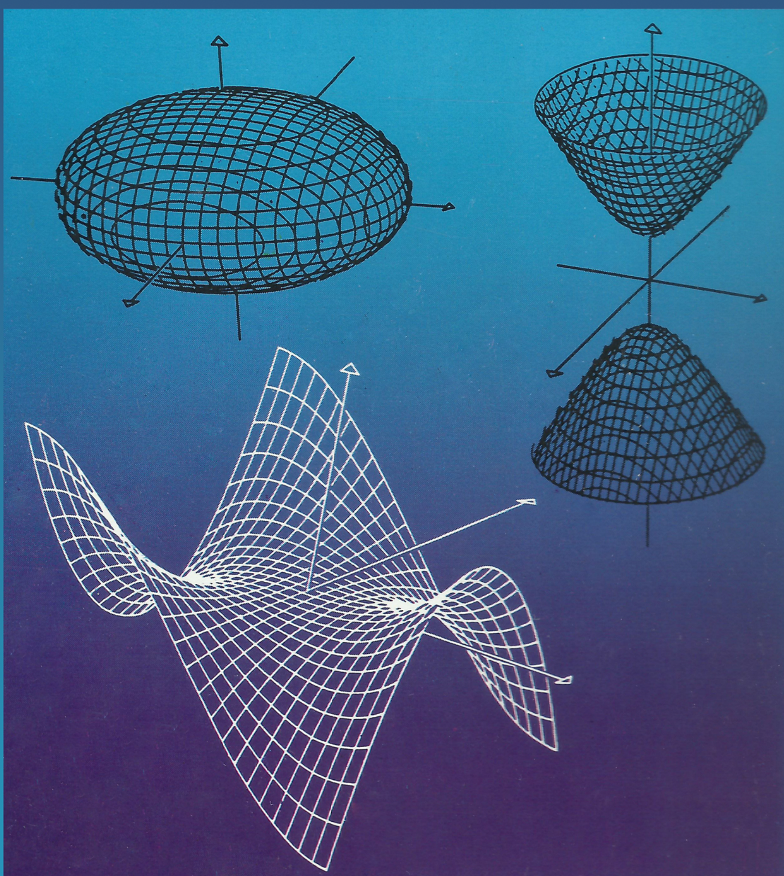
ANALYTICAL GEOMETRY OF TWO AND THREE DIMENSIONS

AND VECTOR ANALYSIS

R. M. KHAN



Analytical Geometry
of
Two and Three Dimensions and Vector Analysis

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Analytical Geometry *of* Two and Three Dimensions and Vector Analysis

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Contents

Preface to the Fifth Edition

In this revised edition some additions and alterations have been made according to the trend of University Questions. I hope that this edition will be more useful to all.

I convey my thanks to students and teachers for the enthusiastic appreciation of the previous editions of this book.

R. M. Khan

Preface to the First Edition

This book is more or less a compilation of all basic topics of Analytical Geometry (two and three dimensions) and Vector (Algebra and Analysis).

Highlights of Some Specialities.

- The book is self-contained.
- The knowledge of school mathematics is sufficient to understand the matter discussed in this book.
- The discussion is simple, lucid, comprehensive and rigorous.
- In some cases matrix theory is used.
- Basic ideas of Vector Algebra and Analysis will be helpful to bridge the treatments of different branches of science.
- To represent vector quantities bar ($\bar{x}$) has been used instead of bold face type to distinguish them from scalar quantities.
- Numerous examples, mostly taken from university questions, are either worked out or given in the exercises with hints to grasp the subject easily and clearly.
- Engineering students and candidates of different competitive examinations will also be benefited by this book.

I convey my gratefulness to my colleagues, students, publisher and printer in bringing out this book, All suggestions for the improvement of the book will be cordially received.

R. M. Khan

Part I

Analytical Geometry of Two Dimensions

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Chapter 1

Coordinates and Locus

1.10 Rectangular Cartesian coordinates

To define the position of a point in space, the idea of coordinates was first invented by French Mathematician Rene Descartes in 1619 and in memory of the inventor coordinates w.r.t. rectangular axes are sometimes called *cartesian coordinates*. First we discuss the coordinates in a plane and subsequently in space.

Let $X'OX$ and $Y'OY$ be two straight lines intersecting at right angles in the plane of the paper [Fig. 1]. The lines $X'OX$ and $Y'OY$ are called the *x-axis* and *y-axis* respectively. The two together is named as axes of coordinates.

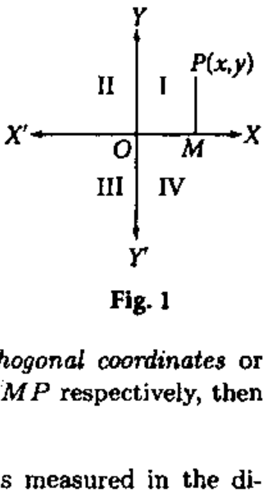


Fig. 1

From a point P in the plane of the axes PM is drawn perpendicular to OX. The position of P referred to OX and OY is known, if the lengths of OM and MP are known. OM and MP are called the *abscissa* or *x-coordinate* and the *ordinate* or *y-coordinate* of P respectively. These lengths with proper sign are termed as *rectangular or orthogonal coordinates* or simply *coordinates of P*. If x and y are lengths of OM and MP respectively, then the coordinates of P are generally denoted by (x, y).

The sign of x-coordinate is positive or negative as it is measured in the direction of OX or OX'. Similarly the y-coordinate is positive or negative according as it is measured in the direction of OY or OY'. If we name the regions XOY, YOX', $X'OY'$ and $Y'OX$ as quadrants I, II, III and IV respectively, then for a point in the quadrant I both the coordinates are positive; for a point in the quadrant II x-coordinate is negative and y-coordinate is positive; for that in the quadrant III both the coordinates are negative and for a point in the quadrant IV x-coordinate is positive but y-coordinate is negative.

By this representation a point can be located definitely when its coordinates are given and conversely if the point is given, its coordinates are defined in magnitude and sign.

1.11 Distance between two points

Let the coordinates of two points P and Q be (x_1, y_1) and (x_2, y_2) referred to rectangular axes OX and OY. PM and QN are perpendiculars to OX and PR is perpendicular to QN [Fig. 2]. Now $OM = x_1$, $MP = y_1$; $ON = x_2$, $NQ = y_2$.

From the figure,

$$PR = MN = ON - OM = x_2 - x_1,$$
$$RQ = NQ - NR = NQ - MP = y_2 - y_1.$$

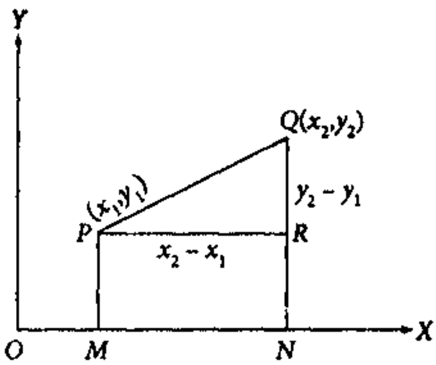


Fig. 2

From the right-angled triangle PRQ,

$$PQ^2 = PR^2 + RQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

$$\therefore\ PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Corollary. The distance of a point $P(x, y)$ from the origin $= \sqrt{x^2 + y^2}$.

Example 1. *Show that the points* $(0, -1)$, $(-2, 3)$ *and* $(6, 7)$ *form a right-angled triangle.*

Let the coordinates of the vertices A, B, C of ΔABC be $(0, -1), (-2, 3)$ and $(6, 7)$ respectively. Then

$$AB^2 = (-2 - 0)^2 + (3 + 1)^2 = 20,$$
$$BC^2 = (6 + 2)^2 + (7 - 3)^2 = 80,$$
$$CA^2 = (0 - 6)^2 + (-1 - 7)^2 = 100.$$
$$\therefore\ CA^2 = AB^2 + BC^2.$$

Hence ΔABC is a right-angled triangle.

Example 2. *Find the coordinates of points whose distances from* $(4, 6)$ *and* $(6, -1)$ *are* 5 *and* $\sqrt{34}$ *respectively.*

Let (x, y) be the coordinates of the point.

Here

$$(x - 4)^2 + (y - 6)^2 = 25 \tag{1}$$
$$\text{and}\quad (x - 6)^2 + (y + 1)^2 = 34. \tag{2}$$

From these two equations we have

$$x = 1,\quad 459/53$$
$$\text{and}\quad y = 2,\quad 222/53.$$

$\therefore$ the required points are $(1, 2)$ and $(459/53,\ 222/53)$.

1.12 Coordinates of a point dividing a line segment

Let (x_1, y_1) and (x_2, y_2) be the coordinates of two points P and Q respectively and the point R divide the line segment PQ in the ratio $m : n$. Let (x, y) be the coordinates of R [Fig. 3].

PL, QM and RN are perpendicular to OX.

PTS is perpendicular to RN and QM.

From the figure,

$$OL = x_1, ON = x, OM = x_2;$$
$$LP = y_1, NR = y, MQ = y_2;$$
$$PT = LN = ON - OL = x - x_1;$$
$$PS = LM = OM - OL = x_2 - x_1;$$
$$TR = NR - NT = NR - LP = y - y_1;$$
$$SQ = MQ - MS = MQ - LP = y_2 - y_1.$$

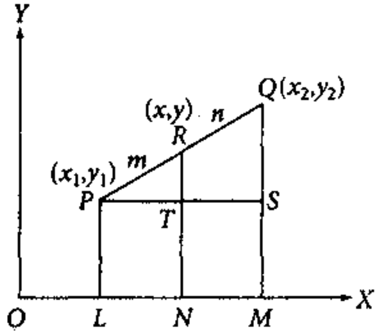


Fig. 3

From similar triangles PTR and PSQ

$$\frac{PT}{PS} = \frac{TR}{SQ} = \frac{PR}{PQ} \quad \text{or,} \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{m}{m + n}.$$

On simplification this gives

$$x = \frac{mx_2 + nx_1}{m + n}, \quad y = \frac{my_2 + ny_1}{m + n}.$$

Corollary I. *If R divides PQ externally in the ratio $m : n$, then*

$$x = \frac{mx_2 - nx_1}{m - n}, \qquad y = \frac{my_2 - ny_1}{m - n}. \qquad \left[Here\ \frac{PR}{PQ} = \frac{m}{m - n}.\right]$$

Corollary II. *If R is the middle point of PQ, then*

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}.$$

Example 3. *Find the coordinates of the centroid of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3).*

Let the coordinates of A, B, C of ΔABC be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively. AD is a median and G divides AD in the ratio $2 : 1$. Therefore, G is the centroid of ΔABC [Fig. 4].

Since D is the middle point of BC, the coordinates of D are

$$\left(\frac{x_2 + x_3}{2}, \quad \frac{y_2 + y_3}{2}\right).$$

The coordinates of G are

$$\left(\frac{1 \cdot x_1 + 2 \cdot \frac{x_2 + x_3}{2}}{1 + 2}, \quad \frac{1 \cdot y_1 + 2 \cdot \frac{y_2 + y_3}{2}}{1 + 2}\right)$$

i.e. $$\left(\frac{x_1 + x_2 + x_3}{3}, \quad \frac{y_1 + y_2 + y_3}{3}\right).$$

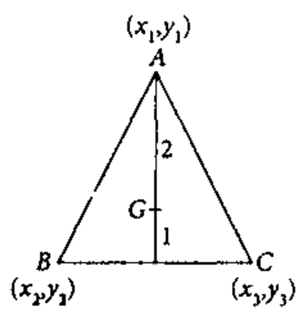


Fig. 4

1.13 Area of a triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC in the frame of co-ordinate axes OX and OY.

AL, BM and CN are perpendicular to OX.

Now ΔABC = trapezium $ABML+$ trapezium $ALNC-$ trapezium $BMNC$.

Area of a trapezium$=\frac{1}{2}$ (the sum of the parallel sides) $\times$ (the perpendicular distance between them).

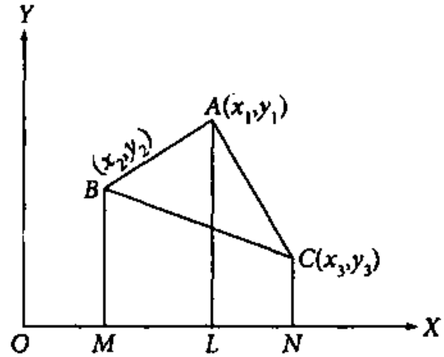


Fig. 5

Thus

$$\begin{aligned}\Delta ABC &= \tfrac{1}{2}(MB+LA)\times ML+\tfrac{1}{2}(LA+NC)\times LN-\tfrac{1}{2}(MB+NC)\times MN\\ &= \tfrac{1}{2}\left[(y_2+y_1)(x_1-x_2)+(y_1+y_3)(x_3-x_1)-(y_2+y_1)(x_3+x_2)\right]\\ &= \tfrac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]\\ &= \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}.\end{aligned}$$

> **Note.** The area of a triangle can be positive or negative. It is reckoned positive if the order of the vertices in anticlockwise. The sign is disregarded when the numerical value of area is required.

Corollary I. *The area of a polygon can be determined by considering it as composed of a number of triangles.*

Corollary II (Condition of collinearity of three points). *If the three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, the area of the triangle formed by these points is zero. Hence the required condition is*

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0.$$

Example 4. *If $(a, 0)$, $(0, b)$ and (x, y) lie on the line show that $\frac{x}{a}+\frac{y}{b}=1$.*

From the condition of collinearity

$$\begin{aligned}& a(b-y)+0\cdot(y-0)+x(0-b)=0\\ \text{or,}\quad & ab-ay-bx=0\\ \text{or,}\quad & bx+ay=ab\\ \text{or,}\quad & \frac{x}{a}+\frac{y}{b}=1.\end{aligned}$$

1.14 Locus

The path traced out by a moving point under certain conditions is called the locus of that point. An algebraical relation is satisfied by the coordinates of all points on the path and the relation is not satisfied by the coordinates of no other points. This algebraical relation is *the equation of this locus.* The locus can also be defined as the curve from an algebraical equation in the coordinates x and y. For example, $x^2 + y^2 = a^2$ is the locus of a point which moves on a circle with centre at origin and radius a.

WORKED-OUT EXAMPLES

1. *Find the coordinates of the incentre of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ and a, b, c are the lengths of the sides opposite to the vertices A, B, C respectively.*

Let the coordinates of A, B, C of ΔABC be (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively. The lengths of BC, CA and AB are a, b, c respectively. Let AD be the bisector of $\angle BAC$ and the point I divide AD in the ratio $b + c : a$.

Thus I is the in centre. Again D divides BC in the ratio $c : b$, i.e.

$$BD : DC = c : b.$$

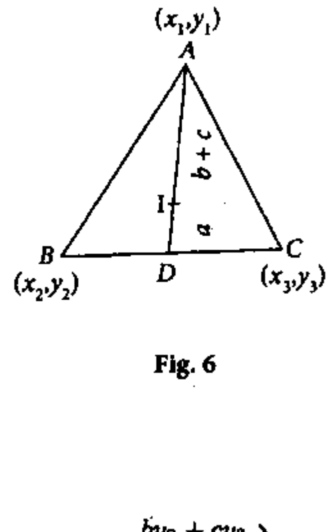


Fig. 6

$\therefore$ the coordinates of I are

$$\left\{\frac{ax_1 + (b+c)\dfrac{bx_2 + cx_3}{b+c}}{a + (b+c)},\quad \frac{ay_1 + (b+c)\dfrac{by_2 + cy_3}{b+c}}{a + (b+c)}\right\},$$

$$\text{i.e.}\quad \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c},\quad \frac{ay_1 + by_2 + cy_3}{a+b+c}\right).$$

2. *If the distance between (x_1, y_1) and (x_2, y_2) be divided into n equal parts, show that the coordinates of the rth point of division from (x_1, y_1) are*

$$\left\{\frac{(n-r)\,x_1 + rx_2}{n},\quad \frac{(n-r)\,y_1 + ry_2}{n}\right\}.$$

The rth point from (x_1, y_1) divides the distance in the ratio $r : (n - r)$.

$\therefore$ the coordinates of the required point are

$$\left\{\frac{(n-r)\,x_1 + rx_2}{r+n-r},\quad \frac{(n-r)\,y_1 + ry_2}{r+n-r}\right\}$$

i.e. $$\left\{\frac{(n-r)\,x_1 + rx_2}{n},\quad \frac{(n-r)\,y_1 + ry_2}{n}\right\}.$$

3. *Find the locus of a point which moves so that the sum of its distances from the points* $(ae,\ 0)$ *and* $(-ae,\ 0)$ *is equal to* $2a$.

Let $(x,\ y)$ be any position of the moving point.

By the given condition

$$\sqrt{(x-ae)^2+y^2}+\sqrt{(x+ae)^2+y^2}=2a$$

or, $$(x-ae)^2+y^2+(x+ae)^2+y^2+2\sqrt{\left\{(x-ae)^2+y^2\right\}\left\{(x+ae)^2+y^2\right\}}=4a^2$$

or, $$\left(x^2+y^2+a^2e^2\right)-2a^2=-\sqrt{\left\{(x-ae)^2+y^2\right\}\left\{(x+ae)^2+y^2\right\}}$$

or, $$\left(x^2+y^2+a^2e^2\right)^2+4a^4-4a^2\left(x^2+y^2+a^2e^2\right)=\left(x^2+y^2+a^2e^2\right)^2-4a^2e^2x^2$$

or, $$x^2+y^2-e^2x^2=a^2-a^2e^2$$

or, $$x^2\left(1-e^2\right)+y^2=a^2\left(1-e^2\right)$$

or, $$\frac{x^2}{a^2}+\frac{y^2}{a^2\left(1-e^2\right)}=1.$$

It is the required locus.

4. *A straight line of length* $2a$ *slides between two rectangular axes* OX *and* OY. *Show that the locus of the middle point is* $x^2+y^2=a^2$.

Let the ends of the line be $(2\alpha,\ 0)$ and $(0,\ 2\beta)$. Therefore, the middle point is $(\alpha,\ \beta)$. By the given condition

$$4\alpha^2+4\beta^2=4a^2$$

or, $$\alpha^2+\beta^2=a^2.$$

Hence the required locus is $x^2+y^2=a^2$.

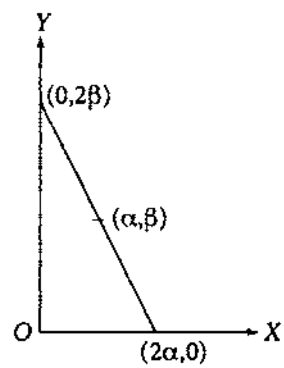


Fig. 7

EXERCISE I

1. Find the distance between the following points:

(a) $(2,\ 1)$ and $(6,\ 9)$,

(b) $(at_1^2,\ 2at_1)$ and $(at_2^2,\ 2at_2)$,

(c) $\left\{a\cos\left(\theta+\frac{\pi}{3}\right),\ a\sin\left(\theta+\frac{\pi}{2}\right)\right\}$ and $\left\{a\cos\left(\theta+\frac{2\pi}{3}\right),\ a\sin\left(\theta+\frac{2\pi}{3}\right)\right\}$.

2. If the distance between $(10,\ \alpha)$ and $(2,\ -3)$ is 10, find α.

3. (a) Show that $(2a,\ 4a)$, $(2a,\ 6a)$ and $(2a+\sqrt{3}\,a,\ 5a)$ form an equilateral triangle. Find the length of each side.

   (b) Show that the points $(-2,\ 3)$, $(-3,\ 10)$ and $(4,\ 11)$ form an isosceles right-angled triangle.

   (c) Show that the points $(-1,\ -2)$, $(3,\ -1)$, $(4,\ 3)$ and $(0,\ 2)$ form a rhombus.

4. (a) If the vertices of parallelogram be $(x_1,\ y_1)$, $(x_2,\ y_2)$, $(x_3,\ y_3)$ and $(x_4,\ y_4)$ in order, show that $x_1+x_3=x_2+x_4$ and $y_1+y_3=y_2+y_4$.

   (b) If the points $(a_1,\ b_1)$, $(a_1+a_2,\ b_1+b_2)$, $(a_1+a_2+a_3,\ b_1+b_2+b_3)$ and $(a_1+a_3,\ b_1+b_3)$ form a square, show that $a_2^2+b_2^2=a_3^2+b_3^2$ and $a_2a_3+b_2b_3=0$.

5. Find the circumcentre and circumradius of the triangle whose vertices are $(1,\ 1)$, $(2,\ 3)$ and $(-2,\ 2)$.

6. (a) Find the coordinates of the points which trisect the distance between $(3,\ 4)$ and $(-1,\ 7)$.

   (b) Show that the same point divides the distance between $(14,\ 2)$ and $(-1,\ -3)$ internally in the ratio $2:3$ and the distance between $(0,\ 4)$ and $(4,\ 2)$ externally in the ratio $2:1$.

7. (a) Find the area of the triangle whose vertices are

      (i) $(-2,\ -7)$, $(0,\ 0)$, $(1,\ 3)$;

      (ii) $(\alpha,\ 1/\alpha)$, $(\beta,\ 1/\beta)$, $(\gamma, 1/\gamma)$.

   (b) Find the area of the quadrilateral whose vertices are $(1,\ 1)$, $(0,\ 5)$, $(-1,\ -3)$ and $(7,\ 0)$.

8. Show that the following points are collinear:

   (a) $(0,\ -3)$, $(3,\ 0)$, $(5,\ 2)$;

   (b) $(a,\ b)$, $(c,\ d)$, $\left(\dfrac{ma+nc}{m+n},\ \dfrac{mb+nd}{m+n}\right)$.

9. In a $\triangle ABC$, prove that

   (a) $AB^2+AC^2=2\left(AD^2+DC^2\right)$ where D is the middle point of BC;

   (b) $\triangle ABC=4\triangle DEF$ where D, E, F are the middle points of BC, CA and AB respectively;

   (c) $AB^2+BC^2+CA^2=3\left(GA^2+GB^2+GC^2\right)$ where G is the centroid of the triangle.

10. (a) If the area of the quadrilateral formed by the points $(k,\ -2)$, $(1,\ 2)$, $(-5,\ 6)$ and $(7,\ -4)$ is zero, show that k is 3.

    (b) If $\triangle ABD = 2\triangle ACD$ and the coordinates of A, B, C, D are $(-3,\ 4)$, $(-1,\ -2)$, $(5,\ 6)$ and $(k,\ -4)$ respectively, show that k is $-71/5$.

11. (a) The coordinates of A, B, C are $(6,\ 3)$, $(-3,\ 5)$ and $(4,\ -2)$ respectively. Find the locus of P when $\triangle PBC/\triangle ABC = 2/3$.

    (b) Find the locus of a point which moves so that its distance from the axis of x is half its distance from the origin.

12. The line joining the points $A(b\cos\theta,\ b\sin\theta)$ and $B(a\cos\phi,\ a\sin\phi)$ is produced to the point $P(x,\ y)$ such that $AP : BP = b : a$, prove that

$$x + y\tan\frac{\theta+\phi}{2} = 0.$$

ANSWERS

1. (a) $4\sqrt{5}$; (b) $a(t_2 - t_1)\ \sqrt{(t_1+t_2)^2+4}$; (c) a.

2. 3, -9. 3. (a) $2a$.

5. $\left(-\dfrac{1}{14},\ \dfrac{39}{14}\right)$, $\dfrac{5}{14}\sqrt{34}$. 6. (a) $\left(\dfrac{5}{3},\ 5\right)$, $\left(\dfrac{1}{3},\ 6\right)$; (b) $(8,\ 0)$.

7. (a) (i) 14; (iii) $\dfrac{1}{2}\left\{\alpha\left(\dfrac{1}{\beta}-\dfrac{1}{\gamma}\right)+\beta\left(\dfrac{1}{\gamma}-\dfrac{1}{\alpha}\right)+\gamma\left(\dfrac{1}{\alpha}-\dfrac{1}{\beta}\right)\right\}$.

   (b) 19.

8. (a) $3x+3y=20$; (b) $3y^2 = x^2$.

1.20 Any linear equation in x and y represents a straight line

Let

$$ax + by + c = 0 \tag{1}$$

be a given linear equation and three points $(x_1,\ y_1)$, $(x_2,\ y_2)$ and $(x_3,\ y_3)$ lie on the locus represented by (1). Then

$$ax_1 + by_1 + c = 0, \tag{2}$$

$$ax_2 + by_2 + c = 0, \tag{3}$$

$$ax_3 + by_3 + c = 0. \tag{4}$$

Eliminating a, b and c from (2), (3) and (4), we have

$$x_1\ (y_2 - y_3) + x_2\ (y_3 - y_1) + x_3\ (y_1 - y_2) = 0$$

which is the condition of collinearity.

This shows that any three points on the locus of (1) are collinear. Hence the equation (2) represents a straight line.

1.21 Different forms of the equation of a straight line

(A) Gradient form

Let a straight line AB makes an angle θ with the x-axis and an intercept of length c $(= OB)$ on the y-axis [Fig. 8].

Let $P(x, y)$ be a point on the line, PL perpendicular to OX and BM perpendicular to PL.

From the figure,

$$BM = OL = x,$$
$$MP = LP - LM = LP - OB = y - c$$
$$\text{and} \quad \angle PBM = \theta.$$

From ΔPBM,

$$\tan\theta = \frac{MP}{BM} = \frac{y-c}{x}.$$

Denoting $\tan\theta$ by m, we get

$$y = mx + c.$$

This relation is satisfied by any point on the line. Hence it is the equation of the straight line.

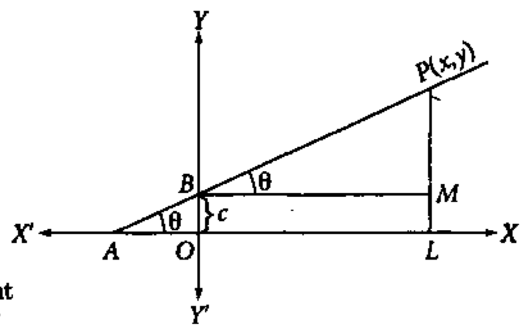


Fig. 8

> **Note.** $m(= \tan\theta)$ is defined as gradient or slope of the line. Also c is positive when the line intersects OY and is negative when it intersects OY'.

Corollary I (Line parallel to x-axis). *If the line is parallel to x-axis, then* $\theta = 0$ *i.e.* $m = 0$. *Therefore, the equation of a line parallel to x-axis is* $y = c$. c *is the distance of the line from the x-axis. If* $c = 0$, *the line coincides with the x-axis. Thus the equation of the x-axis is* $y = 0$.

Corollary II (Line through the origin). *If the line passes through the origin.* $c = 0$. *Thus the equation of a line through the origin is* $y = mx$.

(B) Intercept form

Let the line AB cut off intercepts $a(= OA)$ and $b(= OB)$ on the axes of x and y respectively. $P(x, y)$ is a point on the line and PM is perpendicular to OX.

From similar triangles APM and ABO,

$$\frac{MA}{OA} = \frac{MP}{OB} \quad \text{or,} \quad \frac{a-x}{a} = \frac{y}{b}$$
$$\text{or,} \quad \frac{x}{a} + \frac{y}{b} = 1.$$

It is the required equation of the straight line.

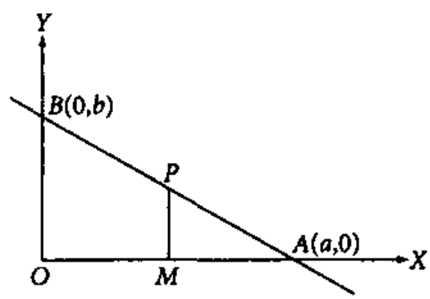


Fig. 9

Corollary III (Line parallel to y-axis). *If the line is parallel to y-axis, then b is infinite and $1/b = 0$. The equation reduces to $x = a$. a is the distance of the line from the y-axis. Evidently $x = 0$ is the equation of the y-axis.*

(C) Normal form

Let $ON(= p)$ be perpendicular to the line AB from the origin O and ON make an angle α with the x-axis. Here OA and OB are the intercepts made by the line on the axes and $OA = p\sec\alpha$, $OB = p\operatorname{cosec}\alpha$.

$\therefore$ the equation of the line is

$$\frac{x}{p\sec\alpha} + \frac{y}{p\operatorname{cosec}\alpha} = 1$$

or, $$x\cos\alpha + y\sin\alpha = p.$$

It is the required equation of the straight line.

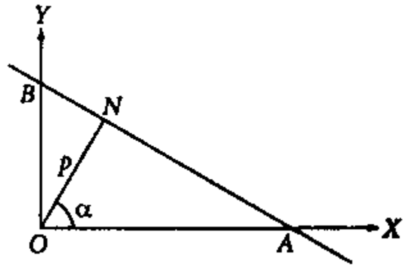


Fig. 10

Note. p is always taken positive and α may be acute or obtuse.

(D) Equation of a straight line through a given point with the gradient m

Let the given point be $(x_1,\ y_1)$ and the equation of the line be

$$y = mx + c. \tag{5}$$

As it passes through $(x_1,\ y_1)$

$$y_1 = mx_1 + c. \tag{6}$$

Eliminating c from (5) and (6),

$$y - y_1 = m\,(x - x_1).$$

It is the required equation of the straight line.

(E) Equation of a straight line through two given points

Let the given points be $(x_1,\ y_1)$ and $(x_2,\ y_2)$. If m is the gradient of this line, then the equation of it can be written as

$$y - y_1 = m\,(x - x_1). \tag{7}$$

As it passes through $(x_2,\ y_2)$

$$y_2 - y_1 = m\,(x_2 - x_1). \tag{8}$$

Eliminating m from (7) and (8), the required equation is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Note. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

(F) General form

The linear equation

$$ax + by + c = 0$$

always represents a straight line (by Sec 1.20). If its normal form be

$$x\cos\alpha + y\sin\alpha = p$$

then

$$\frac{\cos\alpha}{-a} = \frac{\sin\alpha}{-b} = \frac{p}{c} = \frac{1}{\pm\sqrt{a^2+b^2}}.$$

$\therefore$ the perpendicular distance from the origin to the line

$$p = \frac{c}{\pm\sqrt{a^2+b^2}}.$$

p is taken positive. Therefore, the sign of the denominator is taken as that of c. Consequently $\cos\alpha$ and $\sin\alpha$ are fixed.

1.22 Angle between two lines

Let the equations of the lines AC and BC be

$$y = m_1x + c_1$$
$$\text{and} \quad y = m_2x + c_2$$

respectively and ϕ be the angle between them.

If θ_1 and θ_2 be the inclination of the lines with the x-axis, then

$$\tan\theta_1 = m_1,$$
$$\tan\theta_2 = m_2$$
$$\text{and} \quad \phi = \theta_1 - \theta_2.$$

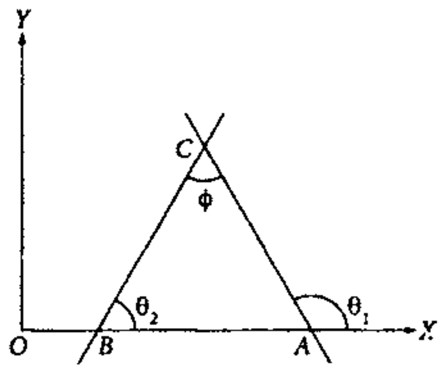


Fig. 11

Now

$$\tan\phi = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2} = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$\text{or,} \qquad \phi = \tan^{-1}\frac{m_1 - m_2}{1 + m_1m_2}.$$

> **Note.** If the two lines do not intersect at right angle, one of the two angles formed by them at the point of intersection must be acute and the other must be obtuse.

Corollary I. *If the lines are in the form of*

$$a_1x + b_1y + c_1 = 0$$
$$\textit{and} \quad a_2x + b_2y + c_2 = 0$$

then $m_1 = -a_1/b_1$ *and* $m_2 = -a_2/b_2$.

In this case

$$\tan\phi = \frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}.$$

Corollary II (Parallel lines). *If the lines are parallel* $\phi = 0$.

$$m_1 = m_2 \quad or, \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}, \quad i.e. \quad a_1b_2 - a_2b_1 = 0.$$

Corollary III (Perpendicular lines). *If the lines are at right angle,* $\phi = 90°$.

$\therefore$ $\cot\phi = 0$ *or,* $1 + m_1m_2 = 0$ *or,* $m_1m_2 = -1$ *or,* $a_1a_2 + b_1b_2 = 0$.

General forms of the equations of a pair of parallel or perpendicular lines

Corollary II suggests that the coefficients of x and y of two parallel lines will be proportional. In simplified forms the constants will only differ from each other. Thus $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ are parallel lines.

Corollary III indicates that $ax+by+c_1 = 0$ and $bx-ay+c_2 = 0$ are perpendicular to each other.

1.23 Perpendicular distance of a point from a line

Let

$$ax + by + c = 0$$

be the equation of a line AB and $P(x_1, y_1)$ be a given point. Draw a line through P, which is parallel to AB. Let its equation be

$$ax + by + k = 0.$$

As it passes through the point (x_1, y_1), $k = -(ax_1 + by_1)$.

$\therefore$ the equation of this line is

$$ax + by - (ax_1 + by_1) = 0.$$

OMM' is perpendicular to both the lines.

Now

$$OM = \frac{c}{\pm\sqrt{a^2+b^2}},$$

$$OM' = \frac{-(ax_1 + by_1)}{\sqrt{a^2+b^2}}.$$

$\therefore$ PN (perpendicular distance from the point to the line AB)

$$= OM' - OM = \frac{-(ax_1 + by_1 + c)}{\pm\sqrt{a^2+b^2}}.$$

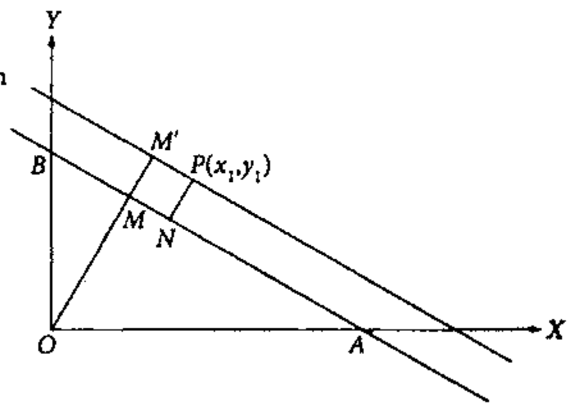


Fig. 12

To fix the sign of the denominator we follow our convention that the perpendicular distance from the origin on a line is positive. Thus the sign of the denominator is same as that of c. It suggests that the point and the origin are on the same side of the line, if the distance is positive and the distance will be negative when the point and the origin are on the opposite sides of the line.

Corollary. Two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$ are on the same or opposite sides of a line $ax+by+c=0$ according as ax_1+by_1+c and ax_2+by_2+c are of the same or opposite signs.

1.24 Bisectors of the angles between two straight lines

Let the equations of AC and BC be $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ and $P(\alpha,\ \beta)$ be point on any of the bisectors. The perpendicular distances of P from the lines are numerically equal.

$$\therefore\ \frac{a_1\alpha+b_1\beta+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2\alpha+b_2\beta+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Thus the locus of P i.e. the equations of the bisectors are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad (9)$$

and

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}. \qquad (10)$$

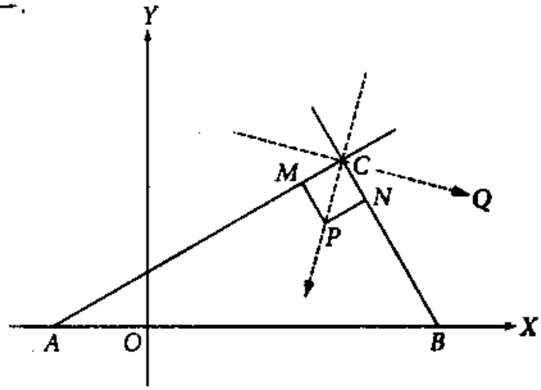


Fig. 13

If c_1 and c_2 are of the same sign, the equation (9) will represent the bisector of the angle in which the origin lies and the equation (10) will represent the other bisector. If c_1 and c_2 have opposite signs, the case will be reversed.

1.25 Point of intersection between two lines

Let the equations of the lines be

$$a_1x+b_1y+c_1=0$$
$$\text{and}\quad a_2x+b_2y+c_2=0.$$

If $(\alpha,\ \beta)$ be the point of intersection,

$$a_1\alpha+b_1\beta+c_1=0$$
$$\text{and}\quad a_2\alpha+b_2\beta+c_2=0.$$

From these two

$$\frac{\alpha}{b_1c_2-b_2c_1}=\frac{\beta}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}.$$

$$\therefore\ \alpha=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\qquad \beta=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

1.26 The equations of lines through the point of intersection of two lines

If

$$a_1x + b_1y + c_1 = 0 \tag{11}$$
$$\text{and} \quad a_2x + b_2y + c_2 = 0 \tag{12}$$

be the equations of two lines then

$$a_1x + b_1y + c_1 + \lambda\ (a_2x + b_2y + c_2) = 0 \tag{13}$$

represents a straight line passing through the common point of (11) and (12). λ is an arbitrary constant.

If $(\alpha,\ \beta)$ be the common point, then

$$a_1\alpha + b_1\beta + c_1 = 0 \quad \text{and} \quad a_2\alpha + b_2\beta + c_2 = 0.$$

$$\therefore\ a_1\alpha + b_1\beta + c_1 + \lambda\ (a_2\alpha + b_2\beta + c_2) = 0.$$

Hence the equation (13) which is linear in x and y represents lines for different values of λ through the common point of (11) and (12).

Conversely

$$a_1x + b_1y + c_1 + \lambda\ (a_2x + b_2y + c_2) = 0,$$

where λ is an arbitrary constant, represents a straight line passing through a fixed point which is the point of intersection of

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2 = 0.$$

1.27 Condition for the concurrence of three lines

(i) Let the lines

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0$$
$$\text{and} \quad a_3x + b_3y + c_3 = 0$$

pass through a point $(\alpha,\ \beta)$.

$$\therefore\ a_1\alpha + b_1\beta + c_1 = 0,$$
$$a_2\alpha + b_2\beta + c_2 = 0$$
$$\text{and} \quad a_3\alpha + b_3\beta + c_3 = 0.$$

Eliminating α and β from the above three equations, the required condition for concurrence is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

(ii) If

$$\lambda(a_1x+b_1y+c_1)+\mu(a_2x+b_2y+c_2)+\nu(a_3x+b_3y+c_3)=0$$

identically for non-zero constants λ, μ, ν, then the lines

$$\begin{aligned} a_1x+b_1y+c_1&=0,\\ a_2x+b_2y+c_2&=0\\ \text{and}\quad a_3x+b_3y+c_3&=0 \end{aligned}$$

must be concurrent.

Since

$$\lambda(a_1x+b_1y+c_1)+\mu(a_2x+b_2y+c_2)+\nu(a_3x+b_3y+c_3)=0$$

identically,

$$\lambda(a_1x+b_1y+c_1)+\mu(a_2x+b_2y+c_2)=-\nu(a_3x+b_3y+c_3)$$

identically. Thus the line

$$\lambda(a_1x+b_1y+c_1)+\mu(a_2x+b_2y+c_2)=0$$

is the same as the line

$$a_3x+b_3y+c_3=0.$$

Again

$$\lambda(a_1x+b_1y+c_1)+\mu(a_2x+b_2y+c_2)=0$$

passes through the common point of

$$\begin{aligned} a_1x+b_1y+c_1&=0\\ \text{and}\quad a_2x+b_2y+c_2&=0. \end{aligned}$$

Hence the lines are concurrent.

WORKED-OUT EXAMPLES

1. *The equations of the sides of a triangle are* $y=m_1x+c_1$, $y=m_2x+c_2$ and $x=0$.

   Prove that the area of the triangle is

$$\frac{1}{2}\frac{(c_1-c_2)^2}{m_1-m_2}.$$

$$y=m_1x+c_1, \tag{1}$$

$$y=m_2x+c_2 \tag{2}$$

$$\text{and}\quad x=0. \tag{3}$$

The point of intersection of the lines of (1) and (2) is

$$\left(\frac{c_1 - c_2}{m_2 - m_1}, \frac{m_2c_1 - m_1c_2}{m_2 - m_1}\right).$$

The points of intersection of pair of lines (2), (3) and (3), (1) are $(0, c_2)$ and $(0, c_1)$. These are vertices of the triangle.

$$\therefore \text{ area of the triangle} = \frac{1}{2}\begin{vmatrix} \dfrac{c_1 - c_2}{m_2 - m_1} & \dfrac{m_2c_1 - m_1c_2}{m_2 - m_1} & 1 \\ 0 & c_2 & 1 \\ 0 & c_1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\frac{(c_1 - c_2)^2}{m_1 - m_2}.$$

2. *Find the locus of the feet of the perpendiculars from the origin to the lines passing through a fixed point.*

Let the fixed point be (h, k). The equation of a line through (h, k) can be written as

$$y - k = m(x - h). \tag{1}$$

The equation of the line passing through the origin and perpendicular to (1) is

$$y = -\frac{x}{m}. \tag{2}$$

The locus of the foot of the perpendicular i.e. the point of intersection between (1) and (2) is obtained by eliminating the variable constant m from (1) and (2). Thus the required locus is

$$(y - k)y = -x(x - h)$$

$$\text{or,} \quad x^2 + y^2 = hx + ky.$$

3. *A line moves in such a way that the sum of the perpendicular distances from the vertices of a triangle is zero. Show that the line always passes through the centroid of the triangle.*

Let

$$ax + by + c = 0$$

be the equation of the line and (x_1, y_1), (x_2, y_2), (x_3, y_3) be the vertices of the triangle. By the given condition

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} + \frac{ax_2 + by_2 + c}{\sqrt{a^2 + b^2}} + \frac{ax_3 + by_3 + c}{\sqrt{a^2 + b_2}} = 0$$

$$\text{or,} \quad a(x_1 + x_2 + x_3) + b(y_1 + y_2 + y_3) + 3c = 0$$

$$\text{or,} \quad a\frac{x_1 + x_2 + x_3}{3} + b\frac{y_1 + y_2 + y_3}{3} + c = 0.$$

It shows that the line passes through the centroid of the triangle.

4. *Show that the perpendicular bisectors of the sides of any triangle are concurrent.*

   Let ABC be a triangle and the coordinates of A, B, C be $(x_1,\ y_1)$, $(x_2,\ y_2)$ and $(x_3,\ y_3)$. If D be the middle point of BC, the coordinates of D are

$$\left(\frac{x_2+x_3}{2},\ \frac{y_2+y_3}{2}\right).$$

   The gradient of $BC = \dfrac{y_2-y_3}{x_2-x_3}$.

   $\therefore$ the equation of the perpendicular bisector of the side BC is

$$y-\frac{y_2+y_3}{2}=-\frac{x_2-x_3}{y_2-y_3}\left(x-\frac{x_2+x_3}{2}\right)$$

$$\text{or,}\quad 2\left(x_2-x_3\right)x+2\left(y_2-y_3\right)y-\left(x_2^2-x_3^2\right)-\left(y_2^2-y_3^2\right)=0. \qquad (1)$$

   Similarly the other two equations are

$$2\left(x_3-x_1\right)x+2\left(y_3-y_1\right)y-\left(x_3^2-x_1^2\right)-\left(y_3^2-y_1^2\right)=0 \qquad (2)$$

$$\text{and}\quad 2\left(x_1-x_2\right)x+2\left(y_1-y_2\right)y-\left(x_1^2-x_2^2\right)-\left(y_1^2-y_2^2\right)=0. \qquad (3)$$

   The sum of (1), (2) and (3) is identically zero. Hence the result follows.

5. *Find the equations of the line passing through $(h,\ k)$ and making an angle ψ with $ax+by+c=0$.*

   The equation of a line through $(h,\ k)$ can be written as

$$y-k=m(x-h)$$

$$\text{or,}\quad mx-y+k-mh=0. \qquad (1)$$

   Since the angle between (1) and $ax+by+c=0$ is ψ

$$\tan\psi=\pm\frac{mb+h}{am-b}$$

$$\therefore\ m=\frac{b\tan\psi+a}{a\tan\psi-b}$$

$$\text{and}\quad m=\frac{b\tan\psi-a}{a\tan\psi+b}.$$

   $\therefore$ the required lines are

$$y-k=\frac{b\tan\psi+a}{a\tan\psi-b}(x-h)$$

$$\text{and}\quad y-k=\frac{b\tan\psi-a}{a\tan\psi+b}(x-h).$$

6. *Show that the distance of the point (x_0, y_0) from the line*

$$ax + by + c = 0$$

measured parallel to a line making an angle θ with the x-axis is

$$\frac{ax_0 + by_0 + c}{a\cos\theta + b\sin\theta}.$$

The line passing through (x_0, y_0) and parallel to the line making an angle θ with the x-axis is

$$\frac{x - x_0}{\cos\theta} = \frac{y - y_0}{\sin\theta} \quad (= r \text{ say}). \tag{1}$$

The distance will be measured along this line and the required distance lies between (x_0, y_0) and the point of intersection between (1) and

$$ax + by + c = 0. \tag{2}$$

To find the point of intersection we put in (2)

$$x = x_0 + r\cos\theta$$
$$\text{and} \quad y = y_0 + r\sin\theta.$$

$$\therefore \quad ax_0 + by_0 + c + r(a\cos\theta + b\sin\theta) = 0$$
$$\text{or,} \quad r = -\frac{ax_0 + by_0 + c}{a\cos\theta + b\sin\theta}.$$

The magnitude of r is the required distance.

7. *If a line moves in such a way that the sum of the reciprocals of the intercepts made by the line on the axes is always constant, then the line passes through a fixed point.*

Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{and} \quad \frac{1}{a} + \frac{1}{b} = k \text{ (constant)}.$$

The imposed condition is obtained from the equation of the line by simply putting $x = \frac{1}{k}$ and $y = \frac{1}{k}$.

Since k is a constant, the point $(1/k,\ 1/k)$ is fixed and the line passes through the fixed point.

8. *If the lines*

$$lx + my + n = 0,$$
$$mx + ny + l = 0$$
$$\textit{and} \quad nx + ly + m = 0$$

are concurrent, show that $l + m + n = 0$.

The lines will be concurrent, if

$$\begin{vmatrix} l & m & n \\ m & n & l \\ n & l & m \end{vmatrix} = 0 \quad \text{or,} \quad \begin{vmatrix} l+m+n & m & n \\ l+m+n & n & l \\ l+m+n & l & m \end{vmatrix} = 0$$

$$\text{or,} \quad (l+m+n)\begin{vmatrix} 1 & m & n \\ 1 & n & l \\ 1 & l & m \end{vmatrix} = 0$$

$$\text{or,} \quad (l+m+n)\ (mn+nl+lm-l^2-m^2-n^2) = 0$$

$$\text{or,} \quad (l+m+n)\ \left\{(l-m)^2+(m-n)^2+(n-l)^2\right\} = 0.$$

$\because$ l, m, n are different $l+m+n=0$.

9. *Show that the locus of the points of intersection of the two variable lines $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$, t_1 and t_2 being parameters connected by the relation $1/t_1 + 1/t_2 = m$ (constant) is a straight line.*

$$t_1y = x + at_1^2 \tag{1}$$

$$t_2y = x + at_2^2. \tag{2}$$

From (1) and (2)

$$\frac{y}{t_1} = \frac{x}{t_1^2} + a \tag{3}$$

$$\text{and} \quad \frac{y}{t_2} = \frac{x}{t_2^2} + a. \tag{4}$$

Subtracting (4) from (3),

$$y\left(\frac{1}{t_1} - \frac{1}{t_2}\right) = x\left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right) \quad \text{or,} \quad y = x\left(\frac{1}{t_1} + \frac{1}{t_2}\right) \quad \text{or,} \quad y = mx.$$

It is the required locus which is a straight line.

EXERCISE II

1. Find the equation of a line making an angle 30° with the x-axis and passing through (1, 2).

2. If the product of the intercepts made by a line on the axes be 6 and (1, 3/2) is a point on the line, find the equation of the line.

3. Find the equation of the line when the normal to the line from the origin is inclined to the x-axis at an angle of 150° and is of length 5.

4. Find the equation of the line passing through the common point of $2x-3y+7=0$ and $4x+5y+7=0$ and (a) parallel to $7x-3y+5=0$ (b) perpendicular to $5x-y+3=0$.

5. Find the equation of a line which passes through (0, 4) and whose slope is equal to the angle between the lines $2y-x-12=0$ and $x+3y+16=0$.

6. Show that the equation of the line whose intercepted portion between the axes is divided internally in the ratio by the point $(-4, 3)$ is $9x-20y+96=0$.

7. Find the equations of the lines drawn through the point (0, 0) on which the perpendiculars let fall from the point $(2a, 2a)$ are each of length a.

8. Find the equations of the bisectors of the angles between the lines $4x-3y+1=0$ and $12x+5y+13=0$ and assign the bisector of the angle in which the origin lies.

9. If p_1 and p_2 be the perpendicular distances from the origin to the lines $x\sec\theta - y\operatorname{cosec}\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$ respectively prove that $4p_1^2+p_2^2 = a^2$.

10. If the lines $p_1x+q_1y=1$, $p_2x+q_2y=1$ and $p_3x+q_3y=1$ be concurrent, prove that (p_1, q_1), (p_2, q_2) and (p_3, q_3) are collinear.

11. If $lx+my+n=0$ where l, m, n are constants, is the equation of a variable line and l, m, n are connected by the relation $al+bm+cn=0$ where a, b, c are constants, show that the variable line passes through a fixed point $(a/c, b/c)$.

12. If the points (a, b), (a', b'), $(a-a', b-b')$ are collinear, show that their join passes through the origin and then $ab'=a'b$.

13. Show that the lines $x\cos\left(\alpha+\frac{r\pi}{3}\right)+\sin\left(\alpha+\frac{r\pi}{3}\right)=a$, $(r=2, 4, 6)$ from an equilateral triangle.

14. Find the coordinates of the middle points of the three diagonals of the complete quadrilateral formed by the lines $x=0$, $y=0$, $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{a'}+\frac{y}{b'}=1$ and show that these points are collinear.

15. Find the ortho-centre of the triangle whose sides are $y=m_rx+\frac{a}{m_r}$, $r=1, 2, 3$.

16. A variable line passes through the fixed point (h, k). Prove that the locus of the middle point of its portion intercepted between the axes is given by $h/x+k/y=2$.

17. Prove that $x=\frac{a+bt}{p+qt}$, $y=\frac{c+dt}{p+qt}$ (t is a parameter) represent a straight line.

18. If the line $\frac{x}{a}+\frac{y}{b}=1$ moves in such a way that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$ where c is a constant, a and b are variables, then show that the locus of the foot of the perpendicular from the origin is $x^2+y^2=c^2$.

19. Through a fixed point O a line is drawn to meet n lines at $P_1, P_2, \ldots, P_n$ and a point P is taken on it such that $\frac{n}{OP}=\frac{1}{OP_1}+\frac{1}{OP_2}+\cdots+\frac{1}{OP_n}$, prove that the locus of P is a straight line.

20. $OACB$ is a rectangle of which the corner O is fixed and the corner C moves on a fixed straight line. Show that the locus of the middle point of the diagonal AB is a straight line parallel to the given straight line.

ANSWERS

1. $x - \sqrt{3}y + 2\sqrt{3} - 1 = 0.$
2. $\frac{x}{2} + \frac{y}{3} = 1.$
3. $\sqrt{3}x - y + 10 = 0.$
4. $77x - 33y + 217 = 0,$
 $11x + 55y - 7 = 0.$
5. $y = x + 4.$
7. $(4 \pm \sqrt{7})\, x - 3y = 0.$
8. $2x + 16y + 13 = 0,$
 $56x - 7y + 39 = 0.$
 The first one.
14. $\left(\frac{a}{2}, \frac{b'}{2}\right), \left(\frac{a'}{2}, \frac{b}{2}\right), \left\{\frac{aa'(b-b')}{2(a'b-ab')}, \frac{bb'(a'-a)}{2(a'b-ab')}\right\}.$
15. $\left\{-a, a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1m_2m_3}\right)\right\}.$

1.30 Transformation of axes

The coordinates of a point depend on the position of axes. Thus the coordinates of a point and consequently the equation of a locus will be changed with the alteration of origin without the alteration of direction axes, or by altering the direction of axes and keeping the origin fixed, or by altering the origin and also the direction of axes. Either of these processes is known as *transformation of axes* or *transformation of coordinates.*

1.31 Change of origin without change of direction of axes

Let (x, y) be the coordinates of P w.r.t. rectangular axes OX and OY and (x', y') be the coordinates of it w.r.t. a new set of axes $O'X'$ and $O'Y'$ which are parallel to the original axes OX and OY respectively.

Let (α, β) be the coordinates of the new origin O' w.r.t. axes OX and OY. PN is perpendicular to OX and it meets $O'X'$ at N'. $O'T$ is perpendicular to OX.

$$\begin{aligned} \therefore \quad ON &= x, & NP &= y \\ O'N' &= x', & N'P &= y' \\ OT &= \alpha, & TO' &= \beta \end{aligned}$$

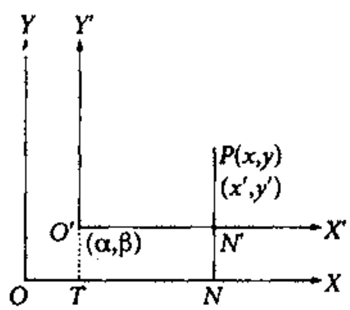


Fig. 14

Now

$$x = ON = OT + TN = OT + O'N' = \alpha + x',$$
$$y = NP = NN' + N'P = TO' + N'P = \beta + y'.$$

Hence, the required transformation formulae are given by $x = x' + \alpha$, $y = y' + \beta$.

This transformation is also known as *translation* or *parallel displacement*.

Note. In the equation of a locus referred to original system of axes $(x,\ y)$ will be replaced by $(x' + \alpha,\ y' + \beta)$ when the equation is referred to new pair of axes. Inversely $(x',\ y')$ will be replaced by $(x - \alpha,\ y - \beta)$.

Example 1. *Find the form of the equation* $3x + 4y = 5$ *due to change of origin to the point* $(3,\ -2)$ *only.*

The transformed equation is $3(x' + 3) + 4(y' - 2) = 5$ or $3x' + 4y' = 4$.

Example 2. *Find the equation of* $\frac{x}{a} + \frac{y}{b} = 2$ *when the origin is shifted to* $(a,\ b)$.

The transformed equation is

$$\frac{x' + a}{a} + \frac{y' + b}{b} = 2, \quad \text{or,} \quad \frac{x'}{a} + \frac{y'}{b} = 0.$$

1.32 Rotation of rectangular axes in their own plane without changing the origin

Let the original axes OX and OY be rotated through an angle θ in the anti-clockwise direction. In the adjoining figure OX' and OY' are the new set of axes. Let $(x,\ y)$ and $(x',\ y')$ be the coordinates of the same point P referred to OX, OY and OX', OY' respectively.

PM and PM' are perpendicular to OX and OX' respectively. PO is joined. Here $\angle XOX' = \theta$.

Let $\angle X'OP = \alpha$.

From the figure

$$OM = x, \quad MP = y,$$
$$OM' = x', \quad M'P = y'.$$

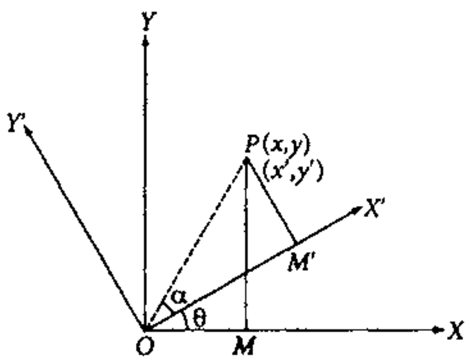


Fig. 15

Now

$$\begin{aligned} x = OM &= OP\cos(\theta + \alpha) \\ &= OP\cos\alpha\cos\theta - OP\sin\alpha\sin\theta \\ &= OM'\cos\theta - M'P\sin\theta \\ &= x'\cos\theta - y'\sin\theta. \end{aligned} \qquad \begin{aligned} y = MP &= OP\sin(\theta + \alpha) \\ &= OP\cos\alpha\sin\theta + OP\sin\alpha\cos\theta \\ &= OM'\sin\theta + M'P\cos\theta \\ &= x'\sin\theta + y'\cos\theta. \end{aligned}$$

Hence the change from $(x,\ y)$ to $(x',\ y')$ is given by

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ \text{and} \quad y &= x'\sin\theta + y'\cos\theta. \end{aligned}\right\} \qquad (1)$$

From (1) we can easily deduce that

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ \text{and}\quad y' &= y\cos\theta - x\sin\theta. \end{aligned}\right\} \qquad (2)$$

Both of the transformations (1) and (2) can be remembered by the scheme

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

Example 1. *Find the equation of the line $y = \sqrt{3}\,x$ when the axes are rotated through an angle $\pi/3$.*

The transformed equation is

$$\begin{aligned} & x'\sin\frac{\pi}{3} + y'\cos\frac{\pi}{3} = \sqrt{3}\left(x'\cos\frac{\pi}{3} - y'\sin\frac{\pi}{3}\right) \\ \text{or,}\quad & \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' = \sqrt{3}\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right) \\ \text{or,}\quad & \frac{1}{2}y' + \frac{3}{2}y' = 0 \\ \text{or,}\quad & y' = 0. \end{aligned}$$

Example 2. *What will be the form of $x\cos\alpha + y\sin\alpha = p$ when the axes are rotated through an angle α.*

The transformed form is

$$\begin{aligned} & (x'\cos\alpha - y'\sin\alpha)\cos\alpha + (x'\sin\alpha + y'\cos\alpha)\sin\alpha = p \\ \text{or,}\quad & x'(\cos^2\alpha + \sin^2\alpha) = p \quad \text{or,}\quad x' = p. \end{aligned}$$

1.33 Combination of translation and rotation

If the origin O of a set of rectangular axes (OX, OY) is shifted to O' $(\alpha,\ \beta)$ [referred to OX and OY] without changing the direction of axes and then the axes are rotated through an angle θ in the anti-clockwise direction, the total effective changes in the coordinates $(x,\ y)$ of a point are given by

$$\left.\begin{aligned} x &= \alpha + x''\cos\theta - y''\sin\theta \\ \text{and}\quad y &= \beta + x''\sin\theta + y''\cos\theta. \end{aligned}\right\}$$

$(x'',\ y'')$ are the coordinates of the point referred to the final set of axes.

1.34 Transformation of coordinates when the equations of new axes are given

Let (x, y) be the coordinates of a point P referred to rectangular axes OX and OY and (x', y') be the coordinates of the same point referred to a new set of rectangular axes $O'X'$ and $O'Y'$ whose equations are $lx + my + n = 0$ and $mx - ly + k = 0$ w.r.t. OX and OY.

Perpendicular distances from P to $mx - ny + k = 0$ is

$$N'P = x' = \pm\frac{mx - ly + k}{\sqrt{l^2 + m^2}}. \quad (3)$$

Perpendicular distance from P to $lx + my + n = 0$ is

$$M'P = y' = \pm\frac{lx + my + n}{\sqrt{l^2 + m^2}}. \quad (4)$$

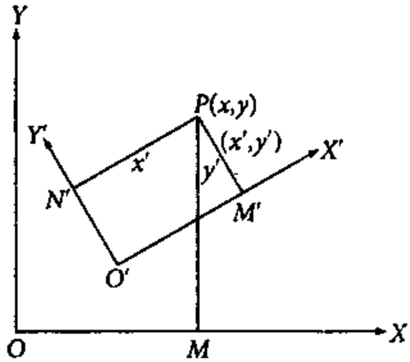


Fig. 16

The same sign is taken in both cases according to convenience. By (3) and (4) values of x and y are found out in terms of x' and y'.

Note. This transformation is helpful when the equation can be expressed in terms of

$$\frac{mx - ly + k}{\sqrt{l^2 + m^2}} \quad \text{and} \quad \frac{lx + my + n}{\sqrt{l^2 + m^2}}$$

Example. Find the transformed equation of the curve

$$(3x + 4y + 7)(4x - 3y + 5) = 50$$

when the axes are $3x + 4y + 7 = 0$ and $4x - 3y + 5 = 0$.

If (x', y') be the coordinates of a point (x, y) referred to the new set of axes, then

$$y' = \frac{3x + 4y + 7}{\sqrt{3^2 + 4^2}} = \frac{3x + 4y + 7}{5}. \qquad x' = \frac{4x - 3y + 5}{\sqrt{4^2 + 3^2}} = \frac{4x - 3y + 5}{5}.$$

The given equation can be written as

$$\frac{3x + 4y + 7}{5} \cdot \frac{4x - 3y + 5}{5} = 2. \quad \therefore\ x'y' = 2.$$

Hence the transformed equation is $xy = 2$.

EXERCISE III

1. Reduce the following in the form $lx + my = 0$ by choice of new origin (i) on the x-axis, (ii) on the y-axis without rotation of axes.

   (a) $2x + 3y - 6 = 0$,

   (b) $y = mx + c$,

   (c) $x \cos\alpha + y \sin\alpha = p$.

2. Reduce the following equation in the form $ax^2 + by^2 = 1$ by proper translation of axes without rotation.

   (a) $5x^2 - 7y^2 + 2x - 3y = 0$,

   (b) $4x^2 + 3y^2 - 2x - 3y - 7 = 0$.

3. Choose a new origin in such a way that the following equations referred to this new set of axes will turn out to be homogeneous in $(x,\ y)$ of second degree.

   (a) $2x^2 + 4xy + 5y^2 - 4x - 22y + 29 = 0$,

   (b) $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$,

   (c) $x^2 - y^2 + (h+k)x - (h-k)y + hk = 0$.

4. Find a new origin without rotating the axes in order that the equation $y^2 + 2y - 8x + 25 = 0$ will turn out to be $y^2 = 4ax$. Determine a. [NH 2010]

5. What does the equation $4x^2 + 2\sqrt{3}\,xy + 2y^2 = 1$ become when the axes are rotated through an angle $30°$?

6. (a) Find the angle of rotation of the axes for which the equation $x^2 - y^2 = a^2$ will reduce to $xy = c^2$. Determine c^2.

   (b) What is the angle of rotation of the axes for which the equation $3x^2 - 5xy + 3y^2 = 1$ will reduce to one being wanted of the xy term.

7. Transform the equation $x^2 - 2xy + y^2 + x - 3y = 0$ to axes through the point $(-1,\ 0)$ parallel to the lines bisecting the angles between the axes.

8. Find the equations of the following when $ax + by + c = 0$ and $bx - ay + d = 0$ are considered as axes of x and y respectively.

   (a) $(ax + by + c)^2 = a^2 + b^2$,

   (b) $(ax + by + c)(bx - ay + d) = a^2 + b^2$.

9. Show that if the origin is transferred to $(0,\ 1)$ and the axes are rotated through $45°$, the equation $5x^2 - 2xy + 5y^2 + 2x - 10y - 7 = 0$ referred to new axes becomes $\frac{x'^2}{3} + \frac{y'^2}{2} = 1$. [CH 2000]

10. If the origin is shifted to $P(7, 3)$ and the coordinate axes are rotated so that the positive direction of the new x-axis coincide with that of the vector $\overline{PQ}$, Q being the point $(10, 7)$, find the formula for coordinate transformation.

    [BH 1998]

11. For the transformation of the origin to the point (α, β) the equation $\psi(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ transforms to $a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c'$.

    Show that $a' = a$, $h' = h$, $b' = b$, $g' = \frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)_{(\alpha,\beta)}$, $f' = \frac{1}{2}\left(\frac{\partial\psi}{\partial y}\right)_{(\alpha,\beta)}$, $c' = \psi(\alpha, \beta)$.

12. If x and y be coordinates of a point referred to a system of oblique axes and x' and y' be its coordinates referred to another system of oblique axes with the same origin and if the formulae of transformation be $x = mx' + ny'$ and $y = m'x' + n'y'$ prove that $\frac{m^2+n^2-1}{m'^2+n'^2-1} = \frac{mn'}{nn'}$.

    [*Hints.* If ω and ω' be the angles between the original and final pairs of axes, then $x^2 + y^2 + 2xy\cos\omega = x'^2 + y'^2 + 2x'y'\cos\omega'$.]

ANSWERS

1. (a) $(3,\ 0)$, $(0,\ 2)$;

   (b) $\left(-\dfrac{c}{m},\ 0\right)$, $(0,\ c)$;

   (c) $\left(\dfrac{p}{\cos\alpha},\ 0\right)$, $\left(0,\ \dfrac{p}{\sin\alpha}\right)$.

2. (a) $\dfrac{980}{17}y^2 - \dfrac{700}{17}x^2 = 1$, $\left(-\dfrac{1}{5},\ -\dfrac{3}{14}\right)$;

   (b) $\dfrac{x^2}{2} + \dfrac{3}{8}y^2 = 1$; $\left(\dfrac{1}{4},\ \dfrac{1}{2}\right)$.

3. (a) $(-2,\ 3)$;

   (b) $\left(-\dfrac{3}{2},\ \dfrac{-5}{2}\right)$;

   (c) $\left(-\dfrac{h+k}{2},\ -\dfrac{h-k}{2}\right)$.

4. $(3, -1)$, $a = 2$.

5. $5x^2 + y^2 = 1$.

6. (a) $-\dfrac{\pi}{2}$, $\dfrac{a^2}{2}$;

   (b) $\pm\dfrac{\pi}{4}$.

7. $\sqrt{2}\,y^2 = x$.

8. (a) $y^2 = 1$

   (b) $xy = 1$.

10. $x = 7 + x'\cos\theta - y'\sin\theta$,

    $y = 3 + x'\sin\theta + y'\cos\theta$,

    $\cos\theta = \dfrac{3}{5}$, $\sin\theta = \dfrac{4}{5}$.

1.40 Conic section

A plane section of a right circular cone is a circle or a parabola or an ellipse or a hyperbola or degenerate conics (pair of straight lines, point) depending on the proper choice of section. Hence these are called *conic sections*.

The parabola, ellipse and hyperbola have a property in common. Each of them is the locus of a point moving on a plane in such a way that its distances from a fixed point and from a fixed straight line on the plane bear a constant ratio. The fixed point is known as *focus*, the fixed straight line is the *directrix* and the constant ratio is the *eccentricity* of the conic.

The eccentricity is generally denoted by e. For a parabola $e = 1$, for an ellipse $0 < e < 1$ and for a hyperbola $e > 1$. These names are due to Apollonius (3rd century BC) and they have been defined as loci of a moving point by Pappus of Alexandria (AD 300). Descartes showed that their equations are of second degree.

1.41 Circle

Definition 1. *A circle is the locus of a point moving on a plane in such a way that its distance from a fixed point on the plane is equal to a given distance. The fixed point is called the centre and the given distance is the radius of the circle.*

Equation of a circle. Let $C(h, k)$ be the centre and a the radius of the circle in the plane of the rectangular axes OX and OY. $P(\alpha, \beta)$ is a point on the circle. CL and PM are perpendicular to OX. CN is perpendicular to PM. From the figure

$$OL = h, \quad OM = \alpha,$$
$$LC = k, \quad MP = \beta,$$

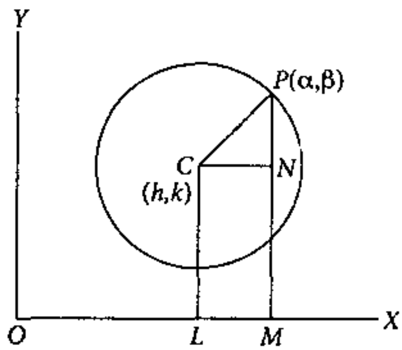


Fig. 17

$$CN = LM = OM - OL = \alpha - h$$

and $NP = MP - MN = MP - LC = \beta - k$.

From the right-angled triangle CNP,

$$CN^2 + NP^2 = CP^2$$

or, $$(\alpha - h)^2 + (\beta - k)^2 = a^2.$$

Hence the locus of (α, β) i.e. the equation of the circle whose centre is (h, k) and radius a is $(x - h)^2 + (y - k)^2 = a^2$.

Corollary I. *If the centre is at the origin, the equation of the circle reduces to* $x^2 + y^2 = a^2$.

Corollary II. *If the origin is on the circle, then* $h^2 + k^2 = a^2$. *In this case, the equation of the circle reduces to* $x^2 + y^2 - 2hx - 2ky = 0$.

Corollary III.

(i) *If the y-axis touches the circle, then* $h = a$ *and the equation of the circle is*

$$x^2 + y^2 - 2hx - 2ky + k^2 = 0.$$

(ii) *If the x-axis touches the circle, then* $k = a$ *and the equation of the circle is*

$$x^2 + y^2 - 2hx - 2ky + h^2 = 0.$$

(iii) *If the circle touches both the coordinate axes, then $h = k = a$ and the equation of the circle is*

$$x^2 + y^2 - 2ax - 2ay + a^2 = 0.$$

Corollary IV. *If the origin is on the circle and the x-axis passes through the centre, then $k = 0, h = a$ and the equation of the circle is of the form $x^2 + y^2 - 2hx = 0$.*

1.42 General equation of a circle

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle. This equation can be written as $(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$.

This shows that the equation represents a circle with the centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

Note. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent a circle, if $a = b$ and $h = 0$.

Example. Find the centre and radius of the circle $x^2 + y^2 - 8x + 10y - 4 = 0$.

The centre is $(4, -5)$ and the radius $= \sqrt{4^2 + 5^2 + 4} = 3\sqrt{5}$.

1.43 Equation of thè circle when two ends of a diameter are given

Let $P(\alpha, \beta)$ be a point on the circle and the coordinates of the ends A and B of the diameter AB be (x_1, y_1) and (x_2, y_2) respectively. Since AB is the diameter, $\angle APB$ is a right angle.

The gradients of PA and PB are

$$\frac{\beta - y_1}{\alpha - x_1} \quad \text{and} \quad \frac{\beta - y_2}{\alpha - x_2}.$$

Here

$$\frac{\beta - y_1}{\alpha - x_1} \cdot \frac{\beta - y_2}{\alpha - x_2} = -1$$

or, $(\alpha - x_1)(\alpha - x_2) + (\beta - y_1)(\beta - y_2) = 0.$

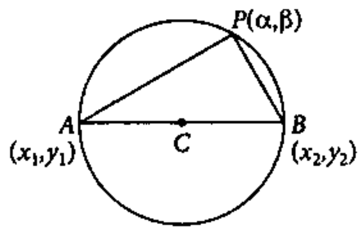


Fig. 18

Thus the locus of (α, β) i.e. the equation of the circle is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

1.44 Position of a point w.r.t. a circle

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of a circle and $P(x_1, y_1)$ be a given point. The distance of the point from the centre $(-g, -f)$ is

$$\sqrt{(x_1 + g)^2 + (y_1 + f)^2}.$$

The point P is inside, on or outside the circle according as $CP <=>$ the radius of the circle,

i.e. $\sqrt{(x_1+g)^2+(y_1+f)^2} \quad <=> \sqrt{g^2+f^2-c}$

or, $(x_1+g)^2+(y_1+f)^2 \quad <=> g^2+f^2-c$

or, $x_1^2+y_1^2+2gx_1+2fy_1+c <=> 0.$

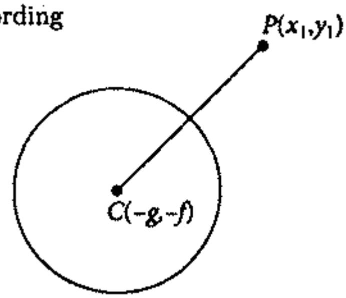


Fig. 19

1.45 Length of the chord

A line may meet a circle in two distinct points, in two coincident points (case of tangent) or two imaginary points (the line is outside the circle). If the line cuts the circle at two distinct points, then the distance between these two points is known as the length of the chord made by the line. Since the line joining the centre and the midpoint of the chord is perpendicular to the chord, the length of the chord can be found out by calculating the distance of the chord from the centre.

Example. Find the length of the chord made by the line

$$3x+4y+7=0$$

on the circle

$$x^2+y^2-6x-8y-50=0.$$

The centre of the circle is (3, 4) and the radius

$$=\sqrt{3^2+4^2+50}=5\sqrt{3}.$$

The distance of the chord from the centre

$$=\frac{3\cdot 3+4\cdot 4+7}{\sqrt{3^2+4^2}}=\frac{32}{5}.$$

Length of the chord

$$=2\sqrt{\left\{\left(5\sqrt{3}\right)^2-\left(\frac{32}{5}\right)^2\right\}}=2\sqrt{\left(75-\frac{1024}{25}\right)}$$

$$=2\sqrt{\frac{851}{25}}=\frac{2}{5}\sqrt{851}\text{ units.}$$

WORKED-OUT EXAMPLES

1. *Find the equation of the circle passing through* $(-3,\ 4)$, $(9,\ -12)$, $(-5,\ 2)$ *and also determine the coordinates of the centre.*

   Let

$$x^2+y^2+2gx+2fy+c=0$$

be the equation of the circle. As it passes through $(-3, 4)$, $(9, -12)$ and $(-5, 2)$,

$$25 - 6g + 8f + c = 0, \tag{1}$$

$$225 + 18g - 24f + c = 0 \tag{2}$$

$$\text{and} \quad 29 - 10g + 4f + c = 0. \tag{3}$$

From these three equations, we have $g = -3$, $f = 4$, $c = -75$.

$\therefore$ the equation of the circle is

$$x^2 + y^2 - 6x + 8y - 75 = 0.$$

The coordinates of the centre are $(3, -4)$.

2. *Find the equation of the circle which touches the x-axis at $(4, 0)$ and makes an intercept of length 6 from the y-axis.*

Since the circle touches the x-axis, its radius is equal to the y—coordinate of the centre in magnitude. Let the equation of the circle be

$$(x - h)^2 + (y - k)^2 = k^2.$$

As it passes through $(4, 0)$,

$$(4 - h)^2 + k^2 = k^2 \quad \text{or,} \quad h = 4.$$

It meets the y-axis $(x = 0)$ at the points given by

$$h^2 + (y - k)^2 = k^2$$
$$\text{or,} \quad y^2 - 2ky + 6 = 0.$$

If y_1 and y_2 are the roots, then $y_1 \sim y_2 = 6$.

$$\therefore \quad 6 = y_1 \sim y_2 = \sqrt{(y_1 + y_2)^2 - 4y_1y_2} = \sqrt{4k^2 - 64}$$
$$\text{or,} \quad 4k^2 - 64 = 36 \quad \text{or,} \quad k^2 = 25 \quad \text{or,} \quad k = \pm 5.$$

Thus the required equation is

$$(x - 4)^2 + (y \pm 5)^2 = 5^2$$
$$\text{or,} \quad x^2 + y^2 - 8x \pm 10y + 16 = 0.$$

3. *Show that the equation of the circle on the chord*

$$x \cos\alpha + y \sin\alpha = p$$

of the circle $x^2 + y^2 = a^2$ as diameter is

$$x^2 + y^2 - a^2 - 2p(x\cos\alpha + y\sin\alpha - p) = 0.$$

The equation of the required circle can be written as

$$x^2 + y^2 - a + \lambda(x\cos\alpha + y\sin\alpha - p) = 0$$

where λ is a constant.

The centre of this circle is

$$\left(-\frac{\lambda}{2}\cos\alpha,\ -\frac{\lambda}{2}\sin\alpha\right).$$

Since $x\cos\alpha + y\sin\alpha = p$ is the diameter, the centre must lie on it. For this

$$-\frac{\lambda}{2}\cos^2\alpha - \frac{\lambda}{2}\sin^2\alpha = p \quad \text{or,} \quad \lambda = -2p.$$

Hence the equation of the circle is

$$x^2 + y^2 - a^2 - 2p(x\cos\alpha + y\sin\alpha - p) = 0.$$

4. *Show that the circumcircle of the triangle formed by the lines* $ax + by + c = 0$, $bx + cy + a = 0$ *and* $cx + ay + b = 0$ *passes through the origin, if*

$$\left(b^2 + c^2\right)\left(c^2 + a^2\right)\left(a^2 + b^2\right) = abc(a+b)(b+c)(c+a).$$

Let us consider the equation

$$\begin{aligned}(ax + by + c)(bx + cy + a) + \lambda(bx + cy + a)(cx + ay + b)&\\ + \mu(cx + ay + b)(ax + by + c) = 0.&\qquad (1)\end{aligned}$$

The above equation is satisfied by the point of intersection of any two lines i.e. it passes through the vertices of the triangle formed by the lines. If this equation represents a circle, then coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

$$\begin{aligned}\therefore\quad & ab + \lambda bc + \mu ca = bc + \lambda ca + \mu ab\\ \text{or,}\quad & ab - bc + \lambda(bc - ca) + \mu(ca - ab) = 0 \qquad (2)\\ \text{and}\quad & ca + b^2 + \lambda(ab + c)^2 + \mu\left(bc + a^2\right) = 0. \qquad (3)\end{aligned}$$

Eliminating λ and μ from (1), (2) and (3),

$$\begin{vmatrix}(ax+by+c)(bx+cy+a) & (bx+cy+a)(cx+ay+b) & (cx+ay+b)(ax+by+c)\\ ab-bc & bc-ca & ca-ab\\ ca+b^2 & ab+c^2 & bc+a^2\end{vmatrix} = 0.$$

This is the equation of the circle. If it passes through the origin, then

$$\begin{vmatrix} ca & ab & bc \\ ab-bc & bc-ca & ca-ab \\ ca+b^2 & ab+c^2 & bc+a^2 \end{vmatrix} = 0$$

$$\text{or,}\quad \begin{vmatrix} ca & ab & bc \\ ab-bc & bc-ca & ca-ab \\ b^2 & c^2 & a^2 \end{vmatrix} = 0$$

$$\text{or,}\quad a^4c^2 + c^4a^2 + a^2b^4 + a^4b^2 + b^2c^4 + b^4c^2$$
$$= a^3b^2c + a^3bc^2 + a^2bc^3 + a^2b^3c + ab^2c^3 + ab^3c^2$$

$$\text{or,}\quad a^2c^2\left(c^2+a^2\right) + b^4\left(c^2+a^2\right) + b^2\left(c^2+a^2\right)^2$$
$$= abc\left[ca(c+a) + b^2(c+a) + b(c+a)^2\right]$$

$$\text{or,}\quad \left(b^2+c^2\right)\left(c^2+a^2\right)\left(a^2+b^2\right)$$
$$= abc(b+c)(c+a)(a+b).$$

5. *Two rods of lengths of a and b slide along the axes, which are rectangular, in such a way that their ends are always con-cyclic; prove that the locus of the centre of the circle passing through these ends is the curve*

$$4\left(x^2-y^2\right) = a^2-b^2.$$

Let the rods AB and CD of lengths a and b slide along the axes OX and OY. $P(\alpha,\ \beta)$ is the centre of the circle passing through A, B, C and D. PL and PM are perpendicular to AB and CD. Therefore, L and M are the middle points of AB and CD. In the figure

$$AL = a/2,\quad LP = \beta,$$
$$CM = b/2,\quad MP = \alpha,$$
$$AP = CP = \text{radius.}$$

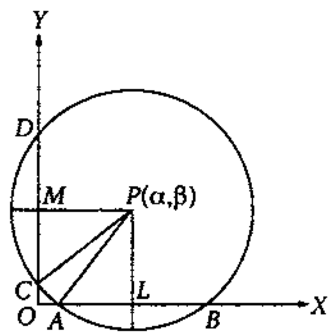


Fig. 20

Now

$$AP^2 = AL^2 + LP^2 = \frac{a^2}{4} + \beta^2, \quad CP^2 = CM^2 + MP^2 = \frac{b^2}{4} + \alpha^2.$$

$$\because\ AP = CP,\quad \frac{b^2}{4} + \alpha^2 = \frac{a^2}{4} + \beta^2 \quad \text{or,}\quad 4\left(\alpha^2-\beta^2\right) = a^2-b^2.$$

Hence the locus of $(\alpha,\ \beta)$ is

$$4\left(x^2-y^2\right) = a^2-b^2.$$

EXERCISE IV

1. Find the equation of the circle passing through

   (a) $(0,\ 1),\ (1,\ 0),\ (1,\ 1)$;

   (b) $(a,\ b),\ (-a,\ b),\ (a-b,\ a+b)$;

   (c) $(1,\ 2),\ (3,\ -4),\ (5,\ -6)$.

2. Find the centre and radius of the following circles.

   (a) $4\left(x^2+y^2\right)-32x+8y+9=0$;

   (b) $\sqrt{1+m^2}\left(x^2+y^2\right)-2cx-2mcy=0$.

3. Find the equation of the circle

   (a) whose centre is $(0,\ 2)$ and radius is 2,

   (b) whose centre is $(a,\ -b)$ and radius is $a+b$.

4. (a) Find the equation of the circle drawn on the line joining $(2,\ 3)$, $(0,\ -5)$ as diameter.

   (b) Find the equation of the circle circumscribing the triangle formed by the axes and the line $3x+4y=12$.

   (c) Find the equation of the circle concentric with $x^2+y^2-6x+8y+7=0$ and passing through $(1,\ -1)$.

   (d) $ABCD$ is a square whose side is a. Taking AB and AD as axes, prove that the equation to the circle circumscribing the square is

   $$x^2+y^2=a(x+y).$$

   (e) Find the equation of the circle which touches the axis of x at a distance 3 from the origin and intercepts a distance 6 on the axis of y.

5. Prove that the centre of the circle $x^2+y^2-6x+6y+14=0$ is the centre of mean position of the centres of circles

   $$\begin{aligned} x^2+y^2-2x+2y-14&=0,\\ x^2+y^2-4x+4y-1&=0\\ \text{and}\quad x^2+y^2-12x+12y+47&=0. \end{aligned}$$

6. Prove that the centres of the circles

   $$\begin{aligned} x^2+y^2&=1,\\ x^2+y^2+6x-2y&=1\\ \text{and}\quad x^2+y^2-12x+4y&=1 \end{aligned}$$

   lie on a right line and find the equation of this line.

7. Find the equation of the nine-point circle of the triangle whose vertices are (0, 0), (4, 0) and (0, 8).

   [*Hint.* Find the circle passing through the midpoints of the sides.]

8. Find the position of the point (1, −2) relative to the circle

$$x^2 + y^2 - 6x + 8y - 9 = 0.$$

9. A point moves so that the sum of the squares of its distances from the four sides of a square is constant; prove that it always lies on a circle.

10. Whatever be the value of θ, prove that the locus of the intersection of the straight lines $x \cos\theta + y \sin\theta = a$ and $x \sin\theta - y \cos\theta = b$ is a circle.

11. Show that the equation of the circle described on the chord $ax + by + c = 0$ of the circle $(a^2 + b^2)(x^2 + y^2) = 2c^2$ as diameter is

$$(a^2 + b^2)(x^2 + y^2) + 2c(ax + by) = 0.$$

12. Show that the circle circumscribing the triangle formed by the lines

$$\begin{aligned} & x\cos\alpha + y\sin\alpha = p, \\ & x\cos\beta + y\sin\beta = q \\ \text{and}\quad & x\cos\gamma + y\sin\gamma = r \end{aligned}$$

will pass through the origin, if

$$qr\sin(\beta - \gamma) + rp\sin(\gamma - \alpha) + pq\sin(\alpha - \beta) = 0.$$

ANSWERS

1. (a) $x^2 + y^2 - x - y = 0$;
   (b) $a(x^2 + y^2) - (a^2 + b^2)y - (a - b)(a^2 + b^2) = 0$;
   (c) $x^2 + y^2 - 22x - 4y + 25 = 0$.

2. (a) (4, 1), $\frac{1}{2}\sqrt{59}$; (b) $\left(\frac{c}{\sqrt{1+m^2}}, \frac{mc}{\sqrt{1+m^2}}\right)$, c.

3. (a) $x^2 + y^2 - 4y = 0$;
   (b) $x^2 + y^2 - 2ax + 2by - 2ab = 0$.

4. (a) $x^2 + y^2 - 2x + 2y - 15 = 0$;
   (b) $x^2 + y^2 - 4x - 3y = 0$;
   (c) $x^2 + y^2 - 6x + 8y + 12 = 0$;
   (e) $x^2 + y^2 - 6x \pm 6\sqrt{2}\,y + 9 = 0$.

6. $x + 3y = 0$.

7. $x^2 + y^2 - 2x - 4y = 0$.

8. Inside the circle.

10. $x^2 + y^2 = a^2 + b^2$.

1.50 Parabola

Standard equation

Let S be the focus and OM be the directrix. SO is drawn perpendicular to OM and let $OS = 2a$. OS and OM are taken as x and y-axes respectively.

Let $P(x, y)$ be a point on the parabola. PM and PN are perpendicular to OY and OX. SP is joined.

Now $ON = x$, $NP = y$.

By definition

$$\frac{SP}{PM} = 1 \quad \text{or,} \quad SP^2 = PM^2$$

$$\text{or,} \quad SN^2 + NP^2 = PM^2$$

$$\text{or,} \quad (x-2a)^2 + y^2 = x^2$$

$$\text{or,} \quad y^2 = 4a(x-a). \qquad (1)$$

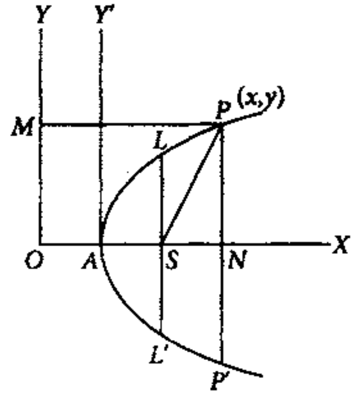


Fig. 21

This is the equation of the curve.

Let the curve meet the x-axis at A where $y = 0$.

By (1) the coordinates of A are $(a, 0)$.

$\therefore OA = a$.

If the origin is transferred to A with the axes remaining parallel, the equation of the parabola reduces to

$$y^2 = 4ax. \qquad (2)$$

If is known as the standard equation of the parabola.

Properties

$y^2 = 4ax$ is the equation of the parabola with the x-axis along the axis of the parabola and y-axis along the tangent at the vertex.

(i) The point A is called the *vertex* (vide Fig. 21).

(ii) The line AS produced indefinitely is known as *axis* of the parabola.

(iii) S is the focus whose coordinates are $(a, 0)$.

(iv) PN is the *ordinate* of P and PNP' is the *double ordinate* of P.

(v) The double ordinate LSL' is called the *latus rectum*. If $SL = b$, the coordinates of L are (a, b). By the equation (2), $b^2 = 4a^2$.

$\therefore SL = 2a$ and $LL' = 4a$. It is the *length of the latus rectum*. The coordinates of L and L' are $(a, 2a)$ and $(a, -2a)$ respectively.

(vi) The y-axis is the tangent to the parabola at the vertex.

(vii) The equation of the *directrix* is $x + a = 0$.

(viii) The curve is *symmetrical* about the x-axis or axis of the parabola.

(ix) The negative value of x would give the imaginary value of y. Thus the curve wholly lies on the positive direction of x-axis. Again x may attain any positive value, however great, so the curve extends to infinity.

(x) *Parametric coordinates* of a point on the parabola are $(at^2,\ 2at)$.

1.51 Equations in different forms

(a) $y^2 = -4ax$ is a parabola whose axis runs in the negative direction of the x-axis. Here the focus is $(-a, 0)$ [See Fig. 22].

(b) $x^2 = 4ay$ is a parabola whose axis runs in the positive direction of y-axis. Here the focus is $(0,\ a)$ [See Fig. 23].

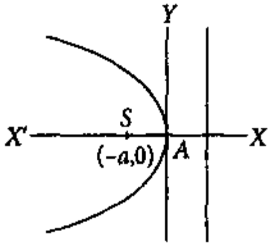


Fig. 22

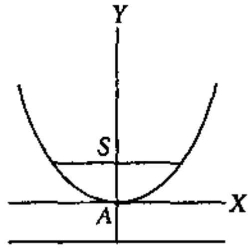


Fig. 23

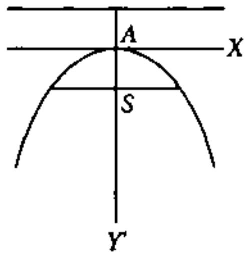


Fig. 24

(c) $x^2 = -4ay$ is a parabola whose axis runs in the negative direction of y-axis. Here the focus is $(0,\ -a)$ [See Fig. 24].

(d) Parabola is the locus of points whose squares of distances from one line vary as their distances from another line which is the tangent to the parabola at the vertex and the first line is the axis of the parabola. Thus

$$(ax + by + c)^2 = k(bx - ay + c')$$

represents a parabola whose axis is $ax + by + c = 0$ and the tangent at the vertex is $bx - ay + c' = 0$.

This equation can be written as

$$\left(\frac{ax + by + c}{\sqrt{a^2 + b^2}}\right)^2 = \frac{k}{\sqrt{a^2 + b^2}} \cdot \frac{bx - ay + c'}{\sqrt{a^2 + b^2}}.$$

$$\text{Length of latus rectum} = \frac{k}{\sqrt{a^2 + b^2}}.$$

WORKED-OUT EXAMPLES

1. *Find the vertex, focus and directrix of the parabola* $y^2 + 2x - 4y + 3 = 0$.

$$y^2 + 2x - 4y + 3 = 0$$

$$\text{or,}\quad y^2 - 4y + 4 = -2x + 1$$

$$\text{or,}\quad (y - 2)^2 = -2\left(x - \frac{1}{2}\right).$$

$\therefore$ the vertex is $(1/2,\ 2)$, the focus is $(0,\ 2)$ and the directrix is $x = 1$.

2. *Find the equation of the parabola whose focus is* $(2,\ 3)$ *and directrix is*

$$4x - 3y + 1 = 0.$$

The equation of the parabola is

$$\begin{aligned} \frac{4x - 3y + 1}{\sqrt{4^2 + 3^2}} &= \sqrt{(x-2)^2 + (y-3)^2} \\ \text{or,}\quad (4x - 3y + 1)^2 &= 25\left(x^2 + y^2 - 4x - 6y + 13\right) \\ \text{or,}\quad 9x^2 + 16y^2 + 24xy - 108x - 144y + 324 &= 0. \end{aligned}$$

3. *If* $\left(at^2,\ 2at\right)$ *be the coordinates of one end of a focal chord of* $y^2 = 4ax$, *find the coordinates of the other end.* **[BH 2001]**

Let the other end be $\left(at_1^2,\ 2at_1\right)$.

The equation of the chord is

$$\begin{aligned} y - 2at &= \frac{2at_1 - 2at}{at_1^2 - at^2}\left(x - at^2\right) \\ \text{or,}\quad y\,(t_1 + t) &= 2\,(x + at_1 t)\,. \end{aligned}$$

As it passes through the focus $(a,\ 0)$,

$$2\,(a + at_1 t) = 0 \quad \text{or,} \quad tt_1 = -1 \quad \text{or,} \quad t_1 = -\frac{1}{t}.$$

$\therefore$ the other end is $\left(\dfrac{a}{t^2},\ \dfrac{-2a}{t}\right)$.

EXERCISE V

1. Find the vertex, focus and directrix of the following.

   (a) $y^2 = 5x + 10$;

   (b) $y^2 + 4x + 2y - 8 = 0$;

   (c) $x^2 + 4ax + 2ay = 0$;

   (d) $y^2 + 2gx + 2fy + c = 0$.

2. (a) Find the equation of the parabola with

   (i) focus $(5,\ 3)$,
   directrix $3x + 2y + 7 = 0$;

   (ii) focus $(a,\ b)$,
   directrix $\frac{x}{a} + \frac{y}{b} = 1$.

   (b) Show that the equation of the parabola whose focus is $(-1,\ 3)$ and vertex is $(4,\ 3)$ is $y^2 + 20x - 6y - 71 = 0$.

3. For what point of the parabola $y^2 = 18x$ is the ordinate equal to three times the abscissa?

4. A double ordinate of the parabola $y^2 = 4ax$ is of the length $8a$. Prove that the line joining the vertex to its two ends are at right angle.

5. Find the equation of the parabola whose latus rectum is 6 and the axis and tangent at the vertex are the lines $3x + 4y + 1 = 0$ and $4x - 3y = 0$.

6. Find the length of the chord intercepted by the parabola $y^2 = 40x$ on the straight line $y = 3x + 2$.

ANSWERS

1. (a) $(-2,\ 0)$, $\left(-\frac{3}{4},\ 0\right)$, $4x + 13 = 0$;

   (b) $\left(\frac{9}{4},\ -1\right)$, $\left(\frac{5}{4},\ -1\right)$, $4x - 13 = 0$;

   (c) $(-2a,\ 2a)$, $\left(-2a,\ \frac{3a}{2}\right)$, $2y - 5a = 0$;

   (d) $\left(\frac{f^2 - c}{2g},\ -f\right)$, $\left(\frac{f^2 - g^2 - c}{2g},\ -f\right)$, $x - \frac{f^2 + g^2 - c}{2g} = 0$.

2. (a) (i) $4x^2 + 9y^2 - 12xy - 172x - 106y + 393 = 0$;

   (ii) $(ax - by)^2 - 2a^3x - 2b^3y + a^4 + b^4 + a^2b^2 = 0$.

3. $(2,\ 6)$.

5. $(3x + 4y + 1)^2 = \pm 30(4x - 3y)$.

6. $\frac{80}{9}$.

1.60 Ellipse

Standard equation

Let S' be the focus, $M'M$ the directrix, e the eccentricity. $S'D$ is perpendicular to $M'M$. Let $DS' = c$. DS' and DM are taken as x and y-axes. Let $P(x,\ y)$ be a point on the ellipse. PM and PN are perpendicular to $M'M$ and DS' (produced). By definition

$$\frac{S'P}{PM} = e \quad \text{or,} \quad S'P^2 = e^2 \cdot PM^2$$

$$\text{or,} \quad S'N^2 + NP^2 = e^2 \cdot DN^2$$

$$\text{or,} \quad (x - c)^2 + y^2 = e^2x^2 \quad [S'N = DN - DS' = x - c,\ NP = y]$$

$$\text{or,} \quad x^2(1 - e^2) - 2cx + y^2 + c^2 = 0$$

$$\text{or,} \quad \left(x - \frac{c}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2c^2}{(1 - e^2)^2}.$$

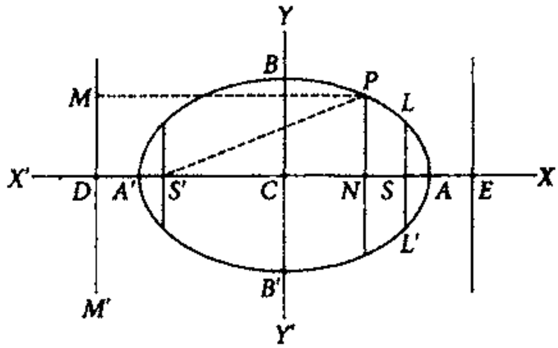


Fig. 25

Changing the origin to $C\left(\frac{c}{1-e^2},\ 0\right)$ with the axes remaining parallel, the equation transforms to

$$x^2 + \frac{y^2}{1-e^2} = \frac{e^2c^2}{(1-e^2)^2}.$$

Let us put

$$\frac{ec}{1-e^2} = a.$$

Then

$$x^2 + \frac{y^2}{1-e^2} = a^2 \quad \text{or,} \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

Putting

$$a^2\left(1-e^2\right) = b^2, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

It is the standard equation of the ellipse.

Properties

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation of the ellipse with the axes as coordinate axes.

(i) $C(0,\ 0)$ is the *centre* of the ellipse.

(ii) $A'A$ and $B'B$ are the *major* and *minor* axes of the ellipse. The coordinates of A and A' are $(a,\ 0)$ and $(-a,\ 0)$.

Length of the major axis $= A'A = 2CA = 2a$.

The coordinates of B and B' are $(0,\ b)$ and $(0,\ -b)$.

Length of the minor axis $= B'B = 2CB = 2b$.

(iii) S and S' are the foci whose coordinates are $(ae,\ 0)$ and $(-ae,\ 0)$.

$[S'A' = e \cdot A'D,\ S'A = e \cdot AD.$

$\therefore S'A' + S'A = e(A'D + AD)$ or, $AA' = e \cdot 2CD$ or, $CA' = e \cdot CD$.

$CA' - S'A' = e(CD - A'D)$ or, $CS' = e \cdot CA' = ea.]$

(iv) The double ordinates of L i.e. LSL' is the *latus rectum.* If $SL = l$ then the coordinates of L are $(ae,\ l)$.

From the equation

$$e^2 + \frac{l^2}{b^2} = 1 \quad \text{or,} \quad l^2 = b^2\left(1 - e^2\right) = \frac{b^4}{a^2} \quad \text{or,} \quad l = \frac{b^2}{a}.$$

$\therefore$ length of latus rectum $= 2b^2/a$.

(v) $SA = e \cdot AE$ or, $CA - CS = e \cdot AE$ or, $a - ae = e \cdot AE$.

$$\therefore\ AE = \frac{a(1-e)}{e}.$$

$$CE = CA + AE = a + \frac{a(1-e)}{e} = \frac{a}{e}.$$

$\therefore$ *the equation of directrices are* $x = \pm a/e$.

(vi) $e^2 = \dfrac{a^2 - b^2}{a^2}$.

(vii) The curve is closed and symmetrical about x-axis and y-axis.

(viii) $(x = a\cos\theta,\ y = b\sin\theta)$ is the *parametric equation.*

Note. If $a = b$, then the equation represents a circle. Consequently the circle is an ellipse with zero eccentricity.

Example 1. *Find the centre, eccentricity and foci of the ellipse*

$$2x^2 + 3y^2 - 4x + 5y + 4 = 0.$$

$$2x^2 + 3y^2 - 4x + 5y + 4 = 0$$

$$\text{or,} \quad 2(x-1)^2 + 3\left(y + \frac{5}{6}\right)^2 = \frac{1}{12} \quad \text{or,} \quad \frac{(x-1)^2}{1/24} + \frac{(y+5/6)^2}{1/36} = 1.$$

If the origin is transferred to $(1,\ -5/6)$, the equation reduces to the standard form

$$\frac{x^2}{1/24} + \frac{y^2}{1/36} = 1.$$

Thus the centre is $(1,\ -5/6)$ and squares of semi-major and minor axes are $1/24$ and $1/36$ respectively.

$$\therefore\ e^2 = \frac{1/24 - 1/36}{1/24} = \frac{1}{3} \quad \text{or,} \quad e = \frac{1}{\sqrt{3}}.$$

The foci are

$$\left(1 \pm \frac{1}{\sqrt{3.24}},\ -\frac{5}{6}\right) \quad \text{or,} \quad \left(1 \pm \frac{1}{6\sqrt{2}},\ -\frac{5}{6}\right).$$

Example 2. *Find the equation of the ellipse which has the point* $(-1,\ 1)$ *for a focus, the line* $4x-3y=0$ *the corresponding directrix and whose eccentricity is* $5/6$.

The equation of the ellipse is

$$
\begin{aligned}
&(x+1)^2+(y-1)^2=\frac{25}{36}\frac{(4x-3y)^2}{25}\\
\text{or,}\quad &36\left(x^2+y^2+2x-2y+2\right)=16x^2+9y^2-24xy\\
\text{or,}\quad &20x^2+27y^2+24xy+72x-72y+72=0.
\end{aligned}
$$

1.61 Auxiliary circle and eccentric angle

Auxiliary circle:

The circle described on the major axis of an ellipse as diameter is called the auxiliary circle.

If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the equation of the ellipse with a as semi-major axis, then the equation of the auxiliary circle is $x^2+y^2=a^2$.

Let $P(x,\ y)$ be any point on the ellipse. PN is the ordinate of P. NP is produced to meet the auxiliary circle at Q. Q is joined with the centre C. Let

$$\angle QCN=\phi$$

ϕ is known as the *eccentric angle* of P. Here $CQ=a$.

$\therefore\ CN=a\cos\phi$ which is the abscissa of P.

$$
\begin{aligned}
\therefore\ &\frac{a^2\cos^2\phi}{a^2}+\frac{y^2}{b^2}=1\\
\text{or,}\quad &y^2=b^2\left(1-\cos^2\phi\right)=b^2\sin^2\phi\\
\text{or,}\quad &y=\pm b\sin\phi.
\end{aligned}
$$

$\therefore$ the coordinates of P are $(a\cos\phi,\ b\sin\phi)$.

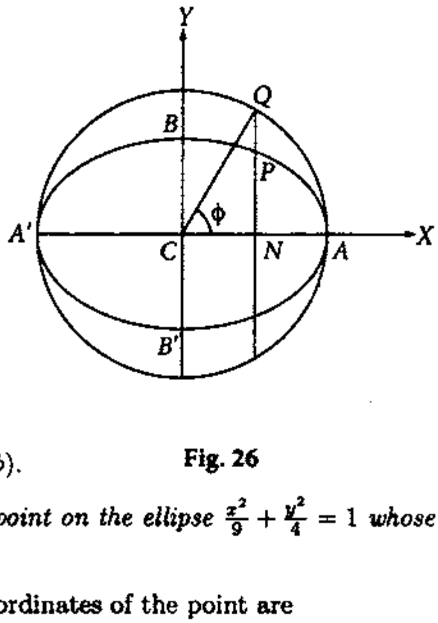


Fig. 26

Example 1. *Find the eccentric angle of the point on the ellipse* $\frac{x^2}{9}+\frac{y^2}{4}=1$ *whose distance from the centre is* $\sqrt{13/2}$.

Let ϕ be the eccentric angle. Then the coordinates of the point are

$$(3\cos\phi,\ 2\sin\phi).$$

Here

$$
\begin{aligned}
9\cos^2\phi+4\sin^2\phi=13/2\quad \text{or,}\quad &5\cos^2\phi=5/2\\
\text{or,}\quad &\cos^2\phi=1/2\\
\text{or,}\quad &\cos\phi=\pm 1/\sqrt{2}.
\end{aligned}
$$

$\therefore\ \phi=\pi/4$ or $3\pi/4$.

Example 2. *The eccentric angles of two points on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *are* α *and* β *and their joint intersects the major axis at a distance* $2a$ *from the centre. Show that* $3\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=1$. **[BH 2006]**

The coordinates of the points are $(a\cos\alpha,\ b\sin\alpha)$ and $(a\cos\beta,\ b\sin\beta)$.

Since the join of these two points meets the major axis (i.e. x-axis) at a distance $2a$ from the centre, the above two points and $(2a,\ 0)$ are collinear.

$$\therefore\quad a\cos\alpha(b\sin\beta-0)+a\cos\beta(0-b\sin\alpha)+2a(b\sin\alpha-b\sin\beta)=0$$

$$\text{or,}\quad \cos\alpha\sin\beta-\cos\beta\sin\alpha+2(\sin\alpha-\sin\beta)=0$$

$$\text{or,}\quad \frac{\sin(\alpha-\beta)}{\sin\alpha-\sin\beta}=2\quad\text{or,}\quad \frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}=2$$

$$\text{or,}\quad \frac{\cos\frac{1}{2}(\alpha-\beta)-\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)+\cos\frac{1}{2}(\alpha+\beta)}=\frac{1}{3}\quad\text{or,}\quad \frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}=\frac{1}{3}$$

$$\text{or,}\quad 3\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=1.$$

EXERCISE VI

1. Find the equation of the ellipse with the axes as coordinate axes, latus rectum 5 and eccentricity 2/3.

2. Find the latus rectum, eccentricity and coordinates of foci of

   (a) $4x^2+9y^2=36$; (b) $9x^2+5y^2-30y=0$.

3. Find the equation of the ellipse whose focus is $(-1,\ 1)$, eccentricity is 1/2 and directrix is $x-y+3=0$.

4. Show that the locus of the point of intersection of the lines

$$\frac{kx}{a}+\frac{y}{b}=k\quad\text{and}\quad\frac{x}{a}-\frac{ky}{b}+1=0$$

   where k is a variable parameter is an ellipse.

5. The distance of a point from the centre of the ellipse

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

   is $5/\sqrt{2}$. Find the eccentric angle of this point.

6. The eccentric angles of two points on an ellipse with semi-major axis a are θ and ϕ. Their join intersects the major axis at a distance c from the centre. Prove that

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2}=\frac{c-a}{c+a}.$$

7. If θ and ϕ be the eccentric angles of the ends of a focal chord of an ellipse of eccentricity e, prove that

   (a) $\pm e\cos\frac{1}{2}(\theta+\phi)=\cos\frac{1}{2}(\theta-\phi)$;

   (b) $\tan\frac{\theta}{2}\tan\frac{\phi}{2}+\frac{1\mp e}{1\pm e}=0.$

8. A rod of length l moves so that its extremities are on two rectangular axes. Prove that the locus of the point which divides the rod in the ratio $\lambda:\mu$ is an ellipse whose equation is
$$\frac{x^2}{\lambda^2}+\frac{y^2}{\mu^2}=\frac{l}{(\lambda+\mu)^2}.$$
   What will be the locus if the point bisects the rod ?

9. Prove that the length of the focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is inclined to the major axis at an angle θ is
$$\frac{2ab^2}{a^2\sin^2\theta+b^2\cos^2\theta}.$$

10. Show that the equation of the ellipse whose axes are of lengths 8 and 6 and equations are $4x+3y-2=0$ and $3x-4y+1=0$ is
$$\frac{(3x-4y+1)^2}{64}+\frac{(4x+3y-2)^2}{36}=25.$$

11. Show that the locus of all points the sum of whose distances from two fixed points is constant is an ellipse.

12. If ASB and $AS'C$ are two chords of an ellipse through the foci S and S', prove that $\frac{SA}{SB}+\frac{S'A}{S'C}$ is a constant. [BH 2001]

ANSWERS

1. $20x^2+36y^2=405.$
2. (a) $\frac{8}{3},\ \frac{\sqrt{5}}{3},\ \left(\pm\sqrt{5},\,0\right)$;
   (b) $\frac{10}{3},\ \frac{2}{3},\ (0,5),\ (0,1).$
3. $7x^2+7y^2+2xy+10x-10y+7=0.$
5. $\frac{\pi}{4},\ \frac{3\pi}{4}.$

1.70 Hyperbola

Standard Equation

Let S be the focus, $M'M$ the directrix and e the eccentricity $(e>1)$.

SD is perpendicular to MM'. Let $SD = c$. DS and DM are taken as x and y-axes. Let $P(x,\ y)$ be a point on the hyperbola. PM and PN are perpendicular to MM' and DS.

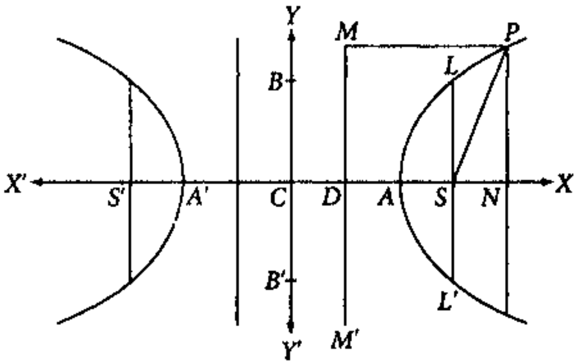


Fig. 27

Now

$$\begin{aligned}\frac{SP}{PM} = e \quad \text{or,} \quad & SP^2 = e^2 \cdot PM^2 \\ \text{or,} \quad & SN^2 + NP^2 = e^2 \cdot DN^2 \\ \text{or,} \quad & (x-c)^2 + y^2 = e^2 \cdot x^2 \\ \text{or,} \quad & x^2\left(e^2-1\right) + 2cx - y^2 = c^2 \\ \text{or,} \quad & x^2 + \frac{2c}{e^2-1}x - \frac{y^2}{e^2-1} = \frac{c^2}{e^2-1} \\ \text{or,} \quad & \left(x + \frac{c}{e^2-1}\right)^2 - \frac{y^2}{e^2-1} = \frac{e^2c^2}{(e^2-1)^2}.\end{aligned}$$

Transferring the origin to the point $C\left(-\frac{c}{e^2-1},\ 0\right)$ and writing $\frac{e^2c^2}{(e^2-1)^2} = a^2$ the equation reduces to

$$x^2 - \frac{y^2}{e^2-1} = a^2, \quad \frac{x^2}{a^2} - \frac{y^2}{a^2\left(e^2-1\right)} = 1.$$

Putting $a^2\left(e^2-1\right) = b^2$, the standard equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Properties

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (Equation of the hyperbola with its axes as coordinate axes.)

(i) $C(0,\ 0)$ is the *centre* of the hyperbola. Any chord through C is bisected at C.

(ii) x-axis meets the curve at A and A'. The coordinates of A and A' are $(a,\ 0)$ and $(-a,\ 0)$. y-axis does not meet the curve in any real point. For any value of x between $(-a,\ a)y$ is imaginary. Therefore, there is no part of the curve between A' and A. x can take up any value which is numerically greater than a. Thus the curve has two infinite branches.

On the y-axis two points B and B' are taken where $CB = CB' = b$. The line $A'A$ is called the *transverse axis* and $B'B$ is called the *conjugate axis*.

(iii) S and S' are two *foci*. Their coordinates are $(\pm ae,\ 0)$.

(iv) The equation of the directrices are $x = \pm a/e$.

(v) LSL' is the double ordinate of L. It is the *latus rectum*. *Length of the latus rectum* $= LSL' = 2b^2/a$.

(vi) $b^2 = a^2\left(e^2 - 1\right),\ e^2 = \left(a^2 + b^2\right)/a^2$.

(vii) *Conjugate hyperbola.* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is known as *conjugate* to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(viii) If $b = a$, $\left[x^2 - y^2 = a^2\right]$ the hyperbola is called *rectangular* or *equilateral*. Its eccentricity is $\sqrt{2}$.

(ix) *Auxiliary circle* is $x^2 + y^2 = a^2$.

(x) *Parametric coordinates.* $(a\sec\phi,\ b\tan\phi)$ are the coordinates of a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(xi) The curve is *symmetrical* about the axes.

(xii) The lines $\frac{x}{a} - \frac{y}{b} = 0$, $\frac{x}{a} + \frac{y}{b} = 0$ are called asymptotes of the hyperbola.

(xiii) The curve extends to infinity on either side $(x \to \infty,\ x \to -\infty)$.

Example 1. *Find the equation of the hyperbola, referred to its axes as axes of coordinates, whose conjugate axis is* 5 *and the distance between the foci is* 13.

Let

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

be the equation of the hyperbola.

Here $b = 5/2$. If e be the eccentricity, the foci are $(\pm ae,\ 0)$.

$$\therefore\ 4a^2e^2 = 169. \tag{1}$$

Again

$$b^2 = a^2\left(e^2 - 1\right) \quad \text{or,} \quad 25 = 4a^2\left(e^2 - 1\right). \tag{2}$$

By (1) and (2)

$$4a^2 = 144 \quad \text{or,} \quad a^2 = 36.$$

$\therefore$ the equation is

$$\frac{x^2}{36} - \frac{y^2}{25/4} = 1 \quad \text{or,} \quad 25x^2 - 144y^2 = 900.$$

Example 2. *Show that the locus of a point which moves such that the difference of its distances from two fixed points is a constant, is a hyperbola.*

Let the fixed points be $(ae, 0)$ and $(-ae, 0)$ and the constant be $2a$. If (α, β) be a point on the locus, then

$$\sqrt{\left[(\alpha - ae)^2 + \beta^2\right]} - \sqrt{\left[(\alpha + ae)^2 + \beta^2\right]} = 2a$$

$$\text{or,}\quad (\alpha - ae)^2 + \beta^2 = 4a^2 + (\alpha + ae)^2 + \beta^2 + 4a\sqrt{\left[(\alpha + ae)^2 + \beta^2\right]}$$

$$\text{or,}\quad -4ae\alpha - 4a^2 = 4a\sqrt{\left[(\alpha + ae)^2 + \beta^2\right]}$$

$$\text{or,}\quad (\alpha e + a)^2 = (\alpha + ae)^2 + \beta^2$$

$$\text{or,}\quad (e^2 - 1)\,\alpha^2 - \beta^2 = a^2\,(e^2 - 1)$$

$$\text{or,}\quad \frac{\alpha^2}{a^2} - \frac{\beta^2}{a^2\,(e^2 - 1)} = 1.$$

Taking $a^2\,(e^2 - 1) = b^2$, the locus is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

It is a hyperbola.

Example 3. *Find the centre, foci and lengths of axes of the hyperbola*

$$x^2 - 2y^2 - 2x + 8y - 1 = 0.$$

$$x^2 - 2y^2 - 2x + 8y - 1 = 0$$

$$\text{or,}\quad (x - 1)^2 - 2(y - 2)^2 = -6$$

$$\text{or,}\quad \frac{(x - 1)^2}{6} - \frac{(y - 2)^2}{3} = -1.$$

Obviously the centre is $(1, 2)$, the transverse axis $= 2\sqrt{3}$, the conjugate axis $= 2\sqrt{6}$, eccentricity $= \sqrt{(3 + 6)/3} = \sqrt{3}$.

Foci are $\left(1,\ 2 + \sqrt{3}\cdot\sqrt{3}\,\right)$ and $\left(1,\ 2 - \sqrt{3}\cdot\sqrt{3}\,\right)$, or $(1,\ 5)$ and $(1,\ -1)$.

Example 4. *If e and e' be the eccentricities of a hyperbola and its conjugate, prove that $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.*

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola.

Its conjugate is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

Now $b^2 = a^2\,(e^2 - 1)$ and $a^2 = b^2\left(e'^2 - 1\right)$.

$$\therefore\quad b^2 \cdot a^2 = a^2\,(e^2 - 1)\cdot b^2\left(e'^2 - 1\right)$$

$$\text{or,}\quad 1 = (e^2 - 1)\left(e'^2 - 1\right)$$

$$\text{or,}\quad e^2 + e'^2 = e^2 e'^2$$

$$\text{or,}\quad \frac{1}{e^2} + \frac{1}{e'^2} = 1.$$

Example 5. *Show that the locus of the point of intersection of the lines $x - y = at$ and $x + y = a/t$ where t is a variable parameter is a rectangular hyperbola.*

Let $(\alpha,\ \beta)$ be the point of intersection. Then $\alpha - \beta = at$ and $\alpha + \beta = a/t$.

To eliminate t,

$$(\alpha - \beta)(\alpha + \beta) = at \cdot (a/t) \quad \text{or,} \quad \alpha^2 - \beta^2 = a^2.$$

Hence the locus of (α, β) is $x^2 - y^2 = a^2$ which is a rectangular hyperbola.

EXERCISE VII

1. Find the centre, foci and lengths of axes of

   (a) $9x^2 - 16y^2 = 144$; (b) $5x^2 - 4y^2 - 20x - 8y - 4 = 0$.

2. Find the equation of the hyperbola whose focus is $(1,\ -1)$, eccentricity is $\sqrt{3}$ and the equation of the directrix is $3x + 4y = 1$.

3. Find the equation of the hyperbola of given transverse axis whose vertex bisects the distance between the centre and the focus.

4. In a rectangular hyperbola, prove that $SP \cdot S'P = CP^2$.

5. Show that $x = \dfrac{a}{2}\left(t + \dfrac{1}{t}\right)$ and $y = \dfrac{a}{2}\left(t - \dfrac{1}{t}\right)$ represents a rectangular hyperbola.

6. If $(a\sec\phi,\ b\tan\phi)$ and $(a\sec\phi',\ b\tan\phi')$ be the ends of a focal chord of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, show that

$$\tan\frac{\phi}{2}\tan\frac{\phi'}{2} + \frac{e-1}{e+1} = 0.$$

7. Show that the equation of the hyperbola whose transverse and conjugate axes are 8 and 12 and equations are $3x - 4y + 5 = 0$ and $4x + 3y - 6 = 0$ respectively is

$$\frac{(4x + 3y - 6)^2}{16} - \frac{(3x - 4y + 5)^2}{36} = 25.$$

8. (a) Show that a hyperbola is the locus of all points the difference of whose distances from two fixed points is a positive constant.

   (b) AOB and COD are two straight lines which bisect one another at right angles. Show that the locus of a point which moves so that

$$PA \cdot PB = PC \cdot PD$$

   is a rectangular hyperbola.

   [*Hints.* Let us consider that AOB and COD as x and y-axis and the coordinates of A, B, C, D be $(a,\ 0)$, $(-a,\ 0)$, $(0,\ c)$, $(0,\ -c)$ respectively.]

9. Show that the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ transforms to $xy = c^2$ where $c^2 = \left(a^2 + b^2\right)/4$ when referred to its asymptotes as coordinate axes.

10. The perpendicular from the centre upon the normal at any point of $x^2/a^2 - y^2/b^2 = 1$ meets it in Q. Show that the locus of Q is

$$\left(x^2 + y^2\right)\left(a^2y^2 - b^2x^2\right) = \left(a^2 + b^2\right)x^2y^2.$$

ANSWERS

1. (a) (0, 0), ($\pm$5, 0), 8, 6;

   (b) (2, -1), (5, -1), (-1, -1), 4, $2\sqrt{5}$.

2. $2x^2 + 23y^2 + 72xy + 32x - 74y - 47 = 0$.

3. $3x^2 - y^2 = 3a^2$.

Chapter 2

Invariants under Orthogonal Transformation

2.10 Orthogonal Transformation

We consider here three types of transformation of axes or transformation of coordinates, namely (i) translation (ii) rotation (iii) translation and rotation.

These are called orthogonal transformation when both the systems of axes are rectangular. The combination of translation and rotation is called a *rigid body motion.*

Let (x, y) be the coordinates of a point in the old system, (x', y') the coordinates of the same point in the new system, (α, β) the coordinates of the new origin w.r.t. the old system and θ the angle of rotation.

Writing $X = \begin{bmatrix} x \\ y \end{bmatrix}, X' = \begin{bmatrix} x' \\ y' \end{bmatrix}, A = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, the formula for coordinate transformation due to (i) translation (ii) rotation and (iii) translation and rotation can be written in matrix notation as (i) $X = IX' + A$, (ii) $X = SX'$ and (iii) $X = SX' + A$ respectively.

From these formulae the inverse relations are

(i) $X' = I^{-1}X - I^{-1}A$, (ii) $X' = S^{-1}X$, (iii) $X' = S^{-1}X - S^{-1}A$.

I^{-1} and S^{-1} are inverses of the matrices I and S.

Thus

$$I^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Note 1. I and S are orthogonal matrices and $|I| = 1 = |S|$. For this property of I and S each transformation is called an ***orthogonal transformation.***

Note 2. The transformation by the formulae

$$\left.\begin{aligned} x &= \lambda x' - \mu y' + \nu \\ y &= \mu x' + \lambda y' + \xi \end{aligned}\right\} \quad \text{where } \lambda^2 + \mu^2 = 1,$$

is an orthogonal transformation.

If $\lambda = 1, \mu = 0,$ then it is translation

If $\nu = 0 = \xi,$ then it is rotation.

2.20 Invariants

Some expressions remain unchanged under an orthogonal transformation. These are known as *invariants of orthogonal transformation.*

(i) *The degree of an equation is an invariant under orthogonal transformation.*

Let a polynomial equation be $f(x, y) = 0$. By an orthogonal transformation of the coordinate axes the equation transforms to

$$f(\lambda x' - \mu y' + \nu, \mu x' + \lambda y' + \xi) = 0$$

where $\lambda^2 + \mu^2 = 1$. If ax^py^q is the highest degree term in $f(x, y) = 0$, then in the transformed equation it becomes $a(\lambda x' - \mu y' + \nu)^p (\mu x' + \lambda y' + \xi)^q$. The highest degree of the term containing x' or $x'y'$ or y' is $p+q$. It is equal to the degree of ax^py^q. Thus the degree of an equation is an invariant under orthogonal transformation.

(ii) *The distance between two points is an invariant under an orthogonal transformation.*

Let (x_1, y_1) and (x_2, y_2) be the coordinates of two points in the old system. In the new system the coordinates of these points are (x_1', y_1') and (x_2', y_2').

The formulae for transformation are

$$\left.\begin{aligned} x &= \lambda x' - \mu y' + \nu \\ y &= \mu x' + \lambda y' + \xi \end{aligned}\right\} \quad \text{where } \lambda^2 + \mu^2 = 1.$$

If d is the distance between the points, then

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

By transformation

$$\begin{aligned} (x_2 - x_1)^2 + (y_2 - y_1)^2 &= \{(\lambda x_2' - \mu y_2' + \nu) - (\lambda x_1' - \mu y_1' + \nu)\}^2 \\ &\quad + \{(\mu x_2' + \lambda y_2' + \xi) - (\mu x_1' + \lambda y_1' + \xi)\}^2 \\ &= \{\lambda(x_2' - x_1') - \mu(y_2' - y_1')\}^2 + \{\mu(x_2' - x_1') + \lambda(y_2' - y_1')\}^2 \\ &= (\lambda^2 + \mu^2)\left\{(x_2' - x_1')^2 + (y_2' - y_1')^2\right\} \\ &= (x_2' - x_1')^2 + (y_2' - y_1')^2. \end{aligned}$$

$\therefore\ d^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2.$

Hence the result follows.

(iii) *The coefficients of x^2, xy and y^2 and Δ obtained from*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

are invariants under translation.

Let the origin be shifted to (α, β). The expression transforms to

$$\begin{aligned}
a(x'+\alpha)^2 + 2h(x'+\alpha)(y'+\beta) + b(y'+\beta)^2 + 2g(x'+\alpha) + 2f(y'+\beta) + c \\
&= ax'^2 + 2hx'y' + by'^2 + 2(a\alpha + h\beta + g)x' + 2(h\alpha + b\beta + f)y' \\
&\quad + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c \\
&= a'x'^2 + 2h'x'y' + b'y^2 + 2g'x' + 2f'y' + c' \text{ (say).}
\end{aligned}$$

Here $a' = a$, $b' = b$, $h' = h$, $g' = a\alpha + h\beta + g$, $f' = h\alpha + b\beta + f$, $c' = a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c$.

We see that the coefficients of x^2, y^2 and xy i.e. a, b and h remain invariant due to translation.

Let us consider the invariance of $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

After translation Δ changes to Δ' (say).

Now

$$\begin{aligned}
\Delta' &= \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} \\
&= \begin{vmatrix} a & h & a\alpha + h\beta + g \\ h & b & h\alpha + b\beta + f \\ a\alpha + h\beta + g & h\alpha + b\beta + f & a\alpha^2 + 2h\alpha\beta + \cdots + c \end{vmatrix} \\
&= \begin{vmatrix} a & h & g \\ h & b & f \\ a\alpha + h\beta + g & h\alpha + b\beta + f & g\alpha + f\beta + c \end{vmatrix}
\end{aligned}$$

(on subtracting α times the elements of the first column and β times the elements of the second column from those of the third column)

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(on subtracting α times the elements of the first row and β times the elements of the second row from those of the third row).

$\therefore$ Δ is also an invariant.

(iv) *$a+b, ab-h^2, f^2+g^2$ and Δ obtained from*

$$ax^2+2hxy+by^2+2gx+2fy+c$$

remain invariant under transformation of rotation.

Let the axes be rotated through an angle θ.

Using $x = x'\cos\theta - y'\sin\theta, y = x'\sin\theta + y'\cos\theta$, the expression $ax^2+2hxy+by^2+2gx+2fy+c$ transforms to

$$\begin{aligned} a(x'\cos\theta - y'\sin\theta)^2 &+ 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) \\ &+ b(x'\sin\theta + y'\cos\theta)^2 + 2g(x'\cos\theta - y'\sin\theta) \\ &+ 2f(x'\sin\theta + y'\cos\theta) + c \\ &= a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c' \text{ (say)}. \end{aligned}$$

Here

$$\begin{aligned} a' &= a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta \\ b' &= a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta \\ h' &= (b-a)\sin\theta\cos\theta + h\left(\cos^2\theta - \sin^2\theta\right) \\ g' &= g\cos\theta + f\sin\theta, f' = g\sin\theta + f\cos\theta, c' = c. \end{aligned}$$

If we put

$$\begin{aligned} a_1 &= a\cos\theta + h\sin\theta, \quad & b_1 &= h\cos\theta + b\sin\theta, \\ a_2 &= -a\sin\theta + h\cos\theta, \quad & b_2 &= -h\sin\theta + b\cos\theta, \end{aligned}$$

then

$$\begin{aligned} a' &= a_1\cos\theta + b_1\sin\theta, \quad b' = -a_2\sin\theta + b_2\cos\theta, \\ h' &= a_2\cos\theta + b_2\sin\theta = -a_1\sin\theta + b_1\cos\theta. \end{aligned}$$

[a_2 and b_2 are the derivatives of a_1 and b_1].

1. $a'+b' = a\left(\cos^2\theta + \sin^2\theta\right) + 2h\sin\theta\cos\theta - 2h\sin\theta\cos\theta + b\left(\sin^2\theta + \cos^2\theta\right)$

$$= a+b.$$

2.
$$\begin{aligned} a'b'-h'^2 &= \begin{vmatrix} a' & h' \\ h' & b' \end{vmatrix} = \begin{vmatrix} a_1\cos\theta + b_1\sin\theta & -a_1\sin\theta + b_1\cos\theta \\ a_2\cos\theta + b_2\sin\theta & -a_2\sin\theta + b_2\cos\theta \end{vmatrix} \\ &= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\ &= \begin{vmatrix} a\cos\theta + h\sin\theta & h\cos\theta + b\sin\theta \\ -a\sin\theta + h\cos\theta & -h\sin\theta + b\cos\theta \end{vmatrix} \\ &= \begin{vmatrix} a & h \\ h & b \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} \\ &= \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2. \end{aligned}$$

3. $g'^2 + f'^2 = (g\cos\theta + f\sin\theta)^2 + (f\cos\theta - g\sin\theta)^2 = g^2 + f^2$.

4. $$\Delta' = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = g' \begin{vmatrix} h' & g' \\ b' & f' \end{vmatrix} - f' \begin{vmatrix} a' & g' \\ h' & f' \end{vmatrix} + c' \begin{vmatrix} a' & h' \\ h' & b' \end{vmatrix}$$

$$= g' \begin{vmatrix} a_2 \cos\theta + b_2 \sin\theta & g\cos\theta + f\sin\theta \\ -a_2 \sin\theta + b_2\cos\theta & -g\sin\theta + f\cos\theta \end{vmatrix}$$

$$- f' \begin{vmatrix} a_1 \cos\theta + b_1 \sin\theta & g\cos\theta + f\sin\theta \\ -a_1 \sin\theta + b_1\cos\theta & -g\sin\theta + f\cos\theta \end{vmatrix} + c\left(ab - h^2\right) \text{ [by (2)]}$$

$$= g' \begin{vmatrix} a_2 & b_2 \\ g & f \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} - f' \begin{vmatrix} a_1 & b_1 \\ g & f \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} + c\left(ab - h^2\right)$$

$$= g' \begin{vmatrix} a_2 & b_2 \\ g & f \end{vmatrix} - f' \begin{vmatrix} a_1 & b_1 \\ g & f \end{vmatrix} + c\left(ab - h^2\right)$$

$$= g'\left(a_2 f - b_2 g\right) - f'\left(a_1 f - b_1 g\right) + c\left(ab - h^2\right)$$

$$= g\left(b_1 f' - b_2 g'\right) - f\left(a_1 f' - a_2 g'\right) + c\left(ab - h^2\right)$$

$$= g \begin{vmatrix} b_1 & b_2 \\ g' & f' \end{vmatrix} - f \begin{vmatrix} a_1 & a_2 \\ g' & f' \end{vmatrix} + c\left(ab - h^2\right)$$

$$= g \begin{vmatrix} h\cos\theta + b\sin\theta & -h\sin\theta + b\cos\theta \\ g\cos\theta + f\sin\theta & -g\sin\theta + f\cos\theta \end{vmatrix}$$

$$- f \begin{vmatrix} a\cos\theta + h\sin\theta & -a\sin\theta + h\cos\theta \\ g\cos\theta + f\sin\theta & -g\sin\theta + f\cos\theta \end{vmatrix} + c\left(ab - h^2\right)$$

$$= g \begin{vmatrix} h & b \\ g & f \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} - f \begin{vmatrix} a & h \\ g & f \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} + c\left(ab - h^2\right)$$

$$= g \begin{vmatrix} h & b \\ g & f \end{vmatrix} - f \begin{vmatrix} a & h \\ g & f \end{vmatrix} + c \begin{vmatrix} a & h \\ h & b \end{vmatrix} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \Delta.$$

$\therefore$ $a + b$, $ab - h^2$, $g^2 + f^2$ and Δ are invariants.

WORKED-OUT EXAMPLES

1. *The equation* $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ *is reduced to* $4x^2 + 2y^2 = 1$ *when referred to rectangular axes through the point* $(2, 3)$. *Find the inclination of the latter axes to the formers.*

   Due to translation the equation transforms to

$$3(x' + 2)^2 + 2(x' + 2)(y' + 3) + 3(y' + 3)^2$$
$$- 18(x' + 2) - 22(y' + 3) + 50 = 0$$
$$\text{or,} \quad 3x'^2 + 2x'y' + 3y'^2 = 1.$$

Let the axes be rotated through an angle θ. The equation changes to

$$3(x''\cos\theta - y''\sin\theta)^2 + 2(x''\cos\theta - y''\sin\theta)(x''\sin\theta + y''\cos\theta)$$
$$+ 3(x''\sin\theta + y''\cos\theta)^2 = 1$$

$$\text{or,}\quad (3+\sin 2\theta)x''^2 + 2\cos 2\theta x''y'' + (3 - \sin 2\theta)y''^2 = 1.$$

To remove the term of $x''y''$ we take $\cos 2\theta = 0$ or $2\theta = 90^\circ$ or $\theta = 45^\circ$. For this value of θ the equation reduces to $4x''^2 + 2y''^2 = 1$.

Thus the required inclination is 45°.

2. *The coordinates of new origin are* $(2, 1)$ *and the axes are rotated through an angle* 60°. *If the coordinates of a point in the new system are*

$$\left(\frac{3-4\sqrt{3}}{2}, -\frac{4+3\sqrt{3}}{2}\right)$$

find the coordinates of it in the old system.

Let (x, y) be the coordinates of the point in the old system. By the formula for translation and rotation

$$x = \frac{3-4\sqrt{3}}{2}\cos 60^\circ + \frac{4+3\sqrt{3}}{2}\sin 60^\circ + 2$$

$$y = \frac{3-4\sqrt{3}}{2}\sin 60^\circ - \frac{4+3\sqrt{3}}{2}\cos 60^\circ + 1$$

or, $x = 5$ and $y = -3$.

3. *The coordinates of two points are* $(3, -2)$ *and* $\left(3 + 3\sqrt{3}, 1\right)$. *The origin is shifted to the point* $(3, -2)$ *and the new x-axis is the line joining the given points. Find the formulae for this orthogonal transformation.*

For translation and rotation

$$x = x'\cos\theta - y'\sin\theta + \alpha,$$
$$y = x'\sin\theta + y'\cos\theta + \beta.$$

By the given condition $\alpha = 3$, $\beta = -2$.

In the new system the other point lies on the x-axis. Therefore, the coordinates of this point are of the form $(c, 0)$. Thus putting $x' = c$, $y' = 0$

$$3 + 3\sqrt{3} = c\cos\theta + 3 \text{ and } 1 = c\sin\theta - 2$$

$$\text{or,}\quad c\cos\theta = 3\sqrt{3} \text{ and } c\sin\theta = 3$$

$$\text{or,}\quad \tan\theta = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} \text{ i.e. } \theta = \frac{\pi}{6}.$$

Thus the required formulae are

$$x = x'\cdot\frac{\sqrt{3}}{2} - y'\cdot\frac{1}{2} + 3,$$

$$y = x'\cdot\frac{1}{2} + y'\cdot\frac{\sqrt{3}}{2} - 2.$$

4. *If $ax + by$ transforms to $a'x' + b'y'$ under rotation of axes, then show that $a^2 + b^2 = a'^2 + b'^2$.* **[NH 01]**

If $A = \begin{bmatrix} a & b \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $x' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, then $ax + by = AX$.

