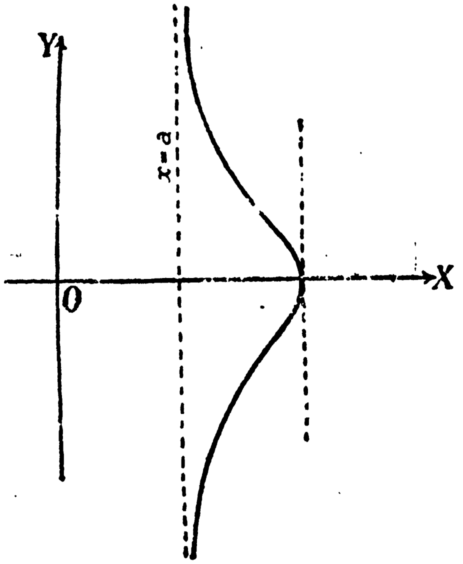


Fig. 147.

When $\theta=0$, $dr/d\theta=0$ so that $\phi=\pi/2$ and, therefore, the tangent is perpendicular to the initial line at the point $(2a, 0)$.

Also, $r=2a$, when $\theta=0$ and $r\to\infty$ as $\theta\to\pi/2$.

Hence we see that when θ increases from 0 to $\frac{1}{2}\pi$, or monotonically increases that $2a$ to ∞ and the point $P(r, \theta)$ describes the part of the curve drawn in the first quadrant.

When θ increases from $\frac{1}{2}\pi$ to π, r, remains negative and decreases in numerical value from ∞ to $2a$ and so the point $P(r, \theta)$ describes the part of the curve as shown in the fourth quadrant.

As the curve is symmetrical about the initial line, no new point will be obtained when θ varies from π to 2π.

18·6. In the case of some curves it is found convenient to make use of the polar as well as the Cartesian form of their equations. Some facts are obtained from the Cartesian and the others from the Polar form.

Curves whose cartesians equations are not solvable for x and y, but whose polar equations are solvable for r, are generally dealt with in this manner.

10. *Trace the curve*

$$x^4+y^4=a^2(x^2-y^2).$$

(*i*) It is symmetrical about both the axes.

(*ii*) Origin is a node on the curve and $y=\pm x$ are the nodal tangents.

(*iii*) It meets X-axis at (0, 0) $(a, 0)$ and $(-a, 0)$, but meets Y-axis at (0, 0) only ; $x=a$ and $x=-a$ are the tangents at $(a, 0)$ and $(-a, 0)$.

(*iv*) It has no asymptotes.

On changing to polar co-ordinates, the equation becomes

$$r^2=\frac{a^2 \cos 2\theta}{\cos^4\theta+\sin^4\theta}.$$

(*v*) We see that

$$r\frac{dr}{d\theta}=-\frac{8a^2 \sin^3\theta \cos^3\theta}{(\cos^4\theta+\sin^4\theta)^2},$$

so that $dr/d\theta$ remains negative as θ varies from 0 to $\pi/4$ and therefore r decreases from a to 0 as θ increases from 0 to $\pi/4$.

(*vi*) As θ changes from $\pi/4$ to $\pi/2$, r^2 remains negative and therefore no point on the curve lies between the lines $\theta=\pi/4$ and $\theta=\pi/2$.

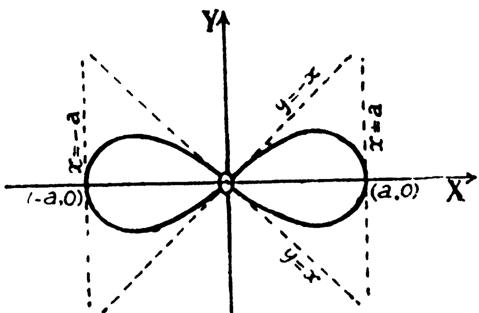


Fig. 148.

As the curve is symmetrical about both the axes, we have its shape as shown. (Fig. 148).

11. *Trace the curve*

$$y^4-x^4+xy=0. \qquad (P.U.)$$

(*i*) It is neither symmetrical about the co-ordinate axes, nor about the line $y=x$.

(*ii*) It passes through the origin : $x=0$, $y=0$ are the two tangents thereat so that the origin is a node.

(*iii*) It cuts the co-ordinate axes at the origin only.

(*iv*) $y=x$, $y=-x$ are its asymptotes.

On transforming to polar co-ordinates, we get

$$r^2=\tfrac{1}{2}\tan 2\theta.$$

(*v*) When θ increases from 0 to $\pi/4$, 2θ increases from θ to $\pi/2$, and, therefore, r^2 monotonically increases from 0 to ∞.

When θ increases from $\pi/4$ to $\pi/2$, $\tan 2\theta$ and therefore also r^2 remains negative and, thus, there is no part of the curve lying between the lines $\theta=\pi/4$ and $=\pi/2$.

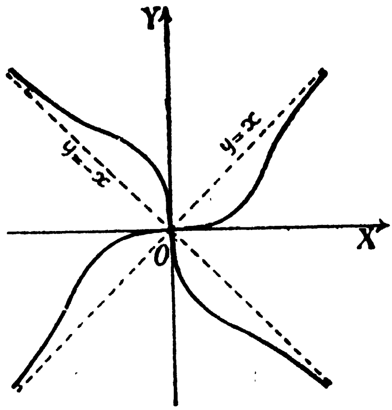


Fig. 149.

When θ increases from $\pi/2$ to $3\pi/4$, r^2 increases from 0 to ∞.

When θ increases from $3\pi/4$ to π, r^2 remains negative and so there is no part of the curve lying between the lines $\theta=3\pi/4$, and $\theta=\pi$.

We can similarly consider the variations of r^2 as θ increases from π to 2π.

Hence we have the curve as drawn. (Fig. 149)

12. *Trace the Folium of Descartes*

$$x^3+y^3=3axy.$$

(*i*) It is symmetrical about the line $y=x$ and meets it in the point $(3a/2, 3a/2)$.

(*ii*) It passes through the origin and $x=0$, $y=0$ are the tangents there so that the origin is a node on the curve.

(*iii*) It meets the co-ordinate axes at the origin only.

(*iv*) $x+y+a=0$ is its only asymptote.

(*v*) x, y cannot both be negative so that no part of the curve lies on the third quadrant.

On transforming to polar co-ordinates, we get

$$r=\frac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}.$$

Now, $r=0$ for $\theta=0$ and $\theta=\pi/2$.

$$\frac{dr}{d\theta}=\frac{3a(\cos\theta-\sin\theta)(1+\sin\theta\cos\theta+\sin^2\theta\cos^2\theta)}{(\cos^3\theta+\sin^3\theta)^2}$$

which vanishes only when

$\cos\theta-\sin\theta=0$, *i.e.*, $\tan\theta=1$. *i.e.*, $\theta=\pi/4$ or $5\pi/4$.

For $\theta=\pi/4$, $r=3a/\sqrt{2}$.

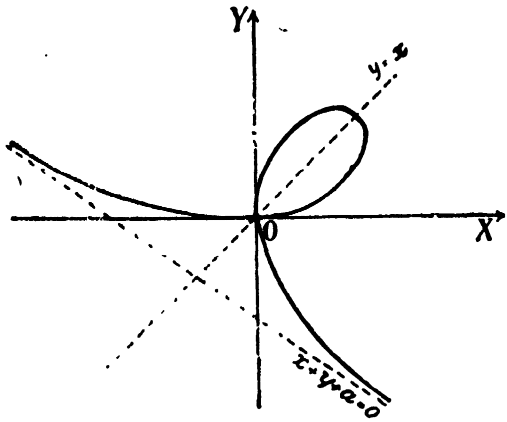


Fig. 150

Thus r monotonically increases from 0 to $3a/\sqrt{2}$, as θ increases from 0 to $\pi/4$, and monotonically decreases from $3a/\sqrt{2}$ to 0, as θ increases from $\pi/4$ to $\pi/2$.

Again as θ increases from $\pi/2$ to $3\pi/4$, r remains negative and numerically increases from 0 to ∞ so that the point (r, θ) describes the part of the curve shown in the fourth quadrant. (Fig. 150).

As θ increases from $3\pi/4$ to π, r remains positive and decreases from ∞ to 0 so that the point describes the part of the curve shown in the second quadrant.

It is easy to see that we do not get any new point on the curve when, θ, increases from π to 2π.

13. *Find the double point of the curve*

$$x^4+y^4=4a^2xy,$$

and trace it. (*P.U. 1955*)

Let

$$f(x, y)=x^4+y^4-4a^2xy.$$

$$\therefore \quad f_x(x, y)=4(x^3-a^2y),$$

$$f_y(x, y)=4(y^3-a^2x).$$

Putting $f_x=0$ and $f_y=0$, we have

$$x^3-a^2y=0,\ y^4-a^2x=0.$$

$$\therefore \quad y^9-a^8y=0,$$

or

$$y=0, \pm a.$$

Thus we see that f_x and f_y vanish at the points (0, 0), (a, a) $(-a, -a)$. Of these only (0, 0) is a point of the curve so that the origin is the only double point of the curve.

To trace the curve we note the following particulars about it :—

(*i*) It is symmetrical about the line $y=x$ and meets it at

$$(\pm\sqrt{2}a,\ \pm\sqrt{2}a).$$

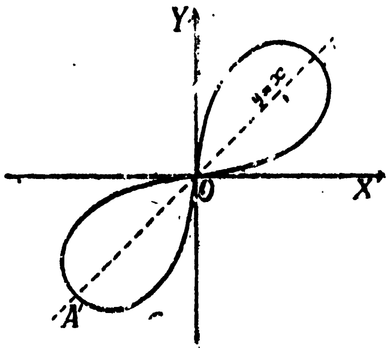


Fig. 151

(*ii*) It passes through the origin and $x=0$, $y=0$, are the two tangents there.

(*iii*) It meets the co-ordinate axes at the origin only.

(*iv*) It has no asymptotes.

(*v*) x and y cannot have values with opposite signs, for such values make $4a^2xy$ negative whereas x^4+y^4 is always positive. Thus the curve lies in the first and fourth quadrants only.

(*vi*) On transforming to polar co-ordinates, we have

$$r^2=4a^2 \sin\theta\cos\theta/(\sin^4\theta+\cos^4\theta).$$

$$\therefore \qquad r\frac{dr}{d\theta}=2a^2\frac{(\cos^2\theta-\sin^2\theta)(\cos^4\theta+\sin^4\theta+4\sin^2\theta\cos^2\theta)}{(\sin^4\theta+\cos^4\theta)^2}$$

so that $dr/d\theta=0$ if and only if

$$\tan\theta=1,\ i.e.,\ \theta=\pi/4 \text{ and } 5\pi/4.$$

Thus we have the curve as drawn (Fig. 151).

18·6. Parametric Equations. The following examples will illustrate the process.

14. *Trace the curve*

$$x=a(\theta+\sin\theta),\ y=a(1+\cos\theta),$$

as θ, *varies in the interval* $(-\pi, \pi)$.

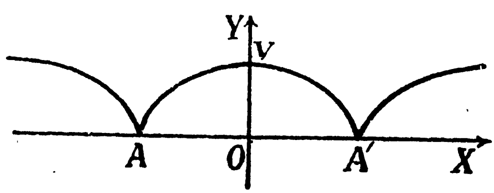


Fig. 152

We have

$$\frac{dx}{d\theta}=a(1+\cos\theta),\ \frac{dy}{d\theta}=-a\sin\theta.$$

$$\therefore \qquad \frac{dy}{dx}=-\tan\frac{\theta}{2}.$$

Since $dx/d\theta$ is positive for every value of θ, we see that as, θ, increases, x will always increase. Also, $dy/d\theta$ is positive when, θ, lies in the interval $(-\pi, 0)$ and negative when, θ, lies in the interval $(0, \pi)$. Therefore y constantly increases as θ increases from $-\pi$ to 0 and constantly decreases as, θ, increases from 0 to π.

The following table gives the corresponding values of θ, x, y, and dy/dx :—

θ	$-\pi$	Intermediate	0	Intermediate	π
x	$-a\pi$	increases	0	increases	$a\pi$
y	0	increases	$2a$	decreases	0
dy/dx	∞		0		∞

Since for $\theta=-\pi$, we have $x=-a\pi$, $y=0$ and $dy/dx=\infty$, we see that the point $A(-a\pi, 0)$ lies on the curve and the tangent thereat is parallel to Y-axis ; when θ increases from $-\pi$ to 0, x and y are both increasing so that the point $P(x, y)$ moves to the right and upwards till it reaches the position $V(0, 2a)$ corresponding to $\theta=0$ where, dy/dx being 0, the tangent is parallel to X-axis.

When θ increases from 0 to π, x increases but y decreases so that the point moves to the right and downwards till it reaches the position $A'(a\pi, 0)$ corresponding to $\theta=\pi$ where, dy/dx being infinite, the tangent is parallel to Y-axis.

Thus the point $P(x, y)$ describes the curve AVA' as θ increases from $-\pi$ to π.

It is easy to see that the point $P(x, y)$ describes congruent portions to the right and to the left of the portion AVA' as θ varies in the intervals

$$\ldots\ldots(-5\pi, -3\pi), (-3\pi, -\pi), (\pi, 3\pi), (3\pi, 5\pi)\ldots\ldots$$

15. *Trace the curve*

$$x=a\sin 2\theta(1+\cos 2\theta),\ y=a\cos 2\theta(1-\cos 2\theta).$$

We have

$$\frac{dx}{d\theta}=4a\cos 3\theta\cos\theta\ ;\ \frac{dy}{d\theta}=4a\cos 3\theta\sin\theta.$$

$$\therefore \qquad \frac{dy}{dx}=\tan\theta.$$

This shows that $\qquad \psi=\theta$

The following table gives corresponding values of θ, x, y and dy/dx.

θ	0	Intermediate	$\pi/6$	Intermediate	$\pi/3$	Intermediate	$\pi/2$
x	0	increases	$\frac{3\sqrt{3}a}{4}$	decreases	$\frac{\sqrt{3}a}{4}$	decreases	0
y	0	increases	$\frac{a}{4}$	decreases	$-\frac{3a}{4}$	decreases	$-2a$
dy/dx	0		$1/\sqrt{3}$		$\sqrt{3}$		∞

θ	$\pi/2$	Intermediate	$\frac{2\pi}{3}$	Intermediate	$\frac{\pi}{6}$	Intermediate	π
x	0	decreases	$-\frac{\sqrt{3}a}{4}$	decreases	$-\frac{3\sqrt{3}a}{4}$	increases	0
y	$-2a$	increases	$-\frac{3a}{4}$	increases	$\frac{a}{4}$	decreases	0
dy/dx	∞		$-\sqrt{3}$		$-\frac{1}{\sqrt{3}}$		0

With the help of this table of values, we have the curve as shown. (Fig. 153).

Since x, y are periodic functions of θ, with π as their period, the values of x, y will repeat themselves as θ varies in the intervals

$$(\pi, 2\pi), (2\pi, 3\pi), \text{ etc.},$$

so that the same curve will be traced.

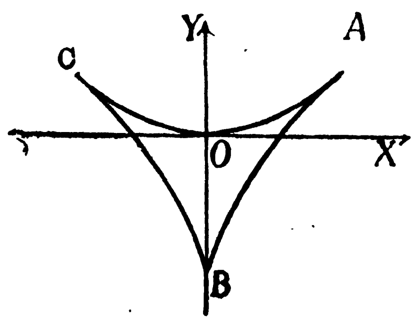


Fig. 153

The curve has 3 cusps *viz.*, A, B and C.