Let us consider the rotation given by $X = SX'$ where

$$S = \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix}, \quad \lambda^2 + \mu^2 = 1.$$

Now

$$\begin{aligned} AX = ASX' &= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= (a\lambda + b\mu)x' + (-a\mu + b\lambda)y' = a'x' + b'y'. \end{aligned}$$

$\therefore\ a' = a\lambda + b\mu$ and $b' = -a\mu + b\lambda$.

$\therefore\ a'^2 + b'^2 = (a\lambda + b\mu)^2 + (-a\mu + b\lambda)^2 = (\lambda^2 + \mu^2)(a^2 + b^2) = a^2 + b^2$.

5. *If $ax^2 + 2hxy + by^2$ and $Ax^2 + 2Hxy + By^2$ transform to $a'x'^2 + 2h'x'y' + b'y'^2$ and $A'x'^2 + 2H'x'y' + B'y'^2$ under rotation of axes, show that $aA + 2hH + bB$ is an invariant.*

If $D = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $\quad D_1 = \begin{bmatrix} A & H \\ H & B \end{bmatrix}$, $\quad X = \begin{bmatrix} x \\ y \end{bmatrix}$,

$$D' = \begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}, \quad D_1' = \begin{bmatrix} A' & H' \\ H' & B' \end{bmatrix}, \quad X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

then $ax^2 + 2hxy + by^2 = X^T DX$, $\qquad Ax^2 + 2Hxy + By^2 = X^T D_1 X$,

$$a'x'^2 + 2h'x'y' + b'y'^2 = X'^T D'X', \ A'x'^2 + 2H'x'y' + B'y'^2 = X'^T D_1' X'.$$

Let us consider the rotation given by $X = SX'$ where

$$S = \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix}, \quad \lambda^2 + \mu^2 = 1.$$

Now $X'^T D'X' = X'^T S^T DSX'$ and $X'^T D_1' X' = X'^T S^T D_1 SX'$.

$\therefore\ D' = S^T DS$ and $D_1' = S^T D_1 S$.

$D'D_1' = (S^T DS)(S^T D_1 S) = S^T DD_1 S$ $(\because\ SS^T = 1)$.

From this

$$\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} \begin{bmatrix} A' & H' \\ H' & B' \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} A & H \\ H & B \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix}$$

or, $$\begin{bmatrix} a'A' + h'H' & a'H' + h'B' \\ h'A' + b'H' & h'H' + b'B' \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix} \begin{bmatrix} aA + hH & aH + hB \\ hA + bH & hH + bB \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix}$$

From the condition of the equality of two matrices

$$a'A' + h'H' = (aA + hH)\lambda^2 + (Ah + bH)\lambda\mu + (aH + Bh)\lambda\mu + (hH + bB)\mu^2$$

and $$h'H' + b'B' = (aA + hH)\lambda^2 - (Ah + bH)\lambda\mu - (aH + Bh)\lambda\mu + (hH + bB)\mu^2.$$

Adding these two we have $a'A' + 2h'H' + b'B' = aA + 2hH + bB$.

Note. Here $S^T DS$ is similar to D.

$\therefore$ trace of $S^T DS$ = trace of D or, trace of D' = trace of D or, $a'+b' = a+b$.

Again $|D'| = |S^T DS| = |S^T|\,|D|\,|S| = |D|$.

$\therefore\ a'b' - h'^2 = ab - h^2$.

6. *Find the rigid motion for which the points* $(4, 2)$ *remains* $(4, 2)$.

Let $4 = \alpha + 4\cos\theta - 2\sin\theta$ and $2 = \beta + 4\sin\theta + 2\cos\theta$.

Now $(4-\alpha)^2 + (2-\beta)^2 = 4^2 + 2^2$ or, $\alpha(\alpha - 8) = -\beta(\beta - 4)$.

For the rigid motion $\alpha \neq 0$, $\beta \neq 0$. By trial $\alpha = 6$, $\beta = 6$ or $\alpha = 2$, $\beta = -2$.

Case I. $\alpha = 6$, $\beta = 6$.

$4 + 2\sin\theta = 6 + 4\cos\theta$ or, $2\cos\theta + 1 = \sin\theta$ or, $(2\cos\theta + 1)^2 = 1 - \cos^2\theta$ or, $5\cos^2\theta + 4\cos\theta$ or, $\cos\theta(5\cos\theta + 4) = 0$ or, $\cos\theta = 0, -\frac{4}{5}$.

If $\cos\theta = 0$, $\sin\theta = 1$. These values do not satisfy $2 = 6 + 4\sin\theta + 2\cos\theta$.

To satisfy $4 = 6 + 4\cos\theta - 2\sin\theta$ and $2 = 6 + 4\sin\theta + 2\cos\theta$,

$\cos\theta = -\frac{4}{5}$ and $\sin\theta = -\frac{3}{5}$.

The rigid motion is $x = 6 - \frac{4}{5}x' + \frac{3}{5}y'$ and $y = 6 - \frac{3}{5}x' + \frac{4}{5}y'$.

Case II. $\alpha = 2$, $\beta = -2$.

Here $4 = 2 + 4\cos\theta - 2\sin\theta$ or, $2\cos\theta - 1 = \sin\theta$ or, $5\cos^2\theta - 4\cos\theta = 0$ or, $\cos\theta(5\cos\theta - 4) = 0$.

If $\cos\theta = 0$, $\sin\theta = 1$. These values satisfy $2 = -2 + 4\sin\theta + 2\cos\theta$.

Also $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$ satisfy $4 = 2 + 4\cos\theta - 2\sin\theta$ and $2 = -2 + 4\sin\theta + 2\cos\theta$.

Thus the rigid motions $x = 2 - 2y'$, $y = -2 + 4x$ and $x = 2 + \frac{4}{5}x' - \frac{3}{5}y'$ and $y = -2 + \frac{3}{5}x' + \frac{4}{5}y'$.

EXERCISE VIII

1. The origin is shifted to the point $(3, -3)$ without changing the directions of axes. If the coordinates of P, Q, R are $(5, 5)$, $(-2, 4)$ and $(7, -7)$ respectively in the new system, find the coordinates of these points in the old system.

2. The axes are rotated through an angle of $60°$ without change of origin. The coordinates of a point are $(4, \sqrt{3})$ in the new system. Find the coordinates of it in the old system.

3. The origin is shifted to the point $(3, -1)$ and the axes are rotated through an angle $\tan^{-1}\frac{3}{4}$. If the coordinates of a point are $(5, 10)$ in the new system, find the coordinates in the old system.

4. The coordinates of A and B are $(5, -1)$ and $(3, 1)$. The origin is shifted to A and the axes are rotated in such a way that the new x-axis coincides with AB. If the rotation is made in the positive direction, find the formulae for transformation.

5. If $ax + by$ and $cx + dy$ are changed to $a'x' + b'y'$ and $c'x' + d'y'$ respectively for rotation of axes, show that $ad - bc = a'd' - b'c'$. **[BH 91; NH 2007]**

6. Show that the radius of a circle remains unchanged due to any rigid body motion.

7. Show that there is only one point whose coordinates do not alter due to a rigid motion. **[CH 08, 09]**

   [*Hints.* Let

$$\left.\begin{aligned} x &= \lambda x' - \mu y' + \nu \\ y &= \mu x' + \lambda y' + \xi \end{aligned}\right\}, \quad \lambda^2 + \mu^2 = 1 \text{ be the rigid motion.}$$

   For unalteration of coordinates,

$$\begin{aligned} x &= \lambda x - \mu y + \nu. \\ y &= \mu x + \lambda y + \xi \end{aligned}$$

   Solving for x and y, the point is

$$\left\{\frac{\nu(1-\lambda) - \mu\xi}{2(1-\lambda)}, \frac{\xi(1-\lambda) + \mu\nu}{2(1-\lambda)}\right\}.]$$

8. If (x_1, y_1), (x_2, y_2), (x_3, y_3) be three non-collinear points in the plane, show that the expression $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ remains invariant under an orthogonal transformation. [Under any rigid motion the area of a triangle is an invariant.]

   [BH 93, 95; NH 2006]

   [*Hints.* Let the orthogonal transformation be

$$\left.\begin{aligned} x &= \lambda x' = \mu y' + \nu \\ y &= \mu x' + \lambda y' + \xi \end{aligned}\right\}, \quad \lambda^2 + \mu^2 = 1.$$

   Now

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} \lambda x_1' - \mu y_1' + \nu & \mu x_1' + \lambda y_1' + \xi & 1 \\ \lambda x_2' - \mu y_2' + \nu & \mu x_2' + \lambda y_2' + \xi & 1 \\ \lambda x_3' - \mu y_3' + \nu & \mu x_3' + \lambda y_3' + \xi & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda & -\mu & \nu \\ \mu & \lambda & \xi \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix} = (\lambda^2+\mu^2) \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}.$$

9. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ transforms to $a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c'$ under rotation of axes, then show that

   (a) $a' + b' + c' = a + b + c$,

   (b) $f'^2 + g'^2 + h'^2 - b'c' - c'a' - a'b' = f^2 + g^2 + h^2 - bc - ca - ab$ and

   (c) $2f'g'h' - a'f'^2 - b'g'^2 = 2fgh - af^2 - bg^2$

10. If $\frac{x^2}{p} + \frac{y^2}{q}$ transforms to $ax'^2 + 2hx'y' + by'^2$ under any transformation of axes. show that $\frac{x^2}{p-\lambda} + \frac{y}{q-\lambda}$ transforms to $\frac{ax'^2+2hx'y'+by'^2-\lambda(ab\ \ h^2)(x'^2+y'^2)}{1-(a+b)\lambda+(ab-h^2)\lambda^2}$

    [*Hints*. Here the transformation is a rotation.

    Let
$$\left.\begin{aligned} x &= lx' \quad my' \\ y &= mx' + ly' \end{aligned}\right\}, \quad l^2 + m^2 = 1.$$

    For this transformation, $\dfrac{l^2}{p} + \dfrac{m^2}{q} = a, \quad lm\left(\dfrac{1}{q} - \dfrac{1}{p}\right) = h, \quad \dfrac{l^2}{q} + \dfrac{m^2}{p} = b.$

    From these $a + b = \dfrac{1}{p} + \dfrac{1}{q}$ and $ab - h^2 = \dfrac{1}{pq}$.

    The transformed form of $\frac{x^2}{p-\lambda} + \frac{y^2}{q-\lambda}$ is

$$\frac{\left(ql^2 + pm^2 - \lambda\right)x'^2 + 2lm(p-q)x'y' + \left(pl^2 + qm^2 - \lambda\right)y'^2}{pq - (p+q)\lambda + \lambda^2}$$

$$= \frac{pq\left(ax'^2 + 2hx'y' + by'^2\right) - \lambda\left(x'^2 + y'^2\right)}{pq - (p+q)\lambda + \lambda^2}$$

$$= \frac{ax'^2 + 2hx'y' + by'^2 - \lambda\left(ab - h^2\right)\left(x'^2 + y'^2\right)}{1 - (a+b)\lambda + \left(ab - h^2\right)\lambda^2}.]$$

ANSWERS

1. $(8, 2), (1, 1), (10, -10)$.

2. $\left(\dfrac{1}{2}, \dfrac{5\sqrt{3}}{2}\right)$.

3. $(1, 10)$.

4. $x = -\dfrac{1}{\sqrt{2}}(x' + y') + 5,$

   $y = \dfrac{1}{\sqrt{2}}(x' - y') - 1.$

Chapter 3

Pair of Straight Lines

3.10 Second degree homogeneous equation

(i) *A second degree homogeneous equation of the form $ax^2+2hxy+by^2=0$ represents a pair of straight lines through the origin.*

Considering $ax^2+2hxy+by^2=0$ as a quadratic equation in x and $a \neq 0$, we have

$$x = \frac{-h \pm \sqrt{h^2-ab}}{a}\, y \quad \text{i.e.} \quad ax + \left(h \pm \sqrt{h^2-ab}\right) y = 0.$$

It shows that the equation will represent

(a) two real and distinct lines through the origin if $h^2-ab>0$;

(b) two coincident lines through the origin if $h^2-ab=0$;

(c) two imaginary lines if $h^2-ab<0$.

The equations of the pair of lines are

$$ax + \left(h+\sqrt{h^2-ab}\right) y = 0$$
$$\text{and} \quad ax + \left(h-\sqrt{h^2-ab}\right) y = 0.$$

These are of the form $l_1x+m_1y=0$ and $l_2x+m_2y=0$.

(ii) *Angle between the lines $ax^2+2hxy+by^2=0$.*

Let $ax^2+2hxy+by^2=(l_1x+m_1y)(l_2x+m_2y)$.

Equating the coefficients of x^2, xy and y^2, we have

$$l_1l_2=a, \quad l_1m_2+l_2m_1=2h, \quad m_1m_2=b.$$

The lines are $l_1x+m_1y=0$ and $l_2x+m_2y=0$.

If θ be the angle between the lines, then

$$\tan^2\theta = \frac{(l_1m_2-l_2m_1)^2}{(l_1l_2+m_1m_2)^2} = \frac{(l_1m_2+l_2m_1)^2-4l_1l_2m_1m_2}{(l_1l_2+m_1m_2)^2} = \frac{4\left(h^2-ab\right)}{(a+b)^2}.$$

$$\therefore \ \tan\theta = \pm\frac{2\sqrt{h^2-ab}}{a+b} \quad \text{or,} \quad \theta = \tan^{-1}\left(\pm\frac{2\sqrt{h^2-ab}}{a+b}\right),$$

$+$ or $-$ sign is taken according as θ is the acute or obtuse angle between the lines.

Corollary I. *Condition of coincidence. If $\theta = 0$, the lines are coincident and in this case $h^2 = ab$.*

Corollary II. *Condition of perpendicularity. If $\theta = \pi/2$, the lines are at right angle and in this case*

$$\cot\theta = 0 = \frac{a+b}{2\sqrt{h^2-ab}} \quad i.e. \quad a+b=0.$$

$$[\text{coefficient of } x^2 + \text{coefficient of } y^2 = 0.]$$

Example 1. *Find the angle between the lines $x^2+xy-6y^2=0$.*

Here $a=1, h=\frac{1}{2}, b=-6$.

$$\therefore \ \theta = \tan^{-1}\frac{2\sqrt{h^2-ab}}{a+b} = \tan^{-1}\frac{2\sqrt{\frac{1}{4}+6}}{1-6} = \tan^{-1}(-1).$$

The obtuse angle between the lines is 135°.

(iii) *Bisectors of the angles between the lines $ax^2+2hxy+by^2=0$.*

Let

$$ax^2+2hxy+by^2 = (l_1x+m_1y)(l_2x+m_2y).$$

Equating the coefficients of x^2, xy and y^2 we have

$$l_1l_2 = a, \quad l_1m_2+l_2m_1 = 2h, \quad m_1m_2 = b.$$

The lines are $l_1x+m_1y=0$ and $l_2x+m_2y=0$.

The equations of the bisectors of the angles between the lines are

$$\frac{l_1x+m_1y}{\sqrt{l_1^2+m_1^2}} = \pm\frac{l_2x+m_2y}{\sqrt{l_2^2+m_1^2}}.$$

In pair form

$$\left(\frac{l_1x+m_1y}{\sqrt{l_1^2+m_1^2}}+\frac{l_2x+m_2y}{\sqrt{l_2^2+m_2^2}}\right)\left(\frac{l_1x+m_1y}{\sqrt{l_1^2+m_1^2}}-\frac{l_2x+m_2y}{\sqrt{l_2^2+m_2^2}}\right)=0$$

$$\text{or,} \quad \frac{(l_1x+m_1y)^2}{l_1^2+m_1^2}-\frac{(l_2x+m_2y)^2}{l_2^2+m_2^2}=0$$

$$\text{or,} \quad (l_2^2+m_2^2)(l_1x+m_1y)^2-(l_1^2+m_1^2)(l_2x+m_2y)^2=0$$

$$\text{or,} \quad (l_1^2m_2^2-l_2^2m_1^2)(x^2-y^2) = 2(l_1m_2-l_2m_1)(l_1l_2-m_1m_2)xy$$

$$\text{or,} \quad (l_1m_2+l_2m_1)(x^2-y^2) = 2(l_1l_2-m_1m_2)xy$$

$$\text{or,} \quad 2h(x^2-y^2) = 2(a-b)xy$$

$$\text{or,} \quad \frac{x^2-y^2}{a-b} = \frac{xy}{h}.$$

Example 2 *Find the equation of the bisectors of the angles between the lines* $x^2 - 4xy - y^2 = 0$.

The equation is

$$\frac{x^2 - y^2}{1-(-1)} = \frac{xy}{-2} \quad \text{or,} \quad \frac{x^2-y^2}{2} = \frac{xy}{-2} \quad \text{or,} \quad x^2 + xy - y^2 = 0.$$

3.11 General second degree equation (non-homogeneous)

(i) *The necessary and sufficient conditions for the general equation of the second degree* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *to represent a pair of real straight lines are* $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ *and* $h^2 - ab \geq 0$.

Proof. (a) *Conditions are necessary.*

If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of real straight lines, the expression on the LHS must be factorised into two linear factors.

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$.

Equating the coefficients, $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$, $l_1n_2 + l_2n_1 = 2g$, $m_1n_2 + m_2n_1 = 2f$, $n_1n_2 = c$.

Conditions are obtained by eliminating l_1, m_1, n_1, l_2, m_2, n_2 from the above relations. For elimination we consider the following product.

$$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} l_2 & l_1 & 1 \\ m_2 & m_1 & 1 \\ n_2 & n_1 & 1 \end{vmatrix} = \begin{vmatrix} 2l_1l_2 & l_1m_2 + l_2m_1 & l_1n_2 + l_2n_1 \\ l_1m_2 + l_2m_1 & 2m_1m_2 & m_1n_2 + m_2n_1 \\ n_1l_2 + n_2l_1 & m_1n_2 + m_2n_1 & 2n_1n_2 \end{vmatrix}$$

$$= \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 8 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

$$\because \begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} = 0, \quad \text{the product is zero.}$$

$$\therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \quad \text{or,} \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

Again $4(h^2 - ab) = (l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2 = (l_1m_2 - l_2m_1)^2$.

$\therefore\ h^2 - ab \geq 0$.

(b) *Conditions are sufficient.*

Let $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 - ab \geq 0$ hold.

The first condition implies that $(fh - bg)^2 = (h^2 - ab)(f^2 - bc)$.

Now

$$
\begin{aligned}
&ax^2 + 2hxy + by^2 + 2gx + 2fy + c \\
&\quad = b\left(y + \frac{hx+f}{b}\right)^2 + ax^2 + 2gx + c - \frac{(hx+f)^2}{b} \\
&\quad = \frac{1}{b}\left[(by+hx+f)^2 - \left\{\left(h^2-ab\right)x^2 + 2(fh-bg)x + f^2 - bc\right\}\right] \\
&\quad = \frac{1}{b}\left[(by+hx+f)^2 - \left\{\left(h^2-ab\right)x^2 \pm 2\sqrt{h^2-ab}\sqrt{f^2-bc}\,x + f^2 - bc\right\}\right] \\
&\quad = \frac{1}{b}\left[(by+hx+f)^2 - \left(\sqrt{h^2-ab}\,x \pm \sqrt{f^2-bc}\right)^2\right] \\
&\quad = \frac{1}{b}\left[\left\{by + \left(h+\sqrt{h^2-ab}\right)x + f \pm \sqrt{f^2-bc}\right\}\right. \\
&\qquad \left.\times \left\{by + \left(h-\sqrt{h^2-ab}\right)x + f \mp \sqrt{f^2-bc}\right\}\right].
\end{aligned}
$$

Thus $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two real lines when the conditions hold.

Note. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ is known as the discriminant of the general equation and it will be denoted by Δ. If $\Delta = 0$, the general equation represents a pair of lines (real or imaginary).

Alternative method. The equation can be written as

$$ax^2 + 2(hy+g)x + by^2 + 2fy + c = 0.$$

If $a \neq 0$, then solving for x we get

$$x = \frac{-(hy+g) \pm \sqrt{\left\{(hy+g)^2 - a\left(by^2 + 2fy + c\right)\right\}}}{a}$$

$$\text{or,}\quad ax + hy + g = \pm\sqrt{\left\{\left(h^2-ab\right)y^2 + 2(gh-af)y + g^2 - ca\right\}}.$$

To represent a pair of real straight lines, the expression under radical sign must be a perfect square. This gives

$$
\begin{aligned}
&4(gh-af)^2 - 4\left(h^2-ab\right)\left(g^2-ca\right) = 0 \\
\text{or,}\quad &a\left(abc + 2fgh - af^2 - bg^2 - ch^2\right) = 0 \quad \text{and} \quad h^2 - ab > 0.
\end{aligned}
$$

$\therefore$ the required conditions are

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{and} \quad h^2 - ab > 0.$$

If $h^2 - ab = 0$, then the lines are parallel.

Note. The conditions obtained are necessary. These are also sufficient, because $(hy+g)^2 - a\left(by^2 + 2fy + c\right)$ is a perfect square when the conditions are satisfied.

Example 3. *Show that* $14x^2 + 29xy + 12y^2 - 31x - 14y - 10 = 0$ *represents a pair of straight lines.*

Since

$$14 \cdot 12 \cdot (-10) + 2 \cdot (-7) \cdot \left(-\frac{31}{2}\right) \cdot \left(\frac{29}{2}\right) - 14 \cdot (-7)^2 - 12 \cdot \left(-\frac{31}{2}\right)^2 - (-10) \cdot \left(\frac{29}{2}\right)^2$$

$$= -1680 + \frac{6293}{2} - 686 - 2883 + \frac{4205}{2} = 5249 - 5249 = 0$$

and $\left(\frac{29}{2}\right)^2 > 14 \times 12$, the equation represents a pair of real straight lines.

Example 4. *For what value of* λ *does* $\lambda xy - 8x + 9y - 12 = 0$ *represent a pair of lines?*

To represent a pair of lines

$$\begin{vmatrix} 0 & \lambda/2 & -4 \\ \lambda/2 & 0 & 9/2 \\ -4 & 9/2 & -12 \end{vmatrix} = 0 \quad \text{or,} \quad \frac{\lambda}{2}(-18 + 6\lambda) - 4 \cdot \frac{\lambda}{2} \cdot \frac{9}{2} = 0$$

$$\text{or,} \quad 3\lambda^2 - 18\lambda = 0 \quad \text{or,} \quad 3\lambda(\lambda - 6) = 0$$

$$\text{or,} \quad \lambda = 0, 6.$$

If $\lambda = 0$, the equation will not be of second degree.

Thus $\lambda = 6$.

(ii) **Angle between the lines.**

Let

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (l_1x + m_1y + n_1)(l_2x + m_2y + n_2).$$

Equating the coefficients, $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$,
$l_1n_2 + l_2n_1 = 2g$, $m_1n_2 + m_2n_1 = 2f$, $n_1n_2 = c$.

The lines are $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$.

If θ be the angle between the lines

$$\tan^2\theta = \frac{(l_1m_2 - l_2m_1)^2}{(l_1l_2 + m_1m_2)^2} = \frac{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}{(l_1l_2 + m_1m_2)^2} = \frac{4\left(h^2 - ab\right)}{(a + b)^2}.$$

$$\therefore \tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b} \quad \text{or,} \quad \theta = \tan^{-1}\left(\pm\frac{2\sqrt{h^2 - ab}}{a + b}\right),$$

$+$ or $-$ sign is taken according as θ is acute or obtuse.

Example 5. *Find the angle between the lines* $y^2 + xy - 2x^2 - 5x - y - 2 = 0$.

The equation is $2x^2 - xy - y^2 + 5x + y + 2 = 0$.

$\therefore$ the required angle

$$= \tan^{-1} \frac{2\sqrt{(-1/2)^2 - 2\cdot(-1)}}{2-1} = \tan^{-1} 3.$$

(iii) (A) **Condition of perpendicularity of the pair of lines.**

The lines will be perpendicular, if $a + b = 0$ i.e. coefficient of x^2+ coefficient of $y^2 = 0$.

(B) **Condition of parallelism of the pair of lines.**

Let the parallel lines be $lx + my + n_1 = 0$ and $lx + my + n_2 = 0$.

In this case

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = \lambda\,(lx + my + n_1)\,(lx + my + n_2)\,,$$

where λ is a constant.

Equating the coefficients, $\lambda l^2 = a$, $\lambda lm = h$, $\lambda m^2 = b$, $\lambda l(n_1 + n_2) = 2g$, $\lambda m\,(n_1 + n_2) = 2f$, $\lambda n_1 n_2 = c$.

Now

$$\frac{a}{h} = \frac{\lambda l^2}{\lambda lm} = \frac{l}{m} = \frac{lm}{m^2} = \frac{h}{b} \quad \text{and} \quad \frac{a}{h} = \frac{\lambda l^2}{\lambda lm} = \frac{l}{m} = \frac{l(n_1 + n_2)}{m\,(n_1 + n_1)} = \frac{g}{f}.$$

$$\therefore\ \frac{a}{h} = \frac{h}{b} = \frac{g}{f}.$$

With $\Delta = 0$, the condition for parallelism can be taken as $h^2 = ab$, $hf = bg$, $bg^2 = af^2$.

(C) **Condition of coincidence of the pair of lines.**

If the lines are coincident, then

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = \lambda(lx + my + n)^2.$$

Equating the coefficients, $\lambda l^2 = a$, $\lambda lm = h$, $\lambda m^2 = b$, $\lambda ln = g$, $\lambda mn = f$, $\lambda n^2 = c$.

Now
$$\frac{a}{h} = \frac{l^2}{lm} = \frac{l}{m} = \frac{lm}{m^2} = \frac{h}{b}, \quad \frac{a}{h} = \frac{l}{m} = \frac{ln}{mn} = \frac{g}{f}.$$

$$\therefore\ \frac{a}{h} = \frac{h}{b} = \frac{g}{f}. \tag{I}$$

$$\frac{h}{g} = \frac{lm}{ln} = \frac{m}{n} = \frac{m^2}{mn} = \frac{b}{f} = \frac{mn}{n^2} = \frac{f}{c}.$$

$$\therefore\ \frac{h}{g} = \frac{b}{f} = \frac{f}{c}. \tag{II}$$

$$\frac{g}{a} = \frac{ln}{l^2} = \frac{n}{l} = \frac{nm}{lm} = \frac{f}{h} = \frac{n^2}{nl} = \frac{c}{g}.$$

$$\therefore\ \frac{g}{a} = \frac{f}{h} = \frac{c}{g}. \tag{III}$$

With $\Delta = 0$, the condition for coincidence can be taken as $h^2 = ab$, $f^2 = bc$, $g^2 = ca$; or $gh - af = 0$, $hf - bg = 0$, $fg - ch = 0$.

Cofactors of all elements of Δ are zero.

Example 6. *Show that* $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ *represents a pair of perpendicular lines.*

Since

$$6 \cdot (-6) \cdot 4 + 2 \cdot \frac{5}{2} \cdot 7 \cdot \left(-\frac{5}{2}\right) - 6 \cdot \left(\frac{5}{2}\right)^2 - (-6) \cdot 7^2 - 4 \cdot \left(-\frac{5}{2}\right)^2$$
$$= -144 - \frac{175}{2} - \frac{75}{2} + 294 - 25 = 294 - 294 = 0,$$

the equation represents a pair of lines.

Again coefficient of x^2 + coefficient of $y^2 = 6 - 6 = 0$. Therefore, the lines are perpendicular to each other.

Example 7. *Show that* $x^2 + 6xy + 9y^2 - 5x - 15y + 6 = 0$ *represents a pair of parallel lines and find the distance between them.*

$$x^2 + 6xy + 9y^2 - 5x - 15y + 6 = 0$$
$$\text{or,}\quad x^2 + (6y - 5)x + 9y^2 - 15y + 6 = 0$$
$$\text{or,}\quad x = \tfrac{1}{2}\left[-(6y-5) \pm \sqrt{(6y-5)^2 - 4(9y^2 - 15y + 6)}\right]$$
$$= \tfrac{1}{2}[-(6y-5) \pm 1].$$

Thus the lines are $x + 3y - 3 = 0$ and $x + 3y - 2 = 0$.

Obviously these are parallel.

Distance between the lines $= \dfrac{3-2}{\sqrt{10}} = \dfrac{1}{\sqrt{10}}$ units.

Example 8. *Show that* $4x^2 + 24xy + 36y^2 + 4x + 12y + 1 = 0$ *represents a pair of coincident lines.*

Since $4x^2 + 24xy + 36y^2 + 4x + 12y + 1 = (2x + 6y + 1)^2$, the equation represents a pair of coincident lines.

(iv) **Point of intersection of the lines.**

Let (α, β) be the point of intersection of the lines represented by the general equation and the origin be transformed to (α, β) without rotation of axes. Replacing x and y by $x' + \alpha$ and $y' + \beta$ the equation takes the form

$$a(x' + \alpha)^2 + 2h(x' + \alpha)(y' + \beta) + b(y' + \beta)^2 + 2g(x' + \alpha)$$
$$+ 2f(y' + \beta) + c = 0$$
$$\text{or,}\quad ax'^2 + 2hx'y' + by'^2 + 2(a\alpha + h\beta + g)x' + 2(h\alpha + b\beta + f)y'$$
$$+ a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0.$$

Since the equation represents a pair of lines through the origin, this must be a second degree homogeneous equation in x' and y'.

For this

$$a\alpha + h\beta + g = 0, \tag{1}$$

$$h\alpha + b\beta + f = 0 \tag{2}$$

$$\text{and} \quad a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0. \tag{3}$$

From (1) and (2)

$$\alpha = \frac{hf - bg}{ab - h^2}, \quad \beta = \frac{hg - af}{ab - h^2}.$$

Here $ab - h^2 \neq 0$, since the lines are not parallel.

Corollary I. Condition for the pair of lines.

Subtracting α times (1) and β times (2) from (3) we have

$$g\alpha + f\beta + c = 0. \tag{1}$$

Eliminating α and β from (1), (2) and (4) we have

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

It is the necessary condition for the pair of lines.

Corollary II. *If $A, H, G, \cdots$ are the cofactors of $a, h, g, \cdots$ of Δ, then $\alpha = \frac{G}{C}$, $\beta = \frac{F}{C}$.*

Corollary III. Distance of the point of intersection from the origin.

If we take the lines as $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$, then the point of intersection is

$$\left(\frac{m_1n_2 - m_2n_1}{l_1m_2 - l_2m_1}, \frac{n_1l_2 - n_2l_1}{l_1m_2 - l_2m_1}\right) \quad \text{or,} \quad \left(\sqrt{\frac{f^2 - bc}{h^2 - ab}}, \sqrt{\frac{g^2 - ca}{h^2 - ab}}\right)$$

$$\text{or,} \quad \left(\sqrt{\frac{A}{C}}, \sqrt{\frac{B}{C}}\right).$$

$$\therefore \text{ the distance} = \sqrt{\left(\frac{A}{C} + \frac{B}{C}\right)} = \sqrt{\frac{A + B}{C}} = \sqrt{\frac{c(a + b) - f^2 - g^2}{ab - h^2}}.$$

Corollary IV. Condition for the point of intersection to lie on the x-axis or the y-axis.

On the x-axis the y-coordinate of a point is zero. Thus if the lines represented by the general equation intersect on the x-axis, then $hg - af = 0$ and $g^2 - ca = 0$ i.e. $\frac{g}{a} = \frac{f}{h} = \frac{c}{g}$. Similarly if the lines intersect on the y-axis, then

$$hf - bg = 0 \quad \text{and} \quad f^2 - bc = 0 \quad \text{i.e.} \quad \frac{f}{b} = \frac{g}{h} = \frac{c}{f}.$$

(v) **Bisectors of the angles between the lines.**

Let (α, β) be the point of intersection and the origin be shifted to this point. The equation will reduce to $ax'^2 + 2hx'y' + by'^2 = 0$ when $x = x' + \alpha$, $y = y' + \beta$.

The equation of the bisectors of the angles between the lines is

$$\frac{x'^2 - y'^2}{a - b} = \frac{x'y'}{h}.$$

Thus the equation in the original set of axes is

$$\frac{(x - \alpha)^2 - (y - \beta)^2}{a - b} = \frac{(x - \alpha)(y - \beta)}{h}.$$

Example 9. *Find the equation of the bisectors of the angles between the lines* $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$.

Let (α, β) be the point of intersection. Changing the origin to (α, β), the equation transforms to

$$(x' + \alpha)^2 - 5(x' + \alpha)(y' + \beta) + 4(y' + \beta)^2 + (x' + \alpha) + 2(y' + \beta) - 2 = 0$$

i.e. $x'^2 - 5x'y' + 4y'^2 = 0$ with $2\alpha - 5\beta + 1 = 0$ and $-5\alpha + 8\beta + 2 = 0$.

Solving for α and β,

$$\frac{\alpha}{-10 - 8} = \frac{\beta}{-5 - 4} = \frac{1}{16 - 25} \Rightarrow \alpha = 2, \beta = 1.$$

Thus the pair of bisectors is

$$\frac{(x - 2)^2 - (y - 1)^2}{1 - 4} = \frac{(x - 2)(y - 1)}{-5/2}$$

$$\text{or,} \quad 5(x - 2)^2 - 5(y - 1)^2 = 6(x - 2)(y - 1)$$

$$\text{or,} \quad 5x^2 - 6xy - 5y^2 - 14x + 22y + 9 = 0.$$

3.20 Equation of a pair of straight lines passing through the origin and the points of intersection of a locus represented by a second degree equation with a straight line

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be the equation of the locus and $lx + my + n = 0$ be the equation of the straight line.

The equation of the line can be written as

$$\frac{lx + my}{-n} = 1. \tag{1}$$

Let us make the equation of the locus homogeneous in x and y of second degree by the help of (1) in the following way.

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\frac{lx + my}{-n} + c\left(\frac{lx + my}{-n}\right)^2 = 0. \tag{2}$$

It is of the form $Ax^2 + 2Hxy + By^2 = 0$.

The equation (2) obviously represents a pair of straight lines through the origin.

Moreover, it passes through the points of intersection of the locus and the line. Hence the equation (2) is the required equation.

Note 1. If the lines are coincident, $H^2 - AB = 0$.
Note 2. If the lines are perpendicular, $A + B = 0$.

Example 10. *Prove that the angle between the lines joining the origin to the intersection of the line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}\frac{2\sqrt{2}}{3}$.*

The equation of the pair of lines through the origin and the points of intersection of the given line and curve is the homogeneous equation

$$x^2 + 2xy + 3y^2 + (4x + 8y)\frac{y - 3x}{2} - 11\left(\frac{y - 3x}{2}\right)^2 = 0$$

or, $\quad 7x^2 - 2xy - y^2 = 0.$

Angle between the lines

$$= \tan^{-1}\frac{2\sqrt{1+7}}{7-1} = \tan^{-1}\frac{2\sqrt{2}}{3}.$$

3.30 A homogeneous equation of the nth degree represents n straight lines, real or imaginary, through the origin

Let the equation be

$$a_0 y^n + a_1 xy^{n-1} + a_2 x^2 y^{n-2} + \cdots + a_n x^n = 0. \tag{1}$$

Dividing each term by x^n,

$$a_0 (y/x)^n + a_1 (y/x)^{n-1} + a_2 (y/x)^{n-2} + \cdots + a_n = 0. \tag{2}$$

It is an equation in y/x of the nth degree and hence must have n roots. If the roots are $m_1, m_2, \ldots, m_n$, then the equation (2) is equivalent to

$$a_0 (y/x - m_1)(y/x - m_2)\cdots(y/x - m_n) = 0. \tag{3}$$

The equation (3) is satisfied by all points which satisfy the separate equations

$$\tfrac{y}{x} - m_1 = 0, \quad \tfrac{y}{x} - m_2 = 0, \quad \cdots, \quad \tfrac{y}{x} - m_n = 0,$$
$$\text{i.e.} \quad y - m_1x = 0, \quad y - m_2x = 0, \quad \cdots, \quad y - m_nx = 0.$$

All these pass through the origin. Moreover, all points which satisfy these n equations satisfy the equation (1).

WORKED-OUT EXAMPLES

1. *Show that the two straight lines $x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ make angles with the x-axis such that the difference of their tangents is 2.*

$$x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$$
$$\text{or,} \quad y^2 - 2xy\sec\theta\operatorname{cosec}\theta + x^2\left(\sec^2\theta + \cot^2\theta\right) = 0.$$

Let $y^2 - 2xy\sec\theta\operatorname{cosec} + x^2\left(\sec^2\theta + \cot^2\theta\right) = (y - m_1x)(y - m_2x)$.

Comparing the coefficients $m_1 + m_2 = 2\sec\theta\operatorname{cosec}\theta$, $m_1m_2 = \sec^2\theta + \cot^2\theta$.

m_1 and m_2 are the tangents of the angles made by the lines with the x-axis.

Now

$$\begin{aligned}(m_1 - m_2)^2 &= (m_1 + m_2)^2 - 4m_1m_2\\ &= 4\sec^2\theta\operatorname{cosec}^2\theta - 4\left(\sec^2\theta + \cot^2\theta\right)\\ &= 4\left\{\sec^2\theta\left(\operatorname{cosec}^2\theta - 1\right) - \cot^2\theta\right\}\\ &= 4\left(\sec^2\theta\cot^2\theta - \cot^2\theta\right) = 4\cot^2\theta\cdot\tan^2\theta = 4\end{aligned}$$
$$\therefore \quad m_1 - m_2 = 2.$$

2. *Show that the equation of the lines through the origin, each of which makes an angle α with the line $y = x$ is $x^2 - 2xy\sec 2\alpha + y^2 = 0$.*

The line $y = x$ makes an angle 45° with the x-axis. The line which makes an angle α with $y = x$ makes an angle $45° + \alpha$ or $45° - \alpha$ with the x-axis. Thus the equation to the pair is

$$\{y - x\tan(45° + \alpha)\}\{y - x\tan(45° - \alpha)\} = 0$$
$$\text{or,} \quad \left\{y - \frac{1 + \tan\alpha}{1 - \tan\alpha}x\right\}\left\{y - \frac{1 - \tan\alpha}{1 + \tan\alpha}x\right\} = 0$$
$$\text{or,} \quad y^2 - \left(\frac{1 + \tan\alpha}{1 - \tan\alpha} + \frac{1 - \tan\alpha}{1 + \tan\alpha}\right)xy + x^2 = 0$$
$$\text{or,} \quad y^2 - 2xy\sec 2\alpha + x^2 = 0.$$

3. *Find the equation to the pair of straight lines through the origin, perpendicular to the pair of straight lines given by $2x^2 + 5xy + 2y^2 + 10x + 5y = 0$.*

$$2x^2 + 5xy + 2y^2 + 10x + 5y = 0$$
$$\text{or,} \quad (2x + y)(x + 2y) + 10x + 5y = 0.$$

The lines are parallel to $2x + y = 0$ and $x + 2y = 0$.

Therefore, the lines passing through the origin and perpendicular to the given lines are $x - 2y = 0$ and $2x - y = 0$.

In pair form the required equation is $(x-2y)(2x-y) = 0$ or, $2x^2 - 5xy + 2y^2 = 0$.

4. *Prove that $bx^2 - 2hxy + ay^2 = 0$ represents two straight lines at right angles to the lines represented by $ax^2 + 2hxy + by^2 = 0$.*

Let $ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$.

Equating the coefficients, $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$.

The lines are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$.

The lines passing through the origin and perpendicular to the lines are

$$m_1x - l_1y = 0 \text{ and } m_2x - l_2y = 0.$$

In pair form the equation is

$$\begin{aligned} & (m_1x - l_1y)(m_2x - l_2y) = 0 \\ \text{or,} \quad & m_1m_2x^2 - (l_1m_2 + l_2m_1)xy + l_1l_2y^2 = 0 \\ \text{or,} \quad & bx^2 - 2hxy + ay^2 = 0. \end{aligned}$$

Hence the result follows.

5. *If pair of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angles between the other pair, prove that $pq + 1 = 0$.*

[NH 2002, 07; BH 2006; CH 2007, 08]

$$x^2 - 2pxy - y^2 = 0 \qquad (1)$$
$$x^2 - 2qxy - y^2 = 0. \qquad (2)$$

Bisectors of the angles between the lines of (1) are

$$\frac{x^2 - y^2}{1-(-1)} = \frac{xy}{-p} \quad \text{or,} \quad x^2 + \frac{2}{p}xy - y^2 = 0.$$

It is identical with (2).

$\therefore\ \frac{2}{p} = -2q$ or, $pq + 1 = 0$.

6. ***Find the product of the perpendiculars from (x_1, y_1) to the lines represented by $ax^2 + 2hxy + by^2 = 0$.***

Let $ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$.

Equating the coefficients $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$.

The lines are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$.

The perpendicular distances from (x_1, y_1) to the lines are

$$\frac{l_1x_1 + m_1y_1}{\sqrt{l_1^2 + m_1^2}} \quad \text{and} \quad \frac{l_2x_1 + m_2y_1}{\sqrt{l_2^2 + m_2^2}}.$$

$$\begin{aligned}
\text{Product of them} &= \frac{(l_1x_1 + m_1y_1)(l_2x_1 + m_2y_1)}{\sqrt{l_1^2 + m_1^2}\sqrt{l_2^2 + m_2^2}} \\
&= \frac{l_1l_2x_1^2 + (l_1m_2 + l_2m_1)x_1y_1 + m_1m_2y_1^2}{\sqrt{(l_1^2l_2^2 + m_1^2m_2^2 + l_1^2m_2^2 + l_2^2m_1^2)}} \\
&= \frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{a^2 + b^2 + (l_1m_2 + l_2m_1)^2 - 2l_1l_2m_1m_2}} \\
&= \frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a^2 + b^2 + 4h^2 - 2ab)}} \\
&= \frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}.
\end{aligned}$$

7. *Show that the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is*

$$\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}.$$ **[BH 2007]**

Let $ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$.

Equating the coefficients $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$.

The triangle is bounded by the lines

$$l_1x + m_1y = 0, \tag{1}$$

$$l_2x + m_2y = 0 \tag{2}$$

$$\text{and} \quad lx + my + n = 0. \tag{3}$$

The lines (1) and (2) meet at $(0, 0)$. The line (3) meets the lines (1) and (2) at

$$\left(\frac{-nm_1}{lm_1 - ml_1}, \frac{nl_1}{lm_1 - ml_1}\right) \quad \text{and} \quad \left(\frac{-nm_2}{lm_2 - ml_2}, \frac{nl_2}{lm_2 - ml_2}\right).$$

The area of the triangle whose vertices are the above points

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{-n^2l_2m_1 + n^2l_1m_2}{(lm_1 - ml_1)(lm_2 - ml_2)}\right] \\
&= \frac{1}{2}\frac{n^2(l_1m_2 - l_2m_1)}{l^2m_1m_2 - lm(l_1m_2 + l_2m_1) + m^2l_1l_2} \\
&= \frac{1}{2}\frac{n^2\sqrt{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}}{bl^2 - 2hlm + am^2} \\
&= \frac{n^2\sqrt{h^2 - ab}}{bl^2 - 2hlm + am^2}.
\end{aligned}$$

8. *Show that the lines* $(A^2 - 3B^2)\,x^2 + 8ABxy + (B^2 - 3A^2)\,y^2 = 0$ *form with the line* $Ax + By + C = 0$ *an equilateral triangle of area* $\dfrac{C^2}{\sqrt{3}\,(A^2 + B^2)}$.

[NH 2002]

Let OLM be the triangle by the given lines and the equation of LM be $Ax+By+C=0$. O is the origin. Let the equation of OL or OM be $y = mx$. The triangle OLM will be equilateral, if each of $\angle L$ and $\angle M$ is $60°$ i.e. the lines OL and OM make angles $60°$ and $60°$ with LM.

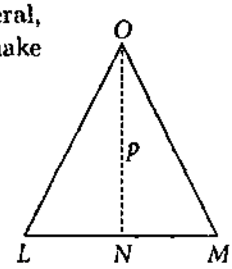


Fig. 28

$$\therefore \quad \tan(\pm 60°) = \frac{m + A/B}{1 - mA/B}$$

($-A/B$ is the slope of LM)

$$\text{or,} \quad \pm\sqrt{3} = \frac{A + Bm}{B - Am}.$$

$$\therefore \quad 3(B - Am)^2 = (A + Bm)^2.$$

The equation of OL and OM in pair form is

$$3(B - Ay/x)^2 = (A + By/x)^2$$

$$\text{or,} \quad (A^2 - 3B^2)\,x^2 + 8ABxy + (B^2 - 3A^2)\,y^2 = 0.$$

Hence the triangle formed by the given lines is an equilateral.

If p is perpendicular from O on LM, then $p = \dfrac{C}{\sqrt{A^2 + B^2}}$.

The area of the triangle

$$\frac{1}{2}LM \cdot p = \frac{1}{2} \cdot 2LN \cdot p = p \cdot \tan 30° \cdot p = p^2 \cdot \frac{1}{\sqrt{3}} = \frac{C^2}{\sqrt{3}\,(A^2 + B^2)}.$$

9. *Show that if one of the lines given by equation* $ax^2 + 2hxy + by^2 = 0$ *be perpendicular to one of the lines given by* $a'x^2 + 2h'xy + b'y^2 = 0$, *then* $(aa' - bb')^2 + 4(ah' + hb')(ha' + bh') = 0$.

$$ax^2 + 2hxy + by^2 = 0 \tag{1}$$

$$a'x^2 + 2h'xy + b'y^2 = 0. \tag{2}$$

Let $y = mx$ and $y = -\frac{1}{m}x$ be one of the lines of (1) and (2) respectively. Putting $y = mx$ in (1) and $y = -\frac{1}{m}x$ in (2) we have $bm^2 + 2hm + a = 0$ and $a'm^2 - 2h'm + b = 0$.

By cross-multiplication

$$\frac{m^2}{2(hb' + h'a)} = \frac{m}{aa' - bb'} = \frac{1}{-2(h'b + ha')}$$

$$\text{or,} \quad m = \frac{2(hb' + h'a)}{aa' - bb'} = \frac{aa' - bb'}{-2(h'b + ha')}.$$

$$\therefore \quad (aa' - bb')^2 + 4(ha' + h'b)(h'a + hb') = 0.$$

10. *Show that the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ is right-angled, if $(a+b)\left(al^2+2hlm+bm^2\right)=0$.* [CH 08]

Let

$$ax^2+2hxy+by^2=(l_1x+m_1y)(l_2x+m_2y).$$

Equating the coefficients $l_1l_2=a$, $l_1m_2+l_2m_1=2h$, $m_1m_2=b$.

The lines are

$$l_1x+m_1y=0 \tag{1}$$

$$l_2x+m_2y=0 \tag{2}$$

$$\text{and}\quad lx+my=1. \tag{3}$$

The triangle will be right-angled, if any two of the above lines are at right-angle.

The lines (1) and (2) will be at right angle if $l_1l_2+m_1m_2=0$.

The lines (2) and (3) will be at right angle if $ll_2+mm_2=0$.

The lines (3) and (1) will be at right angle if $ll_1+mm_1=0$.

Combining these we have

$$\begin{aligned} &(l_1l_2+m_1m_2)(ll_1+mm_1)(ll_2+mm_2)=0\\ \text{or,}\quad &(a+b)\left\{l^2l_1l_2+lm(l_1m_2+l_2m_1)+m^2m_1m_2\right\}=0\\ \text{or,}\quad &(a+b)\left(al^2+2hlm+bm^2\right)=0. \end{aligned}$$

11. *Show that the ortho-centre of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ is given by*

$$\frac{x}{l}=\frac{y}{m}=\frac{a+b}{am^2-2hlm+bl^2}.$$ [BH 2007; CH 2000, 07]

Let $ax^2+2hxy+by^2=(l_1x+m_1y)(l_2x+m_2y)$.

Equating the coefficients $l_1l_2=a$, $l_1m_2+l_2m_1=2h$, $m_1m_2=b$.

The lines are

$$l_1x+m_1y=0, \tag{1}$$

$$l_2x+m_2y=0 \tag{2}$$

$$\text{and}\quad lx+my=1. \tag{3}$$

The lines (1) and (2) meet at $(0,0)$ and the line (3) meets (1) at

$$\left(\frac{m_1}{lm_1-ml_1},\frac{-l_1}{lm_1-ml_1}\right).$$

The equation of the line perpendicular to (3) and passing through the point of intersection between (1) and (2) [i.e. $(0,0)$] is

$$mx-ly=0\quad\text{or,}\quad \frac{x}{l}=\frac{y}{m}=k\ \text{(say)}. \tag{4}$$

The ortho-centre lies on (4). Let the ortho-centre be (kl, km). The line passing through (kl, km) and $\left(\frac{m_1}{lm_1 - ml_1}, \frac{-l_1}{lm_1 - ml_1}\right)$ is perpendicular to (2). The slope of this line is

$$\frac{km + \dfrac{l_1}{lm_1 - ml_1}}{kl - \dfrac{m_1}{lm_1 - ml_1}} \quad \text{and that of (2) is } -\frac{l_2}{m_2}.$$

As these two lines are at right angle

$$\frac{l_2}{m_2} \cdot \frac{km + \frac{l_1}{lm_1 - ml_1}}{kl - \frac{m_1}{lm_1 - ml_1}} = 1$$

or, $l_2 \{km (lm_1 - ml_1) + l_1\} - m_2 \{kl (lm_1 - ml_1) - m_1\} = 0$

or, $k \{lm (l_1 m_2 + l_2 m_1) - l^2 m_1 m_2 - m^2 l_1 l_2\} + l_1 l_2 + m_1 m_2 = 0$

or, $k \{2hlm - l^2 b - m^2 a\} + a + b = 0$

or, $k = \dfrac{a + b}{am^2 - 2hlm + bl^2}.$

$\therefore \quad \dfrac{x}{l} = \dfrac{y}{m} = \dfrac{a + b}{am^2 - 2hlm + bl^2}.$

12. *If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two lines equidistant from the origin, show that $f^4 - g^4 = c\left(bf^2 - ag^2\right)$.*

[CH 1992; BH 1994, 2002, 04; NH 2001, 07]

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (l_1 x + m_1 y + n_1)(l_2 x + m_2 y + n_2)$.

Comparing the coefficients $l_1 l_2 = a$, $m_1 m_2 = b$, $l_1 m_2 + l_2 m_1 = 2h$, $l_1 n_2 + l_2 n_1 = 2g$, $m_1 n_2 + m_2 n_1 = 2f$, $n_1 n_2 = c$.

The lines are $l_1 x + m_1 y + n_1 = 0$ and $l_2 x + m_2 y + n_2 = 0$.

Since the lines are equidistant from the origin

$$\left|\frac{n_1}{\sqrt{l_1^2 + m_1^2}}\right| = \left|\frac{n_2}{\sqrt{l_2^2 + m_2^2}}\right|$$

or, $n_1^2 \left(l_2^2 + m_2^2\right) = n_2^2 \left(l_1^2 + m_1^2\right)$

or, $n_1^2 l_2^2 - n_2^2 l_1^2 = n_2^2 m_1^2 - n_1^2 m_2^2$

or, $(n_1 l_2 + n_2 l_1)(n_1 l_2 - n_2 l_1) = (n_2 m_1 + n_1 m_2)(n_2 m_1 - n_1 m_2)$

or, $2g (n_1 l_2 - n_2 l_1) = 2f (n_2 m_1 - n_1 m_2)$

or, $g^2 \left\{(n_1 l_2 + n_2 l_1)^2 - 4 n_1 n_2 l_1 l_2\right\} = f^2 \left\{(n_2 m_1 + n_1 m_2)^2 - 4 n_1 n_2 m_1 m_2\right\}$

or, $g^2 \left(4g^2 - 4ca\right) = f^2 \left(4f^2 - 4bc\right)$

or, $f^4 - g^4 = c\left(bf^2 - ag^2\right)$.

13. *If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, prove that the equation to the third pair of straight lines passing*

through the points where these meet the axes is $ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4fg}{c}xy = 0$. [CH 2009]

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{1}$$

Combined equation of the axes is

$$xy = 0. \tag{2}$$

The equation of a curve passing through the points of intersection between (1) and (2) is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c + 2\lambda xy = 0$, where λ is a constant. If it represents a pair of straight lines, then

$$abc + 2fg(h + \lambda) - af^2 - bg^2 - c(h + \lambda)^2 = 0. \tag{3}$$

As the equation (1) represents a pair of straight lines,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0. \tag{4}$$

By (3) and (4)

$$\lambda = \frac{2(fg - ch)}{c}.$$

Hence the required equation is

$$ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4fg}{c}xy = 0.$$

14. *Show that the lines joining the origin to the points of intersection of the curves* $ax^2 + 2hxy + by^2 + 2gx = 0$ *and* $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ *will be at right angle if* $g(a' + b') = g'(a + b)$.

$$ax^2 + 2hxy + by^2 + 2gx = 0 \tag{1}$$
$$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0. \tag{2}$$

Multiplying (1) and (2) by g' and g respectively and then subtracting, we have

$$(ag' - a'g)\,x^2 + 2\,(hg' - h'g)\,xy + (bg' - b'g)\,y^2 = 0.$$

This homogeneous equation in x and y is the required equation.

The straight lines will be at right angle if coefficient of x^2 + coefficient of $y^2 = 0$,

$$\begin{aligned} \text{i.e.} \quad & (ag' - a'g) + (bg' - b'g) = 0 \\ \text{or,} \quad & g'(a + b) = g(a' + b'). \end{aligned}$$

15. *Show that all the chords of the curve $3x^2 - y^2 - 2x + 4y = 0$ which subtend a right angle at the origin are concurrent.*

Let the equation of the chord of the given equation be $y = mx + c$. The equation of the lines joining the origin to the points of intersection of the chord and the curve is

$$3x^2 - y^2 + 2(x - 2y)\frac{y - mx}{c} = 0$$
$$\text{or,}\quad (3c + 2m)x^2 - 2(1 + 2m)xy + (4 - c)y^2 = 0.$$

These lines are mutually perpendicular, if coefficient of x^2 + coefficient of $y^2 = 0$ or, $(3c + 2m) + (4 - c) = 0$ or, $c = -m - 2$.

Now the equation of the chord is $y = mx - m - 2$ or, $(y + 2) + m(1 - x) = 0$ which is of the form $P + Q = 0$. For all real values of m, the chord passes through the point of intersection of the lines $y + 2 = 0, 1 - x = 0$, i.e. the point $(1, -2)$.

Hence all chords are concurrent and pass through the fixed point $(1, -2)$.

16. *Find the equations of the lines $x^2 + 8\sqrt{2}xy + 5y^2 = 0$ referred to the bisectors of the angles between them as axes of coordinates.*

$$x^2 + 8\sqrt{2}xy + 5y^2 = 0. \tag{1}$$

The equation of the bisectors is

$$y^2 - x^2 - \frac{1}{\sqrt{2}}xy = 0. \tag{2}$$

Let the lines make angles α and $90° + \alpha$ with the x-axis. The equation of the lines in pair form is

$$(y - \tan\alpha\ x)(y + \cot\alpha\ x) = 0$$
$$\text{or,}\quad y^2 - x^2 - (\tan\alpha - \cot\alpha)xy = 0.$$

Comparing it with (2),

$$\tan\alpha - \cot\alpha = \frac{1}{\sqrt{2}}$$
$$\text{or,}\quad \sqrt{2}\tan^2\alpha - \tan\alpha - \sqrt{2} = 0$$
$$\text{or,}\quad \tan\alpha = \sqrt{2}, -\frac{1}{\sqrt{2}}.$$

Taking $\tan\alpha = \sqrt{2}, \sin\alpha = \frac{\sqrt{2}}{\sqrt{3}}, \cos\alpha = \frac{1}{\sqrt{3}}$.

For rotation of axes through angle α, the equation transforms to

$$(x\cos\alpha - y\sin\alpha)^2 + 8\sqrt{2}(x\cos\alpha - y\sin\alpha)(x\sin\alpha + y\cos\alpha)$$
$$+ 5(x\sin\alpha + y\cos\alpha)^2 = 0$$
$$\text{or,}\quad \left(x - \sqrt{2}y\right)^2 + 8\sqrt{2}\left(x - \sqrt{2}y\right)\left(\sqrt{2}x + y\right) + 5\left(\sqrt{2}x + y\right)^2 = 0$$
$$\text{or,}\quad 27x^2 - 9y^2 = 0 \quad \text{or,}\quad 3x^2 - y^2 = 0.$$

Taking $\tan\alpha = -\frac{1}{\sqrt{2}}$, $\cos\alpha = -\frac{\sqrt{2}}{\sqrt{3}}$ and $\sin\alpha = \frac{1}{\sqrt{3}}$.

For rotation of axes through this angle, the equation (1) will reduce to

$$x^2 - 3y^2 = 0.$$

EXERCISE IX

1. Find the angle between the pair of lines

   (a) $x^2 + 2hxy - y^2 = 0$,

   (b) $4x^2 - 24xy + 11y^2 = 0$,

   (c) $x^2 + 2xy \operatorname{cosec}\alpha + y^2 = 0$.

   (d) $(x^2 + y^2)\sin^2\alpha = (x\cos\theta - y\sin\theta)^2$,

   (e) $(x^2 + y^2)\left(\cos^2\theta\sin^2\alpha + \sin^2\theta\right) = (x\tan\alpha - y\sin\theta)^2$,

   (f) $y^2\left(\cos\alpha + \sqrt{3}\sin\alpha\right)\cos\alpha - xy\left(\sin 2\alpha - \sqrt{3}\cos 2\alpha\right)$
   $+x^2\left(\sin\alpha - \sqrt{3}\cos\alpha\right)\sin\alpha = 0.$

2. Show that the equation of the lines passing through the origin and perpendicular to $5x^2 - 7xy - 3y^2 = 0$ is $3x^2 - 7xy - 5y^2 = 0$.

3. Show that the lines $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ are equally inclined to the lines $ax^2 + 2hxy + by^2 = 0$.

4. Prove that the straight lines $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ have the same pair of bisectors. Interpret the case when $\lambda = -(a+b)$.

5. Show that if one of the lines given by the equation $ax^2 + 2hxy + by^2 = 0$ coincides with one of those given by $a'x^2 + 2h'xy + b'y^2 = 0$, then $(ab' - a'b)^2 = 4(ha' - h'a)(bh' - b'h)$.

6. Prove that the equation $y^3 - x^3 + 3xy(y - x) = 0$ represents three straight lines equally inclined to one another.

   [*Hints.* $y^3 - x^3 + 3xy(y-x) = (y-x)\left(y^2 + x^2 + 4xy\right) = (y-x)\left\{(y+2x)^2 - 3x^2\right\}$
   $= (y-x)\left\{y + \left(2 + \sqrt{3}\right)x\right\}\left\{y + \left(2 - \sqrt{3}\right)x\right\}$]

7. Show that the following equations represent pair of straight lines and find the points of intersections and angles between them.

   (a) $8x^2 + 10xy + 3y^2 + 26x + 16y + 21 = 0$,

   (b) $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$,

   (c) $2x^2 - 7xy + 3y^2 + 26x - 33y + 72 = 0$.

8. Find the value of k so that each of the following equations may represents a pair of straight lines.

   (a) $kx^2 - 3xy - 2y^2 + x + 13y - 15 = 0$,

(b) $x^2 + kxy - 2y^2 + 3y - 1 = 0$,

(c) $xy + 5x + ky + 15 = 0$,

(d) $x^2 + \frac{10}{3}xy + y^2 - 5x - 7y + k = 0$,

(e) $x^2 + 6xy + 9y^2 + kx + 12y - 5 = 0$.

9. Show that each of the following equations represents pair of parallel lines and find the distance between them.

   (a) $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$,

   (b) $4x^2 + 12xy + 9y^2 + 2x + 3y - 42 = 0$.

10. Show that $2x^2 + 3xy - 2y^2 + 7x - y + 3 = 0$ represents a pair of perpendicular lines.

11. Show that

    (a) $x^2 + 6xy + 9y^2 + 4x + 12y + 4 = 0$ and

    (b) $4x^2 + 4xy + y^2 + 4x + 2y + 1 = 0$ represent coincident lines.

12. Find the equation of the lines passing through $(2, 3)$ and perpendicular to the lines $2x^2 + xy - 3y^2 + 3x + 2y + 1 = 0$.

13. The equations to the pair of opposite sides of a rectangle are $x^2 - 7x + 6 = 0$ and $y^2 - 14y + 40 = 0$. Find the equation to its diagonals.

14. Show that the lines $x + y + 1 = 0$ and $(x + y)^2 - 3(x - y)^2 = 0$ form an equilateral triangle.

15. Find the equation of the lines which pass through the origin and whose distances from (h, k) are equal to d. **[BH 2008]**

16. A triangle has the lines $ax^2 + 2hxy + by^2 = 0$ for two of its sides and the point (p, q) for its ortho-centre. Prove that the equation of the third side is $(a + b)(px + qy) = aq^2 - 2hpq + bp^2$. **[BH 2007; CH 2007]**

    [***Hints***. If $lx + my = 1$ is the other side, then

$$p = \frac{l(a+b)}{am^2 - 2hlm + bl^2}, \quad q = \frac{m(a+b)}{am^2 - 2hlm + bl^2}.$$

$$\frac{p}{l} = \frac{q}{m} = \lambda \text{ (say)}, \qquad p = \frac{p(a+b)\lambda}{aq^2 - 2hpq + bp^2}.]$$

17. $ax + by + c = 0$ bisects an angle between a pair of lines of which one is $lx + my + n = 0$. Show that the other line of the pair is

    $(a^2 + b^2)(lx + my + n) - 2(al + bm)(ax + by + c) = 0$. **[CH 2006; BH 1998]**

    [*Hints*. Let the other line be

$$lx + my + n + \lambda(ax + by + c) = 0. \tag{1}$$

$\left(-\frac{c}{a}, 0\right)$ is a point on $ax+by+c=0$. Distance of (1) and $lx+my+n=0$ from $\left(-\frac{c}{a}, 0\right)$ are equal.

$$\therefore \quad \frac{\left|-l\cdot\frac{c}{a}+n\right|}{\sqrt{(l+\lambda a)^2+\sqrt{(m+\lambda b)^2}}} = \frac{\left|-l\cdot\frac{c}{a}+n\right|}{\sqrt{l^2+m^2}}$$

$$\text{or,}\quad l^2+m^2=(l+\lambda a)^2+(m+\lambda b)^2$$

$$\text{or,}\quad \lambda^2\left(a^2+b^2\right)+2\lambda(al+bm)=0,$$

$$\lambda=0 \quad \text{or,} \quad -\frac{2(al+bm)}{a^2+b^2}.$$

For $\lambda=-\frac{2(al+bm)}{a^2+b^2}$, the other bisector is

$$\left(a^2+b^2\right)(lx+my+n)-2(al+bm)(ax+by+c)=0.]$$

18. Show that the pair of straight lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ form a rhombus with the lines $ax^2+2hxy+by^2=0$, if $h\left(g^2-f^2\right)=gf(a-b)$.

[*Hints.* In a rhombus diagonals are at right angle.]

19. Show that one of the bisectors of the angles between the pair of lines $ax^2+2hxy+by^2=0$ will pass through the point of intersection of the two lines $ax^2+2hxy+by^2+2gx+2fy+c=0$, if $h\left(g^2-f^2\right)=fg(a-b)$. [CH 2001]

[*Hints.* The coordinates of the point of intersection of the lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ are $\left(\frac{hf-bg}{ab-h^2}, \frac{hg-af}{ab-h^2}\right)$. The equation of the bisectors of the angles between the pair of lines $ax^2+2hxy+by^2=0$ is

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}. \tag{2}$$

The above point lies on (1),

$$\therefore \quad \frac{(hf-bg)^2-(hg-af)^2}{a-b}=\frac{(hf-bg)(hg-af)}{h}$$

$$\text{or,}\quad h\left(h^2f^2+b^2g^2-2bhfg-h^2g^2-a^2f^2+2ahfg\right)$$

$$=(a-b)\left(h^2fg-ahf^2-bhg^2+abfg\right)$$

$$\text{or,}\quad h^3\left(f^2-g^2\right)+2(a-b)fgh^2=(a-b)\left(h^2+ab\right)fg+abh\left(f^2-g^2\right)$$

$$\text{or,}\quad h\left(f^2-g^2\right)\left(h^2-ab\right)=(a-b)fg\left(ab-h^2\right)$$

$$\text{or,}\quad h\left(g^2-f^2\right)=fg(a-b). \quad \left[\because h^2-ab\neq 0\right].$$

20. Show that the lines joining the origin to the points common to $x^2+hxy-y^2+gx+fy=0$ and $fx-gy=\lambda$ are at right angle whatever λ may be.

21. Find the angle between the chords of the circle $x^2+y^2=ax+by$, obtained by joining the origin to the points of intersection of the straight line $\frac{x}{a}+\frac{y}{b}=1$ and the above circle.

22. If the angle between the lines joining the origin to the points of intersection of $lx + my = 1$ and $x^2 + y^2 = a^2$ be 45°, show that $4\left[a^2\left(l^2+m^2\right)-1\right] = \left[a^2\left(l^2+m^2\right)-2\right]^2$.

23. If the pair of straight lines joining the origin to the points of intersection of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $lx+my+n=0$ are perpendicular to each other, then show that $\frac{a^2+b^2}{l^2+m^2}=\frac{a^2b^2}{n^2}$. [CH 1993, 94]

[*Hints* The equation of the required pair of lines is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{lx+my}{-n}\right)^2$. If these lines are at right angles, then coefficient of x^2+ coefficient of $y^2=0$, i.e. $\frac{1}{a^2}-\frac{l^2}{n^2}+\frac{1}{b^2}-\frac{m^2}{n^2}=0$ or, $\frac{a^2+b^2}{l^2+m^2}=\frac{a^2b^2}{n^2}$.]

24. If the two pairs of straight lines $ax^2+2hxy+by^2=0$ and $a'x^2+2h'xy+b'y^2=0$ have one line in common and θ is the angle between the other two, show that

$$2\cot\theta=\frac{aa'}{ha'-h'a}+\frac{bb'}{h'b-hb'}.$$

[*Hints*. The given pairs have a common line. Therefore, $(ab'-a'b)^2=4(ha'-h'a)(bh'-b'h)$. (See Example 5.)

Let

$$y^2+\frac{2h}{b}xy+\frac{a}{b}x^2=(y-mx)(y-m_1x)$$

$$\text{and}\quad y^2+\frac{2h'}{b'}xy+\frac{a'}{b'}x^2=(y-mx)(y-m_2x).$$

Then

$$m+m_1=-\frac{2h}{b}, \tag{1}$$

$$mm_1=\frac{a}{b}, \tag{2}$$

$$m+m_2=-\frac{2h'}{b'}, \tag{3}$$

$$mm_2=\frac{a'}{b'}. \tag{4}$$

By (1) and (3)

$$m_1-m_2=2\left(\frac{h'}{b'}-\frac{h}{b}\right)=2\frac{h'b-hb'}{bb'}. \tag{5}$$

By (2) and (4)

$$\frac{m_1}{m_2}=\frac{ab'}{a'b}. \tag{6}$$

By (5) and (6)

$$m_2\left(\frac{ab'}{a'b}-1\right)=2\frac{h'b-hb'}{bb'}\quad\text{or,}\quad m_2=2\frac{a'}{b'}\frac{h'b-hb'}{ab'-a'b}$$

$$\text{and}\quad m_1=2\frac{a}{b}\frac{h'b-hb'}{ab'-a'b}.$$

Now

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{2\frac{h'b - hb'}{bb'}}{1 + 4\frac{aa'}{bb'}\frac{(h'b-hb')^2}{(ab'-a'b)^2}}$$

$$= \frac{2(h'b - hb')(ha' - h'a)}{bb'(ha' - h'a) + aa'(h'b - hb')}.$$

$$\therefore\ 2\cot\theta = \frac{aa'}{ha' - h'a} + \frac{bb'}{h'b - hb'}.]$$

25. Show that the distance from the origin to the ortho-centre of the triangle formed by the lines $\frac{x}{\alpha} + \frac{y}{\beta} = 1$ and $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{(a+b)\alpha\beta\left(\alpha^2 + \beta^2\right)^{1/2}}{a\alpha^2 - 2h\alpha\beta + b\beta^2}.$$

26. Show that the distance between the points of intersection of the straight line $x\cos\alpha + y\sin\alpha - p = 0$ with the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{2p\sqrt{h^2 - ab}}{b\cos^2\alpha - 2h\cos\alpha\sin\alpha + a\sin^2\alpha}.$$

27. Show that the distance between the parallel lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$. **[BH 1993, 95; CH 1994, 2007]**

28. A point moves so that the distance between the feet of the perpendiculars from it on the lines $ax^2 + 2hxy + by^2 = 0$ is a constant $2d$. Show that its locus is $\left(x^2 + y^2\right)\left(h^2 - ab\right) = d^2\left[(a-b)^2 + 4h^2\right]$. **[BH 1994]**

[*Hints.* Let OA and OB be the lines represented by $ax^2 + 2hxy + by^2 = 0$ and $P(\alpha, \beta)$ be the moving point. PA and PB are perpendiculars to OA and OB. O, A, P, B are concyclic with OP as a diameter. Taking $\angle AOB = \theta$ and $\angle OBA = \gamma$,

$$\frac{AB}{\sin\theta} = \frac{OA}{\sin\gamma} \quad \text{and} \quad \frac{OP}{\sin 90^\circ} = \frac{OA}{\sin\gamma}.$$

$$\therefore\ OP^2 = AB^2(1 + \cot^2\theta) \quad \text{or,} \quad \alpha^2 + \beta^2 = 4d^2\left\{1 + \frac{(a+b)^2}{4(h^2 - ab)}\right\}.]$$

29. Show that the equation of the line joining the feet of the perpendiculars from the point $(d, 0)$ on the lines $ax^2 + 2hxy + by^2 = 0$ is $(a-b)x + 2hy + bd = 0$. **[CH 1998]**

[*Hints.*

$$ax^2 + 2hxy + by^2 = 0. \tag{1}$$

The equation of the lines drawn perpendicular to the lines of (1) from $(d, 0)$ is

$$b(x-d)^2 - 2h(x-d)y + ay^2 = 0. \tag{2}$$

By (1) $\times\ a$ − (2) $\times\ b$, the result follows.]

30. If one of the lines given by the equation $ax^2 + 2hxy + by^2 = 0$ coincides with one of the lines given by $a'x^2 + 2h'xy + b'y^2 = 0$ and the other lines are perpendicular, then show that

$$\frac{ha'b'}{b'-a'} = \frac{h'ab}{b-a} = \frac{1}{2}\sqrt{(-aa'bb')}.$$

[*Hints.* We may assume that $by^2 + 2hxy + ax^2 = b(y - mx)(y - m_1x)$ and $b'y^2 + 2h'xy + a'x^2 = b'(y - mx)\left(y + \frac{1}{m_1}x\right)$.

Comparing the coefficients

$$m + m_1 = -\frac{2h}{b}, \tag{1}$$

$$mm_1 = \frac{a}{b}, \tag{2}$$

$$m - \frac{1}{m_1} = -\frac{2h'}{b'}, \tag{3}$$

$$\frac{m}{m_1} = -\frac{a'}{b'}. \tag{4}$$

From (2) and (4) $m = \sqrt{-\frac{aa'}{bb'}}$ and $m_1 = \pm\sqrt{-\frac{ab'}{a'b}}$ (+ and − signs for the second and first pair of lines respectively). Putting these values in (1) and (3) the result is obtained.]

31. If the lines $ax^2 - 2hxy + by^2 = 0$ form an equilateral triangle with the line $x\cos\alpha + y\sin\alpha = p$, show that

$$\frac{a}{1 - 2\cos 2\alpha} = \frac{h}{2\sin 2\alpha} = \frac{b}{1 + 2\cos 2\alpha}$$ [CH 2006; NH 2007]

[*Hints.* By the help of worked-out example 8, $ax^2 - 2hxy + by^2 = 0$ is identical with $\left(\cos^2\alpha - 3\sin^2\alpha\right)x^2 + 8\cos\alpha\sin\alpha\ xy + \left(\sin^2\alpha - 3\cos^2\alpha\right)y^2 = 0$ or, $(1 - 2\cos 2\alpha)x^2 - 4\sin 2\alpha\ xy + (1 + 2\cos 2\alpha)y^2 = 0$. Hence the result follows.]

32. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, prove that the area of the triangle formed by their bisectors and the axis of x is

$$\frac{\sqrt{(a-b)^2 + 4h^2}}{2h} \cdot \frac{ca - g^2}{ab - h^2}$$ [NH 08; CH 95, 04]

Hints. The equation of the bisectors is

$$\frac{(x-\alpha)^2 - (y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h}$$

where (α, β) is the point of intersection. Here

$$\alpha = \sqrt{\frac{f^2 - bc}{h^2 - ab}}, \beta = \sqrt{\frac{g^2 - ca}{h^2 - ab}}.$$

If the bisectors meet the x-axis at $(x_1, 0)$ and $(x_2, 0)$, then x_1 and x_2 are the roots of

$$\frac{(x-\alpha)^2-\beta^2}{a-b}=\frac{-\beta\,(x-\alpha)}{h}$$
$$\text{or,}\quad hx^2-\{2h\alpha-(a-b)\beta\}x+h\left(\alpha^2-\beta^2\right)-(a-b)\alpha\beta=0.$$

$$\text{Required area}=\frac{1}{2}\,(x_2-x_1)\,\beta=\frac{\sqrt{(a-b)^2+4h^2}}{2h}\cdot\frac{ca-g^2}{ab-h^2}.]$$

33. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, show that the area of the parallelogram formed by them and the pair of parallel lines through the origin is $\dfrac{c}{2\sqrt{h^2-ab}}$. [NH 2004, 06; CH 1999, 2002]

[*Hints.* Let $OACB$ be the parallelogram. The equation of OA and OB in pair form is $ax^2+2hxy+by^2=0$. If p_1 and p_2 are perpendicular distances from $C(\alpha,\beta)$ on OA and OB respectively and $\angle AOB=\theta$, then the required area $=2\Delta OAC=OA\times p_1=BC\times p_1=\dfrac{p_2}{\sin\theta}\times p_1$.

$$p_1p_2=\frac{a\alpha^2+2h\alpha\beta+b\beta^2}{\sqrt{(a-b)^2+4h^2}}\quad\text{and}\quad\sin\theta=\frac{2\sqrt{h^2-ab}}{\sqrt{(a-b)^2+4h^2}}.$$

Again $a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0$ and $g\alpha+f\beta+c=0$.

$\therefore\; a\alpha^2+2h\alpha\beta+b\beta^2=c$.

Thus the required area $=\dfrac{c}{2\sqrt{h^2-ab}}$.]

34. A parallelogram is formed by the lines $ax^2+2hxy+by^2=0$ and the lines through the point (p,q) parallel to them. Show that the equation of the diagonal which does not pass through the origin is $(2x-p)(ap+hq)+(2y-q)(bq+hp)=0$.

[*Hints.* The equation to the pair of lines through the point (p,q) and parallel to

$$ax^2+2hxy+by^2=0 \tag{1}$$
$$\text{is}\quad a(x-p)^2+2h(x-p)(y-q)+b(y-q)^2=0. \tag{2}$$

Subtracting (1) from (2), the required equation is obtained.]

35. If each of the equation $f_1(x,y)=ax^2+2hxy+by^2+2gx+2fy+c=0$ and $f_2(x,y)=ax^2+2hxy+by^2-2gx-2fy+c=0$ represents a pair of straight lines, prove that the area of the parallelogram enclosed by them is $\dfrac{2c}{\sqrt{h^2-ab}}$.

[BH 1991, 2008; CH 2009]

[*Hints.* Let (α,β) be the point of intersection of the lines of $f_1(x,y)=0$. Then $g\alpha+f\beta+c=0$. The diagonal not passing through (α,β) is f_1-

$f_2 = 0$, i.e. $gx + fy = 0$. It meets the lines of $f_1(x, y) = 0$ at the points $\left(\pm\frac{f}{\sqrt{h^2-ab}}, \mp\frac{g}{\sqrt{h^2-ab}}\right)$. This result is obtained by the condition of the pair of lines. The required area is twice the area of the triangle whose vertices are (α, β) and $\left(\pm\frac{f}{\sqrt{h^2-ab}}, \mp\frac{g}{\sqrt{h^2-ab}}\right)$.]

36. Show that the two of the lines represented by the equation $ay^4+bxy^3+cx^2y^2+dx^3y+ex^4=0$ will be at right angles, if $(b+d)(ad+be)+(a-e)^2(a+c+e)=0$.

[*Hints.* Assume that

$$ay^4+bxy^3+cx^2y^2+dx^3y+ex^4=\left(ay^2+qyx+rx^2\right)\left(y^2+pyx-x^2\right)$$

where p, q and r are constants. Equating the coefficients and then eliminating p, q and r, the result is obtained.]

ANSWERS

1. (a) $\frac{\pi}{2}$;
   (b) $\tan^{-1}\frac{4}{3}$;
   (c) $\frac{\pi}{2}-\alpha$;
   (d) 2α;
   (e) 2α;
   (f) $\frac{\pi}{3}$.

7. (a) $(-1,-1), \tan^{-1}\frac{2}{11}$;
   (b) $\left(-\frac{25}{23}, \frac{43}{23}\right), \tan^{-1}\frac{23}{2}$;
   (c) $(-3, 2), \frac{\pi}{4}$.

8. (a) 2;
   (b) ± 1;
   (c) 3;
   (d) 6;
   (e) 4.

9. (a) $\frac{6}{\sqrt{10}}$;
   (b) $\sqrt{13}$.

12. $3x^2+xy-2y^2-15x+10y=0$.

13. $6x-5y+14=0,\ 6x+5y-56=0$.

21. $\frac{\pi}{2}$.

Chapter 4

General Equation of Second Degree

4.10 Introduction

The general equation of second degree in x and y is usually written in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{1}$$

The curve represented by this equation is a *conic section* or simply a *conic*. The curve is also called a *second order curve*. The nature of the conic is determined by the quantities

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad D = ab - h^2 \quad \text{and} \quad P = a + b.$$

In case of rectangular coordinate axes the quantities Δ, D and P are invariants under any orthogonal transformation.

(i) If $\Delta = 0$, the equation (1) represents a pair of straight lines, which has been discussed in chapter 3.

(ii) If $a = b$ and $h = 0$, the equation represent a circle.

(iii) If $\Delta \neq 0$, the equation represents a proper conic. Here D determines the nature of the conic.

  (a) When $D = 0$, i.e. $ab = h^2$, the conic is a parabola. In this case, the second degree terms form a perfect square.

  (b) When $D > 0$, i.e. $ab > h^2$, the conic is an ellipse.

  (c) When $D < 0$, i.e. $ab < h^2$, the conic is a hyperbola. If $a + b = 0$, the conic is a rectangular hyperbola.

Conditions for proper conics

Let LM be the directrix, $S(\alpha, \beta)$ be the focus and $P(x, y)$ be any point on the conic whose eccentricity is e.

The equation of LM is $lx + my + n = 0$. If PM is perpendicular to the directrix, then by the definition of a conic

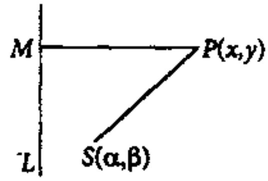


Fig. 29

$$SP^2 = e^2 PM^2$$

or, $$(x-\alpha)^2 + (y-\beta)^2 = e^2 \frac{(lx+my+n)^2}{l^2+m^2}$$

or, $$\left\{l^2\left(1-e^2\right)+m^2\right\}x^2 - 2lme^2xy + \left\{l^2+m^2\left(1-e^2\right)\right\}y^2$$
$$-2\left\{\left(l^2+m^2\right)\alpha + lne^2\right\}x - 2\left\{\left(l^2+m^2\right)\beta + mne^2\right\}y$$
$$+\left(l^2+m^2\right)\left(\alpha^2+\beta^2\right) - n^2e^2 = 0.$$

If this equation represents the equation (1), then we can write that

$$a = l^2\left(1-e^2\right)+m^2,\ b = l^2+m^2\left(l-e^2\right),\ h = -lme^2,$$
$$g = -\left\{\left(l^2+m^2\right)\alpha + lne^2\right\},\ f = -\left\{\left(l^2+m^2\right)\beta + mne^2\right\},$$
$$c = \left(l^2+m^2\right)\left(\alpha^2+\beta^2\right) - n^2e^2.$$

Now

$$D = ab - h^2 = \left\{l^2\left(l-e^2\right)+m^2\right\}\left\{l^2+m^2\left(1-e^2\right)\right\} - l^2m^2e^4$$
$$= \left(l^2+m^2\right)^2\left(1-e^2\right).$$

For the parabola, $e = 1$, i.e. $D = 0$.

For the ellipse, $e < 1$, i.e. $D > 0$.

For the hyperbola, $e > 1$, i.e. $D < 0$.

4.11 Canonical form

The general equation of a second degree can be reduced to the standard equation of a conic by suitable transformation of coordinates. This standard equation is also called the *canonical equation* or *normal canonical form* of the equation.

To find the canonical form from the general equation the following transformations are made successively.

(i) The term in xy is removed by suitable rotation of axes.

(ii) One or both (when possible) the terms in x and y are removed by translation.

(iii) The constant is removed if possible.

4.12 Reduction to canonical form (Tracing of conics)

Let the axes be rotated through an angle θ. The new coordinates are related as $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.

Substituting these values of x and y in the equation (1) of Sec 4.10 we have

$$\begin{aligned} &a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) \\ &\quad + b(x'\sin\theta + y'\cos\theta)^2 + 2g(x'\cos\theta - y'\sin\theta) \\ &\quad + 2f(x'\sin\theta + y'\cos\theta) + c = 0 \\ \text{or,}\quad &\left(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta\right)x'^2 \\ &\quad + 2\left\{h\left(\cos^2\theta - \sin^2\theta\right) - (a-b)\sin\theta\cos\theta\right\}x'y' \\ &\quad + (a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)y'^2 \\ &\quad + 2(g\cos\theta + f\sin\theta)x' + 2(f\cos\theta - g\sin\theta)y' + c = 0. \qquad (2) \end{aligned}$$

Let us choose θ in such a way that the coefficient of $x'y'$ in (2) will be zero. To satisfy this condition, we have

$$h\left(\cos^2\theta - \sin^2\theta\right) = (a-b)\sin\theta\cos\theta$$

$$\text{or,}\quad \tan 2\theta = \frac{2h}{a-b}, \quad \text{i.e.}\quad \theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}.$$

For this value of θ the equation (2) will be of the form

$$Ax'^2 + By'^2 + 2Gx' + 2Fy' + c = 0. \qquad (3)$$

By the property of invariants

$$\Delta = \begin{vmatrix} A & 0 & G \\ 0 & B & F \\ G & F & c \end{vmatrix}, \quad D = AB \quad \text{and} \quad P = A + B.$$

I. If $\Delta \neq 0$ but $D = 0$, then the equation (1) represents a parabola.

There are three possibilities (i) $A = 0, B = 0$.
(ii) $A = 0, B \neq 0$.
(iii) $A \neq 0, B = 0$.

If $A = 0$ and $B = 0$, then $\Delta = 0$. Possibility (i) is thus ruled out. For the possibility (ii) $\Delta \neq 0$, if $G \neq 0$.

In this case, the equation (3) reduces to

$$By'^2 + 2Gx' + 2Fy' + c = 0. \qquad (4)$$

It is a parabola having its axis parallel to the new x-axis.

From (4)

$$B\left(y'^2 + \frac{2F}{B}y'\right) = -2Gx' - c$$

$$\text{or,}\quad \left(y' + \frac{F}{B}\right)^2 = -\frac{2G}{B}\left(x' + \frac{Bc - F^2}{2GB}\right).$$

Changing the origin to

$$\left(-\frac{Bc-F^2}{2GB}, -\frac{F}{B}\right),$$

the equation reduces to

$$y''^2 = -\frac{2G}{B}x''. \tag{5}$$

It is the *canonical form* of the equation (1) when $\Delta \neq 0$ but $D = 0$.

For the possibility (iii) $\Delta \neq 0$, if $F \neq 0$.

In this case, the equation (3) reduces to

$$Ax'^2 + 2Gx' + 2Fy' + c = 0 \tag{6}$$

It is a parabola having its axis parallel to the y'-axis i.e. new y-axis.

From (6)

$$x'^2 + \frac{2G}{A}x' = -\frac{2F}{A}y' - \frac{c}{A} \quad \text{or,} \quad \left(x' + \frac{G}{A}\right)^2 = -\frac{2F}{A}\left(y' + \frac{cA - G^2}{2FA}\right).$$

Changing the origin to

$$\left(-\frac{G}{A}, -\frac{cA - G^2}{2FA}\right),$$

the equation reduces to

$$x''^2 = -\frac{2F}{A}y''. \tag{7}$$

It is the *canonical equation* of the equation (1).

Note

1. Since $h^2 = ab$, the terms of second degree in the equation (1) form a perfect square.
2. If $A = 0 = B$, then $a = 0 = b = h$ and the equation (1) represents a line.
3. If $A = 0 = G$ but $B \neq 0$, then the equation (3) reduces to

$$By'^2 + 2Fy' + c = 0 \quad \text{or,} \quad \left(y' + \frac{F}{B}\right)^2 = \frac{F^2 - Bc}{B^2}.$$

   Here the equation (1) represents a pair of parallel lines, a pair of coincident lines or no geometric locus according as $F^2 - Bc >=< 0$.

4. If $B = 0 = F$ but $A \neq 0$, then the equation (3) reduces to

$$Ax'^2 + 2Gx' + c = 0 \quad \text{or,} \quad \left(x' + \frac{G}{A}\right)^2 = \frac{G^2 - cA}{A^2}.$$

   Here the equation (1) represents a pair of parallel lines, a pair of coincident lines or no geometric locus according as $G^2 - cA >=< 0$.

Example 1. *Discuss the nature of the conic represented by*

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$$

and reduce it to the canonical form.

Here $a = 9, h = -12, b = 16, g = -9, f = -\frac{101}{2}, c = 19,$

$$\Delta = \begin{vmatrix} 9 & -12 & -9 \\ -12 & 16 & -\frac{101}{2} \\ -9 & -\frac{101}{2} & 19 \end{vmatrix} \neq 0, D = 9 \cdot 16 - (-12)^2 = 0.$$

Therefore, the given equation represents a parabola.

Let the axes be rotated through an acute angle θ. The equation transforms to

$$\begin{aligned} & 9(x'\cos\theta - y'\sin\theta)^2 - 24(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) \\ & \quad + 16(x'\sin\theta + y'\cos\theta)^2 - 18(x'\cos\theta - y'\sin\theta) \\ & \quad - 101(x'\sin\theta + y'\cos\theta) + 19 = 0 \\ \text{or,} \quad & \left(9\cos^2\theta - 24\sin\theta\cos\theta + 16\sin^2\theta\right)x'^2 \\ & \quad - 2\{12(\cos^2\theta - \sin^2\theta) - 7\sin\theta\cos\theta\}x'y' \\ & \quad + \left(16\cos^2\theta + 24\sin\theta\cos\theta + 9\sin^2\theta\right)y'^2 - (18\cos\theta + 101\sin\theta)x' \\ & \quad + (18\sin\theta - 101\cos\theta)y' + 19 = 0 \\ \text{or,} \quad & (3\cos\theta - 4\sin\theta)^2 x'^2 - 2\left\{12\left(\cos^2\theta - \sin^2\theta\right) - 7\sin\theta\cos\theta\right\}x'y' \\ & \quad + (4\cos\theta + 3\sin\theta)^2 y'^2 - (18\cos\theta + 101\sin\theta)x' \\ & \quad + (18\sin\theta - 101\cos\theta)y' + 19 = 0. \end{aligned}$$

Let us choose θ in such a way that

$$12\left(\cos^2\theta - \sin^2\theta\right) - 7\sin\theta\cos\theta = 0.$$

From this

$$12\tan^2\theta + 7\tan\theta - 12 = 0 \quad \text{or,} \quad \tan\theta = \frac{3}{4}, -\frac{4}{3}.$$

Since θ is acute,

$$\tan\theta = \frac{3}{4}.$$

Hence

$$\sin\theta = \frac{3}{5} \quad \text{and} \quad \cos\theta = \frac{4}{5}.$$

For these values of $\cos\theta$ and $\sin\theta$ the equation reduces to

$$25y'^2 - 75x' - 70y' + 19 = 0$$

$$\text{or,} \quad \left(y' - \frac{7}{5}\right)^2 = 3\left(x' + \frac{2}{5}\right).$$

Changing the origin to $\left(-\frac{2}{5}, \frac{7}{5}\right)$, the equation further reduces to $y''^2 = 3x''$. It is the required canonical form.

The parabola can be traced now as shown in Fig. 30. In the figure, $\theta = \tan^{-1}\frac{3}{4}$; vertex A is at $\left(-\frac{2}{5}, \frac{7}{5}\right)$ w.r.t. OX', OY' and $AS = \frac{3}{4}$, S being the focus of the parabola.

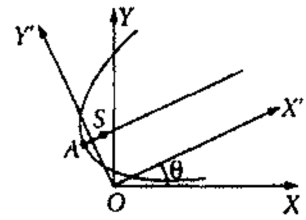


Fig. 30

II The equation (1) represents an ellipse when $\Delta \neq 0$ but $D > 0$

We have $D = AB$. If $D > 0$, none of A and B is zero and both of them are positive or negative. Without any loss of generality we may assume that both of A and B are positive.

From (3)

$$A\left(x' + \frac{G}{A}\right)^2 + B\left(y' + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - c = k \text{ (say)}.$$

By translation

$$x' = x'' - \frac{G}{A}, \quad y' = y'' - \frac{F}{B},$$

the equation reduces to

$$Ax''^2 + By''^2 = k \quad \text{or,} \quad \frac{x''^2}{k/A} + \frac{y''^2}{k/B} = 1. \tag{8}$$

It is the equation of the ellipse in the canonical form.

Note

1. The centre of the ellipse represented by the equation (1) is

$$\left(-\frac{G}{A}\cos\theta + \frac{F}{B}\sin\theta, -\frac{G}{A}\sin\theta - \frac{F}{B}\cos\theta\right).$$

2. The equation (8) represents a real ellipse, a point (null) ellipse or an imaginary (without any real trace) ellipse according as $k >=< 0$.

3. $\Delta = -ABk$. It is $<=> 0$ according as the equation (8) represents a real ellipse, a point ellipse or an imaginary ellipse.

Example 2. *Reduce the equation $3x^2 + 2xy + 3y^2 - 16x + 20 = 0$ to normal form.*

Here $a = 3, h = 1, b = 3, g = -8, f = 0, c = 20$.

$$\Delta = \begin{vmatrix} 3 & 1 & -8 \\ 1 & 3 & 0 \\ -8 & 0 & 20 \end{vmatrix} = -32 \quad \text{and} \quad D = 3 \cdot 3 - 1^2 = 8 > 0.$$

Therefore, the given equation represents an ellipse.

By rotating the axes through an acute angle θ, the equation transforms to

$$\begin{aligned} &3(x'\cos\theta - y'\sin\theta)^2 + 2(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) \\ &\quad + 3(x'\sin\theta + y'\cos\theta)^2 - 16(x'\sin\theta - y'\cos\theta) + 20 = 0 \\ \text{or,}\quad &(3+\sin 2\theta)x'^2 + 2\cos 2\theta x'y' + (3-\sin 2\theta)y'^2 - 16\sin\theta x' \\ &\quad + 16\cos\theta y' + 20 = 0. \end{aligned}$$

θ is chosen in such a way that $\cos 2\theta = 0, 2\theta = 90°$ or, $\theta = 45°$.

For this value of θ, the equation takes the form

$$\begin{aligned} &4x'^2 + 2y'^2 - 8\sqrt{2}y' + 20 = 0 \\ \text{or,}\quad &4\left(x' - \sqrt{2}\right)^2 + 2\left(y' + 2\sqrt{2}\right)^2 = 4. \end{aligned}$$

Changing the origin to $\left(\sqrt{2}, -2\sqrt{2}\right)$, the equation reduces to

$$4x''^2 + 2y''^2 = 4 \quad \text{or,} \quad x''^2 + y''^2/2 = 1.$$

It is the required normal form. The conic is an ellipse with semi-axes 1 and $\sqrt{2}$.

The ellipse can be traced now as shown in Fig. 31. In the figure $\theta = 45°$, the centre C is at $\left(\sqrt{2}, -2\sqrt{2}\right)$ w.r.t. OX', OY'.

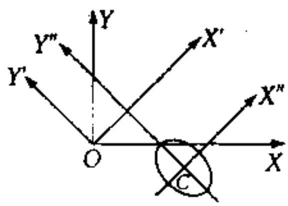


Fig. 31

(III) The equation (1) represents a hyperbola when $\Delta \neq 0$ but $D < 0$.

If $D < 0$, then $AB < 0$. Consequently none of A and B is zero. Moreover, A and B have opposite signs. Without any loss of generality we may assume that $A > 0$ and $B < 0$. Proceeding as (II) the equation (3) reduces to

$$\frac{x''^2}{k/A} + \frac{y''^2}{k/B} = 1 \text{ when } k \text{ is not zero.}$$

If $k > 0$, then it can be written as

$$\frac{x''^2}{\alpha^2} - \frac{y''^2}{\beta^2} = 1. \tag{9}$$

It is the equation of the hyperbola in the canonical form.

If $k < 0$, then the equation can be written as

$$\frac{x''^2}{\alpha^2} - \frac{y''^2}{\beta^2} = -1. \tag{10}$$

It is also the canonical form.

Note

1. Hyperbolas represented by (9) and (10) are conjugate to each other.
2. The centre of the hyperbola represented by the equation (1) is
$$\left(-\frac{G}{A}\cos\theta+\frac{F}{B}\sin\theta,-\frac{G}{A}\sin\theta-\frac{F}{B}\cos\theta\right).$$
3. If $k=0$, the equation (3) reduces to $Ax''^2+By''^2=0$. It represents a pair of straight lines which are asymptotes to the hyperbolas represented by (9) and (10).
4. If $\alpha^2=\beta^2$, the hyperbola is rectangular.
5. Here $\Delta=-ABk$. It may be $<$ or >0 for the real hyperbola.

Example 3. *Show that the equation*

$$7x^2-48xy-7y^2-20x+140y+300=0$$

represents a hyperbola and find its canonical equation.

Here $a=7, h=-24, b=-7, g=-10, f=70, c=300$,

$$\Delta=\begin{vmatrix}7 & -24 & -10\\ -24 & -7 & 70\\ -10 & 70 & 300\end{vmatrix}\neq 0, D=7\cdot(-7)-(-24)^2<0.$$

Therefore, the given equation represents a hyperbola.

Rotating the axes through an acute angle θ, the equation transforms to

$$\begin{aligned}&7(x'\cos\theta-y'\sin\theta)^2-48(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)\\&\quad-7(x'\sin\theta+y'\cos\theta)^2-20(x'\cos\theta-y'\sin\theta)\\&\quad+140(x'\sin\theta+y'\cos\theta)+300=0\\\text{or,}\quad&\left(7\cos^2\theta-7\sin^2\theta-48\sin\theta\cos\theta\right)x'^2\\&\quad-\left\{48\left(\cos^2\theta-\sin^2\theta\right)+28\sin\theta\cos\theta\right\}x'y'\\&\quad-\left(7\cos^2\theta-7\sin^2\theta-48\sin\theta\cos\theta\right)y'^2\\&\quad-20(\cos\theta-7\sin\theta)x'+20(\sin\theta+7\cos\theta)y'+300=0.\end{aligned}$$

To make the coefficient of $x'y'$ zero,

$$\begin{aligned}&48\left(\cos^2\theta-\sin^2\theta\right)+28\sin\theta\cos\theta=0\\\text{or,}\quad&12\tan^2\theta-7\tan\theta-12=0\\\text{or,}\quad&\tan\theta=\frac{4}{3},-\frac{3}{4}.\end{aligned}$$

Taking

$$\tan\theta = \frac{4}{3}, \quad \sin\theta = \frac{4}{5} \quad \text{and} \quad \cos\theta = \frac{3}{5}.$$

For these values of $\sin\theta$ and $\cos\theta$, the above equation takes the form

$$y'^2 - x'^2 + 4x' + 4y' + 12 = 0$$
$$\text{or,} \quad (x'-2)^2 - (y'+2)^2 = 12.$$

By translation,

$$x' = x'' + 2, y' = y'' - 2,$$

the equation reduces to

$$x''^2 - y''^2 = 12.$$

It is the canonical equation of the hyperbola which is a rectangular one in this case. The rectangular hyperbola can be traced now as shown in Fig. 32. In the figure $\theta = \tan^{-1}\frac{4}{3}$, the centre C is at $(2,-2)$ w.r.t. OX', OY'.

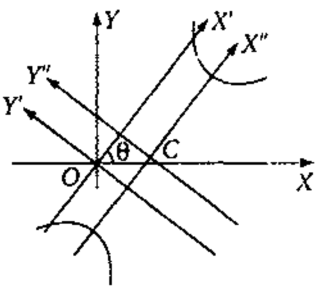


Fig. 32

4.20 Rank and classification of a second order curve

The rank of a second order curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is the rank of the matrix

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.$$

A second order curve is classified as *non-singular* or *non-degenerate* and *singular* or *degenerate* according as its rank is 3 and 2 or 1 respectively. A circle, a parabola, an ellipse and a hyperbola are non-singular. A point ellipse, a pair of intersecting lines, a pair of parallel lines and a pair of coincident lines are singular or degenerate. The rank of a pair of coincident lines is one.

Note. The general second degree equation represents (i) a degenerate conic if $\Delta = 0$ and (ii) a non-degenerate conic if $\Delta \neq 0$. The non-degenerate conics are mainly divided into three classes: (a) *elliptic* if $D > 0$, (b) *parabolic* if $D = 0$ and (c) *hyperbolic* if $D < 0$.

Circle is a special case of the ellipse when the major and minor axes are equal.

4.21 Table for metric classification

Δ	D	Canonical form	Name	Rank	Class
$\Delta < 0$	$D > 0$	$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$	ellipse	3	non-singular
$\Delta < 0$	$D > 0$	$x^2 + y^2 = \alpha^2$	circle	3	non-singular
$\Delta > 0$	$D > 0$	$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = -1$	imaginary ellipse	3	non-singular
$\Delta > 0$	$D < 0$	$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$	hyperbola	3	non-singular
$\Delta < 0$	$D < 0$	$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = -1$	hyperbola	3	non-singular
$\Delta \neq 0$	$D = 0$	$y^2 = 4\alpha x$ $x^2 = 4\beta y$	parabola	3	non-singular
$\Delta = 0$	$D > 0$	$Ax^2 + By^2 = 0$	pair of imaginary lines or point ellipse	2	singular
$\Delta = 0$	$D < 0$	$y^2 - k^2x^2 = 0$	pair of intersecting lines	2	*singular*
$\Delta = 0$	$D = 0$	$y^2 = k^2$ $x^2 = l^2$	pair of parallel lines	2	singular
$\Delta = 0$	$D = 0$	$y^2 = 0$ $x^2 = 0$	pair of coincident lines	1	singular

4.30 Centre of the central conic

Definition. If any chord of a conic through a particular point is bisected by that point, then the conic is said to be *central* and that particular point is called the *centre* of this conic. If the origin is the centre, the extremities of the chord through the origin will be of the form (x, y) and $(-x, -y)$. Thus if the point (x, y) lies on the conic, the point $(-x, -y)$ will also lie on the conic. Consequently there will be no term of odd degree in the equation of the conic.

To determine the centre: Let (α, β) be the centre of the central conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{1}$$

Changing the origin to (α, β) with the axes remaining parallel, the equation transforms to

$$a(x' + \alpha)^2 + 2h(x' + \alpha)(y' + \beta) + b(y' + \beta)^2 + 2g(x' + \alpha) + 2f(y' + \beta) + c = 0$$

or, $$ax'^2 + 2hx'y' + by'^2 + 2(a\alpha + h\beta + g)x' + 2(h\alpha + b\beta + f)y'$$
$$+ \left(a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c\right) = 0. \tag{2}$$

Since (α, β) is the centre, the coefficients of x' and y' must vanish.

$$\therefore \quad a\alpha + h\beta + g = 0 \tag{3}$$

$$\text{and} \quad h\alpha + b\beta + f = 0. \tag{4}$$

From (3) and (4),

$$\alpha = \frac{fh - bg}{ab - h^2}, \; \beta = \frac{gh - af}{ab - h^2}.$$

Thus (α, β) are determined.

$$\begin{aligned}\text{The constant term in (2)} &= a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c\\ &= \alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f) + g\alpha + f\beta + c\\ &= g\alpha + f\beta + c \quad \text{[by (3) and (4)]}\\ &= \frac{g(fh - bg) + f(gh - af) + c\left(ab - h^2\right)}{ab - h^2}\\ &= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = \frac{\Delta}{D}.\end{aligned}$$

Thus the equation (2) can be written as

$$ax'^2 + 2hx'y' + by'^2 + \frac{\Delta}{D} = 0.$$

Note

1. If $S = 0$ is the equation of the conic, then the centre is the point of intersection of the lines $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial y} = 0$.
2. **Non-central conic.** If $ab - h^2 = 0$, $fh - bg \neq 0$ and $gh - af \neq 0$, then the conic represented by the equation (1) is non-central i.e. it has no centre.
3. **Infinitely many-central conic.** If $ab - h^2 = 0$, $fh - bg = 0$ and $gh - af = 0$, then the equation (1) represents an infinitely many-central conic i.e. it has infinitely many centres.
4. For the reduction of a central conic the rotation after translation to the centre is advantageous.

4.40 Common points of two conics

(i) **Through five given points one conic can be drawn in general.**

The general equation

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

contains six constants, namely a, h, b, g, f, c. Dividing by one this number reduces to five and these five constants are independent. Again five linear equations containing these constants are formed by the help of five given points. If these equations are independent, the constants are determined uniquely and only one conic is drawn through five given points.

If three points are collinear, the conic through the five given points is a pair of straight lines, for no straight line can meet a proper conic in three points. If the four points are collinear, the conic will not be unique.

(ii) **Two real conics cut in four points in general.**

Let the conics be $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

and $S' = a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0.$

Eliminating y from these two, a bi-quadratic equation in x is obtained. This equation gives four values of x and these values correspond to four values of y. Thus two conics cut in four points in general. Some of them may be coincident and two or four of them may be imaginary.

Note

1. **Contact of zero order.** If the two conic intersect in four distinct points, then the contact is said to be of zero order.

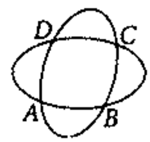


Fig. 33

2. **Contact of first order.** If two points of intersection coincide, then the contact is said to be first order. If contacts of first order take place at two different points, then the conics are said to have *double contact*.

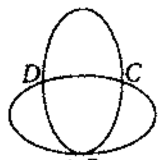


Fig. 34

C,D

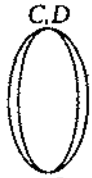


Fig. 35

3. **Contact of second order.** If three points of intersection coincide, then the contact is said to be of second order.

Fig. 36

4. **Contact of third order.** If four points of intersection coincide, then the contact is said to be of third order. In this case, the smaller conic is called *osculating curve*.

A,B,C,D

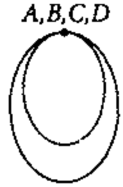


Fig. 37

(iii) **Conic through the common points of two conics.**

If $S = 0$ and $S' = 0$ are two conics, then $S + \lambda S' = 0$ is the equation of a conic through the common points of $S = 0$ and $S' = 0$. λ is an arbitrary constant. It is found out by the given condition.

(iv) **Conic through the common points of a conic and two lines.**

If $S = 0$ is the conic and the lines are $u = 0$ and $v = 0$, then the equation of the required conic is $S + \lambda uv = 0$. λ is an arbitrary constant. It is determined by the given condition.

Example 4. *Find the equation of the conic which passes through the point* $(-1, -1)$ *and also through the intersections of the conic* $25x^2-14xy+25y^2+64x-64y-228 = 0$ *with the lines* $x + 3y - 2 = 0$ *and* $3x + y - 4 = 0$. *Find also the parabolas passing through the same points of intersection.*

The equation of the conic is of the form

$$25x^2 - 14xy + 25y^2 + 64x - 64y - 228 + \lambda(x + 3y - 2)(3x + y - 4) = 0. \qquad (2)$$

If it passes through the point $(-1, -1)$, then

$$25 - 14 + 25 - 64 + 64 - 228 + \lambda(-1 - 3 - 2)(-3 - 1 - 4) = 0 \quad \text{i.e.} \quad \lambda = 4.$$

$\therefore$ the required equation is

$$37x^2 + 26xy + 37y^2 + 24x - 120y - 196 = 0.$$

The equation of the parabola will be also of the form (1), i.e.

$$(25 + 3\lambda)x^2 + (10\lambda - 14)xy + (25 + 3\lambda)y^2 + (64 - 10\lambda)x$$
$$- (64 + 14\lambda)y + 8\lambda - 228 = 0.$$

It represents a parabola, if

$$(5\lambda - 7)^2 - (25 + 3\lambda)^2 = 0$$
$$\text{or,} \quad (5\lambda - 7 + 25 + 3\lambda)(5\lambda - 7 - 25 - 3\lambda) = 0 \quad \text{or,} \quad \lambda = -\frac{9}{4}, 16.$$

For $\lambda = -\frac{9}{4}$, the equation (1) takes the form

$$73x^2 - 146xy + 73y^2 + 346x - 130y - 984 = 0.$$

For $\lambda = 16$, the equation (1) takes the form

$$73x^2 + 146xy + 73y^2 - 96x - 288y - 100 = 0.$$

These are the required equations of parabolas.

WORKED-OUT EXAMPLES

1. *Discuss the nature of the conic represented by*

$$16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0. \qquad \text{[BH 2003; CH 2006]}$$

Here

$$\Delta = \begin{vmatrix} 16 & -12 & -52 \\ -12 & 9 & -86 \\ -52 & -86 & 44 \end{vmatrix} \neq 0.$$

Therefore, the equation does not represent a pair of straight lines. The given equation can be written as

$$(4x - 3y)^2 = 104x + 172y - 44. \tag{1}$$

Since the term of second degree are a perfect square the equation represents a parabola. Let us write the equation in the following way

$$(4x - 3y + \lambda)^2 = 104x + 172y - 44 + 2\lambda(4x - 3y) + \lambda^2$$
$$\text{or,} \quad (4x - 3y + \lambda)^2 = (104 + 8\lambda)x + (172 - 6\lambda)y + \lambda^2 - 44. \tag{2}$$

The constant λ is chosen in such a way that the lines

$$4x - 3y + \lambda = 0$$
$$\text{and} \quad (104 + 8\lambda)x + (172 - 6\lambda)y + \lambda^2 - 44 = 0$$

will be at right angle.

For this

$$4 \cdot (104 + 8\lambda) - 3 \cdot (172 - 6\lambda) = 0 \quad \text{or,} \quad \lambda = 2.$$

Putting $\lambda = 2$ in (2),

$$(4x - 3y + 2)^2 = 40(3x + 4y - 1)$$
$$\text{or,} \quad \left(\frac{4x - 3y + 2}{\sqrt{4^2 + 3^2}}\right)^2 = \frac{40}{\sqrt{4^2 + 3^2}} \cdot \frac{3x + 4y - 1}{\sqrt{3^2 + 4^2}}. \tag{3}$$

Therefore, the axis of the parabola is $4x - 3y + 2 = 0$ and the tangent at the vertex is $3x + 4y - 1 = 0$.

The length of latus rectum $= \frac{40}{\sqrt{4^2+3^2}} = 8$.

Vertex is the point of intersection of the lines

$$4x - 3y + 2 = 0$$
$$\text{and} \quad 3x + 4y - 1 = 0.$$

Thus the vertex is $\left(-\frac{1}{5}, \frac{2}{5}\right)$.

Writing the equation (3) in the form $Y^2 = 4AX$, where

$$A = 2, Y = \frac{4x - 3y + 2}{5}, X = \frac{3x + 4y - 1}{5}$$

the focus is $(A, 0)$, i.e. $X = A$ and $Y = 0$

$$\text{or,} \quad \frac{3x + 4y - 1}{5} = 2 \quad \text{and} \quad \frac{4x - 3y + 2}{5} = 0$$

$$\text{or,} \quad 3x + 4y - 11 = 0 \quad \text{and} \quad 4x - 3y + 2 = 0$$

Solving these two we get the focus as $(1, 2)$.

Equation of the latus rectum is $X = A$

$$\text{or,} \quad \frac{3x + 4y - 1}{5} = 2 \quad \text{or,} \quad 3x + 4y - 11 = 0.$$

Equation of the directrix is $X + A = 0$ or, $3x + 4y + 9 = 0$.

The canonical form of the equation is $Y^2 = 8X$.

Note

1. The canonical form will be $Y^2 = \lambda X$ or $X^2 = \mu Y$ according as the coefficient of $xy <$ or, > 0.
2. In case of parabola the technique adopted in the worked-out example 1 is worthwhile.

2. *Discuss the nature of the conic represented by*

$$11x^2 - 4xy + 14y^2 - 58x - 44y + 71 = 0. \qquad \text{[BH 2002; CH 2004]}$$

Here

$$\Delta = \begin{vmatrix} 11 & -2 & -29 \\ -2 & 14 & -22 \\ -29 & -22 & 71 \end{vmatrix} = -9000.$$

$\because$ $\Delta \neq 0$, the equation does not represent a pair of straight lines.

Again

$$D = 11 \times 14 - 2^2 = 150 > 0.$$

$\therefore$ the equation represents an ellipse.

The centre of the ellipse is obtained from the equations

$$11\alpha - 2\beta - 29 = 0,$$
$$2\alpha - 14\beta + 22 = 0.$$

Solving these two equations, the centre is $(3, 2)$.

Shifting the origin to $(3, 2)$, the equation reduces to

$$\begin{aligned} & 11x^2 - 4xy + 14y^2 - \frac{9000}{150} = 0 \\ \text{or,} \quad & 11x^2 - 4xy + 14y^2 = 60. \end{aligned} \tag{1}$$

Rotating the axes through an angle θ where

$$\tan 2\theta = \frac{2 \cdot (-2)}{11 - 14} = \frac{4}{3},$$

the equation (1) reduces to

$$\begin{aligned} & 11(x\cos\theta - y\sin\theta)^2 - 4(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) \\ & \qquad + 14(x\sin\theta + y\cos\theta)^2 = 60 \\ \text{or,} \quad & \left(11\cos^2\theta - 4\sin\theta\cos\theta + 14\sin^2\theta\right)x^2 \\ & \qquad + \left(11\sin^2\theta + 4\sin\theta\cos\theta + 14\cos^2\theta\right)y^2 = 60 \\ \text{or,} \quad & \left\{11 + \frac{3}{2}(1 - \cos 2\theta) - 2\sin 2\theta\right\}x^2 \\ & \qquad + \left\{11 + \frac{3}{2}(1 + \cos 2\theta) + 2\sin 2\theta\right\}y^2 = 60 \\ \text{or,} \quad & \left(11 + \frac{3}{2} - \frac{3}{2} \cdot \frac{3}{5} - 2 \cdot \frac{4}{5}\right)x^2 + \left(11 + \frac{3}{2} + \frac{3}{2} \cdot \frac{3}{5} + 2 \cdot \frac{4}{5}\right)y^2 = 60 \\ \text{or,} \quad & 10x^2 + 15y^2 = 60 \\ \text{or,} \quad & \frac{x^2}{6} + \frac{y^2}{4} = 1. \end{aligned}$$

It is the canonical form. The semi-axes are $\sqrt{6}$ and 2.

[The equation of the axes are

$$\begin{aligned} & (x - 3)\cos\theta + (y - 2)\sin\theta = 0 \\ \text{and} \quad & (y - 2)\cos\theta - (x - 3)\sin\theta = 0.] \end{aligned}$$

3. *Discuss the nature of the conic represented by*

$$3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0. \qquad \text{[CH 2001]}$$

Here

$$\begin{aligned} \Delta &= 3 \cdot (-3) \cdot 8 + 2 \cdot \left(-\frac{13}{2}\right) \cdot 5 \cdot (-4) - 3 \cdot \left(-\frac{13}{2}\right)^2 - (-3) \cdot 5^2 - 8 \cdot (-4)^2 \\ &= \frac{33}{4} \neq 0. \end{aligned}$$

$$D = 3 \cdot (-3) - (-4)^2 = -9 - 16 = -25 < 0.$$

$\therefore$ the equation represents a hyperbola.

Again coefficient of x^2+ coefficient of $y^2 = 3 - 3 = 0$.

$\therefore$ the hyperbola is rectangular.

Solving the equations $3\alpha - 4\beta + 5 = 0$ and $8\alpha + 6\beta + 13 = 0$, we have the centre $\left(-\frac{41}{25}, \frac{1}{50}\right)$.

Changing the origin to the centre the equation reduces to

$$3x^2 - 8xy - 3y^2 + \frac{\Delta}{D} = 0$$

$$\text{or,}\quad 3x^2 - 8xy - 3y^2 - \frac{33}{100} = 0. \tag{1}$$

The axes are rotated through an angle θ where

$$\tan 2\theta = \frac{-2 \cdot 4}{3-(-3)} = -\frac{4}{3}.$$

If the equation (1) transforms to

$$Ax^2 + By^2 = \frac{33}{100}$$

then $A + B = 0$ and $AB = -25$ by the property of invariants.

$$\therefore\ A = \pm 5, B = \mp 5.$$

Hence the canonical form is either

$$x^2 - y^2 = \frac{33}{500} \quad \text{or,} \quad x^2 - y^2 = -\frac{33}{500}.$$

Here the semi-axis is $\sqrt{\frac{33}{500}}$.

Note. If θ is acute, then the canonical form is $x^2 - y^2 = -\frac{33}{500}$.

4. *Reduce the equation $4x^2 + 4xy + y^2 - 4x - 2y + a = 0$ to the canonical form and determine the type of the conic represented by it for different values of a.*

Here

$$\Delta = \begin{vmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & a \end{vmatrix} = 2 \begin{vmatrix} 2 & 2 & -2 \\ 1 & 1 & -1 \\ -1 & -1 & a \end{vmatrix} = 0$$

($\because$ two column are identical.)

Therefore, the equation represents a pair of straight lines.

The equation can be written as

$$(2x+y)^2 - 2(2x+y) + a = 0$$

$$\text{or,}\quad (2x+y-1)^2 + a - 1 = 0$$

$$\text{or,}\quad \left(\frac{2x+y-1}{\sqrt{5}}\right)^2 + \frac{a-1}{5} = 0.$$

Putting

$$\frac{2x+y-1}{\sqrt{5}} = X,$$

the equation reduces to

$$X^2 + \frac{a-1}{5} = 0. \tag{1}$$

It is the canonical form of the given equation.

If $a < 1$, the given equation represents two parallel lines.

If $a = 1$, the equation (1) reduces to $x^2 = 0$ and the given equation represents two coincident lines.

If $a > 1$, the given equation represents two imaginary lines.

5. *Determine the values of h and g so that the equation $x^2 - 2hxy + 4y^2 + 2gx - 12y + 9 = 0$ may represent* (i) *a conic having no centre,* (ii) *a conic having infinitely many centres. In the last case, find the type of the conic.*

The centre of the conic lies on $x - hy + g = 0$ and $-hx + 4y - 6 = 0$.

By cross-multiplication

$$\frac{x}{6h-4g} = \frac{y}{-gh+6} = \frac{1}{4-h^2}.$$

(i) The given equation will represent a conic having no centre if $4 - h^2 = 0$, $6h - 4g \neq 0$ and $6 - gh \neq 0$.

From $4 - h^2 = 0, h = \pm 2$.

Again $6 - gh \neq 0$ or, $6 \mp 2g \neq 0$ or, $g \neq \pm 3$.

(ii) The given equation will represent a conic having infinitely many centres if $4 - h^2 = 0, 6h - 4g = 0$ and $6 - gh = 0$.

From $4 - h^2 = 0, h = \pm 2$.

For $h = \pm 2, g = \pm 3$.

For these values of h and g, the equation of the conic is

$$x^2 \mp 4xy + 4y^2 \pm 6x - 12y + 9 = 0$$
$$\text{or,} \quad (x \mp 2y \pm 3)^2 = 0.$$

Therefore, the given equation represents a pair of coincident lines.

6. *If the equation $13x^2 + 10xy + by^2 - 62\sqrt{2}x - 46\sqrt{2}y + 170 = 0$ represents a point ellipse, find the value of b and the point.*

Here

$$\Delta = \begin{vmatrix} 13 & 5 & -31\sqrt{2} \\ 5 & b & -23\sqrt{2} \\ -31\sqrt{2} & -23\sqrt{2} & 170 \end{vmatrix} = 0$$

$$\text{or,} \quad (2210b - 13754) + (7130 - 4250) + (7130 - 1922b) = 0$$

$$\text{or,} \quad 288b - 3744 = 0 \quad \text{or,} \quad b = 13.$$

Again $D = 13b - 25 = 169 - 25 > 0$.

$\therefore$ the equation represents a point ellipse when $b = 13$.

The point is the centre. If (α, β) is the centre, then $13\alpha + 5\beta - 31\sqrt{2} = 0$ and $5\alpha + 13\beta - 23\sqrt{2} = 0$.

By cross-multiplication,

$$\frac{\alpha}{288\sqrt{2}} = \frac{\beta}{144\sqrt{2}} = \frac{1}{144} \quad \text{or,} \quad \alpha = 2\sqrt{2} \quad \text{and} \quad \beta = \sqrt{2}.$$

Thus the point is $\left(2\sqrt{2}, \sqrt{2}\right)$.

7. *Find the equation of the conic passing through* $(0,0), (2,3), (0,3), (2,5)$ *and* $(4,5)$.

The equation of the line passing through $(2,3)$ and $(0,3)$ is $y - 3 = 0$.

The equation of the line passing through $(2,5)$ and $(4,5)$ is $y - 5 = 0$.

The pair of lines passing through the four points is $(y-3)(y-5) = 0$.

The equation of the line passing through $(2,3)$ and $(2,5)$ is $x - 2 = 0$.

The equation of the line passing through $(0,3)$ and $(4,5)$ is $x - 2y + 6 = 0$.

The equation of the pair of lines through the same four points is also

$$(x-2)(x-2y+6) = 0.$$

Let the equation of the conic passing through these four points be

$$(y-3)(y-5) + \lambda(x-2)(x-2y+6) = 0.$$

If it passes through $(0,0)$, then $\lambda = \frac{5}{4}$.

Therefore, the equation of the conic passing through the given five points is

$$\begin{aligned} 4(y-3)(y-5) + 5(x-2)(x-2y+6) &= 0 \\ \text{or,}\quad 5x^2 - 10xy + 4y^2 + 20x - 12y &= 0. \end{aligned}$$

8. *Find the equation and the type of the conic which passes through* $(-2,0)$, *touches the y-axis at the origin and has its centre at* $(1,1)$.

Let the equation of the conic be $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

It passes through the origin. $\therefore\ c = 0$.

The tangent at the origin is $gx + fy = 0$. If it is y-axis, i.e. $x = 0$, then $f = 0$.

The centre lies on $ax + hy + g = 0$ and $hx + by = 0$.

The equations are satisfied by $(1,1)$.

$$\therefore \quad a + h + g = 0 \tag{1}$$

$$\text{and} \quad h + b = 0. \tag{2}$$

Again the point $(-2, 0)$ lies on the conic.

$$\therefore\ a - g = 0. \tag{3}$$

By (1), (2) and (3),

$$a = g = -\frac{h}{2},\ b = -h.$$

Putting these values in the equation of the conic,

$$-\frac{h}{2}x^2 + 2hxy - hy^2 - hx = 0$$
$$\text{or,}\quad x^2 - 4xy + 2y^2 + 2x = 0.$$

It is the required equation.

Here

$$\Delta = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -2$$

and $D = 1 \cdot 2 - 4 = -2$.

Therefore, the equation represents a hyperbola.

[**N.B.** The tangent at the origin is obtained by equating the terms of lowest degree to zero.]

EXERCISE X

1. Find the canonical equations and describe the principal properties of the following by proper translation.

   (a) $y^2 + 8x - 6y + 1 = 0$
   (b) $x^2 - 8x - 12y - 20 = 0$
   (c) $9x^2 + 16y^2 - 54x + 64y + 1 = 0$
   (d) $3x^2 - 2y^2 + 6x - 8y - 17 = 0$.

2. Determine the centre of the following conics

   (a) $3x^2 + 4y^2 - 12x + 8y + 4 = 0$
   (b) $3x^2 + 3y^2 - 2xy - 2x + 6y + 2 = 0$
   (c) $2x^2 - 5xy - 3y^2 - x - 4y + 6 = 0$.

3. Determine whether the following have a single centre or infinitely many centres or no centre.

   (a) $12x^2 + 4y^2 + 14xy - 2x - 3y + 7 = 0$
   (b) $4x^2 - 4xy + y^2 - 12x - 10y - 19 = 0$

(c) $9x^2 - 6xy + y^2 + 24x - 8y + 16 = 0$

(d) $4x^2 + 12xy + 9y^2 + 3x + 2y + 1 = 0$.

4. Reduce the following to the canonical form and state the type of the conic

(a) $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ **[BH 1992; NH 2010]**

(b) $8x^2 - 12xy + 17y^2 + 16x - 12y + 3 = 0$ **[CH 2008; BH 2008]**

(c) $4x^2 - 3xy - 18 = 0$ **[CH 2008]**

(d) $8x^2 + 12xy + 13y^2 = 884$

(e) $4x^2 - 12xy + 9y^2 - 52x + 26y + 81 = 0$

(f) $x^2 + y^2 - xy - 6x = 0$ **[NH 2008]**

(g) $x^2 - 2xy + 2y^2 - 4x - 6y + 3 = 0$ **[BH 1991]**

(h) $2x^2 - 4xy - y^2 + 20x - 2y + 17 = 0$ **[CH 2009]**

(i) $x^2 + 4xy + 4y^2 + 4x + y - 15 = 0$ **[CH 2007; NH 2009]**

(j) $3x^2 + 10xy + 3y^2 - 2x - 14y - 5 = 0$ **[NH 2005]**

(k) $12x^2 + 24xy + 19y^2 - 12x - 40y + 31 = 0$

(l) $5x^2 - 20xy - 5y^2 - 16x + 8y - 7 = 0$

(m) $4x^2 + 4xy + y^2 - 12x - 6y - 5 = 0$ **[BH 2007; CH 2007]**

(n) $9x^2 + 82xy + 9y^2 + 40\sqrt{2}x + 360\sqrt{2}y + 800 = 0$.

5. Discuss the nature of the following conics

(a) $9x^2 - 6xy + y^2 - 14x - 2y + 12 = 0$

(b) $3x^2 - 5xy + 6y^2 + 11x - 17y + 13 = 0$

(c) $x^2 - 4xy - 2y^2 + 10x + 4y = 0$

(d) $4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0$

(e) $32x^2 + 52xy - 7y^2 - 64x - 52y - 148 = 0$

(f) $41x^2 + 24xy + 34y^2 + 30x - 40y + 25 = 0$

(g) $3\left(x^2 + y^2\right) + 2xy = 4\sqrt{2}(x + y)$ **[BH 2002, 09; CH 1999, 2003]**

6. (a) Reduce the equation $x^2 - 4xy + 4y^2 + 2x - 4y + c = 0$ to its canonical form and determine the type of the conic represented by it for different values of c.

[*Hints.* Similar to worked-out example 4.]

(b) Find the values of a and g for which the curve

$$ax^2 + 8xy + 4y^2 + 2gx + 4y + 1 = 0$$

represents (i) a conic having no centre, (ii) a conic having infinitely many centres.

[*Hints.* Similar to worked-out example 5.]

(c) Find the values of a and f for which the curve

$$ax^2 - 6xy + 9y^2 + 4x + 2fy + 1 = 0$$

may represent (i) a central conic, (ii) a parabola, (iii) a conic with infinitely many centres.

[*Hints.* The centre lies on $ax - 3y + 2 = 0$ and $3x - 9y - f = 0$.

By cross-multiplication

$$\frac{x}{3(f+6)} = \frac{y}{6+af} = \frac{1}{9(1-a)}.$$

(i) The given equation will represent a central conic if $1 - a \neq 0$, i.e. $a \neq 1$.

(ii) In case of parabola $1 - a = 0, f + 6 \neq 0, 6 + af \neq 0$, i.e. $a = 1$ and $f \neq -6$.

(iii) For infinitely many centres $1 - a = 0, f + 6 = 0, 6 + af = 0$, i.e. $a = 1$ and $f = -6$.]

(d) Find the values of h, f and c for which the equation

$$4x^2 + 2hxy + 36y^2 + 4x + 2fy + c = 0$$

will represent a pair of coincident lines.

[*Hints.* For the coincident lines

$$\begin{vmatrix} 4 & h \\ h & 36 \end{vmatrix} = 0, \quad \begin{vmatrix} 36 & f \\ f & c \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 4 & 2 \\ 2 & c \end{vmatrix} = 0.$$

From these $c = 1, h = \pm 12$ and $f = \pm 6$.

For these values the equation becomes $(2x \pm 6y + 1)^2 = 0$.]

(e) If the equation $5x^2 + 4xy + y^2 + 2gx - 2y + 2 = 0$ represents a point ellipse, find g and then determine the point.

7. (a) Show that the equation of the conic passing through the five points $(2,1), (1,0), (3,-1), (-1,0)$ and $(3,-2)$ is $x^2 + 19xy + 4y^2 - 45y - 1 = 0$.

(b) Find the equation of the conic which passes through the five points $(1,2), (3,-4), (-1,3), (-2,-3)$ and $(5,6)$. Show that it is a hyperbola.

[*Hints.* Let the points be A, B, C, D, E.

The equations of AB and CD are $3x + y - 5 = 0$ and $6x - y + 9 = 0$.

The equation of this pair of lines is $(3x + y - 5)(6x - y + 9) = 0$.

The equations of BC and DA are $7x + 4y - 5 = 0$ and $5x - 3y + 1 = 0$.

The equation of this pair of lines is $(7x + 4y - 5)(5x - 3y + 1) = 0$.

The equation to any conic through the points A, B, C, D is

$$(3x + y - 5)(6x - y + 9) + \lambda(7x + 4y - 5)(5x - 3y + 1) = 0.$$

If it passes through the point E, then $\lambda = -\frac{11}{9}$ and the required equation is $223x^2 - 38xy - 123y^2 - 171x + 83y + 350 = 0$.

Here $\Delta \neq 0, D < 0$. $\therefore$ the conic is a hyperbola.]

(c) Show that the equation of the conic passing through the point $(1, 1)$ and the points of intersection of the conic $x^2 + 2xy + 5y^2 - 7x - 8y + 6 = 0$ with the lines $2x - y - 5 = 0$ and $3x + y - 11 = 0$ is $34x^2 + 55xy + 139y^2 - 233x - 218y + 223 = 0$.

(d) Find the equations of the parabolas passing through the common points of $x^2 + 6xy - y^2 + 2x - 3y - 5 = 0$ and $2x^2 - 8xy + 3y^2 + 2y - 1 = 0$.

8. (a) The centre of a conic is $(25, 15/2)$ and it passes through the point $(1, 1)$. If it touches the y-axis at the origin, show that the equation is

$$x^2 - 6xy + 10y^2 - 5x = 0.$$

[*Hints*. Similar to worked-out example 8.]

(b) By reducing the equation $20x^2 + 15xy + 9x + 3y + 1 = 0$ to its canonical form, show that it represents a pair of intersecting straight lines which are equidistant from the origin. Find their point of intersection.

[CH 1984]

[*Hints*. Here $\Delta = 0$ and $D < 0$. Therefore, the equation represents two intersecting lines. The point of intersection lies on $40x + 15y + 9 = 0$ and $5x + 1 = 0$. It is $\left(-\frac{1}{5}, -\frac{1}{15}\right)$. Changing the origin to this point the equation reduces to $4x'^2 + 3x'y' = 0$.

Now rotating the axes through an acute angle θ where $\tan 2\theta = \frac{3}{4}$, the required canonical form is $9x''^2 - y''^2 = 0$. Obviously it represents two intersecting lines. The bisectors of the angles between these lines are $x'' = 0$ and $y'' = 0$.

The coordinates of the old origin in the new system are

$$\left(\frac{\cos\theta}{5} + \frac{\sin\theta}{15}, \frac{\cos\theta}{15} - \frac{\sin\theta}{5}\right), \quad \text{i.e.} \quad \left(\frac{\sqrt{10}}{5}, 0\right)$$

since $\tan\theta = \frac{1}{3}$. Thus the old origin lies on one of the bisectors. Consequently the lines represented by the given equation are equidistant from the origin.]

(c) Show that the equation $\left(a^2 + 1\right)x^2 + 2(a + b)xy + \left(b^2 + 1\right)y^2 = c$ represents an ellipse whose area is $\frac{\pi c}{ab - 1}$.

[*Hints*. Here $\Delta \neq 0$ and $D = (ab - 1)^2 > 0$.

$\therefore$ the given equation represents an ellipse. Rotating the axes through an acute angle θ where $\tan 2\theta = \frac{2}{a - b}$, the equation transforms to

$$\left\{\frac{a^2 + b^2 + 2}{2} + \frac{a + b}{2}\sqrt{(a - b)^2 + 4}\right\}x'^2$$
$$+ \left\{\frac{a^2 + b^2 + 2}{2} - \frac{a + b}{2}\sqrt{(a - b)^2 + 4}\right\}y'^2 = c.$$

$\therefore$ the area of the ellipse

$$= \frac{\pi c}{\sqrt{\left[\left(\frac{a^2+b^2+2}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2\left\{(a-b)^2+4\right\}\right]}} = \frac{\pi c}{ab-1}.]$$

(d) Show that $(a^2+b^2)(x^2+y^2) = (ax+by-ab)^2$ represents a parabola of latus rectum $\frac{2ab}{\sqrt{a^2+b^2}}$.

(e) Show that any conic passing through the intersections of two rectangular hyperbolas is also a rectangular hyperbola.

[*Hints.* If $S=0$ and $S'=0$ two rectangular hyperbolas, then $a+b=0$ and $a'+b'=0$. In $S+\lambda S'=0$, the coefficient of x^2+ the coefficient of $y^2 = (a+\lambda a')+(b+\lambda b') = a+b+\lambda(a'+b') = 0$. Hence the result follows.]

(f) Prove that in general two parabolas can be drawn to pass through the intersections of the conics

$$S = ax^2+2hxy+by^2+2gx+2fy+c=0$$
$$\text{and}\quad S' = a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$$

and that their axes are at right angles if $h(a'-b') = h'(a-b)$.

[*Hints.* Any conic through the intersections of the given conics is

$$S+\lambda S'=0.$$

It will be a parabola if $(a+\lambda a')(b+\lambda b')-(h+\lambda h')^2=0$.

It is a quadratic equation in λ, so two values of λ are obtained. Hence in general two parabolas can be drawn.

If λ_1 and λ_2 are the roots, then the axes of parabolas are of the forms

$$\sqrt{a+\lambda_1 a'}x+\sqrt{b+\lambda_1 b'}y+k_1=0$$
$$\text{and}\quad \sqrt{a+\lambda_2 a'}x+\sqrt{b+\lambda_2 b'}y+k_2=0.$$

If these are at right angles, then

$$\sqrt{a+\lambda_1 a'}\sqrt{a+\lambda_2 a'}+\sqrt{b+\lambda_1 b'}\sqrt{b+\lambda_2 b'}=0,$$

i.e.

$$(a+\lambda_1 a')(a+\lambda_2 a') = (b+\lambda_1 b')(b+\lambda_2 b')$$
$$\text{or,}\quad a^2+(\lambda_1+\lambda_2)aa'+\lambda_1\lambda_2 a'^2 = b^2+(\lambda_1+\lambda_2)bb'+\lambda_1\lambda_2 b'^2.$$

Eliminating λ_1 and λ_2, the result is obtained.]

ANSWERS

1. (a) Parabola, normal form $y'^2 = -8x'$, length of latus rectum = 8, vertex $(1,3)$, focus $(-1,3)$, axis $y-3=0$, directrix $x-3=0$.

   (b) Parabola, normal form $x'^2 = 12y'$, length of latus rectum = 12, vertex $(4,-3)$, focus $(4,0)$, axis $x=4$, directrix $y+6=0$.

   (c) Ellipse, normal form $\frac{x'^2}{16} + \frac{y'^2}{9} = 1$, centre $(3,-2)$, foci $\left(3 \pm \sqrt{7}, -2\right)$, eccentricity $= \frac{\sqrt{7}}{4}$, length of latus rectum $= 9/2$.

   (d) Hyperbola, normal form $\frac{x'^2}{4} - \frac{y'^2}{6} = 1$, centre $(-1,-2)$, eccentricity $= \sqrt{\frac{5}{2}}$, foci $\left(-1 \pm \sqrt{10}, -2\right)$ length of latus rectum = 6, length of transverse axis = 4, length of conjugate axis $= 2\sqrt{6}$.

2. (a) $(2,-1)$, (b) $(0,-1)$, (c) $(-2/7,-3/7)$.

3. (a) Single centre $(13/2,-11)$, (c) infinitely many centres,

   (b) no centre, (d) no centre.

4. (a) $x''^2 - y''^2 = 0$, pair of intersecting lines;

   (b) $x''^2 + 4y''^2 = 1$, ellipse;

   (c) $x'^2 - 9y'^2 = -36$, hyperbola;

   (d) $\frac{x'^2}{52} + \frac{y'^2}{221} = 1$, ellipse;

   (e) $y''^2 = \frac{8}{\sqrt{13}}x''$, parabola;

   (f) $\frac{x''^2}{24} + \frac{y''^2}{8} = 1$, ellipse;

   (g) $\left(3-\sqrt{5}\right)x''^2 + \left(3+\sqrt{5}\right)y''^2 = 52$, ellipse;

   (h) $\frac{x''^2}{3} - \frac{y''^2}{2} = -1$, hyperbola;

   (i) $x''^2 = \frac{7}{5\sqrt{5}}y''$, parabola;

   (j) $4x''^2 - y''^2 = 0$, pair of intersecting lines;

   (k) $28x''^2 + 3y''^2 = 0$, point ellipse, $(-3/2,2)$;

   (l) $x''^2 - y''^2 = -\frac{351}{125\sqrt{5}}$, rectangular hyperbola;

   (m) $x'^2 = \frac{14}{5}$, two parallel lines;

   (n) $\frac{x''^2}{5} - \frac{y''^2}{5} = 1$, hyperbola, $\theta = -\frac{\pi}{4}$.

5. (a) Parabola, vertex $(1,1)$, focus $\left(\frac{21}{20}, \frac{23}{20}\right)$, axis $3x - y - 2 = 0$, tangent at the vertex $x + 3y - 4 = 0$, directrix $x + 3y = 7/2$, latus rectum $= \frac{2}{\sqrt{10}}$, $x + 3y = 9/2$;

   (b) Ellipse, centre $(-1,-1)$, semi-axes $\sqrt{\frac{2}{9-\sqrt{34}}}, \sqrt{\frac{2}{9+\sqrt{34}}}$;

   (c) Hyperbola, centre $(-1,2)$, semi-axes (transverse $\frac{1}{\sqrt{2}}$, conjugate $\frac{1}{\sqrt{3}}$);

(d) Parabola, axis $2x - y - 1 = 0$, tangent at the vertex $x + 2y - 1 = 0$, vertex $\left(\frac{3}{5}, \frac{1}{5}\right)$, focus $\left(\frac{4}{5}, \frac{3}{5}\right)$, latus rectum$=\frac{4}{\sqrt{5}}$, $x + 2y - 2 = 0$, directrix $x + 2y = 0$;

(e) Hyperbola, centre $(1, 0)$, transverse axis $x - 2y - 1 = 0$, length $= 4$, conjugated axis $2x + y - 2 = 0$, length $= 6$, foci $\left(1 \pm 2\sqrt{\frac{13}{5}}, \pm\sqrt{\frac{13}{5}}\right)$;

(f) Point ellipse, $\left(-\frac{3}{5}, \frac{4}{5}\right)$;

(g) Canonical form $2x^2 + y^2 = 2$, ellipse, centre $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, axes $y - x = 0$, $x + y = \sqrt{2}$.

6. (a) $y''^2 + \frac{c-1}{5} = 0$, parallel, coincident or imaginary lines according as $c <=> 1$;

(b) (i) $a = 4, g \neq 2$ (ii) $a = 4, g = 2$

(e) $g = -3, (1, -1)$ and $g = -1, (-1, 3)$.

7. (d) $5x^2 - 10xy + 5y^2 + 2x + y - 7 = 0, 4x^2 + 4xy + y^2 + 4x + 4y - 9 = 0$.

Chapter 5

Tangent, Normal, Pole, Polar, Chord of Contact, Pair of Tangents

5.10 Equation of the tangent

Definition. If a line (secant) meets a curve at coincident points, then the line is called the tangent to the curve at the meeting point and the point is called the point of contact.

Let

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

be the equation of the conic and (x_1, y_1) be a point in the plane of the conic. The equation of a line through (x_1, y_1) can be written as

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r \text{ (say)} \tag{2}$$

where l and m are the cosines of the angles made by the line with x- and y-axes respectively and r is the algebraical distance between (x, y) and (x_1, y_1) on the line.

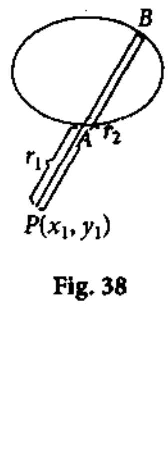


Fig. 38

From (2)

$$x = lr + x_1, \quad y = mr + y_1.$$

To find the point of intersection between (1) and (2) we have

$$\begin{aligned} &a(lr + x_1)^2 + 2h\,(lr + x_1)\,(mr + y_1) + b(mr + y_1)^2 \\ &\qquad + 2g\,(lr + x_1) + 2f\,(mr + y_1) + c = 0 \\ \text{or,}\quad &\left(al^2 + 2hlm + bm^2\right) r^2 + 2\left\{(ax_1 + hy_1 + g)\,l + (hx_1 + by_1 + f)\,m\right\} r \\ &\qquad + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \end{aligned} \tag{3}$$

It is a quadratic equation in r. Let the roots be r_1 and r_2, where r_1 and r_2 are the distances of the two points of intersection between (1) and (2) from (x_1, y_1). If the point (x_1, y_1) is on the conic and the line (2) is the tangent at this point, then $r_1 = 0 = r_2$.

In this case, $r_1 + r_2 = 0$ and $r_1 r_2 = 0$.

By the equation (3)

$$(ax_1 + hy_1 + g)\,l + (hx_1 + by_1 + f)\,m = 0 \tag{4}$$

$$\text{and} \quad ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \tag{5}$$

Eliminating l and m from (4) by (2)

$$\begin{aligned} &(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1) = 0 \\ \text{or,}\quad &axx_1 + h(xy_1 + yx_1) + byy_1 + gx + fy \\ &= ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1 \\ &= -gx_1 - fy_1 - c \;[\text{by } (5)] \\ \text{or,}\quad &axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0. \end{aligned} \tag{6}$$

It is the *standard form* of the equation of the tangent at a point (x_1, y_1) on the conic.

Note. To write down the equation of the tangent at (x_1, y_1) to a conic the following rules are to be remembered.

Change x^2 into xx_1, y^2 into yy_1, $2xy$ into $xy_1 + yx_1$, $2x$ into $x + x_1$ and $2y$ into $y + y_1$.

Tangents of the standard equations

Circle:	Tangent at (x_1, y_1)
$x^2 + y^2 + 2gx + 2fy + c = 0$	$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
$x^2 + y^2 = a^2$	$xx_1 + yy_1 = a^2$
Parabola: $y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
Rectangular hyperbola: $xy = k^2$	$xy_1 + yx_1 = 2k^2$

5.11 To find the condition that a given line may touch the conic

Let the equation of the conic and the line be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{7}$$

$$\text{and} \quad lx + my + n = 0 \quad \text{respectively.} \tag{8}$$

From (2)

$$y = -\frac{lx+n}{m}.$$

Putting this value of y in (1), we have

$$x^2\left(am^2 - 2hlm + bl^2\right) - 2x\left(hmn - bln - gm^2 + flm\right) \\ + bn^2 - 2fmn + cm^2 = 0. \tag{9}$$

If the line (2) be a tangent, the roots of the equation (3) must be equal.

$$\therefore \left(hmn - bln - gm^2 + flm\right)^2 - \left(am^2 - 2hlm + bl^2\right)\left(bn^2 - 2fmn + cm^2\right) = 0$$

or, $$\left(bc - f^2\right)l^2 + \left(ca - g^2\right)m^2 + \left(ab - h^2\right)n^2 + 2(gh - af)mn \\ + 2(hf - bg)nl + 2(fg - ch)lm = 0. \tag{10}$$

It is the required condition.

Deductions

Circle: $x^2 + y^2 = a^2$.

Condition: $a^2\left(l^2 + m^2\right) - n^2 = 0$, i.e. $n = \pm a\sqrt{l^2 + m^2}$.

Parabola: $y^2 = 4ax$.

Condition: $am^2 - ln = 0$, i.e. $n = \frac{am^2}{l}$.

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Condition: $a^2l^2 + b^2m^2 - n^2 = 0$, i.e. $n = \pm\sqrt{a^2l^2 + b^2m^2}$.

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Condition: $a^2l^2 - b^2m^2 - n^2 = 0$, i.e. $n = \pm\sqrt{a^2l^2 - b^2m^2}$.

Note. $y = mx + c$ will touch either (1), (2), (3) or (4), if $c = \pm a\sqrt{1+m^2}$, $c = \frac{a}{m}$, $c = \pm\sqrt{a^2m^2 + b^2}$ or, $c = \pm\sqrt{a^2m^2 - b^2}$ respectively.

5.20 To find the equation of the normal to

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{ at } (x_1, y_1).$$

Definition. The normal to a curve is the straight line perpendicular to the tangent at the point of contact.

The equation of the tangent at (x_1, y_1) is

$$axx_1 + h\left(xy_1 + yx_1\right) + byy_1 + g\left(x + x_1\right) + f\left(y + y_1\right) + c = 0$$

or, $$\left(ax_1 + hy_1 + g\right)x + \left(hx_1 + by_1 + f\right)y + gx_1 + fy_1 + c = 0. \tag{1}$$

Since the normal is perpendicular to (1) and passes through the point (x_1, y_1) its equation is

$$y - y_1 = \frac{hx_1 + by_1 + f}{ax_1 + hy_1 + g}\left(x - x_1\right). \tag{2}$$

5.21 Particular cases

(I) **Parabola:** $y^2 = 4ax$.

The equation of the tangent at (x_1, y_1) is

$$yy_1 = 2a\,(x + x_1) \quad \text{or,} \quad y = \frac{2a}{y_1}\,(x + x_1)\,.$$

$\therefore$ the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}\,(x - x_1)\,.$$

Corollary. If we put $-\frac{y_1}{2a} = m$, then $x_1 = \frac{y_1^2}{4a} = am^2$.

Thus the equation of the normal at $\left(am^2, -2am\right)$ is

$$y = mx - 2am - am^3.$$

(Ia.) *Show that three normals can be drawn to a parabola from a given point and the sum of the ordinates of the feet of the normals is zero.*

Let $(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$. The equation of the normal at this point is

$$y - 2at = -\frac{2at}{2a}\left(x - at^2\right) \quad \text{or,} \quad y + tx = 2at + at^3.$$

If this normal passes through a fixed point (h, k), then

$$k + th = 2at + at^3 \quad \text{or,} \quad at^3 + (2a - h)t - k = 0. \tag{3}$$

It is a cubic equation in t. Therefore, it has three roots. Corresponding to each of these three roots we have, on substitution, the equation of a normal passing through (h, k). Hence in general three normals can be drawn to a parabola through a given point.

Let t_1, t_2, t_3 be the roots of the equation (2). From the relation between roots and coefficients, $t_1 + t_2 + t_3 = 0$. If y_1, y_2, y_3 be the ordinates of the feet of the normals, then $y_1 + y_2 + y_3 = 2a\,(t_1 + t_2 + t_3) = 0$. Hence the result follows.

Note

1. Reality of the normals depends on the nature of the roots of (2).
2. If the normal at (x_1, y_1) passes through (h, k), then $k - y_1 = -\frac{y_1}{2a}\,(h - x_1)$. Hence the *locus of the feet of the normals* through the point (h, k) is $y(h - x) + 2a(k - y) = 0$. It is a rectangular hyperbola.

(II) **Ellipse:** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{or,} \quad y = -\frac{b^2}{a^2}\frac{x_1}{y_1}x + \frac{b^2}{y_1}.$$

$\therefore$ the equation of the normal is

$$y - y_1 = \frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1) \quad \text{or,} \quad \frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2}. \tag{4}$$

(IIa) *Show that four normals can be drawn to an ellipse through a given point and the feet of the normals lie on a rectangular hyperbola.*

Let the normal (1) pass through a given point (h, k).

Then

$$\frac{h - x_1}{x_1/a^2} = \frac{k - y_1}{y_1/b^2} = \lambda \text{ (say)}$$

$$\text{or,} \quad \frac{x_1}{a} = \frac{ah}{a^2 + \lambda}, \quad \frac{y_1}{b} = \frac{bk}{b^2 + \lambda}.$$

Since (x_1, y_1) is on the ellipse,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \quad \text{or,} \quad \frac{a^2h^2}{(a^2 + \lambda)^2} + \frac{b^2k^2}{(b^2 + \lambda)^2} = 1.$$

It is a bi-quadratic equation in λ and gives four values of λ. These four values correspond to four points on the ellipse and the normals at these points pass through the given point.

Again

$$a^2 + \lambda = \frac{a^2h}{x_1}, \quad b^2 + \lambda = \frac{b^2k}{y_1}.$$

Subtracting

$$a^2 - b^2 = \frac{a^2h}{x_1} - \frac{b^2k}{y_1}.$$

The point (x_1, y_1) is the foot of a normal. Hence the locus of the feet of the normals is

$$a^2 - b^2 = \frac{a^2h}{x} - \frac{b^2k}{y} \quad \text{or,} \quad (a^2 - b^2)\,xy = a^2hy - b^2kx,$$

which is a rectangular hyperbola.

(III) **Hyperbola:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

The equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \text{or,} \quad y = \frac{b^2}{a^2}\frac{x_1}{y_1}x - \frac{b^2}{y_1}.$$

$\therefore$ the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1) \quad \text{or,} \quad \frac{x - x_1}{x_1/a^2} = -\frac{y - y_1}{y_1/b^2}. \tag{5}$$

(IIIa) *Show that four normals can be drawn to a hyperbola through a given point and the locus of the feet of these normals is a rectangular hyperbola.*

Let the normal (1) pass through a fixed point (h, k). Then

$$\frac{h - x_1}{x_1/a^2} = -\frac{k - y_1}{y_1/b^2} = \lambda \text{ (say)} \quad \text{or,} \quad \frac{x_1}{a} = \frac{ah}{a^2 + \lambda}, \quad \frac{y_1}{b} = \frac{bk}{b^2 - \lambda}.$$

We have

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1.$$

$$\therefore \quad \frac{a^2h^2}{(a^2 + \lambda)^2} - \frac{b^2k^2}{(b^2 - \lambda)^2} = 1.$$

It is a bi-quadratic equation in λ and gives four values of it. These four values correspond to four points on the hyperbola at which normals pass through the fixed point.

Again

$$a^2 + \lambda = \frac{a^2h}{x_1}, \quad b^2 - \lambda = \frac{b^2k}{y_1}.$$

Adding

$$a^2 + b^2 = \frac{a^2h}{x_1} + \frac{b^2k}{y_1}$$

(x_1, y_1) is a foot of one of the four normals. Hence the locus of the feet of the normals is

$$a^2 + b^2 = \frac{a^2h}{x} + \frac{b^2k}{y} \quad \text{or,} \quad (a^2 + b^2)\, xy = a^2hy + b^2kx.$$

It is a rectangular hyperbola.

Note. Co-normal points. *The points on a curve at which normals pass through a fixed point are known as co-normal points.*

5.30 Pair of Tangents: Director Circle

(a) *A pair of tangents can be drawn to a conic from a point not lying on the conic.*

Let

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

be the equation of the conic and (x_1, y_1) be a point not lying on the conic. Let a line through the point (x_1, y_1) touch the conic at (x_2, y_2). The equation of the tangent at (x_2, y_2) is

$$axx_2 + h(xy_2 + yx_2) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0.$$

As it passes through (x_1, y_1) we have

$$ax_1x_2 + h(x_1y_2 + y_1x_2) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0. \tag{2}$$

Again (x_2, y_2) is on the conic. Therefore

$$ax_2^2 + 2hx_2y_2 + by_2^2 + 2gx_2 + 2fy_2 + c = 0. \tag{3}$$

From (2) and (3) generally two values of x_2 and y_2 are obtained. Thus there will be two points of contact of tangents from (x_1, y_1) but they may not be real in all cases.

(b) **Equation of the pair of tangents.**

The equation of a line through the point (x_1, y_1) can be written as

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} \ (= r \text{ say}). \tag{4}$$

From (4), $x = lr + x_1, y = mr + y_1$. Putting these values of x and y in the equation (1), we have

$$\begin{aligned}\left(al^2 + 2hlm + bm^2\right) r^2 + 2\left\{(ax_1 + hy_1 + g)\, l + (hx_1 + by_1 + f)\, m\right\} r \\ + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \qquad (5)\end{aligned}$$

It is a quadratic equation in r. If the line (4) touches the conic (1), the equation (5) must have equal roots. By the condition of equal roots

$$\begin{aligned}&\left\{(ax_1 + hy_1 + g)\, l + (hx_1 + by_1 + f)\, m\right\}^2 \\ &\quad = \left(al^2 + 2hlm + bm^2\right)\left(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c\right).\end{aligned}$$

Eliminating l and m by (4),

$$\begin{aligned}&\left\{(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1)\right\}^2 \\ &\quad = \left\{a(x-x_1)^2 + 2h(x-x_1)(y-y_1) + b(y-y_1)^2\right\} \times \\ &\qquad \left(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c\right). \qquad (6)\end{aligned}$$

If we write

$$\begin{aligned}S &= ax^2 + 2hxy + by^2 + 2gx + 2fy + c, \\ S_1 &= ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \\ \text{and} \quad T &= axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c,\end{aligned}$$

then the equation (6) can be written as

$$(T - S_1)^2 = (S + S_1 - 2T)\, S_1 \quad \text{or,} \quad SS_1 = T^2. \tag{7}$$

It is the required equation.

(c) **Director circle:** (*Locus of the points of intersection of pair of perpendicular tangents*)

Definition. *If the locus of the points of intersection of pair of perpendicular tangents to a conic is a circle, then that circle is called the director circle of the conic.*

(i) **Circle:** $x^2 + y^2 = a^2$.

The equation of the pair of tangents from (x_1, y_1) to the circle is

$$\left(x^2 + y^2 - a^2\right)\left(x_1^2 + y_1^2 - a^2\right) = \left(xx_1 + yy_1 - a^2\right)^2.$$

If these lines are at right angle, then the coefficient of x^2 + the coefficient of $y^2 = 0$, i.e.

$$\left(x_1^2 + y_1^2 - a^2 - x_1^2\right) + \left(x_1^2 + y_1^2 - a^2 - y_1^2\right) = 0 \quad \text{or,} \quad x_1^2 + y_1^2 = 2a^2.$$

Hence the locus of (x_1, y_1), i.e. the equation of the director circle is $x^2+y^2 = 2a^2$.

(ii) **Parabola:** $y^2 = 4ax$.

The equation of the pair of tangents from (x_1, y_1) to the parabola is

$$\left(y^2 - 4ax\right)\left(y_1^2 - 4ax_1\right) = \left(yy_1 - 2ax - 2ax_1\right)^2.$$

If these lines are at right angle, then

$$4a^2 + y_1^2 - y_1^2 + 4ax_1 = 0 \quad \text{or,} \quad x_1 + a = 0.$$

Hence the locus of (x_1, y_1) is $x + a = 0$. It is a straight line representing the directrix of the parabola. Thus the points of intersection of perpendicular tangents to a parabola lie on the directrix.

(iii) **Ellipse:** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The equation of the pair of tangents from (x_1, y_1) to the ellipse is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2.$$

If these lines are perpendicular to each other, then

$$\frac{1}{a^2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) - \frac{x_1^2}{a^4} + \frac{1}{b^2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) - \frac{y_1^2}{b^4} = 0$$
$$\text{or,} \quad x_1^2 + y_1^2 = a^2 + b^2.$$

Hence the director circle of the ellipse is $x^2 + y^2 = a^2 + b^2$.

(iv) **Hyperbola:** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

The equation of the pair of tangents from (x_1, y_1) to the hyperbola is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2.$$

For the pair of perpendicular tangents

$$\frac{1}{a^2}\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) - \frac{x_1^2}{a^4} - \frac{1}{b^2}\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) - \frac{y_1^2}{b^4} = 0$$
$$\text{or,} \quad x_1^2 + y_1^2 = a^2 - b^2.$$

Hence the equation of the director circle is $x^2 + y^2 = a^2 - b^2$.

5.40 Chord of contact

Definition. *It is a chord joining the points of contact of tangents to a conic from a given point not lying on the conic.*

Let

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

be the equation of the conic and (x_1, y_1) be the point from which tangents are drawn to the conic.

Let (x_2, y_2) and (x_3, y_3) be the points of contact. The equations of tangents at (x_2, y_2) and (x_3, y_3) are

$$axx_2 + h(xy_2 + yx_2) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0$$
$$\text{and} \quad axx_3 + h(xy_3 + yx_3) + byy_3 + g(x + x_3) + f(y + y_3) + c = 0. \tag{2}$$

Since these two tangents pass through the point (x_1, y_1),

$$ax_1x_2 + h(x_1y_2 + y_1x_2) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$$
$$\text{and} \quad ax_1x_3 + h(x_1y_3 + y_1x_3) + by_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0.$$

The above two conditions suggest that the line

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0 \tag{3}$$

passes through (x_2, y_2) and (x_3, y_3). Hence it is the equation of the chord of contact of tangents through (x_1, y_1).

> **Note.** The equation (4) is identical with the equation of the tangent at (x_1, y_1) on the conic. But here (x_1, y_1) does not lie on the conic.

5.50 Pole and Polar

Method I

Definition. *The polar of a point w.r.t. a conic is the locus of the points of intersection of tangents at the extremities of the chords through that point while the point itself is called the pole of its polar.*

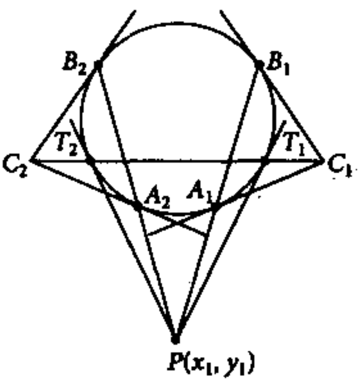


Fig. 39 (i)

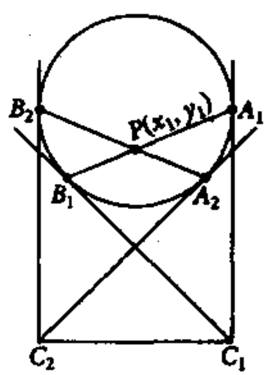


Fig. 39 (ii)

Let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

and (x_1, y_1) be the pole. Let the tangents at the extremities of a chord through (x_1, y_1) meet at (x_2, y_2). According to the definition of the polar (x_2, y_2) lies on the polar of (x_1, y_1).

The chord of contact of (x_2, y_2) is

$$axx_2 + h(xy_2 + yx_2) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0.$$

As (x_1, y_1) lies on it,

$$ax_1x_2 + h(x_1y_2 + y_1x_2) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0.$$

It shows that the locus of (x_2, y_2) is the line

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0. \qquad (2)$$

It is the polar of (x_1, y_1). [In Fig. 39, C_1C_2 is the polar of P]

Method II

Definition. *If a secant of a conic from a point P meets the conic at A and B and the point C on the secant is the harmonic conjugate of P w.r.t. A and B then the locus of C is the polar of P and P is the pole of this polar.*

$$\left[\frac{1}{PA} + \frac{1}{PB} = \frac{2}{PC}\right].$$

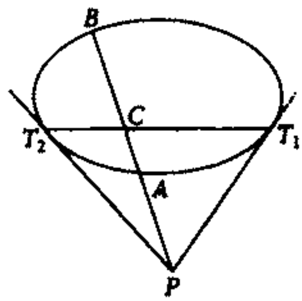


Fig. 40

Let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (3)$$

and the coordinates of P be (x_1, y_1).

Let the equation of the secant through the point P be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r \text{ (say)}. \qquad (4)$$

[l, m and r have the meanings as stated in Sec 5.10.]

If this secant meets the conic at A and B, then the distances PA and PB are the roots of the equation.

$$(al^2 + 2hlm + bm^2)r^2 + 2\{(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\}r$$
$$+ ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \qquad (5)$$

If C is the harmonic conjugate of P w.r.t. A and B, then

$$\frac{2}{PC} = \frac{1}{PA} + \frac{1}{PB}.$$

Let $PC = \rho, PA = r_1$ and $PB = r_2$. [r_1 and r_2 are the roots of (3).]

Now

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{\rho} \quad \text{or,} \quad \frac{r_1 + r_2}{r_1 r_2} = \frac{2}{\rho}.$$

From (3),

$$\frac{-2\{(ax_1 + hy_1 + g)\,l + (hx_1 + by_1 + f)\,m\}}{ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c} = \frac{2}{\rho}. \tag{6}$$

If (α, β) be the coordinates of C, then

$$\frac{\alpha - x_1}{l} = \frac{\beta - y_1}{m} = \rho \quad [\because\ C \text{ lies on } (2).] \tag{7}$$

From (5),

$$l = \frac{\alpha - x_1}{\rho} \quad \text{and} \quad m = \frac{\beta - y_1}{\rho}.$$

Putting these values of l and m in (4),

$$\begin{aligned} &-(ax_1 + hy_1 + g)(\alpha - x_1) - (hx_1 + by_1 + f)(\beta - y_1) \\ &\qquad = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \\ \text{or,}\quad & a\alpha x_1 + h(\alpha y_1 + \beta x_1) + b\beta y_1 + g(\alpha + x_1) + f(\beta + y_1) + c = 0. \end{aligned}$$

Hence the locus of (α, β), i.e. the equation of the polar of (x_1, y_1) is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

[In Fig. 40, T_1T_2 is the polar of P. PT_1 and PT_2 are the pair of tangents through P.]

Note

1. The polar of a point w.r.t. a conic coincides with the chord of contact of tangents from the point to the conic when the point does not lie on the conic.

2. If P is the midpoint of the chord AB, then r_1 and r_2 are equal in magnitude but opposite in sign. In this case

$$\frac{2}{\rho} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_1} - \frac{1}{r_1} = 0. \quad \therefore\ \rho = \infty.$$

   It indicates that the harmonic conjugate point of P is far away from the conic. Therefore, there will be no real polar of a point which is the midpoint of all chords through it. Hence for a central conic the centre has no real polar.

5.51 Properties of Pole and Polar

(i) *If the polar of P passes through Q, then the polar of Q will pass through P. These two points are called conjugate points.*

Let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{8}$$

The polar of $P(x_1, y_1)$ is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0. \tag{9}$$

As $Q(x_2, y_2)$ lies on it

$$ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0. \tag{10}$$

The polar of Q is

$$axx_2 + h(xy_2 + yx_2) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0. \tag{11}$$

The condition (3) suggests that P lies on (4). Hence the result follows.

Note. It is the *reciprocal property* of pole and polar.

(ii) *If the pole of a line L w.r.t. a conic lies on another line L', then the pole of L' w.r.t. this conic lies on L. These two lines are called conjugate lines.*

Let the pole of L w.r.t. the conic (1) be (x_1, y_1) and that of L' w.r.t. this conic be (x_2, y_2).

Therefore, the equation of L is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0 \tag{12}$$

and the equation of L' is

$$axx_2 + h(xy_2 + yx_2) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0. \tag{13}$$

As (x_1, y_1) lies on (6),

$$ax_1x_2 + h(x_1y_2 + y_1x_2) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0. \tag{14}$$

It is also the condition for (x_2, y_2) to lie on (5). Thus the proposition is proved.

(iii) *If two lines L and L' meet at C, then the polar of C w.r.t. a conic passes through the poles of L and L' w.r.t. this conic.*

Let P and Q be the poles of L and L' w.r.t. a conic S. Therefore, the polar of P which is L passes through C and also the polar of Q which is L' passes through C. By property (i) the polar of C passes through P and Q. Hence the proposition is proved.

(iv) **Some special properties for circle:**

(a) *The distances of two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ from the centre of a circle are proportional to the distance of each from the polar of the other.* [Salmon's theorem]

Let $x^2 + y^2 = a^2$ be the equation of the circle whose centre is the origin O.

$$\therefore\ OP = \sqrt{x_1^2 + y_1^2},$$
$$OQ = \sqrt{x_2^2 + y_2^2}.$$

The polar of P and Q are

$$xx_1 + yy_1 - a^2 = 0$$
$$\text{and}\quad xx_2 + yy_2 - a^2 = 0.$$

The distances of P and Q from the polars of Q and P are

$$\frac{x_1x_2 + y_1y_2 - a^2}{\sqrt{x_2^2 + y_2^2}} = d_1 \text{ (say)}$$
$$\text{and}\quad \frac{x_1x_2 + y_1y_2 - a^2}{\sqrt{x_1^2 + y_1^2}} = d_2 \text{ (say)}.$$

Now
$$\frac{d_1}{d_2} = \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}} = \frac{OP}{OQ}.$$

(b) *The polar of a point w.r.t. a circle is perpendicular to the line joining the centre and the point.*

Let $x^2 + y^2 = a^2$ be the equation of the circle. Its centre is $(0,0)$.

Let $P(x_1, y_1)$ be the point. The polar of P is $xx_1 + yy_1 = a^2$. The equation of the line through $(0,0)$ and P is $xy_1 - yx_1 = 0$. Obviously these two lines are perpendicular to each other.

Hence the proposition is proved.

(c) *If O be the centre of a circle, P be any point and OP (produced when necessary) meets the polar of P w.r.t. the circle at Q, then $OP \cdot OQ = (\text{radius})^2$.*

Let $x^2 + y^2 = a^2$ be the equation of the circle. Then O is $(0,0)$. The polar of $P(x_1, y_1)$ is

$$xx_1 + yy_1 = a^2. \tag{15}$$

Since OP is perpendicular to the polar of P, OQ is the perpendicular distance of O from (1).

$$\therefore\ OQ = \pm\frac{a^2}{\sqrt{x_1^2 + y_1^2}} = \pm\frac{a^2}{OP}.\ \therefore\ OP \cdot OQ = (\text{radius})^2.$$

Note. Self conjugate or self-polar triangle: If three points A, B, C are such that any two of them are conjugate points, then the triangle ABC is called a self-conjugate (or self-polar) triangle. Since the polar of A passes through B and C, BC is the polar of A. Therefore, each side is the polar of the opposite vertex and also any two sides are conjugate lines.

5.52 Pole of a given line w.r.t. a conic

Let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{16}$$

and (x_1, y_1) be the pole of the given line

$$lx + my + n = 0 \tag{17}$$

w.r.t. the conic (1).

The polar of (x_1, y_1) w.r.t. the conic (1) is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$
$$\text{or,}\quad (ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + gx_1 + fy_1 + c = 0. \tag{18}$$

It is identical with (2).

$$\therefore\ \frac{ax_1 + hy_1 + g}{l} = \frac{hx_1 + by_1 + f}{m} = \frac{gx_1 + fy_1 + c}{n}.$$

By solving these equations we have the values of x_1 and y_1.

Example 1. *Find the pole of the line $x + 2y + 3 = 0$ w.r.t. the circle*
$$x^2 + y^2 - 2x + 5 = 0.$$ **[NH 2008]**

Let (x_1, y_1) be the pole. The polar of it w.r.t. the given circle is

$$xx_1 + yy_1 - (x + x_1) + 5 = 0$$
$$\text{or,}\quad (x_1 - 1)x + y_1 y + 5 - x_1 = 0.$$

Comparing it with the given line

$$\frac{x_1 - 1}{1} = \frac{y_1}{2} = \frac{5 - x_1}{3}.$$

From these equations

$$3x_1 - 3 = 5 - x_1 \quad \text{or,} \quad x_1 = 2;$$
$$y_1 = 2x_1 - 2 = 4 - 2 = 2.$$

$\therefore$ (2, 2) is the required pole.

5.60 Lengths of sub-tangent, sub-normal, normal and tangent

Definition. *If the tangent and normal at any point P of a conic meet the x-axis at T and N respectively and PM is the ordinate of P, then*

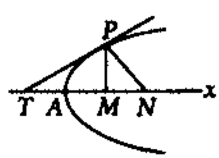


Fig. 41

$$\textit{length of sub-tangent} = MT,$$
$$\textit{length of sub-normal} = MN,$$
$$\textit{length of tangent} = PT$$
$$\textit{and length of normal} = PN.$$

Length of the tangent (in general sense): *If the tangent to a conic through a point P (outside the conic) meets the conic at T, then PT is called the length of the tangent.*

Length of the tangent to a circle:

$$\text{Let} \quad x^2 + y^2 + 2gx + 2fy + c = 0 \tag{1}$$

be the equation of the circle, $P(x_1, y_1)$ be a point outside the circle and PT be the tangent to it. $C(-g, -f)$ is the centre of the circle and the radius CT is perpendicular to PT. From Fig. 42,

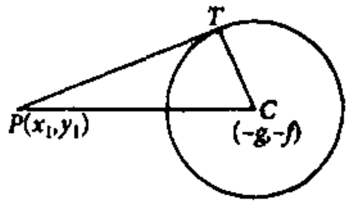


Fig. 42

$$\begin{aligned} PT^2 &= PC^2 - CT^2 \\ &= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c) \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c. \end{aligned}$$

$\therefore\ PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} =$ length of the tangent.

WORKED-OUT EXAMPLES

1. *Find the equation of the tangent and the normal at* $(1, -1)$ *to the conic*

$$y^2 - xy - 2x^2 - 5y + x - 6 = 0.$$

The point $(1, -1)$ is on the conic. Therefore, the equation of the tangent at $(1, -1)$ is

$$\begin{aligned} &y \cdot (-1) - \tfrac{1}{2}\{x \cdot (-1) + y \cdot 1\} - 2x \cdot 1 - \tfrac{5}{2}(y - 1) + \tfrac{1}{2}(x + 1) - 6 = 0 \\ \text{or,} \quad &x + 4y + 3 = 0. \end{aligned}$$

Since the normal is perpendicular to the tangent at $(1, -1)$, its equation is

$$y + 1 = 4(x - 1) \quad \text{or,} \quad 4x - y - 5 = 0.$$

2. *Find the equations to the tangents to the conic*

$$x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0$$

which are parallel to $x + 4y = 0$. [BH 2008]

Let the equation of any such tangent be

$$x + 4y + c = 0 \quad \text{or,} \quad x = -(4y + c).$$

Putting this value of x in the equation of the conic, we have

$$\begin{aligned} &(4y + c)^2 - 4y(4y + c) + 3y^2 + 5(4y + c) - 6y + 3 = 0 \\ \text{or,} \quad &3y^2 + 2(2c + 7)y + c^2 + 5c + 3 = 0. \end{aligned}$$

Since the line is a tangent to the conic, the above equation in y must have equal roots.

$$\therefore\ 4(2c+7)^2 - 4\cdot 3\cdot\left(c^2+5c+3\right) = 0 \quad \text{or,} \quad c^2+13c+40=0,$$
$$\text{or,} \quad (c+5)(c+8)=0.$$
$$\therefore \quad c=-5,-8.$$

$\therefore$ the required tangents are $x+4y-5=0$ and $x+4y-8=0$.

3. *Prove that the normal chord of a parabola at the point whose ordinate is equal to its abscissa, subtends a right angle at the focus.* [BH 2003]

Let

$$y^2 = 4ax \tag{1}$$

be the equation of the parabola. The point, whose ordinate is equal to abscissa, is obtained by putting $y=x$ in (1).

$$\therefore\ x^2 = 4ax \quad \text{or,} \quad x = 0, 4a.$$

Therefore, the point is $(0,0)$ or, $(4a,4a)$. At $(0,0)$ x-axis is the normal which passes through the focus, so it will not serve our purpose. The normal at $(4a,4a)$ is

$$y-4a = -\frac{4a}{2a}(x-4a) \quad \text{or,} \quad y+2x=12a. \tag{2}$$

Solving (1) and (2) we get the ends of this normal chord, which are $(4a,4a)$ and $(9a,-6a)$. The gradient of the line joining the focus $(a,0)$ and $(4a,4a)$

$$= \frac{4a}{4a-a} = \frac{4}{3}.$$

and the gradient of the line joining the focus $(a,0)$ and $(9a,-6a)$

$$= \frac{-6a}{9a-a} = -\frac{3}{4}.$$

The product of these two gradients $=-1$. Hence the result follows.

4. *Show that the tangents at the extremities of a focal chord of a parabola meet at right angle on the directrix.*

Let $y^2=4ax$ be the equation of the parabola and the ends of a focal chord be $\left(at^2, 2at\right)$ and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

The tangents at these points are

$$yt = x + at^2 \tag{1}$$

$$\text{and} \quad -\frac{y}{t} = x + \frac{a}{t^2} \quad \text{or,} \quad -ty = xt^2 + a. \tag{2}$$

Obviously (1) and (2) are at right angle. Again adding (1) and (2)

$$(x+a)\left(1+t^2\right) = 0 \quad \text{or,} \quad x+a=0.$$

$\therefore$ the tangents meet on the directrix $x+a=0$ at right angle.

5. *Find the equation of the common tangents to the circle $x^2 + y^2 = 4ax$ and the parabola $y^2 = 4ax$.*

   Let $y = mx + c$ be the common tangent. If it is tangent to $y^2 = 4ax$, then $c = \frac{a}{m}$.

$$\therefore\ y = mx + \frac{a}{m}. \tag{1}$$

   To find the point of intersection between (1) and the circle,

$$x^2 + \left(mx + \frac{a}{m}\right)^2 = 4ax \quad \text{or,} \quad \left(1 + m^2\right) x^2 - 2ax + \frac{a^2}{m^2} = 0.$$

   If the line (1) be the tangent, the above equation must have equal roots. From the condition of equal roots

$$4a^2 - 4\left(1 + m^2\right) \cdot \frac{a^2}{m^2} = 0 \quad \text{or,} \quad \frac{a^2}{m^2} = 0.$$

   Here $a \neq 0$.

$$\therefore\ \frac{1}{m} = 0.$$

   From (1)

$$\frac{y}{m} = x + \frac{a}{m^2} \quad \text{or,} \quad x = 0.$$

   $\therefore\ x = 0$ is the common tangent.

6. *Show that the sub-normal at any point of a parabola is equal to its semi-latus rectum.*

   Let $y^2 = 4ax$ be the equation of the parabola and $\left(at^2, 2at\right)$ be a point on it.

   The equation of the normal at this point is $y - 2at = -t\left(x - at^2\right)$.

   It meets the x-axis at $\left(2a + at^2, 0\right)$.

   The length of the sub-normal is the distance between $\left(at^2, 0\right)$ and the point where the normal meets the x-axis, i.e. $\left(2a + at^2, 0\right)$. The distance between these points $= 2a$.

   Length of the latus rectum $= 4a$. Hence the length of the sub-normal at any point on the parabola is equal to semi-latus rectum.

7. *Find the pole of the focal chord of the parabola $y^2 = 4ax$, passing through $(9a, 6a)$.*

   The equation of the chord passing through the focus $(a, 0)$ and $(9a, 6a)$ is

$$y = \frac{6a}{9a - a}(x - a) \quad \text{or,} \quad 3x - 4y - 3a = 0. \tag{1}$$

   Let (α, β) be the pole of it. The equation of the polar of (α, β) is

$$y\beta = 2a(x + \alpha) \quad \text{or,} \quad 2ax - \beta y + 2a\alpha = 0.$$

It is identical with (1). Comparing the coefficients

$$\frac{2a}{3}=\frac{\beta}{4}=\frac{2a\alpha}{-3a}.\ \therefore\ \alpha=-a,\beta=\frac{8a}{3}.$$

8. *Show that the locus of the poles of tangents to the parabola $y^2=4ax$ w.r.t. the parabola $y^2=4bx$ is the parabola $y^2=\frac{4b^2}{a}x$.*

Let (α,β) be the pole. The polar of it w.r.t. $y^2=4bx$ is

$$y\beta=2b(x+\alpha)\quad\text{or,}\quad y=\frac{2b}{\beta}x+\frac{2b\alpha}{\beta}.$$

If it is tangent to the parabola $y^2=4ax$, then

$$\frac{2b\alpha}{\beta}=\frac{a}{2b/\beta}\quad\text{or,}\quad \beta^2=\frac{4b^2}{a}\alpha.$$

Hence the required locus is $y^2=\frac{4b^2}{a}x$.

9. *Prove that the two parabolas $y^2=4ax$ and $x^2=4by$ cut one another at an angle*

$$\tan^{-1}\frac{3(ab)^{1/3}}{2\left(a^{2/3}+b^{2/3}\right)}.$$

$$y^2=4ax \tag{1}$$
$$x^2=4by. \tag{2}$$

Let (x_1,y_1) be the common point of (1) and (2). The tangents at this point are

$$yy_1=2a\left(x+x_1\right) \tag{3}$$
$$\text{and}\quad xx_1=2b\left(y+y_1\right). \tag{4}$$

If θ be the angle between (3) and (4), then

$$\tan\theta=\frac{4ab-x_1y_1}{-2\left(ax_1+by_1\right)}. \tag{5}$$

We have $y_1^2=4ax_1$ and $x_1^2=4by_1$. From these two $y_1^4=64a^2by_1$.

$\therefore\ y_1=0$ or, $4a^{2/3}b^{1/3}$.

Similarly $x_1=0$ or, $4a^{1/3}b^{2/3}$.

Putting $x_1=4a^{1/3}b^{2/3}$ and $y_1=4a^{2/3}b^{1/3}$ in (5),

$$\tan\theta=\frac{4ab-16ab}{-8\left(a^{4/3}b^{2/3}+a^{2/3}b^{4/3}\right)}=\frac{3(ab)^{1/3}}{2\left(a^{2/3}+b^{2/3}\right)}.$$

$$\therefore\ \theta = \tan^{-1}\frac{3(ab)^{1/3}}{2\left(a^{2/3}+b^{2/3}\right)}.$$

For $x_1 = 0, y_1 = 0, \tan\theta = \infty$. $\therefore\ \theta = \pi/2$.

Thus the parabolas cut at right angle at the origin.

10. *Show that the line $lx + my = n$ is a normal to the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \text{if}\ \frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{\left(a^2-b^2\right)^2}{n^2}.$$

Let $lx + my = n$ be the normal to the ellipse at $(a\cos\phi, b\sin\phi)$. The equation of the normal at this point is

$$\frac{x-a\cos\phi}{\frac{a\cos\phi}{a^2}}=\frac{y-b\sin\phi}{\frac{b\sin\phi}{b^2}}\quad \text{or,}\quad a\sec\phi\ x - b\operatorname{cosec}\phi\ y = a^2-b^2.$$

It is identical with the given line. Comparing these two we have

$$\frac{a\sec\phi}{l}=\frac{-b\operatorname{cosec}\phi}{m}=\frac{a^2-b^2}{n}.$$

$$\therefore\ \cos\phi=\frac{an}{l\left(a^2-b^2\right)},\ \sin\phi=-\frac{bn}{m\left(a^2-b^2\right)}.$$

Squaring and adding,

$$1=\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right)\frac{n^2}{\left(a^2-b^2\right)^2}\quad \text{or,}\quad \frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{\left(a^2-b^2\right)^2}{n^2}.$$

11. *Show that the foot of the perpendicular from the focus on any tangent to the hyperbola lies on the auxiliary circle.* **[CH 2006; BH 2001, 02]**

Let $(a\sec\theta, b\tan\theta)$ be a point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{1}$$

The equation of the tangent at this point is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1. \tag{2}$$

The equation of the line perpendicular to (2) and passing through the focus $(ae, 0)$ is

$$\frac{x}{b}\tan\theta+\frac{y}{a}\sec\theta=\frac{ae}{b}\tan\theta. \tag{3}$$

Here e is the eccentricity.

To find the locus of the intersection between (2) and (3) we have to eliminate θ from these two equations. Squaring (2) and (3) and then adding, we have

$$\begin{aligned}(x^2+y^2)\left(\frac{\sec^2\theta}{a^2}+\frac{\tan^2\theta}{b^2}\right) &= 1+\frac{a^2e^2}{b^2}\tan^2\theta \\ &= 1+\frac{a^2+b^2}{b^2}\tan^2\theta \\ &= \left(1+\tan^2\theta\right)+\frac{a^2}{b^2}\tan^2\theta \\ &= a^2\left(\frac{\sec^2\theta}{a^2}+\frac{\tan^2\theta}{b^2}\right)\end{aligned}$$

or, $x^2+y^2=a^2$. It is the auxiliary circle.

Similarly considering the other focus $(-ae,0)$ the same result is obtained.

12. *An ellipse slides between two straight lines at right angles to each other. Show that the locus of its centre is a circle.*

Let

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

be the equation of the ellipse.

The lines on which the ellipse slides must be perpendicular tangents to (1). Let the equations of these tangents be

$$y=mx+\sqrt{a^2m^2+b^2} \quad \text{and} \quad my=-x+\sqrt{a^2+b^2m^2}.$$

If (α,β) be the centre of the ellipse referred to these tangents as axes, then α and β are equal to distances from $(0,0)$ to these lines.

$$\therefore\ \alpha=\sqrt{\frac{a^2m^2+b^2}{1+m^2}} \quad \text{and} \quad \beta=\sqrt{\frac{a^2+b^2m^2}{1+m^2}}.$$

Thus

$$\alpha^2+\beta^2=\frac{a^2m^2+b^2+a^2+b^2m^2}{1+m^2}=\frac{\left(a^2+b^2\right)\left(1+m^2\right)}{1+m^2}=a^2+b^2.$$

Hence the required locus is $x^2+y^2=a^2+b^2$.

13. *If the normal at any end of a latus rectum of an ellipse passes through one end of the minor axis, then prove that $e^4+e^2=1$, where e is the eccentricity.*

Let

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

be the equation of an ellipse with a and b as semi-major and semi-minor axes. The coordinates of one end of a latus rectum can be taken as $\left(ae, \frac{b^2}{a}\right)$. The equation of the normal at this point is

$$\frac{x - ae}{ae/a^2} = \frac{y - b^2/a}{\frac{b^2}{a}\Big/b^2} \quad \text{or,} \quad \frac{x}{e} - a = y - \frac{b^2}{a}.$$

This normal passes through $(0, -b)$.

$$\therefore\ -a = -b - \frac{b^2}{a}.$$

We have $b^2 = a^2\left(1 - e^2\right)$.

$$\therefore\ a = b + a\left(1 - e^2\right) \quad \text{or,} \quad ae^2 = b \quad \text{or,} \quad a^2e^4 = b^2$$
$$\text{or,} \quad a^2e^4 = a^2\left(1 - e^2\right) \quad \text{or,} \quad e^4 + e^2 = 1.$$

14. *Prove that the locus of the pole w.r.t. the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of any tangent to the auxiliary circle is the curve $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}$.*

Let (α, β) be the pole. Its polar w.r.t. the given ellipse is

$$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = 1. \tag{1}$$

If it is tangent to the auxiliary circle $x^2 + y^2 = a^2$, then the perpendicular distance from $(0,0)$ to (1) is a.

$$\therefore\ \frac{1}{\sqrt{\left(\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4}\right)}} = a \quad \text{or,} \quad \frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} = \frac{1}{a^2}.$$

Hence the required locus is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}.$$

15. *Find the locus of the poles of the normal chords of*

(i) parabola $y^2 = 4ax$, **[BH 94; NH 2002]**

(ii) ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **[BH 94, 2002; CH 96]**

and (iii) hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **[CH 98, 2001]**

(i) *If (α, β) be the pole, then the polar of it w.r.t. $y^2 = 4ax$ is*

$$y\beta = 2a(x + \alpha). \tag{1}$$

Let it be normal to the parabola at $\left(at^2, 2at\right)$.

The equation of the normal at this point is

$$y - 2at = -t\left(x - at^2\right). \tag{2}$$

Comparing (1) and (2)

$$\frac{\beta}{1} = \frac{2a}{-t} = \frac{2a\alpha}{at^3 + 2at}$$
$$\text{or,}\quad t = -\frac{2a}{\beta} \quad \text{and} \quad at^2 + 2a = -\alpha.$$

Eliminating t, $\frac{4a^3}{\beta^2} + 2a = -\alpha$ or, $\beta^2(\alpha + 2a) + 4a^3 = 0$.

Hence the required locus is $y^2(x + 2a) + 4a^3 = 0$.

(ii) *If (α, β) be the pole, then the polar of it w.r.t.*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = 1. \tag{1}$$

Let it be normal to the ellipse at $(a\cos\theta, b\sin\theta)$.

The equation of the normal at this point is

$$a\sec\theta x - b\,\mathrm{cosec}\,\theta y = a^2 - b^2. \tag{2}$$

Comparing (1) and (2)

$$\frac{a\sec\theta}{\alpha/a^2} = \frac{-b\,\mathrm{cosec}\,\theta}{\beta/b^2} = \frac{a^2 - b^2}{1}$$
$$\text{or,}\quad \cos\theta = \frac{a^3}{\alpha\left(a^2 - b^2\right)}, \quad \sin\theta = \frac{-b^3}{\beta\left(a^2 - b^2\right)}.$$

Squaring and adding

$$1 = \left(\frac{a^6}{\alpha^2} + \frac{b^6}{\beta^2}\right)\frac{1}{\left(a^2 - b^2\right)^2} \quad \text{or,} \quad \frac{a^6}{\alpha^2} + \frac{b^6}{\beta^2} = \left(a^2 - b^2\right)^2.$$

Hence the required locus is

$$\frac{a^6}{x^2} + \frac{b^6}{y^2} = \left(a^2 - b^2\right)^2.$$

(iii) If (α, β) be the pole, then the polar of it w.r.t. the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{x\alpha}{a^2} - \frac{y\beta}{b^2} = 1. \tag{1}$$

Let it be normal to the hyperbola at $(a\sec\theta, b\tan\theta)$.

The equation of the normal at this point is

$$ax\tan\theta + by\sec\theta = \left(a^2+b^2\right)\sec\theta\tan\theta. \tag{2}$$

(1) and (2) are identical.

$$\therefore \quad \frac{a\tan\theta}{\alpha/a^2} = \frac{b\sec\theta}{-\beta/b^2} = \left(a^2+b^2\right)\sec\theta\tan\theta$$

$$\text{or,}\quad \sec\theta = \frac{a^3}{\left(a^2+b^2\right)\alpha}, \quad \tan\theta = -\frac{b^3}{\left(a^2+b^2\right)\beta}.$$

Since $\sec^2\theta - \tan^2\theta = 1$,

$$\frac{a^6}{\left(a^2+b^2\right)\alpha^2} - \frac{b^6}{\left(a^2+b^2\right)\beta^2} = 1 \quad \text{or,} \quad \frac{a^6}{\alpha^2} - \frac{b^6}{\beta^2} = \left(a^2+b^2\right)^2.$$

Hence the locus is

$$\frac{a^6}{x^2} - \frac{b^6}{y^2} = \left(a^2+b^2\right)^2.$$

16. *Find the equation and length of the common tangent to the hyperbolas*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{b^2} - \frac{y^2}{a^2} = -1.$$

The hyperbolas are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

$$\text{and} \quad \frac{x^2}{b^2} - \frac{y^2}{b^2} = -1. \tag{2}$$

Let the tangents at $(a\sec\theta, b\tan\theta)$ and $(b\tan\phi, a\sec\phi)$ to (1) and (2) respectively be identical.

These equations are

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \quad \text{and} \quad \frac{x}{b}\tan\phi - \frac{y}{a}\sec\phi = -1.$$

Comparing these two equations

$$\frac{\sec\theta}{a} = -\frac{\tan\phi}{b} \quad \text{and} \quad \frac{\tan\theta}{b} = -\frac{\sec\phi}{a}.$$

$$\therefore \quad \sec^2\theta - \tan^2\theta = \frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}\sec^2\phi$$

$$\text{or,}\quad 1 = \left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right)\tan^2\phi - \frac{b^2}{a^2}$$

$$\text{or,}\quad \tan^2\phi = \frac{b^2\left(a^2+b^2\right)}{a^4-b^4} = \frac{b^2}{a^2-b^2} \quad \text{and} \quad \sec^2\phi = \frac{a^2}{a^2-b^2}.$$

$$\tan^2\theta = \frac{b^2}{a^2}\sec^2\phi = \frac{b^2}{a^2-b^2} \quad \text{and} \quad \sec^2\theta = \frac{a^2}{a^2-b^2}.$$

$\therefore$ the points of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2-b^2}}\right) \quad \text{and} \quad \left(\mp\frac{b^2}{\sqrt{a^2-b^2}}, \mp\frac{a^2}{\sqrt{a^2-b^2}}\right).$$

Length of the common tangent is the distance between these points of contact. Hence the required length

$$= \sqrt{\frac{(a^2+b^2)^2}{a^2-b^2} + \frac{(a^2+b^2)^2}{a^2-b^2}} = \sqrt{2}\frac{a^2+b^2}{\sqrt{a^2-b^2}}.$$

The equation of the common tangents are $x \pm y = \pm\sqrt{a^2-b^2}$.

17. *If the normal to the rectangular hyperbola $xy = c^2$ at $\left(ct_1, \frac{c}{t_1}\right)$ meets the curve at $\left(ct_2, \frac{c}{t_2}\right)$, then show that $t_1^3 t_2 = -1$.* **[BH 2002]**

The tangent at $(ct_1, c/t_1)$ is

$$\frac{1}{2}\left(x\frac{c}{t_1} + yct_1\right) = c^2 \quad \text{or,} \quad y = -\frac{x}{t_1^2} + \frac{2c}{t_1}.$$

Therefore, the equation of the normal at this point is

$$y - \frac{c}{t_1} = t_1^2(x - ct_1).$$

If it passes through $\left(ct_2, \frac{c}{t_2}\right)$, then

$$\frac{c}{t_2} - \frac{c}{t_1} = t_1^2(ct_2 - ct_1) \quad \text{or,} \quad (t_1 - t_2)\left(\frac{1}{t_1 t_2} + t_1^2\right) = 0,$$

$$\because\ t_1 \neq t_2, \frac{1}{t_1 t_2} + t_1^2 = 0 \quad \text{or,} \quad t_1^3 t_2 = -1.$$

18. *Show that the pole of any tangent to the rectangular hyperbola $xy = c^2$ w.r.t. the circle $x^2 + y^2 = a^2$ lies on a concentric and similar hyperbola.*

Let $(ct, c/t)$ be a point on the hyperbola.

The equation of the tangent at this point is

$$x + t^2 y - 2ct = 0. \tag{1}$$

Let (α, β) be the pole of it w.r.t. the circle. The equation of the polar is

$$x\alpha + y\beta - a^2 = 0. \tag{2}$$

(1) and (2) are identical.

$$\therefore\ \frac{\alpha}{1} = \frac{\beta}{t^2} = \frac{a^2}{2ct}.$$

From these

$$t = \frac{a^2}{2c\alpha} = \frac{2c\beta}{a^2} \quad \text{or,} \quad \alpha\beta = \frac{a^4}{4c^2}.$$

$\therefore$ the locus of (α, β) is $xy = \frac{a^4}{4c^2}$ which is similar to $xy = c^2$.

19. *Tangents are drawn from* (h, k) *to the circle* $x^2 + y^2 = a^2$. *Prove that the area of the triangle formed by them and the straight line joining their points of contact is* $\frac{a(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$. **[BH 2002]**

The equation of the pair of tangents from (h, k) to the given circle is

$$\left(x^2 + y^2 - a^2\right)\left(h^2 + k^2 - a^2\right) = \left(xh + yk - a^2\right)^2$$

$$\text{or,}\quad \left(k^2 - a^2\right)x^2 + \left(h^2 - a^2\right)y^2 - 2hkxy + 2a^2hx + 2a^2ky - a^2\left(h^2 + k^2\right) = 0.$$

If θ be the angle between these tangents, then

$$\tan\theta = \frac{2\sqrt{h^2k^2 - (k^2 - a^2)(h^2 - a^2)}}{h^2 + k^2 - 2a^2} = \frac{2\sqrt{a^2(h^2 + k^2) - a^4}}{h^2 + k^2 - 2a^2}.$$

$$\therefore\ \sin\theta = \frac{2\sqrt{a^2(h^2 + k^2) - a^4}}{\sqrt{4a^2(h^2 + k^2) - 4a^4 + (h^2 + k^2 - 2a^2)^2}} = \frac{2a\sqrt{(h^2 + k^2 - a^2)}}{h^2 + k^2}.$$

Length of the tangent $= \sqrt{(h^2 + k^2 - a^2)}$. Now two sides of the triangle are equal to the length of the tangent and the angle between them is θ.

$$\therefore\ \text{area} = \tfrac{1}{2}\left(\sqrt{h^2 + k^2 - a^2}\right)^2 \sin\theta = \frac{a\left(h^2 + k^2 - a^2\right)^{3/2}}{h^2 + k^2}.$$

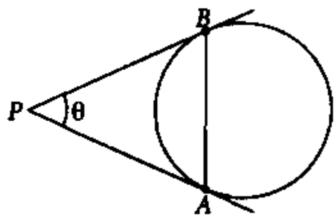


Fig. 43

20. *Two tangents drawn to* $y^2 = 4ax$ *meet at angle* 45°. *Find the locus of their point of intersection.* **[BH 2006; CH 97, 2002]**

Let (h, k) be the point of intersection. The equation of the pair of tangents to $y^2 = 4ax$ through (h, k) is

$$\left(y^2 - 4ax\right)\left(k^2 - 4ah\right) = \{yk - 2a(x + h)\}^2$$

$$\text{or,}\quad a^2x^2 - akxy + ahy^2 + a\left(k^2 - 2ah\right)x - ahky + a^2h^2 = 0.$$

Since the angle between these lines is 45°

$$\tan 45^\circ = \frac{2\sqrt{\left(\frac{a^2k^2}{4} - a^3h\right)}}{a^2 + ah} \quad \text{or,}\quad \left(a^2 + ah\right)^2 = 4\left(\frac{a^2k^2}{4} - a^3h\right)$$

$$\text{or,}\quad (a + h)^2 = k^2 - 4ah.$$

Hence the required locus is $(x + a)^2 = y^2 - 4ax$.

EXERCISE XI

1. Find the equation of the tangents and normals to the following curves.

   (a) $x^2 + y^2 - 4x + 6y - 36 = 0$ at $(2, 4)$.

   (b) $2x^2 + 5xy + 3y^2 + 4x - 10y - 4 = 0$ at $(1, 1)$.

   (c) $13x^2 + 16xy - y^2 - 70x - 40y + 82 = 0$ at $(1, -25)$.

2. Find the points of contact and equations of tangents to

   (a) $x^2 + y^2 + 6x - 10y - 15 = 0$, which are parallel to x-axis.

   (b) $y^2 = 4ax$, which makes angle $60°$ with the x-axis.

3. (a) Show that $3x + 4y + 7 = 0$ touches the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ and find the point of contact.

   (b) Find the condition so that $y = mx+c$ will be the tangent to $y^2 = 4(x+a)$.

   (c) Find the condition for which $lx + my + n = 0$ will be the normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

   (d) Prove that if the straight line $\lambda x + \mu y + \nu = 0$ touches the parabola $y^2 - 4px + 4pq = 0$, then $\lambda^2 q + \lambda\nu - p\mu^2 = 0$.

4. Prove that $x \cos\alpha + y \sin\alpha = a$ and $x \sin\alpha - y \cos\alpha = a$ are tangents to $x^2 + y^2 = a^2$ whatever α may be. Hence find the locus of the point of intersection from which two perpendicular tangents can be drawn to the circle $x^2+y^2 = a^2$.

5. Find the equations to the tangents at the ends of the lateral recta of $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and show that they pass through the intersection of the axis and directrices.

6. (a) The normal at $(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $(at_2^2, 2at_2)$. Show that $t_2 = -t_1 - \frac{2}{t_1}$.

   (b) If a normal chord of a parabola subtends a right angle at the vertex, show that its inclination to the x-axis is $\tan^{-1}\sqrt{2}$.

   [*Hints.* If the chord is normal at $(at^2, 2at)$ to the parabola $y^2 = 4ax$ and the other end of the chord is $(at_1^2, 2at_1)$, then $t_1 = -t - \frac{2}{t}$. The vertex is the origin. Thus the gradient of the line joining the origin and the point $(at^2, 2at)$ is $\frac{2}{t}$. Similarly the gradient of the line joining the vertex and the point $(at_1^2, 2at_1)$ is $2/t_1$. Here

   $$\frac{2}{t} \cdot \frac{2}{t_1} = -1 \quad \text{or,} \quad \frac{4}{t} \cdot \frac{t}{t^2+2} = 1 \quad \text{or,} \quad t^2 = 2.$$

   The equation of the chord is $y = -tx + 2at + at^3$.
   Thus we can say that the inclination of the chord to the x-axis is $\tan^{-1}\sqrt{2}$.]

   (c) If the normal to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ at the point whose eccentric angle is ϕ cuts the curve again at the point whose eccentric angle is 2ϕ, show that $\cos\phi = -2/3$.

[*Hints*. The normal at ϕ is

$$\frac{\sqrt{14}\,(x-\sqrt{14}\cos\phi)}{\cos\phi}=\frac{\sqrt{5}\,(y-\sqrt{5}\sin\phi)}{\sin\phi}.$$

The point $(\sqrt{14}\cos 2\phi, \sqrt{5}\sin 2\phi)$ lies on it.

$$\therefore\quad \frac{14(\cos 2\phi-\cos\phi)}{\cos\phi}=\frac{5(\sin 2\phi-\sin\phi)}{\sin\phi}.$$

On simplification the result is obtained.]

7. If the tangent and normal to a rectangular hyperbola cut off intercepts, a_1 and a_2 on one axis and b_1 and b_2 on the other, show that $a_1a_2+b_1b_2=0$.

8. Prove that the locus of the foot of the perpendicular from the focus to any tangent to $y^2=4ax$ is the tangent at the vertex.

9. Find the locus of the point of intersection of two normals to the parabola which are at right angles to one another.

10. Find the chord of contact of the point

    (a) $(2,3)$ w.r.t. $2x^2+6xy+4y^2-8x+7=0$,
    (b) $(-1,2)$ w.r.t. $7x^2-8xy+5y^2-4x-6y+5=0$.

11. (a) Find the polar of the point

    (i) $(2,3)$ w.r.t. $2x^2-3xy+y^2-2x+3=0$,
    (ii) $(0,-3)$ w.r.t. $xy+2x+3y-1=0$,
    (iii) $(0,0)$ w.r.t. $ax^2+2hxy+by^2+2gx+2fy=0$.

    (b) Prove that the directrix is the polar of the corresponding focus of a hyperbola.

    (c) Prove that the polars of the point $(1,-2)$ w.r.t. the circles whose equations are $x^2+y^2+6y+5=0$ and $x^2+y^2+2x+8y+5=0$ coincide; prove also that there is another point the polars of which w.r.t. these circles are the same and find its coordinates.

    [*Hints*. The polars of the point (x_1,y_1) w.r.t. the given circles are

$$x_1x+(y_1+3)\,y+3y_1+5=0$$
$$\text{and}\quad (x_1+1)\,x+(y_1+4)\,y+x_1+4y_1+5=0.$$

These are identical.

$$\therefore\quad \frac{x_1}{x_1+1}=\frac{y_1+3}{y_1+4}=\frac{3y_1+5}{x_1+4y_1+5}.$$

By the first and second,

$$x_1-y_1-3=0. \tag{1}$$

By the first and third,

$$x_1^2 + x_1 y_1 - 3y_1 - 5 = 0. \qquad (2)$$

By (1) and (2),

$x_1^2 + x_1(x_1 - 3) - 3(x_1 - 3) - 5 = 0$ or, $x_1^2 - 3x_1 + 2 = 0$ or, $x_1 = 1, 2$.

For $x_1 = 1$, $y_1 = -2$ and for $x_1 = 2$, $y_1 = -1$.
Therefore, the other point is $(2, -1)$.]

12. (a) Find the pole of the line
    (i) $5x - 4y - 14 = 0$ w.r.t. $x^2 + y^2 + 6x - 10y - 15 = 0$,
    (ii) $x + y - 1 = 0$ w.r.t. $3x^2 + 4xy - 2y^2 - 5x + 7y - 10 = 0$.

    (b) Show that the points $(1, 3)$ and $(23, 12)$ are conjugate points w.r.t. the conic $x^2 - 4xy + 4y^2 - 6x + 8y + 7 = 0$.

13. Find the length of the tangent to $5x^2 + 5y^2 + 8x + 7y - 10 = 0$ from the point $(-1, 2)$.

14. Find the tangents to

    (a) $x^2 + y^2 - 6x - 6y + 14 = 0$ from $(5, 5)$,
    (b) $3x^2 - 2xy + 7y^2 - 4x = 0$ from $(1, -1)$.

15. The polar of the point P w.r.t. the circle $x^2 + y^2 = a^2$ touches the circle $4x^2 + 4y^2 = a^2$, show that the locus of P is the circle $x^2 + y^2 = 4a^2$.

16. (a) Find the locus of the poles of the chords of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which subtend right angles at the centre. [CH 2002; NH 2007]

    (b) If the sum of the ordinates of two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be b, show that the locus of the pole of the chord which joins them is $b^2x^2 + a^2y^2 = 2a^2by$. [CH 2005; BH 2007, 08; NH 2006]

    [*Hints.* Let the points be $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$.
    Here

$$\sin\theta + \sin\phi = 1. \qquad (1)$$

The equation of the chord joining the two points is

$$b\cos\frac{\theta+\phi}{2}x + a\sin\frac{\theta+\phi}{2}y = ab\cos\frac{\theta-\phi}{2}.$$

If (x_1, y_1) is the pole of it, then it is identical with $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

$$\frac{a\cos\frac{\theta+\phi}{2}}{x_1} = \frac{b\sin\frac{\theta+\phi}{2}}{y_1} = \cos\frac{\theta-\phi}{2}. \qquad (2)$$

Eliminating θ and ϕ from (1) and (2), the result is obtained.]

17. Prove that the locus of the poles of normal chords of the rectangular hyperbola $xy = c^2$ is $(x^2 - y^2)^2 + 4c^2xy = 0$. [CH 2006]

18. (a) Show that the polar of a point on $x^2 + 4ay = 0$ w.r.t. $xy = 2a^2$ will touch the parabola $y^2 = 4ax$. [NH 2008]

    (b) Show that the polar of any point on the circle $x^2 + y^2 - 2ax - 3a^2 = 0$ w.r.t. the circle $x^2+y^2+2ax-3a^2 = 0$ will touch the parabola $y^2 = -4ax$. [CH 2007]

    [*Hints.* If (x_1, y_1) is a point on $x^2 + y^2 - 2ax - 3a^2 = 0$, then

$$x_1^2 + y_1^2 - 2ax_1 - 3a^2 = 0. \tag{1}$$

The polar of the point (x_1, y) w.r.t. the circle $x^2 + y^2 + 2ax - 3a^2 = 0$ is

$$xx_1 + yy_1 + a(x + x_1) - 3a^2 = 0 \quad \text{or,} \quad y = -\frac{x_1 + a}{y_1}x + \frac{3a^2 - ax_1}{y_1}. \tag{2}$$

Now

$$\frac{3a^2 - ax_1}{y_1} - \frac{ay_1}{x_1 + a} = \frac{-a(x_1^2 + y_1^2 - 2ax_1 - 3a^2)}{(x_1 + a)y_1} = 0 \quad [\text{by } (1)].$$

Therefore, the polar touches the given parabola.]

19. Find the locus of the poles of tangents to the director circle of an ellipse w.r.t. this ellipse.

20. (a) Find the locus of the poles of tangents to the auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ w.r.t. the hyperbola itself.

    (b) Show that the locus of the poles of chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtends a right angle at the centre is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2}$. [CH 93; BH 2008]

21. (a) Find the locus of the point of intersection of two tangents to a parabola such that their chord of contact subtends a right angle at the vertex.

    (b) Tangents are drawn to the parabola $y^2 = 4ax$ at the points whose abscissae are in the ratio $p : 1$. Show that the locus of their points of intersection is a parabola.

    (c) Two lines are at right angles to one another, one of them touches $y^2 = 4a(x + a)$ and the other touches $y^2 = 4a'(x + a')$. Show that the point of intersection of the lines will lie on the line $x + a + a' = 0$.

    [*Hints.*

$$y = m(x + a) + \frac{a}{m} \tag{1}$$

touches $y^2 = 4a(x + a)$ and

$$y = m'(x + a') + \frac{a'}{m'}$$

touches $y^2 = 4a'(x+a')$.
Here $mm' = -1$.

$$\therefore\ y = -\frac{1}{m}(x+a') - a'm. \tag{2}$$

Subtracting (2) from (1),

$$0 = \left(m+\frac{1}{m}\right)x + (a+a')\left(m+\frac{1}{m}\right).$$

$\therefore\ x+a+a' = 0.$]

22. Show that the circle described on the line joining any point on the parabola to the focus as diameter touches the tangent at the vertex. [BH 2007]

23. Prove analytically that the difference of the squares of the tangents drawn from a point to two concentric circles is independent of the position of the point.

24. (a) Tangents are drawn from (a, b) to the circle $x^2+y^2=a^2$. Show that the area of the triangle formed by them and the straight line joining their points of contact is $\frac{ab^3}{a^2+b^2}$.

    (b) Prove that the area of the triangle formed by the tangents from (h, k) to the parabola $y^2 = 4ax$ and chord of contact is $\frac{(k^2-4ah)^{3/2}}{2a}$.
    [CH 2005; NH 2007]

25. (a) Show that the sub-tangent at a point on a parabola is bisected at the vertex.

    (b) Show that the sub-tangent and sub-normal of a point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $\left(\frac{a^2}{x_1}-x_1\right)$ and $\frac{b^2}{a^2}x_1$ respectively.

26. Find the common tangents to the circles

$$x^2+y^2+4x+2y-4=0$$
$$\text{and}\quad x^2+y^2-4x-2y+4=0.$$

27. (a) The normal at a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the axes in G and g. Find the equation of the locus of the middle point of Gg and show that the eccentricity of this locus is equal to that of the given ellipse.

    (b) If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of four points on the ellipse such that the normals at them are concurrent, then show that $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.

    [*Hints.* Let $(a\cos\phi, b\sin\phi)$ be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The normal at this point is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2-b^2 = a^2e^2.$$

If it passes through (h, k), then

$$\frac{ah}{\cos\phi} - \frac{bk}{\sin\phi} = a^2e^2$$

$$\text{or,}\quad ah\frac{1+\tan^2\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}} - bk\frac{1+\tan^2\frac{\phi}{2}}{2\tan\frac{\phi}{2}} = a^2e^2$$

$$\text{or,}\quad bkt^4 + 2\left(ah+a^2e^2\right)t^3 + 2\left(ah-a^2e^2\right)t - bk = 0$$

where $\tan\frac{\phi}{2} = t$.
Let t_1, t_2, t_3, t_4 be the roots.
Then

$$S_1 = \sum t_1 = -\frac{2\left(ah+a^2e^2\right)}{bk}, \qquad S_2 = \sum t_1t_2 = 0,$$

$$S_3 = \sum t_1t_2t_3 = -2\frac{ah-a^2e^2}{bk}, \quad S_4 = t_1t_2t_3t_4 = -1.$$

If $\alpha, \beta, \gamma, \delta$ be the eccentric angles corresponding to t_1, t_2, t_3, t_4, then

$$\tan\left(\sum\frac{\alpha}{2}\right) = \frac{S_1 - S_3}{1 - S_2 + S_4} = \infty.$$

$$\therefore \sum\frac{\alpha}{2} = (2n+1)\pi/2 \quad \text{or,} \quad \sum\alpha = (2n+1)\pi].$$

28. (a) Find the condition that the coordinate axes may be conjugate lines w.r.t. the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.
[*Hints*. Let the pole of $y = 0$ be (x_1, y_1). Then $(ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + gx_1 + fy_1 + c = 0$ is identical with $y = 0$.

$$ax_1 + hy_1 + g = 0$$
$$\text{and}\quad gx_1 + fy_1 + c = 0.$$

From these

$$x_1 = \frac{ch - fg}{af - gh}, \quad y_1 = \frac{g^2 - ca}{af - gn}.$$

It satisfies $x = 0$. $\therefore$ the condition is $ch - fg = 0$.]

(b) Find the condition that the straight lines

$$lx + my + n = 0$$
$$\text{and}\quad l'x + m'y + n' = 0$$

may be conjugate lines for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

29. Show that the coordinate axes are tangents to the ellipse

$$x^2 + 3xy + 4y^2 - 28x - 56y + 196 = 0$$

and that the area of it is $8\sqrt{7}\pi$.

[*Hints*. Pair of tangents from $(0,0)$ is

$$\left(x^2+3xy+4y^2-28x-56y+196\right)\cdot 196=(-14x-28y+196)^2 \quad \text{or,} \quad xy=0.$$

The centre of the ellipse is $(8,2)$. Here $\Delta=-49$, $D=\frac{7}{4}$. Shifting the origin to the centre the equation reduces to $x^2+3xy+4y^2=28$.

Now rotating the axes through an acute angle θ where $\tan 2\theta=\frac{3}{1-4}=-1$, the reduced equation transforms to $\left(5+3\sqrt{2}\right)x^2+\left(5-3\sqrt{2}\right)y^2=56$.

Hence the area $=\frac{\pi\cdot 56}{\sqrt{(5+3\sqrt{2})(5-3\sqrt{2})}}=8\sqrt{7}\pi$.]

30. (a) The normal to the rectangular hyperbola $xy=c^2$ at a point P on it meets the curve again at Q and touches the conjugate hyperbola. Show that $(PQ)^4=512c^4$. [NH 2008]

[*Hints*. The normal at $P(ct,c/t)$ is $y-c/t=t^2(x-ct)$. It meets the curve at Q whose coordinates are $\left(-\frac{c}{t^3},-ct^3\right)$. If the normal touches $xy=-c^2$, then

$$t^6+1/t^2-6t^2=0$$

$$\text{or,} \quad \left(\frac{1}{t^3}+t\right)^2=\frac{8}{t^2} \quad \text{and} \quad \left(t^3+\frac{1}{t}\right)^2=8t^2 \quad \text{and} \quad t^2+\frac{1}{t^2}=2\sqrt{2}.$$

Now

$$PQ^2=\left(ct+\frac{c}{t^3}\right)^2+\left(\frac{c}{t}+ct^3\right)^2=c^2\left(\frac{8}{t^2}+8t^2\right)=16\sqrt{2}c^2.$$

$\therefore\ PQ^4=512c^4$.]

(b) If a rectangular hyperbola circumscribes a triangle, prove that it also passes through the ortho-centre of the triangle.

[*Hints*. Let $xy=c^2$ be the equation of the rectangular hyperbola and

$$A\left(ct_1,\frac{c}{t_1}\right),\ B\left(ct_2,\frac{c}{t_2}\right),\ C\left(ct_3,\frac{c}{t_3}\right)$$

be the vertices of the triangle. The equation BC is $x+yt_2t_3=c\,(t_2+t_3)$. The equation of the line passing through A and perpendicular to BC is

$$y+ct_1t_2t_3=t_2t_3\left(x+\frac{c}{t_1t_2t_3}\right). \tag{1}$$

The equation of the line passing through B and perpendicular to CA is

$$y+ct_1t_2t_3=t_3t_1\left(x+\frac{c}{t_1t_2t_3}\right). \tag{2}$$

From (1) and (2) the ortho-centre is $\left(-\frac{c}{t_1t_2t_3},-ct_1t_2t_3\right)$. It lies on $xy=c^2$. If the ortho-centre is $\left(ct_4,\frac{c}{t_4}\right)$, then $t_1t_2t_3t_4=-1$.]

31. If three normals from a point to the parabola $y^2 = 4ax$ cut the axis in points whose distances from the vertex are in A.P., show that the point lies on the curve $27ay^2 = 2(x-2a)^3$.

[*Hints*. The normal at $(at^2, 2at)$ is $y + tx = 2at + at^3$. If it passes through (h,k), then $at^3 + (2a-h)t - k = 0$. Let the roots be t_1, t_2 and t_3. These values correspond to three normals through (h,k). Here $\Sigma t_1 = 0$, $\Sigma t_1t_2 = \frac{2a-h}{a}$, $t_1t_2t_3 = \frac{k}{a}$. The normals meet the axis at the points whose distances from the vertex are $2a + at_1^2, 2a + at_2^2, 2a + at_3^2$.

Here

$$4a + a\left(t_1^2 + t_3^2\right) = 2a\left(2 + t_2^2\right) \quad \text{or,} \quad t_1^2 + t_3^2 = 2t_2^2$$
$$\text{or,} \quad (t_1 + t_3)^2 - 2t_1t_3 = 2t_2^2 \quad \text{or,} \quad t_2^2 = -2t_1t_3.$$

Again
$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a-h}{a} \quad \text{or,} \quad -t_2^2 + t_1t_3 = \frac{2a-h}{a}$$
$$\text{or,} \quad 3t_1t_3 = \frac{2a-h}{a}.$$
$$(t_1t_2t_3)^2 = \frac{k^2}{a^2} \quad \text{or,} \quad -2(t_1t_3)^3 = \frac{k^2}{a^2}.$$

By (1) and (2),

$$-2\cdot\left(\frac{2a-h}{3a}\right)^3 = \frac{k^2}{a^2} \quad \text{or,} \quad 27ak^2 + 2(2a-h)^3 = 0.$$

Therefore, the required locus is $27ay^2 + 2(2a-x)^3 = 0$.]

32. The polars of $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ w.r.t. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet the conic in Q_1, R_1 and Q_2, R_2 respectively. Show that the six points lie on the conic

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)\left(\frac{xx_2}{a^2} + \frac{yy_2}{b^2} - 1\right).$$

[CH 95, 97, 2009]

[*Hints*. Let

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

$$L_1 = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0 \quad \text{and} \quad L_2 = \frac{xx_2}{a^2} + \frac{yy_2}{b^2} - 1 = 0.$$

$S - \lambda L_1L_2 = 0$ passes through Q_1, R_1, Q_2 and R_2. If this passes through P_1 and P_2, then

$$\lambda = \frac{1}{\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1}.$$

Hence the result follows.]

33. Show that the equation of the pair of (imaginary) tangents to a conic drawn from the focus of it satisfies the analytical condition of a circle. Hence find the foci of a conic.

[*Hints.* Let the focus be $(0,0)$ and the directrix be perpendicular to x-axis. Then the equation of the directrix is of the form $x+k=0$ and the equation of the conic is $x^2+y^2=e(x+k)^2$, where e is the eccentricity.

The equation of the pair of tangents from the focus $(0,0)$ is $x^2+y^2=0$. Here the coefficient of x^2 = the coefficient of y^2 and the coefficient of xy is zero. These analytical conditions for a circle remain invariant when the axes and origin of coordinates are changed.

If $\phi(x,y)=ax^2+2hxy+by^2+2gx+2fy+c=0$ be the equation of a conic and (x_1,y_1) be its focus, then the equation of pair of tangents drawn from (x_1,y_1) is

$$\begin{aligned}\phi(x_1,y_1)&\left(ax^2+2hxy+by^2+2gx+2fy+c\right)\\ &=\left[x\left(ax_1+hy_1+g\right)+y\left(hx_1+by_1+f\right)+\left(gx_1+fy_1+c\right)\right]^2.\end{aligned}$$

It must be a circle. Therefore, coefficient of x^2 = coefficient of y^2 and coefficient of $xy=0$. From these two equations, coordinates of foci are obtained.]

ANSWERS

1. (a) $y-4=0, x-2=0$;
   (b) $13x+y-14=0, x-13y+12=0$;
   (c) $222x-13y-547=0, 13x+222y+5537=0$.
2. (a) $y-12=0, y+2=0, (-3,12), (-3,-2)$;
   (b) $y=\sqrt{3}x+\frac{a}{\sqrt{3}}, \left(\frac{a}{3},\frac{2a}{\sqrt{3}}\right)$.
3. (a) $(-1,-1)$; (b) $c=am+\frac{1}{m}$; (c) $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$.
4. $x^2+y^2=2a^2$.
9. $y^2=a(x-3a)$.
10. (a) $9x+18y-1=0$; (b) $17x-11y-1=0$.
11. (a) (i) $3x-2=0$; (ii) $x-3y+11=0$; (iii) $gx+fy=0$.
12. (a) $(2,1)$; (b) $\left(-\frac{34}{7},\frac{19}{7}\right)$.
13. $\sqrt{\frac{21}{5}}$.
14. (a) $x-5=0, y-5=0$; (b) $5x^2+4xy-2y^2-6x-8y-1=0$.
16. (a) $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$.
19. $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$.
20. $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}$.
21. (a) $x+4a=0$.
26. $x=1, y=2, 4x-3y-10=0, 3x+4y-5=0$.

Chapter 6

Equation of a Chord in Terms of Its Middle Point and Diameter

6.10 Introduction

If (x_1, y_1) be the middle point of a chord of the conic

$$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

the equation of the chord is $T = S_1$, where

$$\begin{aligned} T &= axx_1 + h\,(xy_1 + yx_1) + byy_1 + g\,(x + x_1) + f\,(y + y_1) + c \\ \text{and} \quad S_1 &= ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c. \end{aligned}$$

Let the equation of the chord be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = (r \text{ say}). \tag{1}$$

l and m are cosines of the angles made by the line with the axes and r is the distance between (x_1, y_1) and (x, y) on the line.

From (1) $x = lr + x_1$, $y = mr + y_1$.

Putting these values of x and y in S and arranging the terms

$$\begin{aligned} \left(al^2 + 2hlm + bm^2\right) r^2 + 2\left\{(ax_1 + hy_1 + g)\,l + (hx_1 + by_1 + f)\,m\right\} r \\ + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \end{aligned}$$

It is a quadratic equation in r. The roots are the distances of the points of intersection between (1) and S from (x_1, y_1). Since (x_1, y_1) is the midpoint of the chord, the roots are equal in magnitude but opposite in sign, i.e. the sum of the roots is zero.

$$(ax_1 + hy_1 + g)\,l + (hx_1 + by_1 + f)\,m = 0. \tag{2}$$

Eliminating l and m by (1),

$$(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1) = 0.$$

This can be written as

$$axx_1 + h(xy_1 + yx_1) + byy_1 + gx + fy = ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1.$$

Adding $gx_1 + fy_1 + c$ to both sides, we have

$$\begin{aligned} &axx_1 + h(xy_1 + yx_1) + g(x + x_1) + f(y + y_1) + c \\ &\quad = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \\ \text{or,}\quad & T = S_1. \end{aligned}$$

It is the equation of the chord.

Particular cases

(i) **Parabola: $y^2 = 4ax$.**

The equation of the chord is

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1 \quad \text{or,} \quad yy_1 - 2ax = y_1^2 - 2ax_1.$$

(ii) **Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.**

The equation of the chord is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \quad \text{or,} \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

(iii) **Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.**

The equation of the chord is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}.$$

(iv) **Rectangular hyperbola: $xy = c^2$.**

The equation of the chord is $xy_1 + yx_1 = 2x_1y_1$.

6.20 Diameter and conjugate diameters

Definition. *The locus of the middle points of a series of parallel chords of a conic is called a diameter of the conic.*

Let the chords be parallel to

$$\frac{x}{l} = \frac{y}{m} \tag{1}$$

and (x_1, y_1) be the middle point of one of these chords of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

The equation of this chord is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$$

or, $$(ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y = ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1. \tag{2}$$

Since (2) is parallel to (1),

$$\frac{ax_1 + hy_1 + g}{m} = -\frac{hx_1 + by_1 + f}{l}.$$

Therefore, the locus of the middle points or, the equation of the diameter is

$$\frac{ax + hy + g}{m} + \frac{hx + by + f}{l} = 0 \tag{3}$$

or, $$(al + hm)x + (hl + bm)y + gl + fm = 0. \tag{4}$$

Definition (*Conjugate diameters*). *Two diameters of a conic are said to be conjugate when each bisects the chords parallel to the other.*

Let the diameter (4) be parallel to $\frac{x}{l'} = \frac{y}{m'}$.

Then $l'(al + hm) + m'(hl + bm) = 0$ or, $all' + h(lm' + l'm) + bmm' = 0$.

It is the condition for conjugate diameters corresponding to the chords parallel to $\frac{x}{l} = \frac{y}{m}$ and $\frac{x}{l'} = \frac{y}{m'}$.

Corollary. The equation (4) suggests that the diameter passes through the point of intersection of the lines $ax + hy + g = 0$ and $hx + by + f = 0$. This point is the centre of the central conic. Thus the diameter of a central conic passes through the centre of it.

6.21 Diameters of standard conics

(i) **Parabola:** $y^2 = 4ax$.

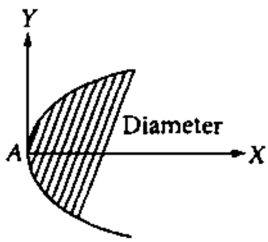


Fig. 44

The equation of the chord whose middle point is (x_1, y_1) is $yy_1 - 2ax = y_1^2 - 2ax_1$.

If it is parallel to

$$y = mx, \text{ then } \frac{2a}{y_1} = m \text{ or, } y_1 = \frac{2a}{m}.$$

$\therefore$ the equation of the diameter corresponding to this system of chords is $y = \frac{2a}{m}$. It is a straight line parallel to the axis of the parabola.

(ii) **Ellipse:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

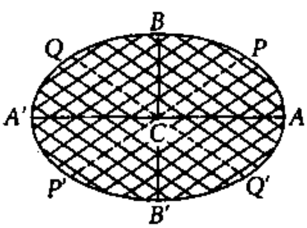


Fig 45

The equation of the chord whose middle point is (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

If it is parallel to $y = mx$, then

$$m = -\frac{b^2}{a^2}\frac{x_1}{y_1} \quad \text{or,} \quad y_1 = -\frac{b^2}{a^2 m}x_1.$$

Hence the equation of the diameter corresponding to this system of chords is

$$y = -\frac{b^2}{a^2 m}x.$$

Let us consider the diameter corresponding to the chords parallel to $y = m'x$ where $m' = -\frac{b^2}{a^2 m}$. The equation of this diameter is $y = mx$. It is a member of the system of chords for the diameter $y = m'x$. Therefore, $y = m'x$ and $y = mx$ are conjugate diameters. Moreover, $mm' = -\frac{b^2}{a^2}$.

Thus $y = mx$ and $y = m'x$ will be two conjugate diameters of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $mm' = -\frac{b^2}{a^2}$. In Fig. 45, PCP' and QCQ' are conjugate diameters.

(iii) **Hyperbola:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The equation of the diameter for a system of chords parallel to $y = mx$ is $y = \frac{b^2}{a^2 m}x$. The condition for which $y = mx$ and $y = m'x$ will be conjugate is that $mm' = \frac{b^2}{a^2}$.

(iv) **Rectangular hyperbola:** $xy = c^2$.

The equation of the diameter for a system of chords parallel to $y = mx$ is $y = -mx$. $y = mx$ and $y = m'x$ will be conjugate diameters, if $m + m' = 0$.

6.22 Properties of diameters and conjugate diameters

Case I. *Ellipse:* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(i) ***The tangent at the extremity of a diameter is parallel to the chords bisected by the diameter.***

Let the chords be parallel to $y = mx$. The equation of the diameter is $y = -\frac{b^2}{a^2 m}x$. If this diameter meets the ellipse at (h, k), then

$$k = -\frac{b^2}{a^2 m}h \quad \text{or,} \quad m = -\frac{b^2 h}{a^2 k}.$$

The equation of the tangent at (h, k) is

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1.$$

Slope of it $= -\frac{b^2 h}{a^2 k}$ which is equal to m. Hence the result follows.

(ii) ***The tangents at the extremities of any chord meet on the diameter which bisects this chord.***

Let the chords be parallel to $y = mx$. The equation of the diameter is $y = -\frac{b^2}{a^2m}x$. If the tangents at the ends of a chord of this system meet at (h, k), then the equation of this chord is $\frac{xh}{a^2} + \frac{yk}{b^2} = 1$ (chord of contact). Since the chord is parallel to

$$y = mx, \ m = -\frac{b^2h}{a^2k} \quad \text{or,} \quad k = -\frac{b^2}{a^2m}h.$$

It shows that (h, k) lies on the diameter.

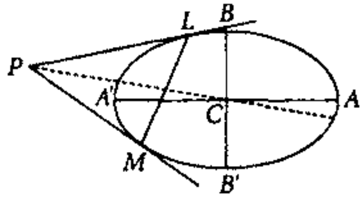


Fig. 46

(iii) *The eccentric angles of the ends of a pair of conjugate diameters differ by an odd multiple of $\pi/2$.*

Let PCP' and QCQ' be the two conjugate diameters and the coordinates of P and Q be $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$ respectively. C is the centre of the ellipse.

Since C is the origin, the gradients of CP and CQ are

$$\frac{b\sin\theta}{a\cos\theta} \quad \text{and} \quad \frac{b\sin\phi}{a\cos\phi}.$$

From the condition of conjugate diameters,

$$\frac{b\sin\theta}{a\cos\theta}\cdot\frac{b\sin\phi}{a\cos\phi} = -\frac{b^2}{a^2}$$

$$\text{or,} \quad \sin\theta\sin\phi = -\cos\theta\cos\phi$$

$$\text{or,} \quad \cos(\phi - \theta) = 0.$$

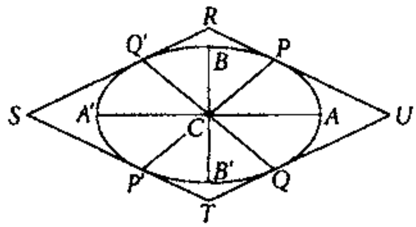


Fig. 47

$\therefore\ \phi - \theta = \pi/2$ and in general $\phi - \theta =$ odd multiple of $\pi/2$.

Corollary. The coordinates of P, Q, P' and Q' are

$$(a\cos\theta, b\sin\theta),$$

$$\left\{a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right\}, \left\{a\cos\left(2\cdot\frac{\pi}{2}+\theta\right), b\sin\left(2\cdot\frac{\pi}{2}+\theta\right)\right\}$$

$$\text{and} \quad \left\{a\cos\left(3\cdot\frac{\pi}{2}+\theta\right), b\sin\left(3\cdot\frac{\pi}{2}+\theta\right)\right\};$$

i.e. $(a\cos\theta, b\sin\theta), (-a\sin\theta, b\cos\theta), (-a\cos\theta, -b\sin\theta)$

and $(a\sin\theta, -b\cos\theta)$ respectively.

(iv) *The sum of the squares of two conjugate semi-diameters is constant.* [See Fig. 47.]

Let CP and CQ be a pair of conjugate semi-diameters. If the coordinates of P be $(a\cos\theta, b\sin\theta)$, then the coordinates of Q are $(-a\sin\theta, \cos\theta)$.

$$CP^2 + CQ^2 = (a^2\cos^2\theta + b^2\sin^2\theta) + (a^2\sin^2\theta + b^2\cos^2\theta)$$
$$= (a^2 + b^2)(\cos^2\theta + \sin^2\theta) = a^2 + b^2 \text{ (constant)}.$$

(v) *The tangents at the ends of a pair of conjugate diameters form a parallelogram of constant area.* [See Fig. 47.]

Let PCP' and QCQ' be a pair of conjugate diameters and the coordinates of P, Q, P' and Q' be $(a\cos\theta, b\sin\theta)$, $(-a\sin\theta, b\cos\theta)$, $(-a\cos\theta, -b\sin\theta)$ and $(a\sin\theta, -b\cos\theta)$ respectively.

The tangents at P, Q, P' and Q' are

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1, \tag{1}$$

$$\frac{x\sin\theta}{a}-\frac{y\cos\theta}{b}=-1, \tag{2}$$

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=-1, \tag{3}$$

$$\frac{x\sin\theta}{a}-\frac{y\cos\theta}{b}=1 \text{ respectively.} \tag{4}$$

Obviously the tangents at P and P' i.e. (1) and (3) and at Q and Q' i.e. (2) and (4) are parallel. Therefore, these tangents form a parallelogram.

By property (i) the tangents at the extremities of a diameter are parallel to the conjugate diameter. Therefore, the tangents at P and P' are parallel to QCQ' and the tangents at Q and Q' are parallel to PCP'.

Thus by symmetry the area of the parallelogram $RSTU$ formed by the tangents

$= 4 \times$ area of the parallelogram $CPRQ$

$= 8 \times \Delta CPQ$

$$= 8\times\frac{1}{2}\begin{vmatrix} 0 & 0 & 1\\ a\cos\theta & b\sin\theta & 1\\ -a\sin\theta & b\cos\theta & 1\end{vmatrix} = 4ab\left(\cos^2\theta+\sin^2\theta\right) = 4ab \text{ (constant).}$$

(vi) *The product of the perpendicular distance from the centre of the ellipse to the tangent at one end of a diameter and the semi-conjugate diameter is constant.*

Let PCP' and QCQ' be the conjugate diameters. If $(a\cos\theta, b\sin\theta)$ be the coordinates of P, then $(-a\sin\theta, b\cos\theta)$ are the coordinates of Q. The tangent at P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$

If p is the perpendicular distance from the centre $(0, 0)$ to the tangent, then

$$p=\frac{1}{\sqrt{\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)}}=\frac{ab}{\sqrt{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}}.$$

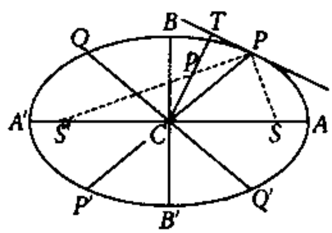


Fig. 48

Semi-conjugate diameter $CQ=\sqrt{\left(a^2\sin^2\theta+b^2\cos^2\theta\right)}$.

$\therefore\ p\cdot CQ = ab =$ constant.

(vii) *The product of the focal distances of a point on an ellipse is equal to the square on the semi-diameter parallel to the tangent at this point.* [See Fig. 48.]

Let $P(a\cos\theta, b\sin\theta)$ be a point on the ellipse.

If S and S' are the foci, then

$$SP = \sqrt{(a\cos\theta - ae)^2 + b^2\sin^2\theta} = a - ae\cos\theta \;\left[\text{using } b^2 = a^2\left(1-e^2\right)\right],$$
$$S'P = \sqrt{(a\cos\theta + ae)^2 + b^2\sin^2\theta} = a + ae\cos\theta.$$
$$\therefore\; SP \cdot S'P = a^2 - a^2e^2\cos^2\theta = a^2 + \left(b^2 - a^2\right)\cos^2\theta$$
$$= a^2\sin^2\theta + b^2\cos^2\theta = CQ^2.$$

CQ is the semi-diameter which is conjugate to the diameter through P.

Note. Equi-conjugate diameters.

If two conjugate diameters of an ellipse are equal in length, then they are said to be equiconjugate diameters.

In this case

$$CP^2 = CQ^2 \quad \text{or,} \quad a^2\cos^2\theta + b^2\sin^2\theta = a^2\sin^2\theta + b^2\cos^2\theta$$
$$\text{or,}\quad \left(a^2 - b^2\right)\cos^2\theta = \left(a^2-b^2\right)\sin^2\theta \quad \text{or,} \quad \tan^2\theta = 1 \quad \text{or,} \quad \tan\theta = \pm 1$$
$$\text{or,}\quad \theta = \tfrac{\pi}{4}, \tfrac{3\pi}{4}, \ldots$$

Case II. ***Hyperbola:*** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(i) *If a pair of diameters be conjugate w.r.t. a hyperbola, then they are also conjugate w.r.t. the conjugate hyperbola.*

Let $y = mx$ and $y = m'x$ be a pair of conjugate diameters of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

Then $mm' = \frac{b^2}{a^2}$.

The equation of the conjugate hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \text{or,} \quad \frac{x^2}{-a^2} - \frac{y^2}{-b^2} = 1. \tag{2}$$

The lines $y = mx$ and $y = m'x$ are the pair of conjugate diameters of (2), since $mm' = \frac{b^2}{a^2} = \frac{-b^2}{-a^2}$.

(ii) *If a diameter meets the hyperbola at real points, then it will meet its conjugate at imaginary points.*

Let

$$y = mx \tag{1}$$

be a diameter of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{2}$$

The conjugate hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \tag{3}$$

The x-coordinates of the points of intersection between (1) and (2) and (1) and (3) are obtained from

$$x^2 = \frac{a^2b^2}{b^2 - a^2m^2} \tag{4}$$

$$\text{and} \quad x^2 = -\frac{a^2b^2}{b^2 - a^2m^2} \tag{5}$$

If (4) gives the real values of x, then (5) must give the imaginary values of x. Hence the result follows.

(iii) *Only one of a pair of conjugate diameters of a hyperbola meets the curve in real points.*

Let

$$y = mx \tag{1}$$

$$\text{and} \quad y = m'x \tag{2}$$

be a pair of conjugate diameters of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{3}$$

Then

$$mm' = \frac{b^2}{a^2}. \tag{4}$$

The x-coordinates of points of intersection between (1) and (3) are obtained from

$$x^2 = \frac{a^2b^2}{b^2 - a^2m^2}. \tag{5}$$

The x-coordinates of points of intersection between (2) and (3) are obtained from

$$\begin{aligned} x^2 &= \frac{a^2b^2}{b^2 - a^2m'^2} = \frac{a^2b^2}{b^2 - a^2 \cdot b^4/a^4m^2} \\ &= \frac{a^4m^2}{a^2m^2 - b^2} = -\frac{a^4m^2}{b^2 - a^2m^2}. \end{aligned} \tag{6}$$

If (5) gives real values of x, then (6) gives imaginary values of x. Hence the result follows.

(iv) *If a pair of conjugate diameters meets the hyperbola and its conjugate at P and Q, then $CP^2 - CQ^2 = a^2 - b^2$ where C is the centre of the hyperbola.*

Let $(a\sec\theta, b\tan\theta)$ be the coordinates of P on

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

Since C is the origin, the slope of

$$CP = \frac{b\tan\theta}{a\sec\theta}. \tag{2}$$

Let $(a\tan\phi, b\sec\phi)$ be the coordinates of Q on the conjugate

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \qquad (3)$$

The slope of

$$CQ = \frac{b\sec\phi}{a\tan\phi}. \qquad (4)$$

Since the diameters are conjugate,

$$\frac{b\tan\theta}{a\sec\theta} \cdot \frac{b\sec\phi}{a\tan\phi} = \frac{b^2}{a^2}$$

or, $\tan\theta \cdot \sec\phi = \sec\theta \cdot \tan\phi$

or, $\sin\phi = \sin\theta$.

$\therefore\ \phi = \theta$ or, $\pi - \theta$.

θ and $\pi - \theta$ for the values of ϕ correspond to the points Q and Q'.

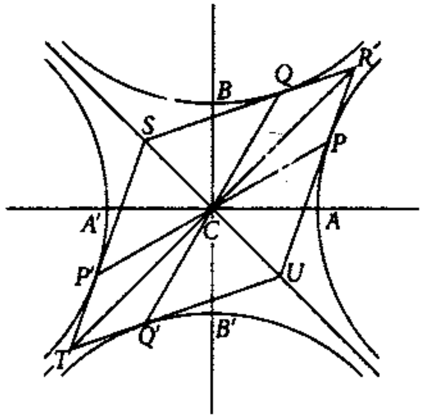


Fig. 49

Now

$$\begin{aligned} CP^2 - CQ^2 &= a^2\sec^2\theta + b^2\tan^2\theta - a^2\tan^2\phi - b^2\sec^2\phi \\ &= a^2\sec^2\theta + b^2\tan^2\theta - a^2\tan^2\theta - b^2\sec^2\theta \\ &= a^2\left(\sec^2\theta - \tan^2\theta\right) - b^2\left(\sec^2\theta - \tan^2\theta\right) = a^2 - b^2. \end{aligned}$$

(v) *The tangents at the points where a diameter and its conjugate meet a hyperbola and its conjugate form a parallelogram of constant area* [See Fig. 49].

Let PCP' and QCQ' be a pair of conjugate diameters of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (1)$$

C is the centre of the hyperbola.

Let PCP' meet (1) at $P(a\sec\theta, b\tan\theta)$ and $P'(-a\sec\theta, -b\tan\theta)$ and QCQ' meet the conjugate

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \qquad (2)$$

at Q and Q'. By property (iv) the coordinates of Q and Q' are $(a\tan\theta, b\sec\theta)$ and $(-a\tan\theta, -b\sec\theta)$ respectively.

The tangents at P and P' to (1) and at Q and Q' to (2) are

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1, \qquad (3)$$

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = -1, \qquad (4)$$

$$\frac{x\tan\theta}{a} - \frac{y\sec\theta}{b} = 1, \qquad (5)$$

$$\frac{x\tan\theta}{a} - \frac{y\sec\theta}{b} = -1. \qquad (6)$$

Obviously (3) and (4) are parallel and (5) and (6) are parallel. Therefore, these tangents form a parallelogram. In Fig. 49, $RSTU$ is the parallelogram formed by the tangents at the ends of conjugate diameters. Since tangents at P and P' are parallel to QCQ' and the tangents at Q and Q' are parallel to PCP',

$$\text{area of parallelogram } RSTU = 8 \times \Delta CPQ = 8 \cdot \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ a\sec\theta & b\tan\theta & 1 \\ a\tan\theta & b\sec\theta & 1 \end{vmatrix}$$

$$= 4ab = \text{a constant.}$$

6.30 Conjugate diameters of a central conic

The necessary and sufficient condition that the diameters $y = mx$ and $y = m'x$ of the central conic $ax^2 + 2hxy + by^2 = 1$ are conjugate to each other is

$$a + h(m + m') + bmm' = 0.$$

Proof. (i) *The condition is necessary.*

Let (x_1, y_1) be the midpoint of a chord parallel to $y = mx$. The equation of this chord is

$$axx_1 + h\,(xy_1 + yx_1) + byy_1 = ax_1^2 + 2hx_1y_1 + by_1^2$$

$$\text{or,}\quad (ax_1 + hy_1)\,x + (hx_1 + by_1)\,y = ax_1^2 + 2hx_1y_1 + by_1^2.$$

Since it is parallel to $y = mx$,

$$(ax_1 + hy_1) + m\,(hx_1 + by_1) = 0.$$

Therefore, the diameter corresponding to the system of chords parallel to $y = mx$ is

$$(a + hm)x + (h + bm)y = 0. \tag{1}$$

If $y = m'x$ is conjugate to $y = mx$, then $y = m'x$ is identical with the equation (1).

$$\therefore\ (a + hm) + m'(h + bm) = 0$$

$$\text{or,}\quad a + h(m + m') + bmm' = 0.$$

(ii) *The condition is sufficient.*

We have

$$a + h(m + m') + bmm' = 0. \tag{2}$$

The equation of the diameter corresponding to the system of chords parallel to $y = mx$ is

$$(a + hm)x + (h + bm)y = 0$$

$$\text{or,}\quad -(hm' + bmm')x + (h + bm)y = 0 \text{ [by (2)]}$$

$$\text{or,}\quad y = m'x.$$

$\therefore\ y = mx$ and $y = m'x$ are conjugate diameters.

WORKED-OUT EXAMPLES

1. *Find the equation of the chord of the parabola $y^2 = 8x$ which is bisected at $(2, -3)$.*

The equation of the chord is

$$y \cdot (-3) - 4(x+2) = (-3)^2 - 8.2$$
$$\text{or,} \quad -3y - 4x - 8 = 9 - 16 \quad \text{or,} \quad 4x + 3y + 1 = 0.$$

2. *Find the equation of the diameter of the ellipse $3x^2 + 4y^2 = 5$, conjugate to $y + 3x = 0$.*

The equation of the ellipse in standard form is

$$\frac{x^2}{5/3} + \frac{y^2}{5/4} = 1.$$

Let $y + mx = 0$ be the required diameter.

Then
$$3 \cdot m = -\frac{5/4}{5/3} = -\frac{3}{4} \quad \text{or,} \quad m = -1/4.$$

Thus $4y - x = 0$ is conjugate to $y + 3x = 0$.

3. *If the line $\frac{l}{a}x + \frac{m}{b}y = n$ cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the ends of conjugate diameters of the ellipse, prove that $l^2 + m^2 = 2n^2$.*

Let the coordinates of the ends of conjugate diameters be $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$. As the line passes through these points

$$l\cos\theta + m\sin\theta = n \tag{1}$$
$$\text{and} \quad -l\sin\theta + m\cos\theta = n. \tag{2}$$

Squaring and adding, $l^2 + m^2 = 2n^2$.

4. *Prove that the locus of the middle points of chords of a parabola which pass through a fixed point is a parabola.*

Let (x_1, y_1) be the middle point of a chord of the parabola $y^2 = 4ax$. The equation of this chord is

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1 \quad \text{or,} \quad yy_1 - 2ax = y_1^2 - 2ax_1.$$

If this chord passes through a fixed point (h, k), then

$$ky_1 - 2ah = y_1^2 - 2ax_1 \quad \text{or,} \quad y_1^2 - ky_1 = 2a(x_1 - h).$$

Hence the locus of (x_1, y_1) is $y^2 - ky = 2a(x - h)$. It is a parabola.

5. *Find the locus of the middle points of the normal chords of the parabola $y^2 = 4ax$.* **[CH 2000]**

Let (x_1, y_1) be the middle point of a chord. The equation of this chord is

$$yy_1 - 2ax = y_1^2 - 2ax_1. \tag{1}$$

If it is normal at $(at^2, 2at)$, then it is identical with

$$y + tx = 2at + at^3. \tag{2}$$

Comparing (1) with (2),

$$\frac{1}{y_1} = \frac{t}{-2a} = \frac{2at + at^3}{y_1^2 - 2ax_1}.$$

$$\therefore\ t = -\frac{2a}{y_1}. \tag{3}$$

and

$$2a + at^2 = \frac{y_1^2 - 2ax_1}{-2a}. \tag{4}$$

Eliminating t from (3) and (4),

$$2a + a \cdot \frac{4a^2}{y_1^2} = \frac{y_1^2 - 2ax_1}{-2a}$$

$$\text{or,}\quad y_1^4 - 2 \cdot ax_1y_1^2 + 4a^2y_1^2 + 8a^4 = 0.$$

$\therefore$ the required locus is $y^4 - 2axy^2 + 4a^2y^2 + 8a^4 = 0$.

6. *If the points of intersection of the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ be the ends of the conjugate diameters of the former, prove that $\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 2$.*

[CH 2001; NH 2008]

Let the ends of the conjugate diameters of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be $(a\cos\theta, b\sin\theta)$, $(-a\sin\theta, b\cos\theta)$, $(-a\cos\theta, -b\sin\theta)$ and $(a\sin\theta, -b\cos\theta)$.

If these points lie on the second ellipse, then

$$\frac{a^2\cos^2\theta}{\alpha^2} + \frac{b^2\sin^2\theta}{\beta^2} = 1 \tag{1}$$

$$\text{and}\quad \frac{a^2\sin^2\theta}{\alpha^2} + \frac{b^2\cos^2\theta}{\beta^2} = 1. \tag{2}$$

Adding (1) and (2), we have $\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 2$.

7. ***Show that the locus of the poles of the line joining the extremities of two conjugate diameters of the ellipse***

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{x^2}{a^2}+\frac{y^2}{b^2}=2. \qquad \textbf{[CH 92, 98, 2001, 03, 04]}$$

Let $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$ be the ends of conjugate diameters of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The equation of the tangents at these points are

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \tag{1}$$

$$\text{and} \quad \frac{x}{a}\sin\theta-\frac{y}{b}\cos\theta=-1. \tag{2}$$

The point of intersection of these tangents is the pole of the line joining the ends of conjugate diameters. Thus eliminating θ from (1) and (2) the required locus is obtained. To eliminate θ we square and add (1) and (2). By this way we have $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.

8. ***Lines drawn through the foci of an ellipse are perpendicular to a pair of conjugate diameters and intersect at Q show that the locus of Q is a concentric ellipse.*** **[CH 2004]**

Let $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$ be the ends of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The foci are $(\pm ae, 0)$ where $b^2=a^2\left(1-e^2\right)$. The slopes of these diameters are

$$\frac{b\sin\theta}{a\cos\theta} \quad \text{and} \quad -\frac{b\cos\theta}{a\sin\theta}.$$

The equation of the line passing through $(ae, 0)$ and perpendicular to the diameter corresponding to the slope $\frac{b\sin\theta}{a\cos\theta}$ is

$$y=-\frac{a\cos\theta}{b\sin\theta}(x-ae). \tag{1}$$

Similarly the equation of the other line is

$$y=\frac{a\sin\theta}{b\cos\theta}(x+ae). \tag{2}$$

(1) and (2) meet at Q. Let (α,β) be the coordinates of Q. Then from (1) and (2),

$$\tan\theta=-\frac{a}{b}\frac{\alpha-ae}{\beta}=\frac{b}{a}\frac{\beta}{\alpha+ae}$$

$$\text{or,} \quad -a^2\left(\alpha^2-a^2e^2\right)=b^2\beta^2.$$

Hence the required locus is $a^2x^2+b^2y^2=a^4e^2$ or, $a^2x^2+b^2y^2=a^2\left(a^2-b^2\right)$.

It is concentric with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

9. ***Show that the locus of the middle points of the normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.*** [NH 2004, 10; CH 2002]

Let (x_1, y_1) be the middle point of a chord of

$$x^2 - y^2 = a^2. \tag{1}$$

The equation of this chord is

$$xx_1 - yy_1 = x_1^2 - y_1^2. \tag{2}$$

Let it be normal at $(a\sec\theta, a\tan\theta)$ to the hyperbola.

The equation of the normal at this point is

$$x\cos\theta + y\cot\theta = 2a. \tag{3}$$

Comparing (2) and (3),

$$\frac{\cos\theta}{x_1} = \frac{\cot\theta}{-y_1} = \frac{2a}{x_1^2 - y_1^2}.$$

$$\therefore\ \cos\theta = \frac{2ax_1}{x_1^2 - y_1^2} \tag{4}$$

and

$$\sin\theta = -\frac{x_1}{y_1}. \tag{5}$$

From (4) and (5),

$$\frac{4a^2x_1^2}{(x_1^2 - y_1^2)^2} + \frac{x_1^2}{y_1^2} = 1$$

$$\text{or,}\quad \frac{4a^2x_1^2}{(x_1^2 - y_1^2)^2} + \frac{x_1^2 - y_1^2}{y_1^2} = 0$$

$$\text{or,}\quad 4a^2x_1^2y_1^2 + (x_1^2 - y_1^2)^3 = 0$$

$$\text{or,}\quad (y_1^2 - x_1^2)^3 = 4a^2x_1^2y_1^2.$$

Hence the locus is $(y^2 - x^2)^3 = 4a^2x^2y^2$.

10. ***Prove that the locus of the middle points of chords of contact of tangents to the hyperbola $x^2 - y^2 = a^2$ from points on the auxiliary circle is the curve $(x^2 - y^2)^2 = a^2(x^2 + y^2)$.*** [NH 2007]

The auxiliary circle is $x^2 + y^2 = a^2$. Let $(a\cos\theta, a\sin\theta)$ be a point on it. The chord of contact of this point w.r.t. $x^2 - y^2 = a^2$ is

$$x\cos\theta - y\sin\theta = a. \tag{1}$$

If (x_1, y_1) is the middle point of this chord, then the equation (1) is identical with

$$xx_1 - yy_1 = x_1^2 - y_1^2. \tag{2}$$

Comparing (1) with (2), we have

$$\frac{\cos\theta}{x_1} = \frac{\sin\theta}{y_1} = \frac{a}{x_1^2 - y_1^2} \quad \text{or,} \quad \cos\theta = \frac{ax_1}{x_1^2 - y_1^2}, \quad \sin\theta = \frac{ay_1}{x_1^2 - y_1^2}.$$

Squaring and adding,

$$1 = \frac{a^2\left(x_1^2 + y_1^2\right)}{\left(x_1^2 - y_1^2\right)^2} \quad \text{or,} \quad \left(x_1^2 - y_1^2\right)^2 = a^2\left(x_1^2 + y_1^2\right).$$

Hence the required locus is $\left(x^2 - y^2\right)^2 = a^2\left(x^2 + y^2\right)$.

EXERCISE XII

1. Find the equation of the chord of
   (a) $x^2 + y^2 = 81$, which is bisected at $(-2, 3)$;
   (b) $y^2 = 4x + 5$, which is bisected at $(1, 4)$;
   (c) $x^2 - y^2 = 9$, which is bisected at $(5, -3)$.

2. Find the locus of the middle points of chords of
   (a) $y^2 = 16x$, which are parallel to $x - 3y = 5$;
   (b) $y^2 = 4ax$, passing through the vertex;
   (c) $y^2 = 4ax$, passing through the focus;
   (d) $xy = c^2$, which are parallel to $2x + 3y = 7$.

3. Find the diameter of
   (a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$, which is conjugate to $5y - 4x = 0$;
   (b) $16x^2 - 9y^2 = 144$, which is conjugate to $x = 2y$.

4. Show that $25x^2 + kxy - 36y^2 = 0$ represents a pair of conjugate diameters of $25x^2 + 36y^2 = 900$ for any value of k.

5. Show that the conjugate diameters of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are equally inclined to each of the axes are $bx + ay = 0$ and $bx - ay = 0$.

   [*Hints.* Let $y = mx$ and $y = m'x$ be the conjugate diameters of the ellipse which are equally inclined to each of the axes.
   Then $m = -m'$ and $mm' = -\frac{b^2}{a^2}$.

$$\therefore\ m^2 = \frac{b^2}{a^2} \quad \text{or,} \quad m = \pm\frac{b}{a}.$$

   Thus the diameters are $y = \pm\frac{b}{a}x$ or, $bx + ay = 0$ and $bx - ay = 0$.]

6. If the pair of lines $Ax^2+2Hxy+By^2=0$ be a pair of conjugate diameters of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that $a^2A+b^2B=0$.

7. If P and Q be the extremities of conjugate diameters of the ellipse $2x^2+3y^2=24$, find the locus of the point of intersection of the tangents at P and Q.

8. If P and Q be the extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that the locus of the middle point of PQ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$ and also PQ touches this ellipse. [CH 2003]

9. If the angle between two equal conjugate diameters of an ellipse is 60°, find the eccentricity of it.

   [*Hints*. Let PCP' and QCQ' be the two equal conjugate diameters of
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

   We have $CP^2+CQ^2=a^2+b^2$.

   $\therefore\ 2CP^2=a^2+b^2,\ \because\ CP=CQ.$

   If ϕ is the eccentric angle of P, then
$$\begin{aligned}
&2\left(a^2\cos^2\phi+b^2\sin^2\phi\right)=a^2+b^2\\
\text{or,}\quad &a^2\cos 2\phi=b^2\cos 2\phi\\
\text{or,}\quad &\left(a^2-b^2\right)\cos 2\phi=0\\
\text{or,}\quad &\cos 2\phi=0,\ (\because\ a\neq b)\\
\therefore\quad &\phi=45^\circ,135^\circ.
\end{aligned}$$

   Now the equation of CP and CQ are $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$.

   By the condition
$$\begin{aligned}
&\tan 60^\circ=\frac{b/a+b/a}{1-b^2/a^2}=\frac{2ab}{a^2-b^2}\\
\text{or,}\quad &\sqrt{3\left(a^2-b^2\right)}=2ab\quad\text{or,}\quad 3\left(a^2-b^2\right)^2=4a^2b^2\\
\text{or,}\quad &3a^4e^4=4a^4\left(1-e^2\right)\quad\text{or,}\quad 3e^4+4e^2-4=0\\
\text{or,}\quad &e^2=\tfrac{2}{3}\quad\text{or,}\quad e=\sqrt{\tfrac{2}{3}}.]
\end{aligned}$$

10. If (x_1,y_1) and (x_2,y_2) be the extremities of a pair of conjugate diameters of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that

    (a) $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=0$; (b) $x_2=\pm\frac{a}{b}y_1, y_2=\mp\frac{b}{a}x_1$; (c) $x_1y_2-x_2y_1=\pm ab$.

11. The normal at a variable point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the diameter CD conjugate to CP at Q. Show that the locus of Q is
$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=\left(\frac{a^2-b^2}{x^2+y^2}\right)^2. \qquad \text{[CH 93]}$$

12. Show that the locus of the middle points of the normal chords of the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=\left(a^2-b^2\right)^2.$$

13. (x_1, y_1) is the middle point of a chord of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If (h,k) is the pole of this chord w.r.t. the ellipse, show that
$$h=\frac{x_1}{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}}, \quad k=\frac{y_1}{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}}.$$

14. If a hyperbola be rectangular and its equation be $xy=c^2$, prove that the locus of the middle points of chords of constant length $2d$ is
$$\left(x^2+y^2\right)\left(xy-c^2\right)=d^2xy.$$ [NH 2000]

15. Find the locus of the middle points of chords of an ellipse, the tangents at the extremities of which intersect at right angles.

16. If PCP' and DCD' be two mutually perpendicular diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that $\frac{1}{CP^2}+\frac{1}{CD^2}=\frac{1}{a^2}+\frac{1}{b^2}$. [NH 2004; CH 95]

[*Hints.* Let the diameters be $y=-\frac{b^2}{a^2m_1}x$ and $y=-\frac{b^2}{a^2m_2}x$.

Here $\frac{b^2}{a^2m_1}\cdot\frac{b^2}{a^2m_2}=-1$ or, $m_1m_2=-\frac{b^4}{a^4}$.

The first diameter meets the ellipse at $P\left(\frac{a^2m_1}{\sqrt{a^2m_1^2+b^2}},-\frac{b^2}{\sqrt{a^2m_1^2+b^2}}\right)$.

The second diameter meets the ellipse at $D\left(\frac{a^2m_2}{\sqrt{a^2m_2^2+b^2}},-\frac{b^2}{\sqrt{a^2m_2^2+b^2}}\right)$.

Here C is the origin.
$$\therefore \frac{1}{CP^2}+\frac{1}{CD^2}=\frac{a^2m_1^2+b^2}{a^4m_1^2+b^4}+\frac{a^2m_2^2+b^2}{a^4m_2^2+b^4}=\frac{a^2m_1^2+b^2}{a^4m_1^2+b^4}+\frac{1}{a^2b^2}\frac{a^6m_1^2+b^6}{a^4m_1^2+b^4}$$
$$=\frac{\left(a^2+b^2\right)\left(a^4m_1^2+b^4\right)}{a^2b^2\left(a^4m_1^2+b^4\right)}=\frac{1}{a^2}+\frac{1}{b^2}.]$$

17. (a) If circles be described on two semi-conjugate diameters of the ellipse $\frac{x^2}{y^2}+\frac{y^2}{b^2}=1$ as diameters, prove that the locus of their second point of intersection besides the origin is $2\left(x^2+y^2\right)^2=a^2x^2+b^2y^2$. [CH 91, 99]

    (b) Show that the locus of the ortho-centre of the triangle formed by the centre and the ends of two conjugate diameters of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $2\left(a^2x^2+b^2y^2\right)=\left(a^2-b^2\right)^2\left(b^2y^2-a^2x^2\right)^2$ and the area of the triangle is constant.

18. Find the equations to the straight lines which are conjugate to the coordinate axes w.r.t. the conic $Ax^2+2Hxy+By^2=1$. Find the condition that they may coincide and interpret the result.

[*Hints.* If $y = mx$ and $y = m'x$ are conjugate diameters, then $A + H(m + m') + Bmm' = 0$. Thus, the diameters conjugate to $y = 0$ and $x = 0$ are $Ax + Hy = 0$ and $Hx + By = 0$ respectively. If these two lines coincide, then $H^2 = AB$. In this case, the conic is a pair of parallel lines.]

19. (a) Prove that the straight lines joining the centres to the intersection of the straight line $y = mx + \sqrt{\frac{a^2m^2+b^2}{2}}$ with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate diameters. [CH 2000]

(b) Show that the lines $(Ha - hA)x^2 + (aB - Ab)xy + (hB - bH)y^2 = 0$ are conjugate diameters of both the conics $ax^2 + 2hxy + by^2 = 1$ and $Ax^2 + 2Hxy + By^2 = 1$.

[*Hints.* Let

$$y^2 + \frac{aB - Ab}{hB - bH}xy + \frac{aH - hA}{hB - bH}x^2 = (y - mx)(y - m'x).$$

Equating the coefficients,

$$m + m' = -\frac{aB - bA}{hB - bH} \quad \text{and} \quad mm' = \frac{aH - hA}{hB - bH}.$$

Now

$$a + h(m + m') + bmm' = a - h\frac{aB - bA}{hB - bH} + b\frac{aH - hA}{hB - bH} = 0$$

$$\text{and} \quad A + H(m + m') + Bmm' = A - H\frac{aB - bA}{hB - bH} + B\frac{aH - hA}{hB - bH} = 0.$$

Hence the given lines are conjugate diameters to the given conics.]

(c) Find the equation to the common conjugate diameters of the conics $x^2 + 4xy + 6y^2 = 1$ and $2x^2 + 6xy + 9y^2 = 1$.

(d) Prove that the equation to the equiconjugate diameters of the conic $ax^2 + 2hxy + by^2 = 1$ is $\frac{ax^2+2hxy+by^2}{ab-h^2} = \frac{2(x^2+y^2)}{a+b}$.

[*Hints.* Here the diameters are equal. If r is the length of semi-diameter, then the equiconjugate diameters are the pair of lines through the centre (origin) and the common points of $ax^2 + 2hxy + by^2 = 1 \cdots (1)$ and the circle $x^2 + y^2 = r^2 \cdots (2)$. Thus the equation of diameters is $ax^2 + 2hxy + by^2 = \frac{x^2+y^2}{r^2}$ or, $(ar^2 - 1)x^2 + 2hr^2xy + (br^2 - 1)y^2 = 0 \cdots (3)$. By the condition of conjugate diameters $a(br^2 - 1) + b(ar^2 - 1) = 2h \cdot hr^2$ or, $r^2 = \frac{a+b}{2(ab-h^2)}$. Putting this value of r^2 in (3), the result is obtained.]

ANSWERS

1. (a) $2x - 3y + 13 = 0$, (b) $x - 2y + 7 = 0$, (c) $5x + 3y = 16$.
2. (a) $y = 24$, (b) $y^2 = 2ax$, (c) $y^2 = 2a(x - a)$, (d) $2x = 3y$.
3. (a) $4x + 5y = 0$, (b) $9y = 32x$.
7. $\frac{x^2}{12} + \frac{y^2}{8} = 2$. 15. $x^2 + y^2 = (a^2 + b^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$. 19. (c) $x^2 + 3xy = 0$.

Chapter 7

Asymptotes

7.10 Introduction

Definition. *A straight line which meets a curve at two points at infinity but which is not wholly at infinity is called an asymptote to this curve.*

Equations of asymptotes to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

Let

$$y = mx + c \tag{2}$$

be an asymptote.

To find the x-coordinates of points of intersection between (1) and (2),

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1 \quad \text{or,} \quad \left(\frac{1}{a^2} - \frac{m^2}{b^2}\right)x^2 - \frac{2mc}{b^2}x - \left(1 + \frac{c^2}{b^2}\right) = 0. \tag{3}$$

Since the line meets the curve at two points at infinity, both of the roots of (3) must be infinite.

$\therefore$ coefficient of x^2 and x are zero i.e. $\frac{1}{a^2} - \frac{m^2}{b^2} = 0$ and $mc = 0$.

$\therefore$ $m = \pm\frac{b}{a}$ and $c = 0$.

Hence the asymptotes are $y = \pm\frac{b}{a}x$.

In pair form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Note:

1. The equation of the pair of asymptotes differs from the equation of the hyperbola by a constant only.
2. The asymptotes pass through the centre of the hyperbola.
3. The bisectors of the angles between the asymptotes are the axes of the hyperbola.

4. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ is the equation of the pair of asymptotes of the conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.
5. The asymptotes of $x^2 - y^2 = a^2$ are $y = \pm x$ which are at right angles.

7.11 Equation of pair of asymptotes to general equation

Let the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a hyperbola.

The equation of the asymptotes differs from that of the curve in the constant term. Thus the equation of the asymptotes will be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0. \tag{1}$$

The equation (1) must represent a pair of straight lines. For this

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+\lambda \end{vmatrix} = 0, \quad \text{i.e.} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} + \lambda \begin{vmatrix} a & h & 0 \\ h & b & 0 \\ g & f & 1 \end{vmatrix} = 0, \quad \lambda = -\frac{\Delta}{D}.$$

Hence the required asymptotes are

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{D} = 0.$$

Note:

1. The asymptotes of

$$\begin{aligned} (lx + my + n)(l'x + m'y + n') &= k \\ \text{are} \quad (lx + my + n)(l'x + m'y + n') &= 0. \end{aligned}$$

2. The equation of the conjugate of hyperbola is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{2\Delta}{ab - h^2} = 0.$$

7.12. Some rules about asymptotes and hyperbola

Let $S = 0$, $S' = 0$ and $A = 0$ be the equations of the hyperbola, conjugate hyperbola and the pair of asymptotes.

1. $S + \lambda = 0$ is the pair of asymptotes. λ is a constant determined by the condition of a pair of straight lines.
2. $A + \lambda' = 0$ is the equation of the hyperbola. λ' is a constant determined by the help of a given point on the hyperbola.

3. $S + S' = 2A$. The equation of the conjugate hyperbola is obtained by this relation.

4. The point of intersection of the asymptotes is the centre of the hyperbola.

5. The bisectors of the angles between the asymptotes are the principal axes of the hyperbola.

6. If the asymptotes are at right angles, the hyperbola is rectangular or equilateral.

.20 The equations of a hyperbola referred to its asymptotes as coordinate axes

Let

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

be the equation of the hyperbola with its axes as coordinate-axes.

The asymptotes are $y = \pm\frac{b}{a}x$. If 2α be the angle between the asymptotes, then

$$\tan\alpha = \frac{b}{a}, \quad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}} \quad \text{and} \quad \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}. \tag{2}$$

In Fig. 50, OX and OY are the axes of the hyperbola, OX' and OY' are the asymptotes, A is the vertex and $\angle X'OA = \angle Y'OA = \alpha$.

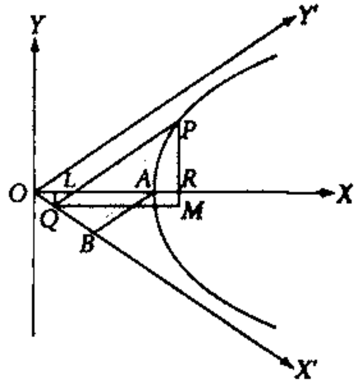


Fig. 50

Method I. If the asymptotes are considered as coordinate-axes, then their equations must be $x = 0, y = 0$. In pair form $xy = 0$. Therefore, the equation of the hyperbola must be of the form

$$xy = c^2. \tag{3}$$

c^2 is a constant.

AB is drawn parallel to OY'. Since A is a point on $xy = c^2$ i.e. $OB \cdot BA = c^2$.

$\because\ \angle BAO = \angle Y'OA = \angle X'OA, OB = BA.$

$\therefore\ OB^2 = c^2$ i.e. $OB = c$.

Again from $\triangle AOB$,

$$\begin{aligned} OA^2 &= OB^2 + AB^2 - 2OB \cdot AB\cos(\pi - 2\alpha) \\ &= 2OB^2 + 2OB^2\cos 2\alpha = 4OB^2\cos^2\alpha \end{aligned}$$

$$\text{or,}\quad OA = 2OB\cos\alpha \quad \text{or,}\quad a = 2c\cos\alpha$$

$$\text{or,}\quad c = \sqrt{\frac{a^2+b^2}{2}} \quad \text{[by (2)]}.$$

$\therefore$ $xy = \frac{a^2+b^2}{4}$ is the equation of the hyperbola referred to asymptotes as axes.

Method II. Let P be a point on the hyperbola and its coordinates be (x, y) w.r.t. OX and OY and (x', y') w.r.t. OX' and OY'. In Fig. 50, PR is perpendicular to OX. QP is parallel to OY' and QM is parallel to OX. PR (produced) meets QM at M. QL is perpendicular to OX.

$\therefore$ $OR = x, RP = y, OQ = x'$ and $QP = y'$.

Now

$$x = OR = OL + LR = OL + QM = OQ\cos\alpha + QP\cos\alpha = (x' + y')\cos\alpha$$

and $$y = RP = MP - MR = MP - QL = QP\sin\alpha - OQ\sin\alpha = (y' - x')\sin\alpha.$$

Putting these values of x and y in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

$$\frac{(x'+y')^2\cos^2\alpha}{a^2} - \frac{(y'-x')^2\sin^2\alpha}{b^2} = 1 \quad \text{or,} \quad \frac{(x'+y')^2}{a^2+b^2} - \frac{(y'-x')^2}{a^2+b^2} = 1$$

or, $$(x'+y')^2 - (y'-x')^2 = a^2+b^2 \quad \text{or,} \quad 4x'y' = a^2+b^2 \quad \text{or,} \quad x'y' = \frac{a^2+b^2}{4}.$$

$\therefore$ the required equation is

$$xy = \frac{a^2+b^2}{4} \quad \text{or,} \quad xy = c^2, \text{ where } c^2 = \frac{a^2+b^2}{4}.$$

7.30 Some properties

(i) *The portion of the tangent to a hyperbola intercepted between the asymptotes is bisected at the point of contact and the area of the triangle formed by the* tangent and the asymptotes is constant.

Let (x_1, y_1) be a point on the hyperbola $xy = c^2$. The equation of the tangent at this point is

$$\tfrac{1}{2}(xy_1 + yx_1) = c^2 \quad \text{or,} \quad \tfrac{1}{2}(xy_1 + yx_1) = x_1y_1 \quad \text{or,} \quad \frac{y}{2x_1} + \frac{x}{2y_1} = 1.$$

The asymptotes are $x = 0$ and $y = 0$. Therefore, the tangent meets the asymptotes at $(0, 2y_1)$ and $(2x_1, 0)$. Obviously the midpoint of the line joining the above two points is (x_1, y_1) which is the point of contact.