Exercises

Trace the following curves :—

1. $3ay^2=x^2(x-a)$.
2. $y^2=x(x+1)^2$.
3. $ay^2=x(a^2-x^2)$.
4. $4a^4y^2=x^5(2a-x)$.
5. $y^2=x^2(4-x^2)$.
6. $x(x^2+y^2)=a(x^2-y^2)$. (*D.U. 1955*)
7. $x^2y^2=(a+y)^2(a^2-y^2)$. (*P.U.*)
8. $y(a^2+x^2)=a^2x$. (*B.U.*)
9. $x^2(y^2+x^2)=a^2(x^2-y^2)$.
10. $a^2y^2=x^2(a^2-x^2)$. (*D.U. 1955*)
11. $y^2(a^2-x^2)=a^3x$.
12. $y^2(x^2-1)=2x-1$.
13. $y^2x=a^2(a-x)$.
14. $ay^2=x(a^2+x^2)$.
15. $y^2(a^2+x^2)=a^2x^2$.
16. $y^2x=a(x^2+a^2)$.
17. $y^2x=a(x^2-a^2)$.
18. $y^2x^2=x^2+1$.
19. $y^2x^2=x^2-1$.
20. $y^2(a^2-x^2)=x^4$.
21. $a^2y^2=x^2(2a-x)(x-a)$.
22. $y^2(x+2)=x+1$.
23. $y^2(2x-1)=x(x-1)$.
24. $y^2=(x^2+1)^2(2-x^2)^3$.
25. $36y^2=(x^2-1)^2(7-x^2)^3$.
26. $x=(y-1)(y-2)(y-3)$. (*P.U.*)

Trace the following curves :—

27. $r=a+b\cos\theta$.
28. $r\cos^3\theta=a\cos 2\theta$. (*L.U.*)
29. $r\cos\theta=a\sin 3\theta$.
30. $r^2\cos\theta=a^2\sin 3\theta$.
31. $r=a\log\theta$.
32. $r\log\theta=a$.
33. $r=ae^{\theta}\sin\theta$.
34. $r=a(\theta-\sin\theta)$.

Trace the following curves :—

35. $x^5+y^5=5ax^2y^2$.
36. $(x^2+y^2)(x-a)^2=a^2x^2$.
37. $x^4=ay^3-axy^2$.
38. $x^4+y^4=4axy^2$.
39. $x(y-x)=(y+x)(y^2+x^2)$.

Trace the following curves :—

40. $y^3=x^2(x+a)$.
41. $x^3+y^3=a^3$.
42. $a^3/x^3-b^3/y^3=1$.
43. $x^3+y^3=a^2x$.
44. $axy=x^3-a^3$.
45. $a^2/x^2-b^2/y^2=1$.

46. Show that the curve $x^2(x+y)-y^2=0$ has a cusp at the origin and a rectilinear asymptote $x+y=1$ and no part of the curve lies between the lines $x=0$ and $x=-4$ and that it consists of two infinite branches, one in the second quadrant and the other in the first and fourth. Give a sketch of the curve.

Trace the following curves :—

47. $xy^2=(x+y)^2$. (*L.U.*)
48. $y(x-y)^2=(x+y)$. (*B.U.*)
49. $x^2(y+1)=y^2(x-4)$. (*P.U.*)
50. $y=x(x^2-m^2y^2)$. (*B.U.*
51. $2xy=x^2+y^3$.
52. $x^4y^2+(y^2+4xy+3x^2)=0$.

53. $y(1-x^2)(2-x)^2=x(1-x)$. 54. $r(1+\theta)=a\theta$.

55. Show that no part of the curve

$$x^2y+xy^2+xy+y^2+3x=0,$$

lies between $x=-4$ and $x=-1$ or between $x=0$ and $x=3$; give a sketch of the curve. (*M.T.*)

56. Show that the cubic

$$xy^2-x^2y-3x^2-2xy+y^2+x-2y+1=0,$$

has double point at (0, 1); find its asymptotes and sketch the curve.

Trace the following curves :—

57. $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$. $(0\leqslant\theta\leqslant 2\pi)$

58. $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$. $(-\pi\leqslant\theta\leqslant\pi)$

59. $x=a\cos^3\theta$, $y=b\sin^3\theta$.

60. $x=a\cos^3\theta$, $y=a(\sin 3\theta+9\sin\theta)$.

61. $x=a(3\cos\theta-\cos^3\theta)$, $y=a(3\sin\theta-\sin^3\theta)$.

62. $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$. $(0\leqslant\theta\leqslant 2\pi)$

63. $x/a=\log(1+\cos\theta)-\cos\theta$, $y/a=\sin\theta$.

64. $x=a\log|(\sec\theta+\tan\theta)|$, $y=a\log|(\operatorname{cosec}\theta+\cot\theta)|$.

65. $x/a=2\sin\theta-\log|(\sec\theta+\tan\theta)|$,
$y/b=2\cos\theta-\log|(\operatorname{cosec}\theta+\cot\theta)|$.

66. $x=a\cos\theta$; $y=a(2+\sin\theta)\sin^2\theta/(3+\cos\theta)$.

67. $x=a(\sin\theta+\frac{1}{3}\sin 3\theta)$, $y=a(\cos\theta-\frac{1}{3}\cos 3\theta)$.

68. $x=a(\theta+\sin\theta\cos\theta-\sin\theta)$, $y=a(\cos^2\theta-\cos\theta)$. $(0\leqslant\theta\leqslant 2\pi)$

CHAPTER XIX

ENVELOPES

19·1. One parameter family of Curves. If $f(x, y, \alpha)$ be any function of three variables, then the equation

$$f(x, y, \alpha)=0,$$

determines a curve corresponding to each particular value of α.

The *totality* of these curves, obtained by assigning different values to α, is said to be a *one-parameter family of curves.*

The variable, α, which is different for different curves is said to be the *parameter* for the family.

Illustration.

(*i*) The equation

$$x^2+y^2-2ax=0,$$

determines a family of curves which are circles with their centres on X-axis and which pass through the origin. Here, a, is the parameter.

(*ii*) The equation

$$y=mx-2am-am^3,$$

determines a family of straight lines which are normals to the parabola

$$y^2=4ax.$$

Here the parameter is m.

We now introduce the concept of the *envelope of a one-parameter family of curves* by means of an example considered in the next article.

19·2. Consider the family of straight lines

$$y=mx+a/m, \qquad \text{...}(i)$$

where, m, is the parameter and, a, is some constant.

The two members of this family corresponding to the parametric values m_1 and $m_1+\delta m$ of the parameter, m, are

$$y=m_1x+\frac{a}{m_1}, \qquad \text{..}(ii)$$

$$y=(m_1+\delta m)x+\frac{a}{m_1+\delta m} \qquad \text{..}(iii)$$

We shall keep m_1 fixed and regard δm as a variable which tends towards 0 so that the line (*iii*) tends to coincide with the line (*ii*).

The two lines (*ii*), (*iii*) intersect at (x, y), where

$$x=\frac{a}{m_1(m_1+\delta m)},\ y=\frac{a(2m_1+\delta m)}{m_1(m_1+\delta m)}$$

As $\delta m \to 0$, this point of intersection goes on changing its position on the line (*ii*) and, in the limit, tends to the point

$$\left(\frac{a}{m_1^2}, \frac{2a}{m_1}\right)$$

which lies on (*ii*).

This point is the limiting position of the point of intersection of the line (*ii*) with another line of the family when the latter tends to coincide with the former.

There will lie a point, similarly, obtained, on every line of the family. The *locus of such points is called the envelope of the given family of lines.*

To find this locus for the family of lines (*ii*), we notice that the co-ordinates (x, y) of such a point lying on the line 'm' are given by

$$x=\frac{a}{m^2},\ y=\frac{2a}{m}.$$

Eliminating m, we obtain

$$y^2=4ax,$$

as the envelope of the given family of lines.

19·3. Definition. *The* **envelope** *of a one-parameter of family of curves is the locus of the limiting position of the points of intersection of any two curves of the family when one of them tends to coincide with the other which is kept fixed.*

19·4. Determination of Envelope. Let

$$f(x, y, \alpha)=0, \qquad ..(i)$$

be any given family of curves.

Consider any two members

$$f(x, y, \alpha)=0$$

and

$$f(x, y, \alpha+\delta\alpha)=0, \qquad ..(ii)$$

corresponding to the parametric values α and $\alpha+\delta\alpha$.

The points common to the two curves (i), (ii) satisfy the equation

$$f(x, y, \alpha+\delta\alpha)-f(x, y, \alpha)=0,$$

or

$$\frac{f(x, y, \alpha+\delta\alpha)-f(x, y, \alpha)}{\delta\alpha}=0. \qquad \ldots(iii)$$

Let $\delta\alpha \to 0$. Therefore the limiting positions of the points, common to (i) and (ii) satisfy the equation which is the limit of (iii) *viz.*,

$$f\alpha(x, y, \alpha)=0. \qquad ..(iv)$$

Thus the co-ordinates of the points on the envelope satisfy the equations (i) and (iv).

Let the elimination of α between (i) and (iv) lead to an equation

$$\phi(x, y)=0.$$

This is, then, the required envelope.

Rule. *To obtain the envelope of the family of curves*

$$f(x, y, \alpha)=0,$$

eliminate, α, *between*

$$f(x, y, \alpha)=0, \text{ and } f\alpha(x, y, \alpha)=0,$$

where $f_\alpha(x, y, \alpha)$ *is the partial derivative of* $f(x, y, \alpha)$ *w.r.* to α.

If, on solving the equations (i) and (iv) for x, y, we obtain

$$x=\phi(\alpha), \qquad ..(v)$$

$$y=\psi(\alpha), \qquad ..(vi)$$

then (v) and (vi) are the parametric equations of the envelope ; α being the parameter.

Illustration. To find the envelope of the family of lines

$$y-\alpha x-a/\alpha=0, \qquad ..(vii)$$

we eliminate, α, between (vii) and

$$-x+a/\alpha^2=0, \qquad ..(viii)$$

which is obtained by differentiating (vii) *w.r.* to α.

The eliminant is

$$y^2=4ax,$$

which is the envelope of the given family of lines.

This conclusion agrees with the one already arrived at '*ab initio*' in § 19·2.

19·5. Theorem. *The evolute of a curve is the envelope of its normals.*

PR, QR are the normals and PT, QT' the tangents at two points P, Q of a curve. L is the point of intersection of the tangents.

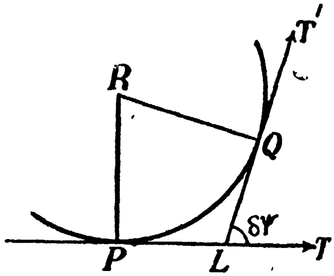


Fig. 154.

$$\angle PRQ = \angle TLT' = \delta\psi,$$

$$\text{arc } PQ = \delta s.$$

Applying sine-formula to the $\triangle PRQ$, we get

$$\frac{PR}{PQ} = \frac{\sin \angle RQP}{\sin \angle PRQ}$$

or $$PR = \sin \angle RQP . \frac{PQ}{\sin \delta\psi}$$

$$= \sin \angle RQP . \frac{\text{chord } PQ}{\text{arc } PQ} . \frac{\delta s}{\delta\psi} . \frac{\delta\psi}{\sin \delta\psi}$$

Let $Q \to P$, so that $\angle RQP \to \angle RPT = \pi/2$.

$$\therefore \qquad \lim_{Q \to P} PR = \sin \frac{\pi}{2} . 1 . \frac{ds}{d\psi} . 1 = \rho.$$

Thus the limiting position of R which is the intersection of the normals at P and Q is the centre of curvature at P.

Hence the theorem.

19·6. Geometrical relation between a family of curves and its envelope.

19·61. *To prove that any singular point of any curve of a given family is a point on its envelope.*

We have to show that if for any point on any member of the family

$$f(x, y, \alpha) = 0, \qquad ..(i)$$

we have

$$f_x = f_y = 0,$$

then, for such a point, we will also have

$$f_\alpha = 0.$$

From (i), we get

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial \alpha} d\alpha = 0.$$

Putting

$$\partial f/\partial x = \partial f/\partial y = 0, \text{ we get}$$

$$\partial f/\partial \alpha = 0.$$

Hence the result.

Thus the locus of the singular points of the curves of a family is a part of its envelope.

19·62. *To prove that, in general, the envelope of a family of curves touches each member of the family.*

Let

$$f(x, y, \alpha)=0, \qquad \ldots(i)$$

be a given family of curves.

Its envelope is obtained by eliminating α between (i) and

$$f_\alpha(x, y, \alpha)=0. \qquad \ldots(ii)$$

Let

$$\left.\begin{array}{l} x=\phi(\alpha) \\ y=\psi(\alpha) \end{array}\right] \qquad \ldots(iii)$$

be the parametric equation of the envelope obtained by solving (i) and (ii) for x and y in terms of α.

The equation (iii) satisfy the equation (i) identically, *i.e.*, for every value of α.

We differentiate (i) *w.r.* to α, regarding x, y as functions of α, so that we obtain

$$\frac{\partial f}{dx} \cdot \frac{dx}{d\alpha}+\frac{\partial f}{dy} \cdot \frac{dy}{d\alpha}+\frac{\partial f}{\partial\alpha}=0,$$

which, with the help of (ii), becomes

$$\frac{\partial f}{\partial x}\phi'(\alpha)+\frac{\partial f}{\partial y}\psi'(\alpha)=0. \qquad \ldots(iv)$$

The slope of the tangent at an ordinary point (x, y) of a curve 'α' of the family is

$$-\frac{\partial f/dx}{\partial f/\partial y}.$$

Also the slope of the tangent at the same point to the envelope (iii) is $\psi'(\alpha)/\phi'(\alpha)$.

We see from (iv) that these two slopes are the same. Thus the slopes of the tangents to the curve and the envelope at the common point are equal. This means that the curve and the envelope have the same tangent at the common point so that they touch.

Note. If for any point on a curve, $\partial f/\partial x$ and $\partial f/\partial y$ are both zero, then the above argument will break down, so that the envelope may not touch a curve at points which are the singular points on the curve.

Cor. 1. We know that a straight line and a conic cannot have a singular point. Hence we can say that *the envelope of a family of straight lines or of conics touches each member of the family at all their common points without exception.*

Examples

1. *Find the envelope of the family of semi-cubical parabolas*

$$y^2-(x+a)^3=0. \qquad ..(i)$$

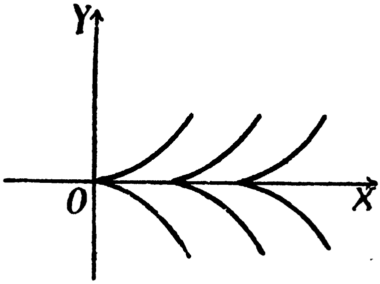


Fig. 155

Differentiating (i) *w.r.* to a, we get

$$3(x+a)^2=0. \qquad ..(ii)$$

Eliminating a between (i) and (ii), we get

$$y=0,$$

which is the required envelope.

We already know that $y=0$ is the locus of singular points *i.e.*, cusps, of (i) and also it touches each member of the family.

2. *Find the envelope of the family*

$$x^2(x-a)+(x+a)(y-m)^2=0,$$

where a is a constant and m is a parameter.

Differentiating *w.r.* to m we get

$$-2(x+a)(y-m)=0.$$

Eliminating m, we get

$$x^2(x-a)=0,$$

which is the envelope.

Thus the envelope consists of two ines

$$x=0 \text{ and } x=a.$$

If we trace the given curves

$$(y-m)^2=\frac{x^2(a-x)}{x+a},$$

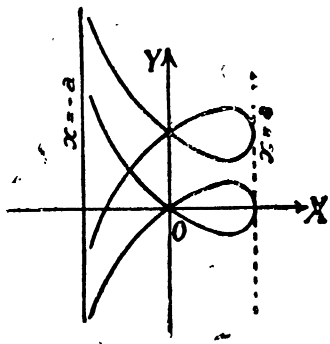


Fig. 156

we will find that y-axis $(x=0)$ is the locus of its singular points and $x=a$ is tangent to each curve.

3. *Considering the evolute of a curve as the envelope of its normals, find the evolute of the ellipse* $x^2/a^2+y^2/b^2=1$.