If the angle between the asymptotes be 2α, the area of the triangle formed by the tangent and the asymptotes is

$$\tfrac{1}{2}\cdot 2x_1\cdot 2y_1\sin 2\alpha = 2x_1y_1\sin 2\alpha = 2c^2\sin 2\alpha (\text{constant}).$$

Note. Area $= 2c^2\sin 2\alpha = 4c^2\sin\alpha\cos\alpha = 4c^2\frac{ab}{a^2+b^2} = ab.$

(ii) *The vertices of the parallelogram formed by the tangents at the extremities of two conjugate diameters of a hyperbola lie on the asymptotes.* [See Fig. 50.]

Let PCP' and QCQ' be a pair of conjugate diameters of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

The coordinates of P, P' and Q, Q' can be taken as

$$(a\sec\theta, b\tan\theta), (-a\sec\theta, -b\tan\theta), (a\tan\theta, b\sec\theta), (-a\tan\theta, -b\sec\theta)$$

respectively.

The tangent at P to (1) is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1. \tag{2}$$

The tangent at Q to the conjugate of (1) i.e. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is

$$\frac{x\tan\theta}{a} - \frac{y\sec\theta}{b} = -1. \tag{3}$$

These two tangents meet on a line given by

$$\frac{x}{a}(\sec\theta + \tan\theta) - \frac{y}{b}(\sec\theta + \tan\theta) = 0 \quad \text{or,} \quad \frac{x}{a} - \frac{y}{b} = 0.$$

It is an asymptote of the curve. Therefore, the tangents at P and Q meet on one asymptote. In this way it can be shown that the tangents at P' and Q' meet on this asymptote, but the tangents at P and Q' meet on the other asymptote on which the tangents at P' and Q also meet.

WORKED-OUT EXAMPLES

1. *If the chord PP' of a hyperbola meets asymptotes at Q and Q', then show that $PQ = P'Q'$.*

Let the equation of the hyperbola be $xy = c^2$. If (x_1, y_1) be the middle point of PP', then the equation of it is

$$xy_1 + yx_1 = 2x_1y_1 \quad \text{or,} \quad \frac{x}{2x_1} + \frac{y}{2y_1} = 1.$$

The intercepts on the axes which are asymptotes are $2x_1$ and $2y_1$. Therefore, the coordinates of Q and Q' are $(2x_1, 0)$ and $(0, 2y_1)$. Consequently the midpoint of QQ' is (x_1, y_1). Hence $QP = P'Q'$.

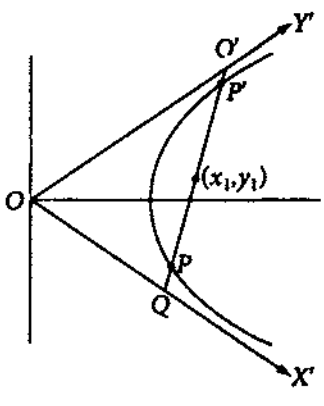


Fig. 51

2. *If e be the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and θ be the angle made by an asymptote with the transverse axis, show that $e = \sec\theta$.*

The asymptotes are $y = \pm\frac{b}{a}x$.

Since the transverse axis is the x-axis, $\tan\theta = \frac{b}{a}$.

Now
$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{b^2}{a^2} = 1 + \frac{a^2(e^2-1)}{a^2} = e^2.$$

3. *Show that the area of a triangle formed by the two asymptotes of the rectangular hyperbola $xy = c^2$ and the normal at the point (x', y') on the hyperbola is $\frac{1}{2}\left(\frac{x'^2-y'^2}{c}\right)^2$.*

The equation of the normal at (x', y') is
$$y - y' = \frac{x'}{y'}(x - x'). \tag{1}$$

Here the asymptotes are coordinate-axes and they are at right angles. If α and β are the intercepts made by the normal on the x-axis and y-axis, then
$$-y' = \frac{x'}{y'}(\alpha - x') \quad \text{or,} \quad \alpha = \frac{x'^2 - y'^2}{x'}$$
$$\text{and} \quad \beta - y' = -\frac{x'^2}{y'} \quad \text{or,} \quad \beta = \frac{y'^2 - x'^2}{y'}.$$

Area of the required triangle
$$= \frac{1}{2}\alpha\beta = -\frac{1}{2}\frac{(x'^2 - y'^2)^2}{x'y'}.$$
$$\therefore \text{ area} = \frac{1}{2}\frac{\left(x'^2 - y'^2\right)^2}{c^2} = \frac{1}{2}\left(\frac{x'^2 - y'^2}{c}\right)^2. \ [\because\ x'y' = c^2.]$$

4. *Show that the length of the focal chord of a hyperbola drawn at right angles to an asymptote is $\frac{2ab^2}{a^2-b^2}$ where a and b are the semi-axes of the hyperbola.*

Let the equation of the hyperbola be
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

The equation of an asymptote is
$$\frac{x}{a} - \frac{y}{b} = 0. \tag{2}$$

The equation of the line perpendicular to (2) and passing through the focus $(ae, 0)$ is
$$y = -\frac{a}{b}(x - ae). \tag{3}$$

By (1) and (3), we have

$$\left(a^4-b^4\right)x^2-2a^5ex+a^6e^2+a^2b^4=0 \tag{4}$$

and $$\left(a^4-b^4\right)y^2+2a^2b^3ey+a^4b^2-a^4b^2e^2=0. \tag{5}$$

If the line (3) meets (1) at (x_1,y_1) and (x_2,y_2), then (x_1,x_2) and (y_1,y_2) are the roots of (4) and (5). If d be the distance between (x_1,y_1) and (x_2,y_2), then

$$\begin{aligned} d^2 &= (x_1-x_2)^2+(y_1-y_2)^2=(x_1+x_2)^2-4x_1x_2+(y_1+y_2)^2-4y_1y_2 \\ &= 4\frac{a^{10}e^2}{(a^4-b^4)^2}-4\frac{a^6e^2+a^2b^4}{a^4-b^4}+4\frac{a^4b^6e^2}{(a^4-b^4)^2}-4\frac{a^4b^2-a^4b^2e^2}{a^4-b^4}. \end{aligned}$$

Putting $a^2e^2=a^2+b^2$ and simplifying we get

$$d^2=\frac{4a^2b^4}{(a^2-b^2)^2}. \qquad \therefore\ d=\frac{2ab^2}{a^2-b^2}.$$

5. ***Find the equation of the asymptotes of the hyperbola** $2x^2-5xy-3y^2-5x-3y-21=0$. **Hence find the centre and the principal axes of the hyperbola and also the equation of the conjugate hyperbola.***

Let the equation of the pair of asymptotes be $2x^2-5xy-3y^2-5x-3y-21+\lambda=0$, where λ is a constant. The equation represents a pair of straight lines. By the condition of the pair of straight lines we get $\lambda=\frac{1011}{49}$.

$$\begin{aligned} \therefore\ &\text{ the asymptotes are } 2x^2-5xy-3y^2-5x-3y-21+\frac{1011}{49}=0 \\ \text{or, }\ & 2x^2-5xy-3y^2-5x-3y-\frac{18}{49}=0 \\ \text{or, }\ & (14x+7y+1)(7x-21y-18)=0. \end{aligned}$$

Thus the asymptotes are

$$14x+7y+1=0 \tag{1}$$

and $$7x-21y-18=0. \tag{2}$$

The point of intersection between (1) and (2) is $\left(\frac{15}{49},-\frac{37}{49}\right)$. It is the centre of the hyperbola.

The principal axes are

$$\begin{aligned} \frac{14x+7y+1}{\sqrt{14^2+7^2}} &= \pm\frac{7x-21y-18}{\sqrt{7^2+21^2}} \\ \text{or, }\ 14x+7y+1 &= \pm\frac{1}{\sqrt{2}}(7x-21y-18). \end{aligned}$$

The equation of the conjugate hyperbola is

$$\begin{aligned} 2x^2-5xy-3y^2-5x-3y-21+2\times\frac{1011}{49} &= 0 \\ \text{or, }\ 49\left(2x^2-5xy-3y^2-5x-3y\right)+1993 &= 0. \end{aligned}$$

6. *The centre of the hyperbola is the point* $(1, 2)$ *and its asymptotes are parallel to* $2x + 3y = 0$ *and* $3x + 2y = 0$. *If the hyperbola passes through the point* $(5, 3)$, *show that its equation is* $(2x + 3y - 8)(3x + 2y - 7) = 154$.

Let the asymptotes be $2x + 3y + c_1 = 0$ and $3x + 2y + c_2 = 0$. Since the asymptotes pass through the centre $(1, 2), c_1 = -8$ and $c_2 = -7$.

$\therefore$ the asymptotes are $2x + 3y - 8 = 0$ and $3x + 2y - 7 = 0$.

Let the equation of the hyperbola be $(2x + 3y - 8)(3x + 2y - 7) + \lambda = 0$.

As the hyperbola passes through the point $(5, 3)$

$$(10 + 9 - 8)(15 + 6 + -7) + \lambda = 0 \quad \text{or,} \quad \lambda = -154.$$

Hence the equation of the hyperbola is

$$(2x + 3y - 8)(3x + 2y - 7) = 154.$$

EXERCISE XIII

1. Show that the foot of the perpendicular from the focus on an asymptote of a hyperbola lies on the auxiliary circle.

2. Prove that the rectangle contained by the intercepts made by any tangent to a hyperbola on its asymptotes is constant.

3. The ordinate through a point P of a hyperbola meets the asymptote in Q and Q'. Show that $PQ \cdot PQ' =$ (semi-conjugate axis)^2.

4. If P and Q are the extremities of a pair of conjugate diameters of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, show that PQ is parallel to one of the asymptotes and bisected by the other.

5. Show that the pair of tangents of a hyperbola from the centre of it are its asymptotes.

6. Prove that the polar of any point on any asymptote of a hyperbola w.r.t. the hyperbola is parallel to that asymptote.

7. The tangent to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(l, m)$ on it intersects the asymptotes of the hyperbola at R and Q respectively. Obtain the coordinates of Q and R and show that P is the midpoint of QR.

8. Find the asymptotes of

   (a) $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$;

   (b) $3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0$;

   (c) $xy - hx - ky = 0$.

9. Find the centre and principal axes of

   (a) $6x^2 - 7xy - 3y^2 - 2x - 8y - 6 = 0$;

(b) $(2x + 3y - 1)(3x - 5y + 2) + 10 = 0$.

10. Find the equation of the hyperbola whose asymptotes are $x + 2y + 3 = 0$ and $3x + 4y + 5 = 0$ and which passes through the point $(1, -1)$.

11. Find the equation of the hyperbola which has the same asymptotes as those of $3x^2 - 2xy - 5y^2 + 7x - 9y = 0$ and which passes through the point $(2, 2)$.

12. Find the equation of the hyperbola conjugate to $6x^2 - 7xy - 3y^2 - 2x - 8y - 6 = 0$.

13. Find the equation of the hyperbola whose centre is $(1, -2)$, whose asymptotes are parallel to $x + 2y = 0$ and $3x + 4y = 0$ and which passes through the point $(1, -1)$.

14. Show that the centres of all hyperbolas whose asymptotes are parallel to the coordinate-axes and which pass through the points $(2, 5)$ and $(3, 2)$ lie on the straight line $3x - y - 4 = 0$.

[*Hints*. Let (h, k) be the centre. The asymptotes are $x - h = 0$ and $y - k = 0$. The hyperbola is $(x - h)(y - k) = \lambda$. Since it passes through the points $(2, 5)$ and $(3, 2)$

$$(2 - h)(5 - k) = \lambda \quad \text{and} \quad (3 - h)(2 - k) = \lambda$$
$$\therefore (2 - h)(5 - k) = (3 - h)(2 - k) \quad \text{or,} \quad 3h - k - 4 = 0.$$

Hence the centres lie on $3x - y - 4 = 0$.]

15. Show that the asymptotes of a hyperbola meet the directrices on its auxiliary circle.

[*Hints*. The asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\frac{x}{a} \pm \frac{y}{b} = 0$. The directrices are $x = \pm\frac{a}{e}$ and the auxiliary circle is $x^2 + y^2 = a^2$. The asymptote $\frac{x}{a} + \frac{y}{b} = 0$ meets the directrices at the points $\left(\frac{a}{e}, -\frac{b}{e}\right)$ and $\left(-\frac{a}{e}, \frac{b}{e}\right)$.

Now
$$\frac{a^2}{e^2} + \frac{b^2}{e^2} = \frac{a^2 + b^2}{e^2} = a^2.$$

$\therefore$ the points $\left(\frac{a}{e}, -\frac{b}{e}\right)$ and $\left(-\frac{a}{e}, \frac{b}{e}\right)$ lie on the auxiliary circle. Similarly the points of intersection of $\frac{x}{a} - \frac{y}{b} = 0$ and $x = \pm\frac{a}{e}$ lie on $x^2 + y^2 = a^2$.]

16. Find the equation of the hyperbola whose asymptotes are parallel to the coordinate–axes and which passes through the points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3).

[*Hints*. Let (h, k) be the centre. The asymptotes are $x - h = 0$ and $y - k = 0$. The equation of the hyperbola is of the form

$$(x - h)(y - k) = \lambda \tag{1}$$
$$\text{or,} \quad xy - kx - hy + hk - \lambda = 0. \tag{2}$$

If it passes through the given points, then

$$x_1y_1 - kx_1 - hy_1 + hk - \lambda = 0, \tag{3}$$

$$x_2y_2 - kx_2 - hy_2 + hk - \lambda = 0, \tag{4}$$

and $$x_3y_3 - kx_3 - hy_3 + hk - \lambda = 0. \tag{5}$$

Eliminating h, k and λ from (1), (2), (3) and (4), the required equation is

$$\begin{vmatrix} xy & x & y & 1 \\ x_1y_1 & x_1 & y_1 & 1 \\ x_2y_2 & x_2 & y_2 & 1 \\ x_3y_3 & x_3 & y_3 & 1 \end{vmatrix} = 0.]$$

17. Prove that the equation of the asymptotes of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is $bX^2 - 2hXY + aY^2 = 0$, where $X = ax + hy + g$ and $Y = hx + by + f$.

18. A series of hyperbolas is drawn having a common transverse axis of length $2a$. Prove that the locus of a point P on each hyperbola, such that its distance from the transverse axis is equal to its distance from an asymptote, is the curve $\left(x^2 - y^2\right)^2 = 4x^2\left(x^2 - a^2\right)$.

[*Hints.* Let (x_1, y_1) be the coordinates of P on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Here $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ and $y_1 = \frac{bx_1 - ay_1}{\sqrt{a^2+b^2}}$. Eliminating b from these two, the result is obtained.]

ANSWERS

7. $\left(\frac{lb+ma}{b}, \frac{lb+ma}{a}\right), \left(\frac{lb-ma}{b}, \frac{ma-lb}{a}\right)$.

8. (a) $x + 2y + 1 = 0, 2x + y + 2 = 0$;

 (b) $x - 2y + 3 = 0, 3x + y - 4 = 0$;

 (c) $x = k$, $y = h$.

9. (a) $\left(-\frac{4}{11}, -\frac{10}{11}\right), \frac{2x-3y-2}{\sqrt{13}} = \pm\frac{3x+y+2}{\sqrt{10}}$;

 (b) $\left(-\frac{1}{19}, \frac{7}{19}\right), \frac{2x+3y-1}{\sqrt{13}} = \pm\frac{3x-5y+2}{\sqrt{34}}$.

10. $(x + 2y + 3)(3x + 4y + 5) - 8 = 0$.

11. $3x^2 - 2xy - 5y^2 + 7x - 9y + 20 = 0$.

12. $6x^2 - 7xy - 3y^2 - 2x - 8y - 2 = 0$.

Chapter 8

System of Circles

8.10 Orthogonal Circles

Definition. *Two circles are said to be orthogonal when the tangents at their common points are at right angles.*

Let us find out the condition for orthogonal circles.

Let the two circles be

$$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \tag{1}$$
$$\text{and} \quad S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0. \tag{2}$$

The coordinates of the centres A and B are $(-g_1, -f_1)$ and $(-g_2, -f_2)$.

A line perpendicular to the tangent at the point of contact always passes through the centre ($\because$ it is the radius of the circle). For this reason the tangents at the common point P being perpendicular to each other must be PA and PB which are the radii of the circles.

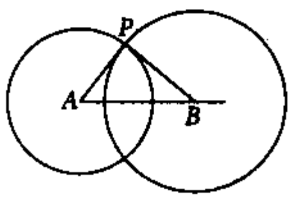


Fig. 52

From ΔPAB,

$$AB^2 = AP^2 + BP^2 \; (\because \; \angle APB = 90^\circ)$$

or, $\quad (g_1 - g_2)^2 + (f_1 - f_2)^2 = (g_1^2 + f_1^2 - c_1) + (g_2^2 + f_2^2 - c_2)$

or, $\quad 2g_1g_2 + 2f_1f_2 = c_1 + c_2.$

It is the required condition for orthogonality of the two circles.

Example 1. *Find the equation of the circle which cuts orthogonally each of the three circles $x^2 + y^2 = 16, x^2 + y^2 - 14x + 40 = 0$ and $x^2 + y^2 - 12y + 32 = 0$.*

$$x^2 + y^2 = 16 \tag{1}$$
$$x^2 + y^2 - 14x + 40 = 0 \tag{2}$$
$$x^2 + y^2 - 12y + 32 = 0. \tag{3}$$

Let $x^2+y^2+2gx+2fy+c=0$ cut each of the above circles orthogonally.

Then $0 = C - 16$ or, $c = 16$. [Considering orthogonality with (1)]

$-2g \cdot 7 = c + 40$ [Considering orthogonality with (2)]

or, $-14g = 56$ or, $g = -4$.

$-2f \cdot 6 = c + 32$ [Considering orthogonality with (3)]

or, $-12f = 48$ or, $f = -4$.

$\therefore$ the required equation is $x^2+y^2-8x-8y+16=0$.

8.20 Power of a point w.r.t. a circle

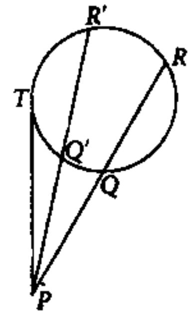


Fig. 53

Let $P(x_1, y_1)$ be a point and

$$x^2+y^2+2gx+2fy+c=0 \tag{1}$$

be the equation of a circle.

The power of P w.r.t. (1) *is defined as the product of distances of P from the point of intersection of any line through P and the circle* (1).

The equation of a line through the point (x_1, y_1) may be written as

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r \text{ (say)} \tag{2}$$

where θ is the angle made by the line with the x-axis and r is the distance between (x, y) and (x_1, y_1) on the line.

To find out the points of intersection between (1) and (2), we put $x = r\cos\theta + x_1$, $y = r\sin\theta + y_1$ in (1).

$$(r\cos\theta + x_1)^2 + (r\sin\theta + y_1)^2 + 2g(r\cos\theta + x_1) + 2f(r\sin\theta + y_1) + c = 0$$

$$\text{or,}\quad r^2 + 2\{(x_1+g)\cos\theta + (y_1+f)\sin\theta\}r + x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0. \tag{3}$$

The equation (3) is a quadratic equation in r. Let the roots be r_1 and r_2. Then r_1 and r_2 are the distances of the points of intersection between (1) and (2) from $P(x_1, y_1)$.

Now

$$r_1 r_2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c. \tag{4}$$

The expression (4) is constant when $P(x_1, y_1)$ and the circle are assigned and it is known as the power of P w.r.t. the circle (1).

From Fig. 52,

$$PQ \cdot PR = PQ' \cdot PR' = PT \cdot PT = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

PT is the length of the tangent.

Thus *the power of a point P w.r.t. a circle is equal to the square of the length of the tangent from P.*

8.30 Radical axis

Definition. *The radical axis of two circles is the locus of a point whose powers w.r.t. the circles are equal.*

Let

$$S = x^2 + y^2 + 2gx + 2fy + c = 0, \qquad (1)$$
$$S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0 \qquad (2)$$

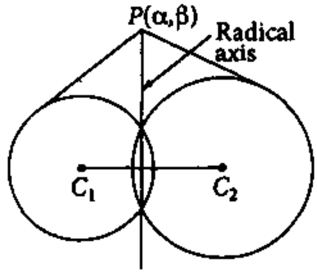


Fig. 54

be the equations of two circles and $P(\alpha, \beta)$ be a point whose powers w.r.t. (1) and (2) are equal.

$$\therefore \quad \alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c$$
$$= \alpha^2 + \beta^2 + 2g'\alpha + 2f'\beta + c'$$
$$\text{or,} \quad 2(g - g')\alpha + 2(f - f')\beta + c - c' = 0.$$

Thus the locus of (α, β) is $2(g - g')x + 2(f - f')y + c - c' = 0$. It is a straight line and is also the radical axis of the circles.

Note

1. The equation of the radical axis can be written as $S - S' = 0$. It suggests that the radical axis passes through the common points of (1) and (2) when the circles intersect.

2. If the circles touch with each other, then the radical axis is the common tangent to these circles.

3. **Radical centre.** Let $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$ be the three circles. From these three circles we get three radical axes and their equations are

$$S_1 - S_2 = 0, \qquad (1)$$
$$S_2 - S_3 = 0 \qquad (2)$$
$$\text{and} \quad S_3 - S_1 = 0. \qquad (3)$$

   Since $(S_1 - S_2) + (S_2 - S_3) + (S_3 - S_1) = 0$ the lines are concurrent. Their common point is known as the *radical centre* w.r.t. these circles. The radical centre is obtained by finding the point of intersection of any two radical axes.

4. The slope of the radical axis of $S = 0$ and $S' = 0$ is $-\frac{g-g'}{f-f'}$. The slope of the line joining the centres $(-g, -f)$ and $(-g', -f')$ is $\frac{f-f'}{g-g'}$. Product of these two slopes is -1. Therefore, *the radical axis is perpendicular to the line joining the centres of the circles.*

5. If two circles cut a third circle orthogonally, then the centre of the third circle lies on the radical axis of the first two circles.

6. The radical axis of two circles bisects each of their common tangents.

Example 2. *Find the radical centre of*

$$x^2 + y^2 - 8x + 6y + 15 = 0, \tag{1}$$

$$x^2 + y^2 - 3x - 11y + 15 = 0 \tag{2}$$

$$\text{and} \quad x^2 + y^2 - 6x - 10y + 15 = 0. \tag{3}$$

The radical axis of (1) and (2) is $5x - 17y = 0$.

The radical axis of (2) and (3) is $3x - y = 0$.

These two lines meet at $(0, 0)$. Therefore, the radical centre is $(0, 0)$.

8.31 Coaxial circles. Limiting point

Definition (*Coaxial circles*). *A system of circles is said to be coaxial, if any two of them have the same radical axis.*

If $S = x^2+y^2+2gx+2fy+c = 0$ is the equation of a circle and $L = lx+my+n = 0$ is the equation of a straight line, then $S+\lambda L = 0$ represents the equation of a coaxial system with λ as a parameter.

In this system $L = 0$ is the radical axis.

$$S + \lambda L = 0 \quad \text{or,} \quad x^2 + y^2 + (2g + \lambda l)x + (2f + \lambda m)y + c + \lambda n = 0.$$

The coordinates of the centre are

$$\left(-\frac{2g + \lambda l}{2}, -\frac{2f + \lambda m}{2}\right)$$

$$\text{and radius} = \sqrt{\left(\frac{2g + \lambda l}{2}\right)^2 + \left(\frac{2f + \lambda m}{2}\right)^2 - (c + \lambda n)}.$$

The circle whose radius is zero is known as a *point circle*. *In a coaxial system the centres of the point circles are called limiting points.*

Note

1. To find out limiting points, we find the value of λ from

$$\left(\frac{2g + \lambda l}{2}\right)^2 + \left(\frac{2f + \lambda m}{2}\right)^2 - (c + \lambda n) = 0.$$

   If λ_1 and λ_2 are the roots of this equation, then the limiting points are

$$\left(-\frac{2g + \lambda_1 l}{2}, -\frac{2f + \lambda_1 m}{2}\right) \quad \text{and} \quad \left(-\frac{2g + \lambda_2 l}{2}, -\frac{2f + \lambda_2 m}{2}\right).$$

2. If $S_1 = 0$ and $S_2 = 0$ be the equations of two circles in the coaxial system, then $S_1 + \lambda S_2 = 0$ is the equation of the coaxial system. (λ is the parameter.)

3. Since the line joining the centres of two circles is perpendicular to the radical axis, the centres of a system of coaxial circles lie on a straight line.

Note

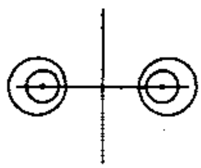
4. If we consider the y-axis ($x = 0$) as the common radical axis and the centres of the circles lie on the x-axis, then the equation of the system can be written as $x^2 + y^2 + 2\lambda x + c = 0$. ($\lambda$ is the parameter.) Here the centre is $(-\lambda, 0)$ and the radius is $\sqrt{\lambda^2 - c}$.

   For the point circle $\lambda^2 - c = 0$ or $\lambda = \pm\sqrt{c}$. Thus the limiting points are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$. If $c \geq 0$, these points are real, otherwise they are imaginary.

   (i) If $c > 0$, there are two distinct limiting points. In this case, the radical axis cuts any member of the system at $\left(0, \pm\sqrt{-c}\right)$. These points are imaginary. Thus the circles of the system do not intersect. This system is said to be *hyperbolic*. [See Fig. 55(a).]

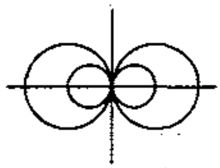
   (ii) If $c = 0$, the two limiting points coincide and the limiting point is $(0, 0)$. The members of the system touch one another at the limiting point and the radical axis is the common tangent. In this case, the system is said to be parabolic. [See Fig. 55(b).]

   (iii) If $c < 0$; the limiting points are imaginary, i.e. there is no limiting point. The radical axis cuts any member of the system at two real points. In this case the system is said to be elliptic. [See Fig. 55(c).]



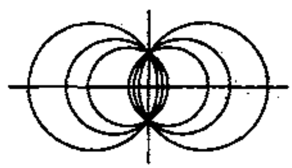
Hyperbolic (two limiting points)

Fig. 55(a)



Parabolic (one limiting points)

Fig. 55(b)



Elliptic (no limiting points)

Fig. 55(c)

5. A circle and any two circles of a coaxial system have the same radical centre.

   Let the equation of a coaxial system be $x^2 + y^2 + 2\lambda x + k = 0$ and the equation of a circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

   The radical axis is

$$2(g - \lambda)x + 2fy + c - k = 0.$$

   This equation holds for all values of λ; $\therefore\ x = 0$.

   When $x = 0, y = \frac{k-c}{2f}$.

   Thus the radical centre is $\left(0, \frac{k-c}{2f}\right)$. It is independent of λ. Hence the result follows.

Example 3. *Find the limiting points of the coaxial system whose two circles are* $x^2 + y^2 - 6x - 2y + 4 = 0$ *and* $3(x^2 + y^2) - 16x - 4y + 12 = 0$.

The radical axis is

$$(x^2 + y^2 - 6x - 2y + 4) - \left(x^2 + y^2 - \frac{16}{3}x - \frac{4}{3}y + 4\right) = 0$$

or, $x + y = 0$.

The equation of the coaxial system is

$$x^2 + y^2 - 6x - 2y + 4 + \lambda(x + y) = 0$$
$$\text{or,}\quad x^2 + y^2 + (\lambda - 6)x + (\lambda - 2)y + 4 = 0,$$

where λ is a parameter.

The centre is

$$\left(\frac{6-\lambda}{2}, \frac{2-\lambda}{2}\right)$$

and the radius is

$$\sqrt{\left(\frac{6-\lambda}{2}\right)^2 + \left(\frac{2-\lambda}{2}\right)^2 - 4}.$$

For the point circle

$$\left(\frac{6-\lambda}{2}\right)^2 + \left(\frac{2-\lambda}{2}\right)^2 - 4 = 0$$
$$\text{or,}\quad 36 - 12\lambda + \lambda^2 + 4 - 4\lambda + \lambda^2 - 16 = 0$$
$$\text{or,}\quad \lambda^2 - 8\lambda - 12 = 0 \quad \text{or,}\quad (\lambda - 2)(\lambda - 6) = 0 \quad \text{or,}\quad \lambda = 2, 6.$$

$\therefore$ the limiting points are $(2, 0)$ and $(0, -2)$.

8.32 Properties of limiting points

(i) *The limiting points of a system of coaxial circles are conjugate points w.r.t. any circle of this system.*

Let the equation of the coaxial system be $x^2 + y^2 + 2\lambda x + c = 0$.

The limiting points are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

The polar of $(\sqrt{c}, 0)$ w.r.t. the circle is

$$x\sqrt{c} + \lambda\left(x + \sqrt{c}\right) + c = 0$$
$$\text{or,}\quad x\left(\sqrt{c} + \lambda\right) + \sqrt{c}\left(\lambda + \sqrt{c}\right) = 0$$
$$\text{or,}\quad \left(x + \sqrt{c}\right)\left(\lambda + \sqrt{c}\right) = 0 \quad \text{or,}\quad x + \sqrt{c} = 0.$$

Obviously this polar passes through $(-\sqrt{c}, 0)$. Hence the result follows.

(ii) *Every circle through the limiting points of a coaxial system is orthogonal to all circles of the system.*

Let us consider the coaxial system in the form

$$x^2 + y^2 + 2\lambda x + c = 0. \tag{1}$$

The limiting points are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

Let the circle $x^2 + y^2 + 2gx + 2fy + c' = 0$ pass through these points. Then $c + 2g\sqrt{c} + c' = 0$ and $c - 2g\sqrt{c} + c' = 0$.

From these two equations $g = 0$ and $c' = -c$.

$\therefore$ the equation of the circle is

$$x^2 + y^2 + 2fy - c = 0. \tag{2}$$

With f as a parameter the equation (2) represents another coaxial system of circles whose centres lie on the y-axis and x-axis is the radical axis.

One can easily verify that (1) and (2) are orthogonal.

Note

1. The centres of the system (1) lie on the radical axis of the system (2) and conversely.
2. The limiting points of the system (2) are $(0, \sqrt{-c})$ and $(0, -\sqrt{-c})$. These points lie on the circles of (1).
3. The points of intersection of any two circles of (1) are $(0, \sqrt{-c})$ and $(0, -\sqrt{-c})$ and those of any two circles of (2) are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$. Hence the limiting points of one system are the points of intersection of the other system.
4. If $c \neq 0$, then c is either positive or negative. Hence if the circles of one system intersect in real points, then the circles of the other system will intersect in imaginary points.
5. The centre of the system (2) is $(0, -f)$. The length of the tangent from this point to any circle of (1) is $\sqrt{f^2 + c}$. It is equal to the radius of the circle of (2). Conversely the length of the tangent from the centre of (1) to (2) is equal to the radius of the circle of (1).

WORKED-OUT EXAMPLES

1. *Find the equation of the circle cutting* $x^2 + y^2 + 2x - 9 = 0$, $x^2 + y^2 - 8x - 9 = 0$ *orthogonally and touching* $y = x + 4$.

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the circle. As it cuts the given circles orthogonally

$$2g = c - 9 \tag{1}$$

$$\text{and} \quad -8g = c - 9. \tag{2}$$

Solving (1) and (2) $g = 0$ and $c = 9$.

The equation of the circle reduces to $x^2 + y^2 + 2fy + 9 = 0$.

To find out the points of intersection of this circle with the given line, we have $x^2 + (x+4)^2 + 2f(x+4) + 9 = 0$ or, $2x^2 + 2(f+4)x + 8f + 25 = 0$. Since the line touches the circle, the above equation in x must have equal roots.

$$\therefore \ 4(f+4)^2 - 4 \cdot 2 \cdot (8f+5) = 0 \quad \text{or,} \quad f^2 - 8f - 34 = 0 \quad \text{or,} \quad f = 4 \pm 5\sqrt{2}.$$

Thus the required equations are $x^2 + y^2 + 2\left(4 \pm 5\sqrt{2}\right)y + 9 = 0$.

2. *Prove that the length of the common chord of the circles* $(x-a)^2 + (y-b)^2 = c^2$ *and* $(x-b)^2 + (y-a)^2 = c^2$ *is* $\sqrt{4c^2 - 2(a-b)^2}$.

$$(x-a)^2 + (y-b)^2 = c^2 \tag{1}$$

$$(x-b)^2 + (y-a)^2 = c^2. \tag{2}$$

Subtracting (2) from (1), the equation of the common chord is

$$(b-a)x + (a-b)y = 0 \quad \text{or,} \quad x - y = 0.$$

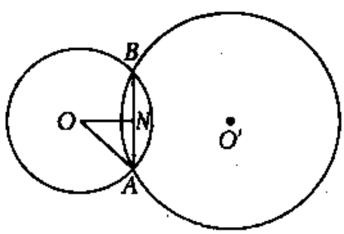


Fig. 56

Let AB be the common chord, O the centre of the circle (1) and ON the perpendicular to AB.

From Fig. 56, $AB = 2AN = 2\sqrt{OA^2 - ON^2}$.

OA = radius of the circle (1)$=c$ and $ON = \frac{a-b}{\sqrt{2}}$.

$$\therefore \ AB = 2\sqrt{c^2 - \frac{(a-b)^2}{2}} = \sqrt{2}\sqrt{2c^2 - (a-b)^2} = \sqrt{4c^2 - 2(a-b)^2}.$$

3. *Prove that the two circles $x^2+y^2+2ax+a^2=0$ and $x^2+y^2+2by+c^2=0$ will touch, if $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$.*

$$x^2+y^2+2ax+a^2=0 \tag{1}$$
$$\text{and } x^2+y^2+2by+c^2=0. \tag{2}$$

The equation of the radical axis is

$$ax-by=0. \tag{3}$$

If the circles touch with each other, then the radical axis must be the common tangent to (1) and (2). If the line (3) be the tangent to (1), the distance of (3) from the centre $(-a,0)$ must be equal to the radius $\sqrt{c^2-a^2}$.

$$\therefore \frac{-a^2}{\sqrt{a^2+b^2}}=\sqrt{a^2-c^2} \quad \text{or,} \quad \frac{a^4}{a^2+b^2}=a^2-c^2$$
$$\text{or,} \quad a^4=\left(a^2+b^2\right)\left(a^2-c^2\right)$$
$$\text{or,} \quad c^2\left(a^2+b^2\right)=a^2b^2 \quad \text{or,} \quad \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}.$$

4. *Prove that the two circles each of which passes through the points $(0,k)$ and $(0,-k)$ and touch the line $y=mx+b$ will cut orthogonally, if $b^2=k^2(2+m^2)$.*

[BH 2007]

Let the equation of one of the circles be

$$x^2+y^2+2gx+2fy+c=0. \tag{1}$$

As it passes through $(0,k)$ and $(0,-k)$

$$k^2+2fk+c=0 \quad \text{and} \quad k^2-2fk+c=0.$$

From these two equations $f=0, c=-k^2$.

$\therefore$ the equation (1) touches the line $y=mx+b$, the distance of it from the centre $(-g,0)$ will be equal to the radius $\sqrt{g^2+k}$.

$$\therefore \frac{-mg+b}{\sqrt{m^2+1}}=\sqrt{g^2+k^2}$$
$$\text{or,} \quad (b-mg)^2=\left(m^2+1\right)\left(g^2+k^2\right)$$
$$\text{or,} \quad g^2+2bmg+k^2\left(m^2+1\right)-b^2=0.$$

It is a quadratic equation in g. Let the roots be g_1 and g_2. Then $g_1g_2=k^2(1+m^2)-b^2$ and the circles are

$$x^2+y^2+2g_1x-k^2=0 \tag{2}$$
$$\text{and} \quad x^2+y^2+2g_2x-k^2=0. \tag{3}$$

If (2) and (3) cut orthogonally

$$2g_1g_2=-2k^2$$
$$\text{or,} \quad k^2\left(1+m^2\right)-b^2=-k^2 \quad \text{or,} \quad b^2=k^2\left(2+m^2\right).$$

5. *If a circle cuts each of the two circles $x^2+y^2+2g_1x+c=0$ and $x^2+y^2+2g_2x+c=0$ orthogonally show that it belongs to a coaxial system.*

Let the equation of the circle be $x^2+y^2+2gx+2fy+c'=0$.

From the condition of orthogonality,

$$2gg_1 = c+c' \tag{1}$$

$$\text{and} \quad 2gg_2 = c+c'. \tag{2}$$

From (1) and (2), $2g(g_1-g_2)=0$. $\quad \because\ g_1 \neq g_2, g=0$.

In this case, $c+c'=0$ or, $c'=-c$.

$\therefore$ the equation of the circle is $x^2+y^2+2fy-c=0$.

Here f is a variable parameter. Therefore, the circle belongs to a coaxial system whose radical axis is $y=0$.

6. *Find the equation of the circle described upon the radical axis of a coaxial system as diameter, the limiting points of the system being* $(1,2)$ *and* $(3,4)$.

The point circles of the coaxial system are

$$(x-1)^2+(y-2)^2=0 \quad \text{or,} \quad x^2+y^2-2x-4y+5=0 \tag{1}$$

$$\text{and} \quad (x-3)^2+(y-4)^2=0 \quad \text{or,} \quad x^2+y^2-6x-8y+25=0. \tag{2}$$

The radical axis is

$$4x+4y-20=0 \quad \text{or,} \quad x+y-5=0. \tag{3}$$

The equation of a circle belonging to the coaxial system is

$$x^2+y^2-2x-4y+5+\lambda(x+y-5)=0$$

$$\text{or,} \quad x^2+y^2-(2-\lambda)x-(4-\lambda)y+5(1-\lambda)=0.$$

The centre is $\left(\frac{2-\lambda}{2}, \frac{4-\lambda}{2}\right)$.

The centre lies on the radical axis when it is the diameter of the circle.

Thus $\frac{2-\lambda}{2}+\frac{4-\lambda}{2}-5=0$ or, $\lambda=2$.

Hence the required circle is $x^2+y^2-4x-6y+15=0$.

7. *If the four points in which the two circles $x^2+y^2+ax+by+c=0$ and $x^2+y^2+a'x+b'y+c'=0$ are intersected by the straight lines $Ax+By+C=0$ and $A'x+B'y+C'=0$ respectively lie on another circle, show that*

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$$

Let $S=x^2+y^2+ax+by+c=0, S'=x^2+y^2+a'x+b'y+c'=0$ and $S''=0$ be the equation of the third circle passing through the four points.

Then $Ax+By+C=0$ is the radical axis of $S=0$ and $S''=0$, $A'x+B'y+C'=0$ is the radical axis of $S'=0$ and $S''=0$ and $(a-a')x+(b-b')y+c-c'=0$ is the radical axis of $S=0$ and $S'=0$. These axes are concurrent.

$$\therefore \begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$$

8. *If the origin be at one of the limiting points of a system of coaxial circles of which $x^2+y^2+2gx+2fy+c=0$ is a member, show that the equation of the system of circles cutting them all orthogonally is $(x^2+y^2)(g+\mu f)+c(x+\mu y)=0$. Show that the other limiting point is*

$$\left(\frac{-gc}{g^2+f^2}, \frac{-fc}{g^2+f^2}\right).$$

The point circle for the limiting point $(0,0)$ is $x^2+y^2=0$ which belongs to the coaxial system. Therefore, the radical axis of this coaxial system is

$$x^2+y^2+2gx+2fy+c-(x^2+y^2)=0 \quad \text{or,} \quad 2gx+2fy+c=0.$$

Thus the equation of the coaxial system is

$$x^2+y^2+2gx+2fy+c+\lambda(2gx+2fy+c)=0$$
$$\text{or,} \quad x^2+y^2+2g(1+\lambda)x+2f(1+\lambda)y+c(1+\lambda)=0 \qquad (1)$$

where λ is a parameter.

Let the circle

$$x^2+y^2+2lx+2my+n=0 \qquad (2)$$

cut each of the circles of (1) orthogonally.

$$\therefore 2lg(1+\lambda)+2mf(1+\lambda)=c(1+\lambda)+n$$
$$\text{or,} \quad 2lg+2mf-c-n+\lambda(2lg+2mf-c)=0.$$

To satisfy this relation for all values of λ, we have

$$2lg+2mfc-n=0 \qquad (3)$$
$$\text{and} \quad 2lg+2mf-c=0. \qquad (4)$$

From (3) and (4), $n=0$ and $2lg+2mf-c=0$ or, $m=\frac{c-2lg}{2f}$.

Putting these values of m and n in (2), we get

$$x^2+y^2+2lx+\frac{c-2lg}{f}y=0 \quad \text{or,} \quad x^2+y^2+2l\left(x+\frac{c-2lg}{2lf}y\right)=0.$$

Putting $2l = \frac{c}{g+\mu f}$, i.e. $\mu = \frac{c-2lg}{2lf}$, we get

$$x^2+y^2+\frac{c}{g+\mu f}(x+\mu y)=0 \quad \text{or,} \quad (x^2+y^2)(g+\mu f)+c(x+\mu y)=0$$

where μ is a parameter.

The radius of (1) is $\sqrt{g^2(1+\lambda)^2+f^2(1+\lambda)^2-c(1+\lambda)}$.

For the point circle,

$$g^2(1+\lambda)^2+f^2(1+\lambda)^2-c(1+\lambda)=0$$
$$\text{or,} \quad (1+\lambda)\{g^2(1+\lambda)+f^2(1+\lambda)-c\}=0.$$

Either $1+\lambda=0$ or, $g^2(1+\lambda)+f^2(1+\lambda)-c=0$.

$$\therefore \lambda=-1 \quad \text{or,} \quad \frac{c-g^2-f^2}{g^2+f^2}.$$

For $\lambda=-1$, the limiting point is $(0,0)$.

For $\lambda=\frac{c-g^2-f^2}{g^2+f^2}$, the limiting point is $\left(\frac{-gc}{g^2+f^2}, \frac{-fc}{g^2+f^2}\right)$.

EXERCISE XIV

1. Prove that the following circles cut orthogonally.

   (a) $x^2+y^2+2x+4y-20=0, x^2+y^2+6x-8y+10=0$;

   (b) $x^2+y^2-2ax+2by+c=0, x^2+y^2+2bx+2ay-c=0$.

2. Show that the three circles $x^2+y^2+3x+6y+12=0, x^2+y^2+2x+8y+16=0$ and $x^2+y^2+12y+24=0$ have a common radical axis.

3. Find the radical centre of the three circles $x^2+y^2-4x-6y-12=0, x^2+y^2+2x+4y-4=0$ and $x^2+y^2-12x+4y+4=0$.

4. Find the equation of the circle which cuts orthogonally the circles

   (a) $x^2+y^2-2x-4y-20=0, x^2+y^2=25$ and $x^2+y^2-6x-4y-23=0$;

   (b) $x^2+y^2=a^2, (x-c)^2+y^2=a^2$ and $x^2+(y-b)^2=a^2$.

5. Find the value of k for which the circle $x^2+y^2-2x+2y+7=0$ will bisect the circumference of $x^2+y^2-6x+8y-k=0$.

6. If $x+iy=a\tan(u+iv)$ where $i=\sqrt{-1}$, prove that u (= constant) and v (= constant) represent two conjugate systems of coaxial circles.

7. Find the equation of the circle whose diameter is the common chord of the circles $x^2+y^2+2x+3y+1=0$ and $x^2+y^2+4x+3y+2=0$.

8. Find the equation of the common chord of the circles $x^2+y^2+2gx+2fy+c=0$ and $x^2+y^2+2fx+2gy+c=0$ and hence find the condition if they touch with each other.

9. Find the coordinates of the limiting points of the coaxial system determined by the pair of circles

(a) $x^2 + y^2 + 2x + 5 = 0, x^2 + y^2 + 2y + 5 = 0$;

(b) $x^2 + y^2 - 6x - 6y + 4 = 0, x^2 + y^2 - 2x - 4y + 3 = 0$.

10. If two circles cut a third circle orthogonally show that the centre of the third circle lies on the radical axis of the two circles.

[*Hints.* Let the two circles be $x^2+y^2+2g_1x+\mu = 0$ and $x^2+y^2+2g_2x+\mu = 0$. If the third circle $x^2+y^2+2gx+2fy+c = 0$ cuts the above circles orthogonally, then $2gg_1 = \mu + c$ and $2gg_2 = \mu + c$. Subtracting, $2g(g_1 - g_2) = 0 \Rightarrow g = 0$, since $g_1 \neq g_2$.

Hence the centre of the third circle is $(0, f)$. It lies on the radical axis $x = 0$.]

11. Find the locus of a point whose power w.r.t. a given circle is constant.

12. Find the equation of the system of coaxial circles of which the points $(0, \pm 3)$ are the limiting points.

13. Prove that the limiting points of the coaxial system determined by $S = x^2 + y^2 + 2gx = 0$ and $S' = x^2 + y^2 + 2fy = 0$ are given by $f^2S^2 + g^2S'^2 = 0$.

14. Show that the limiting points of the system $x^2 + y^2 + 2x - 4y + 5 + \lambda(x^2 + y^2 - 4x + 6y + 13) = 0$ are $(-1, 2)$ and $(2, -3)$.

15. Find the equation of the circle which passes through the origin, has its centre on $x + y - 4 = 0$ and cuts orthogonally $x^2 + y^2 - 4x + 2y + 4 = 0$.

16. Prove that the limiting points of the system $x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + k) = 0$ subtend a right angle at the origin if $\frac{c}{g^2} + \frac{k}{f^2} = 2$.

17. l_1, l_2, l_3 are lengths of tangents from a fixed point to three circles of a coaxial system. If P, Q, R are centres of the circles, prove that $l_1^2QR + l_2^2RP + l_3^2PQ = 0$.

18. The equation of a system of coaxial circles is given by $x^2 + y^2 + 2gx + 2fy + 2\lambda(gx - fy) = 0$, where λ is a parameter. Find the orthogonal system to this system.

[*Hints.* Let the circle $x^2 + y^2 + 2lx + 2my + n = 0$ be orthogonal to $x^2 + y^2 + 2g(1 + \lambda)x + 2f(1 - \lambda)y = 0$. Then $2g(1 + \lambda)l + 2f(1 - \lambda)m = n$. Putting $\lambda = 1$ and -1, we get $2l = \frac{n}{2g}$ and $2m = \frac{n}{2f}$. Thus the required system is $x^2 + y^2 + \frac{n}{2}\left(\frac{x}{g} + \frac{y}{f} + 2\right) = 0$, where n is a parameter.]

19. The polars of a point P w.r.t. two fixed circles meet in the point Q. Prove that the circle on PQ as diameter passes through two fixed points and cuts both the given circles at right angles. Prove also that the radical axis of the circles bisects PQ.

ANSWERS

3. $\left(\frac{4}{7}, -\frac{8}{7}\right)$.

4. (a) $4(x^2 + y^2) + 6x - 13y + 100 = 0$; (b) $x^2 + y^2 - cx - by + a^2 = 0$.

5. $k = -43$. 7. $2(x^2 + y^2) + 2x + 6y + 1 = 0$. 8. $(g + f)^2 = 2c$.

9. (a) $(1, -2), (-2, 1)$; (b) $(-1, 1), \left(\frac{1}{5}, \frac{8}{5}\right)$.

12. $x^2 + y^2 + 2\lambda y + 9 = 0$. 15. $x^2 + y^2 - 4x - 4y = 0$.

Chapter 9

Polar Coordinates and Equations

9.10 Polar coordinates (Fig. 57)

Let O and OX be a fixed point and a fixed line on a given plane. A point P on this plane can be defined with reference to the fixed point and the fixed line. If r be the distance of P from O and θ be the angular distance of P from OX, then the polar coordinates of P are denoted by (r, θ). O and OX are called the *pole* and the *initial line* respectively. If (x, y) are the Cartesian coordinates of P w.r.t. OX and OY, then $x = r\cos\theta$ and $y = r\sin\theta$. r is known as the *radius vector* and θ is the *vectorial angle* of P. Vectorial angle is generally measured in the anticlockwise direction.

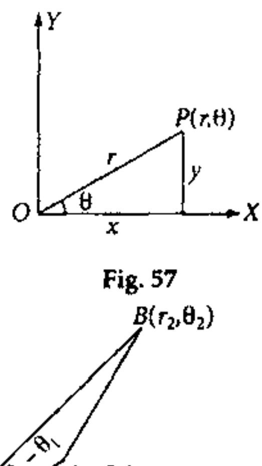


Fig. 57

9.11 Distance between two points (Fig. 58)

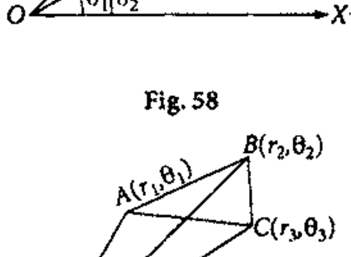


Fig. 58

Let the polar coordinates of A and B be (r_1, θ_1) and (r_2, θ_2) w.r.t. the pole O and the initial line OX. In ΔOAB, $OA = r_1$, $OB = r_2$ and $\angle AOB = \theta_2 - \theta_1$.

Now

$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB\cos AOB$$
$$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1).$$
$$\therefore\ AB = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}.$$

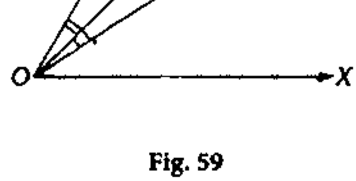


Fig. 59

9.12. Area of a triangle (Fig. 59)

Let the polar coordinates of A, B and C be (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3) respectively w.r.t. the pole O and the initial line OX. From Fig. 59,

$$\begin{aligned}\triangle ABC &= \triangle OAB + \triangle OBC - \triangle OCA \\ &= \tfrac{1}{2}OA \cdot OB \sin A\hat{O}B + \tfrac{1}{2}OB \cdot OC \sin B\hat{O}C - \tfrac{1}{2}OC \cdot OA \sin C\hat{O}A \\ &= \tfrac{1}{2}\left[r_1 r_2 \sin(\theta_1 - \theta_2) + r_2 r_3 \sin(\theta_2 - \theta_3) - r_3 r_1 \sin(\theta_1 - \theta_3)\right] \\ &= \tfrac{1}{2}\left[r_1 r_2 \sin(\theta_1 - \theta_2) + r_2 r_3 \sin(\theta_2 - \theta_3) + r_3 r_1 \sin(\theta_3 - \theta_1)\right].\end{aligned}$$

Note. If the area is zero, the points are collinear.

9.20 Polar equation of a straight line

Let (r, θ) be the coordinates of a point P on the line PN w.r.t. the pole O and the initial line OX. ON is perpendicular to the line.

Let $ON = p$ and $\angle XON = \alpha$.

Now $ON = OP\cos(\theta - \alpha)$ or, $p = r\cos(\theta - \alpha)$. It is the polar equation of the line.

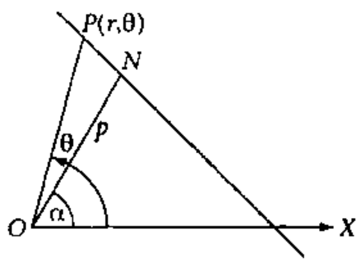


Fig. 60

Corollary I. *If $\alpha = 0, p = r\cos\theta$ is the equation of the line. It is a straight line perpendicular to OX, the initial line.*

Corollary II. *If $\alpha = \frac{\pi}{2}, p = r\sin\theta$ is the equation of the line. It is parallel to OX.*

Corollary III. *If $p = 0, \cos(\theta - \alpha) = 0$ or, $\theta - \alpha = \frac{\pi}{2}$ or, θ = constant. It is the equation of a line passing through the pole.*

Corollary IV. *If the line passes through (r_1, θ_1) and makes an angle β with the initial line, then $\beta = \frac{\pi}{2} + \alpha$ and $p = r_1 \cos\left\{\frac{\pi}{2} - (\beta - \theta_1)\right\}$ or, $p = r_1 \sin(\beta - \theta_1)$. The equation of the line is $r\sin(\beta - \theta) = r_1 \sin(\beta - \theta_1)$.*

Corollary V. *The equation $r\cos(\theta - \alpha) = p$ can be written as $r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p$ or, $\frac{1}{r} = \frac{\cos\alpha}{p}\cos\theta + \frac{\sin\alpha}{p}\sin\theta$ which gives another form of the polar equation of the line as $\frac{1}{r} = A\cos\theta + B\sin\theta$ where A and B are constants. It is the general form of the polar equation of a straight line. The slope of the line is $-A/B$.*

Note. The polar equations of *two parallel lines* are of the form $r\cos(\theta - \alpha) = p$ and $r\cos(\theta - \alpha) = p'$. The polar equations of *two mutually perpendicular lines* are of the form $r\cos(\theta - \alpha) = p$ and $r\cos(\theta - \alpha - \frac{\pi}{2}) = p'$ i.e. $r\sin(\theta - \alpha) = p'$.

9.21 Polar equation of the line passing through (r_1, θ_1) and (r_2, θ_2)

The general form of the polar equation of a straight line is

$$A\cos\theta + B\sin\theta - \frac{1}{r} = 0. \tag{1}$$

If it passes through (r_1, θ_1) and (r_2, θ_2), then

$$A\cos\theta_1 + B\sin\theta_1 - \frac{1}{r_1} = 0 \tag{2}$$

$$\text{and } A\cos\theta_2 + B\sin\theta_2 - \frac{1}{r_2} = 0. \tag{3}$$

Eliminating A and B from (1), (2) and (3), the required equation is

$$\begin{vmatrix} \cos\theta & \sin\theta & \frac{1}{r} \\ \cos\theta_1 & \sin\theta_1 & \frac{1}{r_1} \\ \cos\theta_2 & \sin\theta_2 & \frac{1}{r_2} \end{vmatrix} = 0$$

$$\text{or,}\quad \frac{1}{r}\sin(\theta_1 - \theta_2) + \frac{1}{r_1}\sin(\theta_2 - \theta) + \frac{1}{r_2}\sin(\theta - \theta_1) = 0.$$

Note.

$$A\cos\theta + B\sin\theta = \frac{k}{r} \tag{1}$$

$$\text{and } A\cos\theta + B\sin\theta = \frac{k_1}{r} \tag{2}$$

are parallel lines.

$$A\cos(\frac{\pi}{2} + \theta) + B\sin(\frac{\pi}{2} + \theta) = \frac{k_2}{r} \quad \text{i.e.} \quad -A\sin\theta + B\cos\theta = \frac{k_2}{r}$$

is perpendicular to (1) and (2).

9.30 Polar equation of a circle

Let (c, α) be the polar coordinates of the centre C of the circle of radius a w.r.t. the pole O and the initial line OX. Let $P(r, \theta)$ be a point on the circle.

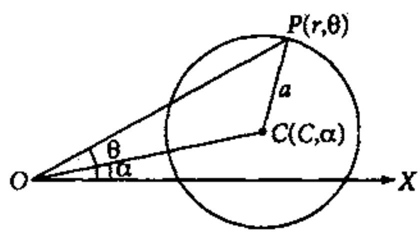


Fig. 61

From ΔOCP,

$$CP^2 = OC^2 + OP^2 - 2OC \cdot OP\cos(\theta - \alpha)$$

$$\text{or,}\quad a^2 = c^2 + r^2 - 2cr\cos(\theta - \alpha)$$

$$\text{or,}\quad r^2 - 2cr\cos(\theta - \alpha) + c^2 - a^2 = 0. \tag{1}$$

It is the required equation.

Corollary I. *If the initial line passes through the centre of the circle, then* $\alpha = 0$. *In this case, the equation* (1) *reduces to*

$$r^2 - 2cr\cos\theta + c^2 - a^2 = 0. \tag{2}$$

Corollary II. *If the pole is on the circle, then* $c = a$ *and the equation* (1) *reduces to*

$$r^2 - 2ar\cos(\theta - \alpha) = 0 \quad or, \quad r = 2a\cos(\theta - \alpha). \tag{3}$$

This equation is of the form

$$r = A\cos\theta + B\sin\theta. \tag{4}$$

Corollary III. *If the pole is on the circle and the initial line passes through the centre, then $c = a, \alpha = 0$ and the equation of the circle is*

$$r = 2a\cos\theta. \tag{5}$$

Corollary IV. *If the pole coincides with the centre, then the equation of the circle is*

$$r = a. \tag{6}$$

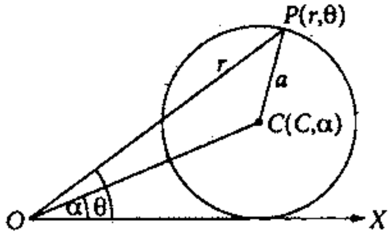


Fig. 62

Corollary V. *If the initial line touches the circle, then $c = a$ cosec.*

Putting this value of c in (1),

$$r^2 - 2ar\operatorname{cosec}\alpha\cos(\theta-\alpha) + a^2\operatorname{cosec}^2\alpha - a^2 = 0$$
$$\text{or, } r^2 - 2ar\operatorname{cosec}\alpha\cos(\theta-\alpha) + a^2\cot^2\alpha = 0. \tag{7}$$

It is the equation of the circle.

Corollary VI. *If the initial line touches the circle at the pole, then $\alpha = 90°$. In this case, the equation (7) reduces to*

$$r^2 - 2ar\cos(\theta - 90°) = 0 \quad or, \quad r = 2a\sin\theta. \tag{8}$$

Corollary VII. *A secant through the pole, given by the equation $\theta = \theta_1$, meet the circle (1) in two points whose distances from the pole are given by the roots of $r^2 - 2cr\cos(\theta_1 - \alpha) + c^2 - a^2 = 0$. If r_1 and r_2 are the roots of this equation, then $r_1 r_2 = c^2 - a^2 =$ a constant. It suggests that if the secant meets the circle at P and Q, then $OP \cdot OQ$=a constant for all positions of OPQ.*

Example 1. *Find the polar coordinates of the centre of the circle $r = 4\cos\theta + 3\sin\theta$.*

Let $4 = a\cos\alpha, 3 = a\sin\alpha$. Then $a = 5$ and $= \tan^{-1}\frac{3}{4}$.

Now $r = 4\cos\theta + 3\sin\theta = a(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = 5\cos(\theta - \alpha)$.

Hence the polar coordinates of the centre are $\left(\frac{5}{2}, \tan^{-1}\frac{3}{4}\right)$.

Example 2. *Find the polar equation of the circle which passes through the pole and two points whose polar coordinates are $(d, 0)$ and $(2d, \pi/3)$. Find also the radius of the circle.*

Let the equation be $r = A\cos\theta + B\sin\theta$.

By the given condition $d = A$ and $2d = \frac{A}{2} + B \cdot \frac{\sqrt{3}}{2}$ or, $B = \sqrt{3}d$.

$\therefore$ the equation is $r = d\cos\theta + \sqrt{3}d\sin\theta = 2d\cos(\theta - \pi/3)$.

Hence the radius of the circle $= d$.

Example 3. *Find the equation of the circle when two ends of a diameter are given.*

Let $P(r,\theta)$ be a point on the circle and the coordinates of the ends A and B of the diameter AB be (r_1, θ_1) and (r_2, θ_2) respectively w.r.t. the pole O and the initial line OX. From Fig. 63,

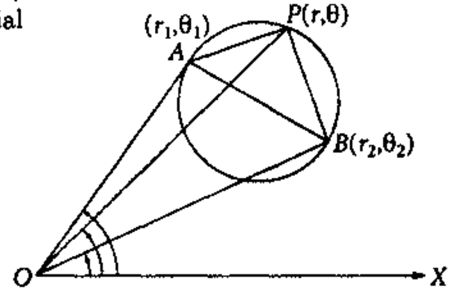


Fig. 63

$$AP^2 = OA^2 + OP^2 - 2OA \cdot OP \cos P\hat{O}A$$
$$= r_1^2 + r^2 - 2r_1 r \cos(\theta),$$
$$BP^2 = OB^2 + OP^2 - 2OB \cdot OP \cos B\hat{O}P$$
$$= r_2^2 + r^2 - 2r_2 r \cos(\theta - \theta_2),$$
$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos A\hat{O}B$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2).$$

Since AB is a diameter, $\angle APB$ is a right angle.

$$\therefore AP^2 + BP^2 = AB^2$$
$$\text{or, } r_1^2 + r^2 - 2r_1 r \cos(\theta_1 - \theta) + r_2^2 + r^2 - 2r_2 r \cos(\theta - \theta_2)$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$$
$$\text{or, } r^2 - r\left[r_1 \cos(\theta - \theta_1) + r_2 \cos(\theta - \theta_2)\right] + r_1 r_2 \cos(\theta_1 - \theta_2) = 0.$$

It is the equation of the circle.

Example 4. *Prove that the equation of the chord joining the points $\theta = \theta_1, \theta = \theta_2$ on the circle $r = 2a\cos\theta$ is $r\cos(\theta - \theta_1 - \theta_2) = 2a\cos\theta_1\cos\theta_2$.*

Let the equation of the chord be $\frac{1}{r} = A\cos\theta + B\sin\theta$. From the equation of the circle, the radius vector corresponding to the vectorial angle θ_1 is $2a\cos\theta_1$ and that for θ_2 is $2a\cos\theta_2$.

The ends of the chord are $(2a\cos\theta_1, \theta_1)$ and $(2a\cos\theta_2, \theta_2)$.

$$\therefore \frac{1}{2a\cos\theta_1} = A\cos\theta_1 + B\sin\theta_1 \quad \text{or,} \quad A2a\cos^2\theta_1 + Ba\sin 2\theta_1 - 1 = 0.$$

Similarly $A2a\cos^2\theta_2 + Ba\sin 2\theta_2 - 1 = 0$.

By cross-multiplication,

$$\frac{A}{a(\sin 2\theta_2 - \sin 2\theta_1)} = \frac{B}{2a(\cos^2\theta_1 - \cos^2\theta_2)}$$
$$= \frac{1}{2a^2(\cos^2\theta_1 \sin 2\theta_2 - \cos^2\theta_2 \sin 2\theta_1)}$$
$$\text{or,} \quad \frac{A}{2a\cos(\theta_1 + \theta_2)\sin(\theta_2 - \theta_1)} = \frac{B}{2a\sin(\theta_1 + \theta_2)\sin(\theta_2 - \theta_1)}$$
$$= \frac{1}{4a^2\cos\theta_1\cos\theta_2\sin(\theta_2 - \theta_1)}$$
$$\text{or,} \quad A = \frac{\cos(\theta_1 + \theta_2)}{2a\cos\theta_1\cos\theta_2}, \quad B = \frac{\sin(\theta_1 + \theta_2)}{2a\cos\theta_1\cos\theta_2}.$$

Thus the equation is

$$\cos\theta\cos(\theta_1+\theta_2)+\sin\theta\sin(\theta_1+\theta_2)=2a\cos\theta_1\cos\theta_2\frac{1}{r}$$

$$\text{or,}\quad r\cos(\theta-\theta_1-\theta_2)=2a\cos\theta_1\cos\theta_2.$$

> **Note.** Tangent at $\theta=\alpha$ on the circle $r=2a\cos\theta$.
> Putting $\theta_1=\theta_2=\alpha$, the tangent at $\theta=\alpha$ is $r\cos(\theta-2\alpha)=2a\cos^2\alpha$.

9.40 Polar equation of a conic with the focus as the pole

Let S be the focus, SX the axis and RM the directrix of the conic in Fig. 64. Let $P(r,\theta)$ be a point on the conic w.r.t. the pole S and the initial line SX. PM is perpendicular to the directrix. LSL' is the latus-rectum. PN and LQ are perpendiculars to the initial line and the directrix respectively.

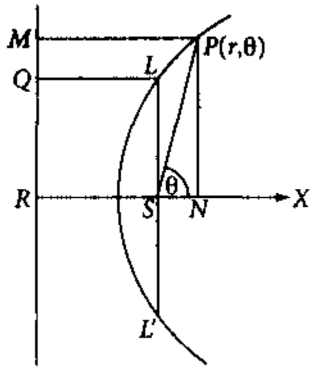


Fig. 64

Now the eccentricity $e=\frac{SP}{PM}$

$$\begin{aligned}\text{or,}\quad SP&=e\cdot PM=e\cdot RN=e\cdot(RS+SN)\\&=e\cdot(LQ+SN)\\&=e\cdot\left(\frac{SL}{e}+SN\right)\quad\left(\because\ \frac{SL}{LQ}=e\right)\\&=SL+e\cdot SP\cos\theta.\end{aligned}$$

If l is the semi-latusrectum, then $r=l+er\cos\theta$ or, $\frac{l}{r}=1-e\cos\theta$.
It is the equation of the conic.

Corollary I. *If the initial line is taken in the direction SR, then the equation of the conic will be $\frac{l}{r}=1-e\cos(\pi-\theta)$, i.e. $\frac{l}{r}=1+e\cos\theta$.*

Corollary II. *If the initial line makes an angle α with the axis, then the equation of the conic is $\frac{l}{r}=1-e\cos(\theta-\alpha)$.*

Corollary III. *If the conic is a parabola, then $e=1$. Thus the equation of a parabola can be expressed as*

$$\frac{l}{r}=1-\cos\theta\quad\textit{or,}\quad\frac{l}{r}=2\sin^2\frac{\theta}{2}\quad\textit{or,}\quad r^{1/2}\sin\frac{\theta}{2}=\left(\frac{l}{2}\right)^{1/2}$$

If the initial line is in the direction SR, then the equation can be written as

$$\frac{l}{r}=1+\cos\theta\quad\text{or,}\quad\frac{l}{r}=2\cos^2\frac{\theta}{2}\quad\text{or,}\quad r^{1/2}\cos\frac{\theta}{2}=\left(\frac{l}{2}\right)^{1/2}$$

> **Note.** If (r,θ) be a point on the directrix, then $r\cos(\pi-\theta)=SR$ or, $\frac{l}{r}=-e\cos\theta$.

Example 5. *Find the nature of the following conics:* (i) $\frac{5}{r}=2-2\cos\theta$, (ii) $\frac{1}{r}=8+5\cos\theta$, (iii) $\frac{8}{r}=4-5\cos\theta$.

(i) The equation $\frac{5}{r} = 2 - 2\cos\theta$ can be written as $\frac{5/2}{r} = 1 - \cos\theta$. Comparing it with $\frac{l}{r} = 1 - e\cos\theta$, we have $l = \frac{5}{2}$ and $e = 1$. Since the eccentricity is 1, the given equation represents a parabola. Its semi-latusrectum is 5/2.

(ii) The equation $\frac{1}{r} = 8 + 5\cos\theta$ can be written as $\frac{1/8}{r} = 1 + \frac{5}{8}\cos\theta$. Comparing it with $\frac{l}{r} = 1 + e\cos\theta$, we have $l = \frac{1}{8}$ and $e = 5/8$. Since the eccentricity is less than 1, the given equation represents an ellipse whose semi-latusrectum is $\frac{1}{8}$.

(iii) The equation $\frac{8}{r} = 4 - 5\cos\theta$ can be written as $\frac{2}{r} = 1 - \frac{5}{4}\cos\theta$. Comparing it with $\frac{l}{r} = 1 - e\cos\theta$, we have $l = 2$ and $e = \frac{5}{4}$. Since $e > 1$, the equation represents a hyperbola whose semi-latusrectum is 2.

Example 6. *Find the points on the conic* $\frac{9}{r} = 2 + \sqrt{2}\cos\theta$ *whose radius vector is* 3.

Putting $r = 3$ in the equation we have $3 = 2 + \sqrt{2}\cos\theta$ or, $\cos\theta = 1/\sqrt{2}$. Therefore, $\theta = \pi/4, -\pi/4$. Thus the required points are $\left(3, \frac{\pi}{4}\right)$ and $\left(3, -\frac{\pi}{4}\right)$.

Example 7. *Find the equation of the directrix of the conic* $r\sin^2\frac{\theta}{2} = 1$.

$$r\sin^2\frac{\theta}{2} = 1 \quad \text{or,} \quad r \cdot 2\sin^2\theta/2 = 2 \quad \text{or,} \quad r(1-\cos\theta) = 2 \quad \text{or,} \quad \frac{2}{r} = 1 - \cos\theta.$$

It is a parabola whose semi-latusrectum is 2.

The equation of the directrix is $\frac{2}{r} = -\cos\theta$ or, $r\cos\theta + 2 = 0$.

Example 8. *Find the polar equation of the left branch of the hyperbola* $9x^2 - 16y^2 = 144$.

The given equation can be written as $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Comparing it with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get $a^2 = 16, b^2 = 9$.

$\therefore$ semi-latusrectum $l = b^2/a = 9/4$ and eccentricity

$$e = \sqrt{\left(1 + \frac{b^2}{a^2}\right)} = \sqrt{\left(1 + \frac{9}{16}\right)} = 5/4.$$

The equation of the left branch in polar coordinates can be written as

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{or,} \quad \frac{9/4}{r} = 1 + \frac{5}{4}\cos\theta \quad \text{or,} \quad \frac{9}{r} = 4 + 5\cos\theta.$$

9.41 Equation of a chord (joining $\theta = \alpha - \beta$ and $\theta = \alpha + \beta$), a tangent and a normal

Chord: Let

$$\frac{l}{r} = 1 - e\cos\theta \tag{1}$$

$$\text{and } \frac{l}{r} = a\cos\theta + b\sin\theta \tag{2}$$

be the equations of the conic and the chord respectively.

For the common points of (1) and (2),

$$1 - e\cos\theta = a\cos\theta + b\sin\theta.$$

At the common points $\theta = \alpha - \beta$ and $\theta = \alpha + \beta$.

$$\therefore\ 1 - e\cos(\alpha - \beta) = a\cos(\alpha - \beta) + b\sin(\alpha - \beta)$$
$$\text{and } 1 - e\cos(\alpha + \beta) = a\cos(\alpha + \beta) + b\sin(\alpha + \beta).$$

From these two,

$$(a + e)\cos(\alpha - \beta) + b\sin(\alpha - \beta) - 1 = 0$$
$$\text{and } (a + e)\cos(\alpha + \beta) + b\sin(\alpha + \beta) - 1 = 0.$$

By cross-multiplication,

$$\frac{a+e}{-\sin(\alpha-\beta)+\sin(\alpha+\beta)} = \frac{b}{-\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$
$$= \frac{1}{\cos(\alpha-\beta)\sin(\alpha+\beta) - \sin(\alpha-\beta)\cos(\alpha+\beta)}$$
$$\text{or, } \frac{a+e}{2\cos\alpha\sin\beta} = \frac{b}{2\sin\alpha\sin\beta} = \frac{1}{2\sin\beta\cos\beta}.$$
$$\therefore a + e = \frac{\cos\alpha}{\cos\beta} \text{ or, } a = \frac{\cos\alpha}{\cos\beta} - e \text{ and } b = \frac{\sin\alpha}{\cos\beta}.$$

$\therefore$ the equation of the chord is

$$\frac{l}{r} = \left(\frac{\cos\alpha}{\cos\beta} - e\right)\cos\theta + \frac{\sin\alpha}{\cos\beta}\sin\theta$$
$$\text{or, } \frac{l}{r} = -e\cos\theta + \sec\beta\cos(\theta - \alpha). \tag{3}$$

Corollary. The equation of the chord through the points γ and δ.

Here $\alpha - \beta = \gamma, \alpha + \beta = \delta$.

$$\therefore\ \alpha = \frac{\gamma+\delta}{2} \text{ and } \beta = \frac{\delta-\gamma}{2}.$$

Putting these values of α and β in (3), we have

$$\frac{l}{r} = -e\cos\theta + \sec\frac{\delta-\gamma}{2}\cos\left(\theta - \frac{\delta+\gamma}{2}\right). \tag{4}$$

It is the equation of the chord.

Tangent. The two points corresponding to vectorial angles $\alpha - \beta$ and $\alpha + \beta$ will coincide, if $\beta = 0$. In this case, the chord will be the tangent at α. Hence the equation of the tangent at the point whose vectorial angle is α is

$$\frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha). \tag{5}$$

Normal. Let the equation of the normal at the point α be

$$\frac{k}{r} = \cos\left(\frac{\pi}{2} + \theta - \alpha\right) - e\cos(\frac{\pi}{2} + \theta)$$

[$\because$ the normal is perpendicular to the tangent]

$$\text{or,} \quad \frac{k}{r} = -\sin(\theta - \alpha) + e\sin\theta.$$

It passes through the point of contact $\left(\frac{l}{1-e\cos}, \alpha\right)$.

$$\therefore \frac{k(1 - e\cos\alpha)}{l} = e\sin\alpha \text{ or, } k = \frac{el\sin\alpha}{1 - e\cos\alpha}.$$

Putting this value of k, the equation of the normal is

$$\frac{el\sin\alpha}{r(1 - e\cos\alpha)} = e\sin\theta - \sin(\theta - \alpha). \tag{6}$$

Note. If the equation of the conic is $\frac{l}{r} = 1 + e\cos\theta$, then the equations of the tangent and normal at the point α are $\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$ and $\frac{le\sin\alpha}{r(1+e\cos\alpha)} = e\sin\theta + \sin(\theta - \alpha)$ respectively.

9.42 To find the condition that the line $\frac{1}{r} = A\cos\theta + B\sin\theta$ may touch the conic $\frac{l}{r} = 1 - e\cos\theta$

Let the line

$$\frac{1}{r} = A\cos\theta + B\sin\theta. \tag{1}$$

touch the conic $\frac{l}{r} = 1 - e\cos\theta$ at $\theta = \alpha$.

The equation of the tangent at $\theta = \alpha$ to the conic is

$$\frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha), \text{ i.e. } \frac{l}{r} = (\cos\alpha - e)\cos\theta + \sin\alpha\sin\theta.$$

Comparing it with the equation (1), we have

$$\frac{l}{1} = \frac{\cos\alpha - e}{A} = \frac{\sin\alpha}{B}.$$

$$\therefore \cos\alpha = Al + e \text{ and } \sin\alpha = Bl.$$

Now $\cos^2\alpha + \sin^2\alpha = (Al + e)^2 + B^2l^2$ or, $(Al + e)^2 + B^2l^2 = 1$.

It is the required condition.

9.43 Chord of contact

Definition. *It is a chord joining the points of contact of tangents to a conic from a given point outside the conic.*

Let

$$\frac{l}{r} = 1 - e\cos\theta \tag{1}$$

be the equation of the conic and (r_1, θ_1) be the point from which tangents are drawn to the conic.

Let $(\alpha - \beta)$ and $(\alpha + \beta)$ be the vectorial angles of the two points of contact of the tangents. Then the equation of the chord joining them is

$$\frac{l}{r} = -e\cos\theta + \sec\beta\cos(\theta - \alpha) \tag{2}$$

[See the equation (3) of Sec 9.41]

The equations of the tangents at the points on the conic having vectorial angles $\alpha - \beta$ and $\alpha + \beta$ are

$$\frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha + \beta)$$

$$\text{and } \frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha - \beta).$$

These pass through the point (r_1, θ_1), if

$$\frac{l}{r_1} = -e\cos\theta_1 + \cos(\theta_1 - \alpha + \beta) \tag{3}$$

$$\text{and } \frac{l}{r_1} = -e\cos\theta_1 + \cos(\theta_1 - \alpha - \beta). \tag{4}$$

Thus $\cos(\theta_1 - \alpha + \beta) = \cos(\theta_1 - \alpha - \beta)$, which is possible if $\theta_1 - \alpha + \beta = \pm(\theta_1 - \alpha - \beta)$.

Since $\beta \neq 0$, the upper sign is inadmissible and

$$\theta_1 - \alpha + \beta = -(\theta_1 - \alpha - \beta), \text{ i.e. } \alpha = \theta_1. \tag{5}$$

Now from (3),

$$\frac{l}{r_1} = -e\cos\theta_1 + \cos\beta \text{ or, } \cos\beta = \frac{l}{r_1} + e\cos\theta_1. \tag{6}$$

Using (5) and (6) in (2), we get

$$\left(\frac{l}{r} + e\cos\theta\right)\left(\frac{l}{r_1} + e\cos\theta_1\right) = \cos(\theta - \theta_1). \tag{7}$$

It is the equation of the chord of contact of tangents through the point (r_1, θ_1).

Note. If the equation of the conic is $\frac{l}{r} = 1 + e\cos\theta$, then the equation of the chord of contact of the tangents to the conic from the point (r_1, θ_1) is

$$\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r_1} - e\cos\theta_1\right) = \cos(\theta - \theta_1).$$

9.44 Pole and Polar

The polar of a point w.r.t. a conic is the locus of the points of intersection of tangents at the extremities of the chords through that point and the point itself is called the pole of its polar.

Let $\frac{l}{r} = 1 - e\cos\theta$ be the equation of the conic and the point (r_1, θ_1) be the pole.

Let the vectorial angles of the extremities of a chord to the conic through the pole be $\alpha-\beta$ and $\alpha+\beta$. Then the equation of this chord is $\frac{l}{r} = -e\cos\theta + \sec\beta\cos(\theta-\alpha)$. Since it passes through the point (r_1, θ_1), we have

$$\frac{l}{r_1} = -e\cos\theta_1 + \sec\beta\cos(\theta_1 - \alpha). \tag{1}$$

The tangents at the extremities are

$$\frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha + \beta)$$
$$\text{and } \frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha - \beta).$$

If these tangents meet at the point (ρ, ϕ), then

$$\frac{l}{\rho} = -e\cos\phi + \cos(\phi - \alpha + \beta) \text{ and } \frac{l}{\rho} = -e\cos\phi + \cos(\phi - \alpha - \beta).$$

Thus $\cos(\phi - \alpha + \beta) = \cos(\phi - \alpha - \beta)$. It is possible if $\phi - \alpha + \beta = \pm(\phi - \alpha - \beta)$. Since $\beta \neq 0$, the upper sign is inadmissible and

$$\phi - \alpha + \beta = -(\phi - \alpha - \beta), \text{ i.e. } \phi = \alpha. \tag{2}$$

$$\therefore \frac{l}{\rho} = -e\cos\phi + \cos\beta \text{ or, } \frac{l}{\rho} + e\cos\phi = \cos\beta. \tag{3}$$

From (1),

$$\frac{l}{r_1} + e\cos\theta_1 = \sec\beta\cos(\theta_1 - \phi). \tag{4}$$

Eliminating β from (3) and (4),

$$\left(\frac{l}{\rho} + e\cos\phi\right)\left(\frac{l}{r_1} + e\cos\theta_1\right) = \cos(\theta_1 - \phi).$$

Hence the locus of (ρ, ϕ) is

$$\left(\frac{l}{r} + e\cos\theta\right)\left(\frac{l}{r_1} + e\cos\theta_1\right) = \cos(\theta - \theta_1). \tag{5}$$

It is the equation of the polar of the pole (r_1, θ_1).

Note. The polar of a point w.r.t. a conic coincides with the chord of contact of tangents from that point to the conic when the point is outside the conic.

9.45 To find the equation of the pair of tangents to the conic $\frac{l}{r} = 1 - e\cos\theta$ from the point (r', θ').

The equation of the tangent to the conic at $\theta = \alpha$ is

$$\frac{l}{r} = \cos(\theta - \alpha) - e\cos\theta. \tag{1}$$

If it passes through the point (r', θ'), then

$$\frac{l}{r'} = \cos(\theta' - \alpha) - e\cos\theta'. \tag{2}$$

From (1),

$$\left(\frac{l}{r} + e\cos\theta\right)^2 = \cos^2(\theta - \alpha) = 1 - \sin^2(\theta - \alpha)$$

$$\text{or,}\quad 1 - \left(\frac{l}{r} + e\cos\theta\right)^2 = \sin^2(\theta - \alpha). \tag{3}$$

Similarly from (2),

$$1 - \left(\frac{l}{r'} + e\cos\theta'\right)^2 = \sin^2(\theta' - \alpha). \tag{4}$$

Again by (1) and (2),

$$\begin{aligned}
&\left(\frac{l}{r} + e\cos\theta\right)\left(\frac{l}{r'} + e\cos\theta'\right) - \cos(\theta - \theta') \\
&= \cos(\theta - \alpha)\cos(\theta' - \alpha) - \cos(\theta - \theta') \\
&= \frac{1}{2}\{\cos(\theta + \theta' - 2\alpha) - \cos(\theta - \theta')\} \\
&= -\sin(\theta - \alpha)\sin(\theta' - \alpha).
\end{aligned} \tag{5}$$

To eliminate α from (3), (4) and (5),

$$\begin{aligned}
&\left\{1 - \left(\frac{l}{r} + e\cos\theta\right)^2\right\}\left\{1 - \left(\frac{l}{r'} + e\cos\theta'\right)^2\right\} \\
&= \sin^2(\theta - \alpha)\sin^2(\theta' - \alpha) \\
&= \left\{\left(\frac{l}{r} + e\cos\theta\right)\left(\frac{l}{r'} + e\cos\theta'\right) - \cos(\theta - \theta')\right\}^2.
\end{aligned}$$

Thus the equation of the pair of tangents to the conic $\frac{l}{r} = 1 - e\cos\theta$ from the point (r', θ') is

$$\begin{aligned}
&\left\{1 - \left(\frac{l}{r} + e\cos\theta\right)^2\right\}\left\{1 - \left(\frac{l}{r'} + e\cos\theta'\right)^2\right\} \\
&= \left\{\left(\frac{l}{r} + e\cos\theta\right)\left(\frac{l}{r'} + e\cos\theta'\right) - \cos(\theta - \theta')\right\}^2.
\end{aligned} \tag{6}$$

9.46 To find the locus of the point of intersection of a pair of perpendicular tangents to the conic $\frac{l}{r} = 1 - e\cos\theta$.

Let the tangents at $\theta = \alpha$ and β to the conic meet at the point (r_1, θ_1) perpendicularly.

The equation of the tangents are

$$\frac{l}{r} = \cos(\theta - \alpha) - e\cos\theta \tag{1}$$

$$\text{and } \frac{l}{r} = \cos(\theta - \beta) - e\cos\theta. \tag{2}$$

Since the lines (1) and (2) meet at (r_1, θ_1),

$$\frac{l}{r_1} = \cos(\theta_1 - \alpha) - e\cos\theta_1 \tag{3}$$

$$\text{and } \frac{l}{r_1} = \cos(\theta_1 - \beta) - e\cos\theta_1. \tag{4}$$

From (3) and (4),

$$\cos(\theta_1 - \alpha) = \cos(\theta_1 - \beta)$$

$$\text{or, } \theta_1 - \alpha = -(\theta_1 - \beta) \ [\because \ \alpha \neq \beta]$$

$$\text{or, } \theta_1 = \tfrac{1}{2}(\alpha + \beta). \tag{5}$$

By (3) and (5),

$$\frac{l}{r_1} = \cos\frac{\alpha - \beta}{2} - e\cos\theta_1$$

$$\text{or, } \frac{l}{r_1} + e\cos\theta_1 = \cos\frac{\alpha - \beta}{2}. \tag{6}$$

The equation (1) and (2) can be written as

$$\frac{l}{r} = (\cos\alpha - e)\cos\theta + \sin\alpha\sin\theta$$

$$\text{and } \frac{l}{r} = (\cos\beta - e)\cos\theta + \sin\beta\sin\theta.$$

Since these two lines are at right angles,

$$(\cos\alpha - e)(\cos\beta - e) + \sin\alpha\sin\beta = 0$$

$$\text{or, } \cos(\alpha - \beta) - e(\cos\alpha + \cos\beta) + e^2 = 0$$

$$\text{or, } 2\cos^2\frac{\alpha - \beta}{2} - 1 - 2e\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2} + e^2 = 0. \tag{7}$$

To eliminate $\cos\frac{\alpha - \beta}{2}$ from (6) and (7),

$$2\left(\frac{l}{r_1} + e\cos\theta_1\right)^2 - 1 + e^2 - 2e\cos\theta_1\left(\frac{l}{r_1} + e\cos\theta_1\right) = 0$$

$$\text{or, } 2\frac{l^2}{r_1^2} + 2\frac{le}{r_1}\cos\theta_1 + (e^2 - 1) = 0$$

$$\text{or, } (e^2 - 1)\, r_1^2 + 2ler_1\cos\theta_1 + 2l^2 = 0.$$

Thus the locus of (r_1, θ_1) is

$$\left(e^2 - 1\right) r^2 + 2ler\cos\theta + 2l^2 = 0. \tag{8}$$

It is the required locus.

Note

1 **Director circle.** The conic is an ellipse or hyperbola according as $e < 1$ or > 1. If $e = 0$, the conic is a circle. In all these cases the equation (8) represents a circle which is known as the *director circle.*

2 For $e = 1$, the conic is a parabola. In this case, the equation (8) reduces to $l = -r\cos\theta$ which is the equation of the directrix. Thus the locus of the point of intersection of two mutually perpendicular tangents to a parabola is the *directrix of the parabola.*

9.47 To find the equation of the asymptotes of the conic $\frac{l}{r} = 1 - e\cos\theta$

Let (r', α) be a point on the conic. Then

$$\frac{l}{r'} = 1 - e\cos\alpha. \tag{1}$$

The tangent at this point is

$$\frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha). \tag{2}$$

Any asymptote is a tangent at infinity. If the point (r', α) is at infinity, then r' is infinite and $1 - e\cos\alpha = 0$, i.e. $\cos\alpha = \frac{1}{e}$ by (1).

Hence $\sin\alpha = \pm\sqrt{1 - \frac{1}{e^2}}$.

The equatión (2) can be written as

$$\frac{l}{r} = -e\cos\theta + \cos\theta\cos\alpha + \sin\theta\sin\alpha.$$

Putting the values of $\cos\alpha$ and $\sin\alpha$,

$$\frac{l}{r} = -e\cos\theta + \cos\theta \cdot \frac{1}{e} \pm \sin\theta\sqrt{1 - \frac{1}{e^2}}$$

$$\text{or, } \frac{l}{r} + \left(e - \frac{1}{e}\right)\cos\theta = \pm\sqrt{1 - \frac{1}{e^2}}\sin\theta$$

$$\text{or, } \frac{el}{r} + \left(e^2 - 1\right)\cos\theta = \pm\sqrt{e^2 - 1}\sin\theta. \tag{3}$$

The equation (3) represent the asymptotes of the conic.

Note

1. Asymptotes are real when $e > 1$, i.e. the conic is a hyperbola.
2. Asymptotes are pair of tangents from the centre $\left(\frac{el}{e^2-1}, \pi\right)$ i.e. the equation of (3) can be deduced from (6) of Sec 9.45 by putting $r' = \frac{el}{e^2-1}$ and $\theta' = \pi$.

WORKED-OUT EXAMPLES

1. *Transform the equations*

   (a) $\left(x^2 + y^2\right)^2 = ax^2y$ to polar form and

   (b) $\frac{1}{r} = 1 + \cos\theta$ to Cartesian form.

Let (r, θ) be the polar coordinates of the point whose Cartesian coordinates are (x, y). Then $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$.

(a) Now $\left(x^2 + y^2\right)^2 = ax^2y$ transforms to $r^4 = ar^2\cos^2\theta \cdot r\sin\theta$ or, $r = a\cos^2\theta\sin\theta$.

It is the equation in polar form.

(b) $\frac{1}{r} = 1 + \cos\theta$ or, $1 = r + r\cos\theta$ or, $1 = \sqrt{x^2 + y^2} + x$ or, $(1 - x)^2 = x^2 + y^2$ or, $y^2 = 1 - 2x$.

It is the equation in Cartesian form.

2. *Show that for the conic* $\frac{l}{r} = 1 + e\cos\theta$, *the equation to the directrix corresponding to the focus other than the pole is* $\frac{l}{r} = -\frac{1-e^2}{1+e^2}e\cos\theta$.

Let $P(r, \theta)$ be a point on the directrix corresponding to the focus S' (other than the pole S). SR' is perpendicular to the directrix. From Fig. 65,

$$\frac{SR'}{SP} = \cos(\pi - \theta)$$

$$\text{or,}\quad r\cos\theta = -SR' = -(RR' - SR) = -\left(\frac{2a}{e} - \frac{l}{e}\right).$$

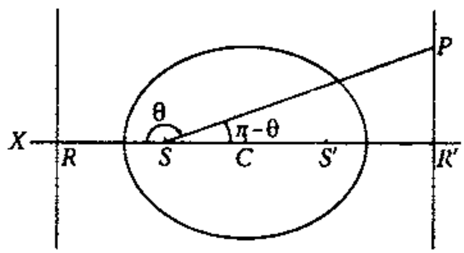


Fig. 65

where l = semi-latusrectum $= \frac{b^2}{a} = \frac{a^2(1-e)^2}{a} = a\left(1 - e^2\right)$, a = semimajor axis.

$$\therefore\ r\cos\theta = -\left[\frac{2}{e}\cdot\frac{l}{1 - e^2} - \frac{l}{e}\right] = -\frac{l}{e}\cdot\frac{1 + e^2}{1 - e^2} \text{ or, } \frac{l}{r} = -\frac{1 - e^2}{1 + e^2}\cdot e\cos\theta.$$

It is the equation of the required directrix.

3. *Show that the tangent at $\theta = \alpha$ to $\frac{2a}{r} = 1 + \cos\theta$ is $r = a\sec\frac{\alpha}{2}\sec(\theta - \frac{\alpha}{2})$.*

   The equation of the tangent at $\theta = \alpha$ is

$$\frac{2a}{r} = \cos\theta + \cos(\theta - \alpha) = 2\cos(\theta - \frac{\alpha}{2})\cos\frac{\alpha}{2}$$

$$\text{or,}\quad r = a\sec\frac{\alpha}{2}\sec\left(\theta - \frac{\alpha}{2}\right).$$

4. *If θ and ϕ are vectorial angles of a point on a conic referred to two foci and they are measured in the same sense w.r.t. the initial line joining the foci, show that the ratio of $\tan\theta/2$ and $\tan\varphi/2$ is constant.*

   Let S and S' be the foci and SX be the initial line. Let (r, θ) be the coordinates of P referred to the pole S and (ρ, ϕ) be those of P referred to the pole S'.

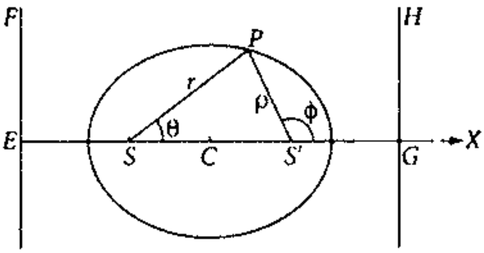


Fig. 66

   The equation of the conic w.r.t. the pole S and the directrix EF is $\frac{l}{r} = 1 - e\cos\theta$, where e is the eccentricity. The equation of the conic w.r.t. the pole S' and the directrix GH is $\frac{l}{\rho} = 1 + e\cos\phi$.

$$\therefore\ \frac{1 - e\cos\theta}{1 + e\cos\phi} = \frac{\rho}{r} = \frac{\sin\theta}{\sin(\pi - \phi)} = \frac{\sin\theta}{\sin\phi}\ [\text{from } \triangle SS'P].$$

$$\therefore\ \sin\phi(1 - e\cos\theta) = \sin\theta(1 + e\cos\phi)$$

$$\text{or, } \sin\phi - \sin\theta = e(\sin\phi\cos\theta + \cos\phi\sin\theta) = e\sin(\phi + \theta)$$

$$\text{or, } 2\cos\frac{\phi + \theta}{2}\sin\frac{\phi - \theta}{2} = e \cdot 2\sin\frac{\phi + \theta}{2}\cos\frac{\phi + \theta}{2}$$

$$\text{or, } \frac{\sin\frac{\phi - \theta}{2}}{\sin\frac{\phi + \theta}{2}} = e \text{ or, } \frac{\sin\frac{\phi + \theta}{2} - \sin\frac{\phi - \theta}{2}}{\sin\frac{\phi + \theta}{2} + \sin\frac{\phi - \theta}{2}} = \frac{1 - e}{1 + e}$$

$$\text{or, } \frac{2\cos\frac{\phi}{2}\sin\frac{\theta}{2}}{2\sin\frac{\phi}{2}\cos\frac{\theta}{2}} = \frac{1 - e}{1 + e} \text{ or, } \frac{\tan\frac{\theta}{2}}{\tan\frac{\phi}{2}} = \frac{1 - e}{1 + e} = \text{constant.}$$

5. *Show that the semi-latusrectum of a conic is the harmonic mean between the segments of a focal chord.*

   [NH 2008; BH 2009; CH 96]

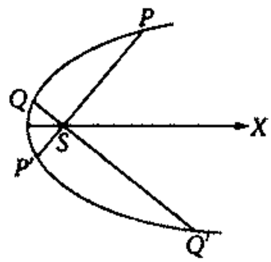


Fig. 67

   Let the equation of the conic be $\frac{l}{r} = 1 - e\cos\theta$ and PSP' be a focal chord. If $\angle XSP = \alpha$, then the vectorial angle of P' is $\pi + \alpha$.

   Now

$$\frac{l}{SP} = 1 - e\cos\alpha$$

$$\text{and } \frac{l}{SP'} = 1 - e\cos(\pi + \alpha) = 1 + e\cos\alpha.$$

Adding these two, we have

$$\frac{l}{SP} + \frac{l}{SP'} = 2 \text{ or, } \frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l}.$$

Hence the result follows.

6. *In any conic, prove that the sum of the reciprocals of two perpendicular focal chords is constant.* [BH 94, CH 2007]

Let

$$\frac{l}{r} = 1 - e\cos\theta \tag{1}$$

be the equation of the conic and PSP' and QSQ' be the two perpendicular focal chords. [See Fig. 67] If α be the vectorial angle of P, then the vectorial angles of Q, P' and Q' are $\frac{\pi}{2} + \alpha$, $\pi + \alpha$ and $\frac{3\pi}{2} + \alpha$ respectively.

From the equation (1)

$$\frac{l}{SP} = 1 - e\cos\alpha \text{ and } \frac{l}{SP'} = 1 + e\cos\alpha.$$

$$\therefore SP + SP' = \frac{l}{1 - e\cos\alpha} + \frac{l}{1 + e\cos\alpha} = \frac{2l}{1 - e^2\cos^2\alpha}$$

$$\text{or, } PP' = \frac{2l}{1 - e^2\cos^2\alpha} \text{ or, } \frac{1}{PP'} = \frac{1 - e^2\cos^2\alpha}{2l}. \tag{2}$$

Again

$$\frac{l}{SQ} = 1 - e\cos(\frac{\pi}{2} + \alpha) = 1 + e\sin\alpha$$

$$\text{and } \frac{l}{SQ'} = 1 - e\cos\left(\frac{3\pi}{2} + \alpha\right) = 1 - e\sin\alpha.$$

$$\therefore SQ + SQ' = \frac{l}{1 + e\sin\alpha} + \frac{l}{1 - e\sin\alpha} = \frac{2l}{1 - e^2\sin^2\alpha}$$

$$\text{or, } QQ' = \frac{2l}{1 - e^2\sin^2\alpha}, \frac{1}{QQ'} = \frac{1 - e^2\sin^2\alpha}{21}. \tag{3}$$

By (2) and (3),

$$\frac{1}{PP'} + \frac{1}{QQ'} = \frac{1 - e^2\cos^2\alpha}{2l} + \frac{1 - e^2\sin^2\alpha}{2l} = \frac{2 - e^2}{2l} = \text{constant.}$$

7. *If PA and PB be the two tangents to the conic $\frac{l}{r} = 1 - e\cos\theta$ at α and β respectively, then show that PS(S is the focus) bisects the angle ASB.*

Equations of tangents at α and β are

$$\frac{l}{r} = \cos(\theta - \alpha) - e\cos\theta$$

$$\text{and } \frac{l}{r} = \cos(\theta - \beta) - e\cos\theta.$$

The vectorial angle of P is obtained from

$$\begin{aligned}
&\cos(\theta-\alpha)-e\cos\theta=\cos(\theta-\beta)-e\cos\theta\\
\text{or, } &\cos(\theta-\alpha)-\cos(\theta-\beta)=0\\
\text{or, } &2\sin\left(\theta-\frac{\alpha-\beta}{2}\right)\sin\frac{\alpha-\beta}{2}=0.\ \because\ \frac{\alpha-\beta}{2}\neq 0, \theta=\frac{\alpha+\beta}{2}.
\end{aligned}$$

Thus PS bisects the angle ASB.

8. *PSP' is a focal chord of a conic; prove that the angle between the tangents at P and P' is $\tan^{-1}\frac{2e\sin\alpha}{1-e^2}$ where α is the angle between the chord and the major axis.* **[NH 2008; CH 92, 93, 2000, 02]**

   Let $\frac{l}{r}=1-e\cos\theta$ be the equation of the conic.

   If α be the vectorial angle of P, then the vectorial angle of P' is $\pi+\alpha$. Tangents at P and P' are

$$\frac{l}{r}=\cos(\theta-\alpha)-e\cos\theta \tag{4}$$

$$\text{and } \frac{l}{r}=\cos(\theta-\pi-\alpha)-e\cos\theta. \tag{5}$$

   The equation (1) can be written as

$$l=r\cos\theta(\cos\alpha-e)+r\sin\theta\sin\alpha.$$

   The slope of this line $=\frac{e-\cos\alpha}{\sin\alpha}$.

   Similarly the slope of (2) $=\frac{e-\cos(\pi+\alpha)}{\sin(\pi+\alpha)}=\frac{e+\cos\alpha}{-\sin\alpha}$.

   Hence the angle between the tangents

$$=\tan^{-1}\frac{\frac{e-\cos\alpha}{\sin\alpha}+\frac{e+\cos\alpha}{\sin\alpha}}{1-\frac{(e-\cos\alpha)(e+\cos\alpha)}{\sin^2\alpha}}=\tan^{-1}\frac{2e\sin\alpha}{\sin^2\alpha-e^2+\cos^2\alpha}=\tan^{-1}\frac{2e\sin\alpha}{1-e^2}.$$

9. *Show that the triangle formed by the pole and the points of intersection of the circle $r=4\cos\theta$ with the line $r\cos\theta=3$ is an equilateral triangle.*

   The vectorial angles of the common points of the line and the circle are obtained by eliminating r from the given equations.

   Thus at the common points

$$4\cos\theta\cdot\cos\theta=3 \text{ or, } \cos^2\theta=\frac{3}{4} \text{ or, } \cos\theta=\pm\frac{\sqrt{3}}{2}.$$

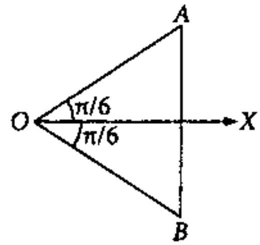


Fig. 68

   Since the line is perpendicular to the initial line and on the right side of the pole,

$$\cos\theta=\frac{\sqrt{3}}{2}.\ \therefore\ \theta=\pm\pi/6.$$

If the line meets the circle at A and B and OX is the initial line with the point O as pole, then $OA = OB = 4\cos(\pm\pi/6) = 2\sqrt{3}$. Again $\angle XOA = \angle BOX = \frac{\pi}{6}$. It follows that $\angle OAB = \angle OBA = \pi/3$. Hence ΔOAB is equilateral.

10. *If S is the focus and P, Q are two points on a conic such that the angle PSQ is constant and equal to δ, prove that the locus of the intersection of the tangents at P and Q is a conic whose focus is S.*

Let the equations of the conic be $\frac{l}{r} = 1 - e\cos\theta$ and the vectorial angles of P and Q be $\alpha - \beta$ and $\alpha + \beta$ respectively. The tangents at P and Q are

$$\frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha + \beta) \tag{1}$$

$$\text{and } \frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha - \beta). \tag{2}$$

If these tangents meet at (r_1, θ_1), then

$$\frac{l}{r_1} = -e\cos\theta_1 + \cos(\theta_1 - \alpha + \beta) \tag{3}$$

$$\text{and } \frac{l}{r_1} = -e\cos\theta_1 + \cos(\theta_1 - \alpha - \beta). \tag{4}$$

From (3) and (4),

$$\frac{l}{r_1} + e\cos\theta_1 = \cos(\theta_1 - \alpha + \beta) = \cos(\theta_1 - \alpha - \beta).$$

$$\therefore\ \cos(\theta_1 - \alpha + \beta) = \cos(\theta_1 - \alpha - \beta) \text{ or, } \sin(\theta_1 - \alpha)\sin\beta = 0.$$

$$\because\ \beta \neq 0, \theta_1 = \alpha.$$

Again

$$\angle PSQ = (\alpha + \beta) - (\alpha - \beta) = 2\beta = \delta \text{ or, } \beta = \frac{\delta}{2}.$$

Now from (3),

$$\frac{l}{r_1} = -e\cos\theta_1 + \cos\left(\alpha - \alpha + \frac{\delta}{2}\right)$$

$$\text{or, } \frac{l}{r_1} = -e\cos\theta_1 + \cos\frac{\delta}{2} \text{ or, } \frac{l\sec\frac{\delta}{2}}{r_1} = 1 - e\sec\frac{\delta}{2}\cos\theta_1.$$

Hence the locus of (r_1, θ_1) is

$$\frac{l\sec\frac{\delta}{2}}{r} = 1 - e\sec\frac{\delta}{2}\cos\theta.$$

It is a conic with the same focus S.

11. *If the normal is drawn at one extremity of the latusrectum PSP' of the conic $\frac{l}{r} = 1 + e\cos\theta$, where S is the pole, show that the distance from the focus S of the other point in which the normal meets the curve is $\frac{l(1+3e^2+e^4)}{1+e^2-e^4}$.*

[CH 2004, 09]

Let the normal be drawn from the point P and the polar coordinates of P be $(l, \frac{\pi}{2})$.

The equation of the normal at P is

$$\frac{el\sin\frac{\pi}{2}}{(1+e\cos\frac{\pi}{2})r} = e\sin\theta + \sin\left(\theta - \frac{\pi}{2}\right) \text{ or, } \frac{el}{r} = e\sin\theta - \cos\theta.$$

At the common points of the normal and the conic

$$e(1+e\cos\theta) = e\sin\theta - \cos\theta \text{ or, } (1+e^2)\cos\theta = e(\sin\theta - 1)$$

$$\text{or, } \frac{\cos\theta}{1-\sin\theta} = -\frac{e}{1+e^2} \text{ or, } \frac{\cos^2\theta}{(1-\sin\theta)^2} = \frac{e^2}{(1+e^2)^2}$$

$$\text{or, } \frac{1+\sin\theta}{1-\sin\theta} = \frac{e^2}{(1+e^2)^2} \text{ or, } \sin\theta = \frac{e^2-(1+e^2)^2}{e^2+(1+e^2)^2}$$

$$\therefore \cos\theta = \frac{e}{1+e^2}\left[\frac{e^2-(1+e^2)^2}{e^2+(1+e^2)^2} - 1\right] = \frac{-2e(1+e^2)}{1+3e^2+e^4}.$$

$\therefore$ the required distance

$$= \frac{l}{1+e\cos\theta} = \frac{l(1+3e^2+e^4)}{1+e^2-e^4}.$$

12. *If PQ is a variable chord of the conic $\frac{l}{r} = 1 - e\cos\theta$ subtending a constant angle 2β at the focus S where S is the pole, show that the locus of the foot of the perpendicular from S on PQ is the curve $r^2(e^2 - \sec^2\beta) + 2elr\cos\theta + l^2 = 0$.*

[BH 93; CH 2008]

Let γ and δ be the vectorial angles of P and Q w.r.t. the pole S and the initial line SX as the axis of the conic.

The equation of PQ is

$$\frac{l}{r} + e\cos\theta = \sec\frac{\delta-\gamma}{2}\cos\left(\theta - \frac{\gamma+\delta}{2}\right). \quad (1)$$

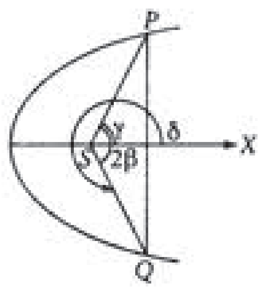


Fig. 69

From Fig. 69,

$$\delta - \gamma + 2\beta = 2\pi \text{ or, } \delta - \gamma = 2(\pi - \beta). \quad (2)$$

The equation of a line perpendicular to (1) is of the form

$$\frac{A}{r} + e\cos\left(\frac{\pi}{2}+\theta\right) = \sec(\pi - \beta)\cos\left(\frac{\pi}{2} + \theta - \frac{\gamma+\delta}{2}\right).$$

If it passes through the focus, then $A = 0$.

Therefore, the equation of the line passing through the point S and perpendicular to (1) is

$$-e\sin\theta = \sec\beta\sin\left(\theta - \frac{\gamma+\delta}{2}\right). \tag{3}$$

From (1),

$$\frac{l}{r} + e\cos\theta = -\sec\beta\cos\left(\theta - \frac{\gamma+\delta}{2}\right). \tag{4}$$

By (3) and (4),

$$\left(\frac{l}{r} + e\cos\theta\right)^2 + e^2\sin^2\theta = \sec^2\beta$$

$$\text{or, } \frac{l^2}{r^2} + \frac{2el}{r}\cos\theta + e^2 = \sec^2\beta$$

$$\text{or, } r^2\left(e^2 - \sec^2\beta\right) + 2elr\cos\theta + l^2 = 0.$$

It is the equation of the required locus.

13. *Find the equation of the circle which passes through the focus of the parabola $r(1+\cos\theta) = 2a$ and touches it at the point $\theta = \alpha$.* **[CH 94; NH 2002]**

Let the equation of the circle be

$$r = 2b\cos(\theta - \gamma). \tag{1}$$

Here $2b$ = diameter, γ = inclination of the diameter through the focus S with the initial line SX. C is the centre of the circle and it touches the parabola at P. The equation of the normal at P to $\frac{2a}{r} = 1 + \cos\theta$ is

$$\frac{2a\sin\alpha}{(1+\cos\alpha)r} = \sin\theta + \sin(\theta - \alpha).$$

It passes through the centre $C(b, \gamma)$.

$$\therefore\ \frac{2a\sin\alpha}{(1+\cos\alpha)b} = \sin\gamma + \sin(\gamma - \alpha). \tag{2}$$

At the common point P,

$$2b\cos(\alpha - \gamma) = \frac{2a}{1+\cos\alpha}$$

$$\text{or, } \frac{2a\sin\alpha}{\sin\gamma + \sin(\gamma-\alpha)} = \frac{a}{\cos(\alpha-\gamma)} \quad [\text{by (2)}]$$

$$\text{or, } 2\sin\alpha\cos(\alpha - \gamma) = \sin\gamma + \sin(\gamma - \alpha)$$

$$\text{or, } \sin(2\alpha - \gamma) + \sin\gamma = \sin\gamma + \sin(\gamma - \alpha)$$

$$\text{or, } \sin(2\alpha - \gamma) = \sin(\gamma - \alpha)$$

$$\text{or, } 2\alpha - \gamma = \gamma - \alpha \text{ or, } \gamma = \frac{3\alpha}{2}. \tag{3}$$

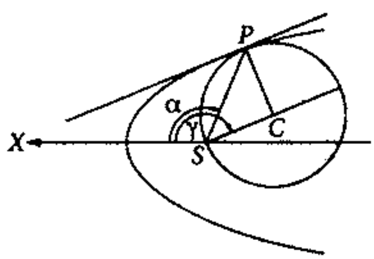


Fig. 70

Again

$$b = \frac{a}{(1+\cos\alpha)\cos(\alpha-\gamma)} = \frac{a}{2\cos^3\frac{\alpha}{2}} \text{ [by (3)]}.$$

$\therefore$ the equation of the circle is

$$r = \frac{a}{\cos^3\frac{\alpha}{2}}\cos\left(\theta - \frac{3\alpha}{2}\right).$$

14. ***Prove that three normals can be drawn to a parabola from a given point.***

If the normals at $\theta_1, \theta_2, \theta_3$ on the parabola $\frac{l}{r} = 1 + \cos\theta$ meet at (ρ, ϕ), prove that $\theta_1 + \theta_2 + \theta_3 = 2\phi$. **[BH 2008]**

The equation of the normal at α to the parabola $\frac{l}{r} = 1+\cos\theta$ is

$$\frac{l}{r}\frac{\sin\alpha}{1+\cos\alpha} = \sin\theta + \sin(\theta-\alpha).$$

If it passes through the point (ρ, ϕ)

$$\frac{l}{\rho}\frac{\sin\alpha}{1+\cos\alpha} = \sin\phi + \sin(\phi-\alpha).$$

From this,

$$\frac{l}{\rho}\tan\frac{\alpha}{2} = 2\sin\left(\phi - \frac{\alpha}{2}\right)\cos\frac{\alpha}{2}$$

$$\text{or, } \frac{l}{\rho}\tan\frac{\alpha}{2} = 2\left(\sin\phi\cos^2\frac{\alpha}{2} - \cos\phi\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right).$$

Multiplying both sides by $\sec^2\frac{\alpha}{2}$,

$$\frac{l}{\rho}\tan\frac{\alpha}{2}\left(1+\tan^2\frac{\alpha}{2}\right) = 2\left(\sin\phi - \cos\phi\tan\frac{\alpha}{2}\right)$$

$$\text{or, } l\tan^3\frac{\alpha}{2} + (l+2\rho\cos\phi)\tan\frac{\alpha}{2} - 2\rho\sin\phi = 0. \tag{1}$$

Putting $\tan\frac{\alpha}{2} = t$, we get

$$lt^3 + (l+2\rho\cos\phi)t - 2\rho\sin\phi = 0. \tag{2}$$

It is a cubic equation in t. Therefore, it has three roots. Corresponding to each of these three roots we have the equation of a normal passing through the point (ρ, ϕ). Hence in general three normals can be drawn to a parabola through a given point.

If t_1, t_2, t_3 are the roots of (2), then $t_1 = \tan\frac{\theta_1}{2}, t_2 = \tan\frac{\theta_2}{2}$ and $t_3 = \tan\frac{\theta_3}{2}$.

Again

$$t_1+t_2+t_3 = 0, t_1t_2+t_2t_3+t_3t_1 = \frac{l+2\rho\cos\phi}{l} \text{ and } t_1t_2t_3 = \frac{2\rho\sin\phi}{l}.$$

We have

$$\tan\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}\right)=\frac{t_1+t_2+t_3-t_1t_2t_3}{1-(t_1t_2+t_2t_3+t_3t_1)}=\frac{-2\rho\sin\phi}{l-(l+2\rho\cos\phi)}=\tan\phi.$$

$$\therefore\ \frac{\theta_1+\theta_2+\theta_3}{2}=n\pi+\phi,$$

where n is any integer.

For $n=0, \theta_1+\theta_2+\theta_3=2\phi$.

15. *If the two conics $\frac{l_1}{r}=1-e_1\cos\theta$ and $\frac{l_2}{r}=1-e_2\cos(\theta-\alpha)$ touch one another, show that*

$$l_1^2\left(1-e_2^2\right)+l_2^2\left(1-e_1^2\right)+2l_1l_2\left(1-e_1e_2\cos\alpha\right).$$

[BH 94; 2004; CH 2001; NH 2008]

Let the conics touch at $\theta=\beta$.

The equation of the tangents at this point are

$$\frac{l_1}{r}=\cos(\theta-\beta)-e_1\cos\theta \text{ or, } (\cos\beta-e_1)\,r\cos\theta+\sin\beta\cdot r\sin\theta=l_1 \tag{1}$$

$$\text{and } \frac{l_2}{r}=\cos(\theta-\beta)-e_2\cos(\theta-\alpha)$$

$$\text{or, } (\cos\beta-e_2\cos\alpha)r\cos\theta+(\sin\beta-e_2\sin\alpha)\,r\sin\theta=l_2. \tag{2}$$

Since (1) and (2) are identical,

$$\frac{l_2}{l_1}=\frac{\cos\beta-e_2\cos\alpha}{\cos\beta-e_1}=\frac{\sin\beta-e_2\sin\alpha}{\sin\beta}.$$

$$\therefore\ (l_1-l_2)\cos\beta=l_1e_2\cos\alpha-l_2e_1$$

$$\text{and } (l_1-l_2)\sin\beta=l_1e_2\sin\alpha.$$

Now squaring and adding,

$$(l_1-l_2)^2=l_1^2e_2^2+l_2^2e_1^2-2l_1l_2e_1e_2\cos\alpha$$

$$\text{or, } l_1^2\left(1-e_2^2\right)+l_2^2\left(1-e_1^2\right)=2l_1l_2\left(1-e_1e_2\cos\alpha\right).$$

16. *If $\theta_1,\theta_2,\theta_3$ be the vectorial angles of A,B,C on the circle $r=2a\cos\theta$, show that the feet of the perpendiculars from the pole on the chords AB, BC, CA lie on the line $r\cos(\theta-\theta_1-\theta_2-\theta_3)=2a\cos\theta_1\cos\theta_2\cos\theta_3$.*

The equation of AB and BC are

$$r\cos(\theta-\theta_1-\theta_2)=2a\cos\theta_1\cos\theta_2 \tag{1}$$

$$\text{and } r\cos(\theta-\theta_2-\theta_3)=2a\cos\theta_2\cos\theta_3. \tag{2}$$

[See the Ex. 4 of Sec 9.30]

The equation of a line perpendicular to (1) is of the form

$$r\cos(\tfrac{\pi}{2}+\theta-\theta_1-\theta_2)=k \text{ or, } r\sin(\theta-\theta_1-\theta_2)=-k.$$

If it passes through the pole, then $k=0$ and consequently

$$\sin(\theta-\theta_1-\theta_2)=0 \text{ or, } \theta=\theta_1+\theta_2.$$

$\therefore$ the foot of the perpendicular is $(2a\cos\theta_1\cos\theta_2, \theta_1+\theta_2)$.

Similarly the foot of the perpendicular on BC is $(2a\cos\theta_2\cos\theta_3, \theta_2+\theta_3)$.

Let the equation of the line through these points be $\rho'=r\cos(\theta-\beta)$.

$$\begin{aligned}
&\therefore\ \cos\theta_1\cos(\theta_1+\theta_2-\beta)=\cos\theta_3\cos(\theta_2+\theta_3-\beta)\\
&\text{or, } \cos(2\theta_1+\theta_2-\beta)+\cos(\theta_2-\beta)=\cos(\theta_2+2\theta_3-\beta)+\cos(\theta_2-\beta)\\
&\text{or, } \cos(2\theta_1+\theta_2-\beta)=\cos(\theta_2+2\theta_3-\beta)\\
&\text{or, } 2\theta_1+\theta_2-\beta=-(\theta_2+2\theta_3-\beta)\ [\because\ \theta_1\neq\theta_3]\\
&\text{or, } \beta=\theta_1+\theta_2+\theta_3.
\end{aligned}$$

$\therefore\ \rho'=2a\cos\theta_1\cos\theta_2\cos(\theta_1+\theta_2+\theta_3-\theta_1-\theta_2)=2a\cos\theta_1\cos\theta_2\cos\theta_3.$

Thus the equation of this line is

$$r\cos(\theta-\theta_1-\theta_2-\theta_3)=2a\cos\theta_1\cos\theta_2\cos\theta_3.$$

The symmetry of the result suggests that the above line passes through the foot of the perpendicular from the pole on CA.

EXERCISE XV

1. Transform the following equations into polar form: (a) $x^2+y^2=9$ (b) $x^2+y^2-ax-by=0$ (c) $xy=c^2$ (d) $y^3+yx^2-2ax^2=0$ (d) $(x^2+y^2)^2=a^2(x^2-y^2)$.

2. Transform the following equations into Cartesian form: (a) $r=a$ (b) $\theta=\pi/3$ (c) $\frac{1}{r}=1-\cos\theta$ (d) $r=\sec\theta\tan^2\theta$ (e) $r=2a\cos(\theta-\alpha)$ (f)$r=a\sin 3\theta$ (g) $r=a-b\cos\theta$ (h) $r(\cos 3\theta+\sin 3\theta)=2a\sin\theta\cos\theta$.

3. Show that the points

$$(a\cos\theta, a\sin\theta),\ \left\{a\cos\left(\theta+\frac{2\pi}{3}\right), a\sin\left(\theta+\frac{2\pi}{3}\right)\right\}$$
$$\text{and } \left\{a\cos\left(\theta-\frac{2\pi}{3}\right), a\sin\left(\theta-\frac{2\pi}{3}\right)\right\}$$

form an equilateral triangle whose circumcentre is the pole.

4. (a) Find the equation of the line joining the points $(3,\pi/6)$ and $(4,\pi/3)$.
   (b) Show that (i) the lines $r\cos(\theta-\alpha)=a$ and $r\cos(\theta-\alpha)=b$ are parallel and (ii) the lines $r\cos(\theta-\alpha)=a$ and $r\sin(\theta-\alpha)=b$ are perpendicular.
   (c) Show that the polar equations of the bisectors of the angles between the lines $\theta=\alpha$ and $\theta=\beta$ are $\theta=\frac{1}{2}(\alpha+\beta)$ and $\theta=\frac{\pi}{2}+\frac{1}{2}(\alpha+\beta)$.

(d) Find the polar coordinates of the foot of the perpendicular from the pole on the line joining the points (r_1, θ_1) and (r_2, θ_2).

(e) Show that the perpendicular distance of the point (r_1, θ_1) from $r\cos(\theta - \alpha) = p$ is $r_1 \cos(\theta_1 - \alpha) - p$.

(f) Show that the four lines $r(\cos\theta + \sin\theta) = \pm 1$ and $r(\cos\theta - \sin\theta) = \pm 1$ form a square whose side is of length $\sqrt{2}$.

(g) Show that the condition for the concurrence of the lines $r\cos(\theta - \alpha) = a$, $r\cos(\theta - \beta) = b$ and $r\cos(\theta - \gamma) = c$ is

$$\begin{vmatrix} \cos\alpha & \sin\alpha & a \\ \cos\beta & \sin\beta & b \\ \cos\gamma & \sin\gamma & c \end{vmatrix} = 0.$$

(h) Show that the lines $r\cos(\theta - \alpha) = p$ and $r\cos(\theta - \beta) = p$ intersect at $\left\{p\sec\frac{1}{2}(\alpha - \beta), \frac{1}{2}(\alpha + \beta)\right\}$.

5. Find the equations of the circles under the following conditions. (a) centre $(4, 0)$, radius 4 (b) centre $\left(5, \frac{\pi}{2}\right)$, radius 5 (c) centre $\left(5, \frac{\pi}{3}\right)$, radius 2 (d) centre $(4, 60°)$, passing through $(7, 45°)$ (e) centre on the line $\theta = \frac{\pi}{3}$, passing through $(0, 0)$ and $\left(5, \frac{\pi}{2}\right)$ (f) extremities of the diameter are $\left(2, \frac{2\pi}{3}\right)$ and $\left(-2, \frac{\pi}{3}\right)$.

6. Find the centre and radius of the following. (a) $r = 3\sin\theta + 3\sqrt{3}\cos\theta$ (b) $r^2 - 5\sqrt{2}r(\cos\theta + \sin\theta) + 16 = 0$.

7. Show that the equation $r^2\cos\theta - ar\cos 2\theta - 2a^2\cos\theta = 0$ represents a straight line and a circle. **[NH 2005]**

8. Show that the line $\frac{l}{r} = A\cos\theta + B\sin\theta$ touches the circle $r = 2a\cos\theta$, if $a^2B^2 + 2Aal = l^2$. **[CH 95; 2006]**

[*Hints.* The equation of the line can be written as

$$r\cos(\theta - \alpha) = \frac{l}{\sqrt{A^2 + B^2}} \text{ where } \tan\alpha = \frac{B}{A}.$$

The centre of the circle is $(a, 0)$ and its radius is a. If the line touches the circle, then its distance from $(a, 0)$ is a.

$$\therefore a\cos\alpha - \frac{l}{\sqrt{A^2 + B^2}} = a \text{ or, } Aa - l = a\sqrt{A^2 + B^2}$$
$$\text{or, } (Aa - l)^2 = a^2\left(A^2 + B^2\right) \text{ or, } a^2B^2 + 2Aal = l^2.]$$

9. Find the points on

(a) $\frac{8}{r} = 3 - \sqrt{2}\cos\theta$ whose radius vector is 4;

(b) $\frac{5}{r} = 1 + 2\cos\theta$ whose radius vector is 5;

(c) $\frac{l}{r} = 1 - \cos\theta$ which has the smallest radius vector.

10. Find the equation of the conic with focus at the pole and having the given e, l or directrix (a) $e = 1, l = 4$ (b) $e = \frac{\sqrt{3}}{2}, l = 6$ (c)$e = 2$, directrix $r\cos\theta = -5$ (d) $e = 1$, directrix $r\cos\left(\theta - \frac{\pi}{4}\right) = 2\sqrt{2}$.

    [If $r\cos\theta = -p$ is the directrix, the conic is $r = ep/(1 - e\cos\theta)$.]

11. Find the eccentricity, latusrectum and the directrix of the following conics. (a) $\frac{3}{r} = 4 - 2\cos\theta$ (b) $r = \frac{6}{1-\cos\theta}$ (c) $r = \frac{9}{2+\cos\theta}$ (d) $\frac{4}{r} = 3 - 5\cos(\theta - \pi/4)$ (e) $\frac{6}{r} = 1 - \sin\theta$ (f) $\frac{3}{r} = 1 + 2\sin\theta$.

12. (a) Find the equation of $\frac{x^2}{25} + \frac{y^2}{16} = 1$ in the polar form with the right-hand focus as the pole and the positive direction of x-axis as the positive direction of the initial line.

    (b) Find the equation of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ in the polar form with the left-hand focus as the pole and the positive direction of the x-axis as the positive direction of the initial line.

    (c) Find the equation of $y^2 = 8x$ in the polar form with the vertex as the pole and the positive direction of the x-axis as the positive direction of the initial line.

13. (a) The latusrectum of a conic is 6 and its eccentricity is $\frac{1}{2}$. Find the length of the focal chord making an angle 45° with the major axis.

    (b) Show that the length of the focal chord of the conic $\frac{l}{r} = 1 - e\cos\theta$ which is inclined to the initial line at an angle α is $2l/\left(1 - e^2\cos^2\alpha\right)$. **[NH 2006]**

14. Find the points of intersection of the parabolas $\frac{1}{3r} = 1 - \cos\theta$ and $\frac{1}{r} = 1 + \cos\theta$.

15. If PSP' and QSQ' are two perpendicular focal chords of a conic, prove that $\frac{1}{PS \cdot SP'} + \frac{1}{QS \cdot SQ'} =$ a constant. **[BH 93; 95]**

16. If r_1 and r_2 are two mutually perpendicular radius vectors of the ellipse $r^2 = \frac{b^2}{1-e^2\cos^2\theta}$, prove that $\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

    [*Hints.* The vectorial angles of the points corresponding to r_1 and r_2 are α and $90° + \alpha$.]

17. Show that the perpendicular focal chords of a rectangular hyperbola are equal. $\left[e = \sqrt{2}\right]$.

18. Show that $r = a\cos(\theta - \alpha)$ and $r = b\sin(\theta - \alpha)$ represent the equations of orthogonal circles.

19. A circle passes through the focus S of a conic and meets it in four points whose distances from S are r_1, r_2, r_3 and r_4. Prove that (a) $r_1 r_2 r_3 r_4 = d^2 l^2/e^2$, where $2l$ and e are the latusrectum and eccentricity of the conic and d is the diameter of the circle and (b) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}$. **[CH 2007, 10; BH 92, 2007, 08]**

    [*Hints.* Let the equation of the conic be

$$\frac{l}{r} = 1 - e\cos\theta \tag{1}$$

and that of the circle be

$$r = 2a\cos(\theta - \alpha). \tag{2}$$

From (2),

$$(r - 2a\cos\alpha\cos\theta)^2 = 4a^2\sin^2\alpha\left(1 - \cos^2\theta\right). \tag{3}$$

From (1),

$$\cos\theta = \frac{1}{e}\left(1 - \frac{l}{r}\right).$$

Putting it in (3), we have

$$e^2r^4 - 4ae\cos\alpha\, r^3 + (4a^2 + 4ael\cos\alpha - 4a^2e^2\sin^2\alpha)r^2 - 8a^2lr + 4a^2l^2 = 0.$$

r_1, r_2, r_3, r_4 are the roots of this equation in r.

$$\therefore r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = \frac{8a^2l}{e^2} \tag{4}$$

$$\text{and } r_1r_2r_3r_4 = \frac{4a^2l^2}{e^2}. \tag{5}$$

Here $2a = d$.

$$\therefore\ r_1r_2r_3r_4 = \frac{d^2l^2}{e^2}.$$

Dividing (4) by (5),

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}.]$$

20. If d is the diameter of the circle passing through the pole and the points (r_1, θ_1) and (r_2, θ_2), show that $d^2\sin^2(\theta_1 - \theta_2) = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$.

21. The tangents at two points P and Q of the parabola $\frac{l}{r} = 1 + \cos\theta$ meet at T. Show that $SP \cdot SQ = ST^2$ where S is the focus. **[BH 2009; CH 94, 03, 2005]**

    [*Hints.* If α and β are the vectorial angles of P and Q, then the vectorial angle of T is $\frac{1}{2}(\alpha + \beta)$.]

22. (a) Prove that the line $\frac{l}{r} = A\cos\theta + B\sin\theta$ will touch the conic $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$, if $A^2 + B^2 - 2e(A\cos\gamma + B\sin\gamma) + e^2 - 1 = 0$.

    (b) Prove that the line $r\cos(\theta - \alpha) = a$ may touch the conic $\frac{l}{r} = 1 - e\cos\theta$, if $l^2 + 2ael\cos\alpha + a^2(e^2 - 1) = 0$. **[BH 2000]**

    [*Hints.* To find the point of intersection between the line and the conic.

    $$l\cos(\theta - \alpha) = a(1 - e\cos\theta) \text{ or, } (l\cos\alpha + ae)\cos\theta + l\sin\alpha\sin\theta - a = 0$$

    $$\text{or, } (l\cos\theta + ae)\left(1 - \tan^2\frac{\theta}{2}\right) + l\sin\alpha \cdot 2\tan\frac{\theta}{2} - a\left(1 + \tan^2\frac{\theta}{2}\right) = 0$$

    $$\text{or, } (l\cos\alpha + ae + a)\tan^2\frac{\theta}{2} - 2l\sin\alpha\tan\frac{\theta}{2} - (l\cos\alpha + ae - a) = 0.$$

    If the line touches the conic, the above equation in $\tan\frac{\theta}{2}$ must have equal roots. By the condition of equal roots the result is obtained.]

23. If g is a variable tangent of the conic $\frac{l}{r} = 1 - e\cos\theta$, show that the locus of the foot of the perpendicular from the pole on g is the circle $r^2\left(e^2-1\right) + 2elr\cos\theta + l^2 = 0$. [It is known as auxiliary circle.]

[NH 2004; CH 2007]

[*Hints.* The equation of the tangent at α is

$$\frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha). \tag{1}$$

The line perpendicular to (1) and passing through the pole is

$$e\sin\theta - \sin(\theta - \alpha) = 0. \tag{2}$$

Eliminating α between (1) and (2)

$$(l/r + e\cos\theta)^2 + e^2\sin^2\theta = 1$$
$$\text{or,}\quad r^2\left(e^2-1\right) + 2ler\cos\theta + l^2 = 0.]$$

24. Show that the equation of the tangent to the conic $\frac{l}{r} = 1 + e\cos\theta$, parallel to the tangent at $\theta = \alpha$, is given by

$$l\left(e^2 + 2e\cos\alpha + 1\right) = r\left(e^2-1\right)\{\cos(\theta-\alpha) + e\cos\theta\}.$$ **[CH 2004]**

[*Hints.* The tangent at α is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \tag{1}$$

and the tangent at β is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \beta). \tag{2}$$

If (1) and (2) are parallel, then

$$\frac{e+\cos\alpha}{e+\cos\beta} = \frac{\sin\alpha}{\sin\beta} = \frac{1}{k}\ \text{(say)}.$$

$$\therefore\ \sin\beta = k\sin\alpha \text{ and } \cos\beta = k(e+\cos\alpha) - e.$$

Squaring and adding,

$$k^2\left(e^2 + 1 + 2e\cos\alpha\right) - 2ke(e+\cos\alpha) + e^2 - 1 = 0$$
$$\text{or, } (k-1)\left\{k\left(e^2 + 2e\cos\alpha + 1\right) - \left(e^2-1\right)\right\} = 0.$$
$$\because k \neq 1,\ k = \frac{e^2-1}{e^2+2e\cos\alpha+1}.$$

Eliminating β from (2)

$$\frac{l}{r} = \frac{e^2-1}{e^2+2e\cos\alpha+1}\{e\cos\theta + \cos(\theta-\alpha)\}$$
$$\text{or, } l\left(e^2+2e\cos\alpha+1\right) = r\left(e^2-1\right)\{e\cos\theta + \cos(\theta-\alpha)\}.]$$

25. If P, Q are variable points on a conic $\frac{l}{r} = 1 - e\cos\theta$ with vectorial angles α and β, where $\alpha - \beta = 2\gamma =$ constant, show that the chord PQ touches the conic $\frac{l\cos\gamma}{r} = 1 - e\cos\gamma\cos\theta$ and that this conic has the same directrix as the original one.

[*Hints.* The equation of the chord PQ is

$$\frac{l}{r} = -e\cos\theta + \sec\frac{\alpha-\beta}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right)$$
$$\text{or, } \frac{l}{r} = -e\cos\theta + \sec\gamma\cos\left(\theta - \frac{\alpha+\beta}{2}\right)$$
$$\text{or, } \frac{l\cos\gamma}{r} = -e\cos\gamma\cos\theta + \cos\left(\theta - \frac{\alpha+\beta}{2}\right).$$

Obviously it is the tangent to the conic $\frac{l\cos\gamma}{r} = 1 - e\cos\gamma\cos\theta$ at $\frac{\alpha+\beta}{2}$.

The directrix of it is $\frac{l\cos\gamma}{r} = -e\cos\gamma\cos\theta$ or, $\frac{l}{r} = -e\cos\theta$. It is also the directrix of the given conic.]

26. If the normals at the points of vectorial angles $\alpha, \beta, \gamma, \delta$ on the conic $\frac{l}{r} = 1 + e\cos\theta$ meet at a point, prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} + \left(\frac{1+e}{1-e}\right)^2 = 0.$$ **[BH 2007]**

[*Hints.* The equation of the normal at α is

$$\frac{l}{r}\cdot\frac{e\sin\alpha}{1+e\cos\alpha} = e\sin\theta + \sin(\theta - \alpha).$$

Let the normals meet at (ρ, ϕ).

$$\therefore\ \frac{l}{\rho}\frac{e\sin\alpha}{1+e\cos\alpha} = e\sin\phi + \sin(\phi - \alpha).$$

Putting

$$\sin\alpha = \frac{2t}{1+t^2},\quad \cos\alpha = \frac{1-t^2}{1+t^2}$$

where $t = \tan\frac{\alpha}{2}$, we have

$$\rho(1-e)^2\sin\phi t^4 + 2\{le + (1-e)\rho\cos\phi\}t^3$$
$$+ 2\{le + (1+e)\rho\cos\phi\}t - \rho(1+e)^2\sin\phi = 0.$$

The roots of this equation in t are $\tan\frac{\alpha}{2}, \tan\beta/2, \tan\gamma/2, \tan\delta/2$.

$$\therefore\ \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} + \left(\frac{1+e}{1-e}\right)^2 = 0.]$$

27. Show that the locus of the middle points of the system of focal chords of a conic is a similar conic. **[BH 2002, 08]**

[*Hints.* Let PSP' be a focal chord of the conic $\frac{l}{r} = 1 - e\cos\theta$. If α is the vectorial angle of P, then that of P' is $\pi + \alpha$.