The equation of the normal at *any* point $(a \cos \theta, b \sin \theta)$ on the ellipse is

$$\frac{ax}{\cos \theta}-\frac{by}{\sin \theta}=a^2-b^2. \qquad ...(i)$$

Thus, (i) is the equation of the family of normals, where θ is the parameter.

Differentiating (i) partially *w.r.* to θ, we have

$$\frac{ax \sin \theta}{\cos^2 \theta}+\frac{by \cos \theta}{\sin^2 \theta}=0. \qquad ...(ii)$$

To obtain the envelope, we have to eliminate θ between (*i*) and (*ii*). From (*ii*), we get

$$\tan\theta=-\frac{(by)^{\frac{1}{3}}}{(ax)^{\frac{1}{3}}}.$$

$$\therefore\quad \sin\theta=\mp\frac{(by)^{\frac{1}{3}}}{\sqrt{[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}]}},\ \cos\theta=\pm\frac{(ax)^{\frac{1}{3}}}{\sqrt{[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}]}}$$

Substituting these values in (*i*), we get

$$\pm[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}]\,[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}]^{\frac{1}{2}}=a^2-b^2,$$

or $$\pm[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}]^{\frac{3}{2}}=(a^2-b^2)],$$

or $$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$$

which is the required evolute.

4. *Find the envelope of the family of ellipses*

$$x^2/a^2+y^2/b^2=1,$$

where the two parameter a, b, are connected by the relation

$$a+b=c\ ;$$

c, being a constant. *(B.U. 1954)*

We will eliminate one parameter and express the equation of given family in terms of the other. We have

$$b=c-a,$$

so that

$$\frac{x^2}{a^2}+\frac{y^2}{(c-a)^2}=1, \qquad \ldots(i)$$

is the equation of the family involving on parameter a.

Differentiating (*i*) partially *w.r.* to a, we have

$$\frac{-2x^2}{a^3}+\frac{2y^2}{(c-a)^3}=0 \text{ or } \frac{c-a}{a}=\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}},$$

which gives

$$a=cx^{\frac{2}{3}}/(x^{\frac{2}{3}}+y^{\frac{2}{3}}),$$

$$\therefore\quad c-a=cy^{\frac{2}{3}}/(x^{\frac{2}{3}}+y^{\frac{2}{3}}).$$

Substituting these values in (*i*), we get

$$x^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})^2+y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})^2=c^2,$$

or $$(x^{\frac{2}{3}}+y^{\frac{2}{3}})^3=c^2$$

or $$x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}},$$

which is the required envelope.

5. *Find the envelope of the family of lines*

$$x/a+y/b=1, \qquad ...(i)$$

where the parameters a and b are connected by the relation

$$a^n+b^n=c^n. \qquad ...(ii)$$

Here, if, as in the example above, we eliminate one parameter, the process of determining the envelope will become rather tedious. This tediousness may be avoided in the following manner.

We consider, b, as a function of, a, as determined from (ii).

Differentiating (i) and (ii) $w.r.$ to the parameter, a, we get

$$\frac{x}{a^2}+\frac{y}{b^2}\cdot\frac{db}{da}=0, \qquad ...(iii)$$

$$a^{n-1}+b^{n-1}\frac{db}{da}=0. \qquad ...(iv)$$

From (iii) and (iv), we eliminate db/da and get

$$\frac{x}{a^{n+1}}=\frac{y}{b^{n+1}}. \qquad ...(v)$$

The equation of the envelope will, now, be obtained by eliminating a and b from (i), (ii) and (v). Now (v) gives

$$\frac{x/a}{a^n}=\frac{y/b}{b^n}=\frac{x/a+y/b}{a^n+b^n}=\frac{1}{c^n}. \quad \text{[From } (i) \text{ and } (ii)]$$

$$a^{n+1}=xc^n \text{ or } a=(xc^n)^{1/(n+1)},$$

$$b^{n+1}=yc^n \text{ or } b=(yc^n)^{1/(n+1)},$$

Substituting these values in (ii), we get

$$(xc^n)^{n/(n+1)}+(yc^n)^{n/(n+1)}=c^n,$$

or

$$x^{n/(n+1)}+y^{n/(n+1)}=c^{n/(n+1)}$$

as the required envelope.

6. *Show that the envelope of a circle whose centre lies on the parabola $y^2=4ax$ and which passes through its vertex is the cissoid*

$$y^2(2a+x)+x^3=0. \qquad (B.U. 1953)$$

Now, $(at^2, 2at)$ is any point on the parabola. Its distance from the vertex $(0, 0)$ is

$$\sqrt{(a^2t^4+4a^2t^2)}.$$

Thus, the equation of the given family of circles is

$$(x-at^2)^2+(y-2at)^2=a^2t^4+4a^2t^2,$$

or

$$x^2+y^2-2at^2x-4aty=0. \qquad ...(vi)$$

Differentiating (i), $w.r.$ to t, we get

$$-4atx-4ay=0, \text{ or } t=-y/x.$$

Substituting this value of t in (i), we get the required envelope.

7. *Find the envelope of straight lines drawn at right angles to the radii vectors of the Cordioide.*

$$r=a(1+\cos\theta).$$

through their extremities.

Let P be any point on the curve. If α be its vectorial angle, then its radius vector $OP=a(1+\cos\alpha)$.

The equation of the line drawn through P at right angles to the radius vector OP is

$$r\cos(\theta-\alpha)=a(1+\cos\alpha). \quad ..(1)$$

The angle α is different for different straight lines.

Differentiating (1) *w.r.* to α, we get

$$r\sin(\theta-\alpha)=-a\sin\alpha. \quad ..(2)$$

To eliminate (α) from (1) and (2), we re-write them as

$$(r\cos\theta-a)\cos\alpha+r\sin\theta\sin\alpha=a. \quad ..(3)$$

$$r\sin\theta\cos\alpha-(r\cos\theta-a)\sin\alpha=0, \quad ..(4)$$

Now, (4) gives

$$\tan\alpha=r\sin\theta/(r\cos\theta-a).$$

$$\therefore \sin\alpha=\frac{r\sin\theta}{\sqrt{(r^2+a^2-2ar\cos\theta)}},\ \cos\alpha=\frac{r\cos\theta-a}{\sqrt{(r^2+a^2-2ar\cos\theta)}}.$$

Substituting these values in (3), we obtain

$$\frac{(r\cos\theta-a)^2+r^2\sin^2\theta}{\sqrt{(r^2+a^2-2ar\cos\theta)}}=a,$$

or

$$r^2+a^2-2ar\cos\theta=a^2,\ i.e.,\ r=2a\cos\theta$$

which is the required envelope.

Exercises

1. Find the envelope of the following families of lines :—

(i) $y=mx+\sqrt{(a^2m^2+b^2)}$, the parameter being m ;

(ii) $x\cos^3\theta+y\sin^3\theta=a$, the parameter being θ ;

(iii) $x\sin\theta-y\cos\theta=a\theta$, the parameter being θ ;

(iv) $x\cos^n\theta+y\sin^n\theta=a$, the parameter being θ ;

(v) $y=mx+am^p$, the parameter being m ;

(vi) $x\operatorname{cosec}\theta-y\cot\theta=c$.

2. Find the envelope of the family of straight lines $x/a+y/b=1$ where a, b are connected by the relation

(i) $a+b=c$. (ii) $a^2+b^2=c^2$. (iii) $ab=c^2$,

c is a constant.

3. Find the envelope of the ellipses, having the axes of co-ordinates as principal axes and the semi-axes a, b connected by the relation

(i) $ab=c^2$. (ii) $a^2+b^2=c^2$.

4. Show that the envelope of the family of the parabolas,

$$\sqrt{(x/a)}+\sqrt{(y/b)}=1$$

under the condition

(i) $ab=c^2$ is a hyperbola having its asymptotes coinciding with axes. (P.U. 1955)

(ii) $a+b=c$ is an astroid.

5. Considering the evolute of a curve as the envelope of its normals, find the evolute of the parabola $y^2=4ax$.

6. Prove that the equation of the normal to the curve

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$

may be written in the form $x \sin \phi - y \cos \phi + a \cos 2\phi = 0$ and find the envelope of the equation of the normals. (*P.U. 1944*)

7. Find the equation of the normal at any point of the curve

$$x=a(3 \cos t - 2 \cos^3 t),\ y=a(3 \sin t - 2 \sin^3 t)$$

and also find the equation of its evolute. [*P.U. (Supp.) 1936*]

8. From any point of the ellipse $x^2/a^2+y^2/b^2=1$, perpendiculars are drawn to the axes and their feet are joined ; show that the straight line thus formed always touches the curve

$$(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1 \qquad (B.U.\ 1952)$$

9. Find the envelope of the curves

$$x^3/a^3 \cos \theta + b^3/y^3 \sin \theta = 1.$$

10. Circles are described on the double ordinates of the parabola $y^2=4ax$ as diameters : prove that the envelope is the parabola $y^2=4a(x+a)$. [*P.U. (Supp.) 1935*]

11. Show that the envelope of the circles whose centres lie on the rectangular hyperbola $x^2-y^2=a^2$ and which pass through the origin is the lemniscate

$$r^2=4a^2 \cos 2\theta. \qquad (B.U.\ 1955)$$

12. Find the envelope of the circles described upon the radii vectors of the ellipse $x^2/a^2+y^2/b^2=1$ as diameter.

13. Find the envelope of a family of parabolas of given latus rectum and parallel axes, when the locus of their foci is a given straight line

$$y=px+q. \qquad (P.U.\ 1937)$$

14. Shnw that the envelope ol the straight line joining the extremities of a pair of semi-conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$ is the ellipse

$$x^2/a^2+y^2/b^2=1/2.$$

15. Find the envelope of straight lines drawn at right angles to the radii vectors of the following curves through their extremities :

(*i*) $r=a+b \cos \theta$. (*ii*) $r^n=a^n \cos n\theta$.

(*iii*) $r=ae^{\theta \cot \alpha}$. (*B.U. 1953*)

16. Find the envelope of the circles described on the radii vectors of the following curves as diameter :

(*i*) $l/r=1+e \cos \theta$. (*ii*) $r^n=a^n \cos n\theta$.

17. Find the envelope of the curves

$$x^m/a^m+y^m/b^m=1,$$

where the parameters a and b are connected by the relation

$$a^p+b^p=c^p. \qquad (P.U.\ 1955)$$

18. Show that the family of circles $(x-a)^2+y^2=a^2$ has no envelope. (*P.U. 1942*)

Miscellaneous Exercises II

1. Show that all the curves represented by the equation

$$\frac{x^{n+1}}{a}+\frac{y^{n+1}}{b}=\left(\frac{ab}{a+b}\right)^n,$$

for different values of n, touch each other at the point $x=y=ab/(a+b)$ and that the radius of curvature at this point is

$$(a^2+b^2)^{\frac{3}{2}} \div n(a+b)^2. \qquad (D.U.\ 1956)$$

2. If the polar equation of a curve be $r=a \sec^2 \frac{1}{2}\theta$, find an expression for its radius of curvature at any point.

3. Show that

$$x=a(\cos \theta+\theta \sin \theta),\ y=a(\sin \theta-\theta \cos \theta),$$

is the involute of a circle.

4. Find the radii of curvature of the curve

$$a^4y^2=a^2x^4-x^6,$$

for the points $x=0$ and $x=a$.

5. If a curve be given by the equations

$$2x=\sqrt{(t^2+2t)}+\sqrt{(t^2-2t)},\ 2y=\sqrt{(t^2+2t)}-\sqrt{(t^2-2t)},$$

find the radius of curvature in terms of t.

6. If a curve passes through the origin at an inclination, α, to the axis of x, show that the diameter of curvature at the origin is the limit of

$$(x^2+y^2)/(x \sin \alpha-y \cos \alpha)$$

when $x \to 0$. Hence show that the radius of curvature at the origin of the curve

$$y^2+2ay-2ax=0,$$

is $-2\sqrt{2}a$.

7. If ρ_1, ρ_2 be the radii of curvature at the extremities of two conjugate semi-diameters of an ellipse semi-axes a, b, prove that

$$(\rho_1^{\frac{2}{3}}+\rho_2^{\frac{2}{3}})\ a^{\frac{2}{3}}\ b^{\frac{2}{3}}=a^2+b^2.$$ (*Delhi Hons. 1948* ; *P.U.*)

8. Show that the projections on the X-axis of radii of curvature at the corresponding points of $y=\log \cos x$ and its evolute are equal.

9. If P is any point on the curve $r^m=a^m \cos m\theta$ and Q is the intersection of the normal at P with the line through O at right angles to the radius vector OP, prove that the centre of curvature corresponding to P divides PQ in the ratio $l : m$.

10. The equation of the equiangular spiral is $r=ae^{\theta \cot \alpha}$. Prove that if O be the pole and P any point on the curve, then a straight line drawn through O at right angles to OP, intersects the normal at P in C such that PC is this radius of curvature at P. (*M.U.*)

11. Show that the normal to the Lemniscate

$$r^2=a^2 \cos 2\theta$$

at the point whose vectorial angle is $\pi/6$ is perpendicular to the initial line and that the centre of curvature at the point at a distance $\sqrt{2}a/12$ below the initial line.

12. Show that a point P on the curve

$$x=3 \cos \theta-\cos^3\theta,\ y=3 \sin \theta-\sin^3\theta$$

where $\theta=\pi/4$, the normal passes through the origin O and that the centre of curvature at P divides OP internally in the ratio 4 : 1. (*B.U.*)

13. Show that the pole lies within the circle of curvature at every point of the cardioide $= a(1+\cos \theta)$, and that its power with respect to it is $r/\sqrt{3}$.

14. Find the co-ordinates of the point in which the circle of curvature of the parabola $y^2=4x$ at the point $(t^2, 2t)$ meets the curve again.

The circles of curvature of a fixed parabola at the extremities of a focal chord meet the parabola again at H and K ; prove that HK passes through a fixed point. (*Indian Police 1935*)

15. ρ, ρ' are the radii of curvature of an ellipse and its evolute at corresponding points P, P' Δ is the area of the triangle which the tangent at P makes with the axes ; show that ρ/ρ' varies as Δ.

16. Prove that the maximum length of the perpendicular from the pole on the normal to the curve $r=a(1-\cos \theta)$ is $4\sqrt{3}a/9$. (*B.U.*)

17. For what points on the curve $xy^2=a^2(a-x)$ is the square of sub-tangent maximum.

18. Show that the tangents at the points of inflexion of the curve

$$y^2(x+a)=x^2(x-a)$$

are

$$5x \pm 3\sqrt{3}y-4a=0, \qquad \text{(B.U.)}$$

19. Find points of inflexion on

$$x(xy-a^2)^2=b^5. \qquad \text{(M.T.)}$$

20. Find the co-ordinates of two real points of inflexion on the curve

$$y^2=(x-2)^2(x-5)$$

and show that they subtend a right angle at the double point.

21. If O is a fixed and Q, a variable point on a given circle whose centre is C and if OQ is produced to P so that $QP=2.\ OC$, prove that the radius of curvature of the locus of P at the point P is

$$\tfrac{4}{3}(OC.CP)^{\frac{1}{2}} \qquad \text{(B.U.)}$$

22. If O is the extremity of the polar sub-tangent at a point P of the curve

$$r=a\tan(\theta/2).$$

prove that the locus of Q is.

$$r=a(1+\sin\theta). \qquad \text{(B.U.)}$$

23. Find polar sub-tangent and polar sub-normal at any point of the curve

$$r=a\theta^2/(1+\theta^2);$$

show that the polar sub-tangent constantly increases as θ increases and the polar sub-normal attains its maximum value $3\sqrt{3}a/8$ at the point $(a/4, 1/\sqrt{3})$.

24. Show that the circles of curvature of the parabola

$$y^2=4ax$$

for the ends of the latus rectum have for their equations

$$x^2+y^2-10ax \pm 4ay-3a^2=0,$$

and that they cut the curve again in the point $(9a, \mp 6a)$. (P.U.)

25. Show that the centre of curvature at every point of the curve

$$r=a(\theta-\sin\theta)$$

where it meets the positive side of initial line is the pole.

26. Show that the co-ordinates of the centre of curvature may be given in any of the following ways :—

(i) $\left[x-\dfrac{dy}{d\psi},\ y+\dfrac{dx}{d\psi}\right]$

(ii) $\left[x+\dfrac{1+(dx/dy)^2}{d^2x/dy^2},\ y+\dfrac{1+(dy/dx)^2}{d^2y/dx^2}\right]$

(iii) $\left[\dfrac{1}{2}\dfrac{d^2r/dx^2}{d^2x/dy^2},\ \dfrac{1}{2}\dfrac{d^2r/dx^2}{d^2y/dx^2}\right].$

27. Show that the area of the triangle formed by a tangent to the cissoid

$$y^2(2a-x)=x^3,$$

its asymptote and X-axis is greatest for the point given by

$$x=3a/2,$$

28. Find the equations of the tangents to the curve

$$x^2(y+1)=y^2(x-4)$$

parallel to the line $y=x$.

29. Trace the curve

$$r=a(1+\cos\theta)$$

and prove that the greatest distance of the tangent to the curve from the middle point of the axis is $\sqrt{2}a$. (P.U.)

30. Show that the length of the perpendicular from the foot of the ordinate to normal at any point of the curve

$$x=a(\sinh 2\theta-2\theta),\ y=4a\cosh\theta,$$

is constant.

31. Show that for the curve

$$x/a=\log(1+\cos\theta)-\cos\theta,\ y/a=\sin\theta,$$

the portion of the X-axis intercepted between the tangent and the normal at any point is constant.

32. Find the nature of the double poiut on the curve

$$(y+1)^2=(x-1)^2(x-4)$$

and show that it has two real points of inflexion. (*P.U.*)

33. Show that the pedal equation of the curve

$$y^2(3a-x)=(x-a)^3$$

is

$$p^2(r^2+15a^2)=9a^2(r^2-a^2).$$

34, Show that the tangential pedal equation of the curve

$$(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1$$

is

$$1/a^2\sin^2\psi+1/b^2\cos^2\psi=1/p^2. \quad (B.U.\ 1955)$$

35. If on the tangent at each point of curve, a constant length be measured from the point of contact, prove that the normal to the locus of the point so formed passes through the corresponding centre of curvature of the given curve.

36. Show that there is a curve for which the circles

$$(x-a)^2+y^2-2y\cosh a+1=0,$$

as the parameter 'a' varies, are the circles of curvature.

37. Prove that the distance from a fixed point $P_0(x_0, y_0)$ to a variable point P on the curve $f(x, y)=0$ is stationary when PP_0 is normal to the curve at P.

38. The tangents at any point 'θ' of

$$x=a(\theta+\sin\theta),\ y=a(1+\cos\theta)$$

is normal to the curve at the point where it meets the next span on the right, prove that $\cot(\theta/2)=\pi/2.$

39. Show that the locus of the intersection of perpendicular tangents to the Astroid

$$x=a\cos^3\theta,\ y=a\sin^3\theta$$

is the curve

$$2(x^2+y^2)^3=a^2(x^2-y^2)^2\ ; \quad (D.U.\ Hons.,\ 1959)$$

and the locus of the intersection of perpendicular normals is the curve

$$(x^2+y^2)^3=8a^2x^2y^2.$$

40. For the curve

$$x=a(2\cos t+\cos 2t),\ y=a(2\sin t-\sin 2t),$$

find the equation of the tangent and normal at the point whose parameter is t and show that

(*i*) the tangent at P meets the curve in the points Q, R whose parameters are $-\frac{1}{2}t$ and $\pi-\frac{1}{2}t$,

(*ii*) QR is constant,

(*iii*) the tangents at Q and R intersect at right angles on a circle,

(*iv*) the normals at P, Q and R are concurrent and intersect on a concentric circle.

41. P, Q are any two points on $ay^2=x^3$ such that PQ always passes through a fixed point (at^2, at^3) lying on the curve ; show that the locus of the point of intersection of the tangents at P, Q is the parabola

$$(3tx+2y)^2+2at^3y=0.$$

42. Tangent at any point P of the Folium $x^3+y^3=3axy$ meets the curve again at Q ; show that

$$\cot^2\angle XOP+\tan\angle XOP=0,$$

where OX is X-axis.

43. Show that $x=a/4$ is a bitangent of the cardioide

$$r=a(1-\cos\theta).$$ (*Birmingham*)

44. The envelope of the straight line

$$x\cos\phi+y\sin\phi=a(\cos n\phi)^{\frac{1}{n}}$$

is the curve whose polar equation is

$$r^{n/(1-n)}=a^{n/(1-n)}\cos\frac{n}{1-n}\theta.$$

45. Show that the radius of curvature of the envelope of the line

$$x\cos\alpha+y\sin\alpha=f(\alpha),$$

is

$$f'(\alpha)+f''(\alpha),$$

and that the centre of curvature is the point

$$\left.\begin{aligned}x&=-f'(\alpha)\sin\alpha-f''(\alpha)\cos\alpha\\ y&=f'(\alpha)\cos\alpha-f''(\alpha)\sin\alpha\end{aligned}\right|$$ (*M.U.*)

46. Discuss the nature of singularity at the origin of the curves,

(*i*) $y^2=x^4(x^2-1)$. (*ii*) $x^5-a(x^2-ay)^2=0$. (*B.U.*)

47. Show that the curve

$$by^2=x^3\sin^2(x/a)$$

has a cusp at the origin and an infinite series of nodes lying at equal distances from each other.

48. Trace the following curves :—

(*i*) $r=a(2\cos\theta+\cos 3\theta)$. (*ii*) $y^4-2c^2y^2+a^2x^2=0$.

(*iii*) $x=a[\cos t+\log\tan\frac{1}{2}t]$. $y=a\sin t$.

49. The tangent to the evolute of a parabola at the point where it meets the parabola is also a normal to the evolute at the point where it meets the evolute again.

50. If ρ be the radius of curvature of a parabola at a point whose distance measured along the curve from a fixed point is, s, prove that

$$3\rho\frac{d^2\rho}{ds^2}-\left[\frac{d\rho}{ds}\right]^2-9=0.$$

51. If ρ and ρ' be the radii of curvature at corresponding points of a cusp and its evolute, and p, q, r first, second and third differential co-efficients of y with respect to x, prove that

$$\frac{\rho'}{\rho}=\frac{3pq^2-r(1+p^2)}{q^2}.$$

[Hint. $\rho'=d^2s/d\psi^2$.]

52. Show that the radius of curvature at a point of the evolute of the curve

$$r^n=a^n\cos n\theta,$$

corresponding to the point (r, θ) is

$$\frac{n-1}{(n+1)^2}r\sec n\theta\tan n\theta.$$

53. (*a*) Show that the curve

$$x^4-2x^2y-xy^2-2x^2-2xy+y^2-x+2y+1=0$$

has a single cusp of the second kind at the point $(0, -1)$.

(*b*) Show that the radius of curvature of the curve $f(r, \theta)=0$ is

$$\frac{r\operatorname{cosec}\varphi}{1+\dfrac{d\varphi}{d\theta}}$$

and the chord of curvature perpendicular to the radius vector is $2\rho\cos\varphi$ where ρ denotes the radius of curvature. (*D.U. 1956*)

ANSWERS

[Pages 4—67.]

Page 4.

Ex. 3. (*i*) 0. (*ii*) a, (*iii*) 1, (*iv*) $n\pi$; n being any integer.

Page 9.

Ex. 2. 1·710.

Page 10.

Ex. 2. (*i*) $| x-1 | \leqslant 2$. (*ii*) $| x-\frac{7}{2} | < \frac{3}{2}$.
(*iii*) $| x-2 | \leqslant 5$. (*iv*) $| x-l | < \varepsilon$.

Ex. 3. (*i*) $-1<x<5$. (*ii*) $-3 \leqslant x \leqslant 1$. (*iii*) $-1<x<3$, $x \neq 1$.

Page 14.

Ex. 4. (*i*) $(-\frac{1}{2}, \infty)$

(*ii*) The entire aggregate of real numbers excluding the numbers $(2n+1)\pi$, where n is any integer.

(*iii*) The set of intervals

$$[(2n-\tfrac{1}{6})\pi, (2n+\tfrac{1}{3})\pi],$$

when n is any integer.

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1. (*i*) $\pi/6$. (*ii*) $-\pi/4$. (*iii*) $\pi/3$. (*iv*) $2\pi/3$.

2. (*i*) (0, 1). (*ii*) The set of intervals $[2n\pi, 2n+1\pi]$. (*iii*) [0, 1). (*iv*) $[-1, 1]$. (*v*) $[-\infty, -1]$, $[1, \infty]$. (*vi*) $[0, \infty]$. (*vii*) The set of intervals $(2n\pi, 2n+1\pi)$, (*viii*) $[1, \infty]$. (*ix*) $[1, \infty]$. (*x*) $[-\infty, \infty]$, (*xi*) $[-\infty, -2]$, $[-1, \infty]$. (*xii*) The entire aggregate of real numbers excluding the numbers $(n+\frac{1}{2})\pi$. (*xiii*) $[-1, 1]$. (*xiv*) $[-\infty, 0]$, $[0, \infty]$.

3. (*i*) Increasing in $[-\infty, \infty]$. (*ii*) Decreasing in $[-\infty, \infty]$. (*iii*) Decreasing in $[-\infty, 0]$ and in $[0, \infty]$. (*iv*) Increasing in $[-\infty, 0]$ and decreasing in $[0, \infty]$. (*v*) Increasing in the set of intervals $[(2n-\frac{1}{2})\pi, (2n+\frac{1}{2})\pi]$ and decreasing in $[(2n+\frac{1}{2})\pi, (2n+\frac{3}{2})\pi]$.

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3. (*i*) continuous. (*ii*) discontinuous.

6. Discontinuous for $x=0$ and $x=1$.

7. Continuous for every value of x.

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2. $-\frac{5}{8}$.

[Pages 66—67.]

2. (*i*) continuous. (*ii*) discontinuous. (*iii*) continuous.

(*iv*) discontinuous. (*v*) continuous. (*vi*) discontinuous.
(*vii*) discontinuous. (*viii*) continuous.

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Ex. 4. (*i*) 13. (*ii*) $-\frac{1}{25}$.

Page 75, §4·12.

Ex. 1. (*i*) $-2x/(x^2+3)^2$. (*ii*) $-1/2x^{\frac{3}{2}}$. (*iii*) $3x^2$. (*iv*) $(2ax+b)$.

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Ex. 1. $2, 2(1-x^2)/(1+x^2)^2$. **Ex. 2.** $-1/\sqrt{2}$.

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Ex. 3. $2x+y+1=0, \quad 4x=y+4$.

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Ex. 1. (*a*) 8, 2 ; 2, 2. (*b*) $-1/16$, $1/32$; -1, 2.
(*c*) $1/4$, $-1/32$; $1/2$, $-1/4$.

Ex. 3. -4, 3, -4 ; -2, 2, 2 ; 2, 7, 8.

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Ex. 2. (*i*) $(x-1)/2x^{\frac{3}{2}}$. (*ii*) $(2x^{-\frac{5}{6}}+5x^{-\frac{7}{12}})/12$.
(*iii*) $(3x^2+x-1)/2x^{\frac{3}{2}}$. (*iv*) $(1+7x)/6^{\frac{5}{6}}$.

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Ex. 2. (*i*) $2x+5$. (*ii*) $2(x+2)(3x+5)$.
(*iii*) $(2x+3)^2(10x-3)/2x^{\frac{3}{2}}$.

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Ex. 2. (*i*) $-(4x+5)^{-2}$. (*ii*) $-x(x+2)^{-3}$.
(*iii*) $[(x^{-\frac{11}{6}}-x^{\frac{1}{6}})+5(x^{\frac{7}{6}}-x^{\frac{5}{6}})]/6x^2/(x^{\frac{1}{3}}+x^{-\frac{1}{3}})^2$.

Page 87.

Ex. 2. (*i*) $an(ax+b)^{n-1}$. (*ii*) $-2x/(1+x^2)^2$.
(*iii*) $(ax+b)/\sqrt{(ax^2+2bx+c)}$.
(*iv*) $-2x/\sqrt{(x^4-1)}[x^2+\sqrt{(x^4-1)}]$.
(*v*) $-\dfrac{2a^2x}{(a^2-x^2)^{\frac{1}{2}}(a^2+x^2)^{\frac{3}{2}}}$. (*vi*) $\dfrac{-4a}{3x^{\frac{1}{3}}(1+ax^{\frac{2}{3}})^2}$.
(*vii*) $\dfrac{x^4-2x^2+4}{(x^2-1)^{\frac{3}{2}}(x^2-4)^{\frac{1}{2}}}$.

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Ex. 2. (*i*) $\dfrac{b}{a} \cdot \dfrac{t^2-1}{2t}$. (*ii*) $\dfrac{2t-t^4}{1-2t^3}$.
(*iii*) $\dfrac{3t+t^3}{2}$. (*iv*) $(t^4+2t^2-1)/2t$.

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Ex. 3. (*i*) $m \sin^{m-1} x \cos x$. (*ii*) $-m \sin mx$.

(*iii*) $mx^{m-1} \cos x^m$. (*iv*) $-\sin 2x$.

(*v*) $(x \cos x - \sin x)/x^2$. (*vi*) $\sin x$.

(*vii*) $-(ax+b)(ax^2+2bx+c)^{-\frac{1}{2}} \sin \sqrt{(ax^2+2bx+c)}$.

(*viii*) $(m \cos^2 x - n \sin^2 x) \sin^{m-1} x \cos^{n-1} x$.

Ex. 4. (*i*) $\tan t$. (*ii*) $\cot t$. (*iii*) $-\tan t$.

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Ex. 1. (*i*) $-4 \operatorname{cosec} 2x \cot 2x$.

(*ii*) $\cos x \tan 2x + 2 \sin x \sec^2 2x$.

(*iii*) $\dfrac{\sin 4x - 8x \sin^2 x}{4 \cos^2 x \sin 2x}$. (*iv*) $2 \sin 2x$.

(*v*) $-\dfrac{1}{\sqrt{[(\cos 2x)]}(\cos x + \sin x)}$. (*vi*) $\frac{1}{2} \sec^2 \dfrac{x}{2}$.

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(*i*) $-9 \operatorname{cosec}^3 3x \cot 3x$. (*ii*) $\frac{1}{2}a\sqrt{[\sec (ax+b)]} \tan (ax+b)$.

(*iii*) $\frac{1}{2}b \sec \sqrt{(a+bx)} \tan \sqrt{(a+bx)}/\sqrt{(a+bx)}$.

(*iv*) $-\sec \cdot (\operatorname{cosec} x) \tan (\operatorname{cosec} x) \cot x \operatorname{cosec} x$.

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Ex. 1. (*i*) $\dfrac{1}{2\sqrt{(x-x^2)}}$. (*ii*) $\dfrac{-1}{4(1+x)\sqrt{x}\sqrt{(\cot^{-1}\sqrt{x})}}$.

(*iii*) $\dfrac{1}{1+x^2}$. (*iv*) $-\dfrac{\sin \sqrt{x}}{2\sqrt{x}} \cdot \dfrac{1}{1+\cos^2\sqrt{x}}$. (*v*) $\dfrac{2}{x\sqrt{(x^4-1)}}$.

(*vi*) $\dfrac{1}{2\sqrt{(x-x^2)}}$. (*vii*) $-\dfrac{2}{1+x^2}$. (*viii*) $\dfrac{x\sqrt{(1-x^2)}+\sin^{-1} x}{(1-x^2)^{3/2}}$.

(*ix*) $\dfrac{2}{\sqrt{x}(1+4x)}$. (*x*) $\dfrac{\sqrt{(b^2-a^2)}}{b+a \cos x}$.

Ex. 2. 1.

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Ex. (*i*) $\cot x$. (*ii*) $-\dfrac{\sin (\log x)}{x}$. (*iii*) $e^{\sin x} . \cos x$.