Now $\frac{l}{SP} = 1 - e\cos\alpha$ and $\frac{l}{SP'} = 1 + e\cos\alpha$. If (r_1, α) are the polar coordinates of the middle point of PP', then $2r_1 = SP - SP'$.

$$\therefore\ 2r_1 = \frac{l}{1 - e\cos\alpha} - \frac{l}{1 + e\cos\alpha} = \frac{2le\cos\alpha}{1 - e^2\cos^2\alpha}.$$

$\therefore$ the required locus is $r\left(1 - e^2\cos^2\theta\right) = le\cos\theta$.

Multiplying both sides by r, we have

$$r^2\left(1 - e^2\cos^2\theta\right) = ler\cos\theta$$
$$\text{or, } x^2 + y^2 - e^2x^2 - lex = 0$$
$$\text{or, } x^2(1 - e^2) + y^2 - lex = 0.$$

It represents an ellipse, a parabola or a hyperbola according as $1 - e^2 >=< 0$ or, $e <=> 1$ which is also the condition for the original conic to represent an ellipse, a parabola or a hyperbola.]

28. Three points on an ellipse are such that their focal radii are in H.P. and their vectorial angles are in A.P. Show that one focal radius is semi-latusrectum and the other two are euqally inclined to that semi-latusrectum.

[*Hints.* Let $\frac{l}{r} = 1 - e\cos\theta$ be the equation of the ellipse and (r_1, θ_1), (r_2, θ_2), (r_3, θ_3) be three points satisfying the given conditions.

Then

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{2}{r_2} \text{ and } \frac{\theta_1 + \theta_3}{2} = \theta_2.$$

Again

$$\frac{l}{r_1} = 1 - e\cos\theta_1,\ \frac{l}{r_2} = 1 - e\cos\theta_2,\ \frac{l}{r_3} = 1 - e\cos\theta_3.$$

$$\therefore\ \frac{l}{r_1} + \frac{l}{r_3} = 2 - e\left(\cos\theta_1 + \cos\theta_3\right) = 2 - 2e\cos\frac{\theta_1 + \theta_3}{2}\cos\frac{\theta_1 - \theta_3}{2}$$
$$\text{or, } \frac{2l}{r_2} = 2 - 2e\cos\theta_2\cos\frac{\theta_1 - \theta_3}{2}$$
$$\text{or, } 1 - e\cos\theta_2 = 1 - e\cos\theta_2\cos\frac{\theta_1 - \theta_3}{2}$$
$$\text{or, } \cos\theta_2\left(1 - \cos\frac{\theta_1 - \theta_3}{2}\right) = 0.$$

It gives that $\theta_2 = \frac{\pi}{2}$ as $\theta_1 \neq \theta_3$.

For $\theta_2 = \frac{\pi}{2}, r_2 = l =$ semi-latusrectum.

$\theta_1 + \theta_3 = 2\theta_2 = \pi$. $\therefore\ \theta_3 - \frac{\pi}{2} = \frac{\pi}{2} - \theta_1$.

It shows that the focal radii corresponding to vectorial angles θ_1 and θ_3 are equally inclined to the semi-latusrectum.]

29. P, Q, R are three points on the parabola $\frac{l}{r} = 1 + \cos\theta$ with vectorial angles α, β, γ respectively. Show that the equation of the circle circumscribing the triangle formed by the tangents at these points to the parabola is

$$r = \frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}\cos\left(\theta - \frac{\alpha+\beta+\gamma}{2}\right).$$

[*Hints.* The tangents at P and Q are

$$\frac{l}{r} = \cos\theta + \cos(\theta - \alpha) \tag{1}$$

$$\text{and } \frac{l}{r} = \cos\theta + \cos(\theta - \beta). \tag{2}$$

To find the point of intersection between (1) and (2) $\cos(\theta-\alpha) = \cos(\theta-\beta)$. It gives that $\theta = \frac{\alpha+\beta}{2}$. The radius vector of this point is

$$\frac{l}{\cos\frac{\alpha+\beta}{2} + \cos\left(\frac{\alpha+\beta}{2} - \alpha\right)} = \frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}.$$

Thus the tangents at P, Q, R meet at

$$\left(\frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}, \frac{\alpha+\beta}{2}\right), \left(\frac{l}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}, \frac{\beta+\gamma}{2}\right), \left(\frac{l}{2}\sec\frac{\gamma}{2}\sec\frac{\alpha}{2}, \frac{\gamma+\alpha}{2}\right).$$

The given equation is satisfied by these points. Hence the result follows.]

30. Find the equation of the circle which passes through the focus of the conic $\frac{l}{r} = 1 - e\cos\theta$ and touches it at the point $\theta = \alpha$. **[NH 2003]**

[*Hints.* Let the circle be $r = 2b\cos(\theta - \gamma)$.

The vectorial angle of the common point of the conic and the circle is α.

$$\therefore\ 2b\cos(\alpha-\gamma) = \frac{l}{1 - e\cos\alpha}. \tag{1}$$

The normal to the conic at α passes through the centre (b, γ).

$$\therefore\ \frac{el\sin\alpha}{b(1 - e\cos\alpha)} = e\sin\gamma - \sin(\gamma - \alpha). \tag{2}$$

By (1) and (2),

$$\tan(\alpha - \gamma) = \frac{e\sin\alpha}{1 - e\cos\alpha}. \tag{3}$$

The equation of the circle can be written as

$$\begin{aligned} r &= 2b\{\cos(\theta-\alpha)\cos(\alpha-\gamma) - \sin(\theta-\alpha)\sin(\alpha-\gamma)\} \\ &= \frac{l}{(1 - e\cos\alpha)\cos(\alpha-\gamma)}\{\cos(\theta-\alpha)\cos(\alpha-\gamma) - \sin(\theta-\alpha)\sin(\alpha-\gamma)\} \\ &= \frac{l}{1 - e\cos\alpha}\left\{\cos(\theta-\alpha) - \sin(\theta-\alpha)\frac{e\sin\alpha}{1 - e\cos\alpha}\right\}. \end{aligned}$$

$\therefore$ the equation is $r(1 - e\cos\alpha)^2 = l\cos(\theta-\alpha) - el\cos(\theta - 2\alpha)$.

31. If the conics $\frac{l}{r} = 1 + e\cos\theta$ and $\frac{l'}{r} = 1 + e\cos(\theta - \gamma)$ touch at $\theta = \alpha$, show that

$$l' = \frac{l(1-e^2)}{e^2+2e\cos\alpha+1}.$$ [CH 99]

[*Hints.* Tangents at α are

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$$

$$\text{or, } l = (e + \cos\alpha)r\cos\theta + \sin\alpha \cdot r\sin\theta \tag{1}$$

$$\text{and } \frac{l'}{r} = e\cos(\theta - \gamma) + \cos(\theta - \alpha)$$

$$\text{or, } l' = (e\cos\gamma + \cos\alpha)r\cos\theta + (e\sin\gamma + \sin\alpha)r\sin\theta. \tag{2}$$

Comparing,

$$\frac{l}{l'} = \frac{e + \cos\alpha}{e\cos\gamma + \cos\alpha} = \frac{\sin\alpha}{e\sin\gamma + \sin\alpha}. \tag{3}$$

From the first and second $el\cos\gamma = (l' - l)\cos\alpha + el'$.

From the first and third $el\sin\gamma = (l' - l)\sin\alpha$.

Squaring and adding, $e^2l^2 = (l' - l)^2 + 2el'(l' - l)\cos\alpha + e^2l'^2$.

From this the result is obtained.]

32. A conic is described having the same focus as the conic $\frac{l}{r} = 1 + e\cos\theta$ and they touch at $\theta = \alpha$. Prove that the least value of the eccentricity of such a conic is

$$\frac{e\sin\alpha}{\sqrt{(1 + 2e\cos\alpha + e^2)}}.$$

[*Hints.* Let the conic be $\frac{l'}{r} = 1 + e'\cos(\theta - \gamma)$.

Tangents to the conics at $\theta = \alpha$ are

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \tag{1}$$

$$\text{and } \frac{l'}{r} = e'\cos(\theta - \gamma) + \cos(\theta - \alpha). \tag{2}$$

Comparing (1) with (2),

$$\frac{e + \cos\alpha}{e'\cos\gamma + \cos\alpha} = \frac{\sin\alpha}{e'\sin\gamma + \sin\alpha}$$

$$\text{or, } (e + \cos\alpha)(e'\sin\gamma + \sin\alpha) = \sin\alpha(e'\cos\gamma + \cos\alpha)$$

$$\text{or, } e' = \frac{-e\sin\alpha}{e\sin\gamma + \sin(\gamma - \alpha)}. \tag{3}$$

Now

$$\frac{de'}{d\gamma} = \frac{e\sin\alpha\{e\cos\gamma + \cos(\gamma - \alpha)\}}{\{e\sin\gamma + \sin(\gamma - \alpha)\}^2}.$$

For the least value of e'

$$e\cos\gamma + \cos(\gamma - \alpha) = 0. \tag{4}$$

From (3),

$$e\sin\gamma + \sin(\gamma - \alpha) = -\frac{e\sin\alpha}{e'}. \tag{5}$$

Squaring and adding, the result is obtained.]

ANSWERS

1. (a) $r = 3$ (b) $r = a\cos\theta + b\sin\theta$ (c) $r^2\sin 2\theta = 2c^2$
(d) $r = 2a\cos\theta\cot\theta$ (e) $r^2 = a^2\cos 2\theta$.

2. (a) $x^2 + y^2 = a^2$ (b) $y = \sqrt{3}x$ (c) $y^2 = 1 + 2x$ (d) $y^2 = x^3$
(e) $x^2 + y^2 = 2a(x\cos\alpha + y\sin\alpha)$ (f) $\left(x^2 + y^2\right)^2 = a\left(3x^2y - y^3\right)$
(g) $\left(x^2 + y^2 + bx\right)^2 = a^2(x^2 + y^2)$ (h) $x^3 - y^3 + 3xy(x - y) = 2axy$.

4. (a) $\frac{12}{r} = \left(4\sqrt{3} - 3\right)\cos\theta - (4 - 3\sqrt{3})\sin\theta$
(d) (p, α) where $p = \frac{r_1 r_2 \sin(\theta_2 - \theta_1)}{AB}, \alpha = \tan^{-1}\frac{r_2\cos\theta_2 - r_1\cos\theta_1}{r_1\sin\theta_1 - r_2\sin\theta_2}$.

5. (a) $r = 8\cos\theta$ (b) $r = 10\sin\theta$ (c) $r^2 - 10r\cos\left(\theta - \frac{\pi}{3}\right) + 21 = 0$
(d) $r^2 - 8r\cos\left(\theta - \frac{\pi}{3}\right) - 49 + 14\left(\sqrt{2} + \sqrt{6}\right) = 0$
(e) $r = \frac{10}{\sqrt{3}}\cos\left(\theta - \frac{\pi}{3}\right)$ (f) $r^2 + 2r\cos\theta - 2 = 0$.

6. (a) $\left(3, \frac{\pi}{6}\right), 3$ (b) $\left(5, \frac{\pi}{4}\right), 3$ 7. $r = 2a\cos\theta, r\cos\theta + a = 0$.

9. (a) $\left(4, \pm\frac{\pi}{4}\right)$ (b) $\left(5, \pm\frac{\pi}{2}\right)$ (c) $\left(\frac{l}{2}, \pi\right)$.

10. (a) $\frac{4}{r} = 1 \pm \cos\theta$ (b) $\frac{12}{r} = 2 \pm \sqrt{3}\cos\theta$ (c) $\frac{10}{r} = 1 - 2\cos\theta$
(d) $r = \frac{-4}{\sqrt{2} - (\cos\theta + \sin\theta)}$.

11. (a) $e = \frac{1}{2}, 2l = \frac{3}{2}, r\cos\theta = -\frac{3}{2}$ (b) $e = 1, l = 6, r\cos\theta = -6$
(c) $e = \frac{1}{2}, l = \frac{9}{2}, r\cos\theta = 9$ (d) $e = \frac{5}{3}, l = \frac{4}{3}, r\cos\left(\theta - \frac{\pi}{4}\right) = -\frac{4}{5}$
(e) $e = 1, l = 6, r\sin\theta = -6$ (f) $e = 2, l = 3, r\sin\theta = \frac{3}{2}$.

12. (a) $\frac{16}{r} = 5 + 3\cos\theta$ (b) $\frac{9}{r} = 4 + 5\cos\theta$ (c) $r\sin^2\theta = 8\cos\theta$

13. (a) $6\frac{6}{7}$. 14. $\left(\frac{2}{3}, \pm\frac{\pi}{3}\right)$.

Part II

Analytical Geometry of Three Dimensions

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Chapter 1

Rectangular Cartesian Coordinates

1.10 Introduction

A point can be defined by the plane geometry, if it lies in the plane of the reference to two axes. If the point does not lie on this plane, then it is not possible to define the point by the plane geometry. Moreover, the bodies, in general, are of three-dimensions, length, breadth and height. Thus the geometry, which can define a point in space or a body of three-dimensions, is known as the geometry of three-dimensions or solid geometry.

1.11 Origin, Axes and Coordinate Planes, Coordinates of a Point, Octants

Let $X'OX, Y'OY$ and $Z'OZ$ be three mutually perpendicular lines in space, which intersects at O. O is called the origin and the lines are called rectangular coordinate axes, namely x, y and z-axes respectively. OX, OY and OZ whose directions are right-handed are taken as positive ($+ve$) directions of the axes. The planes XOY, YOZ and ZOX are known as xy, yz and zx coordinate planes respectively.

It is obvious that

$$z = 0 \text{ for all points in the } xy\text{-plane,}$$
$$y = 0 \text{ for all points in the } zx\text{-plane,}$$
$$\text{and} \quad x = 0 \text{ for all points in the } yz\text{-plane.}$$

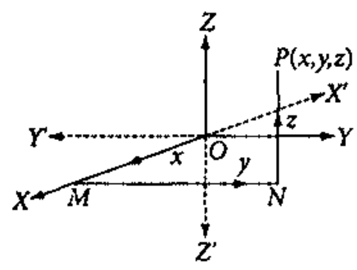

Fig. 1

Let P be a point in space, PN is drawn perpendicular to XOY-plane. NM is perpendicular to OX or, MN is parallel to OY. If $OM = x, MN = y$ and $NP = z$, then (x, y, z) are the coordinates of P. These coordinates are positive or negative as they are measured in the positive directions of axes or negative directions of axes.

The coordinate planes divide space into eight parts called octants and the signs of coordinates of a point lying in the octants are given below:

Octant	$OXYZ$	$OX'YZ$	$OXY'Z$	$OXYZ'$	$OX'Y'Z$	$OXY'Z'$	$OX'YZ'$	$OX'Y'Z'$
x	+	−	+	+	−	+	−	−
y	+	+	−	+	−	−	+	−
z	+	+	+	−	+	−	−	−

1.20 Geometric Vector

Let P and Q be two points in space. The line-segment going from P to Q is called a *directed line segment* or a *geometric vector* or simply a *vector* and it is designated as $\overline{PQ}$. The point P is the *base* or the *initial point* and the point Q is the *head* or the *terminal point*. The base is also known as the point of application. The *modulus* or *magnitude* of the vector $\overline{PQ}$ is the positive number, which gives the measure of its length and is denoted by $|\overline{PQ}|$. $\overline{PQ}$ and $\overline{QP}$ have the same extremities, the same length but opposite directions.

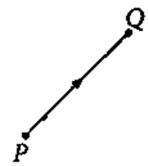

Fig. 2

Null Vector. A vector whose base and head coincide is called a *null-vector* or *zero-vector*. It is denoted by $\bar{0}$.

Unit Vector. A vector whose modulus is unity is called a *unit vector*.

Equal Vectors. Two vectors are said to be equal, if they have the same magnitude and the same direction. For this the point of application of a vector may be arbitrary and the geometric vectors are said to be *free vectors*.

Projection of a vector on an axis. Let l be an axis (a directed line) and $\overline{PQ}$ be a given vector. Through the points P and Q two planes α and β are drawn perpendicular to l. These planes intersect l at A and B. A and B are the projections of P and Q on l. The value of the directed segment $\overline{AB}$ is called the projection of $\overline{PQ}$ on the axis l. If θ is the angle between $\overline{PQ}$ and the axis l, then

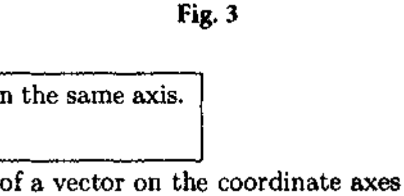

Fig. 3

$$|\overline{AB}| = |\overline{PQ}| \cos\theta.$$

Note 1. Equal vectors have equal projections on the same axis.

Note 2. PA and QB are perpendicular to l.

Coordinates of a vector. The projections of a vector on the coordinate axes are called the coordinates of this vector w.r.t. the coordinate system. If X, Y, Z are the projections of $\overline{PQ}$ on the x-axis, y-axis and z-axis respectively, then it is written as $\overline{PQ} = (X, Y, Z)$.

Note. The projections of the point (x, y, z) on the xy, yz and zx-planes are $(x, y, 0)$, $(0, y, z)$ and $(x, 0, z)$ respectively. Its projections on the x, y and z-axes are $(x, 0, 0)$, $(0, y, 0)$, $(0, 0, z)$ respectively.

Theorem 1.

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points, then the vector $\overline{PQ} = (x_2 - x_1,\, y_2 - y_1,\, z_2 - z_1)$.

Proof. Let $\overline{PQ} = (X, Y, Z)$.

Let the planes through P and Q perpendicular to the x-axis intersect it at P' and Q' respectively. Obviously the x-coordinates of P' and Q' are x_1 and x_2. Therefore, $P'Q' = x_2 - x_1$. $P'Q'$ is the projection of $\overline{PQ}$ on the x-axis.

$\therefore$ $P'Q' = X$. Consequently $X = x_2 - x_1$.

Similarly $Y = y_2 - y_1$ and $Z = z_2 - z_1$.

Thus $\overline{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

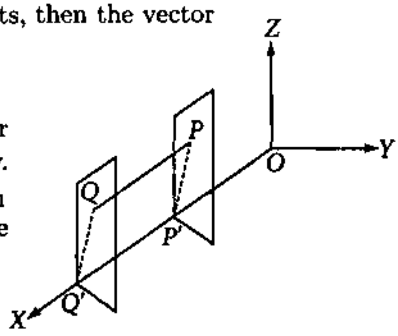

Fig. 4

Theorem 2.

If $P, Q, R, \ldots, T, U$ are any n points in space, the sum of projections of $\overline{PQ}$, $\overline{QR}$, $\ldots$, $\overline{TU}$ on an axis AB is equal to the projection of $\overline{PU}$ on AB.

Let $P', Q', R', \ldots, T', U'$ be the feet of the perpendiculars from $P, Q, R, \ldots, T, U$ on AB respectively.

Now $P'Q' + Q'R' + \ldots + T'U' = P'U'$.

$\therefore$ proj. of $\overline{PQ}$+ proj. of $\overline{QR} + \ldots +$ proj. of $\overline{TU}$ on AB = proj. of $\overline{PU}$ on AB.

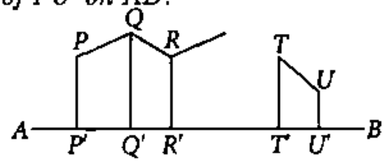

Fig. 5

1.30 Change of Origin

Let OX, OY, OZ be the three rectangular axes and (α, β, γ) be the coordinates of a point O' w.r.t. these axes. $O'X', O'Y'$ and $O'Z'$ are drawn parallel to OX, OY, OZ respectively. O' is the new origin and $O'X', O'Y'$ and $O'Z'$ are the new axes.

Let (x, y, z) be the coordinates of the point P w.r.t. OX, OY and OZ and (x', y', z') be those of this point w.r.t. new axes. PN is perpendicular to XOY-plane. It meets the $X'O'Y'$-plane at N'. NP is parallel to OZ and $O'Z'$. NN' is the distance between two parallel planes XOY and $X'O'Y'$. $O'M$ is perpendicular to XOY-plane.

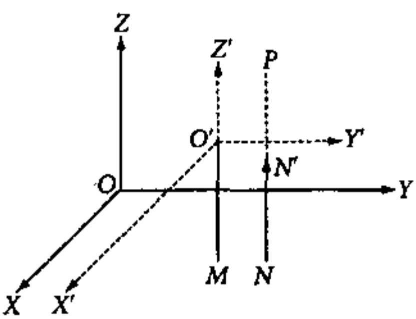

Fig. 6

Now $NN' = MO' = \gamma$.

Again $z = NP = NN' + N'P = \gamma + z'$ or, $z = z' + \gamma$.

Similarly it can be shown that $x = x' + \alpha, y = y' + \beta$.

Note. Coordinates of P w.r.t. new axes are $(x - \alpha, y - \beta, z - \gamma)$.

1.31 Distance of a point from the origin

Let P be a point whose coordinates are (x, y, z) w.r.t. three rectangular axes OX, OY and OZ.

In Fig. 7, PN is perpendicular to XOY-plane and MN is parallel to OY.

$\therefore \quad OM = x, MN = y, NP = z$. ON and OP are joined.

From the right-angled triangle OMN,

$$ON^2 = OM^2 + MN^2 = x^2 + y^2.$$

From the right-angled triangle ONP,

$$OP^2 = ON^2 + NP^2 = x^2 + y^2 + z^2.$$

$$\therefore\ OP = \sqrt{x^2 + y^2 + z^2}.$$

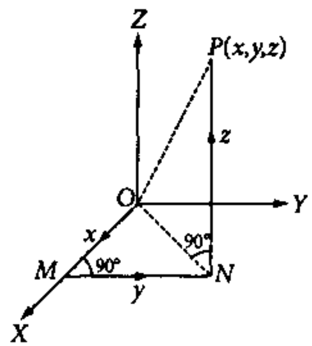

Fig. 7

Note. $\overline{OP} = (x, y, z)$.
$\therefore\ |\overline{OP}| = \sqrt{x^2 + y^2 + z^2}$.

1.32 Distance between Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the two points in the frame of reference of rectangular axes OX, OY and OZ.

Let the origin be transferred to P with the PX', PY' and PZ' parallel to OX, OY and OZ respectively.

Now the coordinates of Q w.r.t. PX', PY', PZ' are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

$\therefore \quad$ the distance of Q from P

$$= \sqrt{\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right\}}.$$

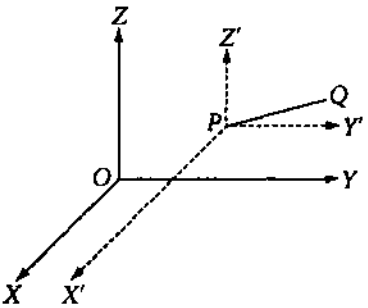

Fig. 8

Since the distance between two points is invariant,

$$PQ = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$$

with reference to OX, OY and OZ.

Example 1. *Find the distance between the points* $(4, 3, -6)$ *and* $(-2, 1, -3)$.

$$\begin{aligned} Distance &= \sqrt{\left\{(-2 - 4)^2 + (1 - 3)^2 + (-3 + 6)^2\right\}} \\ &= \sqrt{(36 + 4 + 9)} = \sqrt{49} = 7. \end{aligned}$$

1.33 To find the Coordinates of the Point which divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in a given ratio $m : n$

Let $R(x, y, z)$ divide the distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$. $(PR : RQ = m : n)$.

PL, QM and RN are drawn perpendicular to XOY-plane. The lines PL, QM and RN lie in the same plane, so the points L, M, N lie on a straight line in the XOY-plane. $PN'M'$ is drawn parallel to LMN. It is perpendicular to RN and QM.

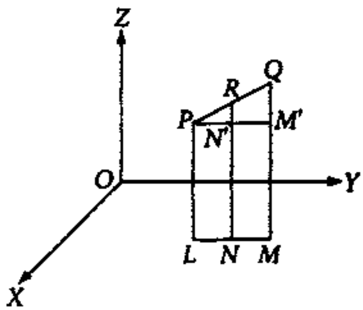

Fig. 9

Here

$$
\begin{aligned}
LP &= MM' = NN' = z_1, \\
N'R &= NR - NN' = z - z_1, \\
M'Q &= MQ - MM' = z_2 - z_1.
\end{aligned}
$$

$\Delta PN'R$ and $\Delta PM'Q$ are similar.

$$\therefore \quad \frac{PR}{PQ} = \frac{N'R}{M'Q} \quad \text{or,} \quad \frac{m}{m+n} = \frac{z - z_1}{z_2 - z_1} \quad \text{or,} \quad z = \frac{mz_2 + nz_1}{m+n}.$$

Similarly, drawing perpendiculars to YOZ and ZOX-planes from P, Q and R, it can be shown that

$$x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2 + ny_1}{m+n}.$$

$\therefore$ the coordinates of R are

$$\left(\frac{mx_2 + nx_1}{m+n}, \quad \frac{my_2 + ny_1}{m+n}, \quad \frac{mz_2 + nz_1}{m+n}\right).$$

Corollary 1. Midpoint: The coordinates of the middle point are

$$\left(\frac{x_1 + x_2}{2}, \quad \frac{y_1 + y_2}{2}, \quad \frac{z_1 + z_2}{2}\right).$$

Corollary 2. If R divides PQ externally in the ratio $m : n$, then the coordinates of R are

$$\left(\frac{mx_2 - nx_1}{m-n}, \quad \frac{my_2 - ny_1}{m-n}, \quad \frac{mz_2 - nz_1}{m-n}\right).$$

Corollary 3. If $\frac{m}{n} = \lambda$ i.e. R divides PQ in the ratio $\lambda : 1$, then

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}, \quad y = \frac{y_1 + \lambda y_2}{1+\lambda}, \quad z = \frac{z_1 + \lambda z_2}{1+\lambda}.$$

Example 2. *Find the coordinates of the points, which divide the distance between two points* $(2, 0, 1)$ *and* $(4, -2, 5)$ *internally and externally in the ratio* $3 : 2$.

The coordinates of the internal point are

$$\left\{\frac{3\cdot 4+2\cdot 2}{3+2}, \frac{3(-2)+2\cdot 0}{3+2}, \frac{3\cdot 5+2\cdot 1}{3+2}\right\}, \text{ i.e. } \left\{\frac{16}{5}, \frac{-6}{5}, \frac{17}{5}\right\}.$$

The coordinates of the external point are

$$\left\{\frac{3\cdot 4-2\cdot 2}{3-2}, \frac{3\cdot(-2)-2\cdot 0}{3-2}, \frac{3\cdot 5-2\cdot 1}{3-2}\right\}, \text{ i.e. } (8,-6,13).$$

WORKED-OUT EXAMPLES

1. *Find the possible octants where the point (x, y, z) may lie for each of the following conditions:*

   (a) $x - y = 0$; (b) $y + z = 0$.

   (a) In this case, x and y have the same sign. The same sign of these two coordinates occur in the octants $OXYZ$, $OXYZ'$, $OX'Y'Z$ and $OX'Y'Z'$. Hence the point may lie in any one of the above four octants.

   (b) Here y and z have opposite signs. This happens in the octants $OXY'Z$, $OXYZ'$, $OX'Y'Z$ and $OX'YZ'$. Therefore, the point may lie in any one of the above octants.

2. *Show that the points* $(0,7,10), (-1,6,6), (-4,9,6)$ *form an isosceles right-angled triangle.*

   Let the points be denoted by A, B, C respectively.

   Now

$$AB = \sqrt{(-1)^2 + (6-7)^2 + (6-10)^2} = \sqrt{1+1+16} = \sqrt{18},$$
$$BC = \sqrt{(-4+1)^2 + (9-6)^2 + (6-6)^2} = \sqrt{9+9+0} = \sqrt{18},$$
$$CA = \sqrt{(4)^2 + (7-9)^2 + (10-6)^2} = \sqrt{16+4+16} = \sqrt{36}.$$

   $\because$ $AB = BC$, the triangle is isosceles.

   Again $AB^2 + BC^2 = 18 + 18 = 36 = CA^2$.

   $\therefore$ the triangle is right-angled.

3. *A, B, C are the points on the x, y and z-axes respectively. The distances of A, B, C from the origin O are a, b, c. Find the coordinates of the point, which is equidistant from O, A, B and C.*

   Let $P(x, y, z)$ be the point, which is equidistant from O, A, B and C. The coordinates of O, A, B, and C are $(0,0,0), (a,0,0), (0,b,0)$ and $(0,0,c)$ respectively.

   Since $OP = PA, x^2 + y^2 + z^2 = (x-a)^2 + y^2 + z^2$ or, $0 = a^2 - 2ax$ or, $x = \frac{a}{2}$.

   Similarly from $OP = PB, y = \frac{b}{2}$ and from $OP = PC, z = \frac{c}{2}$.

   $\therefore$ the required point is $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

4. *Find the point where the line joining* $(5, -2, 3)$ *and* $(3, 0, 1)$ *pierces the xy-plane.*

   Let the point divide the join of $(5, -2, 3)$ and $(3, 0, 1)$ in the ratio $\lambda : 1$. On the XOY-plane, z-coordinate is zero.

$$\therefore \quad 0 = \frac{\lambda \cdot 1 + 1 \cdot 3}{\lambda + 1} \quad \text{or,} \quad \lambda + 3 = 0 \quad \text{or,} \quad \lambda = -3.$$

   Thus the x-coordinate of the point $= \frac{-3 \cdot 3 + 1 \cdot 5}{-3+1} = \frac{4}{2} = 2$

   and the y-coordinate of the point $\frac{-3 \cdot 0 + 1 \cdot (-2)}{-3+1} = \frac{2}{2} = 1$.

   Hence the required point is $(2, 1, 0)$.

5. *Find the coordinates of the centroid of the triangle whose vertices are* (x_1, y_1, z_1), (x_2, y_2, z_2) *and* (x_3, y_3, z_3).

   Let $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) be the coordinates of A, B, C respectively of ΔABC and AD be a median of it.

   Since D is the middle point of BC, the coordinates of D are

$$\left\{\frac{x_2 + x_3}{2}, \quad \frac{y_2 + y_3}{2}, \quad \frac{z_2 + z_3}{2}\right\}.$$

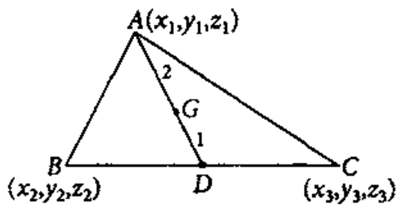

Fig. 10

   Let G be the centroid. G divides AD in the ratio $2 : 1$.

   $\therefore$ the coordinates of G are

$$\left(\frac{2 \cdot \frac{x_2+x_3}{2} + 1 \cdot x_1}{2+1}, \quad \frac{2 \cdot \frac{y_2+y_3}{2} + 1 \cdot y_1}{2+1}, \quad \frac{2 \cdot \frac{z_2+z_3}{2} + 1 \cdot z_1}{2+1}\right),$$

$$\text{i.e.} \quad \left\{\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right\}.$$

6. *Show that the lines joining the midpoints of the opposite edges of a tetrahedron and lines joining the vertices to centroids of the opposite faces of the tetrahedron are concurrent.*

   Let $ABCD$ be a tetrahedron and (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) be the coordinates of A, B, C, D respectively. If E and F are the midpoints of the opposite edges AB and CD, then the coordinates of E and F are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

$$\text{and} \quad \left(\frac{x_3 + x_4}{2}, \frac{y_3 + y_4}{2}, \frac{z_3 + z_4}{2}\right)$$

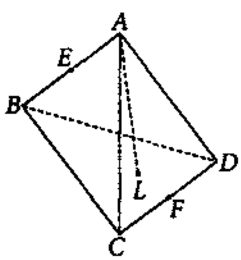

Fig. 11

   respectively.

If G be the midpoint of EF, the coordinates of G are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \quad \frac{y_1+y_2+y_3+y_4}{4}, \quad \frac{z_1+z_2+z_3+z_4}{4}\right).$$

The symmetry of the coordinates of G suggests that the other two lines joining the midpoints of the opposite edges will pass through G.

Let L be the centroid of ΔBCD. Then the coordinates of L are

$$\left(\frac{x_2+x_3+x_4}{3}, \quad \frac{y_2+y_3+y_4}{3}, \quad \frac{z_2+z_3+z_4}{3}\right).$$

Let G' be a point on AL, which divides AL in the ratio $3:1$. The coordinates of G' are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \quad \frac{y_1+y_2+y_3+y_4}{4}, \quad \frac{z_1+z_2+z_3+z_4}{4}\right).$$

$\therefore$ G' coincides with G. Again the symmetry of the coordinates of G' suggests that the other three lines joining the vertices to the centroids of the opposite faces must pass through G'. Hence these seven lines are concurrent.

Note. G is the c.g. of the tetrahedron.

7. *Find the coordinates of a vector whose magnitude is 4 and which makes angles* $60°, 120°, 45°$ *with the* x, y *and* z*-axes respectively.*

Let X, Y, Z be the projections of the vector on the x, y and z-axes respectively.

Then

$$X = 4\cos 60° = 4\cdot\frac{1}{2} = 2,$$
$$Y = 4\cos 120° = 4\cdot\left(-\frac{1}{2}\right) = -2,$$
$$Z = 4\cos 45° = 4\cdot\frac{1}{\sqrt{2}} = 2\sqrt{2}.$$

Hence the coordinates of the vector are $(2, -2, 2\sqrt{2})$.

8. *From the point* $(1, -2, 3)$ *lines are drawn to meet the sphere* $x^2+y^2+z^2=4$ *and they are divided in the ratio* $2:3$. *Prove that the points of section lie on the sphere* $5x^2+5y^2+5z^2-6x+12y-18z+22=0$.

Let the line through $(1, -2, 3)$ meet the sphere at (x_1, y_1, z_1).

$\therefore$ $x_1^2+y_1^2+z_1^2=4.$

If (α, β, γ) divides $(1, -2, 3)$ and (x_1, y_1, z_1) in the ratio $2 : 3$, then

$$\alpha = \frac{2x_1 + 3}{5}, \quad \beta = \frac{2y_1 - 6}{5}, \quad \gamma = \frac{2z_1 + 9}{5}$$

$$\text{or,} \quad x_1 = \frac{5\alpha - 3}{2}, \quad y_1 = \frac{5\beta + 6}{2}, \quad z_1 = \frac{5\gamma - 9}{2}.$$

$$\therefore \quad \left(\frac{5\alpha - 3}{2}\right)^2 + \left(\frac{5\beta + 6}{2}\right)^2 + \left(\frac{5\gamma - 9}{2}\right)^2 = 4$$

$$\text{or,} \quad 5\alpha^2 + 5\beta^2 + 5\gamma^2 - 6\alpha + 12\beta - 18\gamma + 22 = 0.$$

Hence (α, β, γ) lies on the sphere

$$5x^2 + 5y^2 + 5z^2 - 6x + 12y - 18z + 22 = 0.$$

EXERCISE I

1. Find the coordinates of the projections of the points $(2, 1, 3), (-1, 3, 5), (-3, 0, 6)$ and $(0, 0, 4)$ on the coordinate planes and on the coordinate axes.

2. Find the possible octants where the point (x, y, z) may lie for each of the following conditions.

   (a) $x + y = 0$; (b) $z - x = 0$; (c) $z - y = 0$.

3. Find the octants, which contain the points whose coordinates (x, y, z) satisfy each of the following conditions.

   (a) $xy > 0$, (b) $xyz > 0$.

4. (a) If P and Q are points $(3, -2, 5)$ and $(1, 0, -7)$, find the coordinates of $\overline{PQ}$ and $\overline{QP}$.

   (b) If a vector of modulus 6 makes angles $120°, 45°$ and $60°$ with the x, y and z-axes respectively, find the coordinates of it.

5. Find the distance between the following points:

   (a) $(4, -1, 5), (1, 2, -2)$; (b) $(0, 7, 10), (5, 4, -7)$.

6. Find the centre of the sphere passing through $(0, 0, 0), (2, 0, 0), (0, 3, 0)$ and $(0, 0, 4)$.

7. Show that the points $(1, 1, 1), (-2, 4, 1), (-1, 5, 5,)$ and $(2, 2, 5)$ form a square.

8. Show that the following points are collinear.

   (a) $(3, -2, 4), (1, 1, 1), (-1, 4, -2)$.

   [*Hints.* Show that $(1, 1, 1)$ divides the join of the other two in a certain ratio.]

   (b) $(2, 5, -4), (1, 4, -3), (4, 7, -6)$.

9. (a) Find the coordinates of the point, which divides the segment joining the points $(1, 3, -9)$ and $(6, -2, -4)$ in the ratio $2 : 3$.

(b) Find the coordinates of the points which divide the segment joining the points $A(2,7,1)$ and $B(8,-2,5)$ into three equal parts.

10. (a) Find the ratio in which the line joining $(4,6,7)$ and $(-1,2,5)$ is divided by the yz-plane. Find also the coordinates of the point on the yz-plane.

    (b) A point P lies on the line passing through the points $(2,-3,1)$ and $(3,4,-5)$. If the x-coordinates of P is 5, find the other coordinates of it.

11. The coordinates of A, B, C, D are $(1,1,1), (-1,3,-3), (3,-1,2)$ and $(-3,5,-4)$ respectively. Show that the lines AB and CD intersect and find the point of intersection.

    [*Hints.* Let the coordinates of a point P on AB be

$$\left(\frac{-\lambda+1}{\lambda+1}, \quad \frac{3\lambda+1}{\lambda+1}, \quad \frac{-3\lambda+1}{\lambda+1}\right)$$

and those of a point Q on CD be

$$\left(\frac{-3\lambda'+3}{\lambda'+1}, \quad \frac{5\lambda'-1}{\lambda'+1}, \quad \frac{-4\lambda'+2}{\lambda'+1}\right).$$

The lines will intersect if there exist values of λ and λ' such that

$$\frac{-\lambda+1}{\lambda+1} = \frac{-3\lambda'+3}{\lambda'+1}, \quad \frac{3\lambda+1}{\lambda+1} = \frac{5\lambda'-1}{\lambda'+1}, \quad \frac{-3\lambda+1}{\lambda+1} = \frac{-4\lambda'+2}{\lambda'+1}.$$

From these relations it is found that $\lambda = \lambda' = 1$ and the point of intersection is $(0,2,-1)$.]

12. Find the point where the line joining $(2,-3,1)$ and $(3,-4,-5)$ cuts the plane $2x+y+z=7$.

13. Find the locus of the point, which is equidistant from $(0,2,3)$ and $(2,0,4)$,

14. Find the locus of a point whose distance from the y-axis is always 3.

15. Find the ratio in which the sphere $x^2+y^2+z^2=504$ divides the line joining the points $(12,-4,8)$ and $(27,-9,18)$.

16. Show that the points where the line joining the points $(3,0,-2)$ and $(6,6,7)$ cuts the surface represented by $x^2+y^2+z^2-4x+4y+10z-23=0$ are $(4,2,1)$ and $(0,-6,-11)$.

ANSWERS

1. Coordinates on the xy-plane $(2,1,0)$, $(-1,3,0)$, $(-3,0,0)$, $(0,0,0)$; Coordinates on the yz-plane $(0,1,3)$, $(0,3,5)$, $(0,0,6)$, $(0,0,4)$; Coordinates on the zx-plane $(2,0,3)$, $(-1,0,5)$, $(-3,0,6)$, $(0,0,4)$; Coordinates on the x-axis $(2,0,0)$, $(-1,0,0)$, $(-3,0,0)$, $(0,0,0)$; Coordinates on the y-axis $(0,1,0)$, $(0,3,0)$, $(0,0,0)$, $(0,0,0)$; Coordinates on the z-axis $(0,0,3)$, $(0,0,5)$, $(0,0,6)$, $(0,0,4)$.

2. (a) $OX'YZ$, $OXY'Z$, $OXY'Z'$, $OX'YZ'$,
   (b) $OXYZ$, $OXY'Z$, $OX'YZ'$, $OX'Y'Z'$,
   (c) $OXYZ$, $OX'YZ$, $OXY'Z'$, $OX'Y'Z'$.

3. (a) $OXYZ$, $OXYZ'$, $OX'Y'Z'$, $OX'Y'Z'$,
   (b) $OXYZ$, $OXY'Z'$, $OX'Y'Z$, $OX'YZ'$.

4. (a) $\overline{PQ} = (-2, 2, -12)$, $\overline{QP} = (2, -2, 12)$;
   (b) $\left(-3, 3\sqrt{2}, 3\right)$.

5. (a) $\sqrt{67}$,
   (b) $\sqrt{323}$.

6. (a) $\left(1, \frac{3}{2}, 2\right)$.

9. (a) $(3, 1, -7)$,
   (b) $\left(4, 4, \frac{7}{3}\right)$, $\left(6, 1, \frac{11}{3}\right)$.

10. (a) $4 : 1$, $\left(0, \frac{14}{5}, \frac{27}{5}\right)$,
    (b) $(5, 18, -17)$,

11. $(0, 2, -1)$.

12. $(1, -2, 7)$.

13. $4x - 4y + 2z - 7 = 0$.

14. $z^2 + x^2 = 9$.

15. $2 : 3, -2 : 3$.

Chapter 2

Direction Cosines and Angle Between Two Lines

2.10 Angle between Two Non-Coplanar Lines

We have a knowledge about the angle between two lines lying on the same plane. If the lines are non-coplaner, then we define that the angle between them is equal to the angle between the two coplaner straight lines parallel to the non-coplanar straight lines.

2.11 Direction Cosines of a Line

Definition. *If a directed line makes angles α, β, γ with the positive directions of x, y and z-axes respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines (d.cs.) of this line. These numbers are generally denoted by l, m, n.*

In Fig. 12, $\overrightarrow{OP}$ makes angles α, β, γ with the coordinate axes. If (x, y, z) are the coordinates of P and we draw PM perpendicular to x-axis, then we have

$$\cos\alpha = \frac{OM}{|\overline{OP}|} = \frac{x}{|\overline{OP}|}.$$

[$\because$ the coordinates of M are $(x, 0, 0)$].

Similarly

$$\cos\beta = \frac{y}{|\overline{OP}|}, \quad \cos\gamma = \frac{z}{|\overline{OP}|}.$$

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2+y^2+z^2}{OP^2} = \frac{OP^2}{OP^2} = 1,$$

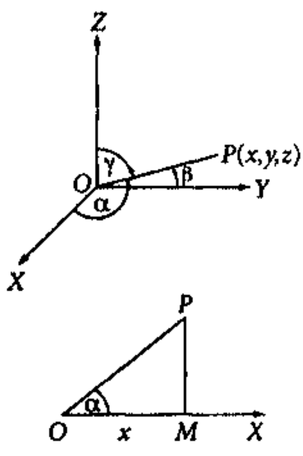

Fig. 12

i.e.

$$l^2 + m^2 + n^2 = 1. \tag{1}$$

Again
$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \tag{2}$$
x, y, z are proportional to the direction cosines. *Any three numbers, which are proportional to the direction cosines of a line are called the direction ratios of the line.*

Note 1. Any three numbers cannot be d.cs. of a line. They will be d.cs. only when the sum of the squares of them is equal to 1.

Note 2. If l, m, n are the d.cs. of $\overline{OP}$, then $-l, -m, -n$ are the d.cs. of $\overline{PO}$.

Corollary 1.

$$\text{From (2), } \frac{x}{l} = \frac{y}{m} = \frac{z}{n} = \pm\sqrt{x^2 + y^2 + z^2}.$$

$\therefore$ the d.cs. of $\overline{OP}$ are
$$\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$
and those of $\overline{PO}$ are
$$-\frac{x}{\sqrt{x^2 + y^2 + z^2}}, -\frac{y}{\sqrt{x^2 + y^2 + z^2}}, -\frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

Thus the d.cs. can be found out when d.rs. (direction ratios) are given.

Corollary 2.
$$\begin{aligned}\sin^2\alpha + \sin^2\beta + \sin^2\gamma &= (1 - \cos^2\alpha) + (1 - \cos^2\beta) + (1 - \cos^2\gamma)\\ &= 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3 - 1 = 2.\end{aligned}$$

2.12 Direction Cosines of a Line Joining Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the two points in the frame of references OX, OY and OZ. Let the origin O be transferred to P with the axes remaining parallel to the original axes.

The coordinates of Q w.r.t. the new set of axes are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. The d.cs. of $\overline{PQ}$ w.r.t. these new axes are
$$\frac{x_2 - x_1}{|\overline{PQ}|}, \quad \frac{y_2 - y_1}{|\overline{PQ}|}, \quad \frac{z_2 - z_1}{|\overline{PQ}|}.$$

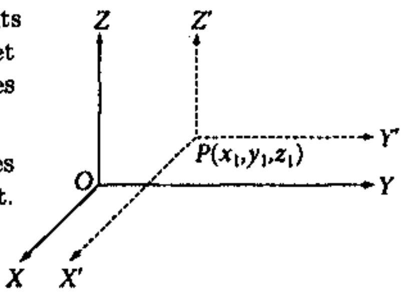

Fig. 13

Since the old set of axes are parallel to the new set of axes, the angles made by $\overline{PQ}$ with the new set of axes are equal to those made by it with OX, OY and OZ. Hence the d.cs. of $\overline{PQ}$ w.r.t. OX, OY and OZ are
$$\frac{x_2 - x_1}{|\overline{PQ}|}, \quad \frac{y_2 - y_1}{|\overline{PQ}|}, \quad \frac{z_2 - z_1}{|\overline{PQ}|}$$
where $|\overline{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Example 1. *Find the d.cs. of the line joining the points* $(2,3,4)$ *and* $(-1,4,6)$.

Distance between the points $= \sqrt{(-1-2)^2+(4-3)^2+(6-4)^2} = \sqrt{14}$.
The d.cs. are

$$\frac{-3}{\sqrt{14}}, \quad \frac{1}{\sqrt{14}}, \quad \frac{2}{\sqrt{14}}.$$

2.20 If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the two given points, then the projection of $\overline{PQ}$ on a line AB whose d.cs. are l, m, n is $l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$.

If the origin is transferred to P with the axes remaining parallel, the coordinates of Q w.r.t. this new set of axes are

$$(x_2-x_1, y_2-y_1, z_2-z_1).$$

In Fig. 14 QN is perpendicular to $X'PY'$-plane and MN is parallel to PY'

$\therefore\ PM = x_2-x_1, MN = y_2-y_1$ and $NQ = z_2-z_1$.

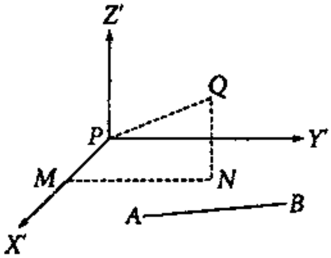

Fig. 14

Now proj. of $\overline{PQ}$ on AB = proj. of PM on AB+ proj. of MN on AB+ proj. of NQ on AB

$$= l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1).$$

[$\because$ cosines of the angles between AB and PX', PY', PZ' are l, m, n respectively.]

2.21 Angle between Two Lines

Let the d.cs. of two lines OA and OB be (l_1, m_1, n_1) and (l_2, m_2, n_2). If O is not the origin, the origin is shifted to O with the axes remaining parallel. For this transformation the d.cs. of the lines will not be changed.

Let us choose $OA = OB = 1$. Then the coordinates of A and B are (l_1, m_1, n_1) and (l_2, m_2, n_2).

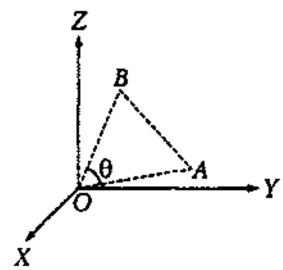

Fig. 15

Method I. If θ be the angle between OA and OB, then from ΔOAB,

$$AB^2 = OA^2 + OB^2 - 2\cdot OA\cdot OB\cdot\cos\theta$$

or, $(l_2-l_1)^2+(m_2-m_1)^2+(n_2-n_1)^2 = 1+1-2\cos\theta$

or, $(l_1^2+m_1^2+n_1^2)+(l_2^2+m_2^2+n_2^2)-2(l_1l_2+m_1m_2+n_1n_2) = 2-2\cos\theta$

or, $1+1-2(l_1l_2+m_1m_2+n_1n_2) = 2-2\cos\theta$

or, $\cos\theta = l_1l_2+m_1m_2+n_1n_2$

or, $\theta = \cos^{-1}(l_1l_2+m_1m_2+n_1n_2)$.

Method II. Considering projection of OB on OA we have

$$OB\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

[$\because$ $(0,0,0)$ and (l_2, m_2, n_2) are the coordinates of O and B and the d.cs. of OA are l_1, m_1, n_1]

$$\text{or,}\quad \cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$
$$\text{or,}\quad \theta = \cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2).$$

Corollary 1. *Angle with direction ratios:*

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are the d.rs. of two lines, then the angle between them is

(i) $\cos^{-1}\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$ or

(ii) $\tan^{-1}\dfrac{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}{a_1a_2 + b_1b_2 + c_1c_2}$.

Corollary 2. *Condition for perpendicularity:*

If the lines are at right angle, then $\theta = 90°$. $\therefore$ $\cos\theta = 0$.

Hence the condition is $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

From Corollary 1. $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Corollary 3. *Condition for parallel lines:*

$$\begin{aligned}\sin^2\theta = 1 - \cos^2\theta &= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2 \\ &= (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2.\end{aligned}$$

[Lagrange's identity]

If the lines are parallel, then $\theta = 0$. $\therefore$ $\sin\theta = 0$.

Hence $(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_1l_1)^2 = 0$.

Since the sum of three squares is zero, each of them is zero.

Thus

$$l_1m_2 - l_2m_1 = 0, m_1n_2 - m_2n_1 = 0, n_1l_2 - n_2l_1 = 0,$$
$$\text{or,}\quad \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{\sqrt{l_1^2 + m_1^2 + n_1^2}}{\sqrt{l_2^2 + m_2^2 + n_2^2}} = 1.$$
$$\therefore\quad l_1 = l_2, m_1 = m_2, n_1 = n_2. \quad \text{[d.cs. are same.]}$$

For direction ratios

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}. \quad \text{[d.rs. are proportional.]}$$

WORKED-OUT EXAMPLES

1. *Can numbers* $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ *be the d.cs. of any directed line?*

Since

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} \neq 1,$$

the given numbers cannot be the d.cs. of a line.

2. *Find the d.cs. of the line, which is equally inclined to the axes.* **[BH 2006]**

Let the line make an angle θ with each of the axes. Therefore, the d.cs. l, m, n are equal and $l = m = n = \cos\theta$.

$$\because \quad l^2 + m^2 + n^2 = 1, 3l^2 = 1 \quad \text{or,} \quad l = \pm\frac{1}{\sqrt{3}}.$$

Hence the d.cs. are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ whence the acute angles made by the line are considered.

3. *A directed line makes angles* $60°$ *and* $45°$ *with the axes of* x *and* y *respectively. What angle does it make with the axis of* z*?*

Here x-d.c. $= \cos 60° = \frac{1}{2}$, y-d.c. $= \cos 45° = \frac{1}{\sqrt{2}}$.

If $n = z$-d.c., then $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + n^2 = 1$ or, $n^2 = 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$.

$\therefore \quad n = \pm\frac{1}{2}$.

Thus the acute angle made by the line with the z-axis is $60°$.

4. *If* l_1, m_1, n_1 *and* l_2, m_2, n_2 *are d.cs. of two mutually perpendicular lines, show that the line whose d. r.s. are* $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$ *is perpendicular to both of them. Find also the d.cs. of this line.*

Let l, m, n be the d.cs. of the line, which is perpendicular to both of the given lines.

Then $ll_1 + mm_1 + nn_1 = 0$ and $ll_2 + mm_2 + nn_2 = 0$.

By cross-multiplication,

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1} = \pm\frac{1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}.$$

If θ be the angle between the given lines, then $\sin^2\theta = \Sigma(m_1n_2 - m_2n_1)^2$.

Here $\theta = 90°$. $\quad 1 = \Sigma(m_1n_2 - m_2n_1)^2$.

$$\therefore \quad \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1} = \pm 1.$$

Hence $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1$ and $l_1m_2 - l_2m_1$ are the d.rs. of the line. The d.cs. are

$$\pm(m_1n_2 - m_2n_1), \pm(n_1l_2 - n_2l_1), \pm(l_1m_2 - l_2m_1).$$

5. *Prove that the acute angle between the lines whose d.cs. are given by the relations $l+m+n=0$ and $l^2+m^2-n^2=0$ is $\pi/3$.*

$$l+m+n=0, \tag{1}$$
$$l^2+m^2-n^2=0. \tag{2}$$

From (1), $n=-(l+m)$.

Putting in (2), $l^2+m^2-(l+m)^2=0$ or, $lm=0$.

Either $l=0$ or, $m=0$.

Taking $l=0, m+n=0$ and $m^2-n^2=0$.

Again

$$l^2+m^2+n^2=1. \tag{3}$$

By (2) and (3), $2n^2=1$ or, $n^2=\frac{1}{2}$ or, $n=\pm\frac{1}{\sqrt{2}}$.

But $m=-n$. $\therefore$ $m=\mp\frac{1}{\sqrt{2}}$.

Considering only one sign, the d.cs. of one line are $0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$.

Taking $m=0$, the d.cs. of the other line are $\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$.

If θ be the angle between the lines

$$\cos\theta = 0\cdot\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cdot 0+\left(-\frac{1}{\sqrt{2}}\right)\cdot\left(-\frac{1}{\sqrt{2}}\right)=\frac{1}{2}. \quad \therefore\ \theta=\frac{\pi}{3}.$$

6. *Show that the straight lines whose d.cs. are given by $al+bm+cn=0$, $fmn+gnl+hlm=0$ are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$ and parallel if $\sqrt{af}\pm\sqrt{bg}\pm\sqrt{ch}=0$.* **[BH 2006; CH 2006, 07]**

$$al+bm+cn=0, \tag{1}$$
$$fmn+gnl+hlm=0. \tag{2}$$

From (1), $l=-\frac{bm+cn}{a}$.

Putting in (2),

$$fmn+(gn+hm)\left(-\frac{bm+cn}{a}\right)=0$$
$$\text{or,}\quad bhm^2+(ch+bg-af)mn+cgn^2=0$$
$$\text{or,}\quad bh\left(\frac{m}{n}\right)^2+(ch+bg-af)\frac{m}{n}+cg=0. \tag{3}$$

If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.cs. of the lines, then $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ are the roots of the equation (3). If the lines are parallel, then $\frac{m_1}{n_1}=\frac{m_2}{n_2}$.

From the condition of equal roots,

$$
\begin{aligned}
& (ch+bg-af)^2 = 4\cdot bh\cdot cg \\
\text{or,}\quad & ch+bg-af = \pm 2\sqrt{bcgh} \\
\text{or,}\quad & af = (\sqrt{ch})^2 + (\sqrt{bg})^2 \pm 2\sqrt{bg.ch} = (\sqrt{ch}\pm\sqrt{bg})^2. \\
\therefore\quad & \sqrt{af} = \pm(\sqrt{ch}\pm\sqrt{bg}) \quad \text{or,} \quad \sqrt{af}\pm\sqrt{bg}\pm\sqrt{ch} = 0.
\end{aligned}
$$

Hence the condition of parallelism is followed.

From (3),

$$\frac{m_1}{n_1}\cdot\frac{m_2}{n_2} = \frac{cg}{bh} \quad \text{or,} \quad \frac{m_1 m_2}{cg} = \frac{n_1 n_2}{bh} \quad \text{or,} \quad \frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c}.$$

Similarly eliminating n from (1) and (2), we get

$$\frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b}.$$

If the lines are perpendicular to each other, then

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0. \quad \therefore \frac{f}{a}+\frac{g}{b}+\frac{h}{c} = 0.$$

7. *If three concurrent lines with d.cs. $l_1, m_1, n_1; l_2, m_2, n_2$ and l_3, m_3, n_3 are coplanar, prove that*

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

[BU 80; CU 89, 92, 2000]

Since the lines are coplanar, they are perpendicular to the normal to the plane of them. If l, m, n be the d.cs. of the normal to the plane, then

$$ll_1 + mm_1 + nn_1 = 0, \tag{1}$$
$$ll_2 + mm_2 + nn_2 = 0, \tag{2}$$
$$ll_3 + mm_3 + nn_3 = 0. \tag{3}$$

Eliminating l, m, n from (1), (2) and (3) we have

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

8. *If a line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, prove that* $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

Let O, A, B, C, D, E, F, G be the corners of a cube whose length of each edge is unity. Let the three mutually perpendicular edges OA, OC and OE be take as x, y and z-axes respectively.

Now the coordinates of the corners are as the following:

$$O(0,0,0),\ A(1,0,0),\ B(1,1,0),\ C(0,1,0),$$
$$D(0,1,1),\ E(0,0,1),\ F(1,0,1),\ G(1,1,1).$$

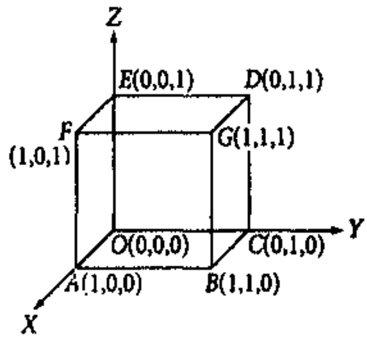

Fig. 16

The diagonals of the cube are OG, AD, BE and CF.

The d.cs. of OG are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

The d.cs. of AD are $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

The d.cs. of BE are $-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

The d.cs. of CF are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Let l, m, n be the d.cs. of the line which makes angles $\alpha, \beta, \gamma, \delta$ with OG, AD, BE and CF respectively.

Then

$$\cos\alpha = \frac{1}{\sqrt{3}}(l+m+n), \quad \cos\beta = \frac{1}{\sqrt{3}}(-l+m+n),$$
$$\cos\gamma = \frac{1}{\sqrt{3}}(-l-m+n), \quad \cos\delta = \frac{1}{\sqrt{3}}(l-m+n).$$

$\therefore\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (-l-m+n)^2 + (l-m+n)^2]$$
$$= \frac{4}{3}(l^2+m^2+n^2) = \frac{4}{3}.$$

9. *If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.cs. of two concurrent lines, show that $l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2$ are the d.rs. of the bisectors of the angles between the lines.*

Let AOA' and BOB' be the given lines.
Let us take O as origin and

$$|OA| = |OA'| = |OB| = 1.$$

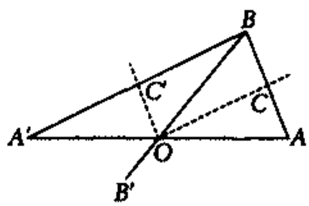

Fig. 17

The d.cs. of OA, OB and OA' are l_1, m, n_1; l_2, m_2, n_2 and $-l_1$, $-m_1$, $-n_1$.

$\therefore$ the coordinates of A, B and A' are (l_1, m_1, n_1) (l_2, m_2, n_2) and $(-l_1, -m_1, -n_1)$ respectively.

Let OC and OC' be the bisectors of the angles $\angle AOB$ and $\angle BOA'$. According to constructions, C and C' are the midpoints of AB and BA'.

$\therefore$ the coordinates of C and C' are

$$\left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right) \quad \text{and} \quad \left(\frac{l_2-l_1}{2}, \frac{m_2-m_1}{2}, \frac{n_2-n_1}{2}\right).$$

Hence the d.rs. of OC and OC' i.e. the bisectors of the angles between the lines are $l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2$.

10. *If a variable line in two adjacent positions has direction cosines $l, m, n; l + \delta l, m + \delta m, n + \delta n$, show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.*

We have

$$\begin{aligned} \cos\delta\theta &= l(l+\delta l) + m(m+\delta m) + n(n+\delta n) \\ &= l^2 + m^2 + n^2 + l\delta l + m\delta m + n\delta n \\ &= 1 + l\delta l + m\delta m + n\delta n. \end{aligned} \tag{1}$$

$$\begin{aligned} \text{Again} \quad & (l+\delta l)^2 + (m+\delta m)^2 + (n+\delta n)^2 = 1 \\ \text{or,} \quad & l^2 + m^2 + n^2 + 2(l\delta l + m\delta m + n\delta n) + \delta l^2 + \delta m^2 + \delta n^2 = 1 \\ \text{or,} \quad & 1 + 2(l\delta l + m\delta m + n\delta n) + \delta l^2 + \delta m^2 + \delta n^2 = 1 \\ \text{or,} \quad & l\delta l + m\delta m + n\delta n = -\frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2). \end{aligned} \tag{2}$$

Expanding $\cos\delta\theta$ in powers of $\delta\theta$,

$$\begin{aligned} \cos\delta\theta &= 1 - \frac{(\delta\theta)^2}{2!} + \frac{(\delta\theta)^4}{4!} - \cdots \\ &= 1 - \frac{(\delta\theta)^2}{2}, \text{ neglecting higher powers of } \delta\theta. \end{aligned}$$

$$\begin{aligned} \therefore\ 1 - \frac{\delta\theta^2}{2} &= 1 + l\delta l + m\delta m + n\delta n \qquad \text{[by (1)]} \\ \text{or,} \quad \delta\theta^2 &= -2(l\delta l + m\delta m + n\delta n) \\ &= \delta l^2 + \delta m^2 + \delta n^2. \qquad \text{[by (2)]} \end{aligned}$$

EXERCISE II

1. What are the d.cs. of the coordinate axes?

2. Find the d.cs. of OP, OQ and PO where O is the origin and the coordinates of P and Q are $(2, 3, 4)$ and $(-6, 5, -1)$.

3. Find the d.cs. of the line joining the points $(3, 2, 5)$ and $(-1, 3.2)$.

4. (a) A line in the XOY-plane makes an angle 30° with the x-axis. Find the d.cs. of this line.

   (b) A line makes angles 60° and 45° with the y-axis and z-axis respectively. Find the d.cs. of the line.

5. If A, B, C, D are the points $(2, 3, -1), (3, 5, -3), (1, 2, 3)$ and $(3, 5, 7)$ respectively, prove by projections that AB is perpendicular to CD.

6. Find the angle between the lines whose d.rs. are $1, 1, 2$ and $\sqrt{3}-1, -\sqrt{3}-1, 1$.

7. (a) A, B, C are $(2, 3, 5), (-1, 3, 2)$ and $(3, 5, -2)$. Find the angles of ΔABC.

   (b) Show that the points $(4, 5, 0), (2, 6, 2), (2, 3, -1)$ are the vertices of an isosseles triangle. Find equal angles of the triangle.

8. Prove by d.cs. that the points $(3, 2, 4), (4, 5, 2), (5, 8, 0)$ and $(2, -1, 6)$ are collinear.

9. Find the d.cs. of the line which is perpendicular to each of the line with d.rs. $(1, -2, -2)$ and $(0, 2, 1)$.

10. Prove that the lines drawn from the origin whose d.cs. are proportional to $(1, -1, 1), (2, -3, 0)$ and $(1, 0, 3)$ are coplanar.

11. Find the point where the line through the origin meets the join of $(-9, 4, 5)$ and $(11, 0, -1)$ perpendicularly.

12. (a) Prove that the angle between two diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

    (b) If the edges of a rectangular parallelopiped be a, b, c, show that the angles between the four diagonals are given by

$$\cos^{-1}\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}.$$

13. (a) If two pairs of opposite edges of a tetrahedron are at right angles, then show that the third pair is also at right angles.

    [*Hints.* Let O be the origin and $OABC$ be the tetrahedron. Let the coordinates of A, B, C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

    If the opposite edges OA, BC and OB, CA are the right angles, then

$$x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 - z_2) = 0$$
$$\text{and} \quad x_2(x_1 - x_3) + y_2(y_1 - y_3) + z_2(z_1 - z_3) = 0.$$

    Adding these two we have

$$x_3(x_1 - x_2) + y_3(y_1 - y_2) + z_3(z_1 - z_2) = 0.$$

    Hence the result follows.]

    (b) If, in a tetrahedron $OABC, OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$, then show that its pairs of opposite edges are at right angles.

14. Show that the figure formed by the points $(5, -1, 1), (7, -4, 7), (1, -6, 10)$ and $(-1, -3, 4)$ as vertices is a rhombus.

15. (a) Prove that the two lines whose d.cs. are given by the relations $2l + 2m - n = 0$ and $lm + mn + nl = 0$ are perpendicular to each other.

(b) The d.cs. of two lines are connected by the relations $l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$. Find the angle between them.

(c) Show that the lines whose d.cs. are given by the relations $a^2l+b^2m+c^2n = 0$ and $mn + nl + lm = 0$ will be parallel, if $a + b + c = 0$.

(d) Show that the straight lines whose d.cs. are given by the equations $al + bm+cn = 0, ul^2 + vm^2 + wn^2 = 0$ are perpendicular or parallel according as $(v + w)a^2 + (w + u)b^2 + (u + v)c^2 = 0$ or, $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$.

(e) Prove that the angle between two straight lines whose d.cs. are given by $l + m + n = 0$ and $fmn + gml + hlm = 0$, i.e., $\frac{\pi}{3}$, if $\frac{1}{f} + \frac{1}{g} + \frac{1}{h} = 0$.

[CH 2010]

[*Hints.* $l + m + n = 0 \cdots$ (1) and $fmn + gnl + hlm = 0 \cdots$ (2).

From (1), $n = -(l+m)$. Putting in (2), we get $g\frac{l^2}{m^2} + (f + g - h)\frac{l}{m} + f = 0$.

If l_1, m_1, n and l_2, m_2, n_2 are d.cs. of the lines, then

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{f}{g} \text{ and } \frac{l_1}{m_1} + \frac{l_2}{m_2} = -\frac{f+g-h}{g}.$$

From symmetry $\frac{l_1 l_2}{f} = \frac{m_1 m_2}{g} = \frac{n_1 n_2}{h} = \lambda$ (say).

$$\text{Again} \quad \left(\frac{l_1}{m_1} - \frac{l_2}{m_2}\right)^2 = \left(\frac{f+g-h}{g}\right)^2 - 4\frac{f}{g}$$

$$\text{or,} \quad (l_1m_2 - l_2m_1)^2 = \lambda^2[(f+g-h)^2 - 4fg].$$

$$\text{Similarly} \quad (m_1n_2 - m_2n_1)^2 = \lambda^2[(g+h-f)^2 - 4gh]$$

$$\text{and} \quad (n_1l_2 - n_2l_1)^2 = \lambda^2[(h+f-g)^2 - 4hf].$$

Now

$$\tan^2\frac{\pi}{3} = \frac{\sum(l_1m_2 - l_2m_1)^2}{(l_1l_2 + m_1m_2 + n_1n_2)^2} = \frac{\sum[(f+g-h)^2 - 4fg]}{(f+g+h)^2}$$

$$\text{or,} \quad 3(f+g+h)^2 = \sum[(f+g-h)^2 - 4fg]$$

$$\text{or,} \quad 12(gh + hf + fg) = 0 \quad \text{or,} \quad \frac{1}{f} + \frac{1}{g} + \frac{1}{h} = 0].$$

16. If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the d.cs. of three mutually perpendicular lines, show that the line whose d.rs. are $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$ makes equal angles with them.

17. If θ is the angle between two lines having d.cs. l_1, m_1, n_1 and l_2, m_2, n_2, then show that the actual d.cs. of one of the bisectors of the angles between the lines are

$$\frac{l_1 + l_2}{2\cos\frac{\theta}{2}}, \frac{m_1 + m_2}{2\cos\frac{\theta}{2}}, \frac{n_1 + n_2}{2\cos\frac{\theta}{2}}.$$

[CH 95]

[*Hints.* See Fig. 17. Here $\angle AOB = \theta$. $\therefore$ $\angle AOC = \frac{\theta}{2}$.

From $\triangle OAC, OC = OA\cos\frac{\theta}{2}$.

The coordinates of C are

$$\left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right) \text{ and } OA = 1.$$

$$\therefore \sqrt{\left(\frac{l_1+l_2}{2}\right)^2 + \left(\frac{m_1+m_2}{2}\right)^2 + \left(\frac{n_1+n_2}{2}\right)^2} = \cos\frac{\theta}{2}.$$

The d.cs. of OC are

$$\frac{l_1+l_2}{\sqrt{\sum(l_1+l_2)^2}}, \frac{m_1+m_2}{\sqrt{\sum(l_1+l_2)^2}}, \frac{n_1+n_2}{\sqrt{\sum(l_1+l_2)^2}},$$

$$\text{i.e.}\quad \frac{l_1+l_2}{2\cos\frac{\theta}{2}}, \frac{m_1+m_2}{2\cos\frac{\theta}{2}}, \frac{n_1+n_2}{2\cos\frac{\theta}{2}}].$$

18. The d.cs. of two intersecting lines are l_1, m_1, n_1 and l_2, m_2, n_2. Prove that all lines passing through the intersection of these two and having d.cs. proportional to $l_1 + \lambda l_2, m_1 + \lambda m_2, n_1 + \lambda n_2$ are coplanar with them.

ANSWERS

1. $1, 0, 0; 0, 1, 0; 0, 0, 1$.

2. $\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}; \frac{-6}{\sqrt{62}}, \frac{5}{\sqrt{62}}, \frac{-1}{\sqrt{62}}; \frac{-2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{-4}{\sqrt{29}}$.

3. $\frac{-4}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{-3}{\sqrt{26}}$.

4. (a) $\frac{\sqrt{3}}{2}, \frac{1}{2}, 0$; (b) $\pm\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}$.

6. $\frac{\pi}{2}$.

7. (a) $\cos^{-1}\frac{1}{\sqrt{3}}, \frac{\pi}{2}, \cos^{-1}\sqrt{\frac{2}{3}}$; (b) $\frac{\pi}{4}$.

9. $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$. 11. $(1, 2, 2)$. 15. (b) $\cos^{-1}\frac{7}{\sqrt{84}}$.

Chapter 3

Plane

3.10 Definition of a plane

A plane is a flat surface such that if any two points are taken on the surface and joined by a straight line, the straight line lies wholly on the surface.

Let us now prove that *every linear equation (i.e. an equation of first degree) represents a plane.*

The most general equation of the first degree in x, y, z is of the form

$$ax + by + cz + d = 0 \tag{1}$$

where a, b, c, d are constants and a, b, c are not all zero.

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be any two points on the surface represented by the equation (1). Then

$$ax_1 + by_1 + cz_1 + d = 0 \tag{2}$$

$$\text{and} \quad ax_2 + by_2 + cz_2 + d = 0. \tag{3}$$

Multiplying (3) by λ and then adding to (2), we get

$$a\,(x_1 + \lambda x_2) + b\,(y_1 + \lambda y_2) + c\,(z_1 + \lambda z_2) + d\,(1 + \lambda) = 0.$$

Dividing by $1 + \lambda$,

$$a\frac{x_1 + \lambda x_2}{1 + \lambda} + b\frac{y_1 + \lambda y_2}{1 + \lambda} + c\frac{z_1 + \lambda z_2}{1 + \lambda} + d = 0.$$

It shows that the point

$$\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}, \frac{z_1 + \lambda z_2}{1 + \lambda}\right)$$

lies on the surface. Again this point is on the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) and divides this line in the ratio $\lambda : 1$. λ may have any value. Therefore, all the points on the line lie on the surface, i.e. the line lies wholly on the surface.

Hence the equation (1) represents a plane.

Note. $ax + by + cz + d = 0$ is known as the *general equation* of a plane. It can be written as

$$\frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + 1 = 0.$$

Therefore, there are three arbitrary constants in the general equation of a plane.

Corollary I. $ax + by + cz = 0$ represents a plane passing through the origin.

Corollary II. $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ represents a plane passing through (x_1, y_1, z_1).

3.11 Equation of a plane in the intercept form

Let $ax + by + cz + d = 0$ be the equation of a plane which meets the coordinate axes at A, B, C. If the coordinates of A, B, C be $(\alpha, 0, 0), (0, \beta, 0)$ and $(0, 0, \gamma)$ respectively, then $a\alpha + d = 0$ or $\alpha = -\frac{d}{a}$.

Similarly $\beta = -\frac{d}{b}, \gamma = -\frac{d}{c}$.

The equation of the plane can be written as

$$ax + by + cz = -d$$

$$\text{or,} \quad \frac{x}{-d/a} + \frac{y}{-d/b} + \frac{z}{-d/c} = 1$$

$$\text{or,} \quad \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1.$$

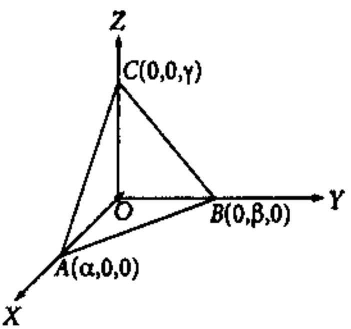

Fig. 18

It is known as the equation of a plane in the intercept form. α, β, γ are the *intercepts* made by the plane on the x, y and z-axes respectively.

Example 1. *Find the equation of the plane which makes intercepts* $2, 3, -4$ *on the axes.*

The equation of the plane is $\frac{x}{2} + \frac{y}{3} - \frac{z}{4} = 1$.

Example 2. *Find the intercepts made by the plane* $x - 2y + 3z - 18 = 0$ *on the axes.*

The equation of the plane can be written as

$$\frac{x}{18} + \frac{y}{-9} + \frac{z}{6} = 1.$$

Therefore, the intercepts on the x, y and z-axes are $18, -9, 6$ respectively.

Example 3. *If a plane meets the axes in* A, B, C *and the centroid of* $\triangle ABC$ *is* (α, β, γ), *show that the equation of the plane is* $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$. **[CH 2007]**

Let $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ be the equation of the plane. The coordinates of A, B, C are $(a,0,0),(0,b,0)$ and $(0,0,c)$ respectively. The centroid of $\triangle ABC$ is $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$. By the given coordinates of the centroid, we have

$$\frac{a}{3}=\alpha \quad \text{or,} \quad a=3\alpha, b=3\beta \quad \text{and} \quad c=3\gamma.$$

$$\therefore \quad \text{the equation of the plane is } \frac{x}{3\alpha}+\frac{y}{3\beta}+\frac{z}{3\gamma}=1$$

$$\text{or,} \quad \frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3.$$

3.12 Equation of a plane passing through three given points

Let

$$ax+by+cz+d=0 \tag{1}$$

represent a plane passing through the points $(x_1,y_1,z_1),(x_2,y_2,z_2)$ and (x_3,y_3,z_3).

Since the plane passes through these points,

$$ax_1+by_1+cz_1+d=0, \tag{2}$$

$$ax_2+by_2+cz_2+d=0 \tag{3}$$

$$\text{and} \quad ax_3+by_3+cz_3+d=0. \tag{4}$$

Eliminating a,b,c,d from (1), (2), (3) and (4), we have

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0.$$

It is the required equation.

Example 4. *Find the equation of the plane passing through* $(0,2,4),(3,1,1)$ *and* $(2,0,-1)$.

The equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 2 & 4 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 0 & -1 & 1 \end{vmatrix}=0$$

$$\text{or,} \quad x\begin{vmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}-y\begin{vmatrix} 0 & 4 & 1 \\ 3 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix}+z\begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}-\begin{vmatrix} 0 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix}=0$$

$$\text{or,} \quad x-9y+4z+2=0.$$

3.13 Equation of the plane in the normal form

Let ABC be a plane in the coordinate system OX, OY and OZ. ON is perpendicular to the plane. ON is called the normal to the plane. Let l, m, n be the d.cs. of ON and $ON = p$.

$P(x, y, z)$ is a point on the plane. PM is perpendicular to XOY-plane. LM is parallel to OY.

$$\therefore \; OL = x, LM = y, MP = z.$$

By the property of projection,

proj. of OP on ON = proj. of OL on ON + proj. of LM on ON + proj. of MP on ON.

Since ON is perpendicular to the plane ABC, ON is perpendicular to NP.

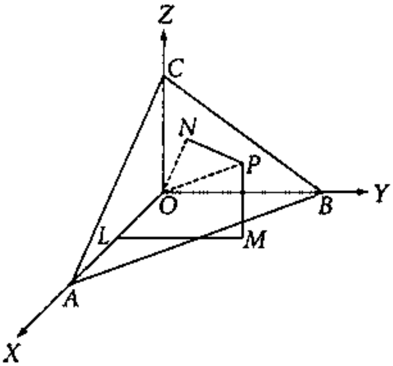

Fig. 19

$\therefore$ projection of OP on $ON = ON = p$.

Again proj. of OL on $ON = lx$,
proj. of LM on $ON = my$.
proj. of MP on $ON = nz$.

$\therefore \; lx + my + nz = p$.

P is any point on the plane. Hence it is the equation of the plane.

Note 1. p = perpendicular distance form the origin to the plane. It is always positive.

Note 2. l, m, n are the d.cs. of the normal to the plane. If the normal makes angles α, β, γ with the coordinate axes, then the equation to the plane is $x\cos\alpha + y\cos\beta + z\cos\gamma = p$.

3.14 Reduction of $ax + by + cz + d = 0$ in the normal form

Let $lx + my + nz = p$ be the normal form of the equation. Comparing these two equations

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{-p}{d}.$$

$$l = -\frac{ap}{d}, m = -\frac{bp}{d}, n = -\frac{cp}{d}.$$

$$l^2 + m^2 + n^2 = \frac{a^2 + b^2 + c^2}{d^2}p^2 \quad \text{or,} \quad 1 = \frac{a^2 + b^2 + c^2}{d^2}p^2$$

$$\text{or,} \quad p^2 = \frac{d^2}{a^2 + b^2 + c^2} \quad \text{or,} \quad p = \pm\frac{d}{\sqrt{a^2 + b^2 + c^2}}.$$

Since p is always positive,

$$p = \frac{d}{\sqrt{a^2+b^2+c^2}}, \text{ when } d \text{ is positive;}$$

$$p = \frac{-d}{\sqrt{a^2+b^2+c^2}}, \text{ when } d \text{ is negative.}$$

Now $\quad l = \mp\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \mp\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \mp\frac{c}{\sqrt{a^2+b^2+c^2}}$

according as d is positive or negative.

Thus the normal form of the general equation is

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = \mp\frac{d}{\sqrt{a^2+b^2+c^2}}$$

according as d is positive or negative.

Note 1. Perpendicular distance from the origin to $ax + by + cz + d = 0$ is $\frac{\pm d}{\sqrt{a^2+b^2+c^2}}$ according as d is positive or negative.

Note 2. The d.cs. of the normal to the plane $ax + by + cz + d = 0$ are

$$\mp\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

according as d is positive or negative. a, b, c are the d.rs. of the normal to the plane.

Corollary I. Equations of the coordinate planes:

The XOY-plane passes through the origin and z-axis is normal to the plane. The d.cs. of z-axis are $0, 0, 1$. Hence the equation of the XOY-plane is $z = 0$. Similarly the equations of the planes YOZ and ZOX are $x = 0$ and $y = 0$.

Corollary II. Equations of planes parallel to the axes.

If a plane is parallel to z-axis, then the normal to the plane is perpendicular to z-axis. Therefore, the z d.c. of this normal is zero. Hence the equation of this plane is $ax + by + d = 0$. Similarly the equations of the planes parallel to x and y-axes are $by + cz + d = 0$ and $ax + cz + d = 0$ respectively.

Example 5. *Express the equation of the plane* $2x + 6y - 3z + 5 = 0$ *in the normal form and hence obtain the length of perpendicular from the origin upon the plane.*

Here $\sqrt{2^2 + 6^2 + (-3)^2} = 7$ and the constant term $= 5$.

Dividing each term of the given equation by -7, the required normal form is

$$-\frac{2}{7}x - \frac{6}{7}y + \frac{3}{7}z = \frac{5}{7}.$$

Perpendicular distance from the origin $= \frac{5}{7}$.

Example 6. *Find the equations of the three planes through the points* $(3,1,1)$, $(1,-2,3)$, *parallel to the coordinate axes.*

(i) Let the equation of the plane parallel to the x-axis be $by+cz+d=0$. If it passes through the given points, then $b+c+d=0$ and $-2b+3c+d=0$. From these relations $b=-\frac{2}{5}d, c=-\frac{3}{5}d$.

$\therefore$ the required equation is $2y+3z-5=0$.

(ii) Let the equation of the plane parallel to the y-axis be $ax+cz+d=0$. If it passes through the given points, then $3a+c+d=0$ and $a+3c+d=0$. From these relations $a=-\frac{d}{4}, c=-\frac{d}{4}$.

$\therefore$ the required equation is $x+z-4=0$.

(iii) Let the equation of the plane parallel to the z-axis be $ax+by+d=0$. If it passes through the given points, then $3a+b+d=0$ and $a-2b+d=0$. From these relations $a=-\frac{3d}{7}, b=\frac{2d}{7}$.

$\therefore$ the required equation is $3x-2y-7=0$.

3.20 Perpendicular distance of a point from a plane

Let

$$ax+by+cz+d=0 \tag{1}$$

be the equation of a plane and (x_1,y_1,z_1) be a given point.

The equation of the plane parallel to (1) and passing through the point (x_1,y_1,z_1) is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \tag{2}$$

$$\text{or,}\quad ax+by+cz-(ax_1+by_1+cz_1)=0. \tag{3}$$

The distances of (1) and (2) from the origin are

$$\pm\frac{d}{\sqrt{a^2+b^2+c^2}} \quad \text{and} \quad \mp\frac{ax_1+by_1+cz_1}{\sqrt{a^2+b^2+c^2}}$$

according as d is positive or negative.

$$\begin{aligned}\therefore \text{ the required distance} &= \left|\pm\frac{d}{\sqrt{a^2+b^2+c^2}}-\left(\mp\frac{ax_1+by_1+cz_1}{\sqrt{a^2+b^2+c^2}}\right)\right| \\ &= \left|\frac{ax_1+by_1+cz_1+d}{\pm\sqrt{a^2+b^2+c^2}}\right| \\ &= \frac{|ax_1+by_1+cz_1+d|}{\pm\sqrt{a^2+b^2+c^2}}.\end{aligned}$$

Corollary. If d is positive, the expression $ax_1+by_1+cz_1+d$ will be positive when (x_1,y_1,z_1) and the origin are on the same side of the plane $ax+by+cz+d=0$ and negative when they are on opposite sides of the plane.

> **Note.** Perpendicular distance of a point (x_1, y_1, z_1) from the plane $x\cos\alpha + y\cos\beta + z\cos\gamma = p$ is $p - x_1\cos\alpha - y_1\cos\beta - z_1\cos\gamma$.

Example 7. *Find the distances of the point* $(2, 3, 4)$ *from the planes* $3x + 4y + 5z + 7 = 0$ *and* $2x - 3y + 5z - 8 = 0$.

The distance of $(2, 3, 4)$ from $3x + 4y + 5z + 7 = 0$ is

$$\frac{3\cdot 2 + 4\cdot 3 + 5\cdot 4 + 7}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{45}{\sqrt{50}} = \frac{9}{\sqrt{2}}.$$

The distance of $(2, 3, 4)$ from $2x - 3y + 5z - 8 = 0$ is

$$\frac{2\cdot 2 - 3\cdot 3 + 5\cdot 4 - 8}{\sqrt{2^2 + 3^2 + 5^2}} = \frac{7}{\sqrt{38}}.$$

3.30 Angle between two planes

Definition. *Angle between two planes is defined as the angle between the normals to the planes drawn through a point.*

Let

$$a_1x + b_1y + c_1z + d_1 = 0 \tag{1}$$

$$\text{and } a_2x + b_2y + c_2z + d_2 = 0 \tag{2}$$

be the equations of two planes. The d.rs. of the normals to the planes are a_1, b_1, c_1 and a_2, b_2, c_2.

If θ be the angle between them, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\cdot\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Corollary I. Condition for perpendicular planes.

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Corollary II. Conditions for parallel planes.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Hence $ax + by + cz + d = 0$ is parallel to $ax + by + cz + d' = 0$.

Example 8. *Find the angle* between the *planes*

$$2x - y + 3z + 7 = 0 \quad \textit{and} \quad x - 2y - 3z + 8 = 0.$$

Angle between the planes $= \cos^{-1}\frac{2\cdot 1 + (-1)\cdot(-2) + 3\cdot(-3)}{\sqrt{2^2+1^2+3^2}\sqrt{1^2+2^2+3^2}} = \cos^{-1}\frac{-5}{14}$.

3.31 Planes bisecting the angles between two planes

Let the equations of the planes be

$$a_1x + b_1y + c_1z + d_1 = 0 \tag{1}$$

$$\text{and} \quad a_2x + b_2y + c_2z + d_2 = 0. \tag{2}$$

If $P(x', y', z')$ be any point on either of the two bisecting planes, then P is equidistant from the two planes.

$$\therefore \quad \frac{a_1x' + b_1y' + c_1z' + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x' + b_2y' + c_2z' + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Hence the equations of the planes are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Note. To distinguish the two bisecting planes.

First write the equations of the planes with the same sign of the constants. Now calculate the angle between the planes. If $\cos\theta$ is negative, i.e. $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then the origin is within the acute angle between the planes. If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then the origin is within the obtuse angle between the planes.

Writing the equations with constants d_1 and d_2 as positive numbers the equation

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

represents the plane bisecting that angle between the planes which contains the origin. If the origin lies in the acute angle, then the above plane bisects the acute angle, otherwise it will bisect the obtuse angle.

Example 9. *Find the equation of the plane which bisects the acute angle between the planes* $x + 2y + 2z = 9$ *and* $4x - 3y + 12z + 13 = 0$.

First we write the equations with the positive sign of the constants in the following way

$$4x - 3y + 12z + 13 = 0 \tag{1}$$

$$-x - 2y - 2z + 9 = 0. \tag{2}$$

Here $4 \cdot (-1) - 3 \cdot (-2) + 12 \cdot (-2) = -22 < 0$.

Therefore, the origin is within the acute angle between the planes. Hence the equation of the plane bisecting the acute angle between the planes is

$$\frac{4x - 3y + 12z + 13}{\sqrt{4^2 + 3^2 + 12^2}} = \frac{-x - 2y - 2z + 9}{\sqrt{1^2 + 2^2 + 2^2}}$$

$$\text{or,} \quad \frac{4x - 3y + 12z + 13}{13} = \frac{-x - 2y - 2z + 9}{3}$$

$$\text{or,} \quad 25x + 17y + 62z - 78 = 0.$$

3.40 Some theorems

Theorem 1. *The necessary and sufficient condition for the equations*

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$\text{and} \quad a_2x + b_2y + c_2z + d_2 = 0$$

to represent the same plane is that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}.$$

Proof. (i) *The condition is necessary.*

Let the equations represent the same plane. Then (a_1, b_1, c_1) and (a_2, b_2, c_2) are d.rs. of the normal to the plane.

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda \text{ (say)}. \tag{1}$$

If (x', y', z') be any point on the plane, then

$$a_1x' + b_1y' + c_1z' + d_1 = 0 \quad \text{or,} \quad \lambda(a_2x' + b_2y' + c_2z') + d_1 = 0 \tag{2}$$
$$\text{and} \quad a_2x' + b_2y' + c_2z' + d_2 = 0 \quad \text{or,} \quad \lambda(a_2x' + b_2y' + c_2z' + d_2) = 0. \tag{3}$$

Subtracting (3) from (2),

$$d_1 - \lambda d_2 = 0 \quad \text{or,} \quad d_1 = \lambda d_2. \tag{4}$$

By (1) and (4),

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}.$$

(ii) *The condition is sufficient.*

If the condition holds, then the coordinates of any point satisfying one of the two equations must satisfy the other. Hence the given equations represent the same and the condition is sufficient.

Theorem 2. *The necessary and sufficient condition for the coplanarity of four points $(x_i, y_i, z_i), i = 1, 2, 3, 4,$ no three of which are collinear, is that*

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

(i) *The condition is necessary.*

Let the points be coplanar and lie on the plane $ax + by + cz + d = 0$.

$$\therefore \; ax_1 + by_1 + cz_1 + d = 0, \tag{1}$$
$$ax_2 + by_2 + cz_2 + d = 0, \tag{2}$$
$$ax_3 + by_3 + cz_3 + d = 0 \tag{3}$$
$$\text{and} \quad ax_4 + by_4 + cz_4 + d = 0. \tag{4}$$

Eliminating a, b, c, d from (1), (2), (3) and (4), we get

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_2 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

(ii) *The condition is sufficient.*

If the condition holds and no three of the given points are collinear, then the equation of the plane through (x_i, y_i, z_i), $i = 2, 3, 4$ is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

and it is satisfied by (x_1, y_1, z_1). Thus the points are coplanar.

Theorem 3. *The necessary and sufficient condition for the equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ to represent two planes is that $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.*

Proof. (i) *The condition is necessary.*

Let the equation represent two planes whose equations are

$$lx + my + nz = 0 \quad \text{and} \quad l'x + m'y + n'z = 0.$$

The constant terms do not appear in the equations; for, otherwise, their joint equation will not be homogeneous.

Now $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = (lx + my + nz)(l'x + m'y + n'z)$.

It implies that $ll' = a, mm' = b, nn' = c, mn' + m'n = 2f, nl' + n'l = 2g, lm' + l'm = 2h$.

To eliminate l, m, n, l', m', n' from the above six relations the following product is considered.

We have

$$\begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix} \times \begin{vmatrix} l' & l & 1 \\ m' & m & 1 \\ n' & n & 1 \end{vmatrix} = 0$$

$$\text{or,} \quad \begin{vmatrix} 2ll' & lm' + l'm & ln' + l'n \\ lm' + l'm & 2mm' & mn' + m'n \\ ln' + l'n & mn' + m'n & 2nn' \end{vmatrix} = 0$$

$$\text{or,} \quad 8\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\text{or,} \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

Thus it is a necessary condition.

(ii) *The condition is sufficient.*

Let

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0. \tag{1}$$

Considering the given equation as a quadratic equation in x and solving for x,

$$x = \frac{-(gz + hy) \pm \sqrt{(gz + hy)^2 - a\,(by^2 + cz^2 + 2fyz)}}{a}, a \neq 0.$$

For the condition (1), the expression within the radical sign is of the form $(\lambda y + \mu z)^2$. Hence the given homogeneous equation represents two planes and the condition (1) is sufficient.

Note. If θ is the angle between the planes, then

$$\tan\theta = \pm\frac{\sqrt{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2}}{ll' + mm' + nn'}$$

$$= \pm\frac{\sqrt{(mn' + m'n)^2 - 4mm'nn' + (nl' + n'l)^2 - 4nn'll' + (lm' + l'm)^2 - 4ll'mm'}}{ll' + mm' + nn'}$$

$$= \pm\frac{\sqrt{(f^2 + g^2 + h^2 - ab - bc - ca)}}{a + b + c}.$$

If the planes are at right angles, then $a + b + c = 0$.

3.41 Any linear combination of two linear equations represents a plane passing through the common line of the planes.

Let

$$a_1x + b_1y + c_1z + d_1 = 0 \tag{1}$$

$$\text{and} \quad a_2x + b_2y + c_2z + d_2 = 0 \tag{2}$$

be the equations of two planes.

The equation

$$a_1x + b_1y + c_1z + d_1 + \lambda\,(a_2x + b_2y + c_2z + d_2) = 0 \tag{3}$$

represents a plane, since it is of first degree. Here λ is a variable parameter. Again any point satisfying the equations (1) and (2) must satisfy the equation (3). Therefore, the equation represents a plane passing through the line of intersection of the planes (1) and (2).

Example 10. *Find the equation of the plane passing through the point* $(-1, 0, 1)$ *and the line of intersection of the planes* $4x - 3y + 1 = 0$ *and* $y - 4z + 13 = 0$.

The equation of a plane passing through the line of intersection of the two planes can be written as

$$4x - 3y + 1 + \lambda(y - 4z + 13) = 0$$

where λ is a constant.

As it passes through $(-1, 0, 1), -4 + 1 + \lambda(-4 + 13) = 0$ or, $\lambda = 1/3$.

$\therefore$ the required equation is

$$4x - 3y + 1 + \tfrac{1}{3}(y - 4z + 13) = 0$$
$$\text{or,}\quad 12x - 8y - 4z + 16 = 0 \quad \text{or,}\quad 3x - 2y - z + 4 = 0.$$

WORKED-OUT EXAMPLES

1. *Find the equation to the plane through the point* $(1, 2, -3)$ *and normal to the straight line joining the points* $(-1, 3, 4)$ *and* $(5, 2, -1)$.

   Let the equation of the plane through $(1, 2, -3)$ be

   $$a(x-1) + b(y-2) + c(z+3) = 0.$$

   The d.rs. of the line joining the given points are $6, -1, -5$.

   Since this line is normal to the plane, $\frac{a}{6} = \frac{b}{-1} = \frac{c}{-5}$.

   $\therefore$ the equation of the plane is

   $$6(x-1) - (y-2) - 5\,(z+3) = 0, \text{ i.e. } 6x - y - 5z - 19 = 0.$$

2. *Find the equation of the plane which is perpendicular to the plane* $x + 2y - z + 1 = 0$ *and which contains the line of intersection of the planes* $x + 2y + 3z - 4 = 0$ *and* $2x + y + z + 2 = 0$.

   The equation of the plane through the line of intersection of the given planes can be written as

   $$x + 2y + 3z - 4 + \lambda\,(2x + y + z + 2) = 0$$
   $$\text{or,}\quad (1 + 2\lambda)x + (2 + \lambda)y + (3 + \lambda)z + 2\lambda - 4 = 0,$$

   where λ is an arbitrary constant.

   Since it is perpendicular to

   $$x + 2y - z + 1 = 0,$$
   $$(1 + 2\lambda)\cdot 1 + (2 + \lambda).2 + (3 + \lambda)\cdot(-1) = 0 \quad \text{or,}\quad \lambda = -\tfrac{2}{3}.$$

   $\therefore$ the required equation is $x - 4y - 7z + 16 = 0$.

3. *Find the equation of the plane passing through the point* $(2, 5, -8)$ *and perpendicular to each of the planes* $2x - 3y + 4z + 1 = 0$ *and* $4x + y - 2z + 6 = 0$.

   Let the equation of the plane through the point $(2, 5, -8)$ be

   $$a(x-2) + b(y-5) + c(z+8) = 0.$$

   Since it is perpendicular to each of the given planes,

   $$2a - 3b + 4c = 0 \quad \text{and} \quad 4a + b - 2c = 0.$$

   By cross-multiplication, $\frac{a}{1} = \frac{b}{10} = \frac{c}{7}$.

   $\therefore$ the equation of the plane is

   $$1\cdot(x-2) + 10(y-5) + 7(z+8) = 0, \quad \text{i.e.}\quad x + 10y + 7z + 4 = 0.$$

4. *Find the equation of the plane which bisects the line joining the points* $(1,2,3)$ *and* $(3,4,5)$ *at right angles and show that it makes equal intercepts on the axes.*

   The d.rs. of the line are $2,2,2$ and the midpoint of the segment is $(2,3,4)$.

   The equation of a plane through the midpoint can be written as

$$a(x-2)+b(y-3)+c(z-4)=0.$$

   It is perpendicular to the line of d.rs. $2,2,2$. $\therefore\ \frac{a}{2}=\frac{b}{2}=\frac{c}{2}$.

   $\therefore$ the equation of the plane is

$$2(x-2)+2(y-3)+2(z-4)=0, \quad \text{i.e.} \quad x+y+z=9.$$

   Obviously the plane makes equal intercepts on the axes and each of them is equal to 9.

5. *Show that the points* $(2,3,-5)$ *and* $(3,4,7)$ *lie on the opposite sides of the plane* $x+2y-2z=9$.

   Perpendicular distances of the plane $x+2y-2z-9=0$ from the given points are

$$\frac{2+2\cdot 3+2\cdot 5-9}{\sqrt{1+4+4}}=3 \quad \text{and} \quad \frac{3+2\cdot 4-2\cdot 7-9}{\sqrt{1+4+4}}=-4.$$

   Since these are of opposite signs the points lie on the opposite sides of the plane.

6. *The plane* $x-2y+3z-4=0$ *is rotated through a right angle about its line of intersection with the plane* $2x+3y-4z-5=0$. *Find the equation of the plane in its new position.*

   The plane

$$x-2y+3z-4=0 \tag{1}$$

   is perpendicular to the plane which is generated by rotating it through a right angle about the line of intersection with

$$2x+3y-4z-5=0. \tag{2}$$

   The equation of a plane through the line of intersection between (1) and (2) is

$$x-2y+3z-4+\lambda(2x+3y-4z-5)=0, \tag{3}$$

$$\text{i.e.} \quad (1+2\lambda)x+(3\lambda-2)y+(3-4\lambda)z-(4+5\lambda)=0. \tag{4}$$

   Since it is perpendicular to (1),

$$(1+2\lambda)-2\cdot(3\lambda-2)+3(3-4\lambda)=0, \quad \text{i.e.} \quad \lambda=\frac{7}{8}.$$

Therefore, the required equation is

$$\left(1+\frac{7}{4}\right)x+\left(\frac{21}{8}-2\right)y+\left(3-\frac{7}{2}\right)z-\left(4+\frac{35}{8}\right)=0$$
$$\text{or,}\quad 22x+5y-4z-67=0.$$

7. *The sum of the squares of the distances of a point from the planes $x+y+z=0$ and $x-2y+z=0$ is equal to the square of its distance from the plane $x-z=0$. Find the locus of the point.*

Let the point be (x_1, y_1, z_1).

Distances of the planes from this point are

$$\frac{x_1+y_1+z_1}{\sqrt{1+1+1}}, \frac{x_1-2y_1+z_1}{\sqrt{1+4+1}}, \frac{x_1-z_1}{\sqrt{1+1}}.$$

By the given conditions, we have

$$\frac{(x_1+y_1+z_1)^2}{3}+\frac{(x_1-2y_1+z_1)^2}{6}=\frac{(x_1-z_1)^2}{2} \quad \text{or,} \quad y_1^2+2z_1x_1=0.$$

Thus the required locus is $y^2+2zx=0$.

8. *A variable plane passes through a fixed point (α, β, γ) and meets the axes of reference in A, B and C. Show that the locus of the point of intersection of the planes through A, B and C parallel to the coordinate planes is*

$$\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z}=1.$$ **[CH 96, 2001]**

Let the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

As it passes through (α, β, γ),

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1. \tag{1}$$

The plane meets the axes at $A(a,0,0), B(0,b,0)$ and $C(0,0,c)$. The planes through A, B and C parallel to the coordinate planes are $x=a, y=b$ and $z=c$. These three planes meet at (a,b,c). (a,b,c) satisfies the relation (1). Hence the required locus is

$$\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z}=1.$$

9. *A variable plane at a constant distance p from the origin meets the axes at A, B, C. Show that the locus of the centroid of the tetrahedron $OABC$ is*

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}.$$ **[BH 2007; CH 97]**

Let the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Since its distance from the origin is p,

$$p = \frac{1}{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}} \quad \text{or,} \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}. \tag{1}$$

The plane meets the axes at $A(a,0,0), B(0,b,0)$ and $C(0,0,c)$. If (α,β,γ) be the centroid of the tetrahedron $OABC$, (O the origin), then

$$\alpha = \frac{a}{4}, \beta = \frac{b}{4}, \gamma = \frac{c}{4}.$$

From (1),

$$\frac{1}{16\alpha^2} + \frac{1}{16\beta^2} + \frac{1}{16\gamma^2} = \frac{1}{p^2}.$$

Hence the locus of (α,β,γ) is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}.$$

10. *Perpendicular PL, PM, PN are drawn from the point $P(a,b,c)$ to the coordinate planes. Show that the equation of the plane LMN is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 2$.*

[BH 91; CH 93, 99]

Coordinates of L, M, N are $(a,b,0), (0,b,c), (a,0,c)$ respectively.

Let the equation of the plane LMN be $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$.

It passes through the above points, so

$$\frac{a}{\alpha} + \frac{b}{\beta} = 1, \tag{1}$$

$$\frac{b}{\beta} + \frac{c}{\gamma} = 1 \tag{2}$$

$$\text{and} \quad \frac{a}{\alpha} + \frac{c}{\gamma} = 1. \tag{3}$$

By (1), (2), (3) we have

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = \frac{3}{2}. \tag{4}$$

Subtracting (1), (2), (3) from (4), $\frac{c}{\gamma} = \frac{1}{2}$, $\frac{a}{\alpha} = \frac{1}{2}$, $\frac{b}{\beta} = \frac{1}{2}$ or, $\alpha = 2a$, $\beta = 2b$, $\gamma = 2c$.

$\therefore$ the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 2$.

11. *A variable plane makes intercepts on the coordinate axes, the sum of whose squares is constant and equal to k^2. Show that the locus of the foot of the perpendicular from the origin to the plane is*

$$\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\left(x^2+y^2+z^2\right)^2=k^2.$$

[BH 94, 96, 2007; CH 92; 97, 2007; NH 2008]

Let the equation of the plane at one position be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Here

$$a^2+b^2+c^2=k^2. \tag{1}$$

Let (α,β,γ) be the foot of the perpendicular from the origin to the plane. This point lies on the plane.

$$\therefore\ \frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1. \tag{2}$$

The d.rs. of the line joining the origin and the point (α,β,γ) are α,β,γ. Since it is perpendicular to the plane,

$$a\alpha=b\beta=c\gamma=\lambda \text{ (say)}. \tag{3}$$

By (1) and (3),

$$\lambda^2\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)=k^2. \tag{4}$$

By (2) and (3),

$$\alpha^2+\beta^2+\gamma^2=\lambda. \tag{5}$$

By (4) and (5),

$$\left(\alpha^2+\beta^2+\gamma^2\right)^2\left(\frac{1}{\alpha^2}+\frac{1}{\gamma^2}+\frac{1}{\gamma^2}\right)=k^2.$$

Hence the locus of (α,β,γ) is

$$\left(x^2+y^2+\gamma^2\right)^2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=k^2.$$

12. *Show that the line of intersection of the planes $x+2y-z-3=0$ and $3x-y+2z-1=0$ is coplanar with the line of intersection of the planes $2x-2y+3z-2=0$ and $x-y+z+1=0$. Obtain the equation of the plane containing the lines.*

The lines will be coplanar, if the planes

$$x+2y-z-3+\lambda(3x-y+2z-1)=0$$
$$\text{and}\quad 2x-2y+3z-2+\lambda'(x-y+z+1)=0$$

are identical for some values of λ and λ'. This requires that

$$\frac{1+3\lambda}{2+\lambda'} = \frac{2-\lambda}{-2-\lambda'} = \frac{-1+2\lambda}{3+\lambda'} = \frac{-3-\lambda}{-2+\lambda'}$$

$$\text{or,}\quad 2\lambda\lambda' + 4\lambda + 3\lambda' + 6 = 0 \tag{1}$$

$$\lambda\lambda' + \lambda + \lambda' + 4 = 0 \tag{2}$$

$$\text{and}\quad 3\lambda\lambda' - \lambda + 2\lambda' + 11 = 0. \tag{3}$$

By (1) and (2), $\lambda = 2, \lambda' = -2$ or $\lambda = -\frac{3}{2}, \lambda' = 5$.

Of these two sets, the equation (3) is satisfied by $\lambda = -\frac{3}{2}, \lambda' = 5$.

Thus the two lines are coplanar and the equation of the plane containing the lines is $7x - 7y + 8z + 3 = 0$.

EXERCISE III

1. Find the intercepts made by (i) $x - y + z = 2$, (ii) $2x + 3y + 6z - 6 = 0$ on the axes.

2. Calculate the d.cs. and the length of the directed normal from the origin to the plane: (i) $2x - 3y + 6z = 7$, (ii) $4x + 3y + 2z + 12 = 0$.

3. Express the following in the normal form:

   (a) $2x - 3y + 4z - 5 = 0$, (b) $x + 5 = 0$.

4. Find the equation of the plane

   (i) passing through $(3, 5, 1), (2, 3, 0)$ and $(0, 6, 0)$;

   (ii) through $(4, 1, 1)$ and parallel to $3x - 4y + 7z + 5 = 0$;

   (iii) through $(2, 0, -1)$ and perpendicular to the line whose d.rs. are $3, 4, -2$;

   (iv) passing through the point $(1, 2, 3)$ and parallel to the zx-plane;

   (v) which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ and which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$;

   (vi) through the line $x + y - 2z + 4 = 0 = 3x - y + 2z + 1$ and parallel to the line with d.rs. $2, 3, -1$;

   (vii) parallel to the plane containing OZ and the point $(2, 2, 1)$ and passing through $(1, -1, 1)$;

   (viii) passing through the points $(2, 3, -4)$, $(1, -1, 3)$ and perpendicular to YOZ-plane;

   (ix) passing through $(2, 1, 4)$ and perpendicular to each of the planes $x + y + 2z - 4 = 0$ and $2x - 3y + z + 5 = 0$;

   (x) through the points $(2, , 2, 1)$ and $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 9$;

   (xi) bisecting the line joining the points $(3, 4, -1)$ and $(3, -1, 5)$ normally;

(xii) through the points $(4, 3, 1)$ and $(1, -3, 4)$ and parallel to the y-axis.

5. (a) The x and y intercepts of a plane are 3 and -5 respectively and the plane passes through $(2, 1, 8)$. Find its equation.

   (b) A plane passes through $(5, 1, 5)$ and $(-1, -2, 4)$ and has its y intercept equal to -4. Find its equation.

6. Show that the four points $(0, -1, 0), (2, 1, -1), (1, 1, 1)$ and $(3, 3, 0)$ are coplanar and obtain the equation of the plane.

7. Find the angle between the planes

   (a) $2x + y + z = 6, x - y + 2z = 3$;

   (b) $x + 2y + 3z - 4 = 0, 2x + 5y - 4z = 3$.

8. (a) Find the distances of the points $(2, 0, 1)$ and $(3, -3, 2)$ from the plane $x - 2y + z - 6 = 0$. Do the points lie on the same side or opposite sides of the plane?

   (b) Find a point on the y-axis such that it is equidistant from $6x + 12y - 4z + 2 = 0$ and $x - 2y + 2z + 5 = 0$.

   (c) Find the distance between the parallel planes $x - 4y + 8z - 9 = 0$ and $x - 4y + 8z + 18 = 0$.

   [*Hints.* Distances of the planes from the origin are $\frac{-9}{9} = -1$ and $\frac{18}{9} = 2$. $\therefore$ The required distance $= 2 - (-1) = 3$.]

9. Find the locus of a point the sum of squares of whose distances from the planes $x + y + z = 0, x - z = 0, x - 2y + z = 0$ is 9.

10. The foot of the perpendicular drawn from the origin to a plane is $(12, -4, -3)$, find the equation of the plane.

11. Find the equations of the two planes through the points $(0, 4, -3)$ and $(6, -4, 3)$ other than the plane through the origin which cut off from the axes intercepts whose sum is zero.

12. (a) Find the equation of the planes bisecting the angle between the planes $3x - 4y + 12z = 26, x + 2y - 2z = 9$ and distinguish them.

    (b) Find the equation of the plane which bisects the obtuse angle between the planes $4x - 3y + 12z + 13 = 0$ and $x + 2y + 2z = 9$.

13. Find the equations of the planes parallel to the plane $16x + 12y - 15z + 75 = 0$ and at a distance 4 from it.

14. (a) Show that the equation of the plane through the point (x_1, y_1, z_1) and perpendicular to both the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ can be expressed as

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

(b) A variable plane moves in such a manner that the sum of the reciprocals made by the plane on the axes is constants. Show that the plane passes through a fixed point.

15. The plane $lx + my = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle α. Prove that its equation in its new position is

$$lx + my \pm z\sqrt{l^2 + m^2}\,\tan\alpha = 0.$$

[*Hints.* Let the required plane be $lx + my + \lambda z = 0$. It makes an angle α with $lx + my = 0$.

$$\therefore\ \cos\alpha = \frac{\sqrt{l^2 + m^2}}{\sqrt{l^2 + m^2 + \lambda^2}} \quad \text{and} \quad \sin^2\alpha = \frac{\lambda^2}{l^2 + m^2 + \lambda^2}.$$

$$\therefore\ \tan^2\alpha = \frac{\lambda^2}{l^2 + m^2} \quad \text{or,} \quad \lambda = \pm\sqrt{l^2 + m^2}\,\tan\alpha.$$

Hence the equation is $lx + my \pm z\sqrt{l^2 + m^2}\tan\alpha = 0$.]

16. Show that $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represents a pair of planes and find the angle between them. [NH 2008]

17. Show that the line of intersection of the planes $2x - 3y + z + 1 = 0$ and $x - y + z - 2 = 0$ is coplanar with the line of intersection of the planes $6x - y - z - 9 = 0$ and $x + 2y - 2z - 3 = 0$. Obtain the equation of the plane containing the lines.

18. The plane $ax + by + cz + d = 0$ bisects an angle between a pair of planes of which one is $lx + my + nz + p = 0$. Show that the equation of the other plane of the pair is

$$(lx + my + nz + p)\left(a^2 + b^2 + c^2\right) = 2(al + bm + cn)(ax + by + cz + d).$$

[*Hints.* The other plane is of the form

$$ax + by + cz + d + \lambda(lx + my + nz + p) = 0. \tag{1}$$

$(-d/a, 0, 0)$ is a point on $ax + by + cz + d = 0$. Distances of the planes $lx + my + nz + p = 0$ and (1) are equal.

$$\therefore\ \frac{\left|p - \frac{dl}{a}\right|}{\sqrt{l^2 + m^2 + n^2}} = \frac{|\lambda|\left|p - \frac{dl}{a}\right|}{\sqrt{\sum(a + \lambda l)^2}}$$

$$\text{or,} \quad \sum(a + \lambda l)^2 = \lambda^2\left(l^2 + m^2 + n^2\right)$$

$$\text{or,} \quad \lambda = -\frac{a^2 + b^2 + c^2}{2(al + bm + cn)}.$$

Hence the result follows.]

19. P, Q, R, S are four points in space. L, M, N, T divide segments PQ, QR, RS, SP in the ratio $l:1, m:1, n:1$ and $t:1$ respectively. If L, M, N, T are coplanar, show that $lmnt = 1$. **[BH 2008]**

[*Hints*. Let the coordinates of P, Q, R, S be $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ and (x_4, y_4, z_4) respectively. Then the coordinates of L, M, N, T are

$$\left(\frac{lx_2+x_1}{l+1}, \frac{ly_2+y_1}{l+1}, \frac{lz_2+z_1}{l+1}\right), \left(\frac{mx_3+x_2}{m+1}, \frac{my_3+y_2}{m+1}, \frac{mz_3+z_2}{m+1}\right)$$
$$\left(\frac{nx_4+x_3}{n+1}, \frac{ny_4+y_3}{n+1}, \frac{nz_4+z_3}{n+1}\right), \left(\frac{tx_1+x_4}{t+1}, \frac{ty_1+y_4}{t+1}, \frac{tz_1+z_4}{t+1}\right)$$

respectively.

Let these points lie on the plane $ax + by + cz + d = 0$. Then

$$a\frac{lx_2+x_1}{l+1} + b\frac{ly_2+y_1}{l+1} + c\frac{lz_2+z_1}{l+1} + d = 0 \quad \text{or,} \quad l = -\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}.$$

Similarly

$$m = -\frac{ax_2+by_2+cz_2+d}{ax_3+by_3+cz_3+d},$$
$$n = -\frac{ax_3+by_3+cz_3+d}{ax_4+by_4+cz_4+d},$$
$$t = -\frac{ax_4+by_4+cz_4+d}{ax_1+by_1+cz_1+d}.$$

Multiplying these we get $lmnt = 1$.]

20. A point P moves on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ which is fixed. The plane through P perpendicular to OP meets the axes in A, B and C. The planes through A, B, C parallel to yz, zx and xy-planes respectively intersect in Q. Prove that if the axes be rectangular, the locus of Q is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}.$$

[*Hints*. Let (α, β, γ) be the coordinates of P. Then

$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1. \tag{2}$$

The equation of the plane through the point P and perpendicular to OP is

$$\alpha x + \beta y + \gamma z = \alpha^2 + \beta^2 + \gamma^2. \tag{3}$$

The coordinates of A, B, C are

$$\left(\frac{\sum\alpha^2}{\alpha}, 0, 0\right), \left(0, \frac{\sum\alpha^2}{\beta}, 0\right) \quad \text{and} \quad \left(0, 0, \frac{\sum\alpha^2}{\gamma}\right)$$

respectively. If (x', y', z') be the coordinates of Q, then

$$x' = \frac{\sum \alpha^2}{\alpha}, y' = \frac{\sum \alpha^2}{\beta}, z' = \frac{\sum \alpha^2}{\gamma}.$$

$$\text{Now} \quad \frac{1}{x'^2} + \frac{1}{y'^2} + \frac{1}{z'^2} = \frac{1}{\sum \alpha^2}$$

$$\text{and} \quad \frac{1}{ax'} + \frac{1}{by'} + \frac{1}{cz'} = \frac{1}{\sum \alpha^2} \quad \text{by (1)}.$$

Hence the locus of Q is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}.]$$

21. Show that $\frac{a}{y-z} + \frac{b}{z-x} + \frac{c}{x-y} = 0$ represents a pair of planes.
[*Hints*. The given equation can be written as

$$ax^2 + by^2 + cz^2 - (b+c-a)yz - (c+a-b)zx - (a+b-c)xy = 0.$$

$$\because \begin{vmatrix} a & -\frac{1}{2}(a+b-c) & -\frac{1}{2}(c+a-b) \\ -\frac{1}{2}(a+b-c) & b & -\frac{1}{2}(b+c-a) \\ -\frac{1}{2}(c+a-b) & -\frac{1}{2}(b+c-a) & c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ -\frac{1}{2}(a+b-c) & b & -\frac{1}{2}(b+c-a) \\ -\frac{1}{2}(c+a-b) & -\frac{1}{2}(b+c-a) & c \end{vmatrix} = 0 \text{ by } R_1 + (R_2 + R_3)$$

the given equation represents a pair of planes.]

22. A triangle, the length of whose sides are a, b, c is placed so that the middle points of the sides are on the axes. Show that the equation to its plane is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$$

where $8\alpha^2 = b^2 + c^2 - a^2, 8\beta^2 = c^2 + a^2 - b^2, 8\gamma^2 = a^2 + b^2 - c^2$. Find also the coordinates of the vertices. [BH 2008, CH 95]

[*Hints*. Let D, E, F be the midpoints of BC, CA, AB of $\triangle ABC$. If the coordinates of D, E, F are $(\alpha, 0, 0)$, $(0, \beta, 0)$ and $(0, 0, \gamma)$, then the equation of the plane of $\triangle ABC$ is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$. Again the line joining the midpoints of two sides of a triangle is half of the length of the third side.

$$\therefore \alpha^2 + \beta^2 = \frac{c^2}{4}, \beta^2 + \gamma^2 = \frac{a^2}{4} \quad \text{and} \quad \gamma^2 + \alpha^2 = \frac{b^2}{4}.$$

Adding these three, we have

$$\alpha^2 + \beta^2 + \gamma^2 = \tfrac{1}{8}\left(a^2 + b^2 + c^2\right) \quad \text{or,} \quad 8\alpha^2 = b^2 + c^2 - a^2, \text{ etc.}$$

Further it can be easily shown that the coordinates of A, B, C are

$$(-\alpha, \beta, \gamma), (\alpha, -\beta, \gamma) \quad \text{and} \quad (\alpha, \beta, -\gamma).]$$

ANSWERS

1. (i) $2, -2, 2$; (ii) $3, 2, 1$.

2. (i) $\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}; 1$; (ii) $\frac{-4}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{-2}{\sqrt{29}}; \frac{12}{\sqrt{29}}$.

3. (a) $\frac{2}{\sqrt{29}}x - \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{5}{\sqrt{29}}$; (b) $-x = 5$.

4. (i) $3x+2y-7z-12=0$; (ii) $3x-4y+7z-15=0$; (iii) $3x+4y-2z-8=0$; (iv) $y=2$; (v) $51x+15y-50z+173=0$; (vi) $20x-8y+16z+3=0$; (vii) $x-y-2=0$; (viii) $7y+4z-5=0$; (ix) $7x+3y-5z+3=0$; (x) $3x+4y-5z-9=0$; (xi) $10y-12z+9=0$; (xii) $x+z-5=0$.

5. (a) $\frac{x}{3} - \frac{y}{5} + \frac{z}{15} = 1$; (b) $\frac{x}{10} - \frac{y}{4} + \frac{3z}{20} = 1$. 6. $4x - 3y + 2z - 3 = 0$.

7. (a) $\frac{\pi}{3}$; (b) $\frac{\pi}{2}$. 8. (a) $\frac{3}{\sqrt{6}}, \frac{5}{\sqrt{6}}$, opposite sides; (b) $(0,1,0), \left(0, -\frac{19}{2}, 0\right)$.

9. $x^2+y^2+z^2=9$. 10. $12x-4y-3z=169$. 11. $6x+3y-2z=18, 2x-3y-6z=6$.

12. (a) $4x+38y-62z-39=0$ (acute angle), $22x+14y+10z-195=0$; (b) $x+35y-10z-156=0$.

13. $16x+12y-15z+175=0, 16x+12y-15z-25=0$.

16. $\tan^{-1}\left(\frac{\sqrt{185}}{16}\right)$. 17. $4x-5y+3z-3=0$.

Chapter 4

Straight Line

4.10 To find the equation of a line passing through a given point with the given direction cosines

Let (x_1, y_1, z_1) be the given point and l, m, n be proportional to d.cs. of the line. If (x, y, z) be any point on the line, then d.cs. are proportional to $x - x_1, y - y_1, z - z_1$.

Hence the equations of the line are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

This is known as the equations of a line in *symmetrical form* or the *canonical equations* or *general form* of a line.

Corollary 1. Equation of a line through the origin.

If the line passes through the origin, the equations of the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

Corollary 2. Equations of the line through two given points.

If the line passes through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2), then

$$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n}.$$

Therefore, the equations of the line are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Corollary 3. Any point on the line.

Let

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r.$$

Then $x = lr + x_1, y = mr + y_1, z = nr + z_1$. These are called the *parametric equations* of the line.

Thus any point on the line is $(lr + x_1, mr + y_1, nr + z_1)$.

Note. If (l, m, n) are the actual direction cosines of the line, then the point $(lr + x_1, mr + y_1, nr + z_1)$ is at a distance r from the point (x_1, y_1, z_1) on the line.

4.11 To find the equation of the line of intersection of two planes in symmetrical form

Let the planes be

$$a_1x + b_1y + c_1z + d_1 = 0 \tag{1}$$

$$\text{and} \quad a_2x + b_2y + c_2z + d_2 = 0. \tag{2}$$

The equations (1) and (2) together represent the straight line of intersection of the planes, since they are satisfied by any point on the line of intersection of the planes. It is the *unsymmetrical form* general form of a line.

Since the line lies on both the planes, it must be perpendicular to the normals to the planes. If l, m, n be the d.cs. of the line, then $la_1 + mb_1 + nc_1 = 0$ and $la_2 + mb_2 + nc_2 = 0$.

From these two relations,

$$\frac{l}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}.$$

If the line meets the plane $z = 0$ at $(x', y', 0)$, then

$$a_1x' + b_1y' + d_1 = 0 \quad \text{and} \quad a_2x' + b_2y' + d_2 = 0.$$

From these two,

$$\frac{x'}{b_1d_2 - b_2d_1} = \frac{y'}{d_1a_2 - d_2a_1} = \frac{1}{a_1b_2 - a_2b_1}.$$

$\therefore$ the d.rs. of the line are

$$b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1$$

and the line passes through

$$\left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, 0\right).$$

Hence the equation of the common line in symmetrical form is

$$\frac{x - \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}}{b_1c_2 - b_2c_1} = \frac{y - \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}.$$

Example 1. *Show that the line $2x + 2y - z - 6 = 0 = 2x + 3y - z - 8$ is parallel to a coordinate plane and find the equation of the plane normal to this line and passing through the point where this line meets the plane $x = 0$.*

Let l, m, n be the d.cs. of the line. Then $2l + 2m - n = 0$ and $2l + 3m - n = 0$. By cross-multiplication, $\frac{l}{1} = \frac{m}{0} = \frac{n}{2}$.

Since the y-d.c. is zero, the line is perpendicular to y-axis or it is parallel to ZOX-plane.

Let the line meet the plane $x = 0$ at $(0, y', z')$.

$\therefore\ 2y' - z' - 6 = 0, 3y' - z' - 8 = 0$.

From these two, $y' = 2, z' = -2$.

Now we find out the equation of the plane passing through $(0, 2, -2)$ and normal to the line whose d.rs. are $1, 0, 2$. The equation of the plane is

$$x + 2(z + 2) = 0 \quad \text{or,} \quad x + 2z + 4 = 0.$$

Example 2. *Find the equations of projection of the line $3x - y + 2z - 1 = 0 = x + 2y - z - 2$ on the plane $3x + 2y + z = 0$ in the symmetrical form.*

Any plane through the given line is

$$3x - y + 2z - 1 + \lambda(x + 2y - z - 2) = 0,$$
$$\text{or,} \quad (3 + \lambda)x + (2\lambda - 1)y + (2 - \lambda)z - (2\lambda + 1) = 0,$$

where λ is an arbitrary constant.

It will be perpendicular to $3x + 2y + z = 0$, if

$$3 \cdot (3 + \lambda) + 2 \cdot (2\lambda - 1) + 1 \cdot (2 - \lambda) = 0, \quad \text{i.e.} \quad \lambda = -\frac{3}{2}.$$

Thus the equation of the plane is

$$3x - 8y + 7z + 4 = 0. \tag{1}$$

The equations of projection are given by (1) and $3x + 2y + z = 0$.

Let l, m, n be the d.cs. of the line of projection. Then

$$3l - 8m + 7n = 0 \quad \text{and} \quad 3l + 2m + n = 0.$$

By cross-multiplication,

$$\frac{l}{-22} = \frac{m}{18} = \frac{n}{30}, \quad \text{i.e.} \quad \frac{l}{11} = \frac{m}{-9} = \frac{n}{-15}.$$

Let the line cut the planes $z = 0$ at $(x_1, y_1, 0)$. Then

$$3x_1 - 8y_1 + 4 = 0 \quad \text{and} \quad 3x_1 + 2y_1 = 0.$$

From these equations $x_1 = -\frac{4}{15}, y_1 = \frac{2}{5}$.

Hence the line of projection in symmetrical form is

$$\frac{x + \frac{4}{15}}{11} = \frac{y - \frac{2}{5}}{-9} = \frac{z}{-15}.$$

4.20 The plane and a straight line

Let the equations of the plane and the straight line be

$$ax + by + cz + d = 0 \tag{1}$$

$$\text{and} \quad \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}. \tag{2}$$

Any point on the line is $(lr + x_1, mr + y_1, nr + z_1)$.

If the line meets the plane at this point, then

$$a(lr + x_1) + b(mr + y_1) + c(nr + z_1) + d = 0 \tag{3}$$

r is proportional to the distance of the point from (x_1, y_1, z_1).

Thus the line will be parallel to the plane, if

$$al + bm + cn = 0.$$

If the line lies on the plane, then

$$al + bm + cn = 0 \quad \text{and} \quad ax_1 + by_1 + cz_1 + d = 0.$$

If the line is perpendicular to the plane, then it is parallel to the normal to the plane. In this case, $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

If θ is the angle between the plane and the straight line, then $90° - \theta$ is the angle between the line and the normal to the plane. Thus

$$\cos(90° - \theta) = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}},$$

$$\text{i.e.} \quad \theta = \sin^{-1}\left[\frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}\right].$$

Example 3. *Prove that the distance of the point of intersection of the line* $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ *and the plane* $x - y + z = 5$ *from the point* $(-1, -5, -10)$ *is* 13.

[CH 2009]

Any point on the line is $(3r + 2, 4r - 1, 12r + 2)$.

If the line meets the plane at this point, then

$$3r + 2 - (4r - 1) + 12r + 2 + 5 = 0 \quad \text{or,} \quad 11r = 0, r = 0.$$

$\therefore$ the point of intersection of the line and the plane is $(2, -1, 2)$.

Now the distance between $(2, -1, 2)$ and $(-1, -5, -10)$

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = \sqrt{169} = 13.$$

4.21 The necessary and sufficient condition for the coplanarity of two non-parallel straight lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \tag{1}$$

and
$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \tag{2}$$

is that

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

(i) **The condition is necessary.**

Let the lines be coplanar and lie on $ax+by+cz+d=0$.

$$\therefore al_1+bm_1+cn_1=0, \tag{3}$$

$$al_2+bm_2+cn_2=0, \tag{4}$$

$$ax_1+by_1+cz_1+d=0, \tag{5}$$

and
$$ax_2+by_2+cz_2+d=0. \tag{6}$$

From (5) and (6)

$$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0. \tag{7}$$

Eliminating a, b, c from (7), (3) and (4), we get

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

(ii) **The condition is sufficient.**

If the condition holds, then the system of homogeneous equations

$$(x_2-x_1)x+(y_2-y_1)y+(z_2-z_1)z=0, \tag{8}$$

$$l_1x+m_1y+n_1z=0 \tag{9}$$

and
$$l_2x+m_2y+n_2z=0 \tag{10}$$

has non-trivial solutions.

Let (α,β,γ) be a non-zero solution.

Then

$$(x_2-x_1)\alpha+(y_2-y_1)\beta+(z_2-z_1)\gamma=0, \tag{11}$$

$$l_1\alpha+m_1\beta+n_1\gamma=0 \tag{12}$$

and
$$l_2\alpha+m_2\beta+n_2\gamma=0. \tag{13}$$

From (11),

$$\alpha x_1 + \beta y_1 + \gamma z_1 = \alpha x_2 + \beta y_2 + \gamma z_2 = k \text{ (say)},$$

$$\text{i.e. } \alpha x_1 + \beta y_1 + \gamma z_1 - k = 0 \tag{14}$$

$$\text{and } \alpha x_2 + \beta y_2 + \gamma z_2 - k = 0. \tag{15}$$

The conditions (12) and (14) suggest that the line (1) lies on

$$\alpha x + \beta y + \gamma z - k = 0. \tag{16}$$

By the conditions (13) and (15) the line (2) also lies on the plane (16). Hence the lines are coplanar.

Corollary. The equation of the plane on which the lines (1) and (2) lie, is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

provided the condition of coplanarity is satisfied.

Note. The condition of coplanarity is the necessary condition for the intersection of two lines but not sufficient. Conversely if two lines intersect, they must be coplanar.

Example 4. *Prove that*

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \textit{and} \quad \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

are coplanar. Find also the equation of the plane.

Since

$$\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 3 & 4 & 5 \end{vmatrix} = 0.$$

(adding the elements of the first row with those of second row, the two rows become identical), the lines are coplanar.

The equation of the plane is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$$\text{or,} \quad -(x-1) + 2(y-2) - (z-3) = 0 \quad \text{or,} \quad x - 2y + z = 0.$$

4.22 Condition for the coplanarity of two lines in general form

Let the two lines be

$$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2 \tag{1}$$

$$\text{and} \quad a_3x + b_3y + c_3z + d_3 = 0 = a_4x + b_4y + c_4z + d_4. \tag{2}$$

Any planes through the lines (1) and (2) are

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

$$\text{and} \quad a_3x + b_3y + c_3z + d_3 + \mu(a_4x + b_4y + c_4z + d_4) = 0$$

$$\text{or,} \quad (a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + d_1 + \lambda d_2 = 0 \tag{3}$$

$$\text{and} \quad (a_3 + \mu a_4)x + (b_3 + \mu b_4)y + (c_3 + \mu c_4)z + d_3 + \mu d_4 = 0. \tag{4}$$

If the lines (1) and (2) are coplanar, then the planes (3) and (4) must be identical for certain values of λ and μ.

Let the planes (3) and (4) be identical. Then

$$\frac{a_1 + \lambda a_2}{a_3 + \mu a_4} = \frac{b_1 + \lambda b_2}{b_3 + \mu b_4} = \frac{c_1 + \lambda c_2}{c_3 + \mu c_4} = \frac{d_1 + \lambda d_2}{d_3 + \lambda d_4} = k \text{ (say)}.$$

From these equalities,

$$a_1 + \lambda a_2 - ka_3 - k\mu a_4 = 0, \tag{5}$$

$$b_1 + \lambda b_2 - kb_3 - k\mu b_4 = 0, \tag{6}$$

$$c_1 + \lambda c_2 - kc_3 - k\mu c_4 = 0 \tag{7}$$

$$\text{and} \quad d_1 + \lambda d_2 - kd_3 - k\mu d_4 = 0. \tag{8}$$

Eliminating $\lambda, -k, -k\mu$ from (5), (6), (7) and (8), we have

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = 0.$$

It is the required condition.

[The condition can be obtained in the following way:

Let the lines intersect at (x_1, y_1, z_1). Then

$$a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0,$$

$$a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0,$$

$$a_3x_1 + b_3y_1 + c_3z_1 + d_3 = 0$$

$$\text{and} \quad a_4x_1 + b_4y_1 + c_4z_1 + d_4 = 0.$$

Eliminating x_1, y_1, z_1 from the above relations,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0 \quad \text{It is the condition.]}$$

Example 5. *Show that the lines*

$$x + 2y - 5z + 9 = 0 = 3x - y + 2z - 5$$
$$\text{and} \quad 2x + 3y - z - 3 = 0 = 4x - 5y + z + 3$$

are coplanar and find the point of intersection.

If l, m, n are the d.cs. of the first line, then

$$l + 2m - 5n = 0 \quad \text{and} \quad 3l - m + 2n = 0.$$

From these $\frac{l}{1} = \frac{m}{17} = \frac{n}{7}$.

Let the line meet the plane $z = 0$ at $(\alpha, \beta, 0)$. This point lies on the planes of the line.

$\therefore\ \alpha + 2\beta + 9 = 0 \quad \text{and} \quad 3\alpha - \beta - 5 = 0.$

From these equations

$$\alpha = \frac{1}{7},\ \beta = -\frac{32}{7}.$$

$\therefore$ the equations of the line in symmetrical form are

$$\frac{x - \frac{1}{7}}{1} = \frac{y + \frac{32}{7}}{17} = \frac{z}{7}.$$

Any point on this line is $\left(r + \frac{1}{7}, 17r - \frac{32}{7}, 7r\right)$.

Let the two lines meet at this point. Then this point should satisfy the equations of the planes of the second line.

$$\therefore 2\left(r + \frac{1}{7}\right) + 3\left(17r - \frac{32}{7}\right) - 7r - 3 = 0 \quad \text{or,} \quad r = \frac{5}{14}$$
$$\text{and} \quad 4\left(r + \frac{1}{7}\right) - 5\left(17r - \frac{32}{7}\right) + 7r + 3 = 0 \quad \text{or,} \quad r = \frac{5}{14}.$$

Since these two values of r are equal, the lines intersect. Consequently they are coplanar. The point of intersection is $\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right)$.

4.30 Perpendicular distance of a point from a line

Let $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ be the equations of the line AN and $P(\alpha, \beta, \gamma)$ be a given point.