(*iv*) $\dfrac{\cot (\log x)}{x}$. (*v*) $\dfrac{2x+1}{2(x^2+x+1)}$. (*vi*) $\sec x$.

(*vii*) $\sec x$. (*viii*) $\dfrac{e^{2x}.2x \log x - e^{2x}}{x(\log x)^2}$. (*ix*) $\dfrac{a^{\sqrt{x}} . \log a}{4\sqrt{x}\sqrt{(a^{\sqrt{x}})}}$.

(*x*) $\dfrac{2x. \log_{10} e}{\sqrt{(1-x^4)} \sin^{-1} x^2}$. (*xi*) $\dfrac{1}{\sqrt{(x^2+a^2)}}$. (*xii*) $m \tanh mx$.

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(*xiii*) $a^{2x}(\sin^2 x \,.\, \log a^2 + \sin 2x)$. (*xiv*) $\dfrac{a}{3} \cdot \dfrac{e^{\sqrt[3]{(ax)}}}{\sqrt[3]{(a^2x^2)}}$.

(*xv*) $\dfrac{2ab}{a^2 \cos^2 x - b^2 \sin^2 x}$.

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Ex. (*i*) $\tanh x$. (*ii*) $\sinh 2x \, e^{\sinh^2 x}$.

(*iii*) $\sec^2 x \tanh x + \tan x \operatorname{sech}^2 x$.

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Ex. 4. (*i*) $(\cos x)^{\log x} [(\log \cos x)/x - \tan x . \log x]$.

(*ii*) $(1+x^{-1})^x[\log(1+x^{-1})-(1+x)^{-1}]$. (*iii*) $e^{x^x} x^x \log ex$.

(*iv*) $(\tan x)^{\cot x} . \operatorname{cosec}^2 x \log (e \cot x) -$

$(\cot x)^{\tan x} . \sec^2 x \log (e \tan x)$.

(*v*) $(\log x)^x \left(\log \log x + \dfrac{1}{\log x}\right) +$

$(\sin^{-1} x)^{\sin x}\left(\cos x . \log \sin^{-1} x + \dfrac{\sin x}{\sqrt{(1-x^2)} \sin^{-1} x}\right)$.

(*vi*) $\dfrac{(1-x)^{1/2}(2-x^2)^3}{(3-x^3)^4(4-x^4)^{4/5}} \cdot \left[-\dfrac{1}{2(1-x)} - \dfrac{4x}{3(2-x^2)} + \dfrac{9x^2}{4(3-x^3)} + \dfrac{16x^3}{5(4-x^4)}\right]$.

(*vii*) $(2x^4+15x^2+36)/3(x^2+3)^{\frac{3}{2}}(x^2+4)^{\frac{2}{3}}$.

(*viii*) $\sin x \,.\, e^x \,.\, \log x \,.\, x^x [\cot x + (x \log x)^{-1} + \log e^2 x]$.

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Ex. 5. (*i*) $\dfrac{2}{1+x^2}$. (*ii*) $\dfrac{3}{\sqrt{(1-x^2)}}$. (*iii*) $\dfrac{1}{2\sqrt{x}(1+x)} - \dfrac{1}{1+x^2}$.

(*iv*) $\pm \frac{1}{2}$. (*v*) $\dfrac{2}{1+x^2}$.

(*vi*) $\dfrac{1}{2\sqrt{x}(1+x)}$. (*vii*) $\dfrac{2\sqrt{x}-\sqrt{(1+x)}}{2\sqrt{x}\sqrt{(1+x^2)}}$.

Ex. 6. (*i*) -1. (*ii*) $3/(1+x^2)$.

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Ex. 3. (*i*) $2x \cos x^2$. (*ii*) $\sin 2x$. (*iii*) $1/2\sqrt{x}$.

(*iv*) $e^{\sin x} \cos x$. (*v*) $e^{\sqrt{x}}/2\sqrt{x}$. (*vi*) $1/(1+x^2)$.

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1. $2x\, e^{x^2}$. **2.** $\dfrac{\sec^2 x\sqrt{(\cot x)}}{2}$. **3.** $(x \cos x + \sin x)$.

4. $2x \sec^2 x^2$. **5.** $\dfrac{1}{|x|\sqrt{(x^2-1)}}$.

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6. $\dfrac{\cos^{-1}x - x\sqrt{(1-x^2)}}{(1-x^2)^{3/2}}$.

7. $\dfrac{3-x^2}{x^4\sqrt{(1+x^2)}}$. **8.** $\dfrac{x^4-2a^2x^2+4a^4}{(x^2-a^2)^{3/2}(x^2-4a^2)^{1/2}}$.

9. $\dfrac{a^2b}{a^2+x^2\left(\tan^{-1}\dfrac{x}{a}\right)^2}\left[\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+\dfrac{x}{a^2+x^2}\right]$.

10. $e^{x^x} \cdot x^x \cdot \log ex$. **11.** $\dfrac{1}{3} \cdot \dfrac{1}{x^{2/3}(1+x)^{2/3}}$.

12. $\frac{1}{2}$. **13.** $\frac{1}{2}$.

14. $\dfrac{(1-2x)^{2/3}(1+3x)^{-3/4}(1-4x)^{4/5}}{(1-6x)^{5/6}(1+7x)^{-6/7}(1-8x)^{7/8}} \times$

$\left[\dfrac{5}{1-6x}+\dfrac{6}{1+7x}+\dfrac{7}{1-8x}-\dfrac{4}{3(1-2x)}-\dfrac{9}{4(1+3x)}-\dfrac{16}{5(1-4x)}\right]$.

15. $x^{(x^2+1)} \log (ex^2)$.

16. $[1+x \log x \log ex]x^{(x^x)+x-1}$.

17. $1-x^2-3x\sqrt{(1-x^2)} \cdot \sin^{-1}x$. **18.** $\operatorname{cosech} x$.

19. $\dfrac{x}{\sqrt{(1-x^4)}}$. **20.** $\dfrac{1}{(\cos x-\sin x)\sqrt{(\cos 2x)}}$.

21. $\dfrac{2a}{\sqrt{(1-a^2x^2)}}$. **22.** $10^{\log \sin x} \cdot \cot x \cdot \log 10$.

23. $\log (e \log x)+\log x \cdot \log \log x$.

24. $\dfrac{(a^2-b^2) \sin x}{(a^2+b^2)(1+\cos^2 x)+4ab \cos x}$.

25. $(\sin x)^{\cos^{-1} x}\left(\cos^{-1} x \cdot \cot x-\dfrac{\log \sin x}{\sqrt{(1-x^2)}}\right)$.

26. $e^{-ax^2}[\cos (x \log x) \cdot \log ex-2ax \sin (x \log x)]$.

27. $a \operatorname{sech} ax$. **28.** $\dfrac{1}{2(1+x^2)}$. **29.** $\dfrac{2a^2x}{a^4+x^4}$.

30. $\dfrac{x^2-1}{x^2-4}$. **31.** $\dfrac{1}{\sqrt{x(1-x)}}$. **32.** $\dfrac{\sin a}{1-2x\cos a+x^2}$.

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33. $9x^4 \sin(3x-7) \, . \log(1-5x) \times$

$$\left[\frac{4}{x}-[\log(1-5x)]^{-1}\frac{5}{(1-5x)}+3\cot(3x-7)\right].$$

34. $e^{ax}[a(1+x^2)\cos(b\tan^{-1}x)-b\sin(b\tan^{-1}x)]/(1+x^2)$.

35. $\dfrac{n \operatorname{cosec} x \, . \cot x \, . e^{-n/\sin x}}{2[1-\sqrt{(1-e^{-n/\sin x})}] \, . \sqrt{[1-e^{-n/\sin x}]}}$. √

36. $-\operatorname{cosec}^2 x \, . \coth x - \operatorname{cosech}^2 x \, . \cot x$.

37. $a^x \sinh x + xa^x \sinh x \log a + xa^x \cosh x$.

38. $\sqrt{3}\sqrt{(\sec x)}\sqrt{(\sec 3x)}$.

39. $\left(1+\dfrac{1}{x}\right)^{x^2}\left[2x\log\left(1+\dfrac{1}{x}\right)-\dfrac{x}{1+x}\right]$.

40. $\dfrac{C\sqrt{(Ac-aC)}}{\sqrt{[(ax^2+c)(Ax^2+C)]}}$.

41. $(1-x^3)^{-\frac{1}{3}}$.

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42. (*i*) $1/(x^4+1)$. (*ii*) $x/(x^3-1)$.

43. (*i*) 0. (*ii*) $-\tan 3t$. (*iii*) $\tan t$.

45. $x \sin x \cos x/\log x$.

46. $\dfrac{x^{\sin x-1}(\sin x + x\cos x \log x)}{(\sin x)^{x-1}(x\cos x+\sin x \log \sin x)}$.

47. $\frac{1}{2}$. **48.** $x(2+2\tan\log x+\sec^2\log x)$. **49.** $-\frac{1}{2}$.

50. $\dfrac{-\sqrt{(1-x^2)}\,(\log x)^{\tan x}[\sec^2 x \log(\log x)+\tan x(x\log x)^{-1}]}{m\cos(m\cos^{-1}x)}$.

Page 111.

1. $f'(0)$ exists but $f''(0)$ does not.

2. Differentiable at $x=0$ for $m \geqslant 2$. $f'(x)$ is continuous at the origin for $m \geqslant 3$.

3. $f(x)$ is continuous but does not have derivative at the origin.

5. $f(x)$ is continuous but not derivable at the origin.

6. Continuous when x is irrational or zero and discontinuous for other values of x. Differentiable for no value.

7. $f'(0)=1$.

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10. Continuous but not differentiable for $x=a$.

11. (*i*) Continuous. (*ii*) Discontinuous.

12. Discontinuous at 0 ; derivable at 1 ; continuous but non-derivable at 2.

Page 115.

5. $-3/2$.

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11. $\dfrac{d^2y}{dz^2}+4y=0.$

Page 119.

1. (*i*) $(-1)^n n!\left[\dfrac{16}{(x-2)^{n+1}}-\dfrac{1}{(x-1)^{n+1}}\right], n>2.$

(*ii*) $\dfrac{(-1)^n n!}{4}\left[\dfrac{3}{(x-2)^{n+1}}+\dfrac{1}{(x+2)^{n+1}}\right].$

(*iii*) $(-1)^n n!\left[\dfrac{1}{(x-1)^{n+1}}-\dfrac{1}{(x+1)^{n+1}}+\dfrac{2(n+1)}{(x-1)^{n+2}}\right].$

2. $10!\left[\dfrac{1}{(x-1)^{11}}+\dfrac{1}{(x+1)^{11}}-\dfrac{1}{(x+2)^{11}}\right].$

$(-1)^n n!\left[\dfrac{1}{(x-1)^{n+1}}+\dfrac{1}{(x+1)^{n+1}}-\dfrac{1}{(x+2)^{n+1}}\right].$

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3. (*i*) $(-1)^n n!\left[\dfrac{1}{(x+1)^{n+1}}-\dfrac{2^n}{(2x+1)^{n+1}}\right].$

(*ii*) $\dfrac{(-1)^n n!}{4a^3}\left[\dfrac{1}{(x-a)^{n+1}}-\dfrac{1}{(x+a)^{n+1}}\right.$

$$\left.-\frac{2\sin[(n+1)\cot^{-1}(x/a)]}{(a^2+x^2)^{(n+1)/2}}\right].$$

(*iii*) $\dfrac{2(-1)^n n!}{\sqrt{3}r^{n+1}}\sin(n+1)\theta$, where

$$r=\sqrt{(x^2+x+1)},\ \theta=\cot^{-1}[2x+1)/\sqrt{3}].$$

(*iv*) $\dfrac{(-1)^n n!}{r^{n+1}}\left[\cos(n+1)\theta-\dfrac{\sin(n+1)\theta}{\sqrt{3}}\right]$, where

$$r=\sqrt{(x^2+x+1)},\ \theta=\cot^{-1}[(2x+1)/\sqrt{3}].$$

7. (*i*) $(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\theta=\cot^{-1}x$.

(*ii*) $(-1)^{n-1}(n-1)!\operatorname{cosec}^n\alpha\sin^n\theta\sin n\theta$, where

$$\theta=\cot^{-1}[(x-\cos\alpha)/\sin\alpha].$$

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2. (*i*) $\frac{3}{4} \sin (x+\frac{1}{2}n\pi)-(3^n/4) \sin (3x+\frac{1}{2}n\pi)$.

(*ii*) $\frac{1}{4}[2^n \cos(2x+\frac{1}{2}n\pi)+4^n \cos(4x+\frac{1}{2}n\pi)+6^n \cos(6x+\frac{1}{2}n\pi)]$.

(*iii*) $\frac{1}{16}[2 \cos (x+\frac{1}{2}n\pi)-3^n \cos (3x+\frac{1}{2}n\pi)-5^n \cos(5x+\frac{1}{2}n\pi)]$.

(*iv*) $[3e^x-4\cdot 5^{\frac{1}{2}n}\cos (x+n \tan^{-1} 2)$

$+17^{\frac{1}{2}n}\cos (4x+n \tan^{-1} 4)]/8$.

(*v*) $\frac{1}{4} e^{2x} [25^{\frac{1}{2}n} \cos (x+n \tan^{-1} \frac{1}{2})-13^{\frac{1}{2}n} \cos (3x+n \tan^{-1} \frac{3}{2})$

$-29^{\frac{1}{2}n} \cos (5x+n \tan^{-1} \frac{5}{2})]$.

Page 123.

1. (*i*) $e^x\left[x^n+\frac{n^2}{1\,!}x^{n-1}+\frac{n^2(n-1)^2}{2\,!}x^{n-2}+\ldots\ldots\right.$

$\left.\frac{n^2(n-1)^2\ldots\ldots 1^2}{n\,!}\right]$.

(*ii*) $x^3 \cos (x+\frac{1}{2}n\pi)+3nx^2 \cos [x+\frac{1}{2}(n-1)\pi]$

$+3n(n-1)x \cos [x+\frac{1}{2}(n-2)\pi]+$

$n(n-1)(n-2) \cos [x+\frac{1}{2}(n-3)\pi]$.

(*iii*) $a^{n+2}.\ e^{ax}.\ x^2$

(*iv*) $e^x[\log x+{}^nc_1x^{-1}-{}^nc_3x^{-2}+{}^nc_3\, 2\,!.\ x^{-3}\ldots$

$+\ldots.+(-1)^{n-1}\ {}^nc_n\ (n-1)\,!\ .\ x^{-n}]$.

4. $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(a^2-n^2)y_n=0$.

5. $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+a^2-m^2)y_n=0$.

Page 125.

1. $u_{2n}(0)=0$; $u_{2n+1}(0)=(-1)^n\ 2n\,!$.

2. $y_{2n}(0)=0$; $y_{2n+1}(0)=m(1^2-m^2)(3^2-m^2)\ldots[(2n-1)^2-m^2]$.

3. $y_{2n}(0)=0$; $y_{2n+1}(0)=(-1)^n\ .\ 1^2\ .\ 3^2\ .\ 5^2\ldots(2n-1)^2$.

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4. $y_{2n}(0)=m^2(m^2-2^2)(m^2-4^2)\ldots.[m^2-(2n-2)^2]$;

$y_{2n+1}(0)=m(m^2-1^2)(m^2-3^2)(m^2-5^2)\ldots[m^2-(2n-1)^2]$.

5. $y_{2n+1}=-e^{m\pi/2}m(1^2+m^2)(3^2+m^2)\ldots.[(2n-1)^2+m^2]$.

$y_{2n}=e^{m\pi/2}m^2(2^2+m^2)(4^2+m^2)\ldots.[(2n-2)^2+m^2]$.

6. $y_{2n+1}=0,\ y_{2n}=(-1)^{n-1}\ 2\ .\ 2^2\ .\ 4^2\ .\ 6^2\ldots.(2n-2)^2$.

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4. $(1+2x^2)/(1-x^2)^{\frac{5}{2}}$.

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9. If n is even, $y_n(0)=0$; if n is odd, $y_n(0)=n\,!\,(3n-5)\dfrac{1}{2^{n+4}}$.

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Ex. 3. $c=(6-\sqrt{21})/6$.

Ex. 4. (*i*) $\frac{1}{7}$. (*ii*) $\sqrt{5}$. (*iii*) $\log(e-1)$.

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2. For $a \geqslant 1$, the function is steadily increasing ; for $a \leqslant -1$, the function is steadily decreasing.

3. Increasing in $[-2, -1]$ and $[0, 1]$; decreasing in $(-\infty, -2]$ $[-1, 0]$, $[1, \infty)$.

4. Increasing in $[-1, 1]$; decreasing in $(-\infty, -1]$, and $[1, \infty)$.

5. Decreasing in $(-\infty, -2]$ and $[0, 2]$; increasing in $[-2, 0]$ and $[2, \infty)$.

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6. 36, 0.

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4. $e^{ax}\cos bx = 1+ax+\dfrac{a^2-b^2}{2\,!}x^2+\dfrac{a(a^2-3b^2)}{3\,!}x^3+\ldots$

$$+\ldots+\frac{x^n}{n\,!}(a^2+b^2)^{\frac{1}{2}n}\; e^{a\theta x}\cos\left(b\theta x+n\tan^{-1}\frac{b}{a}\right).$$

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Ex. 2. 136, -8.

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Ex. 3. max. -8 ; min. -10, 26.

Ex. 5. 55/27, 1 ; 1109, -27.

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1. (*i*) max. 38 ; min. 37. (*ii*) min. 4.
2. Max. for $x=2a$ and minimum for $x=0$ and $x=3a$.
3. Max. for $x=1$, min. for $x=2$.
4. $2\sqrt{3}/9$.
5. max. 54, min. 50 ; least 0, greatest 70.
6. max. for $x=1$, min. for $x=6$.
7. max. for $x=\sqrt[4]{3}$; min. for $x=-\sqrt[4]{3}$.

11. e^{-1}.

12. min. value : $\log[(ae-e)\log a]/\log a$.

13. max. values : $(2\pi+3\sqrt{3})/6$; $(8\pi+3\sqrt{3})/6$.
min. values : $(4\pi-3\sqrt{3})/6$; $(10\pi-3\sqrt{3})/6$.

14. max. for $x=n\pi$ where n is an odd positive or even negative integer, and minimum for $x=n\pi$ where n is an even positive or odd negative integer.

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15. min. values : $-\frac{1}{8}(5\pi+3\sqrt{3}+2)$, $\frac{1}{8}(2\pi-3\sqrt{3}+2)$.
max. value : $\frac{1}{8}(-\pi+3\sqrt{3}-2)$, $\frac{1}{8}(5\pi+3\sqrt{3}+2)$.

17. (*i*) min. value $\frac{\sqrt{3}}{4}$; max. values $\frac{4\sqrt{2}\pm3}{6}$,

least value 0 ; greatest value $\frac{4\sqrt{2}+3}{6}$.

(*ii*) min. values : $-\frac{1}{2}$, $-\frac{5}{6}$; max. values : $\frac{11}{6}$, $-\frac{5}{12}$,
least value : $-\frac{5}{6}$; greatest value : $\frac{11}{6}$.

18. (*i*) min. -1, $-2/3\sqrt{6}$; max. $1, 2/3\sqrt{6}$.

(*ii*) min. $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$; max. $-(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$.

(*iii*) min. 0, $-2/3\sqrt{3}$; max. 0, $2/3\sqrt{3}$.

(*iv*) min. $\frac{-e^{(a+\pi/4)}}{\sqrt{2}} e^{(2n+1)\pi}$, max. $\frac{e^{(a+\pi/4)}}{\sqrt{2}} \cdot e^{2n\pi}$,

where n is any integer.

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20. max. $c^2/(a+b)$.

21. The required values are the roots of the quadratic equation
$$(a-r^{-2})(b-r^{-2})=h^2, \text{ in } r.$$

Page 163.

1. 9, 6. **3.** Sq. whose each side $=\sqrt{2}a$.

4. $\sqrt{(40/3)}$, $\sqrt{(40/3)}$, $\frac{1}{2}\sqrt{(40/3)}$.

5. Diameter of the semi-circle $=40/(\pi+4)$;
Height of the rectangle $=20/(\pi+4)$.

9. Breadth $\sqrt{\frac{4}{3}}a$, depth $\sqrt{\frac{8}{3}}a$.

10. $15(\frac{5}{3})^{\frac{2}{3}}$ miles per hour, $19800(\frac{3}{5})^{\frac{2}{3}}$ rupees.

18. $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

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21. $3\sqrt{3}ab/4$. **22.** $a-b$. **25.** $3\sqrt{3}/4$.

26. $(3v/8\pi)^{\frac{1}{3}}$. **28.** $2\sqrt{3}\pi/27$.

30. Length 2 ft., girth 4 ft. ; yes, length should now be $1\frac{3}{4}$ ft. and girth $4\frac{1}{4}$ ft.

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Ex. 3. (*i*) $-\frac{1}{2}$. (*ii*) $\pi^2/2e$. (*iii*) $\frac{1}{2}$. (*iv*) $\frac{3}{2}$ (*v*) 2.
(*vi*) $2a/b$.

Ex. 4. (*i*) $\frac{3}{2}$. (*ii*) $\frac{1}{2}$. (*iii*) $-\frac{2}{3}$. (*iv*) $\frac{1}{2}$.

Page. 169.

Ex. 5. $a=-2$; limit is -1.

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Ex. 3. (*i*) $\frac{1}{12}$. (*ii*) $-\frac{1}{2}$. (*iii*) 1.
(*iv*) $-2/\pi$. Change $\cot \pi x$ to $\cot \frac{1}{2}\pi x$. (*v*) 4.
(*vi*) $\frac{1}{3}$. (*vii*) $\frac{2}{3}$.

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Ex. 2. (*i*) 0. (*ii*) 3. (*iii*) 1. (*iv*) 1. (*v*) 1. (*vi*) 0.

Page 174.

Ex. 2. (*i*) 0. (*ii*) 1. (*iii*) $2a/\pi$.

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Ex. 2. (*i*) $\frac{1}{2}$. (*ii*) 0. (*iii*) $\frac{2}{3}$. (*iv*) $\pi^2/6$.

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Ex. 3. (*i*) 1. (*ii*) e^{-1}. (*iii*) 1. (*iv*) 1.

Page 177. [§8·7].

(*v*) e. (*vi*) e. (*vii*) $e^{\frac{1}{6}}$. (*viii*) e.

Page 177.

1. $\frac{1}{3}$. **2.** 0. **3.** $-\frac{5}{8}$. **4.** $\frac{1}{12}$. **5.** $\frac{1}{6}$. **6.** -1. **7.** -2.

8 e^{-1}. **9.** $e^{-\frac{1}{6}}$ **10.** $\frac{8}{9}$. **11.** -2. **12.** $e^{-\frac{1}{2}}$.

13. $e^{-a^2b/2}$. **14.** e. **15.** $e^{\frac{1}{12}}$. **16.** $e^{2/\pi}$

17. e^{-a^2/b^2}. **18.** $a \log a$. **19.** 2. **20.** 1.

21. $e^{2/\pi}$. **22.** $\sqrt{3}\pi/6$. **23.** $\log (e/b) \log be$.

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24. $(a_1 a_2 a_3 \ldots a_n)^{\frac{1}{n}}$. **25.** $-e/2$. **26.** $11e/24$. **27.** 16.
28. $-\frac{1}{2}a^2e^a$. **29.** a^2e^a. **30.** $f''(a)/2f'(a)$. **31.** $\frac{1}{15}$.
33. Continuous at the origin.

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3. $\dfrac{(-1)^n x^{2n+1}}{2n+1}$.

7. $y = 2x - x^2 - \dfrac{2}{3}x^3 + \dfrac{3}{2}x^4 - \dfrac{5}{3}x^5 + \dfrac{3}{2}x^6$.

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11. $1 + x + \dfrac{x^2}{2} - \dfrac{x^4}{8}$.

12. $x - \dfrac{1}{2}x^2 + \dfrac{2}{3}x^3$. **13.** $\dfrac{1}{2} + \dfrac{x}{4} - \dfrac{x^3}{48}$.

14. $\log \sin 3 + (x-3) \cot 2 - \dfrac{(x-3)^2}{2} . \operatorname{cosec}^2 3 +$

$\dfrac{(x-3)^3}{3} . \cot 3 \operatorname{cosec}^2 3$.

15. $14+29(x-2)+16(x-2)^2+3(x-2)^3$.

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17. $1+x+\dfrac{x^2}{2}-\dfrac{x^3}{3}$.

(Read $e^x - e^{x} \cos x$ in place of $e-e^x \cos x$).

Page 202.

1. (i) $\dfrac{1}{1+(x+y)^2}, \dfrac{1}{1+(x+y)^2}$.

(ii) $ae^{ax} \sin by, be^{ax} \cos by$.

(iii) $\dfrac{2x}{(x^2+y^2)}, \dfrac{2y}{(x^2+y^2)}$.

2. (i) $e^{x-y}, -e^{x-y}, -e^{x-y}, e^{x-y}$.

(ii) $ye^{(x^y)} x^{y-2}[yx^y+y-1]$.

$e^{(x^y)} x^{y-1}[1+(1+x^y) \log x^y]$.

$e^{(x^y)} x^{y-1}[1+(1+x^y) \log x^y]$.

$e^{(x^y)} x^y(1+x^y)(\log x)^2$.

(iii) $\dfrac{2y(1+y^2)}{(1-xy)^3}, \dfrac{2(x+y)}{(1-xy)^3}, \dfrac{2(x+y)}{(1-xy)^3}, \dfrac{2x(1+x^2)}{(1-xy)^3}$.

4. $1/c^2z$. 8. 0.

Page 203.

17. $-\frac{3}{2}$.

Page 205.

Ex. 1. $-r \sin \theta, \cos \theta, r \cos \theta, \sin \theta, -r \operatorname{cosec} \theta, \operatorname{cosec} \theta$.

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2. 4%. 6. $-3\frac{1}{3}$.

Page 215.

1. $4x+2y$. 2. $(\cos y-xy \sin y)/x^2$

3. $-(u \cot x \cos u \sin y+z \sin v \sin x)/(\cos v \sin y)$.

4. $-(1+x^2)(1+y^2)(2s+r^2)/(1-xy)^2$.

5. (i) $[y+x^2 \cos (x-y)]/[x+x^2 \cos (x-y)]$.

(ii) $(\cot x-yx^{y-1} \log y)y/x^y(y \log x \log y+1)$.

(iii) $\dfrac{\sin y(y \sin x+\cos x \log \sin y)}{\cos x(\sin y \log \cos x-x \cos y)}$.

(iv) $y(y-x \log y)/x(x-y \log x)$.

(v) $-\dfrac{y(\tan x)^{y-1} \sec^2 x-\operatorname{cosec}^2 x \log y(y)^{\cot x}}{\log \tan x (\tan x)^y+\cot x(y)^{\cot x-1}}$.

6. (i) $\dfrac{\partial z}{\partial x}=-\dfrac{\partial F/\partial x}{\partial F/\partial z}; \dfrac{\partial z}{\partial y}=-\dfrac{\partial F/\partial y}{\partial F/\partial z}$.

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14. (*i*) $\dfrac{2a^3xy}{(ax-y^2)^3}$, (*ii*) $\dfrac{2a^2xy(3a^4+x^2y^2)}{(a^2x-y^3)^3}$,

(*iii*) $-\dfrac{6a^3xy(2a^6+x^3y^3)}{(a^3x-y^4)^3}$, (*iv*) $\dfrac{6a^3x^2(a^3+x^3)}{y^9}$.

Page 228.

1. (*i*) min. at $(\frac{2}{3}, -\frac{4}{3})$. (*ii*) min. at $(0, 0)$ if $a > \frac{1}{4}$.

(*iii*) min. at $[\frac{1}{3}(b-2a), \frac{1}{3}(a-2b)]$.

(*iv*) max. at $(2, 1)$. (*v*) min. at $(2, \frac{1}{2})$.

(*vi*) min. at $(0, 0)$, $(-1, 0)$. (*vii*) max. at $(-2, 0)$.

(*viii*) No extreme value.

(*ix*) max. at $[\frac{1}{10}\pi(1+4m+12n), \frac{1}{5}\pi(1+4m+2n)]$,

min. at $[\frac{1}{10}\pi(7+4m+12n), \frac{2}{5}\pi(1+2m+2n)]$,

where m and n are integers.

(*x*) max. at $(\frac{1}{2}, \frac{1}{3})$.

2. max. value 112 at $(4, 0)$; min. value 108 at $(6, 0)$.

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3. $\Sigma(x_1-x_2)(m_1n_2-m_2n_1)/\sqrt{\Sigma(m_1n_2-m_2n_1)^2}$.

Page 232.

1. (*i*) $3a^2$. (*ii*) $3a^2$. (*iii*) $3a^2$.

2. $-a^2$ (min.) and $\frac{1}{3}a^2$ (max.)

3. $2k\sqrt{(a^2-ab+b^2)}$.

4. $(al+bm+cn)/\sqrt{(l^2+m^2+n^2)}$.

5. $\left(-\dfrac{2}{\sqrt{14}}, -\dfrac{1}{\sqrt{14}}, -\dfrac{3}{\sqrt{14}}\right)$.

6. Squares of the semi-axes are the roots of the quadratic

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)-h^2 \text{ in } r^2.$$

7. $\frac{1}{8}$. **8.** $\frac{28}{5}$, 3.

Miscellaneous Exercises

Page 233.

1. Zero is the only point of discontinuity.

2. $n+\frac{1}{2}$, where n is any integer, are the only points of discontinuity.

3. $(2m+1)/(1-2m)$, m being any integer and $x=-1$.

4. $1, -1$.

5. (*i*) All integral values of x.

(*ii*) All integral values of x.

(*iii*) $x=0$.

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6. $x=0, \frac{1}{2}, 1.$

7. Discontinuous at the origin.

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8. $\dfrac{\sin(C-A)+\sin B(\cos A-\cos C)}{\sin(B-C)+\sin A(\cos C-\cos B)}.$ 9. $\dfrac{3a^4}{r^5}.$

11. $\dfrac{(-1)^n\, n\,!}{a^{n+2}\sin^{n+2}\alpha}\sin(n+1)\,\theta\,\sin^{n+1}\theta$, where

$\tan\theta = a\sin\alpha/(x+a\cos\alpha)$.

16. $\frac{1}{2}x-\frac{1}{16}x^3-\frac{1}{256}x^5.$

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30. (*i*) 1 ; (*ii*) -1 ; (*iii*) -1.

31. $\frac{2}{5}$. 32. $\frac{1}{8}$.

33. (*i*) 1. (*ii*) $2a/b$. (*iii*) 0.

34. -2.

35. (*i*) continuous, (*ii*) discontinuous, (*iii*) continuous.

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37. $\dfrac{4-3\sqrt{2}}{4+3\sqrt{2}}$ max., $\dfrac{4+3\sqrt{2}}{4-3\sqrt{2}}$ min.

38. max. at $\pi/3$ and min. at $5\pi/3$.

40. $4/e$ max. and 0 min.

43. $(3d-b)/(a-3c)$ is a max. (min.) and $-(b+d)/(a+c)$ is a min. (max.) if, $ad-bc$, is positive (negative).

47. Height$=2(3v/\pi)^{\frac{1}{3}}$, radius$=(1/\sqrt{2})(3v/\pi)^{\frac{1}{3}}$ where v is the volume of the cone.

50. $2\pi\{1-(\sqrt{2}/3)\}$ radians.

51. The vertices of the rectangle are

$$\left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right), \left(a, \frac{2a}{\sqrt{3}}\right), \left(a, -\frac{2a}{\sqrt{3}}\right), \left(\frac{a}{3}, -\frac{2a}{\sqrt{3}}\right);$$

$y^2=4ax$ being the equation of the parabola.