Let PN be perpendicular to the line AN and the coordinates of A be (x_1, y_1, z_1). The d.cs. of AN are

$$\frac{l}{\sqrt{l^2 + m^2 + n^2}}, \frac{m}{\sqrt{l^2 + m^2 + n^2}}, \frac{n}{\sqrt{l^2 + m^2 + n^2}}.$$

The d.cs. of AP are

$$\frac{\alpha - x_1}{AP}, \frac{\beta - y_1}{AP}, \frac{\gamma - z_1}{AP}.$$

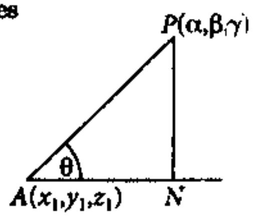

Fig. 20

If θ be the angle between AN and AP, then

$$PN = AP\sin\theta = AP\sqrt{1-\cos^2\theta}$$

$$= AP\sqrt{\left[1 - \frac{\{l(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)\}^2}{(l^2+m^2+n^2)\cdot AP^2}\right]}$$

$$= \sqrt{\left[AP^2 - \frac{\{l(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)\}^2}{l^2+m^2+n^2}\right]}$$

$$= \sqrt{\left[(\alpha - x_1)^2 + (\beta - y_1)^2 + (\gamma - z_1)^2 - \frac{\{l(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)\}^2}{l^2+m^2+n^2}\right]}$$

Example 6. *Find the distance of the point* $(3, 2, 1)$ *from the line*

$$\frac{x-1}{3} = \frac{y}{4} = \frac{z-2}{1}.$$

The required distance

$$= \sqrt{\left[(3-1)^2 + 2^2 + (1-2)^2 - \frac{\{3\cdot(3-1) + 4\cdot 2 + 1\cdot(1-2)\}^2}{3^2+4^2+1^2}\right]}$$

$$= \left[9 - \frac{169}{26}\right]^{\frac{1}{2}} = \left[9 - \frac{13}{2}\right]^{\frac{1}{2}} = \sqrt{\frac{5}{2}}.$$

4.40 Shortest distance between two skew lines

Definition. *If two non-parallel lines do not intersect, then they are called skew lines. These are non-coplnar lines.*

Let the equations of the skew lines LA and MB be

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \tag{1}$$

$$\text{and} \quad \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \text{ respectively.} \tag{2}$$

Let LM be the shortest distance between them. Therefore, LM is perpendicular to both the lines. Let the coordinates of A and B be (x_1, y_1, z_1) and (x_2, y_2, z_2) and the d.cs. of LM be l, m, n.

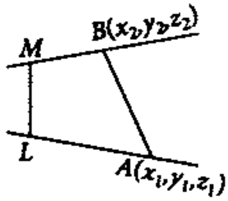

Fig. 21

Since LM is perpendicular to both the lines

$$ll_1 + mm_1 + nn_1 = 0$$

$$\text{and} \quad ll_2 + mm_2 + nn_2 = 0.$$

By cross-multiplication,

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$
$$= \pm\frac{1}{\sqrt{\sum(m_1n_2 - m_2n_1)^2}}. \qquad (3)$$

The d.cs. of AB are $\frac{x_2-x_1}{AB}, \frac{y_2-y_1}{AB}, \frac{z_2-z_1}{AB}$.

LM is the projection of AB on LM. If θ be the angle between AB and LM, then $LM = AB\cos\theta$

$$= AB \cdot \frac{l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)}{AB}$$
$$= \pm\frac{\sum(x_2 - x_1)(m_1n_2 - m_2n_1)}{\sqrt{\sum(m_1n_2 - m_2n_1)^2}} \quad \text{[by(3)]}$$
$$= \pm\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\left[\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2\right]^{\frac{1}{2}}}.$$

[Sign will be adjusted to make the distance positive.]

Corollary I. Equation of the line of s.d.

S.d. is the line of intersection of the planes ALM and BLM. Therefore, the equation of it is.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}.$$

Corollary II. Points of intersection of s.d. and the lines.

Let $(l_1r_1 + x_1, m_1r_1 + y_1, n_1r_1 + z_1)$ and $(l_2r_2 + x_2, m_2r_2 + y_2, n_2r_2 + z_2)$ be any two points on the lines. If these points are the points of intersection, then the d.rs. of the s.d. can be calculated by the help of these two points. Applying the condition of perpendicularity of it with the lines, two equations of r_1 and r_2 are obtained. From these equations r_1 and r_2 are also obtained. Consequently the length of s.d., the points of intersection and its equation are found out.

Corollary III. S.d. between two lines when one is in general form and the other in symmetrical form.

Let the given lines be

$$u = 0 = v \qquad (1)$$

$$\text{and} \quad \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}. \qquad (2)$$

To find the s.d. the plane $\alpha = u + \lambda v = 0$ is determined in such a way that $\alpha = 0$ is parallel to the line (2). The s.d. is the distance of the point (x_1, y_1, z_1) from the plane $\alpha = 0$. The equations of the line of s.d. are the planes through the given lines and perpendicular to $\alpha = 0$.

Note. Here s.d. can be found out by writing the line $u = 0 = v$ in symmetrical form.

Corollary IV. S.d. between two lines when both are in general form.

Let the lines be $u_1 = 0 = v_1$ and $u_2 = 0 = v_2$.

The values of λ_1 and λ_2 are determined in such a way that the planes $u_1 + \lambda_1 v_1 = 0$ and $u_2 + \lambda_2 v_2 = 0$ are parallel. The s.d. is the distance between these two parallel planes. The equations of the line of s.d. are the planes through the given lines and perpendicular to any one of the parallel planes.

Note. S.d. can also be found out by expressing the given lines in symmetrical form.

Example 7. *Find the s.d. between the lines*

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \tag{1}$$

$$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}. \tag{2}$$

Find also the equations and the points of intersection in which it meets the lines.

Let the s.d. meet (1) and (2) at

$$(3r_1 + 3, -r_1 + 8, r_1 + 3) \quad \text{and} \quad (-3r_2 - 3, 2r_2 - 7, 4r_2 + 6)$$

respectively. The d.rs. of the s.d. are $3r_1 + 3r_2 + 6, -r_1 - 2r_2 + 15, r_1 - 4r_2 - 3$.

Since s.d. is perpendicular to (1) and (2),

$$3(3r_1 + 3r_2 + 6) - 1 \cdot (-r_1 - 2r_2 + 15) + 1 \cdot (r_1 - 4r_2 - 3) = 0$$

$$\text{or,} \quad 11r_1 + 7r_2 = 0 \tag{3}$$

$$\text{and} \quad -3 \cdot (3r_1 + 3r_2 + 6) + 2 \cdot (-r_1 - 2r_2 + 15) + 4 \cdot (r_1 - 4r_2 - 3) = 0$$

$$\text{or,} \quad 7r_1 + 29r_2 = 0. \tag{4}$$

From (3) and (4), $r_1 = r_2 = 0$.

$\therefore$ the points of intersection are $(3, 8, 3)$ and $(-3, -7, 6)$.

Hence the length of s.d.$= \sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = 3\sqrt{30}$.

The d.rs. of s.d. are $6, 15, -3$ and the equations of s.d. are

$$\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{2}.$$

Example 8. *Find the magnitude and position of the line of s.d. between the lines*

$$\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2}$$

and $5x - 2y - 3z + 6 = 0 = x - 3y + 2z - 3.$ [BH 2009; CH 2002]

The equation of a plane through the second line is

$$(5x - 2y - 3z + 6) + \lambda(x - 3y + 2z - 3) = 0$$
$$\text{or,}\quad (5+\lambda)x + (-2-3\lambda)y + (-3+2\lambda)z + 6 - 3\lambda = 0.$$

If it is parallel to the first line, then

$$4(5+\lambda) - 3(2+3\lambda) - 2(3-2\lambda) = 0 \quad \text{or,} \quad \lambda = 8.$$

In this case, the equation of the plane is

$$13x - 26y + 13z - 18 = 0. \tag{1}$$

S.d. = distance of $(0,-1,2)$ from (1) $= \frac{26+26-18}{13\sqrt{1+4+1}} = \frac{34}{13\sqrt{6}} = \frac{17\sqrt{6}}{39}$.
The plane through the first line and perpendicular to (1) is

$$\begin{vmatrix} x & y+1 & z-2 \\ 4 & 3 & 2 \\ 13 & -26 & 13 \end{vmatrix} = 0 \quad \text{or,} \quad 7x - 2y - 11z + 20 = 0. \tag{2}$$

The equation of a plane through the second line is

$$(5+\mu)x - (2+3\mu)\,y + (-3+2\mu)z + 6 - 3\mu = 0.$$

If it is perpendicular to (1), then

$$13\cdot(5+\mu) + 26\cdot(2+3\mu) + 13\cdot(-3+2\mu) = 0 \quad \text{or,} \quad \mu = -\frac{2}{3}.$$

The equation of the plane is

$$13x - 13z + 24 = 0. \tag{3}$$

Planes (2) and (3) give the equations of s.d.

Example 9. *Find the magnitude and the position of the line of s.d. between the lines*

$$2x + y - z = 0 = x - y + 2z$$
$$\textit{and}\quad x + 2y - 3z - 4 = 0 = 2x - 3y + 4z - 5.$$

Equations of planes through the given lines are

$$2x + y - z + \lambda(x - y + 2z) = 0 \tag{1}$$
$$\text{and}\quad x + 2y - 3z - 4 + \mu(2x - 3y + 4z - 5) = 0. \tag{2}$$

If these planes are parallel, then

$$\frac{2+\lambda}{1+2\mu} = \frac{1-\lambda}{2-3\mu} = \frac{-1+2\lambda}{-3+4\mu}.$$

Solving for λ and μ, $\lambda = -1, \mu = 0$.

The planes are

$$x + 2y - 3z = 0 \tag{3}$$

$$\text{and} \quad x + 2y - 3z - 4 = 0. \tag{4}$$

A point on (3) is $(0, 0, 0)$.

S.d. = distance between the parallel planes (3) and (4) $= \frac{4}{\sqrt{1+4+9}} = \frac{4}{\sqrt{14}}$.

If the planes (1) and (2) are perpendicular to (3), then

$$(2+\lambda)\cdot 1 + (1-\lambda)\cdot 2 + (-1+2\lambda)\cdot(-3) = 0 \quad \text{or,} \quad \lambda = 1$$

$$\text{and} \quad (1+2\mu)\cdot 1 + (2-3\mu)\cdot 2 + (-3+4\mu)\cdot(-3) = 0 \quad \text{or,} \quad \mu = \frac{7}{8}.$$

For these values of λ and μ, the planes are

$$3x + z = 0 \quad \text{and} \quad 22x - 5y + 4z - 67 = 0.$$

These planes give the equation of s.d.

Example 10. *Show that the s.d. between the lines* $y = az + b, z = \alpha x + \beta$;

$y = a'z + b', z = \alpha' x + \beta'$ *is* $\left|\frac{(\alpha - \alpha')(b - b') - (\alpha'\beta - \alpha\beta')(a - a')}{[\alpha^2\alpha'^2(a - a')^2 + (\alpha - \alpha')^2 + (a\alpha - a'\alpha')^2]^{1/2}}\right|$.

The line $y = az + b, z = \alpha x + \beta$ in symmetrical form is

$$\frac{x + \beta/\alpha}{1/\alpha} = \frac{y - b}{a} = \frac{z}{1}$$

and the line $y = a'z + b', z = \alpha' x + \beta'$ in symmetrical form is

$$\frac{x + \beta'/\alpha'}{1/\alpha'} = \frac{y - b'}{a'} = \frac{z}{1}.$$

$$\text{S.d.} = \begin{vmatrix} \frac{\beta'}{\alpha'} - \frac{\beta}{\alpha} & b - b' & 0 \\ \frac{1}{\alpha} & a & 1 \\ \frac{1}{\alpha'} & a' & 1 \end{vmatrix} \div \left[(a - a')^2 + \left(\frac{1}{\alpha'} - \frac{1}{\alpha}\right)^2 + \left(\frac{a'}{\alpha} - \frac{a}{\alpha'}\right)^2\right]^{1/2}$$

$$= \begin{vmatrix} \alpha\beta' - \alpha'\beta & b - b' & 0 \\ \alpha' - \alpha & a - a' & 0 \\ \alpha & a' & 1 \end{vmatrix} \div \left[\alpha^2\alpha'^2(a - a')^2 + (\alpha - \alpha')^2 + (a'\alpha' - a\alpha)^2\right]^{1/2}$$

$$= \frac{(\alpha - \alpha')(b - b') - (\alpha\beta' - \alpha'\beta)(a - a')}{[\alpha^2\alpha'^2(a - a')^2 + (\alpha - \alpha')^2 + (a'\alpha' - a\alpha)^2]^{1/2}}.$$

It proves the result.

4.41 Equations of two skew lines in simplified form w.r.t. suitable coordinate axes

Let LM be the s.d. between two skew lines AB and CD. Let $LM = 2c$ and the angle between them $= 2\alpha$.

The midpoint of LM is chosen as the origin O. OP and OQ are drawn parallel to AB and CD respectively. The internal and external bisectors of the angle POQ i.e. OX and OY are chosen as the x-axis and y-axis respectively. OL is taken as the z-axis.

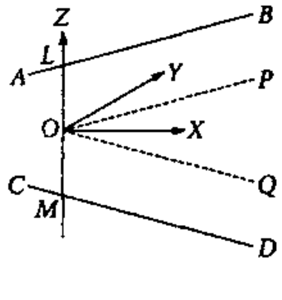

Fig. 22

In this system of axes the line OP or AB makes angles $\alpha, \frac{\pi}{2}-\alpha, \frac{\pi}{2}$ with x, y, z-axes respectively and the line OQ or CD makes angles $-\alpha, \frac{\pi}{2}+\alpha, \frac{\pi}{2}$ with x, y, z-axes respectively. Therefore, the d.cs. of AB are $\cos\alpha, \cos(\pi/2-\alpha), \cos\pi/2$ i.e. $\cos\alpha, \sin\alpha, 0$ and those of CD are $\cos(-\alpha), \cos(\pi/2+\alpha), \cos\pi/2$ i.e. $\cos\alpha, -\sin\alpha, 0$. The coordinates of L and M are $(0,0,c)$ and $(0,0,-c)$. Now the equations of AB and CD are

$$\frac{x-0}{\cos\alpha} = \frac{y-0}{\sin\alpha} = \frac{z-c}{0} \quad \text{and} \quad \frac{x-0}{\cos\alpha} = \frac{y-0}{-\sin\alpha} = \frac{z+c}{0}$$

$$\text{or,} \quad y = x\tan\alpha, z = c \quad \text{and} \quad y = -x\tan\alpha, z = -c \quad \text{respectively.}$$

Note 1. If we put $\tan\alpha = m$, then the equations of the skew lines are $y = mx, z = c$ and $y = -mx, z = -c$.

Note 2. The parametric coordinates of a point on AB are $(r, r\tan\alpha, c)$ and those of a point on CD are $(r', -r'\tan\alpha, -c)$.

Example 11. *Find the surface generated by the lines which intersect the lines $y = mx, z = c; y = -mx, z = -c$ and x-axis.*

The equations of a line intersecting the given skew lines can be written as

$$(y-mx) + \lambda_1(z-c) = 0, \tag{1}$$

$$(y+mx) + \lambda_2(z+c) = 0. \tag{2}$$

Let this line meet the x-axis. On the x-axis $y = 0, z = 0$. Putting $y = 0, z = 0$ in (1) and (2),

$$-mx - \lambda_1 c = 0 \quad \text{and} \quad mx + \lambda_2 c = 0 \quad \text{or,} \quad x = -\frac{c\lambda_1}{m} = -\frac{c\lambda_2}{m}.$$

$$\therefore\ \lambda_1 = \lambda_2. \tag{3}$$

Eliminating λ_1 and λ_2 from (1), (2) and (3), we have

$$\frac{y-mx}{z-c} = \frac{y+mx}{z+c}$$

$$\text{or,} \quad (y-mx)(z+c) = (y+mx)(z-c) \quad \text{or,} \quad mzx = cy.$$

It is the equation of the required surface.

Example 12. *A line of constant length 2k has its extremities on two fixed lines. Prove that the locus of its midpoint is an ellipse whose axes are equally inclined to the fixed line.*

Let the fixed lines be

$$\frac{x}{1} = \frac{y}{m} = \frac{z-c}{0} = r \qquad (1)$$

$$\text{and} \quad \frac{x}{1} = \frac{y}{-m} = \frac{z+c}{0} = r'. \qquad (2)$$

The points on the lines (1) and (2) are (r, mr, c) and $(r', -mr', -c)$. If these are the end points of the variable line at one position and (α, β, γ) are the coordinates of the midpoint of it then

$$2\alpha = r + r', 2\beta = m(r - r'), 2\gamma = 0. \qquad (3)$$

Since the length of the variable line is $2k$,

$$(r - r')^2 + m^2(r + r')^2 + (c + c)^2 = 4k^2$$

$$\text{or,} \quad \frac{4\beta^2}{m^2} + 4m^2\alpha^2 + 4c^2 = 4k^2$$

$$\text{or,} \quad m^2\alpha^2 + \frac{\beta^2}{m^2} = k^2 - c^2.$$

Hence the required locus is

$$m^2x^2 + \frac{y^2}{m^2} = k^2 - c^2, z = 0.$$

It is an ellipse in the xy-plane and its axes (x-axis, y-axis) are equally inclined to the fixed lines.

4.42 (i) To find the locus of the line intersecting two given lines and a given curve

Let the lines be $u_1 = 0 = v_1$ and $u_2 = 0 = v_2$.

Any line intersecting these lines is given by

$$u_1 + k_1v_1 = 0 = u_2 + k_2v_2. \qquad (1)$$

Let the given curve be

$$f(x, y, z) = 0. \qquad (2)$$

If the line (1) intersects this curve, then a relation of the form $\phi(k_1, k_2) = 0$ is obtained by the help of (1) and (2).

Now eliminating k_1 and k_2 by (1), the required locus is obtained.

(ii) To find the locus of a line under given conditions

Let the line be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n},$$

where (x_1, y_1, z_1) is any point on the line.

Eliminating l, m, n by the help of given conditions a relation of the form $f(x_1, y_1, z_1) = 0$ is obtained. Since (x_1, y_1, z_1) is any point on the line, the required locus is $f(x, y, z) = 0$.

Example 13. *Prove that the locus of a line which meets two lines $y = \pm mx, z = \pm c$ and the circle $x^2 + y^2 = a^2, z = 0$ is*

$$c^2m^2(cy-mzx)^2+c^2(yz-cmx)^2=a^2m^2(z^2-c^2)^2.$$

Let the equation of a line which meets the given two skew lines be

$$y-mx+k_1(z-c)=0, \tag{1}$$
$$y+mx+k_2(z+c)=0. \tag{2}$$

This line meets the circle $x^2 + y^2 = a^2, z = 0$.

Putting $z = 0$ in (1) and (2),

$$y-mx=ck_1, y+mx=-ck_2$$
$$\text{or,}\quad y=\frac{c(k_1-k_2)}{2}, x=-\frac{c(k_1+k_2)}{2m}.$$

Putting these values in $x^2 + y^2 = a^2$,

$$\frac{c^2(k_1+k_2)^2}{4m^2}+\frac{c^2(k_1-k_2)^2}{4}=a^2.$$

Eliminating k_1 and k_2 by (1) and (2)

$$\frac{c^2\cdot 4(yz-mcx)^2}{4m^2(z^2-c^2)^2}+\frac{c^2\cdot 4(mzx-cy)^2}{4(z^2-c^2)^2}=a^2$$
$$\text{or,}\quad c^2m^2(cy-mzx)^2+c^2(yz-mcx)^2=a^2m^2(z^2-c^2)^2.$$

Example 14. *Show that the locus of lines which meet the lines*

$$\frac{x+a}{0}=\frac{y}{\sin\alpha}=\frac{z}{-\cos\alpha},\quad \frac{x-a}{0}=\frac{y}{\sin\alpha}=\frac{z}{\cos\alpha}$$

at the same angle is $(xy\cos\alpha - az\sin\alpha)(zx\sin\alpha - ay\cos\alpha) = 0$.

Let the variable line meet the given lines at

$$(-a, r\sin\alpha, -r\cos\alpha)\quad\text{and}\quad(a, r'\sin\alpha, r'\cos\alpha)$$

in one of its position.

Its equations are

$$\frac{x+a}{2a} = \frac{y - r\sin\alpha}{(r'-r)\sin\alpha} = \frac{z + r\cos\alpha}{(r'+r)\cos\alpha}. \tag{1}$$

Since it meets the given lines at the same angle,

$$\begin{aligned}2a\cdot 0 + (r'-r)\sin\alpha\cdot\sin\alpha + (r'+r)\cos\alpha(-\cos\alpha)\\ = \pm\{2a\cdot 0 + (r'-r)\sin\alpha\cdot\sin\alpha + (r'+r)\cos\alpha\cdot\cos\alpha\}.\end{aligned}$$

For + sign, $(r'+r)\cos^2\alpha = 0$ or, $r' = -r$.

For − sign, $(r'-r)\sin^2\alpha = 0$ or $r' = r$.

If $r' = -r$, the equations of (1) reduces to

$$\frac{x+a}{2a} = \frac{y - r\sin\alpha}{-2r\sin\alpha} = \frac{z + r\cos\alpha}{0}.$$

$$\therefore r = -\frac{z}{\cos\alpha} \quad\text{and}\quad r = -\frac{ay}{x\sin\alpha}.$$

$$\therefore \frac{z}{\cos\alpha} = \frac{ay}{x\sin\alpha} \quad\text{or,}\quad zx\sin\alpha - ay\cos\alpha = 0.$$

Putting $r' = r$ in (1) and eliminating r we get that

$$xy\cos\alpha - az\sin\alpha = 0.$$

$\therefore$ the required locus is

$$(xy\cos\alpha - az\sin\alpha)(zx\sin\alpha - ay\cos\alpha) = 0.$$

Example 15. *A variable line intersects the lines*

$$y = 0, z = c; x = 0, z = -c$$

and is parallel to the plane $lx + my + nz = p$. Prove that the surface generated by it is

$$lx(z-c) + my(z+c) + n(z^2 - c^2) = 0. \qquad \text{[CH 2000]}$$

Any line intersecting the given lines is given by the intersection of two planes

$$\left.\begin{aligned} y + k_1(z-c) = 0\\ \text{and}\quad x + k_2(z+c) = 0.\end{aligned}\right\}. \tag{1}$$

D.rs. of this line are $k_2, k_1, -1$.

Since it is parallel to the given plane

$$lk_2 + mk_1 - n = 0. \tag{2}$$

Eliminating k_1 and k_2 by (1),

$$l\frac{x}{z+c} + m\frac{y}{z-c} + n = 0$$

$$\text{or,}\quad lx(z-c) + my(z+c) + n(z^2 - c^2) = 0.$$

It is the equation of the required surface.

Example 16. *Prove that the locus of the lines which are parallel to the plane $x+y=0$ and which intersect the line $x-y=0=z$ and the curve $x^2=2az, y=0$ is $x^2-y^2=2az$.*

Let the variable line at one position meet the line $x-y=0=z$ and the curve $x^2=2az, y=0$ at the points $(\lambda,\lambda,0)$ and $(at,0,\frac{1}{2}at^2)$ respectively.

Therefore, the equations of the line are

$$\frac{x-\lambda}{at-\lambda}=\frac{y-\lambda}{-\lambda}=\frac{z}{\frac{1}{2}at^2}. \tag{1}$$

This line is parallel to the plane $x+y=0$, if $(at-\lambda)\cdot 1-\lambda\cdot 1+\frac{1}{2}at^2\cdot 0=0$ or, $\lambda=\frac{1}{2}at$.

Putting this value of λ in (1),

$$\frac{x-\frac{1}{2}at}{\frac{1}{2}at}=\frac{y-\frac{1}{2}at}{-\frac{1}{2}at}=\frac{z}{\frac{1}{2}at^2}$$

$$\text{or,}\quad x-\frac{1}{2}at=-\left(y-\frac{1}{2}at\right)=\frac{z}{t}. \tag{2}$$

From the first two of the equations (2),

$$t=\frac{x+y}{a}. \tag{3}$$

From the last two,

$$\frac{1}{2}at^2-yt=z. \tag{4}$$

Eliminating t from (3) and (4), we have

$$\frac{1}{2}a\frac{(x+y)^2}{a^2}-y\cdot\frac{x+y}{a}=z \quad\text{or,}\quad x^2-y^2=2az.$$

It is the required locus.

Example 17. *A variable line always intersects the line $z=0, x=y$ and the circles $x=0, y^2+z^2=d^2; y=0, z^2+x^2=d^2$. Show that the equation to its locus is $(x+y)^2\{z^2+(x-y)^2\}=d^2(x-y)^2$.*

Let the variable line at one position meet the circles $x=0, y^2+z^2=d^2; y=0, z^2+x^2=d^2$ at $(0, d\cos\theta, d\sin\theta)$ and $(d\cos\phi, 0, d\sin\phi)$ respectively.

Therefore, the equations of the line are

$$\frac{x}{d\cos\phi}=\frac{y-d\cos\theta}{-d\cos\theta}=\frac{z-d\sin\theta}{d(\sin\phi-\sin\theta)}. \tag{1}$$

It meets the line $z=0, x=y$. Putting $z=0$ in (1),

$$x=\frac{d\sin\theta\cos\phi}{\sin\theta-\sin\phi}, y=\frac{-d\cos\theta\sin\phi}{\sin\theta-\sin\phi}.$$

Since $x = y, \sin\theta\cos\phi = -\cos\theta\sin\phi$ or, $\sin\theta\cos\phi + \cos\theta\sin\phi = 0$ or, $\sin(\theta+\phi) = 0$ or, $\theta = -\phi$.

Putting $\theta = -\phi$ in (1),

$$\frac{x}{\cos\phi} = \frac{y - d\cos\phi}{-\cos\phi} = \frac{z + d\sin\phi}{2\sin\phi} = k \text{ (say)}.$$

$$\therefore\ x = k\cos\phi, y = (d-k)\cos\phi, z = (2k-d)\sin\phi.$$

Now $x - y = (2k-d)\cos\phi$.

$$\therefore (x-y)^2 + z^2 = (2k-d)^2 = \frac{(x-y)^2}{\cos^2\phi} = \frac{d^2(x-y)^2}{(x+y)^2}$$

$$\text{or,}\quad (x+y)^2[z^2 + (x-y)^2] = d^2(x-y)^2.$$

Hence the result follows.

4.50 Equations of a line intersecting two lines

Case 1. Given lines in symmetric form.

Let the given lines be

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} = r_1 \tag{1}$$

$$\text{and}\quad \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} = r_2. \tag{2}$$

Here r_1 and r_2 are parameters.

Any point on (1) is

$$(l_1r_1 + x_1, m_1r_1 + y_1, n_1r_1 + z_1)$$

and any point on (2) is

$$(l_2r_2 + x_2, m_2r_2 + y_2, n_2r_2 + z_2).$$

The line passing through these two points intersects the given lines. r_1 and r_2 are obtained by the conditions of the problem.

Case 2. Given lines in general form.

Let the given lines be

$$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$$

$$\text{and}\quad a_3x + b_3y + c_3z + d_3 = 0 = a_4x + b_4y + c_4z + d_4.$$

The equations of the required lines are

$$a_1x + b_1y + c_1z + d_1 + \lambda_1(a_2x + b_2y + c_2z + d_2) = 0,$$

$$a_3x + b_3y + c_3z + d_3 + \lambda_2(a_4x + b_4y + c_4z + d_4) = 0.$$

λ_1 and λ_2 are determined by the conditions of the problem.

Example 18. *A line with d.cs. proportional to* $2, 7$, *and* -5 *is drawn to intersect the lines*

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1} \quad and \quad \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}.$$

Find the coordinates of the points of intersection and the length of the intercept between the lines. **[CH 2009]**

Let the line with d.rs. $2, 7, -5$ meet the given lines at

$$(3r_1+5, -r_1+7, r_1-2) \quad \text{and} \quad (-3r_2-3, 2r_2+3, 4r_2+6).$$

Then the d.rs. of the line are

$$-3r_2-3r_1-8, 2r_2+r_1-4, 4r_2-r_1+8.$$

$$\therefore \frac{-3r_2-3r_1-8}{2}=\frac{2r_2+r_1-4}{7}=\frac{4r_2-r_1+8}{-5}.$$

$$\therefore -21r_2-21r_1-56=4r_2+2r_1-8, \tag{1}$$

$$\text{or,} \quad 23r_1+25r_2+48=0$$

$$\text{and} \quad -10r_2-5r_1+20=28r_2-7r_1+56$$

$$\text{or,} \quad r_1-19r_2-18=0. \tag{2}$$

From (1) and (2), $r_1=r_2=-1$.

Thus the points of intersection are $(2, 8, -3)$ and $(0, 1, 2)$.

Intercepted length$=\sqrt{(-2)^2+(1-8)^2+(2+3)^2}=\sqrt{78}$.

Example 19. *Find the equation to the line which intersects the lines*

$$z=5x-6=4y+3 \quad and \quad z=2x-4=3y+5$$

and it is parallel to $\frac{x}{4}=\frac{y}{1}=\frac{z}{1}$.

Equations of the line which intersects the given lines are

$$z-5x+6+\lambda_1(z-4y-3)=0,$$

$$z-2x+4+\lambda_2(z-3y-5)=0,$$

$$\text{i.e.} \quad -5x-4\lambda_1 y+(\lambda_1+1)z+6-3\lambda_1=0,$$

$$-2x-3\lambda_2 y+(\lambda_2+1)z+4-5\lambda_2=0.$$

If it is parallel to $\frac{x}{4}=\frac{y}{1}=\frac{z}{1}$,

$$-5\cdot 4-4\lambda_1\cdot 1+(\lambda_1+1)\cdot 1=0 \quad \text{or,} \quad \lambda_1=-\frac{19}{3}$$

$$\text{and} \quad -2\cdot 4-3\lambda_2\cdot 1+(\lambda_2+1)\cdot 1=0 \quad \text{or,} \quad \lambda_2=-\frac{7}{2}.$$

$\therefore$ the equations of the required line are

$$15x-76y+16z+75=0, 4x-21y+5z-43=0.$$

4.51 Intersection of three planes

To find the condition that the planes

$$a_1x + b_1y + c_1z + d_1 = 0, \tag{1}$$

$$a_2x + b_2y + c_2z + d_2 = 0 \tag{2}$$

$$\text{and} \quad a_3x + b_3y + c_3z + d_3 = 0. \tag{3}$$

(a) *may intersect in a common line;*

(b) *may form a triangular prism i.e. may intersect two by two in three distinct lines parallel to one another and*

(c) *may meet at a single point.*

Let

$$\Delta_1 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}, \quad \Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The equations of the line represented by the planes (1) and (2) in canonical form are

$$\frac{x - \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}}{b_1c_2 - b_2c_1} = \frac{y - \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}. \tag{4}$$

(a) If the planes meet along a line, then the line (4) should lie on the plane (3). To satisfy this condition

$$a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0 \tag{5}$$

$$\text{and} \quad a_3(b_1d_2 - b_2d_1) + b_3(a_2d_1 - a_1d_2) + d_3(a_1b_2 - a_2b_1) = 0. \tag{6}$$

From (5),

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0, \quad \text{i.e.} \quad \Delta_4 = 0$$

and from (6),

$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0, \quad \text{i.e.} \quad \Delta_3 = 0.$$

Hence the conditions for the planes to intersect in a line are $\Delta_4 = 0$ and $\Delta_3 = 0$.

> **Note.** If the canonical form of the line (4) is considered on taking $x = 0$, then $\Delta_1 = 0$ in place of $\Delta_3 = 0$. Similarly on considering $y = 0$ we would get $\Delta_2 = 0$ in place of $\Delta_3 = 0$.
> Thus the three planes will intersect in a line if $\Delta_4 = 0$ and either $\Delta_3 = 0$ or, $\Delta_1 = 0$ or, $\Delta_2 = 0$.

(b) If the planes form a triangular prism, then the line (4) must be parallel to the plane (3). In this case, $\Delta_4 = 0$ and none of Δ_1, Δ_2 and Δ_3 is zero.

(c) If the planes meet at a point, then the line (4) should neither be parallel to the plane (3) nor lie on it. The equations of the planes should be solvable for a unique solution of x, y, z. Hence the condition is that $\Delta_4 \neq 0$.

Example 20. *Show that the planes $x-2y+z-3=0, x+y-2z-3=0, x-z-1=0$ form a triangular prism.*

Here

$$\Delta_4 = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{vmatrix} = 1\cdot(-1+0)+1\cdot(0-2)+1\cdot(4-1) = -3+3 = 0$$

and
$$\Delta_3 = \begin{vmatrix} 1 & -2 & -3 \\ 1 & 1 & -3 \\ 1 & 0 & -1 \end{vmatrix} = 1\cdot(-1+0)+1\cdot(0-2)+1\cdot(6+3) = -1-2+9 = 6.$$

$\therefore$ $\Delta_4 = 0$, but $\Delta_3 \neq 0$, the given planes form a triangular prism.

Example 21. *Prove that the planes $x = ry + qz, y = pz + rx, z = qx + py$ pass through one line, if $p^2+q^2+r^2+2pqr = 1$ and show that the equations of the line are*

$$\frac{x}{\sqrt{1-p^2}} = \frac{y}{\sqrt{1-q^2}} = \frac{z}{\sqrt{1-r^2}}.$$ [BH 92, 95; CH 2006]

The planes are $x - ry - qz = 0, rx - y + pz = 0, qx + py - z = 0$.

Here

$$\Delta_4 = \begin{vmatrix} 1 & -r & -q \\ r & -1 & p \\ q & p & -1 \end{vmatrix} = 1 - p^2 - q^2 - r^2 - 2pqr$$

and
$$\Delta_3 = \begin{vmatrix} 1 & -r & 0 \\ r & -1 & 0 \\ q & p & 0 \end{vmatrix} = 0.$$

The planes intersect in a line, if $\Delta_4 = 0$ and $\Delta_3 = 0$.

$\therefore$ the condition is $p^2+q^2+r^2+2pqr = 1$.

If l, m, n be the d.cs. of the line, then

$$l - rm - qn = 0 \quad \text{and} \quad lr - m + pn = 0.$$

By cross-multiplication,

$$\frac{l}{pr+q} = \frac{m}{qr+p} = \frac{n}{1-r^2}.$$

$$\because \quad p^2+q^2+r^2+2pqr = 1, (pr+q)^2 = (1-p^2)(1-r^2)$$

$$\text{and} \quad (qr+p)^2 = (1-q^2)(1-r^2).$$

$$\therefore \quad \frac{l}{\sqrt{(1-p^2)(1-r^2)}} = \frac{m}{\sqrt{(1-q^2)(1-r^2)}} = \frac{n}{1-r^2}$$

$$\text{or,} \quad \frac{l}{\sqrt{1-p^2}} = \frac{m}{\sqrt{1-q^2}} = \frac{n}{\sqrt{1-r^2}}.$$

The line passes through the origin.

Thus the equations of the line are

$$\frac{x}{\sqrt{1-p^2}} = \frac{y}{\sqrt{1-q^2}} = \frac{z}{\sqrt{1-r^2}}.$$

WORKED-OUT EXAMPLES

1. *Determine the equation of the straight line through the point* $(3, 1, -6)$ *and parallel to each of the planes* $x + y + 2z - 4 = 0$ *and* $2x - 3y + z + 5 = 0$.

   Let the equations of the line be

$$\frac{x-3}{l} = \frac{y-1}{m} = \frac{z+6}{n}. \tag{1}$$

   Since it is parallel to the given planes, the line is perpendicular to the normals to the planes.

   $\therefore$ $l + m + 2n = 0$ and $2l - 3m + n = 0$.

   By cross-multiplication, $\frac{l}{7} = \frac{m}{3} = \frac{n}{-5}$.

   Thus the equations of the line are $\frac{x-3}{7} = \frac{y-1}{3} = \frac{z+6}{-5}$.

2. *Find the equation of the line through the point* $(1, 2, 4)$ *and perpendicular to the line* $3x + 2y - z - 4 = 0 = x - 2y - 2z - 5$.

   If l, m, n be the d.cs. of the given line, then

$$3l + 2m - n = 0 \quad \text{and} \quad l - 2m - 2n = 0.$$

   By cross-multiplication, $\frac{l}{6} = \frac{m}{-5} = \frac{n}{8}$.

   The equation of the plane passing through $(1, 2, 4)$ and perpendicular to the line is

$$6(x-1) - 5(y-2) + 8(z-4) = 0,$$

$$\text{i.e.} \quad 6x - 5y + 8z - 28 = 0. \tag{1}$$

The equation of a plane passing through the given line is

$$3x + 2y - z - 4 + \lambda(x - 2y - 2z - 5) = 0,$$
$$\text{i.e.}\quad (3 + \lambda)x + 2(1 - \lambda)y - (1 + 2\lambda)z - (4 + 5\lambda) = 0.$$

If it passes through the point $(1, 2, 4)$, then

$$(3 + \lambda) + 2(1 - \lambda) \cdot 2 - (1 + 2\lambda) \cdot 4 - (4 + 5\lambda) = 0$$
$$\text{or,}\quad -1 - 16\lambda = 0 \quad \text{or,}\quad \lambda = -\frac{1}{16}.$$

$\therefore$ the equation of the plane is

$$47x + 34y - 14z - 59 = 0. \tag{2}$$

The required line is the line of intersection of the planes (1) and (2) and passing through $(1, 2, 4)$. If l_1, m_1, n_1 are the d.cs. of this line, then

$$6l_1 - 5m_1 + 8n_1 = 0 \quad \text{and} \quad 47l_1 + 34m_1 - 14n_1 = 0.$$

From these two,
$$\frac{l_1}{-202} = \frac{m_1}{460} = \frac{n_1}{439}.$$

Hence the equations of the required line are

$$\frac{x - 1}{-202} = \frac{y - 2}{460} = \frac{z - 4}{439}.$$

3. *Determine the value of k so that the lines*

$$\frac{x - 1}{2} = \frac{y - 4}{1} = \frac{z - 5}{2} \quad \textit{and} \quad \frac{x - 2}{-1} = \frac{y - 8}{k} = \frac{z - 11}{4}$$

may intersect. **[CH 2007; BH 2006]**

A point on the first line is $(2r_1 + 1, r_1 + 4, 2r_1 + 5)$ and a point on the second line is $(-r_2 + 2, kr_2 + 8, 4r_2 + 11)$.

The given lines intersect, if the above points are identical.

In this case

$$2r_1 + 1 = -r_2 + 2, \tag{1}$$
$$r_1 + 4 = kr_2 + 8 \tag{2}$$
$$\text{and}\quad 2r_1 + 5 = 4r_2 + 11. \tag{3}$$

By (1) and (3), $\quad r_1 = 1, r_2 = -1.$

From (2), $5 = -k + 8$ or, $k = 3$.

4. *Find the equation of the projection of the line*

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-4}{4}$$

on the plane $x+3y+z+5=0$.

By projection of a line on a plane, we mean the line of intersection of the given plane and the plane passing through the given line and perpendicular to the given plane.

The given line and the plane are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-4}{4} \tag{1}$$

$$\text{and}\quad x+3y+z+5=0. \tag{2}$$

Equation of a plane through the line (1) is

$$a(x-1)+b(y-2)+c(z-4)=0, \tag{3}$$

where

$$2a+3b+4c=0. \tag{4}$$

If the plane (3) is perpendicular to the plane (2), then

$$a+3b+c=0. \tag{5}$$

From (4) and (5), $\frac{a}{-9}=\frac{b}{2}=\frac{c}{3}$.

Thus the equation of the plane containing the line (1) and perpendicular to the plane (2) is

$$-9(x-1)+2(y-2)+3(z-4)=0 \quad \text{or,} \quad 9x-2y-3z+7=0.$$

Hence the equations of the projection are

$$x+3y+z+5=0=9x-2y-3z+7.$$

5. *Find the equation of the perpendicular from the point* $(2,4,-1)$ *to the line* $x+5=\frac{1}{4}(y+3)=-\frac{1}{9}(z-6)$. *Obtain the foot of the perpendicular.*

Let

$$\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}=r. \tag{1}$$

Any point on the line is $(r-5,4r-3,-9r+6)$. This will be the foot of the perpendicular, if the line joining the points $(2,4,-1)$ and $(r-5,4r-3,-9r+6)$ is perpendicular to the given line. The d.rs. of the line joining the points are $r-7,4r-7,-9r+7$.

$\therefore\ 1\cdot(r-7)+4\cdot(4r-7)-9\cdot(-9r+7)=0$ i.e. $r=1$.

$\therefore$ the foot of the perpendicular is $(-4,1,-3)$ and the d.rs. of it are $-6,-3,-2$.

Hence the equations of this line are

$$\frac{x+4}{6}=\frac{y-1}{3}=\frac{z+3}{2}.$$

6. *Find the equation of the image of the line* $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-4}{2}$ *in the plane* $2x - y + z + 3 = 0$.

[By the image of a given line passing through (x_1, y_1, z_1) in a given plane, we mean the line joining the point of intersection of the given line and the given plane to the image of the point (x_1, y_1, z_1) in the plane.]

Let us find the image of the point $(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$. The equations of the line through $(1, 3, 4)$ and perpendicular to the plane are

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = r_1 \text{ (say)}.$$

Any point on the line is $(2r_1 + 1, -r_1 + 3, r_1 + 4)$. If it is the image point, then the midpoint of the line joining the points $(1, 3, 4)$ and $(2r_1 + 1, -r_1 + 3, r_1 + 4)$ lies on the plane i.e.

$$2 \cdot \frac{2r_1 + 1 + 1}{2} - \frac{-r_1 + 3 + 3}{2} + \frac{r_1 + 4 + 4}{2} + 3 = 0 \quad \text{or,} \quad r_1 = -2.$$

Thus the image of the point $(1, 3, 4)$ is the point $(-3, 5, 2)$. Any point on the given line can be taken as $(3r_2 + 1, 5r_2 + 3, 2r_2 + 4)$. If it lies on the plane, then

$$2 \cdot (3r_2 + 1) - (5r_2 + 3) + (2r_2 + 4) + 3 = 0, \quad \text{i.e.} \quad r_2 = -2.$$

Thus the point of intersection of the line and the plane is $(-5, -7, 0)$.

Hence the equations of the image line are

$$\frac{x+3}{-5+3} = \frac{y-5}{-7-5} = \frac{z-2}{0-2}, \quad \text{i.e.} \quad \frac{x+3}{1} = \frac{y-5}{6} = \frac{z-2}{1}.$$

> **Note.** If the given line is parallel to the given plane, then its image will be the line joining the images of two points on the line in the plane.

7. *Prove that the lines* $x = ay + b = cz + d$ *and* $x = \alpha y + \beta = \gamma z + \delta$ *are coplanar, if* $(\gamma - c)(a\beta - b\alpha) - (\alpha - a)(c\delta - d\gamma) = 0$.

The lines in symmetrical form are

$$\frac{x}{1} = \frac{y + b/a}{1/a} = \frac{z + d/c}{1/c} \quad \text{and} \quad \frac{x}{1} = \frac{y + \beta/\alpha}{1/\alpha} = \frac{z + \delta/\gamma}{1/\gamma}.$$

From the condition of coplanarity

$$\begin{vmatrix} 0 & \frac{\beta}{\alpha} - \frac{b}{a} & \frac{\delta}{\gamma} - \frac{d}{c} \\ 1 & \frac{1}{a} & \frac{1}{c} \\ 1 & \frac{1}{\alpha} & \frac{1}{\gamma} \end{vmatrix} = 0 \quad \text{or,} \quad \begin{vmatrix} 0 & \frac{\beta}{\alpha} - \frac{b}{a} & \frac{\delta}{\gamma} - \frac{d}{c} \\ 0 & \frac{1}{a} - \frac{1}{\alpha} & \frac{1}{c} - \frac{1}{\gamma} \\ 1 & \frac{1}{\alpha} & \frac{1}{\gamma} \end{vmatrix} = 0$$

$$\text{or,} \quad \left(\frac{\beta}{\alpha} - \frac{b}{a}\right)\left(\frac{1}{c} - \frac{1}{\gamma}\right) - \left(\frac{\delta}{\gamma} - \frac{d}{c}\right)\left(\frac{1}{a} - \frac{1}{\alpha}\right) = 0$$

$$\text{or,} \quad (\gamma - c)(a\beta - b\alpha) - (\alpha - a)(c\delta - d\gamma) = 0.$$

8. *A variable straight line always intersects the lines $x = k$, $y = 0$; $y = k$, $z = 0$; $z = k$, $x = 0$. Prove that the equation to its locus is*

$$xy + yz + zx - k(x + y + z - k) = 0. \qquad \text{[NH 92; CH 2003]}$$

A line which intersects the given lines is the common line of the planes

$$x - k + \lambda_1 y = 0, \tag{1}$$
$$y - k + \lambda_2 z = 0 \tag{2}$$
$$\text{and} \quad z - k + \lambda_3 x = 0, \tag{3}$$

where $\lambda_1, \lambda_2, \lambda_3$ are variable parameters satisfying the condition

$$\begin{vmatrix} 1 & \lambda_1 & 0 \\ 0 & 1 & \lambda_2 \\ \lambda_3 & 0 & 1 \end{vmatrix} = 0. \tag{4}$$

From (4), $1 + \lambda_1\lambda_2\lambda_3 = 0$.

Eliminating λ_1, λ_2 and λ_3 by (1), (2) and (3)

$$1 - \frac{x-k}{y} \cdot \frac{y-k}{z} \cdot \frac{z-k}{x} = 0,$$
$$\text{or,} \quad xy + yz + zx - k(x + y + z - k) = 0.$$

It is the equation of the required locus.

9. *Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $y + z = 0$, $z + x = 0$, $x + y = 0$, $x + y + z = c$ is $\frac{2c}{\sqrt{6}}$ and the three lines of s.d. intersect at the point $x = y = z = -c$.* [CH 2006]

The equation of the edge of the tetrahedron formed by the planes $y + z = 0$, $z + x = 0$ is

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}. \tag{1}$$

The equation of the opposite edge formed by the planes $x + y = 0, x + y + z = c$ is

$$\frac{x}{1} = \frac{y}{-1} = \frac{z-c}{0}. \tag{2}$$

Let the line of s.d. meet the lines (1) and (2) at $(r_1, r_1, -r_1)$ and $(r_2, -r_2, c)$. The d.rs. of s.d. are $r_2 - r_1, -r_2 - r_1, c + r_1$. Since it is perpendicular to the lines (1) and (2),

$$1 \cdot (r_2 - r_1) - 1 \cdot (r_2 + r_1) - 1 \cdot (c + r_1) = 0$$
$$\text{or,} \quad 3r_1 + c = 0 \quad \text{or,} \quad r_1 = -\frac{c}{3}$$
$$\text{and} \quad 1 \cdot (r_2 - r_1) + 1 \cdot (r_2 + r_1) + 0 \cdot (c + r_1) = 0 \quad \text{or,} \quad r_2 = 0.$$

$\therefore$ the points of intersection are $\left(\frac{-c}{3}, \frac{-c}{3}, \frac{c}{3}\right)$ and $(0, 0, c)$.

Hence the length of s.d.

$$= \sqrt{\left(\frac{c^2}{9} + \frac{c^2}{9} + \frac{4c^2}{9}\right)} = \frac{2c}{\sqrt{6}}.$$

The equations of s.d. are

$$\frac{x}{-c/3} = \frac{y}{-c/3} = \frac{z-c}{c/3-c} \quad \text{or,} \quad \frac{x}{1} = \frac{y}{1} = \frac{z-c}{2}.$$

It passes through the point $x = y = z = -c$.

It can be verified that the length of s.d. between the other pairs of edges is $\frac{2c}{\sqrt{6}}$ and their lines pass through the point $x = y = z = -c$.

10. *Show that the equation to the plane containing the line* $\frac{y}{b} + \frac{z}{c} = 1, x = 0$ *and parallel to the line* $\frac{x}{a} - \frac{z}{c} = 1, y = 0$ *is* $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$ *and if* $2d$ *is the s.d., prove that* $\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

[NH 2008; BH 91, 95, 98, 2000, 08; CH 96, 99, 2005, 07]

The equation of any plane through the first line is

$$\frac{y}{b} + \frac{z}{c} - 1 + \lambda x = 0. \tag{1}$$

It will be parallel to the second line, if it is parallel to the plane

$$\frac{x}{a} - \frac{z}{c} - 1 + \mu y = 0. \tag{2}$$

through the second line.

From the condition of parallel planes

$$\frac{\lambda}{1/a} = \frac{1/b}{\mu} = \frac{1/c}{-1/c}. \quad \therefore \lambda = -\frac{1}{a} \quad \text{and} \quad \mu = -\frac{1}{b}.$$

Putting this value of λ in (1),

$$-\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0 \quad \text{or,} \quad \frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0. \tag{3}$$

It is the equation of the plane.

$(a, 0, 0)$ is a point on the second line. The s.d. is the distance of the plane (3) from the point $(a, 0, 0)$.

$$\therefore 2d = \frac{2}{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}} \quad \text{or,} \quad \frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

11. *Show that all lines which cut the z-axis and the lines*

$$\frac{x+3}{2}=\frac{y-6}{3}=\frac{z-3}{-2};\ \frac{x}{2}=\frac{y-6}{2}=\frac{z}{-1}$$

lie on the surface $7(x-y+6)(x+z)=(3x-2y+21)(x+2z)$. **[NH 2004]**

Let the variable line at one position meet the first and second lines at the points $(2r_1-3, 3r_1+6, -2r_1+3)$ and $(2r_2, 2r_2+6, -r_2)$ respectively. The equations of this line are

$$\frac{x-2r_1+3}{2r_2-2r_1+3}=\frac{y-3r_1-6}{2r_2-3r_1}=\frac{z+2r_1-3}{-r_2+2r_1-3}. \tag{1}$$

On the z-axis, $x=0=y$.

If the line meets the z-axis, then

$$\frac{-2r_1+3}{2r_2-2r_1+3}=\frac{-3r_1-6}{2r_2-3r_1}$$

$$\text{or,}\quad 2r_1r_2-12r_1+18r_2+18=0 \quad\text{or,}\quad r_2=\frac{6r_1-9}{r_1+9}.$$

Putting this value of r_2 in (1), we have

$$\frac{x-2r_1+3}{-2r_1^2-3r_1+9}=\frac{y-3r_1-6}{-3r_1^2-15r_1-18}=\frac{z+2r_1-3}{2r_1^2+9r_1-18}=\frac{x+z}{6r_1-9}$$
$$=\frac{3x-2y+21}{21r_1+63}=\frac{x+2z+2r_1-3}{2r_1^2+15r_1-27}=\frac{x-y+r_1+9}{r_1^2+12r_1+27}.$$

From the last two

$$\frac{x+2z+2r_1-3}{(2r_1-3)(r_1+9)}=\frac{x-y+r_1+9}{(r_1+3)(r_1+9)}$$

$$\text{or,}\quad \frac{x+2z+2r_1-3}{2r_1-3}=\frac{x-y+r_1+9}{r_1+3}$$

$$\text{or,}\quad \frac{x+2z}{2r_1-3}=\frac{x-y+6}{r_1+3} \quad\text{or,}\quad \frac{x+2z}{x-y+6}=\frac{2r_1-3}{r_1+3}.$$

Again

$$\frac{x+z}{3x-2y+21}=\frac{2r_1-3}{7(r_1+3)}.$$

$$\therefore\quad \frac{7(x+z)}{3x-2y+21}=\frac{x+2z}{x-y+6}$$

$$\text{or,}\quad 7(x-y+6)(x+z)=(3x-2y+21)(x+2z).$$

It is the equation of the required surface.

12. *Two straight lines*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$$

are cut by a third line whose d.cs. are λ, μ, v. Show that the length intercepted on the third line is given by

$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & v \end{vmatrix}$$

and deduce the length of s.d.

Let the third line meet the given lines at

$$P(lr+\alpha, mr+\beta, nr+\gamma) \quad \text{and} \quad Q(l'r'+\alpha', m'r'+\beta', n'r'+\gamma').$$

Since λ, μ, v are d.cs. of PQ,

$$\frac{lr-l'r'+\alpha-\alpha'}{\lambda}=\frac{mr-m'r'+\beta-\beta'}{\mu}=\frac{nr-n'r'+\gamma-\gamma'}{v}=d \text{ (say)}.$$

Here d is the length of PQ.

From (1),

$$\begin{aligned} lr-l'r'+\alpha-\alpha'-\lambda d&=0, \\ mr-m'r'+\beta-\beta'-\mu d&=0, \\ nr-n'r'+\gamma-\gamma'-vd&=0. \end{aligned}$$

Eliminating r and r', we have

$$\begin{vmatrix} l & l' & \alpha-\alpha'-\lambda d \\ m & m' & \beta-\beta'-\mu d \\ n & n' & \gamma-\gamma'-vd \end{vmatrix}=0 \quad \text{or,} \quad \begin{vmatrix} l & l' & \alpha-\alpha' \\ m & m' & \beta-\beta' \\ n & n' & \gamma-\gamma' \end{vmatrix}-d\begin{vmatrix} l & l' & \lambda \\ m & m' & \mu \\ n & n' & v \end{vmatrix}=0.$$

$$\therefore\ d=\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & v \end{vmatrix}.$$

PQ will be s.d., if it is perpendicular to both the lines. For this

$$l\lambda+m\mu+nv=0 \quad \text{and} \quad l'\lambda+m'\mu+n'v=0.$$

By cross-multiplication,

$$\frac{\lambda}{mn'-m'n}=\frac{\mu}{nl'-n'l}=\frac{v}{lm'-l'm}=\frac{1}{\sqrt{[\sum(mn'-m'n)^2]}}.$$

Thus

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & v \end{vmatrix} = \lambda(mn' - m'n) + \mu(nl' - n'l) + v(lm' - l'm)$$

$$= \frac{\sum(mn' - m'n)^2}{\sqrt{[\sum(mn' - m'n)^2]}} = \sqrt{\left[\sum(mn' - m'n)^2\right]}.$$

Hence s.d.

$$= \begin{vmatrix} \alpha - \alpha' & \beta - \beta' & \gamma - \gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \sqrt{\left[\sum(mn' - m'n)^2\right]}.$$

EXERCISE IV

1. Find the equation of the line in symmetrical form

   (a) through the origin which has d.rs. $1, -1, 4$;

   (b) through $(-1, -3, 5)$ which has d.rs. $2, 1, -3$;

   (c) through the points $(2, 0, 4)$ and $(1, -2, 3)$;

   (d) through $(5, -7, 8)$ and parallel to $\frac{x-2}{3} = \frac{y-2}{4} = \frac{z+1}{-5}$;

   (e) which is perpendicular to $\frac{x-5}{1} = \frac{y-2}{3} = \frac{z+1}{-3}$ and $\frac{x}{2} = \frac{y-3}{3} = \frac{z+1}{2}$ and passes through $(1, 2, -1)$.

2. Find the equation of the line $8x + 12y - 13z - 32 = 0 = 4x + 4y - 5z - 12$ in symmetrical form.

3. (a) Find the points where the line through the points $(5, -2, 3)$ and $(3, 0, 1)$ pierces the coordinate planes.

   (b) Find the point where the line $x + 3y - z = 6, y - z = 4$ meets the plane $2x + 2y + z = 0$. **[NH 2008]**

4. (a) Find the equations to the line passing through $(-1, -2, -3)$ and parallel to the line $(2x + 3y - 3z + 2 = 0 = 3x - 4y + 2z - 4$.

   (b) Find the equations of the line through the point $(2, -1, 3)$ and perpendicular to the line $x - 2y + 3z - 4 = 0 = 2x - 3y + 4z - 5$.

5. (a) Find the equation to the plane through the line $3x - 4y + 5z = 10, 2x + 2y - 3z = 4$ and parallel to $x = 2y = 3z$.

   (b) Find the equation of the plane which contains the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and is perpendicular to the plane $x + 2y + z = 12$.

   (c) Show that the equation of the plane through the points $(2, -1, 0), (3, -4, 5)$ and parallel to the straight line $2x = 3y = 4z$ is $29x - 27y - 22z = 85$.

(d) Show that the equation of the plane which contains the parallel lines $\frac{x-4}{1} = \frac{y-3}{-4} = \frac{z-2}{5}$ and $\frac{x-3}{1} = \frac{y+2}{-4} = \frac{z}{5}$ is $11x - y - 3z = 35$.

(e) Show that the equation of the plane through $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and perpendicular to the planes containing the lines $\frac{x}{m} = \frac{y}{n} = \frac{z}{l}$ and $\frac{x}{n} = \frac{y}{l} = \frac{z}{m}$ is $(m-n)x + (n-l)y + (l-m)z = 0$. **[BH 2007; CH 2005]**

(f) Prove that the plane through the point (α, β, γ) and the straight line $x = py + q = rz + s$ is given by

$$\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix} = 0.$$ **[CH 2009]**

(g) Show that $x\cos\theta + y\sin\theta = 0$ is the plane which contains the z-axis and is perpendicular to $\frac{x-1}{\cos\theta} = \frac{y+2}{\sin\theta} = \frac{z-3}{0}$.

6. (a) Find the distance of the point of intersection of the line $\frac{x-2}{1} = \frac{y-3}{2} = \frac{z+1}{2}$ and the plane $2x + 3y + 4z + 7 = 0$ from the point $(2, 1, 3)$.

(b) Find the distance of the point $(2, -1, 1)$ from the plane $x + y + z = 3$ measured parallel to the line whose d.rs. are $2, 3, -4$.

7. (a) Prove that the following lines are coplanar and also find the equation of the plane.

(i) $\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}, \frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$.

(ii) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}, 4x - 3y + 1 = 0 = 5x - 3z + 2$.

(iii) $x+2y-5z+9 = 0 = 3x-y+2z-5, 2x+3y-z-3 = 0 = 4x-5y+z+3$.

(iv) $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}, \frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$. **[CH 2008]**

(b) Prove that the lines $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, $ax + by + cz + d = 0 = a_1x + b_1y + c_1z + d_1$ are coplanar, if

$$\frac{a\alpha + b\beta + c\gamma + d}{al + bm + cn} = \frac{a_1\alpha + b_1\beta + c_1\gamma + d_1}{a_1l + b_1m + c_1n}.$$

8. (a) Show that the lines

$$\frac{px}{l} = \frac{qy}{m} = \frac{rz}{n}, \quad \frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{and} \quad \frac{x}{pl} = \frac{y}{qm} = \frac{z}{rn}$$

are coplanar, if $p = q$ or $q = r$ or $r = p$.

(b) Show that the lines $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}, \frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma}, \frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ will lie in the same plane, if $\frac{l}{\alpha}(b-c) + \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(a-b) = 0$.

9. (a) Find the angle between the lines $3x + 2y + z - 5 = 0 = x + y - 2z = 3$ and $2x - y - z = 0 = 7x + 10y - 8z$.

(b) Find the angle between the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}$ and the plane $x + 2y + z - 3 = 0$.

10. Find the equation of the plane passing through the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and the point $(0, 7, -7)$. Show that the line $x = \frac{y-7}{-3} = \frac{z+7}{2}$ lies in this plane.

11. (a) Find the value of k so that the lines $\frac{x+4}{k} = \frac{y+6}{5} = \frac{z-1}{-2}$ and $3x-2y+z+5 = 0 = 2x+3y+4z-4$ may intersect. What is the point of intersection?

    (b) Show that the straight lines whose equations are

$$\frac{x-a_1}{a_2} = \frac{y-b_1}{b_2} = \frac{z-c_1}{c_2} \quad \text{and} \quad \frac{x-a_2}{a_1} = \frac{y-b_2}{b_1} = \frac{z-c_2}{c_1},$$

    where a_1, b_1, c_1 are not proportional to a_2, b_2, c_2, intersect and find their point of intersection.

12. (a) Find the image of the point $(4,-2,3)$ in the plane $2x-3y+z=7$.

    (b) Find the equations of the image of the line $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{4}$ in the plane $3x+y-4z+21=0$.

13. (a) Find the distance of the line $x+2y+3z+4=0=2x+3y+4z+5$ from the origin.

    (b) Prove that the equation of the perpendicular from the point $(3,-1,11)$ to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is $\frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-11}{4}$ and that the coordinates of its foot are $(2,5,7)$.

    (c) Find the coordinates of the point on the join of $(23,19,25)$ and $(5,7,34)$ which is nearest to the intersection of the planes $4x+7y+4z-75=0$ and $2x+4y+z-33=0$.

14. Find the equations of the bisectors of the angles between the lines

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}, \quad \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}.$$

[*Hints*. The lines intersect in the point $(5,-7,6)$. Now the bisectors are

$$\frac{x-5}{2\sqrt{66} \pm \sqrt{77}} = \frac{y+7}{-3\sqrt{66} \mp 4\sqrt{77}} = \frac{z-6}{8\sqrt{66} \pm 7\sqrt{77}}.]$$

15. Show that the lines $x+7y-z-16=0=x-y+z-4$ and $x+11y-2z=0=x-5y+2z-4$ are parallel and find the distance between them.

16. Find the equations of the line which passes through the point $(3,-2,-4)$, is parallel to the plane $3x-2y-3z-7=0$ and intersects the straight line $\frac{x-2}{3} = \frac{y+4}{-2} = \frac{z-1}{2}$.

17. (a) Find the s.d. and its equation between the lines

    (i) $\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}, \frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$;

    (ii) $3x-9y+5z=0=x+y-z, 6x+8y+3z-13=0=x+2y+z-3$;

    (iii) $x=y=z$ and $x+y=2, z-x=2$;

    (iv) $\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2}, 5x-2y-3z+6=0=x-3y+2z-3$.

    (b) (i) Show that the s.d. between the lines $x+a=2y=-12z$ and $x=y+2a=6z-6a$ is $2a$.

(ii) Show that the s.d. between the axis of z and the line $ax+by+cz+d = 0 = a_1x + b_1y + c_1z + d_1$ is

$$\frac{dc_1 - d_1c}{\sqrt{\left[(ac_1 - a_1c)^2 + (bc_1 - b_1c)^2\right]}}$$ [CH 2001; 09]

(iii) Prove that the shortest distances between the diagonals of a rectangular parallelopiped whose sides are a, b, c and the edges not meeting it, are

$$\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}.$$

(iv) Show that the s.d. between an edge of a cube and a diagonal which does not meet it is the join of their midpoints.

18. Find the s.d. between the lines $\frac{x-1}{2} = \frac{y-2}{4} = z - 3$ and $y - mx = z = 0$. For what value of m will the two lines intersect? [BH 2008; CH 2008]

19. Find the equation of the projection of the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ on the plane $x + 2y + z = 6$.

20. (a) Find the equations of the line which can be drawn from the point $(2, -1, 3)$ to intersect the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$, $\frac{x-4}{4} = \frac{y}{5} = \frac{z+3}{3}$.

(b) A line with d.rs. $2, 1, 3$ intersects the lines

$$\frac{x-a}{1} = \frac{y}{1} = \frac{z-a}{1} \quad \text{and} \quad \frac{x+a}{1} = \frac{y}{1} = \frac{z+a}{2}.$$

Find the coordinates of the points of intersection and the length intercepted on it.

(c) A line with d.rs. $(2, 1, 2)$ meets each of the lines given by $x = y+a = z$ and $x + a = 2y = 2z$. Show that the coordinates of the points of intersection are $(3a, 2a, 3a)$ and (a, a, a).

21. Find the equation to the straight line drawn from the origin to intersect the lines

$$3x + 2y + 4z - 5 = 0 = 2x - 3y + 4z + 1$$
$$\text{and} \quad 2x - 4y + z + 6 = 0 = 3x - 4y + z - 3.$$

[*Hints.* The general equation of the line intersecting the given lines can be written as

$$3x + 2y + 4z - 5 + \lambda(2x - 3y + 4z + 1) = 0,$$
$$2x - 4y + z + 6 + \mu(3x - 4y + z - 3) = 0.$$

Since the line passes through $(0, 0, 0)$, $\lambda = 5, \mu = 2$.

$\therefore$ the line is $13x - 13y + 24z = 0 = 8x - 12y + 3z$ or, $\frac{x}{249} = \frac{y}{153} = \frac{z}{-52}$.]

22. Show that the equation to the plane through the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and perpendicular to the plane containing the lines $\frac{x}{m} = \frac{y}{n} = \frac{z}{l}$ and $\frac{x}{n} = \frac{y}{l} = \frac{z}{m}$ is $(m-n)x + (n-l)y + (l-m)z = 0$. **[BH 2007; CH 98, 2007, 08]**

23. (a) Examine the nature of intersection of the following planes:

(i) $2x + y + z + 4 = 0, y - z + 4 = 0, 3x + 2y + z + 8 = 0$.

(ii) $2x - y + z = 4, 5x + 7y + 2z = 0, 3x + 4y - 2z + 3 = 0$.

(iii) $4x - 5y - 2z - 2 = 0, 5x - 4y + 2z + 2 = 0, 2x + 2y + 8z = 1$.

(iv) $3x - y + z = 5, 2x + 4y + z + 10 = 0, 6x - 2y + 2z + 9 = 0$.

(b) (i) Prove that the planes $ny - mz = \lambda, lz - nx = \mu, mx - ly = v$ have a common line, if $l\lambda + m\mu + nv = 0$.

(ii) Show that the planes $ax + hy + gz = 0$, $hx + by + fz = 0$,

$gx + fy + cz = 0$ have a common line, if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

(iii) Show that the planes $x = y\sin\gamma + z\sin\beta$, $y = z\sin\alpha + x\sin\gamma$, $z = x\sin\beta + y\sin\alpha$, intersect in the line $\frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma}$, if 2α, 2β, 2γ are the angles of a triangle.

(c) Prove that the planes

$$x + ay + (b+c)z + d = 0, x + by + (c+a)z + d = 0, x + cy + (a+b)z + d = 0$$

have a common line.

(d) (i) Find the values of k for which the planes

$$kx + 3y + 2z = 3, 3x + ky + z = 2, x - y + z + 1 = 0$$

intersect at a point or intersect in a line or form a prism.

(ii) Find the values of a and b for which the planes

$$x + 2y + z = b, 3x - 5y + 3z = 1, 2x + 7y + az = 8$$

intersect in one point or intersect in a line or form a prism.

24. (a) A variable line intersects the lines

$$x = b, y + c = 0; y = c, z + a = 0; z = a, x + b = 0.$$

Show that the locus of the line is

$$axy + byz + czx + abc = 0.$$ **[BH 2007; NH 2007; CH 2001]**

[*Hints*. Any line intersecting the first two lines is

$$x - b + k_1(y + c) = 0, y - c + k_2(z + a) = 0. \tag{1}$$

It meets the third line. Thus putting $z = a$ and $x = -b$, $-2b + k_1(y+c) = 0$, $y - c + k_2 2a = 0$.

From these $ak_1k_2 - ck_1 + b = 0$.

Eliminating k_1 and k_2 by (1), the result is obtained.]

(b) Prove that the locus of a variable line which intersects the lines

$$y - z = 1, x = 0; z - x = 1, y = 0; x - y = 1, z = 0$$

is the surface whose equation is $x^2 + y^2 + z^2 - 2(yz + zx + xy) = 1$.

25. (a) Prove that the locus of the point which is equidistant from the lines $y = mx, z = c$ and $y = -mx, z = -c$ is the surface $mxy + (1 + m^2)cz = 0$.
[NH 2007, 08; CH 94, 2009]

(b) Find the locus of the midpoints of lines whose extremities are on two skew lines and which are parallel to a given plane.

[*Hints.* Let the skew lines be $y = \pm mx, z = \pm c$. The d.rs. of the variable line are $r - r', m(r + r'), 2c$. Let it be parallel to

$$Ax + By + Cz + D = 0$$
$$\therefore A(r - r') + Bm(r + r') + C \cdot 2c = 0. \qquad (1)$$

If (α, β, γ) be the midpoint, then

$$2\alpha = r + r', 2\beta = m(r - r'), 2\gamma = 0. \qquad (2)$$

Eliminating r and r' from (1) by (2), $Bm^2\alpha + A\beta + mcC = 0$. Thus the locus is $Bm^2x + Ay + mcC = 0$.]

(c) Find the locus of a line which meets the lines $y = \pm mx, z = \pm c$ and the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$.

(d) Show that the locus of a variable line which intersects the three lines $y = mx, z = c; y = -mx, z = -c; y = z, mx = -c$ is the surface $y^2 - m^2x^2 = z^2 - c^2$. **[CH 97, 2008]**

[*Hints.* Planes through the first and second lines are

$$y - mx + \lambda_1(z - c) = 0 \qquad (1)$$
$$\text{and} \quad y + mx + \lambda_2(z + c) = 0. \qquad (2)$$

The line of intersection between (1) and (2) meets the third line.

$\therefore$ $y + c + \lambda_1(y - c) = 0$ and $y - c + \lambda_2(y + c) = 0$. Eliminating y, we have $\lambda_1\lambda_2 = 1$. By (1) and (2),

$$\frac{y - mx}{z - c} \cdot \frac{y + mx}{z + c} = 1 \quad \text{or,} \quad y^2 - m^2x^2 = z^2 - c^2.]$$

(e) Show that the surface generated by a straight line which intersects the lines $y = 0, z = c; x = 0, z = -c$ and the curve $z = 0, xy + c^2 = 0$ is $z^2 - c^2 = xy$.

(f) A straight line is parallel to the yz-plane and intersects the curves $x^2 + y^2 = a^2, z = 0$ and $x^2 = az, y = 0$. Prove that it generates the surface $x^4y^2 = (a^2 - x^2)(x^2 - az)^2$.

[*Hints.* The equations of a line parallel to the plane $x = 0$ are of the form

$$\frac{x-\alpha}{0} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}.$$

It intersects $x^2 + y^2 = a^2, z = 0$ at $x = \alpha, z = 0, y = \beta - \frac{m}{n}\gamma$.

$$\therefore\ \alpha^2 + \left(\beta - \frac{m}{n}\gamma\right)^2 = a^2. \tag{1}$$

Again it intersects $x^2 = ax, y = 0$ at $x = \alpha, y = 0, z = \gamma - \frac{n}{m}\beta$.

$$\therefore\ \alpha^2 = a\left(\gamma - \frac{n}{m}\beta\right). \tag{2}$$

Eliminating m and n from (1) by (2), the result is obtained.]

(g) Prove that the line of s.d. between the z-axis and the variable line

$$\frac{x}{a} + \frac{z}{c} = \lambda\left(1 + \frac{y}{b}\right),\quad \frac{x}{a} - \frac{z}{c} = \frac{1}{\lambda}\left(1 - \frac{y}{b}\right),$$

(where λ varies) generates the surface $abz(x^2+y^2) = (a^2-b^2)cxy$.

[CH 99]

[*Hints.* The given lines in canonical form are

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{1} \quad \text{and} \quad \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z}{n}$$

where

$$x_1 = \frac{2a}{\lambda + \frac{1}{\lambda}},\quad y_1 = -\frac{b\left(\lambda - \frac{1}{\lambda}\right)}{\lambda + \frac{1}{\lambda}} \quad \text{and} \quad \frac{l}{\frac{1}{bc}\left(\lambda - \frac{1}{\lambda}\right)} = \frac{m}{\frac{2}{ca}} = \frac{n}{\frac{1}{ab}\left(\lambda + \frac{1}{\lambda}\right)}.$$

If l_1, m_1, n_1 are d.rs. of s.d., then $ll_1 + mm_1 = 0, n_1 = 0$.
The equations of s.d. are

$$\begin{vmatrix} x & y & z \\ l_1 & m_1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0, \text{ i.e. } m_1x - l_1y = 0 \text{ and } \begin{vmatrix} x-x_1 & y-y_1 & z \\ l_1 & m_1 & 0 \\ l & m & n \end{vmatrix} = 0.$$

$$\text{or, } \begin{vmatrix} x-x_1 & y-y_1 & z \\ x & y & 0 \\ l & m & n \end{vmatrix} = 0 \text{ or, } \begin{vmatrix} -x_1 & -y_1 & z \\ x & y & 0 \\ l & m & n \end{vmatrix} = 0. \tag{3}$$

By $ll_1 + mm_1 = 0$ and $m_1x - l_1y = 0, lx + my = 0$.

$$\therefore\ \frac{1}{bc}\left(\lambda - \frac{1}{\lambda}\right)x + \frac{2}{ca}y = 0 \quad \text{or,} \quad \lambda - \frac{1}{\lambda} = -\frac{2b}{a}\frac{y}{x}.$$

Now,

$$\left(\lambda+\frac{1}{\lambda}\right)^2=\left(\lambda-\frac{1}{\lambda}\right)^2+4=\frac{4b^2}{a^2}\frac{y^2}{x^2}+4=\frac{4}{a^2x^2}\left(a^2x^2+b^2y^2\right)$$

$$\text{or,}\quad \lambda+\frac{1}{\lambda}=\frac{2}{ax}\sqrt{a^2x^2+b^2y^2}\ \text{(considering + sign only).}$$

Consequently,

$$x_1=\frac{a^2x}{\sqrt{a^2x^2+b^2y^2}},\quad y_1=\frac{b^2y}{\sqrt{a^2x^2+b^2y^2}},$$

$$l=-\frac{my}{x},\quad n=\frac{mc}{abx}\sqrt{a^2x^2+b^2y^2}.$$

Putting these values in (1), the result is obtained.]

26. (a) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B and C. Prove that the planes through the axes and the internal bisectors of the angles of the triangle ABC pass through the line

$$\frac{x}{a\sqrt{b^2+c^2}}=\frac{y}{b\sqrt{c^2+a^2}}=\frac{z}{c\sqrt{a^2+b^2}}.$$

[*Hints*. Points A, B, C are $(a,0,0),(0,b,0)$ and $(0,0,c)$.

D.cs. of AB and AC are

$$\frac{a}{\sqrt{a^2+b^2}}\frac{-b}{\sqrt{a^2+b^2}},0\quad\text{and}\quad\frac{a}{\sqrt{c^2+a^2}},0\frac{-c}{\sqrt{c^2+a^2}}.$$

The d.rs. of the internal bisector of $\angle BAC$ are

$$\frac{1}{2}\left(\frac{a}{\sqrt{a^2+b^2}}+\frac{a}{\sqrt{c^2+a^2}}\right),\quad\frac{-b}{2\sqrt{a^2+b^2}},\quad\frac{-c}{2\sqrt{c^2+a^2}}.$$

Let these be denoted by l, m, n.

Any plane through x-axis is $y+\lambda z=0$. If it contains the internal bisector, then $m+\lambda n=0$ or, $\lambda=-m/n$.

$\therefore$ the plane is $y-\frac{m}{n}z=0$ or, $\frac{y}{m}=\frac{z}{n}$. Similarly the other two planes are obtained.]

(b) A line is drawn to meet $y=x\tan\alpha, z=c; y=-x\tan\alpha, z=-c$ so that the length intercepted on it is constant. Show that its equations may be written in the form

$$\frac{x-k\sin\theta\cot\alpha}{k\cos\theta}=\frac{y-k\cos\theta\tan\alpha}{k\sin\theta}=\frac{z}{c},$$

where k is a constant and θ is a parameter. Deduce the equation to the locus of the line.

[*Hints*. Let the line meet the given lines at the points (r, mr, c) and $(r', -mr', -c)$, where $m=\tan\alpha$.

The equations of the line are

$$\frac{x-r}{r-r'} = \frac{y-mr}{m(r+r')} = \frac{z-c}{2c} \quad \text{or,} \quad \frac{x-\frac{r+r'}{2}}{\frac{r-r'}{2}} = \frac{y-\frac{m}{2}(r-r')}{\frac{m}{2}(r+r')} = \frac{z}{c}. \quad (1)$$

From the length between the points

$$(r-r')^2 + m^2(r+r')^2 + 4c^2 = 4d^2 \text{ (say)}$$

$$\text{or,} \quad \left(\frac{r-r'}{2}\right)^2 + \left\{\frac{m}{2}(r+r')\right\}^2 = d^2 - c^2 = k^2 \text{ (say).} \qquad (2)$$

Taking $\frac{r-r'}{2} = k\cos\theta$, $\frac{m}{2}(r+r') = k\sin\theta$, the equations of (1) can be written as

$$\frac{x - k\sin\theta\cot\alpha}{k\cos\theta} = \frac{y - k\cos\theta\tan\alpha}{k\sin\theta} = \frac{z}{c}. \qquad (3)$$

Eliminating θ by (3), the equation to the locus

$$k^2(c^2-z^2)^2 = c^2\{(cx\tan\alpha - yz)^2 + (cy\cot\alpha - zx)^2\}$$

is obtained.]

(c) A line through the origin makes angles α, β, γ with its projections in the coordinate planes. The distances of any point (x, y, z) from the line and its projections are d, a, b, c respectively. Prove that the locus of (x, y, z) is $d^2 = (a^2 - x^2)\cos^2\alpha + (l^2 - y^2)\cos^2\beta + (c^2 - z^2)\cos^2\gamma$. [NH 2006]
[*Hints.* Let OA be the line and its projections on XOY, YOZ and ZOX planes be OL, OM, ON. If A be (x_1, y_1, z_1), then L, M, N are $(x_1, y_1, 0)$, $(0, y_1, z_1)$, $(x_1, 0, z_1)$ respectively. Now

$$\cos\alpha = \frac{OL}{OA}, \quad \cos\beta = \frac{OM}{OA}, \quad \cos\gamma = \frac{ON}{OA}.$$

If P be (x, y, z), then

$$d^2 = OP^2 - \left(\frac{xx_1 + yy_1 + zz_1}{OA}\right)^2, \quad a^2 = OP^2 - \left(\frac{xx_1 + yy_1}{OA}\right)^2,$$

$$b^2 = OP^2 - \left(\frac{yy_1 + zz_1}{OA}\right)^2, \qquad c^2 = OP^2 - \left(\frac{zz_1 + xx_1}{OA}\right)^2.$$

It follows that $d^2 = (a^2 - x^2)\cos^2\alpha + (b^2 - y^2)\cos^2\beta + (c^2 - z^2)\cos^2\gamma$.]

ANSWERS

1. (a) $\frac{x}{1} = \frac{y}{-1} = \frac{z}{4}$; (d) $\frac{x-5}{3} = \frac{y+7}{4} = \frac{z-8}{-5}$;

   (b) $\frac{x+1}{2} = \frac{y+3}{1} = \frac{z-5}{-3}$; (e) $\frac{x-1}{15} = \frac{y-2}{-8} = \frac{z+1}{-3}$.

   (c) $\frac{x-2}{1} = \frac{y}{2} = \frac{z-4}{1}$;

2. $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$.

3. (a) $(2,1,0),(0,3,-2),(3,0,1)$;
   (b) $(2,0,-4)$.
4. (a) $\frac{x+1}{6}=\frac{y+2}{13}=\frac{z+3}{17}$;
   (b) $\frac{x-2}{13}=\frac{y+1}{-10}=\frac{z-3}{7}$.
5. (a) $x-20y+27z=14$;
   (b) $9x-2y-5z+4=0$.
6. (a) $\sqrt{37}$;
   (b) $\sqrt{29}$.
7. (a) (i) $x-2y+z+7=0$;
   (ii) $x-2y+z=0$;
   (iii) $83x-2y-7z=21$;
   (iv) $x-2y+z=0$.
9. (a) $\frac{\pi}{2}$;
   (b) $\sin^{-1}\sqrt{\frac{2}{27}}$.
10. $x+y+z=0$.
11. (a) $k=3,(2,4,-3)$;
    (b) $(a_1+a_2,b_1+b_2,c_1+c_2)$.
12. (a) $\left(\frac{8}{7},\frac{16}{7},\frac{11}{7}\right)$;
    (b) $\frac{x-6}{47}=\frac{y-9}{46}=\frac{z-12}{24}$.
13. (a) $\sqrt{\frac{7}{3}}$;
    (c) $(11,11,31)$.
15. 3.
16. $\frac{x-3}{5}=\frac{y+2}{-6}=\frac{z+4}{9}$.
17. (a) $4\sqrt{3}, x=y=z$;
    (b) $\frac{11}{\sqrt{342}}, 10x-29y+16z=0=13x+82y+55z-109$;
    (c) $\sqrt{2}, z+x=3, 2y=3$;
    (d) $\frac{34}{13\sqrt{6}}, 7x-2y-11z-20=0=13x-13z+24$.
18. $\frac{5m-10}{\sqrt{(5m^2-16m+17)}}, m=2$.
19. $x+2y+z-6=0=9x-2y+5z+4$.
20. (a) $12x+4y-9z+7=0=11x-10y+2z-38$;
    (b) $(a,0,a),(-3a,-2a,-5a),2\sqrt{14}a$.
23. (a) (i) line;
    (ii) point;
    (iii) prism;
    (iv) second plane intersects the other two parallel planes;
    (d) (i) $k\neq 4,-3$; $k=-3$; $k=4$.
    (ii) $a\neq 2$; $a=2$, $b=\frac{91}{31}$; $a=2$, $b\neq\frac{91}{31}$.
25. (b) $b^2c^2(yz-cmx)^2+a^2c^2m^2(cy-mzx)^2=a^2b^2m^2(z^2-c^2)^2$.

Chapter 5

Change of Axes: Volume of a Tetrahedron

5.10 Transformation of one set of rectangular axes to another without changing the position of origin

Let OX, OY, OZ and OX', OY', OZ' be two sets of rectangular axes through the same origin O and $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ be the d.cs. of OX', OY', OZ' respectively referred to OX, OY, OZ.

Let P be the point whose coordinates are

$$(x, y, z) \text{ w.r.t. } OX, OY, OZ$$
$$\text{and} \quad (x', y', z') \text{ w.r.t. } OX', OY', OZ'.$$

In Fig. 23,

$$OK = x, KL = y, LP = z;$$
$$OM = x', MN = y', NP = z'.$$

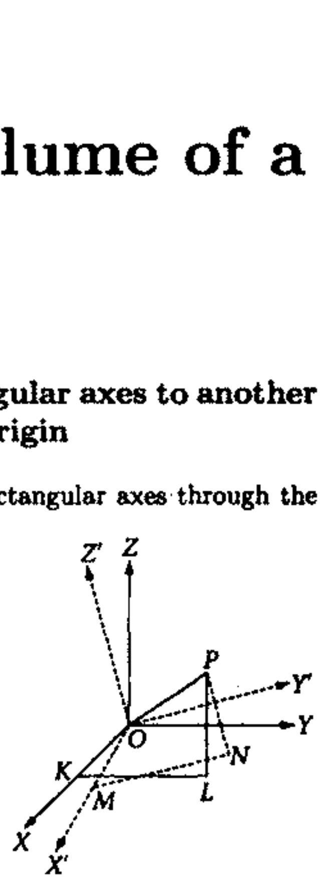

Fig. 23

Projection of OP on OX = proj. of OM on OX+ proj. of MN on OX + proj. of NP on OX or, $x = l_1x' + l_2y' + l_3z'$.

Similarly $y = m_1x' + m_2y' + m_3z', z = n_1x' + n_2y' + n_3z'$.

Again projecting OP and OK, KL, LP on OX', OY' and OZ' in turn we have

$$x' = l_1x + m_1y + n_1z, y' = l_2x + m_2y + n_2z, z' = l_3x + m_3y + n_3z.$$

The above rules of transformation can be remembered in a compact form. To obtain the value of x in terms of x', y', z' multiply each element of the row of x with the corresponding element of the first row and add i.e. $x = l_1x' + l_2y' + l_3z'$. Similarly to obtain the value of x' in terms of x, y, z multiply each element of its column with the corresponding element of the first column and add i.e. $x' = l_1x + m_1y + n_1z$.

	x'	y'	z'
x	l_1	l_2	l_3
y	m_1	m_2	m_3
z	n_1	n_2	n_3

5.11 Relations between the d.cs. of the three mutually perpendicular lines

Let $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ be the d.cs. of the three mutually perpendicular lines OX', OY', OZ' respectively referred to a set of a rectangular axes OX, OY and OZ.

Then

$$\left.\begin{array}{l} l_1^2+m_1^2+n_1^2=1 \\ l_2^2+m_2^2+n_2^2=1 \\ l_3^2+m_3^2+n_3^2=1 \end{array}\right\}, \qquad (1) \qquad \left.\begin{array}{l} l_2l_3+m_2m_3+n_2n_3=0 \\ l_3l_1+m_3m_1+n_3n_1=0 \\ l_1l_2+m_1m_2+n_1n_2=0 \end{array}\right\}. \qquad (2)$$

Again $l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3$ are the d.cs. of OX, OY, OZ respectively referred to OX', OY' and OZ'.

$$\left.\begin{array}{l} l_1^2+l_2^2+l_3^2=1 \\ m_1^2+m_2^2+m_3^2=1 \\ n_1^2+n_2^2+n_3^2=1 \end{array}\right\}, \qquad (3) \qquad \left.\begin{array}{l} m_1n_1+m_2n_2+m_3n_3=0 \\ n_1l_1+n_2l_2+n_3l_3=0 \\ l_1m_1+l_2m_2+l_3m_3=0 \end{array}\right\}. \qquad (4)$$

The relations (1), (2), (3) and (4) can be remembered in a compact form. Considering the elements of horizontal rows and vertical columns as the d.cs. of two sets of three mutually perpendicular lines, the results of (1), (2), (3) and (4) are easily obtained.

l_1	m_1	n_1
l_2	m_2	n_2
l_3	m_3	n_3

Corollary. Show that

$$\begin{aligned} l_1 &= \pm(m_2n_3 - m_3n_2), \\ m_1 &= \pm(n_2l_3 - n_3l_2), \\ n_1 &= \pm(l_2m_3 - l_3m_2). \end{aligned}$$

We have $\quad l_1l_2 + m_1m_2 + n_1n_2 = 0,$

$$l_1l_3 + m_1m_3 + n_1n_3 = 0.$$

$$\therefore\ \frac{l_1}{m_2n_3 - m_3n_2} = \frac{m_1}{n_2l_3 - n_3l_2} = \frac{n_1}{l_2m_3 - l_3m_2} = \pm\frac{\sqrt{(l_1^2 + m_1^2 + n_1^2)}}{\sqrt{[\sum(m_2n_3 - m_3n_2)^2]}} = \pm 1.$$

$$\left[\because\ \sin\theta = \sqrt{\left[\sum(m_2n_3 - m_3n_2)^2\right]} \text{ and here } \theta = 90°.\right]$$

$$\therefore\ l_1 = \pm(m_2n_3 - m_3n_2), m_1 = \pm(n_2l_3 - n_3l_2), n_1 = \pm(l_2m_3 - l_3m_2).$$

5.12.(a) Transformation formulae under matrix notation

Let

$$\xi = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \eta = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \zeta = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, R = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let (x', y', z') be the transformed coordinates of (x, y, z) due to translation or, rotation or, translation and rotation.

(i) *Translation*. If the origin is shifted to (α, β, γ) only, then $\xi = I\eta + \zeta$.

(ii) *Rotation*. If the axes are rotated only as Sec 5.10, then $\xi = R\eta$.

(iii) *Translation and rotation* (*rigid motion*). In this case, $\xi = R\eta + \zeta$.

(b) Show that R is orthogonal

$$R^TR = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \times \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2 + m_1^2 + n_1^2 & l_1l_2 + m_1m_2 + n_1n_2 & l_3l_1 + m_3m_1 + n_3n_1 \\ l_1l_2 + m_1m_2 + n_1n_2 & l_2^2 + m_2^2 + n_2^2 & l_2l_3 + m_2m_3 + n_2n_3 \\ l_3l_1 + m_3m_1 + n_3n_1 & l_2l_3 + m_2m_3 + n_2n_3 & l_3^2 + m_3^2 + n_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \text{ by (1) and (2) of Sec 5.11.}$$

$\therefore$ R is orthogonal.

Corollary I (R is non-singular). $|R^TR| = |I|$ or, $|R^T||R| = 1$ or, $|R|^2 = 1$ or, $|R| = \pm 1$. Hence the result follows.

Corollary II (Inverse relations). Since R and I are orthogonal, the inverse relations corresponding to (i), (ii) and (iii) are $\eta = I^T\xi - I^T\zeta = I\xi - \zeta, \eta = R^T\xi$ and $\eta = R^T\xi - R^T\zeta$.

> **Note.** I and R are orthogonal matrices. For this each of the above transformations is called an orthogonal transformation.

5.13. Invariants

(I) *The square of the distance between two points is an invariant.*

Let the coordinates of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ become (x_1', y_1', z_1') and (x_2', y_2', z_2') respectively under the orthogonal transformation $\xi = R\eta + \zeta$.

Then by this transformation,

$$\begin{aligned} x_1 &= l_1x_1' + l_2y_1' + l_3z_1' + \alpha, & y_1 &= m_1x_1' + m_2y_1' + m_3z_1' + \beta, \\ z_1 &= n_1x_1' + n_2y_1' + n_3z_1' + \gamma, & x_2 &= l_1x_2' + l_2y_2' + l_3z_2' + \alpha, \\ y_2 &= m_1x_2' + m_2y_2' + m_3z_2' + \beta, & z_2 &= n_1x_2' + n_2y_2' + n_3z_2' + \gamma. \end{aligned}$$

Therefore

$$\begin{aligned} x_1 - x_2 &= l_1(x_1' - x_2') + l_2(y_1' - y_2') + l_3(z_1' - z_2'), \\ y_1 - y_2 &= m_1(x_1' - x_2') + m_2(y_1' - y_2') + m_3(z_1' - z_2'), \\ z_1 - z_2 &= n_1(x_1' - x_2') + n_2(y_1' - y_2') + n_3(z_1' - z_2'). \end{aligned}$$

Now $(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

$$\begin{aligned} &= (l_1^2 + m_1^2 + n_1^2)(x_1' - x_2')^2 + (l_2^2 + m_2^2 + n_2^2)(y_1' - y_2')^2 \\ &\quad + (l_3^2 + m_3^2 + n_3^2)(z_1' - z_2')^2 + 2(l_1l_2 + m_1m_2 + n_1n_2)(x_1' - x_2')(y_1' - y_2') \\ &\quad + 2(l_2l_3 + m_2m_3 + n_2n_3)(y_1' - y_2')(z_1' - z_2') \\ &\quad + 2(l_3l_1 + m_3m_1 + n_3n_1)(z_1' - z_2')(x_1' - x_2') \\ &= (x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2. \end{aligned}$$

Hence the result follows.

(II) *If, by change of set of rectangular axes to another with the same origin, $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ transforms to $a_1x_1^2 + b_1y_1^2 + c_1z_1^2 + 2f_1y_1z_1 + 2g_1z_1x_1 + 2h_1x_1y_1$, then*

(i) $a + b + c = a_1 + b_1 + c_1$,

(ii) $ab + bc + ca - f^2 - g^2 - h^2 = a_1b_1 + b_1c_1 + c_1a_1 - f_1^2 - g_1^2 - h_1^2$ and

(iii) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} a_1 & h_1 & g_1 \\ h_1 & b_1 & f_1 \\ g_1 & f_1 & c_1 \end{vmatrix}$.

Let

$$\begin{aligned} Q &= ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy, \\ Q_1 &= a_1x_1^2 + b_1y_1^2 + c_1z_1^2 + 2f_1y_1z_1 + 2g_1z_1x_1 + 2h_1x_1y_1, \end{aligned}$$

$$\xi = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \eta = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, D = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, D_1 = \begin{bmatrix} a_1 & h_1 & g_1 \\ h_1 & b_1 & f_1 \\ g_1 & f_1 & c_1 \end{bmatrix}.$$

Then $Q = \xi^T D\xi, Q_1 = \eta^T D_1\eta$.

If $\xi = R\eta$ be a rotation, then

$$Q_1 = (R\eta)^T D(R\eta) = \eta^T R^T DR\eta = \eta^T D_1\eta.$$

$\therefore\ D_1 = R^T DR.$

Now

$$D_1 - \lambda I = R^T DR - \lambda I, \text{ where } \lambda \text{ is a scalar and } I \text{ is the unit matrix,}$$
$$= R^T(D - \lambda I)R. \quad [\because\ R \text{ is orthogonal.}]$$

$$\therefore\ |D_1 - \lambda I| = |R^T(D - \lambda I)R| = |R^T||D - \lambda I||R| = |D - \lambda I|.$$

$[\because\ |R^T| = |R|$ and $|R|^2 = 1.]$

Expanding both sides, we have

$$\lambda^3 - (a+b+c)\lambda^2 + (A+B+C)\lambda - |D|$$
$$= \lambda^3 - (a_1+b_1+c_1)\lambda^2 + (A_1+B_1+C_1)\lambda - |D_1|$$

A, B, C and A_1, B_1, C_1 are cofactors of a, b, c and a_1, b_1, c_1 in $|D|$ and $|D_1|$ respectively.

Now equating the coefficients, we get

$$a+b+c = a_1+b_1+c_1,$$
$$A+B+C = A_1+B_1+C_1, \text{ i.e.}$$
$$ab+bc+ca-f^2-g^2-h^2 = a_1b_1+b_1c_1+c_1a_1-f_1^2-g_1^2-h_1^2$$
$$\text{and}\quad |D| = |D_1|.$$

5.20

(i) If A_x, A_y and A_z be the projections of an area A on the coordinate planes yz, zx, xy respectively, then $A^2 = A_x^2 + A_y^2 + A_z^2$.

Let l, m, n be the d.cs. of the normal to the plane of A. Then l = cosine of the angle between the plane of A and yz-plane. Therefore, $A_x = lA$. Similarly $A_y = mA, A_z = nA$. Thus $(l^2+m^2+n^2)A^2 = A_x^2+A_y^2+A_z^2$ or, $A^2 = A_x^2+A_y^2+A_z^2$.

(ii) Area of a triangle whose vertices are given.

Let $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ be the coordinates of the vertices of triangle of area A. The projection of these points on the yz-plane are $(0, y_1, z_1), (0, y_2, z_2)$ and $(0, y_3, z_3)$.

These three points on the yz-plane form a triangle whose area is equal to the projection of A on the yz-plane. If A_x, A_y, A_z be the projections of A on the yz, zx and xy-planes respectively, then

$$A_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, A_y = \frac{1}{2}\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}, A_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Now $A^2 = A_x^2 + A_y^2 + A_z^2$. Hence A is obtained.

5.21 Volume of a tetrahedron in terms of coordinates of the vertices

Let $ABCD$ be a tetrahedron and the coordinates of the vertices A, B, C, D be

$$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) \text{ and } (x_4, y_4, z_4)$$

respectively.

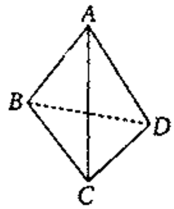

Fig. 24

If V, Δ and p be the volume of the tetrahedron, area of the triangle BCD and perpendicular distance from A to the opposite face BCD respectively, then $V = \frac{1}{3}\Delta p$.

If $\Delta_x, \Delta_y, \Delta_z$ are projections of Δ on yz, zx and xy-planes (coordinate planes) respectively, then

$$2\Delta_x = \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}, 2\Delta_y = \begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}, 2\Delta_z = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

and $\Delta^2 = \Delta_x^2 + \Delta_y^2 + \Delta_z^2$.

The equation of the plane BCD is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

$$\text{or,} \quad x\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} + y\begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix} + z\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0.$$

$\therefore$ the coefficients of x, y, z are $2\Delta_x, 2\Delta_y$ and $2\Delta_z$ respectively.

$$p = \frac{x_1\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} + y_1\begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix} + z_1\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}}{2\sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}}$$

$$= \frac{1}{2\Delta}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

$$\therefore\ V=\frac{1}{3}\Delta p=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1\\ x_4 & y_4 & z_4 & 1\end{vmatrix}=\frac{1}{6}\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2\\ x_2-x_3 & y_2-y_3 & z_2-z_3\\ x_3-x_4 & y_3-y_4 & z_3-z_4\end{vmatrix}.$$

Corollary. If one vertex is the origin i.e. $x_4=y_4=z_4=0$, then

$$V=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}.$$

WORKED-OUT EXAMPLES

1. *Find the form of the equation $x^2+y^2+z^2+12x-6y+8z-3=0$ referred to the new origin $(-6,3,-4)$ without changing the direction of axes.*

   If (x,y,z) and (x',y',z') be the coordinates of the same points w.r.t. old system and new system, then $x=x'-6, y=y'+3, z=z'-4$.

   For this transformation the equation changes to

   $$(x'-6)^2+(y'+3)^2+(z'-4)^2+12(x'-6)-6(y'+3)+8(z'-4)-3=0$$
   or, $x'^2+y'^2+z'^2=64$.

   In the usual symbol of current coordinates this equation is $x^2+y^2+z^2=64$.

2. *Show that the equation $lx+my+nz=0$ becomes $z=0$ when referred to new axes through the origin with d.rs. $-m,l,0;-ln,-mn,l^2+m^2;l,m,n\cdot[l^2+m^2+n^2=1]$*

   The d.cs. of the new axes are

   $$\frac{-m}{\sqrt{l^2+m^2}},\frac{l}{\sqrt{l^2+m^2}},0;\frac{-ln}{\sqrt{l^2+m^2}},\frac{-mn}{\sqrt{l^2+m^2}},\sqrt{l^2+m^2};l,m,n.$$

   By the rule of transformation

   $$x=\frac{-m}{\sqrt{l^2+m^2}}x'-\frac{ln}{\sqrt{l^2+m^2}}y'+lz',$$
   $$y=\frac{l}{\sqrt{l^2+m^2}}x'-\frac{mn}{\sqrt{l^2+m^2}}y'+mz'$$
   $$\text{and } z=\sqrt{l^2+m^2}y'+nz'.$$

   Now the equation changes to

   $$l\left(-\frac{m}{\sqrt{l^2+m^2}}x'-\frac{ln}{\sqrt{l^2+m^2}}y'+lz'\right)+m\left(\frac{l}{\sqrt{l^2+m^2}}x'-\frac{mn}{\sqrt{l^2+m^2}}y'+mz'\right.$$
   $$\left.+n\left(\sqrt{l^2+m^2}y'+nz'\right)=0\right.$$
   or, $(l^2+m^2+n^2)z'=0$ or, $z'=0$.

   In the usual symbol of current coordinates this is $z=0$.

3. *Prove that the three planes $2x + y + z = 3, x - y + 2z = 4, x + z = 2$ form a triangular prism and find the area of a normal section of the prism.*

If l, m, n are the d.cs. of the line of intersection of the first two planes, then

$$2l + m + n = 0 \quad \text{and} \quad l - m + 2n = 0.$$

From these two $\frac{l}{1} = \frac{m}{-1} = \frac{n}{-1}$.

If the line meets the plane $z = 0$ at $(x_1, y_1, 0)$, then

$$2x_1 + y_1 = 3 \quad \text{and} \quad x_1 - y_1 = 4.$$

From these $x_1 = \frac{7}{3}, y_1 = -\frac{5}{3}$.

The point $(x_1, y_1, 0)$ does not satisfy the equation $x + z = 2$.

Again $1 \cdot 1 + 0 \cdot (-1) + 1 \cdot (-1) = 0$.

Therefore, the line represented by the first two planes is parallel to the third plane.

Thus the planes form a prism.

The plane of the normal section through the origin is $x - y - z = 0$.

Let A be the area of the normal section and A_x be the projection of it on the plane $x = 0$.

If θ is the angle between the planes $x - y - z = 0$ and $x = 0$, then $A_x = A\cos\theta$.

Here $\cos\theta = \frac{1}{\sqrt{3}}$.

Solving the sets of equations

$$\left.\begin{aligned} x &= 0 \\ 2x + y + z &= 3 \\ x - y + 2z &= 4 \end{aligned}\right\}, \qquad \left.\begin{aligned} x &= 0 \\ 2x + y + z &= 3 \\ x + z &= 2 \end{aligned}\right\}, \qquad \left.\begin{aligned} x &= 0 \\ x - y + 2z &= 4 \\ x + z &= 2 \end{aligned}\right\},$$

the vertices of A_x are $\left(0, \frac{2}{3}, \frac{7}{3}\right), (0, 1, 2), (0, 0, 2)$.

$$\therefore A_x = \frac{1}{2}\begin{vmatrix} \frac{2}{3} & \frac{7}{3} & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{vmatrix} = -\frac{1}{6}. \quad \text{Magnitude of } A_x = \frac{1}{6}.$$

$$\therefore A = \frac{\sqrt{3}}{6}.$$

4. *The coordinates of P are (a, b, c). The plane being perpendicular to OP meets the axes OX, OY, OZ at A, B, C respectively. Show that the area of the triangle ABC is $r^5/2abc$, where $OP = r$.*

The d.rs. of OP are a, b, c. Let the equation of the plane be $ax + by + cz = k$, k is a constant. The plane passes through the point (a, b, c). $\therefore\ k = a^2 + b^2 + c^2 = r^2$. Now the equation of the plane is $ax + by + cz = r^2$.

The coordinates of A, B, C are $\left(\frac{r^2}{a}, 0, 0\right)$, $\left(0, \frac{r^2}{b}, 0\right)$, $\left(0, 0, \frac{r^2}{c}\right)$ respectively. Let Δ_x, Δ_y, Δ_z be the projections of the triangle ABC on YOZ, ZOX and XOY respectively.

$$\Delta_x = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{r^2}{b} & 0 & 1 \\ 0 & \frac{r^2}{c} & 1 \end{vmatrix} = \frac{r^4}{2bc}.$$

Similarly

$$\Delta_y = \frac{r^4}{2ca}, \Delta_z = \frac{r^4}{2ab}.$$

If Δ is the area of the triangle ABC, then

$$\begin{aligned} \Delta^2 &= \Delta_x^2 + \Delta_y^2 + \Delta_z^2 = \frac{r^8}{4}\left(\frac{1}{b^2c^2} + \frac{1}{c^2a^2} + \frac{1}{a^2b^2}\right) \\ &= \frac{r^8}{4}\frac{a^2+b^2+c^2}{a^2b^2c^2} = \frac{r^{10}}{4a^2b^2c^2} \\ \therefore\ \Delta &= \frac{1}{2}\frac{r^5}{abc}. \end{aligned}$$

5. *Find the volume of the tetrahedron whose vertices are* $(0,1,2), (1,0,2), (1,2,0)$ *and* $(1,2,1)$.

$$\text{Volume} = \frac{1}{6}\begin{vmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 0 & 1 & 2 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & -1 & 0 \end{vmatrix} \quad \text{(Subtracting the first row from the rest)}$$

$$= -\frac{1}{6}\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \\ 1 & 1 & -1 \end{vmatrix} = -\frac{1}{6}\begin{vmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix} \quad \text{(Subtracting the second column from the first)}$$

$$= -\frac{2}{6} = -\frac{1}{3}.$$

$$\therefore\ \text{Volume} = \frac{1}{3}.$$

6. *Prove that the volume of the tetrahedron formed by the four planes* $lx + my + nz = p, lx + my = 0, my + nz = 0$ *and* $nz + lx = 0$ *is* $\frac{2}{3}\frac{p^3}{lmn}$.

$$lx + my + nz = p, \tag{1}$$

$$lx + my = 0, \tag{2}$$

$$my + nz = 0, \tag{3}$$

$$nz + lx = 0. \tag{4}$$

The planes (1), (2) and (3) meet at $\left(\frac{p}{l}, \frac{-p}{m}, \frac{p}{n}\right)$.

The planes (1), (3) and (4) meet at $\left(\frac{p}{l}, \frac{p}{m}, -\frac{p}{n}\right)$.

The planes (1), (2) and (4) meet at $\left(-\frac{p}{l}, \frac{p}{m}, \frac{p}{n}\right)$.

The planes (2), (3) and (4) meet at $(0, 0, 0)$.

These are the vertices of the tetrahedron. Hence the volume

$$= \frac{1}{6}\begin{vmatrix} -\frac{p}{l} & \frac{p}{m} & \frac{p}{n} \\ \frac{p}{l} & -\frac{p}{m} & \frac{p}{n} \\ \frac{p}{l} & \frac{p}{m} & -\frac{p}{n} \end{vmatrix} = \frac{p^3}{6lmn}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \frac{2}{3}\frac{p^3}{lmn}.$$

7. *The lengths of the edges OA, OB, OC of a tetrahedron $OABC$ are a, b, c and the angles BOC, COA, AOB are λ, μ, ν. Find the volume.*

Let us consider O as origin and the d.cs. of OA, OB, OC as

$$l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$$

respectively.

Since $OA = a$, the coordinates of A are $(l_1 a, m_1 a, n_1 a)$. Similarly the coordinates of B and C are $(l_2 b, m_2 b, n_2 c)$ and $(l_3 c, m_3 c, n_3 c)$.

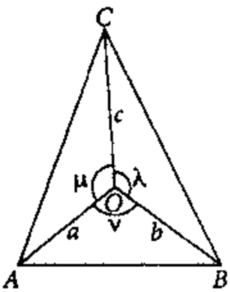

Fig. 25

$$\text{Volume of the tetrahedron} = \frac{1}{6}\begin{vmatrix} l_1 a & m_1 a & n_1 a \\ l_2 b & m_2 b & n_2 b \\ l_3 c & m_3 c & n_3 c \end{vmatrix} = \frac{abc}{6}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}.$$

Again

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1^2 + m_1^2 + n_1^2 & l_1 l_2 + m_1 m_2 + n_1 n_2 & l_3 l_1 + m_3 m_1 + n_3 n_1 \\ l_1 l_2 + m_1 m_2 + n_1 n_2 & l_2^2 + m_2^2 + n_2^2 & l_2 l_3 + m_2 m_3 + n_2 n_3 \\ l_3 l_1 + m_3 m_1 + n_3 n_1 & l_2 l_3 + m_2 m_3 + n_2 n_3 & l_3^2 + m_3^2 + n_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}.$$

$$\therefore \text{ volume} = \frac{abc}{6}\begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{1/2} \quad \text{in magnitude.}$$

EXERCISE V

1. Find the form of the equation $x^2 + y^2 + z^2 - 4x - 6y - 8z - 20 = 0$ referred to $(2, 3, 4)$ as new origin without changing the direction of axes.

2. Find the translation which will transform the equation $x^2 + y^2 - 4z^2 - 2x + 4y + 24z - 31 = 0$ into $x'^2 + y'^2 - 4z'^2 = 0$.

3. If OA, OB, OC are three mutually perpendicular lines through the origin with d.cs. $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ and if $OA = OB = OC = a$, show that the equation of the plane ABC is

$$(l_1 + l_2 + l_3)x + (m_1 + m_2 + m_3)y + (n_1 + n_2 + n_3)z = 0.$$

4. Two systems of rectangular axes have the same origin. If a plane cuts them at distances a, b, c and a_1, b_1, c_1 from the origin, show that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a_1^2} + \frac{1}{b_1^2} + \frac{1}{c_1^2}.$$

[*Hints.* The length of the perpendicular from the origin to the plane remains same.]

5. If $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d$ is changed into $a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y' + 2u'x' + 2v'y' + 2w'z' + d'$ by an orthogonal transformation through the same origin, then prove that

$$\text{(i)}\quad a + b + c = a' + b' + c', \quad \text{(ii)}\quad u^2 + v^2 + w^2 = u'^2 + v'^2 + w'^2.$$

[*Hints.* Put $x = l_1x' + l_2y' + l_3z', y = m_1x' + m_2y' + m_3z', z = n_1x' + n_2y' + n_3z'$.]

6. Show that in an orthogonal transformation of coordinate axes the expression

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

is an invariant, where $(x_i, y_i, z_i), i = 1, 2, 3, 4$ are the coordinates of four non-coplanar points.
[*Hints.* The expression is six times the volume of the tetrahedron with the given points as vertices. Hence the result follows.]

7. Find the volume of the tetrahedron whose vertices are

   (i) $(0, 0, 0), \left(\frac{d}{a}, 0, 0\right), \left(0, \frac{d}{b}, 0\right), \left(0, 0, \frac{d}{c}\right)$;

   (ii) $(1, 0, 4), (1, 1, 1), (4, 1, 0), (0, 2, 3)$.

8. Find the volume of the tetrahedron formed by the planes

$$x + y = 0, y + z = 0, z + x = 0 \quad \text{and} \quad x + y + z = 1.$$

9. Show that the area of the triangle made by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ with the coordinate axes is $\frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$.
[*Hints.* If the plane cuts the x, y, z-axes at the points A, B, C respectively, then the coordinates of A, B, C are $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$.
Let $\triangle$ be the area of the triangle ABC and $\triangle_x, \triangle_y, \triangle_z$ be the areas of projections of $\triangle$ on the coordinate planes $x=0, y=0$ and $z=0$.
The vertices of $\triangle_x$ are $(0,0,0), (0,b,0)$ and $(0,0,c)$. $\therefore\ \triangle_x = \frac{1}{2}bc$.
Similarly $\triangle_y = \frac{1}{2}ca, \triangle_2 = \frac{1}{2}ab$.
Now $\triangle^2 = \triangle_x^2 + \triangle_y^2 + \triangle_z^2$. $\quad\therefore\ \triangle = \frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$.]

10. Find the area of the triangle whose vertices are the points $(1,2,3), (-2,1,5)$ and $(3,4,2)$.

11. If the coordinates of A, B, C are $(2,1,-3), (0,-3,2), (0,-2,0)$, prove that the locus of the point P such that the volume of the tetrahedron $PABC$ is 2 is $3x-4y-2z=20$.

12. If the axes x, y and z are rectangular, prove that the substitutions

$$x=\frac{x'}{\sqrt{6}}+\frac{y'}{\sqrt{2}}+\frac{z'}{\sqrt{3}},\ y=\frac{-2}{\sqrt{6}}x'+\frac{z'}{\sqrt{3}},\ z=\frac{x'}{\sqrt{6}}-\frac{y'}{\sqrt{2}}+\frac{z'}{\sqrt{3}}$$

give a transformation to another set of rectangular axes in which the plane $x+y+z=0$ becomes the plane $z'=0$ and hence prove that the intersection of the surface $yz+zx+xy+1=0$ by the plane $x+y+z=0$ is a circle of radius $\sqrt{2}$.

13. The lengths of two opposite edges of a tetrahedron are a, b, their s.d. is equal to d and the angle between them is θ. Prove that the volume is $\frac{abd}{6}\sin\theta$.
[*Hints.* Let AB and CD be the opposite edges.
If A is the origin and l, m, n are d.cs. of AB, then the coordinates of B are (al, am, an). If (α, β, γ) are the coordinates of C and l', m', n' are the d.cs. of CD, then the coordinates of D are $(\alpha+bl', \beta+bm', \gamma+bn')$.

$$\because\ \sin\theta=\sqrt{\left[\sum(mn'-m'n)^2\right]},\ d\sin\theta=\begin{vmatrix}\alpha & \beta & \gamma\\ l & m & n\\ l' & m' & n'\end{vmatrix}.$$

$$\therefore\ \text{volume}=\frac{1}{6}\begin{vmatrix}al & am & an\\ \alpha+bl' & \beta+bm' & \gamma+bn'\\ \alpha & \beta & \gamma\end{vmatrix}=\frac{1}{6}\begin{vmatrix}al & am & an\\ bl' & bm' & bn'\\ \alpha & \beta & \gamma\end{vmatrix}=\frac{1}{6}abd\sin\theta.]$$

14. Show that the planes

$$3x-6y-5z+3=0, \tag{1}$$

$$6x-9y-8z+3=0 \tag{2}$$

$$\text{and}\quad x-y-z+2=0 \tag{3}$$

form a triangular prism. Find the area and the lengths of the edges of its normal section.

[*Hints*. The equations of the line of intersection of (1) and (2) in symmetrical

$$\text{form are } \frac{x-1}{1} = \frac{y-1}{-2} = \frac{z}{3}.$$

It is parallel to the plane (3) and it does not lie on the plane (3). Hence the planes form a prism. Normal sections are congruent triangles. The equation of the normal section through the origin is

$$x - 2y + 3z = 0. \tag{4}$$

The vertices of the triangle ABC made by this normal section are the points of intersection of the plane (4) with three pairs of the given planes. The vertices are

$$A\left(-\frac{41}{14}, -\frac{16}{14}, \frac{3}{14}\right), B\left(-\frac{71}{14}, -\frac{40}{14}, -\frac{3}{14}\right), C\left(\frac{15}{14}, \frac{12}{14}, \frac{3}{14}\right).$$

$$\therefore \text{ the edges } AB, BC, CA \text{ are } \frac{\sqrt{1512}}{14}, \frac{\sqrt{1036}}{14}, \frac{\sqrt{3920}}{14}.$$

Area of $\triangle ABC = \frac{3\sqrt{14}}{7}$.]

15. (i) **A variable plane makes with the coordinate planes a tetrahedron of constant volume k^3. Show that the locus of the foot of the perpendicular from the origin to the plane is $(x^2+y^2+z^2)^3 = 6k^3xyz$.**

[*Hints*. Let the plane at one position be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. (1)

The vertices of the tetrahedron formed by the plane $x = 0, y = 0, z = 0$ and the plane (1) are $(0,0,0), (a,0,0), (0,b,0), (0,0,c)$.

$$\text{Volume of this tetrahedron} = \frac{1}{6}\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \frac{abc}{6} = k^3. \tag{2}$$

If (α, β, γ) be the foot of the perpendicular from the origin to the plane (1), then

$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1$ (3) and $a\alpha = b\beta = c\gamma$. (4).

By (4) and (3), $\frac{\alpha^2}{\alpha/a} = \frac{\beta^2}{\beta/b} = \frac{\gamma^2}{\gamma/c} = \frac{\alpha^2+\beta^2+\gamma^2}{(\alpha/a)+(\beta/b)+(\gamma/c)} = \alpha^2+\beta^2+\gamma^2$.

$\therefore\ a = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, b = \frac{\alpha^2+\beta^2+\gamma^2}{\beta}, c = \frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$.

Putting these values of a, b, c in (2),

$(\alpha^2+\beta^2+\gamma^2)^3 = 6k^3\alpha\beta\gamma$. Hence the locus is $(x^2+y^2+z^2)^3 = 6k^3xyz$.]

(ii) Three mutually perpendicular lines PA, PB, PC meet the axes OX, OY, OZ at A, B, C respectively. If the volume of the tetrahedron $OABC$ is equal to $a^3/6$, show that the locus of P is $(x^2+y^2+z^2)^3 = 8a^3xyz$.

[*Hints*. Let P be (α, β, γ) and $l_i, m_i, n_i (i = 1, 2, 3)$ be the d.cs. of PA, PB, PC respectively.

$$\text{Here} \quad \sum l_i^2 = 1, \sum l_i l_j = 0 (i \neq j, j = 1, 2, 3)$$

$$\text{and} \quad \sum l_i m_i = 0 = \sum m_i n_i = \sum n_i l_i.$$

If the coordinates of A, B, C are

$$(x_1, 0, 0), (0, y_1, 0) \text{ and } (0, 0, z_1), \text{ then } \frac{\alpha - x_1}{l_1} = \frac{\beta}{m_1} = \frac{\gamma}{n_1},$$
$$\frac{\alpha}{l_2} = \frac{\beta - y_1}{m_2} = \frac{\gamma}{n_2} \text{ and } \frac{\alpha}{l_3} = \frac{\beta}{m_3} = \frac{\gamma - z_1}{n_3}.$$

Taking $x_1 = \alpha - \frac{l_1}{m_1}\beta, y_1 = \beta - \frac{m_2}{n_2}\gamma$ and $z_1 = \gamma - \frac{n_3}{l_3}\alpha$, volume of $OABC = \frac{1}{6}x_1y_1z_1 = \frac{a^3}{6}$.

$$\therefore \left(\alpha - \frac{l_1}{m_1}\beta\right)\left(\beta - \frac{m_2}{n_2}\gamma\right)\left(\gamma - \frac{n_3}{l_3}\alpha\right) = a^3. \qquad (1)$$

Again $\alpha\left(\alpha - \frac{l_1}{m_1}\beta\right) + \beta\left(\beta - \frac{m_2}{n_2}\gamma\right)$

$$= \alpha^2 + \beta^2 + \gamma^2 - \gamma\left(\frac{l_1n_2\alpha + m_2n_1\beta + n_1n_2\gamma}{n_1n_2}\right)$$
$$= \alpha^2 + \beta^2 + \gamma^2 - \frac{\gamma^2}{n_1n_2}(l_1l_2 + m_1m_2 + n_1n_2) = \alpha^2 + \beta^2 + \gamma^2.$$

Similarly $\quad \beta\left(\beta - \frac{m_2}{n_2}\gamma\right) + \gamma\left(\gamma - \frac{n_3}{l_3}\alpha\right) = \alpha^2 + \beta^2 + \gamma^2$

and $\quad \gamma\left(\gamma - \frac{n_3}{l_3}\alpha\right) + \alpha\left(\alpha - \frac{l_1}{m_1}\beta\right) = \alpha^2 + \beta^2 + \gamma^2.$

From these,

$$2\alpha\left(\alpha - \frac{l_1}{m_1}\beta\right) = 2\beta\left(\beta - \frac{m_2}{n_2}\gamma\right) = 2\gamma\left(\gamma - \frac{n_3}{l_3}\alpha\right) = \alpha^2 + \beta^2 + \gamma^2.$$

$$\therefore 8\alpha\beta\gamma\left(\alpha - \frac{l_1}{m_1}\beta\right)\left(\beta - \frac{m_2}{n_2}\gamma\right)\left(\gamma - \frac{n_3}{l_3}\alpha\right) = (\alpha^2 + \beta^2 + \gamma^2)^3$$

or, $(\alpha^2 + \beta^2 + \gamma^2)^3 = 8a^3\alpha\beta\gamma$ by (1). Hence the result follows.]

ANSWERS

1. $x^2 + y^2 + z^2 = 49$. 7. (a) $\frac{1}{6}\frac{d^3}{abc}$; 8. $\frac{2}{3}$.

2. $(1, -2, 3)$. (b) $\frac{7}{3}$. 10. $\frac{\sqrt{13}}{2}$.

Chapter 6

Quadric Surfaces Sphere, Cylinder, Cone, Surfaces of Revolution, Conicoids

6.10 Surface, Curve and Symmetry

Surface in space

An equation $f(x, y, z) = 0$ is said to be the equation of a surface S in a coordinate frame if

(i) every point on S is a solution of $f(x, y, z) = 0$ and

(ii) every solution of $f(x, y, z) = 0$ is a point on S.

Otherwise a surface or the locus of a point is the totality of all points satisfying some geometrical conditions at different positions in space. By these conditions the equation of the surface can be constructed in a coordinate frame.

An equation of the form $f(x, y) = 0$ represents a surface where a point (x_0, y_0, z) lies on the surface, if $f(x_0, y_0) = 0$. Here z may take up any value on the line parallel to z-axis and passing through the point $(x_0, y_0, 0)$. It indicates that the surface is generated by lines parallel to z-axis and passing through the points (x_i, y_i) which are solutions of $f(x, y) = 0$. This type of surface is called a cylindrical surface.

Curve in space

A curve in space may be regarded as the intersection of two surfaces. Thus the equation of the curve C corresponding to the surfaces $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$ is written as $f_1(x, y, z) = 0 = f_2(x, y, z)$.

If $\phi(x, y) = 0$ is obtained from the above two equations by eliminating z, then $\phi(x, y) = 0, z = 0$ is the equation of projection of C on the plane $z = 0$.

Symmetry of two distinct points

(i) Two points A and B are said to be symmetric about a point O, if the line segment AB is bisected at O.

(ii) Two points A and B are said to be symmetric about a line l, if the line segment AB is bisected by l at right angles.

(iii) Two points A and B are said to be symmetric about a plane α, if the line segment AB is bisected by α at right angles.

Symmetry of a surface

(i) A surface S is said to be symmetric about a point O, if for every point P on S there is a point P' on S such that the line segment PP' is bisected at O. The point O is called a *centre* of S.

(ii) A surface S is said to be symmetric about a line l, if for every point P on S there is a point P' on S such that the line segment PP' is bisected by l at right angles. The line l is called an *axis* of S.

(iii) A surface S is said to be symmetric about a plane α, if for every point P on S there is a point P' on S such that the line segment PP' is bisected by α at right angles. The plane α is called the *plane of symmetry* of S.

Note 1. The origin O is the centre of a surface S, if the equation of S remains unchanged when all the variables x, y, z are changed in sign, and conversely.

Note 2. A coordinate axis is the axis of S, if the equation of S remains unchanged when the variables which are zero on the axis are changed in sign and conversely.

Note 3. A coordinate plane is the plane of symmetry of S, if the equation of S remains unchanged when the variable which is zero on the coordinate plane is changed in sign, and conversely.

Quadric surface

A surface defined in space by an equation of the second degree in x, y, z is called a quadric surface. Here we discuss a few types of quadric surfaces by deriving their equations or analysing their equations.

A. Sphere

6.11 Definition

A sphere is the locus of a point which moves in such a way that its distance from a fixed point is constant. The fixed point is called the centre and the distance of the moving point from the centre is known as radius.

Equation of a sphere

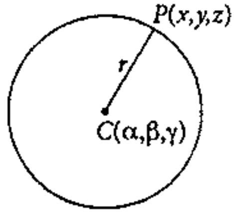

Fig. 26

If (α, β, γ) be the centre and r be the radius, then by definition the equation of the sphere is

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2.$$

This can be written as

$$x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z + \alpha^2 + \beta^2 + \gamma^2 - r^2 = 0.$$

Putting $\alpha = -u, \beta = -v, \gamma = -w$ and $\alpha^2+\beta^2+\gamma^2-r^2 = d$, the equation reduces to $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. It is the general equation of a sphere. The coordinates of the centre are $(-u, -v, -w)$ and radius $= \sqrt{u^2 + v^2 + w^2 - d}$.

Corollary. If the centre is at the origin, the equation of the sphere is $x^2+y^2+z^2 = r^2$. (Putting $\alpha = \beta = \gamma = 0$)

Example 1. *Find the equation of the sphere whose centre is* $(2, 3, -4)$ *and radius is* 5.

The equation of the sphere is

$$(x-2)^2 + (y-3)^2 + (z+4)^2 = 5^2$$
$$\text{or,}\quad x^2 + y^2 + z^2 - 4x - 6y + 8z + 4 = 0.$$

6.12 Diameter form

To find the equation of the sphere having (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *as the ends of a diameter.*

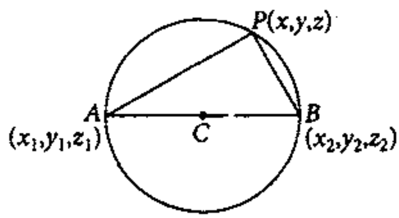

Fig. 27

Let the given points be A and B and $P(x, y, z)$ be a point on the sphere. Since AB is a diameter, $\angle APB = 90°$, i.e. PA and PB are at right angle.

The d.rs. of PA and PB are $x - x_1$, $y - y_1$, $z - z_1$ and $x - x_2$, $y - y_2$, $z - z_2$ respectively.

$\because\ PA \perp PB, (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$

It is the required equation.

[The centre of the sphere is at the middle point of the line segment AB, i.e. at the point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.]

Example 2. *Find the equation of the sphere which has* $(3, 4, -1)$ *and* $(-1, 2, 3)$ *as the ends of a diameter and find its centre and radius.*

The equation of the sphere is

$$(x-3)(x+1) + (y-4)(y-2) + (z+1)(z-3) = 0$$
$$\text{or,}\quad x^2 + y^2 + z^2 - 2x - 6y - 2z + 2 = 0.$$

The centre is $(1, 3, 1)$ and radius $= \sqrt{1+9+1-2} = 3$.

6.13 Through four non-coplanar points one and only one sphere passes.

Let the points be $(x_i, y_i, z_i), i = 1, 2, 3, 4$.

Since the points are non-coplanar

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \neq 0.$$

This implies that the system of linear equations

$$\left.\begin{aligned} 2\alpha x_1 + 2\beta y_1 + 2\gamma z_1 + \delta &= -(x_1^2 + y_1^2 + z_1^2), \\ 2\alpha x_2 + 2\beta y_2 + 2\gamma z_2 + \delta &= -(x_2^2 + y_2^2 + z_2^2), \\ 2\alpha x_3 + 2\beta y_3 + 2\gamma z_3 + \delta &= -(x_3^2 + y_3^2 + z_3^2), \\ 2\alpha x_4 + 2\beta y_4 + 2\gamma z_4 + \delta &= -(x_4^2 + y_4^2 + z_4^2) \end{aligned}\right\} \tag{1}$$

has a unique solution. Let the solution be

$$\alpha = u, \beta = v, \gamma = w, \delta = d.$$

Thus the four points lie on the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \tag{2}$$

If $$x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0 \tag{3}$$

is another sphere which passes through the given points, then the equations (1) have solution (u', v', w', d'). But the system (1) has unique solution. So $u' = u, v' = v, w' = w, d' = d$. Hence only one sphere passes through the given four points.

Note 1. If the four points are coplanar and any three of them are not collinear, many spheres can pass through them.

Note 2. If the four points are coplanar and three of them are collinear, no sphere passes through them.

Example 3. *Find the equation of the sphere through the four points* $(4, -1, 2)$, $(0, -2, 3)$, $(1, -5, -1)$ *and* $(2, 0, 1)$.

Since

$$\begin{vmatrix} 4 & -1 & 2 & 1 \\ 0 & -2 & 3 & 1 \\ 1 & -5 & -1 & 1 \\ 2 & 0 & 1 & 1 \end{vmatrix} = 42 \neq 0,$$

the points are non-coplanar and the sphere passing through these four points is unique.

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

Since the given points lie on it

$$\begin{aligned} 8u-2v+4w+d&=-21,\\ -4v+6w+d&=-13,\\ 2u-10v-2w+d&=-27,\\ 4u+2w+d&=-5. \end{aligned}$$

These are in matrix form

$$\begin{bmatrix} 8 & -2 & 4 & 1\\ 0 & -4 & 6 & 1\\ 2 & -10 & -2 & 1\\ 4 & 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} u\\ v\\ w\\ d\end{bmatrix}=\begin{bmatrix}-21\\ -13\\ -27\\ -5\end{bmatrix}.$$

By elementary operations we get

$$\begin{bmatrix} 8 & -2 & 4 & 1\\ 0 & 0 & 0 & 14\\ 0 & 0 & -3 & 11/2\\ 0 & 1 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} u\\ v\\ w\\ d\end{bmatrix}=\begin{bmatrix}-21\\ 70\\ 61/2\\ 11/2\end{bmatrix}.$$

This gives that $8u-2v+4w+d=-21, 14d=70, -3w+11/2d=61/2, v+1/2d=11/2$. From these $d=5, w=-1, v=3, u=-2$.

Thus the equation is $x^2+y^2+z^2-4x+6y-2z+5=0$.

6.14 Plane section of a sphere

Any section of a sphere by a plane is a circle.

Let PQ be a section of the sphere having the centre C. CN is perpendicular to the plane of the section. N is the centre of the circle of this section and NP is the radius of it.

From Fig. 28, $CP^2=CN^2+NP^2$.

If the coordinates of C and N are (u,v,w) and (α,β,γ), then the equation of the plane of section is $(u-\alpha)(x-\alpha)+(v-\beta)(y-\beta)+(w-\gamma)(z-\gamma)=0$.

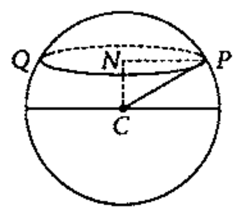

Fig. 28

Note 1. Let $S=0$ and $U=0$ be the equations of a sphere and a plane respectively. If the plane intersects the sphere, then the two equations taken together represent a circle. Again, $S+\lambda U=0$ represents a sphere through the circle $S=0, U=0$.

Note 2. The radius of the circular section of the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ by the plane $x\cos\alpha+y\cos\beta+z\cos\gamma=p$ is $[(u^2+v^2+w^2-d)-(p+u\cos\alpha+v\cos\beta+w\cos\gamma)^2]^{1/2}$.

Note 3. The equation of the plane of the circle through the two given spheres $S = 0$ and $S' = 0$ is $S - S' = 0$. It is known as the *radical plane*. The equation of any sphere through the circle $S = 0, S' = 0$ is $S + \lambda S' = 0$, where λ is a parameter.

Note 4. If the centre of the sphere lies on the section, then the circle is called a *great circle*, otherwise it is a *small circle*.

WORKED-OUT EXAMPLES

1. *Find the equation of the sphere through the four points* $(0,0,0), (a,0,0), (0,b,0)$ *and* $(0,0,c)$.

Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

As it passes through the points $(0,0,0), (a,0,0), (0,b,0)$ and $(0,0,c)$,

$$\begin{aligned} d = 0, \quad a^2 + 2ua &= 0 \text{ or, } 2u = -a \quad (\because\ a \neq 0), \\ b^2 + 2vb &= 0 \text{ or, } 2v = -b \quad (\because\ b \neq 0), \\ c^2 + 2wc &= 0 \text{ or, } 2w = -c \quad (\because\ c \neq 0). \end{aligned}$$

$\therefore$ the equation of the sphere is $x^2 + y^2 + z^2 - ax - by - cz = 0$.

2. *Prove that the sphere* $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ *cuts the sphere* $x^2+y^2+z^2+2u'x+2v'y+2w'z+d' = 0$ *in a great circle, if* $2(uu'+vv'+ww') = 2r'^2 + d + d'$, *where* r' *is the radius of the second sphere.*

The equation of the plane of the circle through the given spheres is

$$2(u - u')x + 2(v - v')y + 2(w - w')z + d - d' = 0.$$

If this circle is a great circle of the second sphere, then the centre $(-u', -v', -w')$ lies on it.

$$\begin{aligned} \therefore \quad & -2(u - u')u' - 2(v - v')v' - 2(w - w')w' + d - d' = 0 \\ \text{or,} \quad & 2(uu' + vv' + ww') = 2(u'^2 + v'^2 + w'^2 - d') + d + d' \\ \text{or,} \quad & 2(uu' + vv' + ww') = 2r'^2 + d + d'. \end{aligned}$$

3. *Find the greatest and the least distance from the point* $(2, -1, 1)$ *to the sphere* $x^2 + y^2 + z^2 - 8x + 4y - 6z + 4 = 0$.

The centre and radius of the sphere are $(4, -2, 3)$ and 5. The distance between the point $(2, -1, 1)$ and the centre

$$= \sqrt{(4-2)^2 + (-2+1)^2 + (3-1)^2} = 3.$$

Since $3 < 5$, the point $(2, -1, 1)$ is within the sphere.

The equation of the line through the point and the centre is

$$\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-1}{2} = r \quad \text{(say)}.$$

If the line meets the sphere at $(2r+2, -r-1, 2r+1)$, then

$$(2r+2)^2 + (r+1)^2 + (2r+1)^2 - 8(2r+2) - 4(r+1) - 6(2r+1) + 4 = 0$$

$$\text{or,}\quad 9r^2 - 18r - 16 = 0 \quad \text{or,}\quad r = \frac{8}{3}, -\frac{2}{3}.$$

$\therefore$ the line meets the sphere at points $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$ and $\left(\frac{22}{3}, -\frac{11}{3}, \frac{19}{3}\right)$. These points lie on the diameter through the given points.

Hence the least distance $= \sqrt{(2-2/3)^2 + (-1+1/3)^2 + (1+1/3)^2} = 2$ and the greatest distance $= 10 - 2 = 8$.

4. *Prove that the equqtion to a sphere circumscribing the tetrahedron whose faces are*

$$\frac{y}{b}+\frac{z}{c} = 0, \frac{z}{c}+\frac{x}{a} = 0, \frac{x}{a}+\frac{y}{b} = 0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \text{ is } \frac{x^2+y^2+z^2}{a^2+b^2+c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0.$$

$$\frac{y}{b} + \frac{z}{c} = 0 \qquad (1) \qquad\qquad \frac{z}{c} + \frac{x}{a} = 0 \qquad (2)$$

$$\frac{x}{a} + \frac{y}{b} = 0 \qquad (3) \qquad\qquad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (4)$$

If we solve these four equations taken three at a time, then we get the vertices of the tetrahedron. These are $(0,0,0), (-a,b,c), (a,-b,c)$ and $(a,b,-c)$.

Let the equation of the sphere through these four points be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

The vertices will be on the sphere, if $d = 0$,

$$a^2 + b^2 + c^2 - 2ua + 2vb + 2wc = 0, \tag{5}$$
$$a^2 + b^2 + c^2 + 2ua - 2vb + 2wc = 0, \tag{6}$$
$$a^2 + b^2 + c^2 + 2ua + 2vb - 2wc = 0. \tag{7}$$

Adding (5) and (6), $2w = -\frac{a^2+b^2+c^2}{c}$.