52. The greatest value is 1 and the least value is $(1/e)^{1/e}$.

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1. (*i*) $x+y+a=0,\ x-y=3a.$
(*ii*) $b^2x'x+a^2y'y=a^2b^2$; $a^2y'(x-x')=b^2x'(y-y')$.
(*iii*) $x+p^2y=2cp$; $p^3x-py=c(p^4-1)$.
(*iv*) $4x\pm2y-a=0$; $2x\mp4y=3a$.
(*v*) $31x\pm8y+9a=0$; $8x\mp31y+42a=0$.
(*vi*) $x+y=0$; $x-y=0$.
(*vii*) $x+y=3a$; $x-y+a=0$.
(*viii*) $x\cos^3\theta+y\sin^3\theta=c$; $x\sin^3\theta-y\cos^3\theta+2c\cot 2\theta=0$.
(*ix*) $13x-16y=2a$; $16x+13y=9a$.

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5. (*i*) $(0, a)$; $(a, 0)$. (*ii*) $(\sqrt[3]{2a}, \sqrt[3]{4a})$; $(\sqrt[3]{4a}, \sqrt[3]{2a})$

(*iii*) $\left(\pm\frac{3}{20}, \mp\frac{5}{8}\right)$, $\left(\pm\frac{1}{4}, \mp\frac{3}{8}\right)$.

6. $x-20y=7$, $20x+y=140$.

Page 262.

20. $\left(\frac{1-2x_1}{2}, \frac{2x_1-7x_1^2+7x_1^3}{2}\right)$.

Page 264.

1. (*i*) $\pi/4$. (*ii*) $\tan^{-1}\sqrt[3]{16}$.

3. (*i*) $m^2h+m(a-b)-h=0$. The roots of this quadratic equation in, m, are the slopes of the two axes.

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3. $a\sin^2\theta$, $a\tan\theta\sin^2\theta$, $a\sin^2\theta\cos\theta$, $a\sin^3\theta\tan\theta$.

Page 270.

1. (*i*) $\theta/2$. (*ii*) $\pi/2+m\theta$. (*iii*) $\pi/2-\theta/2$. (*iv*) $\pi/4+m\theta$.

4. (*i*) $\pi/2$. (*ii*) $\tan^{-1}3$. (*iii*) $\pi/2$. (*iv*) $\tan^{-1}[2e/(e^2-1)]$
(*v*) $2\pi/3$. (*vi*) $\pi/2$.

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4. (*i*) $2a\cos^3\frac{1}{2}\theta\operatorname{cosec}\frac{1}{2}\theta$. (*ii*) $\frac{2a}{e\sin\theta}$. (*iii*) $\frac{a\theta^3}{\theta-2}$.

6. (*i*) $-a/\theta^2$. (*ii*) $-b\sin\theta$.

7. $p=ae/(1+\theta)\sqrt{(2+2\theta^2)}$.

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9. (*i*) $1/p^2=1/r^2+1/a^2$. (*ii*) $p=r\sin\alpha$.

(*iii*) $\frac{1}{p^2}=\frac{(e^2-1)}{l^2}+\frac{2}{lr}$.

(*iv*) $1/p^2=1/r^2+a^2/r^4$. (*v*) $r^3=2ap^2$

(*vi*) $r^4=p^2[a^2m^2+(1-m^2)r^2]$. (*vii*) $ar^3=4p^4$.

(*viii*) $r^4=(b^2-a^2+2ar)p^3$. (*ix*) $r^{m+1}=p\sqrt{(a^{2m}+b^{2m})}$.

Page 279.

1. (*i*) $\cosh\frac{x}{c}$. (*ii*) $\frac{a^2+x^2}{a^2-x^2}$.

(*iii*) $\frac{3x-4a}{\sqrt{[12(a^2-ax)]}}$. (*iv*) $\sqrt{\left(\frac{4a+9x}{4a}\right)}$

(*vi*) $\frac{2x^2-3a^2}{\sqrt{[8a^2(a^2-x^2)]}}$. (*vi*) $\frac{3x+a}{\sqrt{(12ax)}}$.

(*viii*) $(x^2+a^2)/2ax$.

2. $a^{\frac{2}{3}}(a^{\frac{2}{3}}+2y^{\frac{2}{3}})/4xy^{\frac{1}{3}}$.

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4. (*i*) $\sqrt{(a^2 \sin^2\theta + b^2 \cos^2\theta)}$.

(*ii*) $3 \sin\theta \cos\theta \sqrt{(a^2 \cos^2\theta + b^2 \sin^2\theta)}$. (*iii*) $2a \sin \frac{1}{2}\theta$.

(*iv*) $\sqrt{2}a\, e^{\theta}$. (*v*) $a\theta$.

5. (*i*) $2a \cos \theta/2$. (*ii*) $a\sqrt{(\sec 2\theta)}$. (*iii*) $a \operatorname{cosec} a\, e^{\theta \cot \alpha}$.

(*iv*) $a\sqrt{(1+\theta^2)}$. (*v*) $a(\theta^2+1)$. (*vi*) $a(\theta^2+1)/(\theta^2-1)^2$.

(*vii*) a^m/r^{m-1}. (*viii*) $\sqrt{2}a^m/r^{m-1}$.

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2. Concave downwards in $[0, \pi]$ and upwards in $[\pi, 2\pi]$.

3. Concave upwards in $[-2, 0]$ and $[2, \infty)$; concave downwards in $(-\infty, -2]$ and $[0, 2]$. Inflexions at $(-2, 198)$, $(0, -20)$, $(2, -238)$.

4. Concave upwards in $(-\infty, -1]$ and $[1, \infty]$; concave downwards in $[-1, 1]$. Inflexions at $(-1, 2e)$ and $(1, 10/e)$.

5. Concave upwards in $[0, \pi/4]$ and $[5\pi/4, 2\pi]$ and concave downwards in $[\pi/4, 5\pi/4]$.

6. (*i*) For $x = -b/3a$. (*ii*) $(0, 0)$, $(1, 0)$, $(-1, 0)$.

(*iii*) $(5, 0)$, $(\frac{119}{27}, \frac{2}{3})$. (*iv*) $(0, 2)$.

(*v*) For $x = -a$ and $a(2 \pm \sqrt{3})$.

(*vi*) For $x = 0$ and $\pm a\sqrt{3}$. (*vii*) For $x = -2$ and $1 \pm \sqrt{3}$.

(*viii*) For $x = \pm a/\sqrt{2}$. (*ix*) $(\frac{3}{2}ae^{-\frac{3}{2}}, ae^{\frac{3}{2}})$.

(*x*) $(1, 3)$. (*xi*) $\left(\frac{1}{3}, \pm\frac{4}{3\sqrt{3}}\right)$.

(*xii*) $(0, 0)$.

Inflexional tangents to (*ii*) are $x+y=0$, $x-2y \pm 1=0$.

Inflexional tangents to (*iii*) are $x=5$ and $3(9x+6y)=151$.

Inflexional tangents to (*xi*) are $9x \mp 3\sqrt{3}y + 1 = 0$.

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9. For $x = -2$, $(-4 \pm \sqrt{18})$.

11. (*i*) $\left[a\left(4\pi n + \frac{2\pi}{3} \mp \frac{\sqrt{3}}{2}\right), \frac{3a}{2}\right]$.

(*iii*) $\left(\pm\sqrt{3}a, \pm\frac{\sqrt{3}a}{4}\right)$, $(0, 0)$. (*iii*) $\left(\pm\frac{2a}{\sqrt{3}}, \frac{3a}{2}\right)$.

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Ex. (*i*) $c \sec^2 \psi$. (*ii*) $4a \cos \psi$. (*iii*) $\frac{4}{3} a \cos \frac{1}{3}\psi$.

(*iv*) $c \tan \psi$. (*v*) $2a \sec^3\psi$.

Page 296.

1. (*i*) y^2/c. (*ii*) at. (*iii*) $3(axy)^{\frac{1}{3}}$. (*iv*) $-a(t^2+1)^{\frac{3}{2}}/t^4$.

2. (*i*) $-1/2$. (*ii*) 1.

Page 297.

10. $2a^2/b,\ 2b^2/a.$

14. $[-(\log 2)/2,\ 1/\sqrt{2}].$

20. $(\frac{9}{8},\ \pm 3).$

Page 298.

23. $(c^2+s^2)/c.$ **25.** $\sqrt{[8a(2a-y)]}.$

Page 302.

1. $\dfrac{(r^2+a^2n^2-n^2r^2)^{\frac{3}{2}}}{r^2-r^2n^2+2a^2n^2}.$ **2.** (*i*) $\dfrac{a(1+\theta^2)^{\frac{3}{2}}}{(2+\theta^2)}$ (*ii*) $\dfrac{a(1+\theta^2)^{\frac{3}{2}}}{\theta^4}$

(*iii*) $a^n/(n+1)r^{n-1}.$ (*iv*) $\sqrt{(r^2-a^2)}.$ (*v*) $\sqrt{(8r^3)}/\sqrt{a}.$

3. $an/2.$ **5.** (*i*) $2p^3/a^2.$ (*ii*) $r^4/Ap^3.$ (*iii*) a^2b^2/p^3

6. $a^2b^2/p^3.$

7. (*i*) $3p.$ (*ii*) $\dfrac{p}{m+1}\ \text{cosec}^2\left(\dfrac{m}{m+1}\psi\right)$

Page 303

10. $\dfrac{2\sqrt{3}}{3}a.$ **12.** (*i*) $(3a/2,\ \pm\sqrt{3}).$ (*ii*) $(\pm\sqrt{2}a,\ 1/2).$

Page 309.

9. $(a+b)(x^2+y^2)=2x+2y.$

11. (*a*) $x^2+y^2-\dfrac{2(b^2-a^2)}{b}y+(b^2-2a^2)=0.$ **13.** $y^2/c.$

Page 315.

Ex. 2. (*i*) $x-y+a=0.$ (*ii*) $x+y=a.$ (*iii*) $y=x+a/3.$

Page 320.

1. $x=0,\ y=\pm a.$ **2.** $x=2,\ x=3,\ y=3.$

3. $x=\pm 1,\ y=0.$ **4.** $x=\pm a,\ y=\pm b.$

5. $x=0,\ y=x,\ y=x+1.$ **6.** $y=x\pm a,\ x=\pm a.$

7. $y=x-2,\ y=x-3.$ **8.** $x+1=0,\ y=0,\ x+y=0$

9. $y=x-3,\ y=x+2,\ x+y=0.$

10. $y=0,\ y+x=\pm 1.$

11. $y=0,\ y=x\pm\sqrt{2}.$

12. $x+y=\pm\sqrt{2},\ x=\pm 1,\ y=\pm 1.$

Page 321

13. $y=0,\ y\pm x=1.$ **14.** $x=1.$

15. $x+y=2,\ y=x+2,\ y=2x-4.$

16. $y+3=0,\ x+1=0,\ y=x+4.$

17. $x=0,\ y=0,\ 2y-3x=6.$ **18.** $x=\pm a,\ y\pm x=a.$

19. $x+y\pm a=0.$

20. $4(y-x)+1=0,\ 2(y+x)=3,\ 4(y+3x)+9=0.$

Page 321.

21. $x=y+a,\ x=y+2a,\ 2y=x+14a,\ 3y=x-13a.$
22. $x=\pm a.$
27. $x+y=4a.$

Page 325.

1. $y=a,\ x=0,\ x+y+a=0.$ **2.** $x=1,\ x=2,\ x+y+1=0.$
3. $3(y-x)+2=0.$ **4.** $y-x=0.$
5. $x+y+1=0,\ 3y+x-1=0,\ 2y+x+1=0.$
6. $3y+x=(-5\pm\sqrt{106})/9.$
7. $y+2x+2=0,\ y+2x-1=0\ ;\ y-x+3=0,\ y-x-2=0.$
8. $x=0,\ y=0,\ 2y=4x+3,\ 4x+2y=15.$
9. $x\pm y+a=0,\ x=0.$
10. $y=x,\ y=2x+1,\ y=2x+2.$

Page 328.

5. $3y+x=1.$ **6.** $y=\pm x+1,\ y=\pm\frac{2}{3}x+\frac{4}{3}.$
7. $6(y-x)+7=0,\ 2(y-3x)+3=0.$
$3(2y+x)+5=0,\quad 106y-381x+105=0.$
8. $x^3-6x^2y+11xy^3-6y^3-x=0.$
9. $y=2x^3+1,\ y=-2x-1,\ x=2y,\ x+2y=0.$
10. $x+y=0,\ 2x-3y-1=0,\ 2x-3y+3=0,\ 4x-6y+9=0.$

Page 331.

Ex. 2. (*i*) $x+y=a$. The curve lies above or below the asymptote according as x is positive or negative.

(*ii*) $x+y+a=0$. The curve lies above the asymptote both for positive as well as negative values of x.

(*iii*) $y=x+1$. The curve lies above or below the asymptote accordiug as x is positive or negative.

Page 334.

1. $-a=r\sin(1-\theta).$ **2.** $a+\sqrt{2}r\sin(\theta+\frac{1}{4}\pi)=0.$
3. $r\cos\theta\pm b=a.$ **4.** $r\cos\theta\pm a=0,\ r\sin\theta\pm a=0.$
5. $2r\sin\theta=a,\ 2\theta=\pi.$ **6.** $\theta=\pm\pi/4.$
7. $r\sin\theta=a,\ \pi(r\cos\theta)+2a=0.$

8. $r\sin\left(\theta-\dfrac{m\pi}{n}\right)=\dfrac{a}{n\cos m\pi}$ where m is any integer.

9. $n\theta=m\pi$ where m is any integer.
10. $r\cos\theta=\pm a.$ **11.** $\theta=0.$
12. $a=r\sin(\theta-1).$ **13.** $y+a=0.$ **14.** $y+a=0.$

15. System of parallel lines $y=ae^{n\pi}$ where n is any integer or zero.

16. $ye^{\pi}+a=0.$

17. $\dfrac{f_{n-1}(\theta)}{f_n(\theta_1)}=r\sin(\theta_1-\theta)$, where θ_1 is any root of the equation $f_n(\theta)=0$.

19. $a=2r(\cos\theta-\sin\theta),\ a+2r(\cos\theta+\sin\theta)=0.$

Page 338.

1. $x=0, y=0$. 2. $bx=\pm ay$. 3. $x=0$.
4. $y=\pm x$. 5. $y=\pm x$.

Page 339.

1. $x=\pm a$. 2. $(y-1)=\pm(x-2)$.
3. $\pm\sqrt{3}y=\sqrt{2}(x-a)$; $2(x-2a)=\pm\sqrt{3}(y-a)$.

Page 341.

1. Cusp at (0, 0). 2. Cusp at (0, 0).
3. Node at (0, 0).

Page 342.