Similarly

$$2v = -\frac{a^2+b^2+c^2}{b}, 2u = -\frac{a^2+b^2+c^2}{a}.$$

$\therefore$ the required equation is

$$x^2 + y^2 + z^2 - (a^2+b^2+c^2)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

$$\text{or,}\quad \frac{x^2+y^2+z^2}{a^2+b^2+c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0.$$

5. *Find the equation of the sphere which passes through the origin and touches the sphere* $x^2+y^2+z^2=56$ *at the point* $(2,-4,6)$.

   $(2,-4,6)$ is a point on the given sphere and the origin is the centre of it. Since the required sphere passes through the origin, it touches the given sphere internally. Moreover, it is described on the line segment joining the points $(0,0,0)$ and $(2,-4,6)$ as a diameter. Hence the required equation is

$$x(x-2)+y(y+4)+z(z-6)=0$$
$$\text{or,}\quad x^2+y^2+z^2-2x+4y-6z=0.$$

6. *Find the coordinates of the centre and the radius of the circle* $x-2y-2z+7=0, x^2+y^2+z^2-2x+6y+4z-35=0$.

   The centre of the given sphere is $(1,-3,-2)$ and the radius is 7. The distance of the given plane from the centre $=\frac{1+6+4+7}{\sqrt{1^2+2^2+2^2}}=6$.

   $\therefore$ radius of the circle $=\sqrt{7^2-6^2}=\sqrt{13}$.

   The equations of the line perpendicular to the plane and passing through the centre $(1,-3,-2)$ are $\frac{x-1}{1}=\frac{y+3}{-2}=\frac{z+2}{-2}$.

   Any point on the line is $(r+1,-2r-3,-2r-2)$. If this point satisfies the equation of the plane, then

$$r+1+2(2r+3)+2(2r+2)+7=0$$
$$\text{or,}\quad 9r=-18 \text{ or, } r=-2.$$

   $\therefore$ the coordinates of the centre are $(-1,1,2)$.

7. *Obtain the equation of the circle lying on the sphere*

$$x^2+y^2+z^2-2x+2y-4z+3=0$$

   and having its centre at the point $(2,2,-3)$.

   The equations of the circle are the given equation of the sphere and a plane passing through $(2,2,-3)$ and normal to the line joining $(2,2,-3)$ and the centre of the given sphere.

   The centre of the sphere is $(1,-1,2)$. The d.rs. of the line joining the points $(2,2-3)$ and $(1,-1,2)$ are $1,3,-5$.

   $\therefore$ the required plane is

$$x-2+3(y-2)-5(z+3)=0$$
$$\text{or,}\quad x+3y-5z-23=0.$$

   Hence the equations of the circle are

$$x^2+y^2+z^2-2x+2y-4z+3=0,$$
$$x+3y-5z-23=0.$$

8. *Find the equation of the sphere for which the circle $x^2+y^2+z^2+2x-4y+2z+5=0, x-2y+3z+1=0$ is a great circle.*

The equation of a sphere through the given circle is

$$x^2+y^2+z^2+2x-4y+2z+5+\lambda(x-2y+3z+1)=0,$$

where λ is a parameter

$$\text{or,}\quad x^2+y^2+z^2+2(1+\lambda/2)x-2(2+\lambda)y+2(1+3\lambda/2)z+5+\lambda=0.$$

Its centre is
$$\left\{-\left(1+\frac{\lambda}{2}\right), 2+\lambda, -\left(1+\frac{3\lambda}{2}\right)\right\}.$$

If the circle is a great circle, this centre lies on the given plane.

$$\therefore\ -\left(1+\frac{\lambda}{2}\right)-2(2+\lambda)-3\left(1+\frac{3\lambda}{2}\right)+1=0$$
$$\text{or,}\quad -7\lambda-7=0 \quad \text{or,}\quad \lambda=-1.$$

Hence the equation is $x^2+y^2+z^2+x-2y-z+4=0$.

9. *Show that the circles*

$$x^2+y^2+z^2-2x+3y+4z-5=0, 5y+6z+1=0$$
$$\textit{and}\quad x^2+y^2+z^2-3x-4y+5z-6=0, x+2y-7z=0$$

lie on the same sphere and find its equation. [CH 95]

The equation of any sphere through the first circle is

$$x^2+y^2+z^2-2x+3y+4z-5+\lambda_1(5y+6z+1)=0 \tag{1}$$

and that of any sphere through the second circle is

$$x^2+y^2+z^2-3x-4y+5z-6+\lambda_2(x+2y-7z)=0. \tag{2}$$

These equations will be identical, if

$$-2=\lambda_2-3, \quad (3) \qquad 3+5\lambda_1=2\lambda_2-4, \quad (4)$$
$$4+6\lambda_1=5-7\lambda_2, \quad (5) \qquad \lambda_1-5=-6. \quad (6)$$

From (3) and (6), $\lambda_1=-1, \lambda_2=1$.

These values of λ_1 and λ_2 satisfy the equations (4) and (5). Thus the equations (3), (4), (5) and (6) are consistent. Consequently the two circles lie on the same sphere and the equation of it is

$$x^2+y^2+z^2-2x+3y+4z-5-(5y+6z+1)=0$$
$$\text{or,}\quad x^2+y^2+z^2-2x-2y-2z-6=0.$$

10. *A plane passes through a fixed point (p, q, r) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2$.*

[NH 2004; BH 91; 2002; 08]

Let the equation of the plane ABC be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{1}$$

The equation of the sphere $OABC$ is $x^2 + y^2 + z^2 - ax - by - cz = 0$.
If (α, β, γ) be the centre of this sphere, then

$$\alpha = \frac{a}{2}, \beta = \frac{b}{2} \quad \text{and} \quad \gamma = \frac{c}{2}. \tag{2}$$

Since the plane (1) passes through the point (p, q, r),

$$\frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1. \tag{3}$$

Putting the values of a, b and c from (2) in (3), we have $\frac{p}{\alpha} + \frac{q}{\beta} + \frac{r}{\gamma} = 2$.
Hence the required locus is $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2$.

EXERCISE VIA

1. Find the centre and radius of the following sphere:

   (i) $x^2 + y^2 + z^2 - 8y + 10z - 10 = 0$.

   (ii) $2(x^2 + y^2 + z^2) - 2x + 4y - 6z = 15$.

2. (a) Find the equation of the sphere passing through the following points.

   (i) $(0,0,0), (0,1,-1), (-1,2,0), (1,2,3)$.

   (ii) $(1,1,1), (-2,1,2), (3,-3,1), (-1,2,-1)$.

   (b) Is there a sphere passing through the points $(1,2,3), (2,5,-4), (1,4,-3)$ and $(4,7,-6)$?

   [*Hints.* Three points $(2,5,-4), (1,4,-3)$ and $(4,7,-6)$ lie on the line $\frac{x-2}{1} = \frac{y-5}{1} = \frac{z+4}{-1}$. Hence there is no sphere passing through the given points.]

3. Find the equation of the sphere which has the line segment joining the points $(2,3,4)$ and $(0,-1,2)$ as diameter.

4. Discuss the position of the point $(2,-3,0)$ w.r.t. the sphere $x^2 + y^2 + z^2 + 2x - 4y - 4z + 8 = 0$.

5. Find the equation of the sphere which passes through the origin and makes equal intercepts of unit length of the axes.

6. Find the equation of the sphere circumscribing the tetrahedron whose faces are
$$\frac{y}{3}+\frac{z}{4}=0, \frac{z}{4}+\frac{x}{2}=0, \frac{x}{2}+\frac{y}{3}=0, \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1.$$

7. Show that the equation to the sphere through the circle $x^2+y^2+z^2=9, 2x+3y+4z=5$ and the point $(1,2,3)$ is $3(x^2+y^2+z^2)-2x-3y-4z-22=0$.

8. (a) Show that the equation of the sphere for which the circle $x^2+y^2+z^2+7y-2z+2=0, 2x+3y+4z-8=0$ is a great circle is $x^2+y^2+z^2-2x+4y-6z+10=0$.

   (b) A sphere S has points $(0,1,0)$ and $(3,-5,2)$ as the ends of a diameter. Show that the equation of the sphere on which the intersection of the plane $5x-2y+4z+7=0$ with the given sphere S is a great circle is $x^2+y^2+z^2+2x+2y+2z+2=0$. **[CH 2000]**

9. Find the coordinates of the centre and radius of the circle $x^2+y^2+z^2-2y-4z-11=0, x+2y+2z=15$.

10. Show that the equations of that circle on the sphere $x^2+y^2+z^2-2x+4y-6z+3=0$ whose centre is $(2,3,-4)$ are $x^2+y^2+z^2-2x+4y-6z+3=0$ and $x+5y-7z-45=0$.

11. Find the equation of the sphere which passes through the circle $x^2+y^2=4, z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3.

    [*Hints.* Let the sphere through the circle $x^2+y^2=4, z=0$ be $x^2+y^2+z^2-4+2\lambda z=0$. The centre is $(0,0,-\lambda)$ and radius $=\sqrt{\lambda^2+4}$. The distance of the plane from the centre $=-\frac{2\lambda}{3}$. Here $(\lambda^2+4)-\frac{4\lambda^2}{9}=9$ or, $\lambda=\pm 3$.

    Hence there are two spheres with the required property and the equations of them are $x^2+y^2+z^2+6z-4=0$ and $x^2+y^2+z^2-6z-4=0$.]

12. (i) A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the locus of the centroid of the triangle ABC is the sphere $9(x^2+y^2+z^2)=4k^2$. **[BH 2006]**

    (ii) A sphere of constant radius r passes through the origin and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is given by $(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2$.

    [BH 94, 96, 2009; CH 97, 2003]

13. Show that the greatest and the least distances from the point $(1,-1,2)$ to the sphere $x^2+y^2+z^2-4x+6y-8z-71=0$ are 13 and 7 respectively.

14. Show that the two circles
$$x^2+y^2+z^2+3x-4y+3z=0, x-y+2z-4=0$$
$$\text{and}\quad 2(x^2+y^2+z^2)+8x-13y+17z-17=0, 2x+y-3z+1=0$$
lie on the same sphere and find its equation.

15. Find the equation of the circle passing through the points $(2,0,1), (-2,1,0)$ and $(0,3,5)$.

    [*Hints.* It is the plane section of the sphere passing through the given points and the origin by the plane through the given points.]

16. Find the equation of the sphere passing through the points $(2,0,0)$, $(0,2,0)$, $(0,0,2)$ and having the least possible radius.

    [*Hints.* Let the sphere be $(x-a)^2+(y-b)^2+(z-c)^2=r^2$.

    If it passes through the given points, then

$$(2-a)^2+b^2+c^2=r^2, a^2+(2-b)^2+c^2=r^2, a^2+b^2+(2-c)^2=r^2.$$

    From these three equations $a=b=c$.

$$\therefore\ (2-a)^2+a^2+a^2=r^2 \quad \text{or,} \quad 3\left[\left(a-\frac{2}{3}\right)^2+\frac{8}{9}\right]=r^2.$$

    Therefore, the least value of r^2 is $\frac{8}{3}$ when $a=\frac{2}{3}$. Thus the required equation is $3(x^2+y^2+z^2)-4(x+y+z)-4=0$.]

17. Find the smallest sphere (i.e. the sphere of smaller radius) which touches the lines $\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z-6}{1}$ and $\frac{x+3}{7}=\frac{y+3}{-6}=\frac{z+3}{1}$.

    [*Hints.* Since $\begin{vmatrix} 2+3 & 1+3 & 6+3 \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix} \neq 0$, the lines are skew lines.

    Hence the smallest sphere is described on the *s.d.* as diameter. The line of *s.d.* meets the lines at the points $(1,3,5)$ and $(-3,-3,-3)$. Thus the required sphere is $(x-1)(x+3)+(y-3)(y+3)+(z-5)(z+3)=0$.]

18. Obtain the equation of the sphere passing through four non-coplanar points (x_i, y_i, z_i), $i=1,2,3,4$.

    [*Hints.* Let the sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$. Since the given points lie on the sphere,

$$x_i^2+y_i^2+z_i^2+2ux_i+2vy_i+2wz_i+d=0, i=1,2,3,4.$$

    Eliminating u, v, w, d from the above five equations, we have

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix}=0.$$

    It is the required equation.]

19. Show that if all the plane sections of a surface represented by the equation of second degree are circles, the surface must be a sphere.

[*Hints.* Let the equation of the surface be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

The section made by the plane $z = 0$ is the conic $ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0$. Since it is a circle, $a = b, h = 0$.

Similarly for sections by the planes $x = 0$ and $y = 0$, we get $b = c, f = 0$ and $a = c, g = 0$. $\therefore\ a = b = c, f = g = h = 0$.

Consequently the equation of the surface reduces to

$$a(x^2 + y^2 + z^2) + 2ux + 2vy + 2wz + d = 0.$$

It is the equation of a sphere.]

20. Find the locus of the centre of the sphere which passes through the points $(0, 0, \pm c)$ and cuts the lines $y = \pm x \tan\alpha, z = \pm c$ at two points A and B, where AB has a constant length $2a$. **[CH 92, 94, 99, 2002, 08; NH 2002, 05]**

[*Hints.* Let the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. If it passes through $(0, 0, \pm c)$, $w = 0$ and $d = -c^2$.

Any points on the lines may be taken as $(r, r\tan\alpha, c)$ and $(r', -r'\tan\alpha, -c)$. If these points lie on the sphere, then

$$r = -2(u + v\tan\alpha)\cos^2\alpha \quad \text{and} \quad r' = -2(u - v\tan\alpha)\cos^2\alpha,$$
$$\text{i.e.}\quad r - r' = -4v\sin\alpha\cos\alpha \quad \text{and} \quad r + r' = -4u\cos^2\alpha.$$

Again $(r - r')^2 + (r + r')^2\tan^2\alpha + 4c^2 = 4a^2$.

$\therefore\ 16v^2\sin^2\alpha\cos^2\alpha + 16u^2\sin^2\alpha\cos^2\alpha + 4c^2 = 4a^2$ or, $u^2 + v^2 = (a^2 - c^2)\operatorname{cosec}^2 2\alpha$. Hence the locus of the centre is $x^2 + y^2 = (a^2 - c^2)\operatorname{cosec}^2 2\alpha$, $z = 0$.]

ANSWERS

1. (i) $(0, 4, -5), \sqrt{51}$; (ii) $(1/2, -1, 3/2), \sqrt{11}$.

2. (a) (i) $7(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0$;
(ii) $5(x^2 + y^2 + z^2) + 38x - 79y + 24z - 186 = 0$.

3. $x^2 + y^2 + z^2 - 2x - 2y - 6z + 5 = 0$. 4. Outside.

5. $x^2 + y^2 + z^2 - x - y - z = 0$. 6. $x^2 + y^2 + z^2 - 29\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{4}\right) = 0$.

9. $(1, 3, 4), \sqrt{7}$. 14. $x^2 + y^2 + z^2 + 5x - 6y + 7z - 8 = 0$.

15. $2(x^2 + y^2 + z^2) - 3x - 16y - 4z = 0, 7x + 18y - 10z - 4 = 0$.

B. Cylinder

6.20 Definition

A cylinder is a surface generated by a variable straight line which moves parallel to a fixed line and intersects a fixed curve not lying in a plane parallel to the fixed line or touches a given surface.

The given curve is called the *guiding curve* or *directrix* and the variable line is known as *generator*. If the guiding curve is a circle and the fixed line is normal to the plane of the circle through the centre of it, then the cylinder is *right circular* and the fixed line is called the *axis* of this cylinder. The distance between the axis and any generator is known as the *radius of the right circular cylinder* and it is equal to the radius of the guiding circle. Section of a right circular cylinder by a plane perpendicular to the axis is called a *normal section*. It is a circle of the same radius as that of the cylinder.

6.21 Equation of the cylinder whose guiding curve is $f(x, y) = 0, z = 0$ and generators are parallel to $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

Let (α, β, γ) be a point on the cylinder. The equation of the generating line through (α, β, γ) is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

On the guiding curve $z = 0$.

$\therefore$ this line meets the curve where

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n} \quad \text{or,} \quad x = \alpha - \frac{l\gamma}{n}, y = \beta - \frac{m\gamma}{n}, z = 0.$$

This point satisfies the equation of the guiding curve.

$$\therefore \; f\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}\right) = 0.$$

Hence the equation of the cylinder which is the locus of (α, β, γ) is

$$f\left(x - \frac{lz}{n}, y - \frac{mz}{n}\right) = 0.$$

Corollary I. $f(x, y) = 0$ represents a cylinder when the fixed line is the z-axis and the guiding curve is $f(x, y) = 0, z = 0$.

Corollary II. The equation of the cylinder whose generators are parallel to z-axis and which intersect the curve $f(x, y, z) = 0, \phi(x, y, z) = 0$ is obtained by eliminating z between these equations.

6.22 Equation of the right circular cylinder whose radius is a and axis is given by $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$.

Let (α, β, γ) be a point on the cylinder and the perpendicular from this point to the axis meet the axis at $(lr + x_1, mr + y_1, nr + z_1)$.

The d.rs. of the perpendicular are $lr + x_1 - \alpha, mr + y_1 - \beta, nr + z_1 - \gamma$.
From the condition of perpendicularity

$$l(lr + x_1 - \alpha) + m(mr + y_1 - \beta) + n(nr + z_1 - \beta) = 0$$

$$\text{or,} \quad r = \frac{l(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)}{l^2 + m^2 + n^2}. \tag{1}$$

Again this perpendicular distance is equal to the radius a.

$$\therefore \quad (lr + x_1 - \alpha)^2 + (mr + y_1 - \beta)^2 + (nr + z_1 - \gamma)^2 = a^2$$

$$\text{or,} \quad (l^2 + m^2 + n^2)r^2 + 2r\{l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)\}$$
$$+ (\alpha - x_1)^2 + (\beta - y_1)^2 + (\gamma - z_1)^2 = a^2$$

$$\text{or,} \quad (\alpha - x_1)^2 + (\beta - y_1)^2 + (\gamma - z_1)^2$$
$$- \frac{\{l(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)\}^2}{l^2 + m^2 + n^2} = a^2 \quad [\text{by}(1)].$$

Hence the locus of (α, β, γ) or the equation of the cylinder is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - \frac{\{l(x - x_1) + m(y - y_1) + n(z - z_1)\}^2}{l^2 + m^2 + n^2} = a^2.$$

Corollary. $x^2 + y^2 = a^2$ represents a right circular cylinder with the z-axis as the axis of the cylinder.

WORKED-OUT EXAMPLES

1. *Find the equation of the quadric cylinder with generators parallel to z-axis and passing through the curve*

$$ax^2 + by^2 + cz^2 = 1, lx + my + nz = p.$$

Eliminating z between $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = p$, we get

$$ax^2 + by^2 + c\left(\frac{p - lx - my}{n}\right)^2 = 1.$$

It represents a cylinder whose generators are parallel to z-axis and intersect the given curve.

2. *Find the equation of the cylinder whose directrix is $x^2 + y^2 = 9, z = 1$ and the fixed line is $\frac{x}{2} = \frac{y}{3} = \frac{z}{-1}$.*

Let (α, β, γ) be a point on the cylinder. The generating line through this point is $\frac{x-\alpha}{2} = \frac{y-\beta}{3} = \frac{z-\gamma}{-1}$.

It meets the directrix where

$$\frac{x - \alpha}{2} = \frac{y - \beta}{3} = \frac{1 - \gamma}{-1} \quad \text{or,} \quad x = \alpha + 2\gamma - 2, y = \beta + 3\gamma - 3, z = 1.$$

This point lies on the guiding curve.

$\therefore\ (\alpha+2\gamma-2)^2+(\beta+3\gamma-3)^2=9.$

Hence the locus of (α,β,γ), i.e. the equation of the cylinder is

$$(x+2z-2)^2+(y+3z-3)^2=9.$$

3. ***Find the equation of the right circular cylinder of radius 2 whose axis is the straight line** $\frac{x}{1}=\frac{y}{-2}=\frac{z}{2}$.*

The axis passes through the origin. The d.cs. of the line joining the origin $(0,0,0)$ and the point (α,β,γ) are $\frac{\alpha}{OP},\frac{\beta}{OP},\frac{\gamma}{OP}$. The d.cs. of the axis are $\frac{1}{3},\frac{-2}{3},\frac{2}{3}$.

The distance of (α,β,γ) from the axis

$$\begin{aligned}
&= OP\sin\theta \quad (\theta \text{ is the angle between } OP \text{ and the axis})\\
&= OP\sqrt{1-\cos^2\theta}\\
&= OP\sqrt{1-\left(\frac{\alpha-2\beta+2\gamma}{3\cdot OP}\right)^2}\\
&= \tfrac{1}{3}\sqrt{9\cdot OP^2-(\alpha-2\beta+2\gamma)^2}\\
&= \tfrac{1}{3}\sqrt{9(\alpha^2+\beta^2+\gamma^2)-(\alpha-2\beta+2\gamma)^2}.
\end{aligned}$$

$\therefore\ 2=\frac{1}{3}\sqrt{9(\alpha^2+\beta^2+\gamma^2)-(\alpha-2\beta+2\gamma)^2}$

or, $\ 8\alpha^2+5\beta^2+5\gamma^2+4\alpha\beta+8\beta\gamma-4\gamma\alpha=36.$

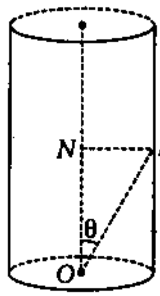

Fig. 29

Hence the required equation is $8x^2+5y^2+5z^2+4xy+8yz-4zx=36$.

EXERCISE VIB

1. Find the equation of the cylinder whose generators are parallel to x-axis and pass through the curve of intersection of $2x+3y-4z=5$ and $3x^2-4y^2+5z^2=4$.

2. Obtain the equation of the cylinder whose generators intersect the plane curve $ax^2+by^2=1, z=0$ and are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$. [BH 2005]

3. (i) Find the equation of the right circular cylinder of radius 5, whose axis passes through $(1,2,3)$ and is parallel to $\frac{x-4}{2}=\frac{y-3}{-1}=\frac{z-2}{2}$.

   (ii) Find the equation of the right circular cylinder whose axis is the straight line which passes through the point $(1,3,4)$ and has $1,-2,3$ as its direction ratios and radius equal to 3.

4. Show that the equation of the right circular cylinder whose guiding curve is the circle through the points $(1,0,0),(0,1,0),(0,0,1)$ is $x^2+y^2+z^2-yz-zx-xy=1$.

5. Show that $2x^2+5y^2+5z^2+4xy+2yz-4zx+16x+22y-10z-18=0$ is the equation of the cylinder which passes through the point $(3,-1,1)$ and has the axis $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-2}{1}$. [BH 2007, 09; NH 2008]

6. Find the equation of the circular cylinder whose guiding circle is $x^2+y^2+z^2=9,\ x-y+z=3$.

ANSWERS

1. $11y^2 + 68z^2 - 72yz - 90y + 120z + 59 = 0$.
2. $a(nx - lz)^2 + b(ny - mz)^2 = n^2$.
3. (i) $5x^2 + 8y^2 + 5z^2 + 4yz - 8zx + 4xy + 6x - 48y - 30z - 135 = 0$.
   (ii) $13x^2 + 10y^2 + 5z^2 + 4xy + 12yz - 6zx - 14x - 112y - 70z + 189 = 0$.
4. $x^2 + y^2 + z^2 + xy + yz - zx = 9$.

C. Cone

6.30 Definition

A cone is a surface generated by a straight line passing through a fixed point and intersecting a curve or touching a given surface.

The fixed point is known as the *vertex* and the given curve is called the *guiding curve* or *directrix* or *base.* Any line lying on the cone is called its generator.

6.31 The equation of a cone with its vertex as origin is homogeneous in x, y, z and conversely.

Let $f(x, y, z) = 0$ be the equation of a cone with the vertex at the origin O. If $P(x', y', z')$ be a point on the cone, the line OP will wholly lie on the cone. Any point on this line can be written as (rx', ry', rz'). Now (rx', ry', rz') satisfies the equation $f(x, y, z) = 0$ for all values of r. It is possible only when $f(x, y, z)$ is homogeneous in x, y, z.

Hence the proposition follows.

Conversely, if $f(x, y, z) = 0$ is an homogeneous equation in x, y, z, then (rx', ry', rz') will satisfy the equation when $P(x', y', z')$ satisfies the equation. Therefore, the line OP wholly lies on $f(x, y, z) = 0$. Hence $f(x, y, z) = 0$ represents a cone with the vertex as origin.

Corollary. If $f(x, y, z) = 0$ be a homogeneous equation of a cone and $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be a generator, then $f(l, m, n) = 0$.

Note. If $f(x, y, z)$ is factorizable into two linear factors, $f(x, y, z) = 0$ will represent a pair of planes. Otherwise it represents a cone.

6.32 Equation of a cone with given vertex and directrix

(i) *To find the equation of the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + cz^2 = 1$.*

Since the vertex is the origin, the equation of the cone is homogeneous. Thus making $ax^2 + by^2 + cz^2 = 1$ homogeneous by $lx + my + nz = p$, the required equation is obtained.

The equation is

$$ax^2+by^2+cz^2=\left(\frac{lx+my+nz}{p}\right)^2 \text{ or, } p^2(ax^2+by^2+cz^2)=(lx+my+nz)^2.$$

(ii) *To find the equation of the cone whose vertex is the point* (α,β,γ) *and the base is the conic* $z=0, f(x,y)=0$.

The equations of a line through the point (α,β,γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \tag{1}$$

It meets the plane $z=0$, where $x=\alpha-\frac{l\gamma}{n}, y=\beta-\frac{m\gamma}{n}$.

If the line meets the curve $f(x,y)=0$, then

$$f\left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n}\right)=0. \tag{2}$$

Eliminating l,m,n from (1) and (2),

$$f\left(\alpha-\gamma\frac{x-\alpha}{z-\gamma},\beta-\gamma\frac{y-\beta}{z-\gamma}\right)=0 \quad \text{or,} \quad f\left(\frac{\alpha z-\gamma x}{z-\gamma},\frac{\beta z-\gamma y}{z-\gamma}\right)=0.$$

It is the required equation.

6.33 Right circular cone

Definition. *A cone formed by a variable line passing through a fixed point (vertex) and making a constant angle with a given line through the vertex is called a right circular cone.* The given line and the constant angle are called the *axis* and the *semi-vertical angle* of the cone respectively. The section of a right circular cone by a plane perpendicular to the axis is circular.

To find the equation of the right circular cone whose vertex is the point (α,β,γ), *the semi-vertical angle is* θ *and the axis is* $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$.

Let A be the vertex, AN the axis and $P(x_1,y_1,z_1)$ a point on the cone.

The d.cs. of the axis are

$$\frac{l}{\sqrt{l^2+m^2+n^2}},\frac{m}{\sqrt{l^2+m^2+n^2}},\frac{n}{\sqrt{l^2+m^2+n^2}}.$$

The d.cs. of the line AP are

$$\frac{x_1-\alpha}{AP},\frac{y_1-\beta}{AP},\frac{z_1-\gamma}{AP}.$$

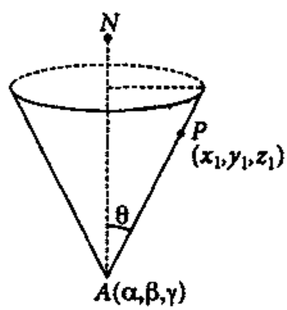

Fig. 30

Since θ is the angle between these two lines

$$\cos\theta=\frac{l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)}{AP\cdot\sqrt{l^2+m^2+n^2}}$$

or, $AP^2(l^2+m^2+n^2)\cos^2\theta$

$$=\{l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)\}^2$$

or, $\{(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2\}(l^2+m^2+n^2)\cos^2\theta$

$$=\{l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)\}^2.$$

Hence the locus of P, i.e. the equation of the cone is

$$\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}(l^2+m^2+n^2)\cos^2\theta$$
$$=\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\}^2.$$

Corollary I. If the vertex is the origin, then $(\alpha,\beta,\gamma)\equiv(0,0,0)$. In this case, the equation of the cone is

$$(x^2+y^2+z^2)(l^2+m^2+n^2)\cos^2\theta=(lx+my+nz)^2.$$

Note that it is a homogeneous equation of second degree.

Corollary II. If the vertex is the origin and the z-axis is the axis of the cone, then $(\alpha,\beta,\gamma)\equiv(0,0,0)$ and $l=0, m=0, n=1$.

$\therefore$ the equation of the cone is

$$(x^2+y^2+z^2)\cos^2\theta=z^2 \quad \text{or,} \quad x^2+y^2=z^2\tan^2\theta.$$

6.34 Condition for the general equation of second degree to represent a cone and coordinates of vertex

Let
$$F(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux$$
$$+2vy+2wz+d=0. \tag{1}$$

represent a cone with the vertex at (x',y',z').

Changing the origin to (x',y',z') with the axes remaining parallel the equation (1) transforms to

$$a(x+x')^2+b(y+y')^2+c(z+z')^2+2f(y+y')(z+z')+2g(z+z')(x+x')$$
$$+2h(x+x')(y+y')+2u(x+x')+2v(y+y')+2w(z+z')+d=0$$

or,
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2(ax'+hy'+gz'+u)x$$
$$+2(hx'+by'+fz'+v)y+2(gx'+fy'+cz'+w)z$$
$$+ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y'$$
$$+2ux'+2vy'+2wz'+d=0. \tag{2}$$

The equation of a cone with the vertex at the origin is a second degree homogeneous. Therefore, the equation (2) should be homogeneous. For this the coefficients of x,y,z and the absolute term should be zero.

Thus
$$ax'+hy'+gz'+u=0, \tag{3}$$
$$hx'+by'+fz'+v=0, \tag{4}$$
$$gx'+fy'+cz'+w=0 \tag{5}$$

and $ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y'+2ux'+2vy'+2wz'+d=0$

or, $x'(ax'+hy'+gz'+u)+y'(hx'+by'+fz'+v)+z'(gx'+fy'+cz'+w)$
$+(ux'+vy'+wz'+d)=0$

or, $ux'+vy'+wz'+d=0$ [by (3), (4), (5)]. (6)

Eliminating x', y', z' from (3), (4), (5) and (6), the required condition is

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0. \tag{7}$$

The coordinates of the vertex are obtained by solving any three of the equations (3), (4), (5) and (6) when the condition (7) holds.

Note 1. If $F(x, y, z, t) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2uxt + 2vyt + 2wzt + dt^2$

then
$$\frac{\partial F}{\partial x} = 2(ax + hy + gz + ut), \frac{\partial F}{\partial y} = 2(hx + by + fz + vt)$$
$$\frac{\partial F}{\partial z} = 2(gx + fy + cz + wt), \frac{\partial F}{\partial t} = 2(ux + vy + wz + dt).$$

Putting $t = 1$, the equations $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial t} = 0$ are satisfied by the vertex (x', y', z'). Elimination of (x', y', z') from these equations gives the condition for representing a cone.

Note 2. The equation $F(x, y, z) = 0$ represents a cone only when $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial t} = 0$ are consistent.

In the case of consistency, the vertex is obtained from any three of the above four equations.

Note 3. If $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ with the condition (7), then the general equation represents a pair of planes.

Example. Show that the equation $x^2 + 2y^2 + z^2 - 4yz - 6zx - 2x + 8y - 2z + 9 = 0$ represents a cone. Find the vertex.

Making the expression of the equation homogeneous, we have

$$F(x, y, z, t) \equiv x^2 + 2y^2 + z^2 - 4yz - 6zx - 2xt + 8yt - 2zt + 9t^2.$$

Putting $t = 1$ in $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \frac{\partial F}{\partial t}$ and then equating to zero, the four equations are

$$x - 3z - 1 = 0, \quad (1) \qquad y - z + 2 = 0, \quad (2)$$
$$3x + 2y - z + 1 = 0, \quad (3) \qquad x - 4y + z - 9 = 0. \quad (4)$$

From (1), (2) and (3), $x = 1, y = -2, z = 0$.

These values satisfy the equation (4). Therefore, the equations are consistent. Consequently the given equation represents a cone with the vertex at $(1, -2, 0)$.

6.35 Angle between lines in which a plane cuts a cone

Let $$\phi(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad (1)$$

be a cone and

$$ux+vy+wz+d=0 \quad (2)$$

be a plane which cuts the cone along two lines.

Let a line in which the plane cuts the cone be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}. \quad (3)$$

Since the line (3) lies on the plane (2),

$$ul+vm+wn=0. \quad (4)$$

This line is a generator of the cone (1).

$$\therefore\ al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0. \quad (5)$$

Eliminating n from (5) by (4),

$$al^2+bm^2+c\left(\frac{ul+vm}{w}\right)^2-2fm\frac{ul+vm}{w}-2gl\frac{ul+vm}{w}+2hlm=0$$

or, $$(cu^2+aw^2-2gwu)l^2+2(hw^2+cuv-fuw-gvw)lm$$
$$+(bw^2+cv^2-2fvw)m^2=0$$

or, $$(cu^2+aw^2-2gwu)\left(\frac{l}{m}\right)^2+2(hw^2+cuv-fuw-gvw)\frac{l}{m}$$
$$+(bw^2+cv^2-2fvw)=0. \quad (6)$$

Since (6) is quadratic in $\frac{l}{m}$, the plane (2) cuts the cone in two lines. If their d.rs. are l_1,m_1,n_1 and l_2,m_2,n_2, then $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ are the roots of (6). Getting the values of $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ from (6), the values of n_1 and n_2 in terms of l_1 and l_2 are obtained from (4).

If $m_1=\lambda_1 l_1, n_1=\mu_1 l_1$ and $m_2=\lambda_2 l_2, n_2=\mu_2 l_2$, then the lines of section are $\frac{x}{1}=\frac{y}{\lambda_1}=\frac{z}{\mu_1}$ and $\frac{x}{1}=\frac{y}{\lambda_2}=\frac{z}{\mu_2}$.

Thus the angle between the lines is

$$\cos^{-1}\frac{1+\lambda_1\lambda_2+\mu_1\mu_2}{\sqrt{1+\lambda_1^2+\mu_1^2}\sqrt{1+\lambda_2^2+\mu_2^2}}.$$

Corollary I. *Condition for two perpendicular generators made by a plane.*

From (6),

$$\frac{l_1l_2}{m_1m_2}=\frac{bw^2+cv^2-2fvw}{cu^2+aw^2-2gwu} \quad \text{(as product of the roots).}$$

From symmetry

$$\frac{l_1 l_2}{n_1 n_2} = \frac{bw^2 + cv^2 - 2fvw}{av^2 + bu^2 - 2huv}.$$

$$\therefore \frac{l_1 l_2}{bw^2 + cv^2 - 2fvw} = \frac{m_1 m_2}{cu^2 + aw^2 - 2gwu} = \frac{n_1 n_2}{av^2 + bu^2 - 2huv} = k \text{ (say)}.$$

$$\begin{aligned} \therefore\ & l_1 l_2 + m_1 m_2 + n_1 n_2 \\ &= k[a(v^2 + w^2) + b(w^2 + u^2) + c(u^2 + v^2) - 2fvw - 2gwu - 2huv] \\ &= k[(a + b + c)(u^2 + v^2 + w^2) - \phi(u, v, w)]. \end{aligned} \tag{7}$$

If the plane cuts the cone in two perpendicular generators, then $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$, i.e.

$$(a + b + c)(u^2 + v^2 + w^2) - \phi(u, v, w) = 0. \tag{8}$$

Corollary II. *Necessary and sufficient condition for three mutually perpendicular generators.*

Let the normal to the plane (2) through the origin, i.e. $\frac{x}{u} = \frac{y}{v} = \frac{z}{w}$ is a generator of the cone.

Then $\phi(u, v, w) = 0$.

In this case, $l_1 l_2 + m_1 m_2 + n_1 n_2 = k(a + b + c)(u^2 + v^2 + w^2)$.

If the plane now cuts the cone in two perpendicular generators, then

$$a + b + c = 0. \quad [\because\ u^2 + v^2 + w^2 \neq 0.] \tag{9}$$

It is independent of u, v, w. Therefore, $a + b + c = 0$ is the necessary condition for three mutually perpendicular generators of the cone (1).

From the condition of $a + b + c = 0$ it follows that the plane perpendicular to an arbitrary generator cuts the cone in two perpendicular generators. Therefore, $a + b + c = 0$ is the sufficient condition for three mutually perpendicular generators.

[**Alternative approach:** If the cone (1) passes through the coordinate axes, then the d.cs. of the axes, i.e. $1, 0, 0; 0, 1, 0; 0, 0, 1$ satisfy the equation (1). Therefore, $a = 0 = b = c$. Consequently $fyz + gzx + hxy = 0$ represents a cone passing through the coordinate axes. If the equation (1) has three mutually perpendicular generators, let us transform the equation (1) into a new coordinate system whose axes are these three generators. The equation (1) will transform into the form $f'yz + g'zx + h'xy = 0$.

By the property of invariant under transformation of axes the sum of the coefficients of x^2, y^2 and z^2 remains unchanged. $\therefore$ $a + b + c = 0$ is the required condition.]

Corollary III. *If $a + b + c = 0$, the cone (1) has an infinite number of triads of mutually perpendicular generators.*

WORKED-OUT EXAMPLES

1. *Find the equation of the cone, whose vertex is the origin and base is the circle* $x = a, y^2 + z^2 = b^2$.

In Fig. 31, $OC = a, AC = b$ (radius of the base).

If θ is the semi-vertical angle, then $\tan\theta = \frac{b}{a}$. The equation of the right circular cone whose vertex, semi-vertical angle and axis are the origin, θ and the x-axis respectively is

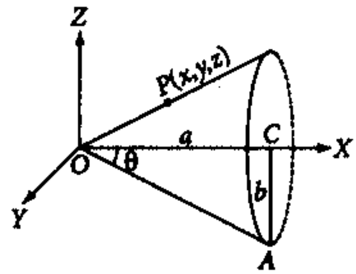

Fig. 31

$$y^2 + z^2 = x^2\tan^2\theta \text{ or, } y^2 + z^2 = x^2 \cdot \frac{b^2}{a^2}$$
$$\text{or, } a^2(y^2 + z^2) = b^2x^2.$$

2. *The axis of a right cone, vertex O, makes equal angles with the coordinates axes, and the cone passes through the line drawn from O, with d.cs. proportional to* $1, -2, 2$. *Find the equation to the cone.*

The d.cs. of the axis are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and the d.cs. of the given generating line are $\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}$.

If θ be the semi-vertical angle, $\cos\theta = \frac{1-2+2}{3\sqrt{3}} = \frac{1}{3\sqrt{3}}$.

$\therefore$ the equation of the cone is

$$(x^2 + y^2 + z^2) \cdot \tfrac{1}{27} = \tfrac{1}{3}(x + y + z)^2 \quad \text{[by Cor. 1 of Sec 6.33]}$$
$$\text{or,} \quad 9(x + y + z)^2 = x^2 + y^2 + z^2.$$

3. *Find the equations of the lines of intersection of the plane* $3x + 4y + z = 0$ *and the cone* $15x^2 - 32y^2 - 7z^2 = 0$.

Let the equation of the generating line in which the plane cuts the cone be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

The line lies on the cone and also on the plane.

$$\therefore 15l^2 - 32m^2 - 7n^2 = 0 \tag{1}$$
$$\text{and} \quad 3l + 4m + n = 0. \tag{2}$$

Eliminating n from (1) and (2),

$$15l^2 + 32m^2 - 7(3l + 4m)^2 = 0$$
$$\text{or,} \quad 2l^2 + 7mn + 6m^2 = 0$$
$$\text{or,} \quad (2l + 3m)(l + 2m) = 0 \quad \text{or,} \quad m = -\frac{2}{3}l, -\frac{1}{2}l.$$

$$\text{For}\quad m = -\frac{2}{3}l, n = -(3l+4m) = -\left(3l - \frac{8}{3}l\right) = -\frac{l}{3}.$$

$$\text{For}\quad m = -\frac{1}{2}l, n = -(3l+4m) = -(3l-2l) = -l.$$

$$\therefore\quad \text{the d.cs. are proportional to } 1, -\frac{2}{3}, -\frac{1}{3} \text{ or, } 1, -\frac{1}{2}, -1.$$

Hence the equations of the required generators are

$$\frac{x}{3} = \frac{y}{-2} = \frac{z}{-1} \quad \text{and} \quad \frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}.$$

4. *The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A, B, C. Prove that the equation to the cone generated by lines drawn from O to meet the circle ABC is*

$$\left(\frac{b}{c}+\frac{c}{b}\right)yz + \left(\frac{c}{a}+\frac{a}{c}\right)zx + \left(\frac{a}{b}+\frac{b}{a}\right)xy = 0.$$

[BH 93, 95, 96, 98, 2003, 05, 10; CH 96, 2005]

The plane meets the axes at $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$. The equation of the sphere through $(0,0,0)$ and the above three points is $x^2+y^2+z^2-ax-by-cz=0$.

$\therefore$ the equations of the circle are

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1, \qquad (1) \qquad\qquad x^2+y^2+z^2-ax-by-cz=0. \qquad (2)$$

The equation of the cone whose vertex is the origin is homogeneous. Thus the equation to the cone is obtained by making (2) homogeneous with the help of (1). Hence the required equation is

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

$$\text{or,}\quad \left(\frac{b}{c}+\frac{c}{b}\right)yz + \left(\frac{c}{a}+\frac{a}{c}\right)zx + \left(\frac{a}{b}+\frac{b}{a}\right)xy = 0.$$

5. *Show that the condition that the plane $ax+by+cz=0$ may cut the cone $yz+zx+xy=0$ in perpendicular lines is $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.*

[BH 94, 98, 2003; CH 91, 97; 2001; NH 2007]

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the generators of the cone made by the plane. Then

$$al+bm+cn=0 \qquad (1)$$

$$\text{and}\quad mn+nl+lm=0. \qquad (2)$$

Eliminating n from (1) and (2),

$$-(l+m)\frac{al+bm}{c}+lm=0$$

$$\text{or,}\quad al^2+(a+b-c)lm+bm^2=0$$

$$\text{or,}\quad a\left(\frac{l}{m}\right)^2+(a+b-c)\frac{l}{m}+b=0.$$

Let the roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$. Then $\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b}{a}$.

From symmetry $\frac{l_1}{n_1} \cdot \frac{l_2}{n_2} = \frac{c}{a}$. $\therefore$ $al_1l_2 = bm_1m_2 = cn_1n_2$.

If the generators are at right angle, then

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \quad \text{or,} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

6. *Find the equation of the cone whose vertex is* $(1, 0, -1)$ *and which passes through the circle* $x^2 + y^2 + z^2 = 4, x + y + z = 1$.

Let $\frac{x-1}{l} = \frac{y}{m} = \frac{z+1}{n}$ be a generator. If it meets the base at the point $(lr + 1, mr, nr - 1)$, then

$$lr + 1 + mr + nr - 1 = 1 \quad \text{or,} \quad (l + m + n)r = 1 \tag{1}$$

$$\text{and} \quad (lr + 1)^2 + m^2r^2 + (nr - 1)^2 = 4$$

$$\text{or,} \quad (l^2 + m^2 + n^2)r^2 + 2(l - n)r = 2. \tag{2}$$

By (1) and (2), $(l^2 + m^2 + n^2) + 2(l - n)(l + m + n) = 2(l + m + n)^2$.

Now eliminating l, m, n by the equation of the generator the equation of the cone is

$$(x - 1)^2 + y^2 + (z + 1)^2 + 2(x - z - 2)(x + y + z) = 2(x + y + z)^2.$$

7. *Lines are drawn through the origin to meet the circle in which the plane* $x + y + z = 1$ *cuts the sphere* $x^2 + y^2 + z^2 - 4x - 6y - 8z + 4 = 0$. *Show that they lie on the cone* $x^2 - y^2 - 3z^2 - 6yz - 4zx - 2xy = 0$ *and meet the sphere again at points on the plane* $y + 2z = 2$. **[NH 2005; BH 91]**

The circle is

$$x + y + z = 1, \quad (1) \qquad x^2 + y^2 + z^2 - 4x - 6y - 8z + 4 = 0. \quad (2)$$

The equation of the cone is obtained by making (2) homogeneous with the help of (1). Hence the equation is

$$x^2 + y^2 + z^2 - 2(2x + 3y + 4z)(x + y + z) + 4(x + y + z)^2 = 0$$

$$\text{or,} \quad x^2 - y^2 - 3z^2 - 6yz - 4zx - 2xy = 0. \tag{3}$$

Again by making (2) homogeneous with the help of $y + 2z = 2$, the equation is

$$x^2 + y^2 + z^2 - 2(2x + 3y + 4z)\frac{y + 2z}{2} + 4\left(\frac{y + 2z}{2}\right)^2 = 0$$

$$\text{or,} \quad x^2 - y^2 - 3z^2 - 6yz - 4zx - 2xy = 0. \tag{4}$$

Since (3) and (4) are identical, the result follows.

8. *The section of a cone whose guiding curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ by the plane $x=0$ is a rectangular hyperbola. Show that the locus of the vertex is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$.* **[NH 2004; BH 2007; CH 92; 2003; 07]**

Let (α,β,γ) be the vertex and the equation of a generator be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \tag{1}$$

It meets the guiding curve at the point $\left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n},0\right)$.

Putting these values in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

$$\frac{1}{a^2}\left(\alpha-\frac{l\gamma}{n}\right)^2+\frac{1}{b^2}\left(\beta-\frac{m\gamma}{n}\right)^2=1. \tag{2}$$

Eliminating l,m,n by (1),

$$\frac{1}{a^2}\left(\alpha-\gamma\frac{x-\alpha}{z-\gamma}\right)^2+\frac{1}{b^2}\left(\beta-\gamma\frac{y-\beta}{z-\gamma}\right)^2=1$$

$$\text{or,}\quad \frac{1}{a^2}(\alpha z-\gamma x)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2. \tag{3}$$

It is the equation of the cone.

The section by $x=0$ is the curve

$$\frac{\alpha^2}{a^2}z^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2, x=0.$$

It will be a rectangular hyperbola, if the coefficient of y^2+ the coefficient of $z^2=0$,

$$\text{i.e.}\quad \frac{\gamma^2}{b^2}+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)=0\quad\text{or,}\quad \frac{\alpha^2}{a^2}+\frac{\beta^2+\gamma^2}{b^2}=1.$$

Hence the locus of the vertex is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$.

9. *Find the equation of the cone through the coordinate axes and the lines in which $lx+my+nz=0$ cuts the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.*

Any cone through coordinate axes is

$$Fyz+Gzx+Hxy=0. \tag{1}$$

The given cone and the cone (1) intersect in four lines. Two of them are contained in the plane $lx+my+nz=0$.

Let the other two lie on $l'x+m'y+n'z=0$.

$$\begin{aligned}\text{Thus}\quad & ax^2+by^2+cz^2+2fyz+2gzx+2hxy+\lambda(Fyz+Gzx+Hxy)\\ &=(lx+my+nz)(l'x+m'y+n'z).\end{aligned}$$

Equating the coefficients,

$$ll' = a \quad \text{or,} \quad l' = \frac{a}{l}, mm' = b \quad \text{or,} \quad m' = \frac{b}{m}, nn' = c \quad \text{or,} \quad n' = \frac{c}{n},$$

$$2f + \lambda F = mn' + m'n = \frac{cm}{n} + \frac{bn}{m}, \text{ i.e.} \lambda F = \frac{cm^2 + bn^2 - 2fmn}{mn}. \quad (2)$$

$$\text{Similarly} \quad \lambda G = \frac{an^2 + cl^2 - 2gnl}{ln} \quad \text{and} \quad \lambda H = \frac{bl^2 + am^2 - 2hlm}{lm}.$$

Putting these values in (1), the equation of the required cone is

$$l(cm^2 + bn^2 - 2fmn)yz + m(an^2 + cl^2 - 2gnl)zx + n(bl^2 + am^2 - 2hlm)xy = 0.$$

10. *Two cones are described with guiding curves $zx = a^2, y = 0$ and $yz = b^2, x = 0$ and with any vertex. Show that if their four common generators meet the plane $z = 0$ in four concyclic points, the vertex lies on the surface $z(x^2 + y^2) = a^2x + b^2y$.*

Let us find the equation of the cone with the vertex (α, β, γ) and base $zx = a^2$, $y = 0$.

The equations of a line through (α, β, γ) are

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}. \quad (1)$$

It meets the plane $y = 0$ at $\left(\alpha - \frac{l\beta}{m}, 0, \gamma - \frac{n\beta}{m}\right)$.

If this point lies on the base, then $\left(\gamma - \frac{n\beta}{m}\right)\left(\alpha - \frac{l\beta}{n}\right) = a^2$.

Eliminating l, m, n by (2),

$$\left(\gamma - \beta\frac{z - \gamma}{y - \beta}\right)\left(\alpha - \beta\frac{x - \alpha}{y - \beta}\right) = a^2$$

$$\text{or,} \quad (\gamma y - \beta z)(\alpha y - \beta x) = a^2(y - \beta)^2. \quad (2)$$

It is the equation of the cone.

Similarly the equation of the cone with vertex (α, β, γ) and base $yz = b^2, x = 0$ is

$$(\beta x - \alpha y)(\gamma x - \alpha z) = b^2(x - a)^2. \quad (3)$$

(2) and (3) meet the plane $z = 0$ in two conics whose equations are

$$\gamma y(\alpha y - \beta x) - a^2(y - \beta)^2 = 0, z = 0 \quad (4)$$

$$\text{and} \quad (\beta x - \alpha y)\gamma x - b^2(x - \alpha)^2 = 0, z = 0. \quad (5)$$

Any curve through (4) and (5) is

$$\gamma y(\alpha y - \beta x) - a^2(y - \beta)^2 + \lambda\{\gamma x(\beta x - \alpha y) - b^2(x - \alpha)^2\} = 0, z = 0.$$

If it represents a circle, then

coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$, i.e. $\lambda(\beta\gamma - b^2) = \gamma\alpha - a^2$ and $\beta\gamma + \lambda\gamma\alpha = 0$.

$\because\ \gamma \neq 0, \lambda = -\frac{\beta}{\alpha}$.

$\therefore\ -\frac{\beta}{\alpha}(\beta\gamma - b^2) = \gamma\alpha - a^2$ or, $\gamma(\alpha^2 + \beta^2) = a^2\alpha + b^2\beta$.

Hence the vertex lies on $z(x^2 + y^2) = a^2x + b^2y$.

EXERCISE VIC

1. Find the equation to the right circular cone, whose vertex is the origin, semi-vertical angle $\pi/4$ and the axis is $x = y = z$.

2. Find the equation of a right circular cone which passes through the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and whose axis is $\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$.

3. Find the equation of a right circular cone whose vertex is $(1, 2, 3)$, the semi-vertical angle $\pi/3$ and the axis is parallel to $\frac{x}{2} = \frac{y}{5} = \frac{z}{7}$.

4. (i) Find the equation to the cone with the vertex at the origin and which passes through the curve of intersection of $x^2 + 5y^2 - 7z^2 = 1$ and $2x - 3y + 4z = 1$.

   (ii) Find the equation of the cone generated by lines drawn through the origin to meet the circle through the points $(3, 0, 0), (0, 2, 0), (0, 0, 1)$.

   [BH 2007, 09]

5. (i) Find the equation of the cone whose vertex is $(2, 3, 4)$ and the base is $x^2 + y^2 = 25, z = 0$.

   (ii) Find the equation of the cone whose vertex is $(1, -2, 3)$ and generators having d.cs. satisfy the relation $3l^2 + 5m^2 + 7n^2 = 0$.

6. (i) Determine the angle between the lines of intersection of the plane $x - 3y + z = 0$ and a quadric cone $x^2 - 5y^2 + z^2 = 0$.

   (ii) Find the equation of the lines in which the plane $2x - 6y - 5z = 0$ cuts the cone $xy + yz + zx = 0$.

   (iii) Show that the plane $3x - 2y - z = 0$ cuts the cones $21x^2 - 4y^2 - 5z^2 = 0$ and $3yz - 2zx + 2xy = 0$ in the same pair of perpendicular lines.

   (iv) If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equations to the other two generators.

   [CH 93, 98, 2009]

7. If a right circular cone has three mutually perpendicular generators, prove that the semi-vertical angle is $\tan^{-1}\sqrt{2}$.

8. Find the equation of the cone whose vertex is (α, β, γ) and base is $ax^2 + by^2 = 1, z = 0$.

9. Find the equation of the cone obtained by revolving the straight line $x - 2y = 1$ about x-axis and find the coordinates of its vertex.

10. (i) Prove that the angle between the lines given by $x + y + z = 0, ayz + bzx + cxy = 0$ is $\pi/2$, if $a + b + c = 0$.

(ii) Prove that the plane $ux+vy+wz = 0$ may cut the cone $ax^2+by^2+cz^2 = 0$ in perpendicular generators if $(b+c)u^2 + (c+a)v^2 + (a+b)w^2 = 0$.

[NH 2002, 05; CH 94, 2002]

11. (i) Show that the equation

$$x^2 - 2y^2 + 3z^2 - 4xy + 5yz - 6zx + 8x - 19y - 2z - 20 = 0$$

represents a cone with vertex $(1, -2, 3)$. **[NH 2001]**

(ii) Show that the equation

$$7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26x - 2y + 2z - 17 = 0$$

represents a cone whose vertex is $(1, -2, 2)$.

12. Show that the locus of the points from which three mutually perpendicular lines can be drawn to intersect the conic $z = 0, ax^2 + by^2 = 1$ is $ax^2 + by^2 + (a+b)z^2 = 1$.

13. Show that the equation of the cone with vertex $(5, 4, 3)$ and base $3x^2 + 2y^2 = 6, y + z = 0$ is

$$3(5y + 5z - 7x)^2 + 2(4z - 3y)^2 = 6(y + z - 7)^2.$$

14. Show that the equation of the cone whose vertex is the origin and base is the curve $f(x, y) = 0, z = k$ is given by $f\left(\frac{kx}{z}, \frac{ky}{z}\right) = 0$.

15. Show that the equation of the cone whose vertex is the origin and base is the curve $ax^2+by^2+cz^2 = 1, \alpha x^2+\beta y^2 = 2z$ is $4(ax^2+by^2+cz^2)z^2 = (\alpha x^2+\beta y^2)^2$.

16. Prove that $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represents a cone, if $bcu^2 + cav^2 + abw^2 = abcd$ and $abc \neq 0$.

17. Prove that a cone of second degree can be found to pass through any five concurrent lines.

[*Hints.* Let the lines and the cone through the origin be

$$\frac{x}{l_i} = \frac{y}{m_i} = \frac{z}{n_i} (i = 1, 2, 3, 4, 5)$$
$$\text{and} \quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

The equation of the cone can be written as

$$x^2 + \frac{b}{a}y^2 + \frac{c}{a}z^2 + 2\frac{f}{a}yz + 2\frac{g}{a}zx + 2\frac{h}{a}xy = 0$$
$$\text{or,} \quad x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0. \qquad (6)$$

There are five arbitrary constants and these can be determined by five independent conditions. D.cs. of any generator of (1) satisfy the equation (1), so any five lines through the origin are sufficient to determine the five arbitrary constants. Thus a cone of second degree can be found to pass through five concurrent lines.]

18. A cone has for its guiding curve the circle $x^2+y^2+2ax+2by=0, z=0$ and passes through a fixed point $(0,0,c)$. If the section of the cone by the plane $x=0$ is a rectangular hyperbola, prove that the vertex lies on the fixed circle $x^2+y^2+z^2+2ax+2by=0, 2ax+2by+cz=0$.

19. Find the locus of the vertices of the right circular cones that pass through the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$.

[*Hints.* The equation of the right circular cone with vertex (α,β,γ), semi-vertical angle θ and axis having d.cs. l,m,n is

$$\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}\cos^2\theta=(lx+my+nz-l\alpha-m\beta-n\gamma)^2.$$

The section at $z=0$ is

$$\begin{aligned}x^2(\cos^2\theta-l^2)+y^2(\cos^2\theta-m^2)-2lmxy&\\ +2\{l(l\alpha+m\beta+n\gamma)-\alpha\cos^2\theta\}x&\\ +2\{m(l\alpha+m\beta+n\gamma)-\beta\cos^2\theta\}y&\\ +(\alpha^2+\beta^2+\gamma^2)\cos^2\theta-(l\alpha+m\beta+n\gamma)^2&=0.\end{aligned}$$

Comparing it with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$lm=0, \tag{1}$$

$$l\sum l\alpha-\alpha\cos^2\theta=0, \tag{2}$$

$$m\sum l\alpha-\beta\cos^2\theta=0, \tag{3}$$

$$(\cos^2\theta-l^2)a^2=(\cos^2\theta-m^2)b^2=(\sum l\alpha)^2-\sum\alpha^2\cos^2\theta. \tag{4}$$

If $l=0$, then $\alpha=0$ by (2). In this case

$$\begin{aligned}a^2\cos^2\theta&=(\cos^2\theta-m^2)b^2\\ &=(m\beta+n\gamma)^2-(\beta^2+\gamma^2)\cos^2\theta \quad \text{and} \quad m(m\beta+n\gamma)=\beta\cos^2\theta.\end{aligned}$$

Eliminating θ, we have

$$\frac{\beta^2}{a^2-b^2}+\frac{\gamma^2}{a^2}+1=0.$$

For $m=0, \beta=0$. In this case, $\frac{\alpha^2}{a^2-b^2}-\frac{\gamma^2}{b^2}=1$.

Hence the locus is $x=0, \frac{y^2}{a^2-b^2}+\frac{z^2}{a^2}+1=0$ or, $y=0, \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2}=1$.]

20. Show that the plane $3x + 2y - 4z = 0$ passes through a pair of common generators of the cones $27x^2 + 20y^2 - 32z^2 = 0$ and $2yz + zx - 4xy = 0$. Also show that the plane containing the other two generators is $9x + 10y + 8z = 0$.

[*Hints*. Any surface through the intersection of the given cones is of the type $27x^2 + 20y^2 - 32z^2 + \lambda(2yz + zx - 4xy) = 0$. λ is a constant. It will represent a pair of planes, if

$$\begin{vmatrix} 27 & -2\lambda & \lambda/2 \\ -2\lambda & 20 & \lambda \\ \lambda/2 & \lambda & -32 \end{vmatrix} = 0$$

$$\text{or,} \quad \lambda^3 - 48\lambda^2 + 8640 = 0 \quad \text{or,} \quad \lambda = -12, 30 \pm 6\sqrt{5}.$$

For $\lambda = -12$, the pair of planes is given by

$$27x^2 + 20y^2 - 32z^2 - 24yz - 12zx + 48xy = 0$$

$$\text{or,} \quad (3x + 2y - 4z)(9x + 10y + 8z) = 0.$$

Hence the result follows.]

21. Show that the equation of the cone which passes through the common generators of the cones $-2x^2 + 4y^2 + z^2 = 0$ and $10xy - 2yz + 5zx = 0$ and the line with d.rs. $1, 2, 3$ is $2x^2 - 4y^2 - z^2 + 10xy - 2yz + 5zx = 0$.

22. Prove that the equation to the cone through the coordinate axes and the lines of section of the cone $11x^2 - 5y^2 + z^2 = 0$ and the plane $7x - 5y + z = 0$ is $14yz - 30zx + 3xy = 0$. Find the equation of the plane containing the remaining two common generators of the cones.

ANSWERS

1. $x^2 + y^2 + z^2 - 4(yz + zx + xy) = 0$.

2. $13x^2 + 100y^2 + 100z^2 - 232yz + 116zx - 116xy = 0$.

3. $39\{(x-1)^2 + (y-2)^2 + (z-3)^2\} = 2(2x + 5y + 7z - 33)^2$.

4. (i) $3x^2 + 4y^2 + 23z^2 - 24yz + 16zx - 12xy = 0$.
   (ii) $13xy + 15yz + 20zx = 0$.

5. (i) $16x^2 + 16y^2 - 12z^2 - 24yz - 16zx + 200z - 400 = 0$.
   (ii) $3(x-1)^2 + 5(y+2)^2 + 7(z-3)^2 = 0$.

6. (i) $\cos^{-1}\frac{5}{6}$.
   (ii) $\frac{x}{2} = \frac{y}{-1} = \frac{z}{2}, \frac{x}{15} = \frac{y}{10} = \frac{z}{-6}$.
   (iv) $x = y = -z, \frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$.

8. $a(\alpha z - \gamma x)^2 + b(\beta z - \gamma y)^2 = (z - \gamma)^2$.

9. $(x-1)^2 = 4(y^2+z^2), (1,0,0)$.

22. $11x + 7y + 7z = 0$.

D. Surface of revolution

6.40 Surface of revolution

A sphere is generated by revolving a circle about its diameter. A right circular cylinder is formed when a straight line revolves about a parallel line. A right circular cone is obtained by revolving a straight line about a fixed line and intersecting the fixed line at a given point. These types of surfaces are defined by geometric conditions and known as surfaces of revolution.

Definition. *The surface produced by the rotation of a plane curve (or a straight line) about a given line in its plane is called a surface of revolution.*

The plane curve is called the *generatix* of the surface and the given line is called the *axis* or *axis of revolution*. Each circle described by a point on the generatix is called a *parallel* of the surface and the generatix at any position is called a *meridian section* of the surface. In Fig. 32, the parabola $y^2 = 4az, x = 0$ is the generatix, z-axis is the axis of revolution. The circles C_1 and C_2 are parallel and the parabola is the meridian section. Each parallel has its centre on the axis and each meridian section passes through the axis.

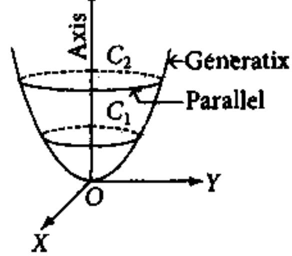

Fig. 32

6.41 Equation of the surface of revolution

(i) *Generatix is $f(y,z) = 0, x = 0$ and the axis is the z-axis.*

Let $P(0, y_1, z_1)$ be a point on the generatix. As P moves on a parallel due to rotation of the generatix about the z-axis, its distance from the XOY-plane remains unaltered and the distance from the z-axis is equal to y_1. If $Q(x,y,z)$ be the position of P at any other position and QN is the distance of it from OZ, then $QN = \sqrt{x^2+y^2}$.

$\therefore\ y_1 = \pm\sqrt{x^2+y^2}$ and $z_1 = z$.

We have $f(y_1, z_1) = 0$. Thus the equation of the surface of revolution is

$$f\left(\pm\sqrt{x^2+y^2}, z\right) = 0.$$

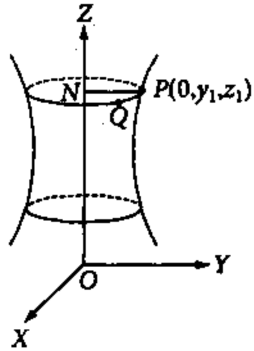

Fig. 33

Note 1. If the z-axis is the axis of revolution, then the equation of the surface of revolution is obtained by changing y to $\pm\sqrt{x^2+y^2}$. z remains unaltered.

(ii) *Generatix is $f(y,z) = 0, x = 0$ and the axis is $x = 0, y = k$.*

Let $P(0, y_1, z_1)$ be a point on the generatix. As P moves on a parallel due to rotation of the generatix about a line parallel to the z-axis, its distance from the XOY-plane remains unaltered and the distance from the axis of revolution is always equal to $y_1 - k$. If $Q(x, y, z)$ be the position of P at any subsequent position, then $y_1 - k = \pm\sqrt{x^2 + (y-k)^2}$ and $z_1 = z$.

We have $f(y_1, z_1) = 0$. Thus the equation of the surface of revolution is

$$f\left(k \pm \sqrt{x^2 + (y-k)^2}, z\right) = 0.$$

WORKED-OUT EXAMPLES

1. *Find the equation of the surface of revolution whose generatix is $x^2 - 4z = 0, y = 0$ and the axis is the z-axis.*

   The equation of the surface of revolution is

$$\left(\pm\sqrt{x^2 + y^2}\right)^2 - 4z = 0 \quad \text{or,} \quad x^2 + y^2 - 4z = 0.$$

2. *Find the equation of the surface of revolution generated by revolving the curve $y^2 = 2x, z = 0$ about the line $x = 5, z = 0$.*

   Here the curve revolves about a line parallel to the y-axis. Thus x will be replaced by $5 \pm \sqrt{(x-5)^2 + z^2}$.

   $\therefore$ the required equation is

$$y^2 = 2\left(5 \pm \sqrt{(x-5)^2 + z^2}\right)$$
$$\text{or,} \quad (y^2 - 10)^2 = 4\{(x-5)^2 + z^2\}.$$

3. *Show that the equation $x^4 = 16(y^2 + z^2)$ represents a surface of revolution.*

   Here $x^2 = \pm 4\sqrt{y^2 + z^2}$.

   This equation is obtained for the generatix $x^2 = 4y, z = 0$ and the axis of revolution as the x-axis. Consequently the given equation represents a surface of revolution.

EXERCISE VID

1. Find the equation of the surface of revolution for the revolving line $x = a, y = 0$ about the z-axis.

2. Find the equation of the surface of revolution of the straight line $y = z\tan\alpha, x = 0$ about the z-axis.

3. Find the equation of the surface of revolution generated by

   (i) the generatix $x^2 = 2y, z = 0$ about y-axis;

   (ii) the generatix $x^2 + 4y^2 = 4, z = 0$ about x-axis;

   (iii) the generatix $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, x = 0$ about z-axis;

(iv) the generatix $y^2 = 4z, x = 0$ about $z = 2, x = 0$;

(v) the generatix $x^2 + y^2 - 2by + b^2 = a^2, z = 0$ about x-axis;

(vi) the generatix $x^2 + 2y^2 = 1, z = 0$ about $x = 3, z = 0$.

4. Show that each of the following surfaces represents a surface of revolution and find the axis of revolution in each case:

(i) $y^4 = 25(z^2 + x^2)$, (ii) $x^2z^2 + y^2z^2 = 1$,

(iii) $\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{9} = 1$, (iv) $x^4 - 4x^2 = 4y^2 - 8y + 4z^2$.

ANSWERS

1. $x^2 + y^2 = a^2$.

2. $x^2 + y^2 = z^2 \tan^2 \alpha$.

3. (i) $z^2 + x^2 = 4y^2$,

(ii) $x^2 + 4(y^2 + z^2) = 4$,

(iii) $\frac{x^2+y^2}{b^2} - \frac{z^2}{c^2} = 1$,

(iv) $(y^2 - 8)^2 = 16\{x^2 + (z - 2)^2\}$,

(v) $(x^2 + y^2 + z^2 + b^2 - a^2)^2 = 4b^2(y^2 + z^2)$,

(vi) $\{(x - 3)^2 + 2y^2 + z^2 + 2\}^2 = 36\{(x - 3)^2 + z^2\}$.

4. (i) y-axis, (ii) z-axis, (iii) z-axis, (iv) the line $y = 1, z = 0$.

E. Ellipsoid, Hyperboloid, Paraboloid

6.50 Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{1}$$

The equation (1) represents a surface which is known as ellipsoid.

Some properties

(i) **Centre of the surface.** If (x, y, z) is a point on the surface of (1), then $(-x, -y, -z)$ is also a point on it. Thus the surface is symmetrical about the origin. Origin is the *centre* of it, so it is called a *central conicoid.*

(ii) **Principal axes.** The equation (1) remains unchanged on changing (x, y, z) to $(x, -y, -z)$. Thus the surface is symmetrical about the x-axis. Similarly the surface is symmetrical about the y-axis and the z-axis. These are called the ***principal axes*** of the ellipsoid.

The x-axis ($y = 0, z = 0$) meets the surface at the points $(\pm a, 0, 0)$. Similarly the y-axis and the z-axis meet the surface at the points $(0, \pm b, 0)$ and $(0, 0, \pm c)$ respectively. These points are A, A', B, B', C, C' on Fig. 34 and these are known as *vertices* of the ellipsoid. The line segments AA', BB', CC' are lengths of axes and these are equal to $2a, 2b, 2c$ respectively. If $a > b > c$, i.e. $OA > OB > OC$, then $2a, 2b, 2c$ are called lengths of *major axis, mean axis* and *minor axis* and the ellipsoid is referred to as triaxial.

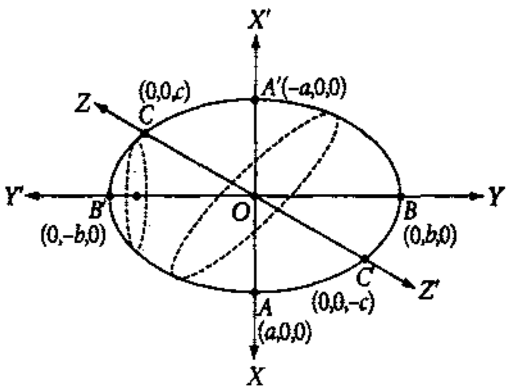

Fig. 34

If any two of a, b, c are equal, then the surface will be an *ellipsoid of revolution* or *spheroid*. As by rotating $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$ about the y-axis, the generated surface is $\frac{x^2+z^2}{a^2} + \frac{y^2}{b^2} = 1$. The ellipsoid of revolution obtained by revolving an ellipse about its major axis is known as *prolate spheroid* and that obtained by revolving the ellipse about minor axis is known as *oblate spheroid.*

(iii) **Principal planes.** If the point (α, β, γ) satisfies the equation (1), then the point $(\alpha, \beta, -\gamma)$ satisfies it. Therefore, the surface is symmetrical about the plane $z = 0$. The surface is also symmetrical about other two coordinate planes. These planes are called the *principal planes* of the surface.

(iv) **The surface is closed.** The equation of the section of the ellipsoid (1) by the plane $z = k$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, z = k$. The ellipse is real if $1 - \frac{k^2}{c^2} \geq 0$, i.e. $-c \leq k \leq c$, otherwise the ellipse becomes imaginary. It indicates that no part of the surface lies beyond $z = \pm c$. Similarly no part of the surface lies beyond $x = \pm a$ and $y = \pm b$. Hence the surface is closed.

Note 1. The equation (1) is called the canonical equation of the ellipsoid. a, b, c are positive non-zero numbers.

Note 2. The section $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, z = k$ is an ellipse. For different values of k from $-c$ to c different ellipses are obtained with their centres on the x-axis. Thus *an ellipsoid is generated by a variable ellipse.*

6.51 Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{1}$$

The equation (1) represents a surface known as hyperboloid of one sheet.

Some properties

(i) **Symmetry.** The surface is symmetrical about the origin, the coordinate axes and the coordinate planes. Thus it is a central conicoid with the centre at the origin. The coordinate axes and the coordinate planes are the principal axes and the principal planes.

(ii) **Intersection with axes.** The x-axis meets the surface at two points $A(a,0,0)$ and $A'(-a,0,0)$ and the y-axis meets the surface at two points $B(0,b,0)$ and $B'(0,-b,0)$. The z-axis does not meet the surface at any real point. Two points C and C' are taken on the z-axis, where $OC = OC' = c$ in magnitude. The points A, A', B and B' are called vertices. The line segments AA', BB' and CC' of lengths $2a, 2b, 2c$ are lengths of axes. OA, OB and OC are semi-axes.

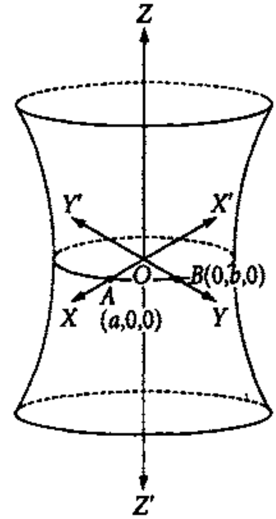

Fig. 35

(iii) **Sections by planes parallel to principal planes.** The section made by the plane $z = k$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, z = k. \tag{2}$$

It is an ellipse whose centre lies on the z-axis and whose semi-axes are $a\sqrt{1+\frac{k^2}{c^2}}$ and $b\sqrt{1+\frac{k^2}{c^2}}$. The size of the ellipse (2) increases indefinitely as $|k|$ increases. Thus the surface is generated by variable ellipse (2) as k varies from $-\infty$ to ∞.

Sections made by the plane $x = k$ or, $y = k$ are

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}, x = k \quad \text{or,} \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{b^2}, y = k.$$

These are hyperbolas except when $k = \pm a$ or, $k = \pm b$. When $k = \pm a$ or, $k = \pm b$, the corresponding sections are pair of straight lines $z = \pm\frac{c}{b}y, x = k$ or, $z = \pm\frac{c}{a}x$, $y = k$.

If the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, y = 0$ revolve about its conjugate axis, i.e. z-axis, the surface whose equation is $\frac{x^2+y^2}{a^2} - \frac{z^2}{c^2} = 1$ is produced. It is a **hyperboloid of revolution of one sheet.**

Note 1. The equation (1) is called the **canonical equation** of the hyperboloid of one sheet. There are two other canonical equations, namely

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Any hyperboloid of one sheet is not a **closed surface.**

6.52 Hyperboloid of two sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{1}$$

The equation (1) represents a surface known as hyperboloid of two sheets.

Some properties

(i) **Symmetry.** The surface is symmetrical about the origin, the coordinate axes and the coordinate planes. Thus it is a **central conicoid** with the **centre** at the origin. The coordinate axes and the coordinate planes are the **principal axes** and the **principal planes.**

(ii) **Intersection with axes.** The z-axis meets the surface at two real points, namely $(0,0,c)$ and $(0,0,-c)$. The other two axes do not meet it at real points. The point C and C' whose coordinates are $(0,0,c)$ and $(0,0-c)$ are called vertices. The points A and A′ are taken on the x-axis with coordinates $(a,0,0)$ and $(-a,0,0)$. Similarly the points $B(0,b,0)$ and $(0,-b,0)$ are taken on the y-axis. The line segments AA', BB' and CC' of lengths $2a, 2b$ and $2c$ are called lengths of axes. OA, OB, OC are semi-axes.

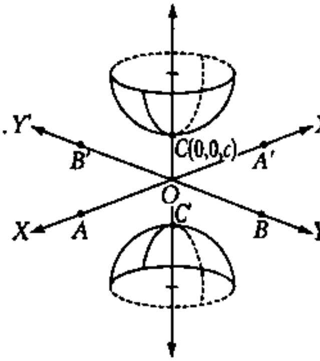

Fig. 36

(iii) **Sections by planes parallel to principal planes.** The section made by the plane $z=k$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}-1, z=k$. It is an ellipse, whose centre lies on the z-axis. Its semi-axes are $a\sqrt{\frac{k^2}{c^2}-1}, b\sqrt{\frac{k^2}{c^2}-1}$. These are imaginary when $|k|<c$. This means that the $z=k$ does not intersect the surface, so there is no portion of the surface between the planes $z=-c$ and $z=c$. Each of the planes $z=-c$ and $z=c$ has only one point in common with the surface. The size of the elliptical section increases as $|k|$ increases from c. Thus the surface is generated by variable ellipse. Moreover, the surface consists of two detached portions.

The section by a plane parallel to YOZ or, ZOX-plane is a hyperbola. If the hyperbola $\frac{z^2}{c^2}-\frac{x^2}{a^2}=1, y=0$ revolves about its transverse axis, i.e. z-axis, the surface whose equation is

$$\frac{z^2}{c^2}-\frac{x^2+y^2}{a^2}=1,$$

is produced. It is known as a hyperboloid of two sheets.

> **Note.** The equation (1) is called the **canonical equation** of the hyperboloid of two sheets. There are two other canonical equations, namely
>
> $$\frac{x}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad \text{and} \quad -\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$
>
> The surface is **not closed.**

6.53 Paraboloid

(a) **Elliptic paraboloid.**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}. \tag{1}$$

The equation (1) represents a surface known as elliptic paraboloid.

Some properties

(i) **Symmetry.** The surface is symmetrical about the z-axis and the planes $x = 0$ and $y = 0$. Thus the z-axis is the **principal axis** and the coordinate planes $x = 0$ and $y = 0$ are **the principal planes.**

(ii) **Vertex.** The coordinate axes meet the surface at the origin. This point is the **vertex.**

(iii) **Section by the plane $z = k$.**
The section made by such a plane is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2k}{c}, z = k. \qquad (2)$$

(k and c are of the same sign).

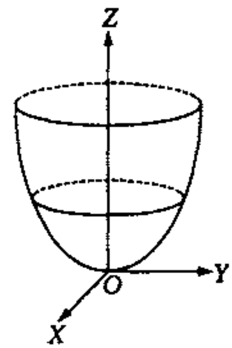

Fig. 37

If $c > 0$, then $k > 0$. The section is then an ellipse with the centre $(0, 0, k)$. The section lies above the plane $z = 0$ and at the vertex it is a point ellipse. Moreover, the size of the elliptical section increases as k increases. Thus the surface is generated by the variable ellipse (2) and it is wholly above the plane $z = 0$.

If $c < 0$, then $k < 0$. In this case, the surface wholly lies below the plane $z = 0$.

(iv) **Sections by planes parallel to $x = 0$ and $y = 0$.**

The section by the plane $x = k$ is

$$\frac{y^2}{b^2} = \frac{2z}{c} - \frac{k^2}{a^2}, \; x = k$$

$$\text{or,} \quad y^2 = \frac{2b^2}{c}\left(z - \frac{ck^2}{2a^2}\right), \; x = k.$$

It is a parabola of latus rectum $\frac{2b^2}{c}$. Its vertex is the point $\left(k, 0, \frac{ck^2}{2a^2}\right)$ and it is symmetrical about a line parallel to the z-axis and passing through the vertex. If $c > 0$, the concavity is upwards.

Similarly the section by the plane parallel to $y = 0$ is a parabola of latus rectum $\frac{2a^2}{c}$.

Revolving the parabola $y^2 = 2\lambda z, x = 0$ about its axis, i.e. z-axis, a surface of equation $x^2 + y^2 = 2\lambda z$ is obtained. It is a **paraboloid of revolution.**

Note. The equation (1) is the **canonical equation** of the elliptic paraboloid. Two other canonical equations are

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = \frac{2y}{b} \quad \text{and} \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2x}{a}.$$

(b) **Hyperbolic paraboloid.**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}. \tag{1}$$

The equation (1) represents a surface known as hyperbolic paraboloid.

Some Properties

(i) **Symmetry.** The equation is symmetric w.r.t. the z-axis and the planes $x = 0$ and $y = 0$. Thus the z-axis is the **principal axis** and the above two coordinate planes are the **principal planes** of the surface.

(ii) **Vertex.** The coordinate axes meet the surface at the origin. This is the vertex of the surface.

(iii) **Section by planes parallel to $z = 0$.**
The section is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2k}{c}, z = k$. These are similar hyperbolas. If k and c have the same sign, then the transverse and conjugate axes are parallel to the x and y-axes respectively. If k and c have opposite signs, then the transverse and the conjugate axes are parallel to the y and x-axes respectively. Centres of these hyperbolas lie on the z-axis and the asymptotes are $\frac{x}{a} \pm \frac{y}{b} = 0, z = k$.
The section made by the plane $z = 0$ is a pair of straight lines $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, z = 0$.

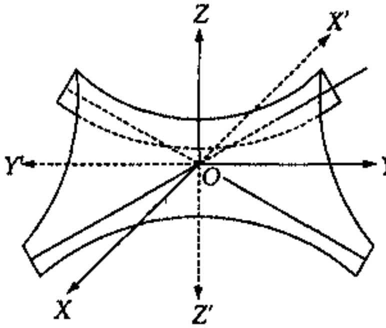

Fig. 38

(iv) **Sections by planes parallel to $x = 0$ and $y = 0$.**
The section by the plane $x = k$ is $y^2 = -\frac{2b^2}{c}\left(z - \frac{ck^2}{2a^2}\right), x = k$. It is a parabola whose vertex lies on the parabola $x^2 = \frac{2a^2}{c}z, y = 0$. Its concavity is downwards when $c > 0$. The section by the plane $y = k$ is $x^2 = \frac{2a^2}{c}\left(z + \frac{ck^2}{2b^2}\right), y = k$. It is also a parabola. Its vertex lies on $y^2 = -\frac{2b^2}{c}z, x = 0$. The concavity of the parabolic section is upwards, when $c > 0$.

Note. The equation (1) is called the **canonical equation** of the hyperbolic paraboloid. The other canonical equations are

$$\frac{x^2}{a^2} - \frac{z^2}{b^2} = \frac{2y}{c} \quad \text{and} \quad \frac{y^2}{a^2} - \frac{z^2}{b^2} = \frac{2x}{c}.$$

6.54 General equation

The general equation of the second degree is of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

It contains nine disposable constants. Thus a quadric or a conicoid can be determined by nine conditions each of which gives rise to one relation between the constants. By transformation of coordinate axes the general equation can be reduced to its standard form.

Note. (i) $ax^2+by^2+cz^2=1$ represents a **central conicoid** and (ii) $ax^2+by^2=2cz$ represents a **paraboloid.**

WORKED-OUT EXAMPLES

1. *Find the nature of the conicoid* $3x^2-2y^2-12x-12y-6z=0$.

The given equation can be written as

$$3(x-2)^2-2(y+3)^2=6(z+1) \quad \text{or,} \quad \frac{(x-2)^2}{2}-\frac{(y+3)^2}{3}=z+1.$$

If the origin is shifted to $(2,-3,-1)$, the equation will reduce to $\frac{x^2}{2}-\frac{y^2}{3}=z$. It is a hyperbolic paraboloid with the principal planes $x=0$ and $y=0$. Hence the original equation is a hyperbolic paraboloid with the principal planes $x-2=0$ and $y+3=0$.

2. *Find the nature of the quadric surface given by the equation*

$$2x^2+5y^2+3z^2-4x+20y-6z-5=0.$$

The given equation can be written as

$$2(x-1)^2+5(y+2)^2+3(z-1)^2=30 \quad \text{or,} \quad \frac{(x-1)^2}{15}+\frac{(y+2)^2}{6}+\frac{(z-1)^2}{10}=1.$$

Shifting the origin to $(1,-2,1)$ the equation reduces to

$$\frac{x^2}{15}+\frac{y^2}{6}+\frac{z^2}{10}=1.$$

It is an ellipsoid with the centre at the origin and the semi-axes are $\sqrt{15}$, $\sqrt{6}$, $\sqrt{10}$. Hence the original equation represents an ellipsoid whose centre is $(1,-2,1)$ and the principal axes of it are parallel to coordinate axes. The principal planes are $x=1, y=-2, z=1$.

3. *Find the equation of the curve in which the plane* $z=h$ *cuts the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

and find the area enclosed by the curve.

The section made by the plane $z=h$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{h^2}{c^2}, z=h$. It is an ellipse.

The equation of the ellipse can be written as $\frac{x^2}{\frac{a^2(c^2-h^2)}{c^2}}+\frac{y^2}{\frac{b^2(c^2-h^2)}{c^2}}=1$.

Hence the area of this section

$$= \pi \sqrt{\frac{a^2(c^2 - h^2)}{c^2}} \cdot \sqrt{\frac{b^2(c^2 - h^2)}{c^2}} = \frac{\pi ab}{c^2}(c^2 - h^2).$$

EXERCISE VIE

1. Find the nature of the following surface given by the equations

   (i) $9x^2 + 36y^2 + 4z^2 - 36x + 216y + 32z + 388 = 0$,

   (ii) $x^2 + 4y^2 - 9z^2 = 36$,

   (iii) $-3x^2 + 4y^2 - 5z^2 = 60$,

   (iv) $4x^2 + 6y^2 - 16x + 12y - 24z - 22 = 0$.

2. (i) Find the equation of the ellipsoid whose four vertices are $(\pm 1, 0, 0)$, $(0, \pm 2, 0)$ and which passes through $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{3\sqrt{5}}{2\sqrt{2}}\right)$.

   (ii) Obtain the equation of the surface generated by the rotation of $z^2 = 4x, y = 0$ about the z-axis.

3. Determine if the line $\frac{x-2}{2} = \frac{y-3}{-6} = \frac{z-1}{1}$ intersects the quadric $\frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{4} = 1$ and in case it does, find the points of intersection.

4. Find the value of k for which the plane $x + kz - 2 = 0$ intersects the hyperboloid of two sheets $x^2 + y^2 - z^2 = -1$ in a hyperbola.

   [*Hints.* Eliminating x from the two equations we have

   $$(2 - kz)^2 + y^2 - z^2 = -1 \quad \text{or,} \quad y^2 + (k^2 - 1)z^2 - 4kz + 1 = 0.$$

   It is the projection of the section on the plane $x = 0$. It will be a hyperbola, if $1 \cdot (k^2 - 1) < 0$ or, $-1 < k < 1$.]

ANSWERS

1. (i) Ellipsoid, centre $(2, -3, -4)$, semi-axes $2, 1, 3$, principal planes $x - 2 = 0, y + 3 = 0, z + 4 = 0$;

   (ii) Hyperboloid of one sheet, centre $(0, 0, 0)$, semi-axes $6, 3, 2$, principal planes $x = 0, y = 0, z = 0$;

   (iii) Hyperboloid of two sheets, centre $(0, 0, 0)$, semi-axes $2\sqrt{5}, \sqrt{15}, 2\sqrt{3}$, principal planes $x = 0, y = 0, z = 0$;

   (iv) Elliptic paraboloid, vertex $(2, -1, 0)$.

2. (i) $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$,

   (ii) $y^2 + z^2 = 4x$.

3. $(2, 3, 1), (4, -3, 2)$.

Chapter 7

Tangent, Normal, Enveloping Cone and Cylinder, Pole and Polar

7.10 To find the equation of the tangent at (x_1, y_1, z_1) to the quadric surface represented by

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

[In the subsequent discussion we shall denote

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d,$$
$$\phi(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

$\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial y_1}, \frac{\partial F}{\partial z_1}, \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial y_1}, \frac{\partial \phi}{\partial z_1}$ as partial derivatives of $F(x, y, z)$ and $\phi(x, y, z)$ w.r.t. x, y and z respectively at the point (x_1, y_1, z_1).]

The equations of the line having d.cs. l, m, n and passing through the point (x_1, y_1, z_1) are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{(say)}. \tag{1}$$

Any point on this line is $(lr + x_1, mr + y_1, nr + z_1)$. If the line meets the surface at this point, then

$$F(lr + x_1, mr + y_1, nr + z_1) = 0$$
$$\text{or,} \quad \phi(l, m, n)r^2 + \left(l\frac{\partial F}{\partial x_1} + m\frac{\partial F}{\partial y_1} + n\frac{\partial F}{\partial z_1}\right)r + F(x_1, y_1, z_1) = 0. \tag{2}$$

It is a quadratic equation in r. The two roots are the distances of the two points of intersection between the line (1) and the surface from the point (x_1, y_1, z_1). If the line is the tangent to the given surface at (x_1, y_1, z_1), both the roots will be zero. In this case, the sum of the roots and product of the roots are zero. By the equation (2),

$$l\frac{\partial F}{\partial x_1} + m\frac{\partial F}{\partial y_1} + n\frac{\partial F}{\partial z_1} = 0,$$

$$\text{i.e.}\quad (ax_1 + hy_1 + gz_1 + u)l + (hx_1 + by_1 + fz_1 + v)m + (gx_1 + fy_1 + cz_1 + w)n = 0 \tag{3}$$

$$\text{and}\quad F(x_1, y_1, z_1) = 0. \tag{4}$$

The conditions (3) and (4) ensure that the line (2) is the tangent to the surface of $F(x, y, z) = 0$ at the point (x_1, y_1, z_1) on the surface. The tangent plane at (x_1, y_1, z_1) is the locus of the line (1) with the condition (3). Thus the equation of this tangent plane is obtained by eliminating l, m, n from (3) by (1) and its equation is

$$(ax_1 + hy_1 + gz_1 + u)(x - x_1) + (hx_1 + by_1 + fz_1 + v)(y - y_1) + (gx_1 + fy_1 + cz_1 + w)(z - z_1) = 0$$

$$\text{or,}\quad axx_1 + byy_1 + czz_1 + f(yz_1 + zy_1) + g(zx_1 + xz_1) + h(xy_1 + yx_1) + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0 \text{ [by (4)]} \tag{5}$$

Special cases

Quadric surface	**Equation of the tangent plane at (x_1, y_1, z_1)**
(1) Sphere	
(i) $x^2 + y^2 + z^2 = a^2$	$xx_1 + yy_1 + zz_1 = a^2$
(ii) $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$	$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$
(2) Cone	
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$	$axx_1 + byy_1 + czz_1 + f(yz_1 + zy_1) + g(zx_1 + xz_1) + h(xy_1 + yx_1) = 0$
(3) Central conicoid	
$ax^2 + by^2 + cz^2 = 1$	$axx_1 + byy_1 + czz_1 = 1$
(4) Paraboloid	
$ax^2 + by^2 = 2cz$	$axx_1 + byy_1 = c(z + z_1)$

Note

(i) If the roots of (2) are real and unequal and they are r_1 and r_2, then the line (1) cuts the surface at two distinct points $(lr_1 + x_1, mr_1 + y_1, nr_1 + z_1)$ and $(lr_2 + x_1, mr_2 + y_1, nr_2 + z_1)$. In this case the line is called a secant of the surface.

(ii) If the roots of (2) are real and equal, then the line is a tangent to the surface at the common point $(lr + x_1, mr + y_1, nr + z_1)$. This point is called the **point of contact.**

(iii) If the roots of (2) are complex, the line does not intersect the surface.

7.11 To find the condition that the plane $lx + my + nz = p$ is a tangent plane to (i) the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d = 0$, (ii) the central conicoid $ax^2 + by^2 + cz^2 = 1$ and (iii) the paraboloid $ax^2 + by^2 = 2cz$.

(i) (a) **The necessary condition.**

The plane will touch the sphere, if its distance from the centre of the sphere is equal to the radius of the sphere.

$$\therefore \quad \text{the condition is} \quad \frac{-lu - mv - nw - p}{\sqrt{l^2 + m^2 + n^2}} = \sqrt{(u^2 + v^2 + w^2 - d)}$$

$$\text{or,} \quad (lu + mv + nw + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d). \tag{1}$$

This condition is necessary.

(b) **The sufficient condition.**

Let the condition (1) be satisfied.

$$\text{Then} \quad \left(\frac{lu + mv + nw + p}{\sqrt{l^2 + m^2 + n^2}}\right)^2 = u^2 + v^2 + w^2 - d$$

$$\text{or,} \quad \left(\frac{-lu - mv - nw - p}{\sqrt{l^2 + m^2 + n^2}}\right)^2 = u^2 + v^2 + w^2 - d$$

$$\text{or,} \quad \frac{-lu - mv - nw - p}{\sqrt{l^2 + m^2 + n^2}} = \sqrt{u^2 + v^2 + w^2 - d}.$$

It shows that the distance of the plane $lx + my + nz = p$ from the centre $(-u, -v, -w)$ is equal to the radius $\sqrt{u^2 + v^2 + w^2 - d}$. Therefore, the plane touches the sphere. Hence the condition (1) is sufficient.

Note. Point of contact. Let the plane touch the sphere at (x_1, y_1, z_1). The equation of the tangent plane at this point is

$$(x_1 + u)x + (y_1 + v)y + (z_1 + w)z + ux_1 + vy_1 + wz_1 + d = 0.$$

It is identical with the given plane.

$$\therefore \frac{x_1+u}{l} = \frac{y_1+v}{m} = \frac{z_1+w}{n} = \frac{-(ux_1+vy_1+wz_1+d)}{p}$$
$$= \frac{u(x_1+u)+v(y_1+v)+w(z_1+w)-(ux_1+vy_1+wz_1+d)}{lu+mv+nw+p}$$
$$= \frac{u^2+v^2+w^2-d}{lu+mv+nw+p}.$$

Thus the point of contact is $(l\lambda - u, m\lambda - v, n\lambda - w)$ where

$$\lambda = \frac{u^2+v^2+w^2-d}{lu+mv+nw+p}.$$

(ii) (a) **The necessary condition.**

Let the plane touch the central conicoid at (x_1, y_1, z_1). The equation of the tangent plane at this point is $axx_1 + byy_1 + czz_1 = 1$.

It is identical with the given plane $lx + my + nz = p$.

Comparing the coefficients,

$$\frac{ax_1}{l} = \frac{by_1}{m} = \frac{cz_1}{n} = \frac{1}{p} \quad \text{or,} \quad x_1 = \frac{l}{ap}, y_1 = \frac{m}{bp}, z_1 = \frac{n}{cp}.$$

Since (x_1, y_1, z_1) is a point on the conicoid,

$$ax_1^2 + by_1^2 + cz_1^2 = 1 \quad \text{or,} \quad a \cdot \frac{l^2}{a^2p^2} + b \cdot \frac{m^2}{b^2p^2} + c \cdot \frac{n^2}{c^2p^2} = 1$$
$$\text{or,} \quad \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2. \tag{2}$$

It is the necessary condition for the tangency of the plane.

(b) **The sufficient condition.**

Let the condition (2) hold.

$$\text{Then from (2),} \quad a \cdot \left(\frac{l}{ap}\right)^2 + b \cdot \left(\frac{m}{bp}\right)^2 + c \cdot \left(\frac{n}{cp}\right)^2 = 1.$$

Putting $\frac{l}{ap} = x_1, \frac{m}{bp} = y_1, \frac{n}{cp} = z_1$, we have $ax_1^2 + by_1^2 + cz_1^2 = 1$.

It shows that the point $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$ lies on the conicoid.

Now the equation of the tangent plane at this point is

$$ax \cdot \frac{l}{ap} + by \cdot \frac{m}{bp} + cz \cdot \frac{n}{cp} = 1 \quad \text{or,} \quad lx + my + nz = p.$$

Therefore, the given plane is a tangent plane to the central conicoid. Hence the condition (2) is sufficient.

Note. The point of contact is $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$.

The planes $lx + my + nz = \pm\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$ touch the conicoid.

(iii) (a) **The necessary condition.** Let the plane touch the paraboloid at (x_1, y_1, z_1). The equation of the tangent plane at this point is $axx_1 + byy_1 = c(z + z_1)$. It is identical with $lx + my + nz = p$.

Comparing the coefficients,

$$\frac{ax_1}{l} = \frac{by_1}{m} = \frac{-c}{n} = \frac{cz_1}{p} \quad \text{or,} \quad x_1 = -\frac{cl}{an}, y_1 = -\frac{cm}{bn}, z_1 = -\frac{p}{n}.$$

Since the point (x_1, y_1, z_1) is on the paraboloid,

$$ax_1^2 + by_1^2 = 2cz_1 \quad \text{or,} \quad a\frac{c^2l^2}{a^2n^2} + b\frac{c^2m^2}{b^2n^2} = -2c\frac{p}{n}$$

$$\text{or,} \quad \frac{l^2}{a} + \frac{m^2}{b} + \frac{2np}{c} = 0. \tag{3}$$

It is the necessary condition.

(b) **The sufficient condition.**

Let the condition (3) hold. From (3) $a\frac{c^2l^2}{a^2n^2} + b\frac{c^2m^2}{b^2n^2} = -2c\frac{p}{n}$.

Putting $x_1 = -\frac{cl}{an}, y_1 = -\frac{cm}{bn}, z_1 = -\frac{p}{n}$, we have $ax_1^2 + by_1^2 = 2cz_1$.

Therefore, the point $\left(-\frac{cl}{an}, -\frac{cm}{bn}, -\frac{p}{n}\right)$ lies on the paraboloid.

Again the equation of the tangent plane at this point is

$$ax \cdot \left(-\frac{cl}{an}\right) + by \cdot \left(-\frac{cm}{bn}\right) = c\left(z - \frac{p}{n}\right) \quad \text{or,} \quad lx + my + nz = p.$$

Thus the given plane is a tangent plane to the paraboloid.

Hence the condition (3) is sufficient.

Note. The point of contact is $\left(-\frac{cl}{an}, -\frac{cm}{bn}, -\frac{p}{n}\right)$.

The plane $lx + my + nz = -\frac{c}{2n}\left(\frac{l^2}{a} + \frac{m^2}{b}\right)$ touches the paraboloid.

7.12 To find the condition that the plane $lx + my + nz = 0$ is a tangent plane to the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

(i) **The necessary condition.**

If (x_1, y_1, z_1) be the point of contact, the tangent plane is

$$(ax_1 + by_1 + gz_1)x + (hx_1 + by_1 + fz_1)y + (gx_1 + fy_1 + cz_1)z = 0.$$

It is identical with $lx + my + nz = 0$.

Comparing the coefficients,

$$\frac{ax_1+hy_1+gz_1}{l}=\frac{hx_1+by_1+fz_1}{m}=\frac{gx_1+fy_1+cz_1}{n}=k \text{ (say)}.$$

$$\text{Then,}\quad ax_1+hy_1+gz_1-kl=0, \tag{1}$$

$$hx_1+by_1+fz_1-km=0, \tag{2}$$

$$gx_1+fy_1+cz_1-kn=0. \tag{3}$$

Again

$$lx_1+my_1+nz_1=0. \tag{4}$$

Eliminating x_1, y_1, z_1 from (1), (2), (3) and (4), we have

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix}=0 \tag{5}$$

$$\text{or,}\quad Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0. \tag{6}$$

It is the necessary condition. Here $A, B, C, \ldots$ are the cofactors of $a, b, c, \ldots$ of the determinant

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

(ii) The sufficient condition.

Let the condition (5) hold.

Again

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\neq 0.$$

For these two conditions the system of equations

$$ax+hy+gz+lw=0,\quad hx+by+fz+mw=0,$$
$$gx+fy+cz+nw=0,\quad lx+my+nz=0$$

has a non-trivial solution. If this solution be x_1, y_1, z_1, w_1, then

$$ax_1+hy_1+gz_1=-lw_1,\quad hx_1+by_1+fz_1=-mw_1,$$
$$gx_1+fy_1+cz_1=-nw_1,\quad lx_1+my_1+nz_1=0.$$

Now the equation $lx+my+nz=0$ can be written as $x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0$. It is the equation of the tangent to the cone at the point (x_1, y_1, z_1). Hence the condition (5) is sufficient for the tangency of the plane to the cone.

Corollary I. Point of contact.

Here $\triangle \neq 0$. Therefore, the equations (1), (2), (3) and (4) in x_1, y_1 and z_1 are consistent. Thus x_1, y_1, z_1 are obtained from (1), (2) and (3).

Corollary II. Reciprocal cone.

The equations of the line through the origin and perpendicular to the plane $lx + my + nz = 0$ are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \tag{7}$$

Eliminating l, m, n between (6) and (7), we get

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0. \tag{8}$$

Since $\triangle^2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ and $\triangle \neq 0$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \neq 0$.

Therefore, the equation (8) represents a cone with the vertex at the origin and generators are normals to the tangent planes to the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \tag{9}$$

at the origin. The cone represented by (8) is called the reciprocal to the cone (9).

The reciprocal to the cone (8) is

$$\bar{A}x^2 + \bar{B}y^2 + \bar{C}z^2 + 2\bar{F}yz + 2\bar{G}zx + 2\bar{H}xy = 0, \tag{10}$$

where $\bar{A} = BC - F^2 = a\triangle, \bar{B} = CA - G^2 = b\triangle, \bar{C} = AB - H^2 = c\triangle,$
$\bar{F} = GH - AF = f\triangle, \bar{G} = HF - BG = g\triangle, \bar{H} = FG - CH = h\triangle.$

For these relations the equation (10) is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \tag{11}$$

Hence the cones (9) and (8) are reciprocal to each other and they are called reciprocal cones.

Corollary III. Condition for three perpendicular tangent planes.

The condition for the cone (9) to possess three mutually perpendicular tangent planes is $A + B + C = 0$, i.e. $bc + ca + ab = f^2 + g^2 + h^2$. It implies that the cone (8) has three mutually perpendicular generators.

Example 1. *Prove that the tangent planes to the cone* $x^2 - y^2 + 2z^2 - 3yz + 4zx - 5xy = 0$ *are perpendicular to the generators of the cone* $17x^2 + 8y^2 + 29z^2 + 28yz - 46zx - 16xy = 0$.

Let us find the reciprocal cone to

$$x^2 - y^2 + 2z^2 - 3yz + 4zx - 5xy = 0.$$

Here $a = 1, b = -1, c = 2, f = -\frac{3}{2}, g = 2, h = -\frac{5}{2}$.

$$\therefore A = bc - f^2 = -2 - \frac{9}{4} = -\frac{17}{4}, B = ca - g^2 = 2 - 4 = -2,$$
$$C = ab - h^2 = -1 - \frac{25}{4} = -\frac{29}{4}, F = gh - af = -5 + \frac{3}{2} = -\frac{7}{2},$$
$$G = hf - bg = \frac{15}{4} + 2 = \frac{23}{4}, H = fg - ch = -3 + 5 = 2.$$

$\therefore$ the reciprocal cone is

$$-\frac{17}{4}x^2 - 2y^2 - \frac{29}{4}z^2 - 7yz + \frac{23}{2}zx + 4xy = 0$$
$$\text{or,} \quad 17x^2 + 8y^2 + 29z^2 + 28yz - 46zx - 8xy = 0.$$

Hence the result follows.

Example 2. *Prove that the equation* $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$ *represents a cone that touches the coordinate planes.*

The equation of the cone whose generators are the coordinate axes is of the form

$$fyz + gzx + hxy = 0 \quad \text{or,} \quad 2fyz + 2gzx + 2hxy = 0.$$

Its reciprocal cone is

$$-f^2x^2 - g^2y^2 - h^2z^2 + 2ghyz + 2hfzx + 2fgxy = 0$$
$$\text{or,} \quad (fx + gy - hz)^2 = 4fgxy$$
$$\text{or,} \quad fx + gy - hz = \pm 2\sqrt{fgxy}$$
$$\text{or,} \quad fx + gy \pm 2\sqrt{fgxy} = hz$$
$$\text{or,} \quad \left(\sqrt{fx} \pm \sqrt{gy}\right)^2 = hz$$
$$\text{or,} \quad \sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0.$$

Hence the result follows.

7.20 To find the equation of the normal at (x_1, y_1, z_1) to (i) the central conicoid $ax^2 + by^2 + cz^2 = 1$ and (ii) the paraboloid $ax^2 + by^2 = 2cz$.

The normal at (x_1, y_1, z_1) is a straight line passing through this point and perpendicular to the tangent plane. Therefore, the d.rs. of this normal are ax_1, by_1, cz_1. Hence the equations of the normal at (x_1, y_1, z_1) are

$$\frac{x-x_1}{ax_1} = \frac{y-y_1}{by_1} = \frac{z-z_1}{cz_1}.$$

(ii) The equation of the tangent plane at (x_1, y_1, z_1) to the paraboloid is $axx_1 + byy_1 = c(z + z_1)$.

The normal is a straight line perpendicular to the tangent plane at the point of contact. Therefore, the d.rs. of this normal are $ax_1, by_1, -c$. Hence the equations of the normal are

$$\frac{x-x_1}{ax_1} = \frac{y-y_1}{by_1} = \frac{z-z_1}{-c}.$$

7.21 Number of normals through a given point

(a) Central conicoid. The equations of the normal at (x_1, y_1, z_1) to the central conicoid $ax^2 + by^2 + cz^2 = 1$ are

$$\frac{x-x_1}{ax_1} = \frac{y-y_1}{by_1} = \frac{z-z_1}{cz_1}.$$

If this normal passes through the point (α, β, γ), then

$$\frac{\alpha-x_1}{ax_1} = \frac{\beta-y_1}{by_1} = \frac{\gamma-z_1}{cz_1} = \lambda \text{ (say)}.$$

From these relations

$$x_1 = \frac{\alpha}{1+a\lambda}, y_1 = \frac{\beta}{1+b\lambda}, z_1 = \frac{\gamma}{1+c\lambda}.$$

We have $ax_1^2 + by_1^2 + cz_1^2 = 1$.

$$\therefore \frac{a\alpha^2}{(1+a\lambda)^2} + \frac{b\beta^2}{(1+b\lambda)^2} + \frac{c\gamma^2}{(1+c\lambda)^2} = 1.$$

This equation gives six values of λ. Each value corresponds to a normal at a point on the conicoid. Hence six normals can be drawn in general from a point to a central conicoid.

Corollary I. The feet of the normals from (α, β, γ) lie on the intersection of the conicoid and a certain cubic curve.

[A curve is called a cubic curve, if a plane, in general, meets it in three points.]

The feet of the normals from the point (α, β, γ) lie on the curve whose parametric coordinates are

$$x = \frac{\alpha}{1+a\lambda}, y = \frac{\beta}{1+b\lambda}, z = \frac{\gamma}{1+c\lambda}.$$

Let the plane $Ax+By+Cz+D=0$ intersect the curve at a point $\left(\frac{\alpha}{1+ar}, \frac{\beta}{1+br}, \frac{\gamma}{1+cr}\right)$ where r is given by

$$\frac{A\alpha}{1+ar}+\frac{B\beta}{1+br}+\frac{C\gamma}{1+cr}+D=0.$$

It is a cubic in r. Therefore, the plane intersects the curve in three points. Hence the curve is a cubic one.

Corollary II. The six normals through the point (α, β, γ) lie on a cone.

Any line through the point (α, β, γ) has equations

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \tag{1}$$

It will intersect the cubic

$$x=\frac{\alpha}{1+a\lambda},\ y=\frac{\beta}{1+b\lambda},\ z=\frac{\gamma}{1+c\lambda},$$

$$\text{if}\quad \frac{\frac{\alpha}{1+a\lambda}-\alpha}{l}=\frac{\frac{\beta}{1+b\lambda}-\beta}{m}=\frac{\frac{\gamma}{1+c\lambda}-\gamma}{n}.$$

From these $\quad \lambda=\dfrac{bn\beta-cm\gamma}{bc(m\gamma-n\beta)}=\dfrac{am\alpha-bl\beta}{ab(l\beta-m\alpha)}.$

It implies that $\quad \dfrac{a\alpha(b-c)}{l}+\dfrac{b\beta(c-a)}{m}+\dfrac{c\gamma(a-b)}{n}=0.$

Eliminating, l, m, n by (1),

$$\frac{a\alpha(b-c)}{x-\alpha}+\frac{b\beta(c-a)}{y-\beta}+\frac{c\gamma(a-b)}{z-\gamma}=0.$$

The normals lie on this cone.

(b) **Paraboloid.** The equations of the normal at (x_1, y_1, z_1) to the paraboloid $ax^2+by^2=2cz$ are

$$\frac{x-x_1}{ax_1}=\frac{y-y_1}{by_1}=\frac{z-z_1}{-c}.$$

If this normal passes through the point (α, β, γ), then

$$\frac{\alpha-x_1}{ax_1}=\frac{\beta-y_1}{by_1}=\frac{\gamma-z_1}{-c}=\lambda \text{ (say)}.$$

From these relations

$$x_1=\frac{\alpha}{1+a\lambda}, y_1=\frac{\beta}{1+b\lambda}, z_1=\gamma+c\lambda.$$

We have $ax_1^2+by_1^2=2cz_1$.

$$\therefore\ \frac{a\alpha^2}{(1+a\lambda)^2}+\frac{b\beta^2}{(1+b\lambda)^2}=2c(\gamma+c\lambda).$$

This equation gives five values of λ. Each value corresponds to a normal to a point on the paraboloid. Hence five normals can be drawn in general to a paraboloid through a given point.

Corollary III. The five normals through the point (α,β,γ) lie on a cone.

Any line through the point (α,β,γ) has equations

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \tag{1}$$

It will intersect the cubic

$$x=\frac{\alpha}{1+a\lambda},\ y=\frac{\beta}{1+b\lambda},\ z=\gamma+c\lambda,$$

$$\text{if}\qquad \frac{\frac{\alpha}{1+a\lambda}-\alpha}{l}=\frac{\frac{\beta}{1+b\lambda}-\beta}{m}=\frac{\gamma+c\lambda-\gamma}{n}.$$

From these

$$-\lambda=\frac{na\alpha+cl}{cal}=\frac{bn\beta+cm}{cbm}.$$

It implies that

$$\frac{ab\alpha}{l}-\frac{ab\beta}{m}+\frac{c(b-a)}{n}=0.$$

Eliminating l,m,n by (1),

$$\frac{ab\alpha}{x-\alpha}-\frac{ab\beta}{y-\beta}+\frac{c(b-a)}{z-\gamma}=0.$$

The normals lie on this cone.

7.30 To find the locus of the point of intersection of three mutually perpendicular tangent planes to a conicoid.

(a) Central conicoid. Let $ax^2+by^2+cz^2=1$ be the equation of the central conicoid and the three mutually perpendicular tangent planes be

$$\left.\begin{aligned} l_1x+m_1y+n_1z&=\sqrt{\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}\right)},\\ l_2x+m_2y+n_2z&=\sqrt{\left(\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}\right)}\\ \text{and}\quad l_3x+m_3y+n_3z&=\sqrt{\left(\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}\right)}.\end{aligned}\right\}. \tag{1}$$

Here

$$\left.\begin{aligned} \sum_{r=1}^{3}l_r^2=1=\sum_{r=1}^{3}m_r^2&=\sum_{r=1}^{3}n_r^2\\ \text{and}\quad \sum_{r=1}^{3}l_rm_r=0=\sum_{r=1}^{3}m_rn_r&=\sum_{r=1}^{3}n_rl_r.\end{aligned}\right\}. \tag{2}$$

Now squaring and adding the equations of (1) and using the conditions (2), the locus of the point of intersection of the planes of (1) is obtained.

The locus is $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. It is a sphere called the *director sphere.*

(b) **Paraboloid.** Let $ax^2+by^2=2cz$ be the equation of the paraboloid and the three mutually perpendicular tangents to it be

$$\left.\begin{aligned} l_1x+m_1y+n_1z+\frac{c}{2n_1}\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}\right)&=0,\\ l_2x+m_2y+n_2z+\frac{c}{2n_2}\left(\frac{l_2^2}{a}+\frac{m_2^2}{b}\right)&=0\\ \text{and}\quad l_3x+m_3y+n_3z+\frac{c}{2n_3}\left(\frac{l_3^2}{a}+\frac{m_3^2}{b}\right)&=0.\end{aligned}\right\}. \tag{3}$$

Here

$$\left.\begin{aligned} \sum_{r=1}^{3} l_r^2=1=\sum_{r=1}^{3} m_r^2&=\sum_{r=1}^{3} n_r^2\\ \text{and}\quad \sum_{r=1}^{3} l_rm_r=0=\sum_{r=1}^{3} m_rn_r&=\sum_{r=1}^{3} n_rl_r.\end{aligned}\right\}. \tag{4}$$

Now multiplying the equations of (1) by n_1,n_2,n_3 respectively and then adding and using the conditions (2) the required locus is obtained.

This locus is $2z+c\left(\frac{1}{a}+\frac{1}{b}\right)=0$.

7.40 Enveloping cone

Definition. *The locus of tangent lines to a quadric from a given point is called an enveloping cone or tangent cone to that quadric with the given point as its vertex.*

Equation. *To find the equation of the enveloping cone of the quadric* $F(x,y,z)=0$ *with* (x_1,y_1,z_1) *as its vertex.*

The line through the point (x_1,y_1,z_1) with d.cs. l,m,n will touch the quadric, if the roots of the quadratic equation (2) of Sec 7.10 in r are equal. The condition for this is

$$\left(l\frac{\partial F}{\partial x_1}+m\frac{\partial F}{\partial y_1}+n\frac{\partial F}{\partial z_1}\right)^2=4\phi(l,m,n)F(x_1,y_1,z_1). \tag{1}$$

Eliminating l,m,n between (1) and the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}, \tag{2}$$

the locus of the line (2) is

$$\left\{(x-x_1)\frac{\partial F}{\partial x_1}+(y-y_1)\frac{\partial F}{\partial y_1}+(z-z_1)\frac{\partial F}{\partial z_1}\right\}^2$$
$$=4\phi(x-x_1,y-y_1,z-z_1)F(x_1,y_1,z_1). \tag{3}$$

Putting $S=F(x,y,z), S_1=F(x_1,y_1,z_1)$,

$$T=\frac{1}{2}\left(x\frac{\partial F}{\partial x_1}+y\frac{\partial F}{\partial y_1}+z\frac{\partial F}{\partial z_1}\right)+ux_1+vy_1+wz_1+d,$$

the equation (3) can be written as

$$(T - S_1)^2 = S_1(S + S_1 - 2T) \quad \text{or,} \quad SS_1 = T^2. \tag{4}$$

It is the equation of the enveloping cone of $F(x, y, z) = 0$ with the point (x_1, y_1, z_1) as its vertex.

Example 3. *Find the equation of the cone whose vertex is the point* $(6, 0, 0)$ *and whose generators touch the surface* $x^2 + y^2 + z^2 - 25 = 0$.

$$\text{Here} \quad S = x^2 + y^2 + z^2 - 25, S_1 = 6^2 + 0 + 0 - 25 = 11,$$
$$T = x \cdot 6 + y \cdot 0 + z \cdot 0 - 25 = 6x - 25.$$

The required equation of the enveloping cone is

$$SS_1 = T^2 \quad \text{or,} \quad (x^2 + y^2 + z^2 - 25) \cdot 11 = (6x - 25)^2$$
$$\text{or,} \quad 25x^2 - 11(y^2 + z^2) - 300x + 900 = 0.$$

Example 4. *Find the locus of a point from which three mutually perpendicular tangent lines can be drawn to the paraboloid* $ax^2 + by^2 = 2cz$.

Let the point be (x_1, y_1, z_1).

The equation of the enveloping cone with the point (x_1, y_1, z_1) as vertex is

$$(ax^2 + by^2 - 2cz)(ax_1^2 + by_1^2 - 2cz_1) = (axx_1 + byy_1 - cz - cz_1)^2.$$

If it has three mutually perpendicular generators, then coefficient of x^2+ coefficient of y^2+ coefficient of $z^2 = 0$.

$$\therefore \quad a(by_1^2 - 2cz_1) + b(ax_1^2 - 2cz_1) - c^2 = 0$$
$$\text{or,} \quad ab(x_1^2 + y_1^2) - 2c(a + b)z_1 - c^2 = 0.$$

Hence the locus of (x_1, y_1, z_1) is $ab(x^2 + y^2) - 2c(a + b)z - c^2 = 0$.

Example 5. *Find the locus of luminous point of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *that casts a parabolic shadow on the plane* $z = 0$.

The enveloping cone of the ellipsoid with the luminous point (x_1, y_1, z_1) as vertex is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1\right)^2.$$

The section of it by $z = 0$ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2.$$

If it represents a parabola, then

$$\frac{x_1^2y_1^2}{a^4b^4} = \frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)\cdot\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$$

$$\text{or,}\quad \left(\frac{z_1^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=0.$$

Since (x_1, y_1, z_1) does not lie on the ellipsoid, $\frac{z_1^2}{c^2} - 1 = 0$.

Hence the required locus is $\frac{z^2}{c^2} - 1 = 0$ or, $z = \pm c$.

7.41 Enveloping cylinder

Definition. *The locus of tangent lines to a quadric parallel to a given line is called an enveloping cylinder of that quadric.*

Equation. *To find the enveloping cylinder of the quadric $F(x, y, z) = 0$ with generators parallel to $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.*

Let (x_1, y_1, z_1) be a point on the enveloping cylinder. The equations of the generator through this point are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}. \tag{1}$$

If the line (1) is a tangent line to $F(x, y, z) = 0$, then from the condition of equal roots of the equation (2) of Sec 7.10 it is obtained that

$$\left(l\frac{\partial F}{\partial x_1}+m\frac{\partial F}{\partial y_1}+n\frac{\partial F}{\partial z_1}\right)^2 = 4\phi(l,m,n)F(x_1,y_1,z_1). \tag{2}$$

The locus of (x_1, y_1, z_1) is

$$\left[\sum l(ax+hy+gz+u)\right]^2 = \phi(l,m,n)F(x,y,z). \tag{3}$$

It is the equation of the enveloping cylinder of $F(x, y, z) = 0$.

Example 6. *Find the equation of the cylinder whose generators touch the sphere $x^2+y^2+z^2=9$ and are perpendicular to the plane $x+y-3z=5$.*

Here $F(x,y,z) = x^2+y^2+z^2-9$, $\phi(x,y,z) = x^2+y^2+z^2$.

The d.rs. of the generators are $1, 1, -3$.

$\therefore$ the enveloping cylinder is

$$(1\cdot x+1\cdot y-3\cdot z)^2 = (1+1+9)(x^2+y^2+z^2-9)$$
$$\text{or,}\quad (x+y-3z)^2 = 11(x^2+y^2+z^2-9)$$
$$\text{or,}\quad 10x^2+10y^2+2z^2+6yz+6zx-2xy-99=0.$$

Example 7. *Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose generators are parallel to the lines $\frac{x}{0} = \frac{y}{\pm\sqrt{a^2-b^2}} = \frac{z}{c}$ meet the plane $z=0$ in circles.*

Here $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1, \phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ and the d.rs. of the generators are $0, \pm\sqrt{a^2-b^2}, c$.

$\therefore$ the enveloping cylinder is

$$\left(0 \cdot \frac{x}{a^2} \pm \sqrt{a^2-b^2}\frac{y}{b^2} + c \cdot \frac{z}{c^2}\right)^2 = \left(0 + \frac{a^2-b^2}{b^2} + \frac{c^2}{c^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

or, $$\left(\pm\sqrt{a^2-b^2}\frac{y}{b^2} + \frac{z}{c}\right)^2 = \frac{a^2}{b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right).$$

It meets the plane $z = 0$ in the conic

$$z = 0, \frac{a^2-b^2}{b^4}y^2 = \frac{a^2}{b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \quad \text{or,} \quad z = 0, x^2 + y^2 = a^2.$$

It is a circle.

7.50 Plane of contact

The tangent plane at the point (x_1, y_1, z_1) to the quadric $F(x,y,z) = 0$ is

$$\begin{aligned} &axx_1 + byy_1 + czz_1 + f(yz_1 + zy_1) + g(zx_1 + xz_1) \\ &\quad + h(xy_1 + yx_1) + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0. \end{aligned}$$

It passes through the point (α, β, γ), if

$$\begin{aligned} &a\alpha x_1 + b\beta y_1 + c\gamma z_1 + f(\beta z_1 + \gamma y_1) + g(\gamma x_1 + \alpha z_1) \\ &\quad + h(\alpha y_1 + \beta x_1) + u(\alpha + x_1) + v(\beta + y_1) + w(\gamma + z_1) + d = 0. \end{aligned}$$

This shows that the points on the quadric at which the tangent planes pass through the point (α, β, γ) lie on the plane

$$\begin{aligned} &a\alpha x + b\beta y + c\gamma z + f(\beta z + \gamma y) + g(\gamma x + \alpha z) + h(\alpha y + \beta x) \\ &\quad + u(\alpha + x) + v(\beta + y) + w(\gamma + z) + d = 0, \end{aligned}$$

which is called the plane of contact for the point (α, β, γ).

7.51 Pole and Polar

Definition. *If a secant of a conicoid from a point P meets the conicoid at A and B and the point R on the secant is the harmonic conjugate of P w.r.t. A and B, then the locus of R is the polar plane or polar of P and P is the pole of this polar plane.*

Let the coordinates of P be (x_1, y_1, z_1) and the equation of the conicoid be $F(x,y,z) = 0$.

If the secant $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} (= r \text{ say})$ through the point P with d.cs. l, m, n meets the conicoid at the point $(lr + x_1, mr + y_1, nr + z_1)$, then

$$\phi(l,m,n)r^2 + \left(l\frac{\partial F}{\partial x_1} + m\frac{\partial F}{\partial y_1} + n\frac{\partial F}{\partial z_1}\right)r + F(x_1, y_1, z_1) = 0. \tag{1}$$

It is a quadratic equation in r. If r_1 and r_2 are the roots, then r_1 and r_2 stand for PA and PB.

$$\text{Here} \quad \frac{2}{PR} = \frac{1}{PA} + \frac{1}{PB} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2}$$

$$\text{or,} \quad \frac{2}{PR} = -\frac{l\frac{\partial F}{\partial x_1} + m\frac{\partial F}{\partial y_1} + n\frac{\partial F}{\partial z_1}}{F(x_1, y_1, z_1)} \quad \text{by (1).} \tag{2}$$

If (x, y, z) be the coordinates of R, then

$$x - \alpha = l \cdot PR, y - \beta = m \cdot PR, z - \gamma = n \cdot PR. \tag{3}$$

Eliminating l, m, n from (2) by (3), we have

$$(ax_1 + hy_1 + gz_1 + u)(x - x_1) + (hx_1 + by_1 + fz_1 + v)(y - y_1)$$
$$+ (gx_1 + fy_1 + cz_1 + w)(z - z_1) + F(x_1, y_1, z_1) = 0$$
$$\text{or,} \quad axx_1 + byy_1 + czz_1 + f(yz_1 + zy_1) + g(zx_1 + xz_1)$$
$$+ h(xy_1 + yx_1) + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0. \tag{4}$$

It is the locus of R, so the polar plane of P.

Pole of a given plane. If (x_1, y_1, z_1) be the pole of the plane $lx + my + nz = p$, then it is same as the equation (4).

Comparing, we get

$$\frac{ax_1 + hy_1 + gz_1 + u}{l} = \frac{hx_1 + by_1 + fz_1 + v}{m} = \frac{gx_1 + fy_1 + cz_1 + w}{n}$$
$$= \frac{ux_1 + vy_1 + wz_1 + d}{-p}.$$

From these the pole (x_1, y_1, z_1) is obtained.

Note 1. Conjugate points and conjugate planes. If the polar plane of a point P passes through the point Q, then the polar plane of Q passes through the point P.

Two such points are known as *conjugate points* and two such planes are called *conjugate planes*.

Note 2. The polar of a point on a quadric is not defined. The polar of the centre for the central conicoid is not defined.

Example 8. *Prove that the locus of the poles of the tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ w.r.t. the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$ is the conicoid*

$$\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1.$$

Any tangent plane to $ax^2 + by^2 + cz^2 = 1$ is

$$lx + my + nz = p, \tag{1}$$

$$\text{where} \qquad \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2. \tag{2}$$

If (x_1, y_1, z_1) be the pole of (1) w.r.t. $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$, then it is identical with

$$\alpha x x_1 + \beta y y_1 + \gamma z z_1 = 1. \tag{3}$$

Comparing (1) with (3),

$$\frac{\alpha x_1}{l} = \frac{\beta y_1}{m} = \frac{\gamma z_1}{n} = \frac{1}{p}. \tag{4}$$

Putting the values of l, m, n from (4) in (2), we have

$$\frac{(p\alpha x_1)^2}{a} + \frac{(p\beta y_1)^2}{b} + \frac{(p\gamma z_1)^2}{c} = p^2 \quad \text{or,} \quad \frac{\alpha^2 x_1^2}{a} + \frac{\beta^2 y_1^2}{b} + \frac{\gamma^2 z_1^2}{c} = 1.$$

Hence the locus is $\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1$.

7.52 Polar lines

Definition. *Polar of a given line AB w.r.t. a conicoid is another line CD such that the polar planes of all points on AB pass through CD.*

Let the equations of AB be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ (say) and the equation of the conicoid be $F(x, y, z) = 0$.

Any point P on the line AB is $(lr + \alpha, mr + \beta, nr + \gamma)$. The polar of this point w.r.t. the conicoid is

$$\begin{aligned} &ax(lr+\alpha) + by(mr+\beta) + cz(nr+\gamma) + f\{y(nr+\gamma) + z(mr+\beta)\} \\ &\quad + g\{z(lr+\alpha) + x(nr+\gamma)\} + h\{x(mr+\beta) + y(lr+\alpha)\} \\ &\quad + u\{x + (lr+\alpha)\} + v\{y + (mr+\beta)\} + w\{z + (nr+\gamma)\} + d = 0 \\ \text{or,}\quad & ax\alpha + by\beta + cz\gamma + f(y\gamma + z\beta) + g(z\alpha + x\gamma) + h(x\beta + y\alpha) \\ &\quad + u(x+\alpha) + v(y+\beta) + w(z+\gamma) + d \\ &\quad + r\{l(ax + hy + gz + u) + m(hx + by + fz + v) \\ &\quad + n(gx + fy + cz + w)\} = 0. \end{aligned}$$

This plane passes through the line

$$\begin{aligned} &ax\alpha + by\beta + cz\gamma + f(y\gamma + z\beta) + g(z\alpha + x\gamma) + h(x\beta + y\alpha) \\ &\quad + u(x+\alpha) + v(y+\beta) + w(z+\gamma) + d = 0, \\ &l(ax + hy + gz + u) + m(hx + by + fz + v) + n(gx + fy + cz + w) = 0. \end{aligned}$$

Now the polar plane of P passes through every point on CD. Therefore, the polar plane of any point on CD passes through P. Since P is arbitrary, the polar plane of any point on CD must pass through every point of AB. Thus AB and CD are polar lines.

Note 1. The polar line of any given line is the line of intersection of the polar planes of any two points on the given line.

Note 2. Conjugate lines. If a line AB intersects the polar of a line CD, then the line CD intersects the polar of AB. Two such lines AB and CD are called conjugate lines.

Example 9. *Find the condition that the lines*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \textit{and} \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$$

will be polar lines w.r.t. the conicoid $ax^2+by^2+cz^2=1$.

Let the first line be AB and the second line be CD. The polar line of AB is the line of intersection of the planes

$$a\alpha x+b\beta y+c\gamma z=1 \tag{1}$$

$$\text{and} \quad alx+bmy+cnz=0. \tag{2}$$

The line CD lies on these planes.

$$\therefore \quad a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'=1, \quad a\alpha l'+b\beta m'+c\gamma n'=0,$$
$$al\alpha'+bm\beta'+cn\gamma'=0, \quad all'+bmm'+cnn'=0.$$

These are the conditions.

WORKED-OUT EXAMPLES

1. *Find the equation of the tangent plane to* $2x^2-5y^2+6z=0$ *at* $(1,2,3)$.

   The point $(1,2,3)$ is on the conicoid. Therefore, the equation of the tangent plane at this point is

$$2x\cdot 1-5y\cdot 2+3(z+3)=0 \quad \text{or,} \quad 2x-10y+3z+9=0.$$

2. *Find the equation of the normal to the hyperboloid* $3x^2-5y^2+7z^2=30$ *at* $(2,3,3)$.

   The point $(2,3,3)$ is on the conicoid. Therefore, the equation of the tangent plane at this point is

$$3x\cdot 2-5y\cdot 3+7z\cdot 3=30 \quad \text{or,} \quad 2x-5y+7z=10.$$

   The normal is a straight line perpendicular to this plane at $(2,3,3)$. Hence the required equations of the normal are

$$\frac{x-2}{2}=\frac{y-3}{-5}=\frac{z-3}{7}.$$

3. *Find the equation of the sphere that passes through the points* $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ *and touches the plane* $2x+2y-z=15$.

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

As it passes through the given points

$$1+2u+d=0, \quad (1) \qquad 1+2v+d=0, \quad (2) \quad \text{and} \quad 1+2w+d=0. \quad (3)$$

Since the plane $2x+2y-z=15$ touches the sphere, the distance of the centre $(-u,-v,-w)$ from the plane is equal to the radius $\sqrt{u^2+v^2+w^2-d}$.

$$\therefore \quad \frac{-2u-2v+w-15}{\sqrt{2^2+2^2+1}} = \sqrt{u^2+v^2+w^2-d}. \tag{4}$$

From (1), (2) and (3), $u=v=w$.

Putting in (4)

$$\frac{-2u-2u+u-15}{3} = \sqrt{3u^2-d}$$

$$\text{or,} \quad -(u+5) = \sqrt{3u^2-d} \quad \text{or,} \quad u^2+10u+25 = 3u^2-d$$

$$\text{or,} \quad 2u^2-10u-d-25=0.$$

Again $d=-(2u+1)$.

$$\therefore \quad 2u^2-10u+2u+1-25=0 \quad \text{or,} \quad 2u^2-8u-24=0$$

$$\text{or,} \quad u^2-4u-12=0 \quad \text{or,} \quad (u-6)(u+2)=0 \quad \text{or,} \quad u=6,-2.$$

Corresponding to these values of $u, d=-13, 3$.

$\therefore$ the equations of the spheres are

$$x^2+y^2+z^2+12(x+y+z)-13=0$$

$$\text{and} \quad x^2+y^2+z^2-4(x+y+z)+3=0.$$

4. *Find the equation of the tangent plane to the hyperbolic paraboloid* $5x^2-2y^2=2z$, *parallel to* $10x-6y-z=7$ *and also find the point of contact.*

Let the equation of the tangent plane be $10x-6y-z=k$. Since it touches the conicoid, $\frac{10^2}{5}+\frac{6^2}{-2}+\frac{2\cdot(-1)\cdot k}{1}=0$ or, $k=1$.

$\therefore$ the equation of the tangent plane is $10x-6y-z=1$.

The point of contact is $\left(-\frac{1\cdot 10}{5\cdot -1}, -\frac{1\cdot -6}{-2\cdot -1}, -\frac{1}{-1}\right)$, i.e. $(2,3,1)$.

5. *Find the equations of the tangent planes to* $2x^2-6y^2+3z^2=5$, *which pass through the line* $x+9y-3z=0=3x-3y+6z-5$. [CH 2001]

Let the equation of the tangent plane be

$$(3x - 3y + 6z - 5) + \lambda(x + 9y - 3z) = 0 \tag{1}$$

$$\text{or,} \quad (3 + \lambda)x + (9\lambda - 3)y + (6 - 3\lambda)z - 5 = 0. \tag{2}$$

The equation of the conicoid can be written as

$$\frac{2x^2}{5} - \frac{6y^2}{5} + \frac{3z^2}{5} = 1.$$

From the condition of the tangency of a plane to a conicoid

$$\frac{5}{2}(3 + \lambda)^2 - \frac{5}{6}(9\lambda - 3)^2 + \frac{5}{3}(6 - 3\lambda)^2 = 25$$

$$\text{or,} \quad (3 + \lambda)^2 - 3(3\lambda - 1)^2 + 6(2 - \lambda)^2 = 10$$

$$\text{or,} \quad \lambda^2 = 1 \quad \text{or,} \quad \lambda = \pm 1.$$

Putting these values of λ in (1), the tangent planes are $4x + 6y + 3z = 5$ and $2x - 12y + 9z = 5$.

6. *Show that the points* $(12, -18, 8)$ *and* $(-6, 18, -10)$ *are at the feet of the normals to the ellipsoid* $x^2 + 2y^2 + 3z^2 = 984$ *which lie in the plane* $x + y + z = 2$.

[BH 91; CH 92]

The points $(12, -18, 8)$ and $(-6, 18, -10)$ are on the surface of the given quadric. Therefore, the tangent planes at these points are

$$12x - 2 \cdot 18y + 3 \cdot 8z = 984 \quad \text{or,} \quad x - 3y + 2z = 82 \tag{1}$$

$$\text{and} \quad -6x + 2 \cdot 18y - 3 \cdot 10z = 984 \quad \text{or,} \quad x - 6y + 5z = -164. \tag{2}$$

Consequently the normals at $(12, -18, 8)$ and $(-6, 18, -10)$ are

$$\frac{x - 12}{1} = \frac{y + 18}{-3} = \frac{z - 8}{2} \tag{3}$$

$$\text{and} \quad \frac{x + 6}{1} = \frac{y - 18}{-6} = \frac{z + 10}{5}. \tag{4}$$

Since

$$\begin{vmatrix} -6 - 12 & 18 - (-18) & -10 - 8 \\ 1 & -3 & 2 \\ 1 & -6 & 5 \end{vmatrix} = \begin{vmatrix} -18 & 36 & -18 \\ 1 & -3 & 2 \\ 1 & -6 & 5 \end{vmatrix}$$

$$= -18 \begin{vmatrix} 1 & -2 & 1 \\ 1 & -3 & 2 \\ 1 & -6 & 5 \end{vmatrix} = -18 \begin{vmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 5 \end{vmatrix} \quad [\text{by } C_2 + C_3]$$

$= 0$, the lines (3) and (4) are coplanar.