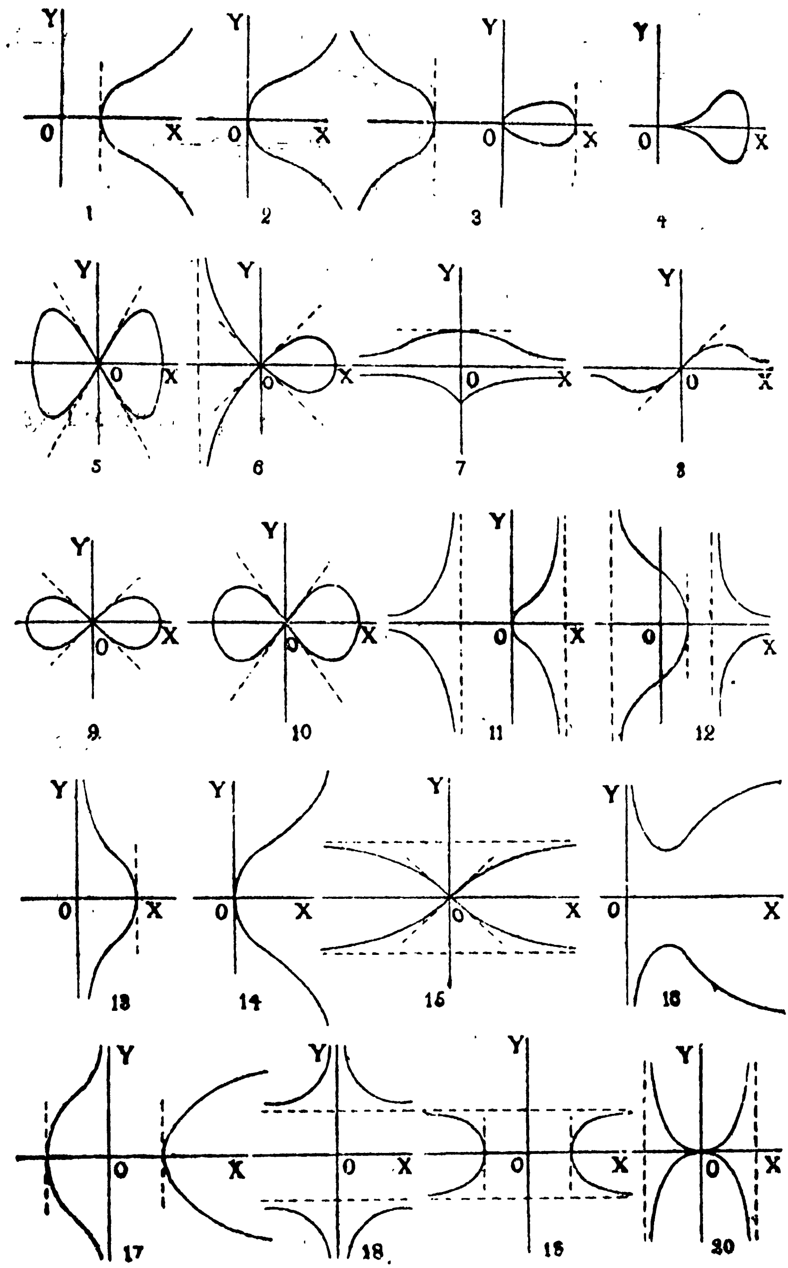
4. Node at $(a, 0)$.
5. Node at (2, 0).
6. $(a, 0)$ is a node, cusp or an isolated point according as b/a is less than, equal to or greater than 1.
7. Conjugate point at $(a, 0)$.
8. Conjugate point at (2, 3).
9. Conjugate point at $(-a, -b)$.
10. Cusp at $(0, -4a)$.
11. Cusp at $(-1, -2)$.
12. Cusp at $(1, -1)$.
13. No multiple point.
14. $(0, \pm a)$ are triple point : $(-\frac{3}{2}a, \pm a)$ are cusps : node at $(-a, 0)$.
15. $(0, -a)$ is a double point ; being a node, cusp or conjugate point according as $b>a$, $b=a$ or $b<a$.
16. $y-a=\pm(2/\sqrt{3})x$; $y=\pm\sqrt{(2/3)}(x-a)$; $y-a=\pm(2/3)(x-2a)$.
17. $y-2=\pm(2\sqrt{2})(x-2)$.
18. $x-y+1=0$, $x+y=3$.

Page 345.

1. Single cusp of first species.
2. Single cusp of first species.
3. Single cusp of first species.
4. Single cusp of first species.
5. Isolated point.
6. Double cusp of second species.
7. Oscu-inflexion.
8. Oscu-inflexion.

Page 347.

2. $2\sqrt{2}$. 3. $2\sqrt{2}$ for each. 4. $3a/2$ for each.
5. $\frac{85}{2}\sqrt{17}$, $5\sqrt{2}$. 6. $a, -a/2, -5^{\frac{3}{2}}a$.
7. $\frac{2}{3}\sqrt{2}a$, $\frac{2}{3}\sqrt{2}a$, $\frac{2}{3}a$.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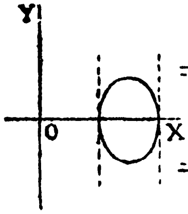

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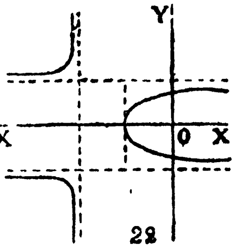

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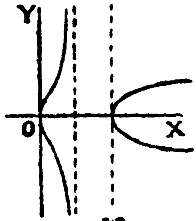

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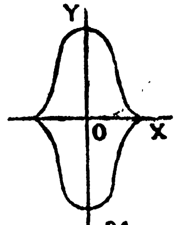

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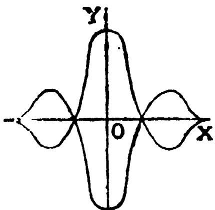

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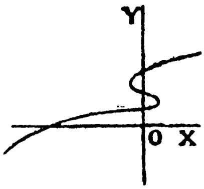

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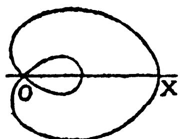
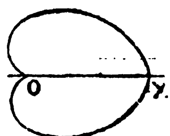


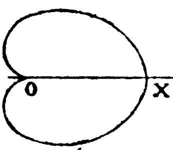
$a<b$

27



$a>b$

27



$a=b$

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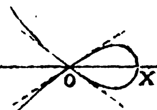

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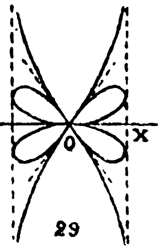

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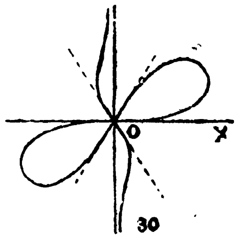

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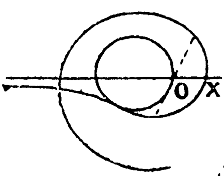

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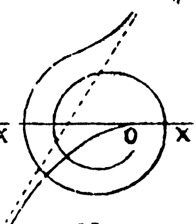

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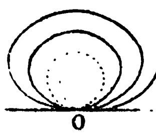

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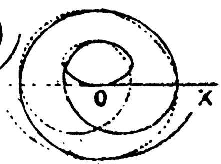

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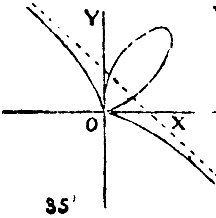

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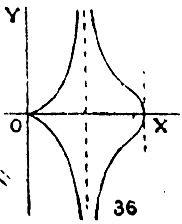

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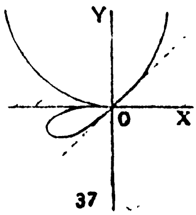

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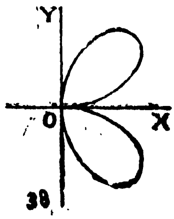

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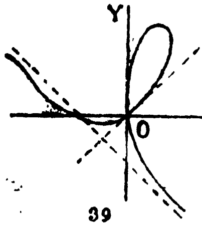

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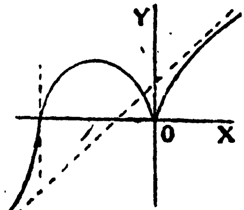

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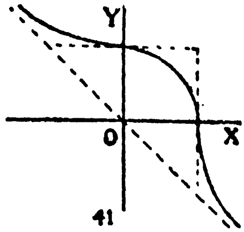

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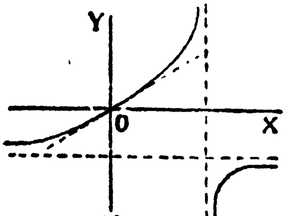

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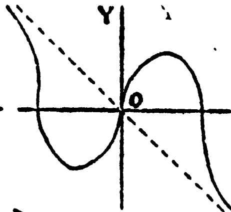

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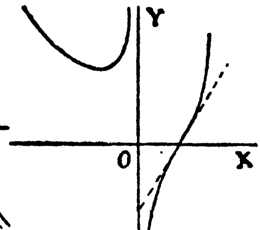

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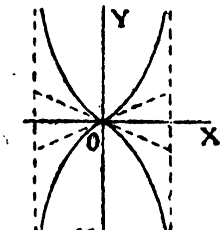

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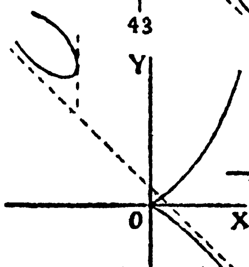

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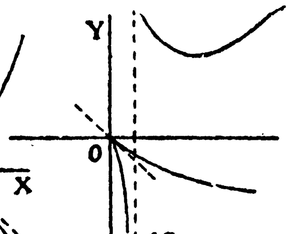

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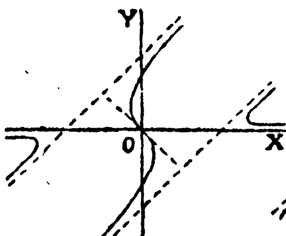

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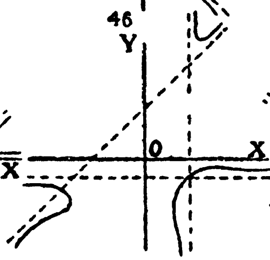

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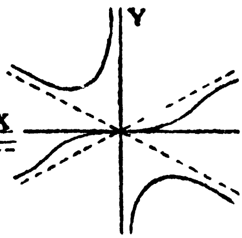

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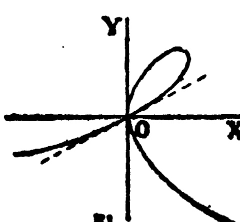

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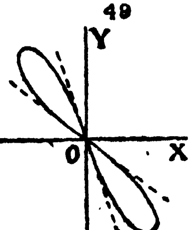

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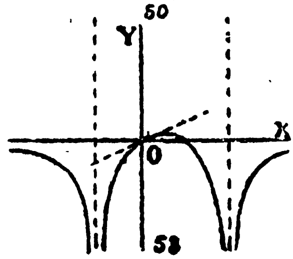

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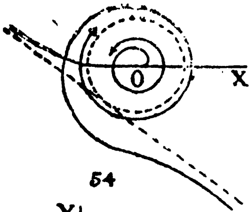

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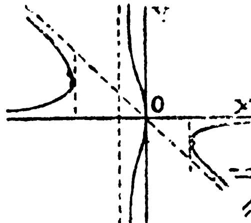

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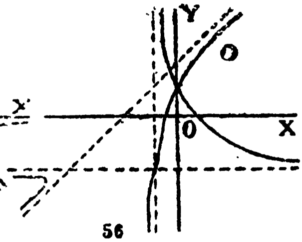

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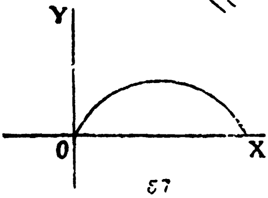

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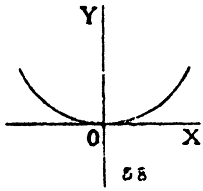

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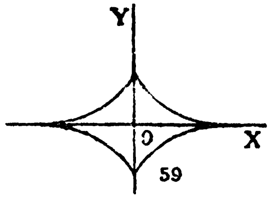

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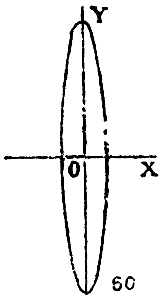

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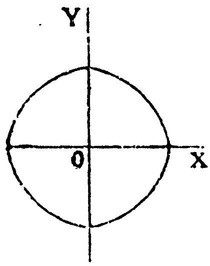

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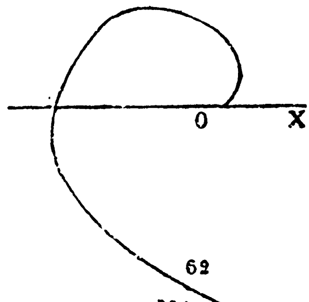

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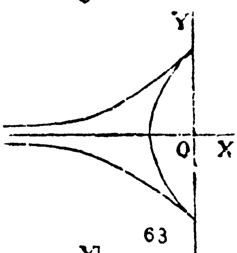

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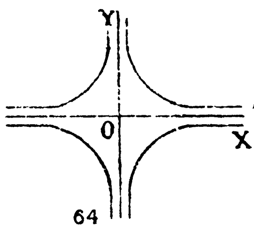

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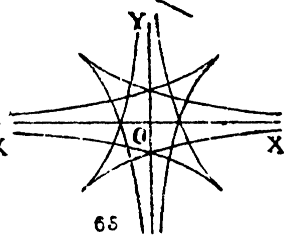

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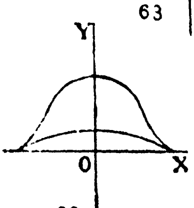

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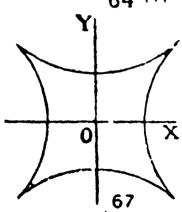

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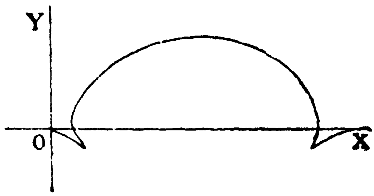

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1. (*i*) $x^2/a^2+y^2/b^2=1$. (*ii*) $c^2(x^2+y^2)=x^2y^2$.

(*iii*) $x=a\cos\theta+a\theta\sin\theta$, $y=a\sin\theta-a\theta\cos\theta$.

(*iv*) $x^{2/(2-n)}+y^{2/(2-n)}=a^{2/(2-n)}$.

(*v*) $(p-1)^{p-1}x^p+p^p\,ay^{p-1}=0$.

(*vi*) $x^2-y^2=c^2$.

2. (*i*) $x^{\frac{1}{2}}+y^{\frac{1}{2}}=c^{\frac{1}{2}}$. (*ii*) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$. (*iii*) $4xy=c^2$.

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3. (*i*) $2xy=c^2$. (*ii*) $x\pm y\pm c=0$.
5. $27ay^2=4(x-2a)^3$.
6. $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$. 7. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=(4a)^{\frac{2}{3}}$.
9. $x^2/a^2+y^2/b^2=1$.
12. $(x^2+y^2)^2=a^2x^2+b^2y^2$.
13. The straight line $p^2x-py+(a+ap^2+pq)=0$.
15. (*i*) $r^2-2br\cos\theta+(b^2-a^2)=0$.

(*ii*) $r^{n/(1-n)}=a^{n/(1-n)}\cos\left(\dfrac{n}{1-n}\,\theta\right)$.

(*iii*) $r\sin\alpha=ae^{(\alpha-\pi/2)\cot\alpha}\,e^{\theta\cot\alpha}$.

16. (*i*) $r^2(e^2-1)-2l\,er\cos\theta+2l^2=0$.

(*ii*) $r^{n/(n+1)}=a^{n/(n+1)}\cos\dfrac{n\theta}{1+n}$.

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17. $x^{mp/(m+p)}+y^{mp/(m+p)}=c^{mp/(m+p)}$.

Miscellaneous Exercises II.

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2. $2a\sec^3(\theta/2)$. 4. $\pm a/2$; $-a$.
5. $\frac{1}{3}t(t^2-3)^{\frac{3}{2}}$.

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14. $(9t^2, -6t)$.
17. $(\frac{1}{2}a, \pm a)$.
19. $\left(\dfrac{225b^5}{64a^4}, \dfrac{448a^6}{3375b^5}\right)$. 20. $(6, \pm 4)$.

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28. $y=x\pm 4$.
32. $(1, -1)$ is an isolated point; $(5, 3)$ and $(5, -5)$ are the two points of inflexion.
36. $x=a-\tanh a$, $y=\operatorname{sech} a$ is the required curve.

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40. $x\sin\frac{1}{2}t+y\cos\frac{1}{2}t=a\sin\frac{3}{2}t$, $x\cos\frac{1}{2}t-y\sin\frac{1}{2}t=3a\cos\frac{3}{2}t$.
47. (*i*) Isolated point. (*ii*) Single cusp of second species.

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