The equation of the plane containing these normals is

$$\begin{vmatrix} x-12 & y+18 & z-8 \\ 1 & -3 & 2 \\ 1 & -6 & 5 \end{vmatrix} = 0$$

or, $\quad -3(x-12)-3(y+18)-3(z-8)=0 \quad$ or, $\quad x+y+z=2.$

7. *If the normal at a point P on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the principal planes in G_1, G_2, G_3, then show that $PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$.* **[CH 95]**

Let the coordinates of P be (x_1, y_1, z_1). The equations of the normal at this point are

$$\frac{x-x_1}{\frac{x_1}{a^2}} = \frac{y-y_1}{\frac{y_1}{b^2}} = \frac{z-z_1}{\frac{z_1}{c^2}}.$$

The coordinate planes are the principal planes. Let this normal meet YOZ, ZOX and XOY-planes at G_1, G_2 and G_3 respectively.

To find the coordinates of G_1,

$$\frac{0-x_1}{\frac{x_1}{a^2}} = \frac{y-y_1}{\frac{y_1}{b^2}} = \frac{z-z_1}{\frac{z_1}{c^2}}.$$

From this $y = y_1\left(1-\frac{a^2}{b^2}\right), z = z_1\left(1-\frac{a^2}{c^2}\right)$.

$\therefore$ coordinates of G_1 are $\left\{0, y_1\left(1-\frac{a^2}{b^2}\right), z_1\left(1-\frac{a^2}{c^2}\right)\right\}$.

Similarly coordinates of G_2 and G_3 are

$$\left\{x_1\left(1-\frac{b^2}{a^2}\right), 0, z_1\left(1-\frac{b^2}{c^2}\right)\right\} \quad \text{and} \quad \left\{x_1\left(1-\frac{c^2}{a^2}\right), y_1\left(1-\frac{c^2}{b^2}\right), 0\right\}.$$

Now

$$PG_1^2 = x_1^2 + \frac{a^4}{b^4}y_1^2 + \frac{a^4}{c^4}z_1^2 = a^4\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right),$$
$$PG_2^2 = \frac{b^4}{a^4}x_1^2 + y_1^2 + \frac{b^4}{c^4}z_1^2 = b^4\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right),$$
$$PG_3^2 = \frac{c^4}{a^4}x_1^2 + \frac{c^4}{b^4}y_1^2 + z_1^2 = c^4\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right).$$

$$\therefore \; PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2.$$

8. *Show that the locus of the line of intersection of tangent planes to the cone $ax^2+by^2+cz^2=0$ which touch along the perpendicular generators is*

$$a^2(b+c)x^2 + b^2(c+a)y^2 + c^2(a+b)z^2 = 0.$$ **[BH 2008]**

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be any two points on two perpendicular generators and the tangent planes along these generators meet at a line

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}. \tag{1}$$

The tangent plane at (x_1, y_1, z_1) is

$$axx_1 + byy_1 + czz_1 = 0. \tag{2}$$

Since the line (1) lies on the plane (2),

$$a\lambda x_1 + b\mu y_1 + c\nu z_1 = 0. \tag{3}$$

Similarly by considering the tangent plane at (x_2, y_2, z_2) it is obtained that

$$a\lambda x_2 + b\mu y_2 + c\nu z_2 = 0.$$

The conditions (3) and (4) suggest that the generators through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the plane

$$a\lambda x + b\mu y + c\nu z = 0.$$

If this plane cuts the cone at two perpendicular generators, then

$$a(b^2\mu^2 + c^2\nu^2) + b(c^2\nu^2 + a^2\lambda^2) + c(a^2\lambda^2 + b^2\mu^2) = 0$$
$$\text{or,}\quad a^2(b+c)\lambda^2 + b^2(c+a)\mu^2 + c^2(a+b)\nu^2 = 0.$$

[See (8) of Sec 6.35.]

Eliminating λ, μ, ν by (1), the required locus is

$$a^2(b+c)x^2 + b^2(c+a)y^2 + c^2(a+b)z^2 = 0.$$

9. *Find the equation to the normal plane of the cone $ax^2 + by^2 + cz^2 = 0$ which passes through the generator $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.*

If (x', y', z') is a point on the cone and the generator, then $\frac{x'}{l} = \frac{y'}{m} = \frac{z'}{n}$.
The equation of the tangent plane at this point is

$$axx' + byy' + czz' = 0 \quad \text{or,} \quad alx + bmy + cnz = 0.$$

If $ux + vy + wz = 0$ is the normal plane through the given generator, then it is perpendicular to the tangent plane and also the generator lies on this plane.

$$\therefore\ alu + bmv + cnw = 0 \quad \text{and} \quad lu + mv + nw = 0.$$

By cross-multiplication,

$$\frac{u}{mn(b-c)} = \frac{v}{nl(c-a)} = \frac{w}{lm(a-b)}.$$

Thus the normal plane through the given generator is

$$mn(b-c)x + nl(c-a)y + lm(a-b)z = 0 \quad \text{or,} \quad \frac{b-c}{l}x + \frac{c-a}{m}y + \frac{a-b}{n}z = 0.$$

10. *Prove that the lines through the point (α, β, γ) at right angles to their polar lines w.r.t.* $\frac{x^2}{a+b} + \frac{y^2}{2a} + \frac{z^2}{2b} = 1$ *generate the curve* $(y - \beta)(\alpha z - \gamma x) + (z - \gamma)(\alpha y - \beta x) = 0$.

   What is the peculiarity of the case when $a = b$ *?*

   A line through the point (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}.$$

Its polar line w.r.t. the given conicoid is the line of intersection of the planes

$$\frac{\alpha x}{a+b} + \frac{\beta y}{2a} + \frac{\gamma z}{2b} = 1 \quad \text{and} \quad \frac{lx}{a+b} + \frac{my}{2a} + \frac{nz}{2b} = 0.$$

The d.rs. of this line are

$$\frac{n\beta - m\gamma}{4ab}, \frac{l\gamma - n\alpha}{2b(a+b)}, \frac{m\alpha - l\beta}{2a(a+b)}.$$

In order that $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ may be perpendicular to the polar line, we must have

$$l\frac{n\beta - m\gamma}{4ab} + m\frac{l\gamma - n\alpha}{2b(a+b)} + n\frac{m\alpha - l\beta}{2a(a+b)} = 0$$
$$\text{or,} \quad (a-b)(-2mn\alpha + nl\beta + lm\gamma) = 0.$$

In case $a \neq b, 2mn\alpha - nl\beta - lm\gamma = 0$ or, $m(n\alpha - l\gamma) + n(m\alpha - l\beta) = 0$.

Hence the locus of the line (1) is

$$(y-\beta)\{(z-\gamma)\alpha - (x-\alpha)\gamma\} + (z-\gamma)\{(y-\beta)\alpha - (x-\alpha)\beta\} = 0,$$
$$\text{i.e.} \quad (y-\beta)(\alpha z - \gamma x) + (z-\gamma)(\alpha y - \beta x) = 0.$$

In case $a = b$, the conicoid becomes a sphere of radius $\sqrt{2}a$. For a sphere the polar lines are perpendicular to each other. Hence the locus of such lines is the whole space.

EXERCISE VII

1. Find the equation of the tangent plane to
   (i) $2x^2 + 4y^2 + 5z^2 = 63$ at $(1, 2, 3)$; (ii) $3x^2 - 9y^2 + 11z = 0$ at $(1, 2, 3)$.

2. Find the equation of the normal to the hyperbolic paraboloid $2x^2 - 3y^2 = 10z$ at the point $(2, 4, -4)$.

3. Show that the tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 4 = 0$ at $(4, -2, 2)$ and $(0, 0, -2)$ are parallel.

4. Find the equation of the sphere with the centre at $(1, -1, 3)$ and touching the plane $2x + y - 3z = 5$.

5. Show that the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$ and find the point of contact.

6. Obtain the equation of the sphere touching the three coordinate planes and passing through the point $(2, 1, 5)$.

7. Find the equation of the sphere passing through the points $(4, 1, 0)$, $(2, -3, 4)$, $(1, 0, 0)$ and touching the plane $2x + 2y - z = 11$.

8. Show that the equations to the spheres which pass through the circle $x^2 + y^2 + z^2 = 5$, $x + 2y + 3z = 3$ and touch the plane $4x + 3y - 15 = 0$ are $x^2+y^2+z^2+2x+4y+6z-11 = 0$ and $5(x^2+y^2+z^2)-4x-8y-12z-13 = 0$.

9. Find the equations of the tangent planes to $\frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{4} = 1$, parallel to $x + y + z = 0$.

10. Find the values of k for each of which $x + y + z = k$ is a tangent plane to the ellipsoid $x^2 + 2y^2 + 3z^2 = 66$.

11. Find the equations of the tangent planes to $3x^2-7y^2+5z^2 = 12$, perpendicular to $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$.

12. Find the equations of the pair of tangent planes to the parabolid $2x^2 - y^2 = 4z$ which pass through the line $x + 1 = 0 = y + z - 1$ and find their points of contact.

13. (i) Find the equations of the two tangent planes to the quadric $\frac{x^2}{42} - \frac{y^2}{18} + \frac{z^2}{3} = 1$ which pass through the line $\frac{x-8}{-7} = \frac{y+4}{5} = \frac{z-1}{1}$. **[BH 97, 99]**
[*Hints.* The given line can be written as $5x + 7y - 12 = 0 = y - 5z + 9$. Any plane through the line is $5x + 7y - 12 + \lambda(y - 5z + 9) = 0$. If it is the tangent plane to the given quadric, then

$$5^2 \cdot 42 - (7 + \lambda)^2 \cdot 18 + 25\lambda^2 \cdot 3 = (9\lambda - 12)^2$$

$$\text{or,}\quad 2\lambda^2 + 3\lambda - 2 = 0 \quad \text{or,}\quad (\lambda + 2)(2\lambda - 1) = 0 \quad \text{or,}\quad \lambda = -2, \frac{1}{2}.$$

$\therefore$ the tangent planes are $x + y + 2z - 6 = 0$ and $2x + 3y - z - 3 = 0$.].

(ii) Find the points of contact of the tangent planes to the conicoid $2x^2 - 25y^2+2z^2 = 1$ which pass through the line joining the points $(-12, 1, 12)$ and $(13, -1, -13)$.

14. Find the equation of the tangent plane to the quadric $3x^2+2y^2-6z^2 = 6$ which passes through the point $(3, 4, -3)$ and is parallel to the line $x = y = -z$.

15. A tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ meets the coordinate axes in points A, B, C. Prove that the centroid of $\triangle ABC$ lies on $\frac{a^2}{x^2} + \frac{b^2}{y^2} + \frac{c^2}{z^2} = 9$.

16. (i) Find the equation of the plane which touches the cone $x^2 + 4y^2 - 5z^2 + 4xy - 6yz + 2zx = 0$ along the generator with d.rs. $1, 1, 1$.

[*Hints.* The equations of the generator with d.rs. 1, 1, 1 are $x = y = z$. A point on this generator is $(\lambda, \lambda, \lambda)$. The equation of the tangent plane at this point, i.e. the plane touching the cone along the generator is $x\lambda+4y\lambda-5z\lambda+2(x\lambda+y\lambda)-3(y\lambda+z\lambda)+(z\lambda+x\lambda) = 0$ or, $4x+3y-7z = 0$].

(ii) Find the equation to the normal plane of the cone $ax^2 + by^2 + cz^2 = 0$ which passes through the generator $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

17. Show that the normals to the ellipsoid $x^2 + 2y^2 + 3z^2 = 50$ at the points $(6, 1, 2)$ and $(6, -1, 2)$ lie in the plane $x - z = 4$.

18. If the normal at a point P of the paraboloid $x^2 + y^2 = 4z$ meets the planes YOZ and ZOX at Q and R respectively, then show that $PQ = PR$.

19. Find the other two of the set of three mutually perpendicular generators of the cone $11yz + 6zx - 14xy = 0$ when one is $x = \frac{y}{2} = z$.

20. (i) Show that the reciprocal cone of $ax^2+by^2+cz^2 = 0$ is $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = 0$.

(ii) Find the equation to the cone whose vertex is $(0, 0, c)$ and base $z = 0, x^2 + y^2 - 2ax = 0$. Find the equation to its reciprocal cone. **[CH 2008]**

21. Find the enveloping cone of the sphere $x^2+y^2+z^2 = 25$ with $(10, 0, 0)$ as the vertex.

22. Find the equation of the cone whose vertex is the point $(4, 5, -5)$ and whose generators touch the quadric $x^2 + 2y^2 + 3z^2 = 6$.

23. Prove that the equation of the cylinder whose generators touch the sphere $x^2 + y^2 + z^2 = 4$ and are perpendicular to the plane $x + y - 2z = 8$ is $5x^2 + 5y^2 + 2z^2 + 4yz + 4zx - 2xy = 24$.

24. Find the equation of the enveloping cylinder whose generators touch the quadric $5x^2 + 7y^2 = 2z$ and are parallel to the line with d.rs. $2, -2, 1$.

25. Find the locus of the points from which three mutually perpendicular generators can be drawn to the quadric $ax^2 + by^2 + cz^2 = 1$. **[BH 2007]**

26. Prove that the plane $z = 0$ cuts the enveloping cone of the sphere $x^2+y^2+z^2 = 11$ with $(2, 4, 1)$ as vertex in a rectangular hyperbola.

27. Find the locus of the centres of spheres of constant radius which pass through a given point and touch a given line.

[*Hints.* Let $(0, 0, c)$ and the x-axis be the given point and the given line.

If $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is the sphere, then

$$c^2 + 2wc + d = 0 \tag{1}$$
$$\text{and} \quad u^2 + v^2 + w^2 - d = \lambda^2. \tag{2}$$

The sphere meets the x-axis where $x^2 + 2ux + d = 0$.

This equation must have equal roots.

$$\therefore\ u^2 = d. \tag{3}$$

By (1) and (3) $\quad u^2 + 2cw + c^2 = 0.$

By (2) and (3) $\quad v^2 + w^2 = \lambda^2.$

Thus the required locus is the curve $x^2 - 2cz + c^2 = 0, y^2 + z^2 = \lambda^2$.]

28. Find the locus of luminous points of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ that casts a circular shadow on the plane $z = 0$. **[BH 94; CH 91, 94, 99]**

[*Hints.* The enveloping cone with the luminous point (x_1, y_1, z_1) as vertex is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1\right)^2.$$

The section of it by $z = 0$ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2.$$

If it is a circle, then the coefficient of $xy = 0$ and the coefficient of $x^2 =$ the coefficient of y^2.

Now the result is obtained.]

29. Prove that the locus of the points from which three mutually perpendicular tangent planes can be drawn to touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$ is the sphere $x^2 + y^2 + z^2 = a^2 + b^2$.

[*Hints.* The equation of the cone with the vertex (α, β, γ) and the guiding curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$ is

$$\frac{\gamma^2}{a^2}x^2 + \frac{\gamma^2}{b^2}y^2 + \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right)z^2 - 2\gamma z\left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - 1\right) - \gamma^2 = 0.$$

If it has three mutually perpendicular tangent planes, then

$$\frac{\gamma^2}{b^2}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right) + \frac{\gamma^2}{a^2}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right) + \frac{\gamma^2}{a^2}\cdot\frac{\gamma^2}{b^2} = \frac{\beta^2\gamma^2}{b^4} + \frac{\gamma^2\alpha^2}{a^4}$$

or, $\quad \alpha^2 + \beta^2 + \gamma^2 = a^2 + b^2.$

Hence the locus of (α, β, γ) is $x^2 + y^2 + z^2 = a^2 + b^2$.]

30. Prove that the general equation of a cone which touches the three coordinate planes is

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0.$$

[*Hints.* $ayz + bzx + cxy = 0$ represents a cone having the coordinate axes as generators. Its reciprocal cone is the required cone.]

31. Show that the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ with generators perpendicular to the z-axis may meet the plane $z = 0$ in a pair of lines.

[*Hints.* $\frac{x}{l} = \frac{y}{m} = \frac{z}{o}$ is a line perpendicular to the z-axis. The enveloping cylinder of the conicoid with generators parallel to the above line is $(ax^2 + by^2 + cz^2 - 1)(al^2 + bm^2) = (alx + bmy)^2$.

Its section by the plane $z = 0$ is $ab(mx - ly)^2 = al^2 + bm^2$.

If $\frac{l^2}{b} + \frac{m^2}{a} > 0$, then $mx - ly = \pm\sqrt{\frac{l^2}{b} + \frac{m^2}{a}}, z = 0$ represent two lines.]

32. Prove that the feet of the normals from the point (x', y', z') to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lie on the curve of intersection of the ellipsoid and the cone

$$\frac{a^2(b^2 - c^2)x'}{x} + \frac{b^2(c^2 - a^2)y'}{y} + \frac{c^2(a^2 - b^2)z'}{z} = 0. \qquad \text{[CH 2000]}$$

[*Hints.* The coordinates of the foot of the normal to the ellipsoid from the point (x', y', z') are

$$x = \frac{a^2x'}{a^2 + \lambda}, y = \frac{b^2y'}{b^2 + \lambda}, z = \frac{c^2z'}{c^2 + \lambda}.$$

$$\therefore \lambda = a^2\frac{x'}{x} - a^2, \lambda = b^2\frac{y'}{y} - b^2, \lambda = c^2\frac{z'}{z} - c^2.$$

Multiplying these by $b^2 - c^2, c^2 - a^2$ and $a^2 - b^2$ respectively and adding we have $\sum \frac{a^2(b^2 - c^2)x'}{x} = 0$. Hence the result follows.]

33. If the feet of the six normals from (α, β, γ) are $(x_r, y_r, z_r)(r = 1, 2, \cdots, 6)$, prove that

$$a^2\alpha\sum\frac{1}{x_r} + b^2\beta\sum\frac{1}{y_r} + c^2\gamma\sum\frac{1}{z_r} = 0. \qquad \text{[CH 98, 2002, 06]}$$

34. (i) If $P, Q, R; P', Q', R'$ are the feet of the six normals from a point to the ellipsoid, and the plane PQR is given by $lx + my + nz = p$, then the plane $P'Q'R'$ is given by

$$\frac{x}{a^2l} + \frac{y}{b^2m} + \frac{z}{c^2n} + \frac{1}{p} = 0. \qquad \text{[NH 06]}$$

[*Hints.* Here the ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

If (x', y', z') is the given point, the feet of the six normals through the above point are given by

$$\left(\frac{a^2x'}{a^2 + \lambda}, \frac{b^2y'}{b^2 + \lambda}, \frac{c^2z'}{c^2 + \lambda}\right),$$

where

$$\frac{a^2x'^2}{(a^2 + \lambda)^2} + \frac{b^2y'^2}{(b^2 + \lambda)^2} + \frac{c^2z'^2}{(c^2 + \lambda)^2} = 1. \qquad (1)$$

Let the equation of the plane $P'Q'R'$ be $l'x + m'y + n'z = p'$.

The six feet of the normals lie on the combined equation

$$(lx + my + nz - p)(l'x + m'y + n'z - p') = 0.$$

$$\therefore \left(\frac{a^2x'l}{a^2+\lambda} + \frac{b^2y'm}{b^2+\lambda} + \frac{c^2z'n}{c^2+\lambda} - p\right)\left(\frac{a^2x'l'}{a^2+\lambda} + \frac{b^2y'm'}{b^2+\lambda} + \frac{c^2z'n'}{c^2+\lambda} - p'\right) = 0.$$

It is identical with (1).

Comparing the coefficients,

$$\frac{a^4ll'}{a^2} = \frac{b^4mm'}{b^2} = \frac{c^4nn'}{c^2} = \frac{pp'}{-1}.$$

From these

$$l' = -\frac{pp'}{a^2l}, m' = -\frac{pp'}{b^2m}, n' = -\frac{pp'}{c^2n}.$$

Hence the plane $P'Q'R'$ is $\frac{x}{a^2l} + \frac{y}{b^2m} + \frac{z}{c^2m} + \frac{1}{p} = 0$.]

(ii) Show that there are six points on the ellipsoid $2x^2 + 3y^2 + 6z^2 = 1$, the normals at which pass through a given point. If three of the feet of the normals lie on the plane $x - 2y + 2z = 1$, show that the feet of the remaining three lie on the plane $4x - 3y + 6z + 2 = 0$.

35. Show that the planes which cut $ax^2+by^2+cz^2 = 0$ in perpendicular generators, touch the cone $\frac{x^2}{b+c} + \frac{y^2}{c+a} + \frac{z^2}{a+b} = 0$.

[*Hints.* Let the plane $ux + vy + wz = 0$ cut the given cone in perpendicular generators. Then $(b+c)u^2 + (c+a)v^2 + (a+b)w^2 = 0$. Therefore, the normal to the plane generates the cone $(b+c)x^2 + (c+a)y^2 + (a+b)z^2 = 0$. The required cone is reciprocal to this cone.]

36. Show that the feet of the normals from the point (α, β, γ) to the paraboloid $x^2 + y^2 = 2az$ lie on the sphere $x^2 + y^2 + z^2 - (a+\gamma)z - (\alpha^2 + \beta^2)\frac{y}{2\beta} = 0$ when $\beta \neq 0$. **[CH 2004; BH 2007]**

[*Hints.* Let (x_1, y_1, z_1) be a point on the paraboloid. The normal at this point is $\frac{x-x_1}{x_1/a} = \frac{y-y_1}{y_1/a} = \frac{z-z_1}{-1}$. If it passes through (α, β, γ), then

$$\frac{a(\alpha - x_1)}{x_1} = \frac{a(\beta - y_1)}{y_1} = z_1 - \gamma.$$

It implies that

$$x_1 = \frac{a\alpha}{a+\lambda}, y_1 = \frac{a\beta}{a+\lambda}, z_1 = \gamma + \lambda. \tag{1}$$

Since (x_1, y_1, z_1) lies on the paraboloid,

$$a(\alpha^2 + \beta^2) = 2(\gamma + \lambda)(a + \lambda)^2. \tag{2}$$

By the help of (1) and (2), it can be shown that

$$x_1^2 + y_1^2 + z_1^2 - (a+\gamma)z_1 - (\alpha^2 + \beta^2)\frac{y_1}{2\beta} = 0.$$

Hence the result follows.]

37. Show that the lines drawn through the origin at right angles to normal planes of the cone $ax^2 + by^2 + cz^2 = 0$ generates the cone

$$\frac{a(b-c)^2}{x^2} + \frac{b(c-a)^2}{y^2} + \frac{c(a-b)^2}{z^2} = 0.$$

[*Hints.* The equation of the normal plane through the generator

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \text{ is } \sum \frac{x(b-c)}{a} = 0.$$

The equation of the line perpendicular to the plane is

$$\frac{x}{(b-c)/l} = \frac{y}{(c-a)/m} = \frac{z}{(a-b)/n}.$$

Again $al^2 + bm^2 + cn^2 = 0$. Hence the locus is

$$\frac{a(b-c)^2}{x^2} + \frac{b(c-a)^2}{y^2} + \frac{c(a-b)^2}{z^2} = 0.]$$

38. Two perpendicular tangent planes to the paraboloid $\frac{x^2}{a} + \frac{y^2}{b} = 2z$ intersect in a line lying in the plane $x = 0$. Show that the line touches the parabola $x = 0, y^2 = (a+b)(2z+a)$.

[*Hints.* Any line in the plane $x = 0$ has equations $py + qz = k, x = 0$. Any plane through this line is $py + qz - k + \lambda x = 0$ where λ is a parameter. If it is tangent to the paraboloid, then $a\lambda^2 + bp^2 + 2qk = 0$. This is a quadratic equation in λ. If λ_1 and λ_2 are the roots, then $\lambda_1\lambda_2 = \frac{bp^2+2qk}{a}$ and the tangent planes are $py + qz - k + \lambda_1 x = 0$ and $py + qz - k + \lambda_2 x = 0$. These planes will be perpendicular to each other, if $p^2 + q^2 + \lambda_1\lambda_2 = 0$ or, $p^2 + q^2 + \frac{bp^2+2qk}{a} = 0$.

Again in the plane $x = 0$, the line $py + qz = k$ will touch the parabola $y^2 = (a+b)(2z+a)$, if $4\{kq + p^2(a+b)\}^2 = 4q^2\{k^2 - ap^2(a+b)\}$, i.e. $p^2 + q^2 + \frac{bp^2+2kq}{a} = 0$. Hence the result follows.]

39. Prove that the locus of the pole of the plane $lx + my + nz = p$ w.r.t. the system of conicoids

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1$$

is a straight line perpendicular to the plane, λ being a parameter.

40. Show that the locus of lines drawn through a fixed point (α, β, γ) at right angles to their polars w.r.t. the conicoid $ax^2 + by^2 + cz^2 = 1$ is

$$\frac{\alpha}{x-\alpha}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{\beta}{y-\beta}\left(\frac{1}{c} - \frac{1}{a}\right) + \frac{\gamma}{z-\gamma}\left(\frac{1}{a} - \frac{1}{b}\right) = 0.$$

ANSWERS

1. (i) $2x + 8y + 15z = 63$;

   (ii) $6x - 36y + 11z + 33 = 0$.

2. $\frac{x-2}{4} = \frac{y-4}{-12} = \frac{z+4}{-5}$.

4. $14(x^2 + y^2 + z^2) - 28x + 28y - 84z - 15 = 0$.

5. $\left(-1, 2, \frac{2}{3}\right)$.

6. $x^2 + y^2 + z^2 - 10(x + y + z) + 50 = 0$.

7. $x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0, 16(x^2 + y^2 + z^2) - 102x + 50y - 49z + 86 = 0$.

9. $x + y + z = \pm 3$.

10. ± 11.

11. $2x + 3y + 4z = \pm\sqrt{\frac{1364}{35}}$.

12. $y + z = 1, 2x + y + z + 1 = 0, (0, 2, -1), (-2, 2, 1)$.

13. (ii) $(-2, -1, -3), (3, -1, 2)$.

14. $x + y + 2z = 1$.

16. (ii) $\sum \frac{x(b-c)}{l} = 0$.

19. $\frac{x}{2} = \frac{y}{-3} = \frac{z}{4}, \frac{x}{-11} = \frac{y}{2} = \frac{z}{7}$.

20. (ii) $c(x^2 + y^2) + 2ax(z - c) = 0, a^2y^2 - c^2(z - c)^2 + 2cax(z - c) = 0$.

21. $x^2 - 3y^2 - 3z^2 - 20x + 100 = 0$.

22. $135(x^2 + 2y^2 + 3z^2 - 6) = (4x + 10y - 15z - 6)^2$.

24. $35(x + y)^2 + 5x - 7y - 24z - \frac{1}{4} = 0$.

25. $a(b + c)x^2 + b(c + a)y^2 + c(a + b)z^2 = a + b + c$.

28. $\frac{y^2}{b^2 - a^2} + \frac{z^2}{c^2} = 1, x = 0; \frac{x^2}{a^2 - b^2} + \frac{z^2}{c^2} = 1, y = 0$.

Chapter 8

System of Spheres

8.10 Angle of intersection of two spheres

Definition. *The angle of intersection of two spheres is defined as the angle between their tangent planes at a common point of intersection.* Since the radii of the spheres to the common point are perpendicular to the tangent planes, the angle between these radii is equal to the angle between the tangent planes i.e. the angle of intersection of the spheres.

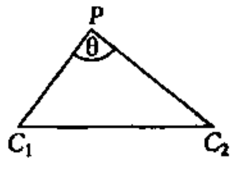

Fig. 39

Let P be the common point of two spheres whose centres are C_1 and C_2. The angle of intersection of two spheres is the angle between PC_1 and PC_2. If $C_1P = r_1, C_2P = r_2$ and $\angle C_1PC_2 = \theta$, then $C_1C_2^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta$.

Orthogonal spheres. *Two spheres are said to be orthogonal if the angle of intersection is a right angle.*

In this case, $C_1C_2^2 = r_1^2 + r_2^2$.

8.11 Condition for the orthogonality of two spheres

Let the equations of the spheres be

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \tag{1}$$

$$\text{and} \quad x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0. \tag{2}$$

If C_1 and C_2 are the centres of (1) and (2) and P is the common point, then the coordinates of C_1 and C_2 are $(-u_1, -v_1, -w_1)$ and $(-u_2, -v_2, -w_2)$ respectively. C_1P and C_2P are radii of (1) and (2) and $\angle C_1PC_2 = 90°$ for orthogonal intersection.

$$\therefore \quad C_1P^2 + C_2P^2 = C_1C_2^2,$$

$$\text{i.e.} \quad (u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2 = (u_1^2 + v_1^2 + w_1^2 - d_1) + (u_2^2 + v_2^2 + w_2^2 - d_2)$$

$$\text{or,} \quad 2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2. \tag{3}$$

It is the required condition.

Example 1. *Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$.*

Let the spheres be

$$x^2+y^2+z^2=r_1^2 \tag{1}$$

$$\text{and}\quad x^2+y^2+z^2+2ux+2vy+2wz+d=0. \tag{2}$$

Radius of (2) is r_2.

$$\therefore\ r_2^2=u^2+v^2+w^2-d. \tag{3}$$

Since (1) and (2) cut orthogonally,

$$d-r_1^2=0. \tag{4}$$

The plane of the common circle is

$$(x^2+y^2+z^2+2ux+2vy+2wz+d)-(x^2+y^2+z^2-r_1^2)=0,$$

$$\text{i.e.}\quad 2ux+2vy+2wz+2r_1^2=0\quad [\text{by (4)}]$$

$$\text{or,}\quad ux+vy+wz+r_1^2=0.$$

If p is the perpendicular distance of the plane from the centre of (1), then

$$p^2=\frac{r_1^4}{u^2+v^2+w^2}=\frac{r_1^4}{d+r_2^2}=\frac{r_1^4}{r_1^2+r_2^2}\quad [\text{by (3) and (4)}].$$

Now the radius of the circle

$$=\sqrt{r_1^2-p^2}=\sqrt{r_1^2-\frac{r_1^4}{r_1^2+r_2^2}}=\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}.$$

Example 2. *Two points P, Q are conjugate w.r.t. a sphere S, prove that the sphere on PQ as diameter cuts S orthogonally.*

Let

$$S\equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0 \tag{1}$$

and the coordinates of P and Q be (x_1,y_1,z_1) and (x_2,y_2,z_2). Since P and Q are conjugate points, the polar plane of P passes through Q.

$$\therefore\ x_1x_2+y_1y_2+z_1z_2+u(x_1+x_2)+v(y_1+y_2)+w(z_1+z_2)+d=0. \tag{2}$$

The equation of the sphere on PQ as diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

$$\text{or,}\quad x^2+y^2+z^2-x(x_1+x_2)-y(y_1+y_2)-z(z_1+z_2)$$
$$+x_1x_2+y_1y_2+z_1z_2=0. \tag{3}$$

Now (1) will cut (3) orthogonally if

$$-u(x_1+x_2)-v(y_1+y_2)-w(z_1+z_2)=d+x_1x_2+y_1y_2+z_1z_2.$$

It is true by (2). Hence the result follows.

8.20 Power of point or length of tangent

$$\text{Let} \quad x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \tag{1}$$

$$\text{and} \quad \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \text{ (say)} \tag{2}$$

be the equations of a sphere and a line respectively.

Any point on the line is $(lr + x_1, mr + y_1, nr + z_1)$.

If this point lies on the sphere, then

$$(lr + x_1)^2 + (mr + y_1)^2 + (nr + z_1)^2 + 2u(lr + x_1) + 2v(mr + y_1) + 2w(nr + z_1) + d = 0$$

$$\text{or,} \quad (l^2 + m^2 + n^2)r^2 + 2(lx_1 + my_1 + nz_1)r + (x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d) = 0. \tag{3}$$

It is a quadratic equation in r. Let the roots be r_1 and r_2. If l, m, n are the d.cs. of the line (2), the r_1 and r_2 are the distances of the points of intersection between (1) and (2) from the point (x_1, y_1, z_1).

If P is the point (x_1, y_1, z_1), A and B are the points of intersection, then

$PA \cdot PB = r_1 r_2 = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$=constant. It is called the *power of the point* P w.r.t. the sphere.

In case of the tangency of the line, roots of (3) are equal. In Fig. 40, PT is the tangent (T is the point of contact) and PT^2=product of the roots=power of the point P.

PT is called the length of the tangent.

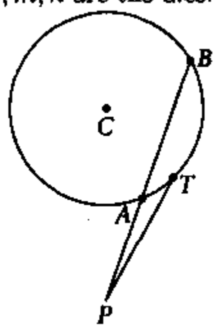

Fig. 40

8.21 Radical plane

Definition. *Radical plane of two spheres is the locus of points whose powers w.r.t. the spheres are equal.*

Let the spheres be

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$\text{and} \quad S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0.$$

Let the powers of the point $P(x_1, y_1, z_1)$ w.r.t. the spheres be equal.

$$\text{Then} \quad x_1^2 + y_1^2 + z_1^2 + 2u_1x_1 + 2v_1y_1 + 2w_1z_1 + d_1 = x_1^2 + y_1^2 + z_1^2 + 2u_2x_1 + 2v_2y_1 + 2w_2z_1 + d_2$$

$$\text{or,} \quad 2(u_1 - u_2)x_1 + 2(v_1 - v_2)y_1 + 2(w_1 - w_2)z_1 + d_1 - d_2 = 0.$$

$\therefore$ the locus of P, i.e. the equation of the radical plane is

$$2(u_1 - u_2)x + 2(v_1 - v_2)y + 2(w_1 - w_2)z + d_1 - d_2 = 0.$$

Note. The radical plane of two spheres $S_1 = 0, S_2 = 0$ is $S_1 - S_2 = 0$ when the coefficients of second degree terms in each equation of the spheres are unity.

8.22 Some properties of radical plane

(i) The radical plane of two spheres is at right angles to the line joining their centres.

Let the spheres by

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \tag{1}$$

$$\text{and} \quad x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0. \tag{2}$$

Let radical plane is

$$2(u_1 - u_2)x + 2(v_1 - v_2)y + 2(w_1 - w_2)z + d_1 - d_2 = 0. \tag{3}$$

The d.rs. of the line joining the centres $(-u_1, -v_1, -w_1)$ and $(-u_2, -v_2, -w_2)$ are $u_1 - u_2, v_1 - v_2, w_1 - w_2$. These are the d.rs. of the normal to the plane (3). Hence the line joining the centres is perpendicular to the radical plane.

(ii) The radical plane of two spheres passes through their points of intersection.

The radical plane of two spheres $S_1 = 0$ and $S_2 = 0$ is $S_1 - S_2 = 0$.

This equation is satisfied by the points which satisfy both $S_1 = 0$ and $S_2 = 0$. Hence the radical plane passes through the common points of the corresponding spheres.

Corollary. If the two spheres intersect in a circle, the radical plane is the plane of that circle.

8.30 Radical axis and radical centre

Radical axis. *The radical planes of three spheres taken two by two intersect along a line which is called the radical axis or radical line of the spheres.*

Let the spheres be $S_1 = 0, S_2 = 0, S_3 = 0$.

Then the radical planes are $S_1 - S_2 = 0, S_2 - S_3 = 0, S_3 - S_1 = 0$.

These planes clearly intersect in the line $S_1 = S_2 = S_3$.

This line is the radical axis.

Radical centre. *The four radical axes of four spheres taken three at a time intersect at a point which is called the radical centre.*

Let the spheres be $S_1 = 0, S_2 = 0, S_3 = 0, S_4 = 0$.

Then the point common to the three planes $S_1 = S_2 = S_3 = S_4$ is also the common point of the radical axes. This point is the radical centre.

8.31 Coaxial spheres

Definition. *A system of spheres is said to be coaxial when they have a common radical plane.*

If $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is the equation of a sphere and $L \equiv lx + my + nz - p = 0$ is the equation of a plane, then $S + \lambda L = 0$ represents a system of coaxial spheres with λ as a parameter.

In this system $L = 0$ is the radical plane

$$S + \lambda L = 0$$

or, $$x^2 + y^2 + z^2 + (2u + \lambda l)x + (2v + \lambda m)y + (2w + \lambda n)z + d - \lambda p = 0.$$

The coordinates of the centre are

$$\left(-\frac{2u+\lambda l}{2}, -\frac{2v+\lambda m}{2}, -\frac{2w+\lambda n}{2}\right)$$

and $$\text{radius} = \sqrt{\left(\frac{2u+\lambda l}{2}\right)^2 + \left(\frac{2v+\lambda m}{2}\right)^2 + \left(\frac{2w+\lambda n}{2}\right)^2 - (d - \lambda p)}.$$

The sphere whose radius is zero is known as a *point sphere*. For two values of λ the above radius is zero. Thus there are two point spheres in a coaxial system. The centres of these point spheres are called *limiting points*.

Note 1. If $S_1 = 0$ and $S_2 = 0$ are the equations of two spheres in a coaxial system, then $S_1 + \lambda S_2 = 0$ is the equation of the coaxial system. λ is a parameter.

Note 2. Since the line joining the centres of two spheres is perpendicular to their radical plane, centres of all spheres of a coaxial system lie on a straight line.

Note 3. If the centres of all spheres in a coaxial system lie on the x-axis, then the equation of the system can be written as $x^2 + y^2 + z^2 + 2\lambda x + d = 0$. ($\lambda$ is a parameter.) Here the centre is $(-\lambda, 0, 0)$ and radius is $\sqrt{\lambda^2 - d}$. For the point sphere $\lambda^2 - d = 0$ or, $\lambda = \pm\sqrt{d}$. Thus the limiting points are $\left(\sqrt{d}, 0, 0\right)$ and $\left(-\sqrt{d}, 0, 0\right)$. If $d \geq 0$, these points are real, otherwise they are imaginary.

Note 4. In the coaxial system $x^2 + y^2 + z^2 + 2\lambda x + d = 0$, the members intersect one another, touch one another or do not intersect one another according as $d <=> 0$.

Here the radical plane is $x = 0$. It intersects the system in a circle $x = 0, y^2 + z^2 + d = 0$. The radius of this circle is $\sqrt{-d}$.

Now the spheres of the system will intersect or touch or not intersect according as the radius of the above circle is positive, zero or negative,

i.e. according as $\sqrt{-d} >=< 0$,

i.e. according as $d <=> 0$.

WORKED-OUT EXAMPLES

1. *Find the equation of the sphere which touches the plane* $3x+2y-z+2 = 0$ *at the point* $(1, -2, 1)$ *and also cuts orthogonally the sphere* $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

Since the plane $3x + 2y - z + 2 = 0$ touches the sphere at the point $(1, -2, 1)$, the centre lies on the line perpendicular to the plane and passes through the point $(1, -2, 1)$.

The equations of this line are $\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1} = r$ (say).

Let the centre be $(3r + 1, 2r - 2, -r + 1)$.

Radius of the sphere = distance between the centre and the point $(1, -2, 1) = \sqrt{14}r$.

Centre of the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ is $(2, -3, 0)$.

Radius of this sphere $= \sqrt{4 + 9 - 4} = 3$.

Since the spheres cut orthogonally

$$(3r + 1 - 2)^2 + (2r - 2 + 3)^2 + (-r + 1)^2 = 14r^2 + 9$$

$$\text{or,} \quad 14r^2 - 4r + 3 = 14r^2 + 9$$

$$\text{or,} \quad r = -3/2.$$

Hence the centre of the sphere is $\left(-\frac{7}{2}, -5, \frac{5}{2}\right)$ and the radius$=\frac{3}{2}\sqrt{14}$.

Thus the required sphere is

$$\left(x + \frac{7}{2}\right)^2 + (y + 5)^2 + \left(z - \frac{5}{2}\right)^2 = \frac{9}{4} \times 14$$

$$\text{or,} \quad x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0.$$

2. *Prove that the equation of the sphere which cuts orthogonally each of the spheres*

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2, x^2 + y^2 + z^2 + 2ax = a^2,$$
$$x^2 + y^2 + z^2 + 2by = b^2, x^2 + y^2 + z^2 + 2cz = c^2$$
$$\text{is } x^2 + y^2 + z^2 + \frac{b^2 + c^2}{a}x + \frac{c^2 + a^2}{b}y + \frac{a^2 + b^2}{c}z + a^2 + b^2 + c^2 = 0.$$

Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. It cuts the spheres orthogonally.

$$\therefore 2u \cdot 0 + 2v \cdot 0 + 2w \cdot 0 = d - a^2 - b^2 - c^2, \tag{1}$$
$$2u \cdot a + 2v \cdot 0 + 2w \cdot 0 = d - a^2, \tag{2}$$
$$2u \cdot 0 + 2v \cdot b + 2w \cdot 0 = d - b^2, \tag{3}$$
$$2u \cdot 0 + 2v \cdot 0 + 2w \cdot c = d - c^2. \tag{4}$$

From these relations, $d = a^2 + b^2 + c^2$,

$$2u = \frac{b^2 + c^2}{a}, 2v = \frac{c^2 + a^2}{b}, 2w = \frac{a^2 + b^2}{c}.$$

$\therefore$ the equation of the sphere is

$$x^2 + y^2 + z^2 + \frac{b^2 + c^2}{a}x + \frac{c^2 + a^2}{b}y + \frac{a^2 + b^2}{c}z + a^2 + b^2 + c^2 = 0.$$

3. *Show that the equation $x^2+y^2+z^2+2\mu y+2\nu z-d=0$ where μ and ν are parameters, represent a system of spheres passing through the limiting points of the system $x^2+y^2+z^2+2\lambda x+d=0$ and cutting every member of that system at right angles.*

   The limiting points of the coaxial system $x^2+y^2+z^2+2\lambda x+d=0$ are $\left(\pm\sqrt{d},0,0\right)$.

   These points satisfy the equation

$$x^2+y^2+z^2+2\mu y+2\nu z-d=0.$$

   Again $2\lambda\cdot 0+2\cdot 0\cdot\mu+2\cdot 0\cdot\nu=d-d$.

   $\therefore$ the first sphere cuts each member of the coaxial system orthogonally.

EXERCISE VIII

1. Find the angle of intersection of the spheres

$$x^2+y^2+z^2-2x-4y-6z+10=0$$
$$\text{and}\quad x^2+y^2+z^2-6x-2y+2z+2=0.$$

2. Find the radical line of the spheres

$$x^2+y^2+z^2-2x+2y-2z+4=0,\quad x^2+y^2+z^2+3x+4y+2z-2=0,$$
$$x^2+y^2+z^2-4x-2y+6z+8=0.$$

3. Find the radical centre of the four spheres whose equations are

$$x^2+y^2+z^2=10,\qquad x^2+y^2+z^2-10x=0,$$
$$x^2+y^2+z^2+5y+5=0,\quad x^2+y^2+z^2+2x+4z-4=0.$$

4. Find the limiting points of the coaxial system defined by the spheres

$$x^2+y^2+z^2+3x-3y+6=0$$
$$x^2+y^2+z^2-6y-6z+6=0.$$

5. Prove that a sphere which cuts two spheres $S_1=0, S_2=0$ orthogonally will cut $S_1+\lambda S_2=0$ orthogonally.

6. Show that the spheres which cut two given spheres along great circles all pass through two fixed points.

   [*Hints.* Let the two spheres be

$$x^2+y^2+z^2+2\lambda_1 x+d=0 \tag{1}$$
$$\text{and}\quad x^2+y^2+z^2+2\lambda_2 x+d=0. \tag{2}$$

Let the other sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + c = 0. \tag{3}$$

The centre of (1) lies on the radical plane of (1) and (3).

It gives

$$2\lambda_1^2 - 2u\lambda_1 + c - d = 0. \tag{4}$$

Similarly for (2) and (3),

$$2\lambda_2^2 - 2u\lambda_2 + c - d = 0. \tag{5}$$

From (4) and (5),

$$u = \lambda_1 + \lambda_2, c = 2\lambda_1\lambda_2 + d.$$

Now the equation (3) is

$$x^2 + y^2 + z^2 + 2(\lambda_1 + \lambda_2)x + 2vy + 2wz + 2\lambda_1\lambda_2 + d = 0.$$

It meets the x-axis where $x^2 + 2(\lambda_1 + \lambda_2)x + 2\lambda_1\lambda_2 + d = 0$.

This equation gives two values of x in terms of λ_1, λ_2 and d which are constants. Hence the result follows.]

7. Prove that the general equation of all spheres through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ is

$$x^2 + y^2 + z^2 - ax - by - cz - \lambda\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1\right) = 0.$$

Find the value of λ so that this sphere may cut orthogonally the sphere represented by $x^2 + y^2 + z^2 - 2ax - 2by - 2cz = 0$.

8. Show that the circles

$$\begin{aligned} &x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0, \quad 5y + 6z + 1 = 0; \\ &x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0, \quad x + 2y - 7z = 0; \end{aligned}$$

lie on the same sphere, and find its equation.

ANSWERS

1. $\cos^{-1}\left(-\frac{2}{3}\right)$.
2. $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z}{-1}$.
3. $(1, -3, -2)$.
4. $(-1, 2, 1), (-2, 1, -1)$.
7. $-\frac{a^2+b^2+c^2}{2}$.
8. $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.

Chapter 9

Generating Lines

9.10 Ruled surface

Definition. The surfaces of cones and cylinders are generated by the motion of a straight line. The hyperboloid of one sheet and the hyperbolic paraboloid are also generated by the motion of a straight line. Such surfaces are called *ruled surfaces*.

If through every point on a surface a straight line can be drawn in such a way that the straight line lies wholly on the surface, then this surface is called a ruled surface and the straight lines are named as generating lines (or shortly generators or rulings).

If consecutive generators intersect, the surface is a *developable surface*; if they do not intersect, it is a *skew surface*. In the first case, the whole surface may be developed into a plane without tearing. Cones and cylinders are developable conicoids but hyperboloid of one sheet and paraboloid are skew conicoids.

9.11 To find the condition that a given straight line should be a generator of a given conicoid.

Let the equation to the conicoid and the line be

$$ax^2 + by^2 + cz^2 = 1 \tag{1}$$

$$\text{and} \quad \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \tag{2}$$

respectively.

Any point on the line (2) is $(lr+\alpha, mr+\beta, nr+\gamma)$. If this point lies on the conicoid, then

$$(al^2 + bm^2 + cn^2)r^2 + 2(al\alpha + bm\beta + cn\gamma)r + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0. \tag{3}$$

The line lies entirely on the surface, if the equation (3) is an identity.

For this

$$al^2 + bm^2 + cn^2 = 0, \tag{4}$$
$$al\alpha + bm\beta + cn\gamma = 0, \tag{5}$$
$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1. \tag{6}$$

The condition (6) shows that the point (α, β, γ) lies on the conicoid; condition (5) shows that a generating line must lie on the tangent plane at (α, β, γ) to the conicoid and the condition (4) implies that the lines parallel to the generating lines and passing through the centre of the conicoid generate the asymptotic cone $ax^2 + by^2 + cz^2 = 0$.

By (4) and (6),

$$\begin{aligned} &(al^2 + bm^2)(a\alpha^2 + b\beta^2) + cn^2(1 - c\gamma^2) = 0 \\ \text{or,}\quad &(al^2 + bm^2)(a\alpha^2 + b\beta^2) + cn^2 - (a\alpha l + b\beta m)^2 = 0 \text{ [by (5)]} \\ \text{or,}\quad &ab(\beta l - \alpha m)^2 = -cn^2 \\ \text{or,}\quad &(\beta l - \alpha m)^2 = -\frac{c}{ab}n^2. \end{aligned} \tag{7}$$

It shows that c and ab are of opposite signs. It is possible when two of a, b, c are positive and the other is negative. Therefore, of all real central conicoids hyperboloid of one sheet is the only ruled surface.

From (7),

$$\beta l - \alpha m \pm \sqrt{-\frac{c}{ab}}\, n = 0. \tag{8}$$

By (5) and (8),

$$\frac{l}{c\gamma\alpha \pm b\beta\sqrt{-\frac{c}{ab}}} = \frac{m}{c\beta\gamma \mp a\alpha\sqrt{-\frac{c}{ab}}} = \frac{n}{-(a\alpha^2 + b\beta^2)}. \tag{9}$$

The relations of (9) exhibit that two and only two sets of values of l, m, n are admissible. Hence through a point on the hyperboloid of one sheet two and only two generators may be drawn. Thus it is a doubly ruled surface.

Note. By similar considerations it can be shown that of all paraboloids hyperbolic paraboloid is the only ruled surface and is a doubly ruled surface.

9.20 Generating lines of a hyperboloid of one sheet

Let the equation of the hyperboloid of one sheet be $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

We rewrite it as

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$$

$$\text{or,}\quad \left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right).$$

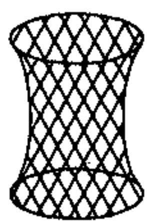

Fig. 41

This can be put in either of the two forms

$$\frac{\frac{x}{a}-\frac{z}{c}}{1-\frac{y}{b}}=\frac{1+\frac{y}{b}}{\frac{x}{a}+\frac{z}{c}}=\lambda\,(\text{say})$$

$$\text{and}\quad \frac{\frac{x}{a}-\frac{z}{c}}{1+\frac{y}{b}}=\frac{1-\frac{y}{b}}{\frac{x}{a}+\frac{z}{c}}=\mu\ (\text{say}),$$

$$\text{i.e.}\quad \frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),\frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right) \tag{1}$$

$$\text{and}\quad \frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right), \tag{2}$$

where λ and μ are variable parameters.

Each of (1) and (2) represents a line of intersection of the pair of planes. These are two systems of generator. Let us call the generators of (1) as the first system and those of (2) as the second system.

Properties.

(i) *Every point on the lines of the first system or the second system lies on the hyperboloid.*

Let (α,β,γ) be a point on the line (1). Then

$$\frac{\alpha}{a}-\frac{\gamma}{c}=\lambda\left(1-\frac{\beta}{b}\right)\quad\text{and}\quad\frac{\alpha}{a}+\frac{\gamma}{c}=\frac{1}{\lambda}\left(1+\frac{\beta}{b}\right).$$

Multiplying we have

$$\left(\frac{\alpha}{a}-\frac{\gamma}{c}\right)\left(\frac{\alpha}{a}+\frac{\gamma}{c}\right)=\left(1-\frac{\beta}{b}\right)\left(1+\frac{\beta}{b}\right)\quad\text{or,}\quad\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\frac{\gamma^2}{c^2}=1.$$

Hence (α,β,γ) is a point on the hyperboloid.

Similarly any point on the line (2) lies on the hyperboloid.

(ii) *Through every point of the hyperboloid there passes one generator of each system.*

If (α,β,γ) be a point on $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, then

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\frac{\gamma^2}{c^2}=1. \tag{3}$$

A generator of the first system will pass through the point (α,β,γ), if λ has a value equal to each of

$$\frac{\frac{\alpha}{a}-\frac{\gamma}{c}}{1-\frac{\beta}{b}}\quad\text{and}\quad\frac{1+\frac{\beta}{b}}{\frac{\alpha}{a}+\frac{\gamma}{c}}.$$

For this value of λ,

$$\frac{\frac{\alpha}{a}-\frac{\gamma}{c}}{1-\frac{\beta}{b}}=\frac{1+\frac{\beta}{b}}{\frac{\alpha}{a}+\frac{\gamma}{c}}\quad\text{or,}\quad\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\frac{\gamma^2}{c^2}=1.$$

It is true by virtue of (3).

Similarly it can be shown that a generator of the second system will pass through the point (α, β, γ) corresponding to the value of

$$\mu = \frac{\frac{\alpha}{a} - \frac{\gamma}{c}}{1 + \frac{\beta}{b}} = \frac{1 - \frac{\beta}{b}}{\frac{\alpha}{a} + \frac{\gamma}{c}}.$$

Hence the result follows.

(iii) *No two generators of the same system intersect.*

Let the two generators of the first system be

$$\frac{x}{a} - \frac{z}{c} = \lambda_1 \left(1 - \frac{y}{b}\right), \frac{x}{a} + \frac{z}{c} = \frac{1}{\lambda_1}\left(1 + \frac{y}{b}\right) \tag{4}$$

and
$$\frac{x}{a} - \frac{z}{c} = \lambda_2 \left(1 - \frac{y}{b}\right), \frac{x}{a} + \frac{z}{c} = \frac{1}{\lambda_2}\left(1 + \frac{y}{b}\right). \tag{5}$$

From (4) and (5),

$$(\lambda_1 - \lambda_2)\left(1 - \frac{y}{b}\right) = 0 \quad \text{or,} \quad y = b \; (\because \; \lambda_1 \neq \lambda_2)$$

$$\text{and} \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\left(1 + \frac{y}{b}\right) = 0 \quad \text{or,} \quad y = -b \; (\because \; \lambda_1 \neq \lambda_2).$$

If (4) and (5) intersect, then on solving the equations of (4) and (5) we must have the same value of y. There is a contradiction since the values of y are different. Hence the two generators of the same system will not intersect.

(iv) *Any two generators belonging to two different systems intersect.*

Let the two generators belonging to two systems be

$$\frac{x}{a} - \frac{z}{c} = \lambda \left(1 - \frac{y}{b}\right), \frac{x}{a} + \frac{z}{c} = \frac{1}{\lambda}\left(1 + \frac{y}{b}\right) \tag{6}$$

and
$$\frac{x}{a} - \frac{z}{c} = \mu \left(1 + \frac{y}{b}\right), \frac{x}{a} + \frac{z}{c} = \frac{1}{\mu}\left(1 - \frac{y}{b}\right). \tag{7}$$

From the first two equations of (6) and (7),

$$\lambda\left(1 - \frac{y}{b}\right) = \mu\left(1 + \frac{y}{b}\right) \quad \text{or,} \quad y = b\frac{\lambda - \mu}{\lambda + \mu}.$$

Putting this value of y in the equations of (6)

$$\frac{x}{a} - \frac{z}{c} = \frac{2\lambda\mu}{\lambda + \mu}, \frac{x}{a} + \frac{z}{c} = \frac{2}{\lambda + \mu}.$$

From these

$$x = a\frac{1 + \lambda\mu}{\lambda + \mu}, z = c\frac{1 - \lambda\mu}{\lambda + \mu}.$$

These values of x, y, z satisfy the equations of (7). Thus the two generators of two systems intersect. The point of intersection is

$$\left(a\frac{1 + \lambda\mu}{\lambda + \mu}, b\frac{\lambda - \mu}{\lambda + \mu}, c\frac{1 - \lambda\mu}{\lambda + \mu}\right).$$

Note 1. The point of intersection satisfies the equation of the hyperboloid.

Note 2. $x = a\frac{1+\lambda\mu}{\lambda+\mu}, y = b\frac{\lambda-\mu}{\lambda+\mu}, z = c\frac{1-\lambda\mu}{\lambda+\mu}$ are the parametric equations of the hyperboloid.

Note 3. $x = a\cos\theta\sec\phi, y = b\sin\theta\sec\phi, z = c\tan\phi$ are also the parametric equations of the hyperboloid. This point is known as (θ, ϕ).

(v) *Two intersecting generators belonging to two systems lie on a plane which is tangent to the hyperboloid at their point of intersection.*

Let us consider the planes

$$\frac{x}{a} - \frac{z}{c} - \lambda\left(1 - \frac{y}{b}\right) - k\left\{\left(\frac{x}{a} + \frac{z}{c}\right) - \frac{1}{\lambda}\left(1 + \frac{y}{b}\right)\right\} = 0$$

$$\text{and} \quad \frac{x}{a} - \frac{z}{c} - \mu\left(1 + \frac{y}{b}\right) - k'\left\{\left(\frac{x}{a} + \frac{z}{c}\right) - \frac{1}{\mu}\left(1 - \frac{y}{b}\right)\right\} = 0.$$

If $k = k' = -\lambda\mu$, both of the equations reduce to the same form

$$\frac{x}{a}\frac{1+\lambda\mu}{\lambda+\mu} + \frac{y}{b}\frac{\lambda-\mu}{\lambda+\mu} - \frac{z}{c}\frac{1-\lambda\mu}{\lambda+\mu} = 1.$$

Therefore, the two generators lie on this plane which is obviously the tangent plane to the conicoid at their common point.

(vi) *Any plane through a generator is the tangent plane at some point of the generator.*

Any plane through a generator of λ system is

$$\frac{x}{a} - \frac{z}{c} - \lambda\left(1 - \frac{y}{b}\right) + k\left\{\frac{x}{a} + \frac{z}{c} - \frac{1}{\lambda}\left(1 + \frac{y}{b}\right)\right\} = 0$$

$$\text{or,} \quad \frac{x}{a}(1+k) + \frac{y}{b}\left(\lambda - \frac{k}{\lambda}\right) + \frac{z}{c}(k-1) - \left(\lambda + \frac{k}{\lambda}\right) = 0. \tag{8}$$

Since

$$a^2\left(\frac{1+k}{a}\right)^2 + b^2\left(\frac{\lambda^2-k}{\lambda b}\right)^2 - c^2\left(\frac{k-1}{c}\right)^2 = \left(\frac{\lambda^2+k}{\lambda}\right)^2,$$

the plane (1) is the tangent plane to the hyperboloid. If (α, β, γ) be the point of contact, then the plane (1) is identical with $\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - \frac{\gamma z}{c^2} = 1$.

$$\therefore \frac{\alpha}{a(1+k)} = \frac{\beta}{b\frac{\lambda^2-k}{\lambda}} = \frac{\gamma}{c(k-1)} = \frac{\lambda}{\lambda^2+k}.$$

It gives that

$$\alpha = \frac{a\lambda(1+k)}{\lambda^2+k}, \beta = \frac{b(\lambda^2-k)}{\lambda^2+k}, \gamma = \frac{c\lambda(k-1)}{\lambda^2+k}.$$

Obviously this point lies on the generator

$$\frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right), \frac{x}{a} + \frac{z}{c} = \frac{1}{\lambda}\left(1 + \frac{y}{b}\right).$$

Hence the plane (1) is the tangent plane to the hyperboloid at some point of the generator. Similarly any plane through a generator of μ system is the tangent plane to the hyperboloid at some point of the generator.

(vii) *Tangent planes at different points of a generator are different.*

Let P and Q be two points on the generator

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right). \tag{9}$$

The generators of μ system through the points P and Q are

$$\frac{x}{a}-\frac{z}{c}=\mu_1\left(1+\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu_1}\left(1-\frac{y}{b}\right) \tag{10}$$

$$\text{and} \quad \frac{x}{a}-\frac{z}{c}=\mu_2\left(1+\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu_2}\left(1-\frac{y}{b}\right) \tag{11}$$

respectively. Here $\mu_1 \neq \mu_2$.

The plane through (9) and (10) is the tangent plane at P and the plane through (9) and (11) is the tangent plane at Q to the hyperboloid. If these two planes are same, then the generators (10) and (11) lie on that plane. In this case, these two lines must be parallel, otherwise they will intersect. The d.rs. of the lines (10) and (11) are

$$\frac{1-\mu_1^2}{bc\mu_1}, \frac{-2}{ac}, \frac{1+\mu_1^2}{ab\mu_1} \quad \text{and} \quad \frac{1-\mu_2^2}{bc\mu_2}, \frac{-2}{ac}, \frac{1+\mu_2^2}{ab\mu_2} \text{ respectively.}$$

These are proportional.

$$\therefore \frac{1-\mu_1^2}{\mu_1}=\frac{1-\mu_2^2}{\mu_2} \quad \text{and} \quad \frac{1+\mu_1^2}{\mu_1}=\frac{1+\mu_2^2}{\mu_2}.$$

It implies that $\mu_1=\mu_2$. It is contradictory to our assumption. Hence the tangent planes at P and Q are different.

Note 1. On a skew surface the tangent planes at different points of a generator are different but the tangent planes at different points of a generator on a developable surface are same.

9.21 Generating lines of a hyperbolic paraboloid

Let the equation of the hyperbolic paraboloid be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$.

It can be written as

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=2z.$$

This can be put in either of the two forms

$$\frac{\frac{x}{a}-\frac{y}{b}}{z}=\frac{2}{\frac{x}{a}+\frac{y}{b}}=\lambda \text{ (say)} \quad \text{and} \quad \frac{\frac{x}{a}+\frac{y}{b}}{z}=\frac{2}{\frac{x}{a}-\frac{y}{b}}=\mu \text{ (say)},$$

$$\text{i.e.} \quad \frac{x}{a}-\frac{y}{b}=\lambda z, \frac{x}{a}+\frac{y}{b}=\frac{2}{\lambda} \tag{1}$$

$$\text{and} \quad \frac{x}{a}+\frac{y}{b}=\mu z, \frac{x}{a}-\frac{y}{b}=\frac{2}{\mu}. \tag{2}$$

Eliminating λ from the equations of (1) or eliminating μ from the equations of (2) we get the equation of the hyperbolic paraboloid. Hence any point on either the line (1) or (2) lies on the hyperbolic paraboloid. λ and μ are two variable parameters. For different values of λ and μ we get two families of straight lines such that each line of either family lies entirely on the conicoid. The lines of these two families are known as two systems of generators of the hyperbolic paraboloid.

Properties.

(i) *Through every point of the hyperbolic paraboloid there passes one generator of each system.*

(ii) *No two generators of the same system intersect.*

[Proofs of (i) and (ii) are similar to those of Sec 9.20.]

(iii) *Any two generators belonging to two different systems intersect.*

Let the two generators of the two systems be given by (1) and (2).

From the last two equations of (1) and (2) $x = a\frac{\lambda+\mu}{\lambda\mu}, y = b\frac{\mu-\lambda}{\lambda\mu}$. Putting these values of x and y in the first equation of (1) and (2) $z = \frac{2}{\lambda\mu}$. Hence the point of intersection is

$$\left(a\frac{\lambda+\mu}{\lambda\mu}, b\frac{\mu-\lambda}{\lambda\mu}, \frac{2}{\lambda\mu}\right).$$

For all values of λ and μ the above point satisfies the equation of the hyperbolic paraboloid. Thus *parametric equations* of it are

$$x = a\frac{\lambda+\mu}{\lambda\mu}, y = b\frac{\mu-\lambda}{\lambda\mu}, z = \frac{2}{\lambda\mu}.$$

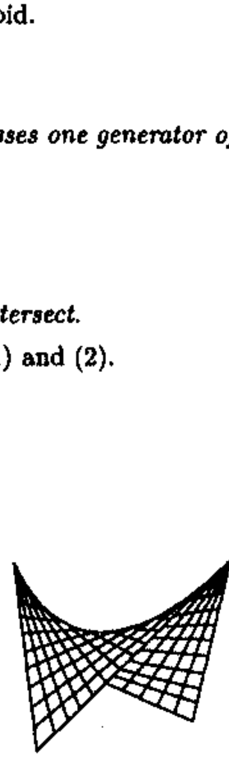

Fig. 42

(iv) *Two intersecting generators belonging to two systems lie on the tangent plane to the hyperbolic paraboloid at their point of intersection.*

Let us consider the planes

$$\frac{x}{a} - \frac{y}{b} - \lambda z + k\left\{\left(\frac{x}{a} + \frac{y}{b}\right) - \frac{2}{\lambda}\right\} = 0$$

$$\text{and}\quad \frac{x}{a} - \frac{y}{b} - \frac{2}{\mu} + k'\left\{\left(\frac{x}{a} + \frac{y}{b}\right) - \mu z\right\} = 0.$$

If $k = k' = \frac{\lambda}{\mu}$, both of the planes reduce to

$$\frac{x}{a}\frac{\lambda+\mu}{\lambda\mu} - \frac{y}{b}\frac{\mu-\lambda}{\lambda\mu} = z + \frac{2}{\lambda\mu}. \tag{3}$$

Therefore, the generators (1) and (2) lies on the same plane whose equation is (3). Obviously it is the tangent plane to the hyperbolic paraboloid at the point of intersection of the generators.

(v) *Any plane through a generator is the tangent plane at some point of the generator.*

(vi) *Tangent planes at different points of a generator are different.*

[Proofs of (v) and (vi) are similar to those of Sec 9.20.]

Note. Parametric equations.

1. $x = ar\cos\theta, y = br\sin\theta, z = \frac{r^2}{2}\cos 2\theta$. The point is called (r, θ).
2. $x = a\cosh\phi\cos\theta, y = b\sinh\phi\cos\theta, z = \frac{1}{2}\cos^2\theta$. The point is called (θ, ϕ).
3. $x = ae^{\phi}\cosh\theta, y = be^{\phi}\sinh\theta, z = \frac{1}{2}e^{2\phi}$. The point is called (θ, ϕ).

9.30 Locus of the point of intersection of perpendicular generators

(a) Hyperboloid of one sheet.

Let

$$\frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right), \frac{x}{a} + \frac{z}{c} = \frac{1}{\lambda}\left(1 + \frac{y}{b}\right) \tag{1}$$

$$\text{and} \quad \frac{x}{a} - \frac{z}{c} = \mu\left(1 + \frac{y}{b}\right), \frac{x}{a} + \frac{z}{c} = \frac{1}{\mu}\left(1 - \frac{y}{b}\right) \tag{2}$$

be the two generators of two systems of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{3}$$

If l_1, m_1, n_1 are the d.cs. of the line (1), then

$$\frac{l_1}{a} + \frac{\lambda m_1}{b} - \frac{n_1}{c} = 0 \quad \text{and} \quad \frac{l_1}{a} - \frac{m_1}{\lambda b} + \frac{n_1}{c} = 0.$$

From these

$$\frac{l_1/a}{\lambda - 1/\lambda} = \frac{m_1/b}{-1-1} = \frac{n_1/c}{-1/\lambda - \lambda}$$

$$\text{or,} \quad \frac{l_1/a}{\lambda^2 - 1} = \frac{m_1/b}{-2\lambda} = \frac{n_1/c}{-(\lambda^2+1)}. \tag{4}$$

Similarly if l_2, m_2, n_2 are the d.cs. of the line (2), then

$$\frac{l_2/a}{\mu^2 - 1} = \frac{m_2/b}{2\mu} = \frac{n_2/c}{-(\mu^2+1)}. \tag{5}$$

For perpendicularity of the generators

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\text{or,} \quad a^2(\lambda^2 - 1)(\mu^2 - 1) - 4b^2\lambda\mu + c^2(\lambda^2+1)(\mu^2+1) = 0.$$

It can be written as

$$a^2(1+\lambda\mu)^2 + b^2(\lambda-\mu)^2 + c^2(1-\lambda\mu)^2 = (\lambda+\mu)^2(a^2+b^2-c^2)$$

$$\text{or,}\quad a^2\left(\frac{1+\lambda\mu}{\lambda+\mu}\right)^2 + b^2\left(\frac{\lambda-\mu}{\lambda+\mu}\right)^2 + c^2\left(\frac{1-\lambda\mu}{\lambda+\mu}\right)^2 = a^2+b^2-c^2.$$

This relation shows that the point of intersection of (1) and (2), namely $\left(a\frac{1+\lambda\mu}{\lambda+\mu}, b\frac{\lambda-\mu}{\lambda+\mu}, c\frac{1-\lambda\mu}{\lambda+\mu}\right)$, lies on the director sphere $x^2+y^2+z^2 = a^2+b^2-c^2$.

Hence the required locus is the curve of intersection of the hyperboloid and the director sphere.

(b) Hyperbolic paraboloid.

$$\text{Let}\quad \frac{x}{a}-\frac{y}{b} = \lambda z, \frac{x}{a}+\frac{y}{b} = \frac{2}{\lambda} \tag{1}$$

$$\text{and}\quad \frac{x}{a}-\frac{y}{b} = \frac{2}{\mu}, \frac{x}{a}+\frac{y}{b} = \mu z \tag{2}$$

be the two generators of two systems of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 2z. \tag{3}$$

If l_1, m_1, n_1, be the d.cs. of the line (1), then

$$\frac{l_1}{a}-\frac{m_1}{b}-\lambda n_1 = 0, \frac{l_1}{a}+\frac{m_1}{b} = 0 \quad \text{or,}\quad \frac{l_1}{a\lambda} = \frac{m_1}{-b\lambda} = \frac{n_1}{2}.$$

Similarly if l_2, m_2, n_2 are the d.cs. of the line (2), then

$$\frac{l_2}{a\mu} = \frac{m_2}{b\mu} = \frac{n_2}{2}.$$

For perpendicularity of the generators $l_1l_2+m_1m_2+n_1n_2 = 0$

$$\text{or,}\quad a^2\lambda\mu - b^2\lambda\mu + 4 = 0 \quad \text{or,}\quad a^2-b^2+\frac{4}{\lambda\mu} = 0. \tag{4}$$

The point of intersection between (1) and (2) is

$$\left(a\frac{\lambda+\mu}{\lambda\mu}, b\frac{\mu-\lambda}{\lambda\mu}, \frac{2}{\lambda\mu}\right).$$

The relation (4) shows that this point of intersection lies on $a^2-b^2+2z = 0$.

Hence the required locus is the curve of intersection of the hyperbolic paraboloid and the plane $a^2-b^2+2z=0$.

WORKED-OUT EXAMPLES

1. *Find the equations to the generating lines of the hyperboloid* $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$, *which pass through the point* $(2,3,-4)$. [BH '93, '96]

Let l, m, n be the d.cs. of a generator through the point $(2,3,-4)$. The equation of it is

$$\frac{x-2}{l}=\frac{y-3}{m}=\frac{z+4}{n}=r \text{ (say)}.$$

Any point on it is $(lr+2, mr+3, nr-4)$. If it lies on the hyperboloid, then

$$\frac{(lr+2)^2}{4}+\frac{(mr+3)^2}{9}-\frac{(nr-4)^2}{16}=1$$

$$\text{or,}\quad r^2\left(\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}\right)+2r\left(\frac{2l}{4}+\frac{3m}{9}+\frac{4n}{16}\right)=0.$$

Since the generator lies wholly on the hyperboloid,

$$\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}=0 \quad\text{and}\quad \frac{l}{2}+\frac{m}{3}+\frac{n}{4}=0.$$

Eliminating n, $\frac{l^2}{4}+\frac{m^2}{9}-\left(\frac{l}{2}+\frac{m}{3}\right)^2=0$ or, $lm=0$.

When $l=0$, then $\frac{m}{3}+\frac{n}{4}=0$ or, $\frac{m}{3}=\frac{n}{-4}$.

When $m=0$, then $\frac{l}{2}+\frac{n}{4}=0$ or, $l=\frac{n}{-2}$.

Hence the two generators are

$$\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4} \quad\text{and}\quad \frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2}.$$

2. *Find the equations to the generating lines of the hyperboloid* $yz+2zx+3xy+6=0$ *which pass through the point* $(-1,0,3)$. [BH 98, 2000]

Let the equation of a generator be

$$\frac{x+1}{l}=\frac{y}{m}=\frac{z-3}{n}=r \text{ (say)}.$$

Any point on it is $(lr-1, mr, nr+3)$. If it lies on the hyperboloid, then

$$mr(nr+3)+2(nr+3)(lr-1)+3(lr-1)mr+6=0$$

$$\text{or,}\quad r^2(mn+2nl+3lm)+2r(3l-n)=0.$$

Since the generator lies wholly on the hyperboloid

$$mn+2nl+3lm=0 \quad\text{and}\quad 3l-n=0 \quad\text{or,}\quad n=3l.$$

Eliminating n we have $l^2+lm=0$ or, $l(l+m)=0$.

$\therefore\ l=0$ or, $l=-m$.

When $l=0$, then $n=0$ and when $l=-m$, then $n=-3m$.

Therefore, the generators are $x+1=0=z-3$ and $\frac{x+1}{1}=\frac{y}{-1}=\frac{z-3}{3}$.

3. *Find the equations to the generating lines of the paraboloid $(x+y+z)(2x+y-z)=6z$, which pass through the point $(1,1,1)$. Hence find the angle between these generators.* [CH 2004; NH 2008]

Let the equation of a generator be

$$\frac{x-1}{l}=\frac{y-1}{m}=\frac{z-1}{n}=r \text{ (say)}.$$

Any point on the line is

$$(lr+1, mr+1, nr+1).$$

If it lies on the paraboloid, then

$$(lr+1+mr+1+nr+1)(2lr+2+mr+1-nr-1)=6(nr+1)$$
$$\text{or,}\quad (l+m+n)(2l+m-n)r^2+(8l+5m-7n)r=0.$$

Since the generator lies wholly on the paraboloid

$$(l+m+n)(2l+m-n)=0 \quad\text{and}\quad 8l+5m-7n=0.$$

For one generator $l+m+n=0, 8l+5m-7n=0$.
From these $\frac{l}{4}=\frac{m}{-5}=\frac{n}{1}$.
For the other generator $2l+m-n=0, 8l+5m-7n=0$.
From these $\frac{l}{1}=\frac{m}{-3}=\frac{n}{-1}$.
Therefore, the generators are

$$\frac{x-1}{4}=\frac{y-1}{-5}=\frac{z-1}{1} \quad\text{and}\quad \frac{x-1}{1}=\frac{y-1}{-3}=\frac{z-1}{-1}.$$

The angle between these lines

$$=\cos^{-1}\frac{4\cdot 1+5\cdot 3-1\cdot 1}{\sqrt{(4^2+5^2+1^2)}\cdot\sqrt{(1^2+3^2+1^2)}}=\cos^{-1}\frac{18}{\sqrt{462}}.$$

4. *Find the equations of the hyperboloid through the three lines $y-z=1, x=0; z-x=1, y=0; x-y-1=0, z=0$. Also obtain the equations of the two systems of generators.*

Any line intersecting the first two lines is given by

$$y-z-1+\lambda_1 x=0=z-x-1+\lambda_2 y. \tag{1}$$

It intersects the third line, if

$$x-2+\lambda_1 x=0 \quad\text{and}\quad -x-1+\lambda_2(x-1)=0,$$
$$\text{i.e.}\quad x=\frac{2}{1+\lambda_1}=\frac{\lambda_2+1}{\lambda_2-1}. \tag{2}$$

Eliminating λ_1 and λ_2 by (1) and (2),

$$\frac{2}{1+\frac{z+1-y}{x}} = \frac{1+\frac{1+x-z}{y}}{\frac{1+x-z}{y}-1}$$

$$\text{or,} \quad \frac{2x}{x-y+z+1} = \frac{x+y-z+1}{x-y-z+1}$$

$$\text{or,} \quad 2x(x-y-z+1) = (x+y-z+1)(x-y+z+1)$$

$$\text{or,} \quad x^2+y^2+z^2-2xy-2yz-2zx = 1. \tag{3}$$

It is the required hyperboloid.

The equation (3) can be written as

$$(x-y)^2 - 1 - z(2x+2y-z) = 0$$

$$\text{or,} \quad (x-y+1)(x-y-1) - z(2x+2y-z) = 0. \tag{4}$$

The equation (4) can be arranged in the following forms

$$\frac{x-y+1}{z} = \frac{2x+2y-z}{x-y-1} = \lambda \text{ (say)}$$

$$\text{or,} \quad \frac{x-y+1}{2x+2y-z} = \frac{z}{x-y-1} = \mu \text{ (say)}.$$

Thus the two systems of generators are

$$x-y+1 = \lambda z, 2x+2y-z = \lambda(x-y-1)$$

$$\text{and} \quad x-y+1 = \mu(2x+2y-z), z = \mu(x-y-1).$$

5. *Obtain the equations to the generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ through a point of the principal elliptic section. Hence show that the projections of the generators of a hyperboloid on a coordinate plane are tangents to the section of the hyperboloid by that plane.*

[NH 2005; BH 94, '95, 2007; CH 91, 93, 99]

Any point on the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ can be taken as $(a\cos\theta, b\sin\theta, 0)$.

Let

$$\frac{x-a\cos\theta}{l} = \frac{y-b\sin\theta}{m} = \frac{z}{n} \tag{1}$$

be a generator through this point. Any point on (1) is $(lr+a\cos\theta, mr+b\sin\theta, nr)$. It is a point on the hyperboloid.

$$\therefore \frac{(lr+a\cos\theta)^2}{a^2} + \frac{(mr+b\sin\theta)^2}{b^2} - \frac{n^2r^2}{c^2} = 1$$

$$\text{or,} \quad \left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)r^2 + 2\left(\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}\right)r = 0.$$

It is true for all values of r. For this

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0 \tag{2}$$

and $$\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}=0. \tag{3}$$

By (2) and (3),

$$\frac{l}{a\sin\theta}=\frac{m}{-b\cos\theta}=\frac{\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}}}{\sqrt{\sin^2\theta+\cos^2\theta}}=\sqrt{\frac{n^2}{c^2}}=\frac{n}{\pm c}.$$

Hence the two generators through the point $(a\cos\theta, b\sin\theta, 0)$ are

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{\pm c}.$$

The section of the hyperboloid made by the coordinate plane $z=0$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0. \tag{4}$$

The projection of one generator of (4) on the plane $z=0$ is

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}, z=0$$

or, $$\frac{x}{a\sin\theta}+\frac{y}{b\cos\theta}=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}, z=0$$

or, $$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1, z=0. \tag{5}$$

Clearly it is the tangent to the section (5) at $(a\cos\theta, b\sin\theta, 0)$. Similarly the projections of the generator on the planes $x=0$ and $y=0$ are tangents to the sections of the hyperboloid by these planes respectively.

6. *Show that the plane* $6x+4y+3z-12=0$ *intersects the hyperboloid* $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ *in two generators.*

The equations of the generators can be written as

$$\frac{x-2\cos\theta}{2\sin\theta}=\frac{y-3\sin\theta}{-3\cos\theta}=\frac{z}{\pm 4}. \tag{1}$$

If these generators lie on the given plane, then

$$12\cos\theta+12\sin\theta-12=0, \quad \text{i.e.} \quad \cos\theta+\sin\theta=1 \tag{2}$$

and $$6.2\sin\theta-4.3\cos\theta\pm 3.4=0, \quad \text{i.e.} \quad \sin\theta-\cos\theta=\mp 1. \tag{3}$$

By (2) and (3), $\sin\theta=0$ or 1, i.e. $\theta=0$ or, $\frac{\pi}{2}$.

Thus the generators are

$$x-2=0, \frac{y}{-3}=\frac{z}{4} \quad \text{and} \quad \frac{x}{2}=\frac{z}{-4}, y-3=0.$$

7. *Obtain the equations to the generators of the paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$ through a point of the principal parabolic section. Hence show that the projections of the generators of the paraboloid on the coordinate plane $x = 0$ or, $y = 0$ are tangents to the section of the paraboloid by that plane.*

Any point on the principal parabolic section $y^2 = -2b^2z, x = 0$ can be taken as $(0, 2bt, -2t^2)$.

Let

$$\frac{x}{l} = \frac{y - 2bt}{m} = \frac{z + 2t^2}{n} \tag{1}$$

be a generator through the point. Any point on (1) is $(lr, mr + 2bt, nr - 2t^2)$. It lies on the paraboloid.

$$\therefore \quad \frac{l^2r^2}{a^2} - \frac{(mr + 2bt)^2}{b^2} = 2(nr - 2t^2)$$

$$\text{or,} \quad \left(\frac{l^2}{a^2} - \frac{m^2}{b^2}\right) r^2 - 2\left(\frac{2mt}{b} + n\right) r = 0.$$

It is true for all values of r.

$$\therefore \quad \frac{l^2}{a^2} - \frac{m^2}{b^2} = 0 \tag{2}$$

$$\text{and} \quad \frac{2mt}{b} + n = 0. \tag{3}$$

By (2) and (3), $\frac{l}{a} = \frac{m}{\pm b} = \frac{n}{\mp 2t}$.

Thus the generators are $\frac{x}{a} = \frac{y-2bt}{\pm b} = \frac{z+2t^2}{\mp 2t}$.

The projection of one generator on the plane $x = 0$ is

$$\frac{y - 2bt}{b} = \frac{z + 2t^2}{-2t} \quad \text{or,} \quad y = -\frac{b}{2t}z + bt.$$

It is tangent to $y^2 = -2b^2z, x = 0$ at $(0, 2bt, -2t^2)$.

In the same way this can be proved for the section $y = 0, x^2 = 2a^2z$.

8. *Prove that points of intersection of generators of $xy = az$ which are inclined at a constant angle α lie on the curve of intersection of the paraboloid and the hyperboloid $x^2 + y^2 - z^2 \tan^2 \alpha + a^2 = 0$.*

The two generators of λ, μ system can be taken as

$$x = a\lambda, y = \frac{z}{\lambda} \tag{1}$$

$$\text{and} \quad y = a\mu, x = \frac{z}{\mu}. \tag{2}$$

The d.rs. of (1) and (2) are $(0, 1, \lambda)$ and $(1, 0, \mu)$ respectively. Since these are inclined at an angle α, we have

$$\cos\alpha = \frac{\lambda\mu}{\sqrt{1+\lambda^2}\sqrt{1+\mu^2}} \quad \text{or,} \quad (1 + \lambda^2)(1 + \mu^2)\cos^2\alpha = \lambda^2\mu^2. \tag{3}$$

The point of intersection between (1) and (2) is $(a\lambda, a\mu, a\lambda\mu)$.

Let

$$x_1 = a\lambda, y_1 = a\mu, z_1 = a\lambda\mu. \tag{4}$$

Eliminating λ, μ from (3) by (4),

$$\left(1 + \frac{x_1^2}{a^2} + \frac{y_1^2}{a^2} + \frac{z_1^2}{a^2}\right)\cos^2\alpha = \frac{z_1^2}{a^2}$$

$$\text{or,} \quad a^2 + x_1^2 + y_1^2 + z_1^2 = z_1^2\sec^2\alpha \quad \text{or,} \quad x_1^2 + y_1^2 - z_1^2\tan^2\alpha + a^2 = 0.$$

Hence the locus of (x_1, y_1, z_1) is the curve of intersection of $x^2+y^2-z^2\tan^2\alpha+a^2=0$ and the paraboloid.

9. *If R is the point (θ, ϕ), show that the equations of PQ are $z = 0, \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = \cos\phi$ and deduce that $\theta - \phi = \alpha, \theta + \phi = \beta$ where α, β are eccentric angles of P and Q respectively. Deduce also that the equations of the generating lines through (θ, ϕ) are*

$$\frac{x - a\cos\theta\sec\phi}{a\sin(\theta \pm \phi)} = \frac{y - b\sin\theta\sec\phi}{-b\cos(\theta \pm \phi)} = \frac{z - c\tan\theta}{\pm c},$$

where, of course, P, Q are points on principle elliptic section, such that the λ-generator at P and μ-generator at Q meet at R.

Let the equations of λ-generator through the point $P(a\cos\alpha, b\sin\alpha, 0)$ and those of μ-generator through the point $Q(a\cos\beta, b\sin\beta, 0)$ be

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha} = \frac{z}{-c} \tag{1}$$

$$\text{and} \quad \frac{x - a\cos\beta}{a\sin\beta} = \frac{y - b\sin\beta}{-b\cos\beta} = \frac{z}{c} \tag{2}$$

respectively.

Since (1) and (2) pass through the point (θ, ϕ), there exist k and k' such that

$$ka\sin\alpha + a\cos\alpha = k'a\sin\beta + a\cos\beta = a\cos\theta\sec\phi, \tag{3}$$

$$-kb\cos\alpha + b\sin\alpha = -k'b\cos\beta + b\sin\beta = b\sin\theta\sec\phi \tag{4}$$

$$\text{and} \quad -kc = k'c = c\tan\phi. \tag{5}$$

From (5)

$$k = -\tan\phi, k' = \tan\phi.$$

From (3)

$$-\tan\phi\sin\alpha + \cos\alpha = \cos\theta\sec\phi$$

$$\text{or,} \quad \cos\theta = \cos(\phi + \alpha), \quad \text{i.e.} \quad \theta = \phi + \alpha \quad \text{or,} \quad \theta - \phi = \alpha \tag{6}$$

$$\text{and} \quad \tan\phi\sin\beta + \cos\beta = \cos\theta\sec\phi$$

$$\text{or,} \quad \cos\theta = \cos(\beta - \phi), \quad \text{i.e.} \quad \theta = \beta - \phi \quad \text{or,} \quad \theta + \phi = \beta. \tag{7}$$

Now equations of PQ are

$$z = 0, \qquad \frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$$

$$\text{or,} \quad z = 0, \quad \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = \cos\phi \quad [\text{by (6) and (7)}].$$

The direction ratios of λ-generator through R are $a\sin(\theta-\phi), -b\cos(\theta-\phi), -c$ and those of μ-generator through R are $a\sin(\theta+\phi), -b\cos(\theta+\phi), c$.

Hence the generating lines through R are

$$\frac{x - a\cos\theta\sec\phi}{a\sin(\theta \pm \phi)} = \frac{y - b\sin\theta\sec\phi}{-b\cos(\theta \pm \phi)} = \frac{z - c\tan\phi}{\pm c}.$$

EXERCISE IX

1. Find the equations to the generating lines of the hyperboloid $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$, which pass through the point $(2, -1, 4/3)$.

2. Find the equations of the generators of the hyperbolic paraboloid $16x^2 - 9y^2 = 4z$, which pass through the point $(-2, 2, 7)$.

3. Show that the line $\frac{x+2}{2} = \frac{y}{3} = \frac{z-1}{-2}$ is a generator of the quadratic $\frac{x^2}{4} - \frac{y^2}{9} = z$.
[NH 2002, 08]

4. Find the generators of the paraboloid $\frac{x^2}{a} - \frac{y^2}{b} = 4z$ through $(\alpha, 0, \gamma)$ and prove that the angle between them is $\cos^{-1}\frac{a-b+\gamma}{a+b+\gamma}$.

5. If $\frac{x-2}{3} = \frac{y+1}{6} = \frac{z-4/3}{10}$ is a generator of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, then show that $a = 2, b = 3, c = 4$.

6. Show that the line $x - 1 = y - 2 = z + 1$ lies entirely on the surface $z^2 - xy + 2x + y + 2z - 1 = 0$.

7. Show that the generators of the hyperboloid $\frac{x^2}{25} + \frac{y^2}{16} - \frac{z^2}{4} = 1$ which are parallel to the plane $4x - 5y - 10z + 7 = 0$ are $x + 5 = 0, y + 2z = 0$ and $y + 4 = 0, 2x = 5z$.

8. Show that the plane $12y + z - 2x - 16 = 0$ intersects the paraboloid $x^2 - 4y^2 = 2z$ in two generators

$$\frac{x}{2} = \frac{y-2}{1} = \frac{z+8}{-8} \quad \text{and} \quad \frac{x}{2} = \frac{y-4}{-1} = \frac{z+32}{16}.$$

9. Show that the perpendiculars from the origin on the generators of the paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$ lie on the cones $\left(\frac{x}{a} \pm \frac{y}{b}\right)(ax \pm by) + 2z^2 = 0$.
[CH 2002; NH 2003]

[*Hints.* The generator of λ-system is

$$\frac{x}{a} - \frac{y}{b} = \frac{\lambda z}{c}, \frac{x}{a} + \frac{y}{b} = \frac{2}{\lambda} \quad \text{or,} \quad \frac{x - \frac{a}{\lambda}}{a\lambda} = \frac{y - \frac{b}{\lambda}}{-b\lambda} = \frac{z}{2c} = r_1 \text{ (say)}. \qquad (1)$$

A line through the origin is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r_2 \text{ (say)}. \tag{2}$$

If (2) is perpendicular to (1), then

$$a\lambda l - b\lambda m + 2cn = 0. \tag{3}$$

(1) and (2) intersect. At the common point

$$a\lambda r_1 + \frac{a}{\lambda} = lr_2, -b\lambda r_1 + \frac{b}{\lambda} = mr_2, 2cr_1 = nr_2,$$

Eliminating r_1 and $r_2, \lambda = \frac{(bl-am)c}{nab}$.

From (3), $acl(bl - am) - bcm(bl - am) + 2abcn^2 = 0$

$$\text{or,} \quad l^2 + m^2 + 2n^2 - \frac{a^2+b^2}{ab}lm = 0.$$

$\therefore$ the required locus is $x^2 + y^2 + 2z^2 - \frac{a^2+b^2}{ab}xy = 0$

$$\text{or,} \quad \left(\frac{x}{a} - \frac{y}{b}\right)(ax - by) + 2z^2 = 0.]$$

10. Show that the perpendiculars from the origin to the generators of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ lie upon the cone

$$\frac{a^2(b^2+c^2)^2}{x^2} + \frac{b^2(c^2+a^2)^2}{y^2} = \frac{c^2(a^2-b^2)^2}{z^2}.$$ [NH 08; CH 89, 04]

[*Hints.* Consider the generator

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta} = \frac{z}{c}. \tag{1}$$

Let

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \tag{2}$$

be perpendicular to (1).

Then

$$al\sin\theta - bm\cos\theta + cn = 0. \tag{3}$$

(1) and (2) are coplanar.

$$\therefore \; lbc\sin\theta - mca\cos\theta - nab = 0. \tag{4}$$

Eliminating θ from (3) and (4), we get

$$\frac{a^2(b^2+c^2)^2}{l^2} + \frac{b^2(c^2+a^2)^2}{m^2} = \frac{c^2(a^2-b^2)^2}{n^2}.$$

$\therefore$ the locus of (2) is

$$\frac{a^2(b^2+c^2)^2}{x^2} + \frac{b^2(c^2+a^2)^2}{y^2} = \frac{c^2(a^2-b^2)^2}{z^2}.]$$

11. Determine the two sets of generators of the paraboloid $(2x+3y+7z)^2-(x+y+z)^2=4x-y+3z$ and prove that they are parallel respectively to the two planes

$$(2x+3y+7z)+(x+y+z)=0 \quad \text{and} \quad (2x+3y+7z)-(x+y+z)=0.$$

12. Prove that the generators of the surface $yz+zx+xy+a^2=0$ through $\left(0, am, -\frac{a}{m}\right)$ are $x(1\pm m)=am-y=\mp(mz+a)$.

[*Hints.* The generators are

$$(y-\lambda z)+a(1+\lambda)=0, (\lambda+1)x+(y+\lambda a)=0$$
$$\text{and} \quad \mu(x+y)-(y+a)=0, \mu(x+a)+z-a=0.]$$

13. Prove that the equations $2x=ae^{2\phi}, y=be^{\phi}\cosh\theta, z=ce^{\phi}\sinh\theta$ determine a hyperbolic paraboloid and that $\theta+\phi$ is constant for points of a given generator of one system and $\theta-\phi$ is constant for a given generator of the other. [CH 95]

[*Hints.* Since $\frac{y^2}{b^2}-\frac{z^2}{c^2}=e^{2\phi}=\frac{2x}{a}$, the given equations represent a hyperbolic paraboloid.

Let $\frac{y}{b}+\frac{z}{c}=\frac{x}{\lambda a}, \frac{y}{b}-\frac{z}{c}=2\lambda$ and $\frac{y}{b}-\frac{z}{c}=\frac{x}{\mu a}, \frac{y}{b}+\frac{z}{c}=2\mu$ be two generators. Now the given point lies on these generators.

For the first generator,

$$e^{\phi}(\cosh\theta+\sinh\theta)=\frac{e^{2\phi}}{2\lambda},$$
$$e^{\phi}(\cosh\theta-\sinh\theta)=2\lambda \quad \text{or,} \quad e^{\phi}\cdot e^{\theta}=\frac{e^{2\phi}}{2\lambda} \quad \text{or,} \quad e^{\phi}\cdot e^{-\theta}=2\lambda,$$
$$\text{i.e.} \quad e^{\phi-\theta}=2\lambda. \quad \therefore\ \phi-\theta=\text{constant}.$$

For the second generator $\phi+\theta=$ constant.]

14. Prove that the s.d. between the generators of the same system drawn at one end of each of the major and minor axes of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is $\frac{2abc}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$.

15. Prove that the angle between the generators through any point P on the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is given by $\tan\theta=\frac{2abc}{p(a^2+b^2-c^2-OP^2)}$ where p is the distance from the centre on the tangent plane at P. Hence find the locus of the point of intersection of perpendicular generators. [NH 2000]

[*Hints.* The generators through the point P $(a\cos\theta\sec\phi, b\sin\theta\sec\phi, c\tan\phi)$ are

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta\pm\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta\pm\phi)}=\frac{z-c\tan\phi}{\pm c}.$$

Tangent plane at P is

$$\frac{x}{a}\cos\theta\sec\phi + \frac{y}{b}\sin\theta\sec\phi - \frac{z}{c}\tan\phi = 1.$$

$p =$ distance of this plane from the origin.

$$\frac{1}{p} = \left(\frac{\cos^2\theta\sec^2\phi}{a^2} + \frac{\sin^2\theta\sec^2\phi}{b^2} + \frac{\tan^2\phi}{c^2}\right)^{1/2}.$$

$$\therefore \quad \frac{abc\cos\phi}{p} = (b^2c^2\cos^2\theta + c^2a^2\sin^2\theta + a^2b^2\sin^2\phi)^{1/2}.$$

If α is the angle between the generators, then

$$\cos\alpha = \frac{a^2\sin(\theta+\phi)\sin(\theta-\phi) + b^2\cos(\theta+\phi)\cos(\theta-\phi) - c^2}{k}$$

$$= \frac{\cos^2\phi}{k}(a^2+b^2-c^2-OP^2)$$

and $\quad \sin\alpha = \dfrac{2abc\cos\phi}{k}\cdot\dfrac{1}{p} \quad$ [by Lagrange's identity]

where $\quad k = \sqrt{[\{a^2\sin^2(\theta+\phi) + b^2\cos^2(\theta+\phi) + c^2\} \times \{a^2\sin^2(\theta-\phi) + b^2\cos^2(\theta-\phi) + c^2\}]}.$

$$\therefore \quad \tan\alpha = \frac{2abc}{p(a^2+b^2-c^2-OP^2)}.]$$

16. Show that the most general quadric surface which has the lines $x = 0, y = 0; x = 0, z = c; y = 0, z = -c$ as generators is $fy(z-c) + gx(z+c) + hxy = 0$, where f, g, h are arbitrary constants.

Hints. Let the equation of the surface be

$$a_1x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy + 2u_1x + 2v_1y + 2w_1z + d_1 = 0.$$

The line $x = 0, y = 0$ lies on the surface. $\therefore\ c_1z^2 + 2w_1z + d_1 = 0.$

By the condition of generator $c_1 = 0, w_1 = 0, d_1 = 0.$

The line $x = 0,\ z = c$ lies on the surface. $\therefore\ b_1y^2 + 2(f_1c + v_1)y = 0.$

It implies that $b_1 = 0, f_1c + v_1 = 0.$

Similarly for the generator $y = 0, z = -c, a_1 = 0$ and $-g_1c + u_1 = 0.$

Now the equation of the surface is $f_1y(z-c) + g_1x(z+c) + h_1xy = 0.$

It is of the form $fy(z-c) + gx(z+c) + hxy = 0.$]

ANSWERS

1. $x - 2 = 0, \frac{y+1}{3} = \frac{z-4/3}{-4}; \frac{x-2}{3} = \frac{y+1}{6} = \frac{z-4/3}{10}.$

2. $\frac{x+2}{3} = \frac{y-2}{4} = \frac{z-7}{-84}; \frac{x+2}{3} = \frac{y-2}{-4} = \frac{z-7}{-12}.$ 4. $\frac{x-a}{2a} = \frac{y}{\pm 2\sqrt{ab}} = \frac{z-\gamma}{\alpha}.$

11. $(2x+3y+7z) - (x+y+z) = \lambda(4x-y+3z), (2x+3y+7z) + (x+y+z) = \frac{1}{\lambda};$ $(2x+3y+7z) - (x+y+z) = \frac{1}{\mu}, (2x+3y+7z) + (x+y+z) = \mu(4x-y+3z).$

Chapter 10

Reduction of General Equation of Second Degree

10.10 General equation

The general equation of second degree is generally written as

$$F(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

$$\text{or,} \quad \phi(x,y,z) + 2ux + 2vy + 2wz + d = 0. \tag{1}$$

It contains nine disposable constants and therefore, nine conditions are required to find these constants. The equation (1) may represent a pair of planes or a quadric and that representation depends on the values of the nine disposable constants. Further, a conicoid can be determined which passes through nine given points, no four of which are coplanar.

Any point on the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \tag{2}$$

can be taken as $(lr + x_1, mr + y_1, nr + z_1)$. If this line meets the surface (1) at this point, then

$$F(lr + x_1, mr + y_1, nr + z_1) = 0$$

$$\text{or,} \quad \phi(l,m,n)r^2 + \left(l\frac{\partial F}{\partial x_1} + m\frac{\partial F}{\partial y_1} + n\frac{\partial F}{\partial z_1}\right) r + F(x_1, y_1, z_1) = 0. \tag{3}$$

It is a quadratic equation in r. If the roots are r_1 and r_2, then the points of intersection between (1) and (2) are $(lr_1 + x_1, mr_1 + y_1, nr_1 + z_1)$ and $(lr_2 + x_1, mr_2 + y_1, nr_2 + z_1)$.

10.11 Section with a given centre

If (x_1, y_1, z_1) is the midpoint of the chord made by the line (2) on the quadric (1) of Sec 10.10, then the roots of (3) are equal and opposite.

In this case

$$l\frac{\partial F}{\partial x_1} + m\frac{\partial F}{\partial y_1} + n\frac{\partial F}{\partial z_1} = 0. \tag{4}$$

Hence all chords which are bisected at the point (x_1, y_1, z_1) lie on the plane

$$(x - x_1)\frac{\partial F}{\partial x_1} + (y - y_1)\frac{\partial F}{\partial y_1} + (z - z_1)\frac{\partial F}{\partial z_1} = 0$$

or, $\sum (x - x_1)(ax_1 + hy_1 + gz_1 + u) = 0$

or, symbolically $T = S_1 \quad [S_1 = F(x_1, y_1, z_1)]$. (5)

> **Note.** The plane (5) meets the quadric represented by (1) in a conic whose centre is (x_1, y_1, z_1).

Diametral plane. The locus of the midpoints of chords parallel to $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0$$

or, $(al + hm + gn)x + (hl + bm + fn)y + (gl + fm + cn)z + (ul + vm + wn) = 0.$ (6)

It is a plane called *the diametral plane conjugate to the direction l, m, n.*

It must be noted that there is no diametral plane conjugate to the direction l, m, n, if the coefficients of x, y, z in the equation (6) are all zero, i.e.

$$al + hm + gn = 0,$$
$$hl + bm + fn = 0,$$
$$gl + fm + cn = 0.$$

For non-zero solution of l, m, n, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$

10.12 Conjugate diameters and diametral planes

(a) Central conicoid. $ax^2 + by^2 + cz^2 = 1.$

Let $P(x_1, y_1, z_1)$ be a point on the conicoid. The equation of the diametral plane bisecting chords parallel to OP is

$$axx_1 + byy_1 + czz_1 = 0. \tag{1}$$

If $Q(x_2, y_2, z_2)$ is a point on the plane (1) and the conicoid, then the line OQ lies on the plane and

$$ax_1x_2 + by_1y_2 + cz_1z_2 = 0. \tag{2}$$

The symmetry of the condition (2) implies that the diametral plane of OQ passes through the point P. Thus *if the diametral plane of OP passes through Q, then the diametral plane of OQ passes through P.*

The line of intersection of the diametral planes of OP and OQ meets the conicoid at two points. Let one point be $R(x_3, y_3, z_3)$. Since the point R is on the diametral planes of OP and OQ, the diametral plane $axx_3 + byy_3 + czz_3 = 0$ passes through the points P and Q.

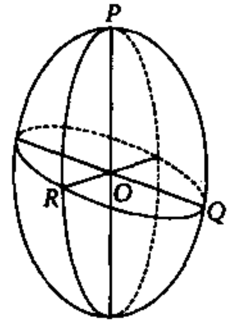

Fig. 43

From these properties we have

$$\left.\begin{aligned} ax_1^2 + by_1^2 + cz_1^2 &= 1 \\ ax_2^2 + by_2^2 + cz_2^2 &= 1 \\ ax_3^2 + by_3^2 + cz_3^2 &= 1 \end{aligned}\right\} \quad \text{(I)} \quad \text{and} \quad \left.\begin{aligned} ax_1x_2 + by_1y_2 + cz_1z_2 &= 0 \\ ax_2x_3 + by_2y_3 + cz_2z_3 &= 0 \\ ax_3x_1 + by_3y_1 + cz_3z_1 &= 0 \end{aligned}\right\} \quad \text{(II)}$$

Here OP, OQ, OR are three *semi-diameters* and the plane containing any two of them is the diametral plane of the third. These three semi-diameters are called *conjugate semi-diameters* and the diametral planes POQ, QOR, ROP are called *conjugate planes.*

Corollary I. *The relations* (I) *and* (II) *state that* $\sqrt{a}\,x_1$, $\sqrt{b}\,y_1$, $\sqrt{c}\,z_1$; $\sqrt{a}\,x_2$, $\sqrt{b}\,y_2$, $\sqrt{c}\,z_2$; $\sqrt{a}\,x_3$, $\sqrt{b}\,y_3$, $\sqrt{c}\,z_3$ *are d.cs. of three mutually perpendicular straight lines. Therefore,* $\sqrt{a}\,x_1$, $\sqrt{a}\,x_2$, $\sqrt{a}\,x_3$; $\sqrt{b}\,y_1$, $\sqrt{b}\,y_2$, $\sqrt{b}\,y_3$; $\sqrt{c}\,z_1$, $\sqrt{c}\,z_2$, $\sqrt{c}\,z_3$ *are d.cs. of three mutually perpendicular straight lines. Consequently*

$$\left.\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= \tfrac{1}{a} \\ y_1^2 + y_2^2 + y_3^2 &= \tfrac{1}{b} \\ z_1^2 + z_2^2 + z_3^2 &= \tfrac{1}{c} \end{aligned}\right\} \quad \text{(III)} \quad \textit{and} \quad \left.\begin{aligned} x_1y_1 + x_2y_2 + x_3y_3 &= 0 \\ y_1z_1 + y_2z_2 + y_3z_3 &= 0 \\ z_1x_1 + z_2x_2 + z_3x_3 &= 0 \end{aligned}\right\} \quad \text{(IV)}$$

From (III),

$$OP^2 + OQ^2 + OR^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Hence *the sum of the squares of conjugate diameters is constant.*

Corollary II. *From* (II),

$$\frac{\sqrt{a}\,x_1}{\sqrt{bc}(y_2z_3 - y_3z_2)} = \frac{\sqrt{b}\,y_1}{\sqrt{ca}(z_2x_3 - z_3x_2)} = \frac{\sqrt{c}\,z_1}{\sqrt{ab}(x_2y_3 - x_3y_2)}$$
$$= \frac{\pm\sqrt{ax_1^2 + by_1^2 + cz_1^2}}{\sqrt{\Sigma bc(y_2z_3 - y_3z_2)^2}} = \pm 1.$$

since $\sqrt{\Sigma bc(y_2z_3 - y_3z_2)^2}$ is the sine of the angle between two perpendicular lines.

$$\begin{aligned} \therefore\ \sqrt{a}\,x_1 &= \pm\sqrt{bc}(y_2z_3 - y_3z_2), \\ \sqrt{b}\,y_1 &= \pm\sqrt{ca}(z_2x_3 - z_3x_2), \\ \sqrt{c}\,z_1 &= \pm\sqrt{ab}(x_2y_3 - x_3y_2). \end{aligned}$$

Now the volume of the tetrahedron $OPQR$

$$= \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$
$$= \frac{1}{6}\{x_1(y_2z_3 - y_3z_2) + y_1(z_2x_3 - z_3x_2) + z_1(x_2y_3 - x_3y_2)\}$$
$$= \frac{1}{6}\left(\frac{\sqrt{a}}{\sqrt{bc}}x_1^2 + \frac{\sqrt{b}}{\sqrt{ca}}y_1^2 + \frac{\sqrt{c}}{\sqrt{ab}}z_1^2\right)$$
$$= \frac{1}{6\sqrt{abc}}(ax_1^2 + by_1^2 + cz_1^2) = \frac{1}{6\sqrt{abc}}.$$

Hence *the volume of the tetrahedron with three conjugate semi-diameters as coterminous edges is constant.*

Let A_1, A_2, A_3 be the areas of $\triangle OQR, \triangle ORP, \triangle OPQ$ and let l_i, m_i, n_i be the d.cs. of the normals to the planes of the triangles respectively.

Now the projection of $\triangle OQR$ on the yz-plane is the triangle with vertices $(0,0,0)$, $(0, y_2, z_2)$, $(0, y_3, z_3)$.

$$\therefore\ l_1A_1 = \frac{1}{2}(y_2z_3 - y_3z_2) = \pm\frac{\sqrt{a}\,x_1}{2\sqrt{bc}}.$$

Similarly we get $m_1A_1 = \pm\dfrac{\sqrt{b}\,y_1}{2\sqrt{ca}}, \quad n_1A_1 = \pm\dfrac{\sqrt{c}\,z_1}{2\sqrt{ab}}.$

Squaring and adding,

$$A_1^2 = \frac{1}{4}\left(\frac{ax_1^2}{bc} + \frac{by_1^2}{ca} + \frac{cz_1^2}{ab}\right).$$

Considering the projections of $\triangle ORP$ and $\triangle OPQ$ on the coordinate planes, we have

$$A_2^2 = \frac{1}{4}\left(\frac{ax_2^2}{bc} + \frac{by_2^2}{ca} + \frac{cz_2^2}{ab}\right), \quad A_3^2 = \frac{1}{4}\left(\frac{ax_3^2}{bc} + \frac{by_3^2}{ca} + \frac{cz_3^2}{ab}\right).$$

Thus

$$A_1^2 + A_2^2 + A_3^2 = \frac{1}{4}\left(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}\right).$$

Hence *the sum of the squares of the areas of the faces of the tetrahedron having three conjugate semi-diameters of a central conicoid as coterminous edges is constant.*

Let l, m, n be the d.cs. of a line.

The projections of OP, OQ, OR on this line are

$$lx_1 + my_1 + nz_1, lx_2 + my_2 + nz_2, lx_3 + my_3 + nz_3.$$

Now sum of the squares of these projections

$$= (lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2 + (lx_3 + my_3 + nz_3)^2$$
$$= l^2 \sum x_1^2 + m^2 \sum y_1^2 + n^2 \sum z_1^2 + 2lm \sum x_1y_1 + 2mn \sum y_1z_1 + 2nl \sum z_1x_1$$
$$= \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}.$$

Hence *the sum of the squares of the projections of three conjugate semi-diameters on a line is constant.*

Similarly *the sum of the squares of projections of the three conjugate semi-diameters on any plane is constant.*

(b) **Paraboloid.** $ax^2 + by^2 = 2cz$.

Let the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ meet the paraboloid at the point $(lr + \alpha, mr + \beta, nr + \gamma)$. Then

$$(al^2 + bm^2)r^2 + 2(al\alpha + bm\beta - cn)r + (a\alpha^2 + b\beta^2 - 2c\gamma) = 0. \tag{1}$$

It is a quadratic equation in r. Thus, in general, a line meets the paraboloid in two points. If $l = m = 0$, then one point of intersection will be at infinity. Such lines are parallel to the axis of the paraboloid and these are called *diameters of the paraboloid.*

If (α, β, γ) is the midpoint of the chord made by the line, then $al\alpha + bm\beta - cn = 0$. Consequently the equation of the *diametral plane* of OP with d.cs. l, m, n is $alx + bmy - cn = 0$. The equation indicates that *all diametral planes of a paraboloid are parallel to its axis.*

Converse is also true. Let $px + qy + r = 0$ be any plane parallel to the axis of the paraboloid. The diametral plane corresponding to the chords parallel to

$$\frac{x}{p/a} = \frac{y}{q/b} = \frac{z}{-r} \text{ is } a \cdot \frac{p}{a}x + b \cdot \frac{q}{b}y - (-r) = 0, \text{ i.e. } px + qy + r = 0.$$

Hence the result follows.

Let OP and OQ be the lines $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and $\frac{x}{l'} = \frac{y}{m'} = \frac{z}{n'}$.

OQ will be parallel to the diametral plane of OP, if $all' + bmm' = 0$. It shows that the diametral plane of OQ is parallel to OP. Hence the *two diametral planes* $alx + bmy - cn = 0$ and $al'x + bm'y - cn' = 0$ *are conjugate, if* $all' + bmm' = 0$.

Example 1. *Find the locus of the equal conjugate diameters of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Let OP, OQ, OR be three equal conjugate semi-diameters.

We have $OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2$.

Here $OP^2 = OQ^2 = OR^2$.

$$\therefore\ OP^2 = \tfrac{1}{3}(a^2 + b^2 + c^2).$$

Let (x_1, y_1, z_1) be the coordinates of P. We are to find the locus of the line

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}, \tag{1}$$

where

$$x_1^2 + y_1^2 + z_1^2 = \tfrac{1}{3}(a^2 + b^2 + c^2) \tag{2}$$

and

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1. \tag{3}$$

By (2) and (3),

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = \frac{3(x_1^2 + y_1^2 + z_1^2)}{a^2 + b^2 + c^2}.$$

Eliminating x_1, y_1, z_1 by the help of (1), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{3(x^2 + y^2 + z^2)}{a^2 + b^2 + c^2}.$$

It is the required locus.

Example 2. *Prove that the middle points of chords of* $ax^2 + by^2 + cz^2 = 1$ *which are parallel to* $x = 0$ *and touch* $x^2 + y^2 + z^2 = r^2$ *lie on the surface*

$$by^2(bx^2 + by^2 + cz^2 - br^2) + cz^2(cx^2 + by^2 + cz^2 - cr^2) = 0.$$

Let $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ be one of the chords with the middle point (α, β, γ). If it is parallel to $x = 0$, then $l \cdot 1 + m \cdot 0 + n \cdot 0 = 0$, i.e. $l = 0$.

Let the chord meet the conicoid at the point $(\alpha, m\lambda + \beta, n\lambda + \gamma)$.

Then

$$\begin{aligned} &a\alpha^2 + b(m\lambda + \beta)^2 + c(n\lambda + \gamma)^2 = 1 \\ \text{or,}\quad &(bm^2 + cn^2)\lambda^2 + 2(bm\beta + cn\gamma)\lambda + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0. \end{aligned}$$

It is a quadratic equation in λ. Two roots correspond to two points of intersection of the chord and the conicoid.

As (α, β, γ) is the midpoint, the sum of the roots is zero.

$$\therefore\ bm\beta + cn\gamma = 0. \tag{1}$$

Let the chord meet the sphere at the point $(\alpha, mt + \beta, nt + \gamma)$.

Then

$$\begin{aligned} &\alpha^2 + (mt + \beta)^2 + (nt + \gamma)^2 = r^2 \\ \text{or,}\quad &(m^2 + n^2)t^2 + 2(m\beta + n\gamma)t + \alpha^2 + \beta^2 + \gamma^2 - r^2 = 0. \end{aligned}$$

As the chord is the tangent to the sphere, the roots of the above equation in t are equal.

$$\therefore\ (m\beta + n\gamma)^2 - (m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 - r^2) = 0. \tag{2}$$

Eliminating m and n between (1) and (2), we have

$$\beta^2\gamma^2(c-b)^2 - (b^2\beta^2 + c^2\gamma^2)(\alpha^2+\beta^2+\gamma^2-r^2) = 0.$$

On simplification

$$b\beta^2(b\alpha^2+b\beta^2+c\gamma^2-br^2) + c\gamma^2(c\alpha^2+b\beta^2+c\gamma^2-cr^2) = 0.$$

Hence the locus of (α,β,γ) is

$$by^2(bx^2+by^2+cz^2-br^2) + cz^2(cx^2+by^2+cz^2-cr^2) = 0.$$

Example 3. *The plane $3x+4y=1$ is a diametral plane of the paraboloid $5x^2+6y^2 = 2z$. Find the equations of the chord through $(3,4,5)$ which it bisects.*

Let the chord be

$$\frac{x-3}{l} = \frac{y-4}{m} = \frac{z-5}{n}. \tag{1}$$

The equation of the diametral plane bisecting the chords parallel to (1) is $5lx + 6my - n = 0$.

This is same as $3x+4y-1=0$.

On comparison $\frac{5l}{3} = \frac{6m}{4} = \frac{n}{1}$. $\therefore\ l = \frac{3n}{5}, m = \frac{2n}{3}$.

Putting these values in (1), the equations of the chord through $(3,4,5)$ is

$$\frac{x-3}{9} = \frac{y-4}{10} = \frac{z-5}{15}.$$

10.20 Principal planes and principal directions

A diametral plane conjugate to the direction l,m,n is said to be a *principal plane*, if the chords bisected by it are at right angles to it. Here the direction corresponding to l,m,n is the principal direction.

Since the diametral plane (6) of Sec 10.11 is at right angle to $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$,

$$\frac{al+hm+gn}{l} = \frac{hl+bm+fn}{m} = \frac{gl+fm+cn}{n} = \lambda \text{ (say)}.$$

$$\left.\begin{aligned} \therefore\ (a-\lambda)l + hm + gn &= 0,\\ hl + (b-\lambda)m + fn &= 0,\\ gl + fm + (c-\lambda)n &= 0. \end{aligned}\right\} \tag{1}$$

For non-zero solutions of l,m,n,

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

$$\text{or,}\quad \lambda^3 - (a+b+c)\lambda^2 + (A+B+C)\lambda - \Delta = 0. \tag{2}$$

A, B, C are the cofactors of a, b, c of the determinant Δ. The equation (2) in λ is called the *discriminating cubic* and each root is called a *characteristic root*. Each of three characteristic roots corresponds to a set of values for l, m, n. Substituting these sets in (6) of 10.11, the equation by means of (1) reduces to

$$\lambda(lx + my + nz) + ul + vm + wn = 0. \tag{3}$$

This gives three principal planes.

Note 1. Corresponding to zero value of a characteristic root there is no principal plane.

Note 2. The principal directions corresponding to two distinct roots of the discriminating cubic are at right angle.

If λ_1 and λ_2 are two distinct roots and $l_1, m_1, n_1; l_2, m_2, n_2$ are the corresponding principal directions, then

$$l_1 \frac{\partial \phi}{\partial l_2} + m_1 \frac{\partial \phi}{\partial m_2} + n_1 \frac{\partial \phi}{\partial n_2} = 2\sum a l_1 l_2 + 2\sum f(m_1 n_2 + m_2 n_1)$$
$$= 2\lambda_2 \sum l_1 l_2$$

and $$l_2 \frac{\partial \phi}{\partial l_1} + m_2 \frac{\partial \phi}{\partial m_1} + n_2 \frac{\partial \phi}{\partial n_1} = 2\sum a l_1 l_2 + 2\sum f(m_1 n_2 + m_2 n_1)$$
$$= 2\lambda_1 \sum l_1 l_2 \quad \text{[by (1)]}.$$

$\therefore\ (\lambda_1 - \lambda_2)\sum l_1 l_2 = 0$ or, $\sum l_1 l_2 = 0$. It proves the proposition.

Note 3. *Principal directions corresponding to the nature of the roots of the discriminating cubic.*

(i) The roots of the discriminating cubic are all real.

(ii) If all the roots are distinct, then the three principal directions are unique and they are mutually perpendicular.

(iii) If two roots are equal, then the set of three mutually perpendicular principal directions is not unique.

(iv) If three roots are equal, then every direction is a principal direction and a set of three mutually perpendicular directions is a set of three principal directions.

Note 4. *Transformation of $\phi(x, y, z)$ when the coordinate axes are parallel to a set of three principal directions.*

Let $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ be the d.cs. corresponding to the roots $\lambda_1, \lambda_2, \lambda_3$ of the cubic (2).

If (x, y, z) transform to (ξ, η, ζ) for the change of axes, then

$$x = l_1\xi + l_2\eta + l_3\zeta,\ y = m_1\xi + m_2\eta + m_3\zeta,\ z = n_1\xi + n_2\eta + n_3\zeta$$

and $$\xi = l_1 x + m_1 y + n_1 z,\ \eta = l_2 x + m_2 y + n_2 z,\ \zeta = l_3 x + m_3 y + n_3 z.$$

We have

$$l_1\frac{\partial\phi}{\partial x}+m_1\frac{\partial\phi}{\partial y}+n_1\frac{\partial\phi}{\partial z}=x\frac{\partial\phi}{\partial l_1}+y\frac{\partial\phi}{\partial m_1}+z\frac{\partial\phi}{\partial n_1}=2\lambda_1(l_1x+m_1y+n_1z)=2\lambda_1\xi.$$

Similarly

$$l_2\frac{\partial\phi}{\partial x}+m_2\frac{\partial\phi}{\partial y}+n_2\frac{\partial\phi}{\partial z}=2\lambda_2\eta, l_3\frac{\partial\phi}{\partial x}+m_3\frac{\partial\phi}{\partial y}+n_3\frac{\partial\phi}{\partial z}=2\lambda_3\zeta.$$

Now multiplying by ξ,η,ζ respectively and adding, we have

$$x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}+z\frac{\partial\phi}{\partial z}=2(\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2)$$

$$\text{or,}\quad \phi(x,y,z)=\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2.$$

Example 4. *Find the principal directions and the corresponding principal planes for the quadric* $3x^2-y^2-z^2+6yz-6x+6y-2z-2=0$. **[BH 99]**

Comparing the given equation with the equation $F(x,y,z)=0$, $a=3$, $b=-1$, $c=-1$, $f=3$, $g=0$, $h=0$, $u=-3$, $v=3$, $w=-1$, $d=-2$.

The discriminating cubic is

$$\begin{vmatrix}3-\lambda & 0 & 0\\ 0 & -1-\lambda & 3\\ 0 & 3 & -1-\lambda\end{vmatrix}=0$$

$$\text{or,}\quad (3-\lambda)\{(1+\lambda)^2-9\}=0$$

$$\text{or,}\quad (3-\lambda)(\lambda-2)(\lambda+4)=0$$

$$\text{or,}\quad \lambda=3,2,-4.$$

For $\lambda=3$, the principal direction l,m,n are given by $-4m+3n=0, 3m-4n=0$. Therefore, $l:m:n=1:0:0$.

Thus the principal direction is $1,0,0$ and the corresponding principal plane is $3x-3=0$ or, $x=1$.

For $\lambda=2$, the principal directions l,m,n are given by $l=0,-3m+3n=0$, $3m-3n=0$. Therefore, $l:m:n=0:1:1$.

Thus the principal direction is $0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}$ and the principal plane is

$$2\left(\frac{y}{\sqrt{2}}+\frac{z}{\sqrt{2}}\right)+\frac{3}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0\quad\text{or,}\quad y+z+1=0.$$

For $\lambda=-4, 7l=0, 3m+3n=0$.

Therefore, $l:m:n=0:1:-1$. Consequently the principal direction is $0,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}$ and the principal plane is

$$-4\left(\frac{y}{\sqrt{2}}-\frac{z}{\sqrt{2}}\right)+\frac{3}{\sqrt{2}}+\frac{1}{\sqrt{2}}=0\quad\text{or,}\quad y-z-1=0.$$

10.21 Centre

If (x, y, z) is the midpoint of a chord of the quadric $F(x, y, z) = 0$ with d.rs. l, m, n, then $l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0$.

If it holds for all values of l, m, n, i.e. any chord is bisected at (x, y, z), then $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$, i.e.

$$ax + hy + gz + u = 0, \tag{1}$$

$$hx + by + fz + v = 0, \tag{2}$$

$$\text{and} \quad gx + fy + cz + w = 0. \tag{3}$$

The point (x, y, z) satisfying the above equation is known as *centre* and the planes represented by (1), (2) and (3) are called the *central planes*.

Multiplying (1), (2), (3) by A, H, G (cofactors a, h, g in Δ) respectively and then adding, we get that

$$\left.\begin{array}{l} x = \dfrac{Au + Hv + Gw}{-\Delta}. \\ \text{Similarly } y = \dfrac{Hu + Bv + Fw}{-\Delta}, \quad z = \dfrac{Gu + Fv + Cw}{-\Delta}. \end{array}\right\}. \tag{4}$$

Note 1. *A quadric may have a single centre, a line of centres or a plane of centres.*

Centre	Conditions	Quadrics
Single centre at a finite distance	$\Delta \neq 0$	ellipsoid, hyperboloid, cone
Single centre at an infinite distance	$\Delta = 0$, $Au + Hv + Gw \neq 0$	paraboloid
Line of centres at a finite distance (Central planes pass through one line)	$\Delta = 0$, $Au + Hv + Gw = 0$, $A \neq 0$	elliptic or hyperbolic cylinder, pair of intersecting planes
Line of centres at an infinite distance (Central planes are parallel but not coincident.)	$A = 0 = B = C$ $= F = G = H$ $fu \neq gv$	parabolic cylinder
Plane of centres (Central planes are coincident.)	$A = 0 = B = C = F$ $= G = H$ $fu = gv = wh$	pair of parallel planes

Note 2. Transformation of $F(x,y,z)=0$ when the origin is shifted to the centre (x_1,y_1,z_1).

The equation transforms to

$$F(x+x_1,y+y_1,z+z_1)=0$$

$$\text{or, } \phi(x,y,z)+\left(x\frac{\partial F}{\partial x_1}+y\frac{\partial F}{\partial y_1}+z\frac{\partial F}{\partial z_1}\right)+F(x_1,y_1,z_1)=0$$

$$\text{or, } \phi(x,y,z)+F(x_1,y_1,z_1)=0 \qquad (5)$$

$$\left[\because\ \frac{\partial F_1}{\partial x_1}=0=\frac{\partial F}{\partial y_1}=\frac{\partial F}{z_1}\right]$$

$$\text{or, } \phi(x,y,z)+ux_1+vy_1+wz_1+d=0.$$

$$\left[\because\ x_1\frac{\partial F}{\partial x_1}+y_1\frac{\partial F}{\partial y_1}+z_1\frac{\partial F}{\partial z_1}=0\right] \qquad (6)$$

Putting the values of x_1,y_1,z_1 from (4)

$$\phi(x,y,z)=\frac{Au^2+Bv^2+Cw^2+2Fvw+2Huv-\Delta d}{\Delta}$$

$$=\frac{-S}{\Delta},\text{ where } S=\begin{vmatrix} a & h & g & u\\ h & b & f & v\\ g & f & c & w\\ u & v & w & d\end{vmatrix}.$$

Example 5. *Find the transformed form of the equation* $14x^2+14y^2+8z^2-4yz-4zx-8xy+18x-18y+5=0$ *referred to the centre as origin.*

The centre lies on the planes

$$\frac{\partial F}{\partial x}=0 \quad\text{or,}\quad 14x-4y-2z+9=0,$$

$$\frac{\partial F}{\partial y}=0 \quad\text{or,}\quad 4x-14y+2z+9=0$$

$$\text{and}\quad \frac{\partial F}{\partial z}=0 \quad\text{or,}\quad x+y-4z=0.$$

$$\text{Here } \Delta=\begin{vmatrix} 14 & -4 & -2\\ 4 & -14 & 2\\ 1 & 1 & -4\end{vmatrix}=648.$$

Since $\Delta\neq 0$, the quadric has a single centre at a finite distance. To solve the above equations

$$\frac{x}{\begin{vmatrix} -9 & -4 & -2\\ -9 & -14 & 2\\ 0 & 1 & -4\end{vmatrix}}=\frac{y}{\begin{vmatrix} 14 & -9 & -2\\ 4 & -9 & 2\\ 1 & 0 & -4\end{vmatrix}}=\frac{z}{\begin{vmatrix} 14 & -4 & -9\\ 4 & -14 & -9\\ 1 & 1 & 0\end{vmatrix}}=\frac{1}{648}$$

$$\text{or,}\quad \frac{x}{-18\times 18}=\frac{y}{18\times 18}=\frac{z}{0}=\frac{1}{648}$$

$$\text{or,}\quad x=-\tfrac{1}{2},\ y=\tfrac{1}{2},\ z=0.$$

Thus the centre of the quadric is $\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$.

If the origin is shifted to this centre, the given equation reduces to

$$14x^2 + 14y^2 + 8z^2 - 4yz - 4zx - 8xy + 9 \cdot (-1/2) - 9 \cdot 1/2 + 5 = 0,$$

i.e. $$14x^2 + 14y^2 + 8z^2 - 4yz - 4zx - 8xy - 4 = 0.$$

10.22 Reduction to canonical form

For reduction and classification the conicoids are divided into two groups according as the centres lie at the finite distance or they lie at infinite distance.

(a) Centres at finite distance.

In this case, the general equation $F(x, y, z) = 0$ of second degree can be reduced to the form $\phi(x, y, z) + k = 0$, where $k = \frac{S}{\Delta}$ by shifting the origin to the centre.

This transforms to the canonical form by rotating the axes to a new system parallel to the set of three mutually perpendicular principal directions. Principal directions depend on the roots $\lambda_1, \lambda_2, \lambda_3$ of the discriminating cubic.

Different canonical forms in relation to these roots are given below:

Case I. *None of* $\lambda_1, \lambda_2, \lambda_3$ *vanishes.*

(i) All the roots are distinct. (Single centre)

1. $Ax^2 + By^2 + Cz^2 = 1$ $\cdots$ ellipsoid
2. $Ax^2 + By^2 - Cz^2 = 1$ $\cdots$ hyperboloid of one sheet
3. $Ax^2 - By^2 - Cz^2 = 1$ $\cdots$ hyperboloid of two sheets
4. $Ax^2 + By^2 + Cz^2 = 0$ $\cdots$ point ellipsoid
5. $Ax^2 + By^2 - Cz^2 = 0$ $\cdots$ cone

(ii) Two roots are equal. (Line of centres)

6. $A(x^2 + y^2) + Bz^2 = 1$ $\cdots$ ellipsoid of revolution
7. $A(x^2 + y^2) - Bz^2 = 1$ $\cdots$ hyperboloid of revolution
8. $A(x^2 + y^2) - Bz^2 = 0$ $\cdots$ cone of revolution

(iii) Three roots are equal.

9. $x^2 + y^2 + z^2 = r^2$ $\cdots$ sphere

Case II. *One of the roots vanishes.*

(iv) Other two roots are distinct. (Line of centres)

10. $Ax^2 + By^2 = C$ $\cdots$ elliptic cylinder
11. $Ax^2 - By^2 = C$ $\cdots$ hyperbolic cylinder
12. $Ax^2 - By^2 = 0$ $\cdots$ pair of intersecting planes

(v) Other two roots are equal. (Line of centres)

13. $A(x^2 + y^2) = D$ $\cdots$ right circular cylinder

Case III. *Two roots vanish.*

14. $Ax^2 = B$ $\cdots$ pair of parallel planes

15. $x^2 = 0$ $\cdots$ pair of coincident planes

(b) **Centres at infinite distance.**

Here the axes are rotated through the same origin to the new axes which are parallel to the set of three mutually perpendicular directions. Then the origin is shifted to a suitable point to make the equation in the canonical form.

Case I. *One root of the discriminating cubic vanishes.*

(i) Other two roots are distinct. (Single centre at infinity)

16. $Ax^2 + By^2 + Cz = 0$ $\cdots$ elliptic paraboloid

17. $Ax^2 - By^2 + Cz = 0$ $\cdots$ hyperbolic paraboloid

(ii) Other two roots are equal. (Single centre at infinity)

18. $A(x^2 + y^2) + Bz = 0$ $\cdots$ paraboloid of revolution

Case II. *Two roots are zero. (Line of centres at infinity)*

19. $Ax^2 + Bz = 0$ $\cdots$ parabolic cylinder

Note 1. If at least two of the characteristic roots are equal and non-zero, then the quadric is a *surface of revolution.*

Note 2. If only one characteristic root is zero, then the line of intersection of the principal planes corresponding to non-zero roots is the *axis of the paraboloid or cylinder or the line of intersection of the pair of intersecting planes.*

Note 3. If the second degree terms of $F(x, y, z) = 0$ form a perfect square, then the equation represents a pair of *parallel planes or a parabolic cylinder.*

Note 4. Reduction of numerical equation.

(i) If no characteristic root is zero and (α, β, γ) is the centre, then the reduced form of $F(x, y, z) = 0$ is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + (u\alpha + v\beta + w\gamma + d) = 0.$$

(ii) If one characteristic root is zero, the principal direction l, m, n corresponding to this root is determined by solving any two of the equations

$$al + hm + gn = 0, hl + bm + fn = 0, gl + fm + cn = 0.$$

The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2(ul + vm + wn)z = 0$ when $ul + vm + wn \neq 0$ and two non-zero characteristic roots are λ_1 and λ_2.

If $ul + vm + wn = 0$, then the centre (α, β, γ) is determined from any two independent centre giving equations. Here the reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + (u\alpha + v\beta + w\gamma + d) = 0$.

Note 5. Axis and vertex of the paraboloid.

Let l_3, m_3, n_3 be the principal directions corresponding to $\lambda_3 = 0$.

Here the reduced equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0, \text{ where } k = ul_3 + vm_3 + wn_3.$$

Vertex of the paraboloid is obtained by solving any two of the equations

$$\begin{aligned} ax + hy + gz + u - l_3 k &= 0, \\ hx + by + fz + v - m_3 k &= 0, \\ gx + fy + cz + w - n_3 k &= 0 \end{aligned}$$

with the equation $k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$.

The axis is the line through the vertex and parallel to the principal direction corresponding to the zero characteristic root.

Note 6. Axis of revolution when unequal characteristic root is not zero.

It is the line through the centre and parallel to the principal direction corresponding to the unequal characteristic root.

WORKED-OUT EXAMPLES

1. *Show that the equation*

$$11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy + 72x - 72y + 36z + 150 = 0$$

represents an ellipsoid. Find the centre and the equations of the axes.

The coordinates of the centre satisfy the equations

$$\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z},$$

$$\text{i.e.} \quad \begin{aligned} 11x - 6y + 2z + 36 &= 0, \\ 3x - 5y + 2z + 18 &= 0, \\ x - 2y + 3z + 9 &= 0. \end{aligned}$$

Solving these equations the centre is $(-2, 2, -1)$.

Shifting the origin to this centre the equation reduces to

$$\begin{aligned} &\phi(x, y, z) + 36 \cdot (-2) - 36 \cdot 2 + 18 \cdot (-1) + 150 = 0 \\ \text{or,} \quad &\phi(x, y, z) - 12 = 0. \end{aligned} \tag{1}$$

The discriminating cubic is

$$\begin{vmatrix} 11 - \lambda & -6 & 2 \\ -6 & 10 - \lambda & -4 \\ 2 & -4 & 6 - \lambda \end{vmatrix} = 0$$

$$\text{or,} \quad \lambda^3 - 27\lambda^2 + 180\lambda - 324 = 0.$$

Roots are $3, 6, 18$.

Rotating the coordinate axes to the principal directions corresponding to the characteristic roots, the equation (1) reduces to

$$3x^2 + 6y^2 + 18z^2 - 12 = 0 \quad \text{or,} \quad x^2 + 2y^2 + 6z^2 = 4.$$

It is an ellipsoid.

The principal directions (l, m, n) corresponding to the root 3 are obtained from

$$\begin{aligned} &8l - 6m + 2n = 0, \\ &-6l + 7m - 4n = 0, \\ &2l - 4m + 3n = 0. \end{aligned}$$

From the first two $\frac{l}{10} = \frac{m}{20} = \frac{n}{20}$ or, $\frac{l}{1} = \frac{m}{2} = \frac{n}{2}$.

The equations of the axis are $\frac{x+2}{1} = \frac{y-2}{2} = \frac{z+1}{2}$.

Similarly the other two axes are

$$\frac{x+2}{2} = \frac{y-2}{1} = \frac{z+1}{-2} \quad \text{and} \quad \frac{x+2}{-2} = \frac{y-2}{2} = \frac{z+1}{-1}.$$

2. *Determine the nature of the following quadric* $2yz + 2zx + 2xy = 1$.

The discriminating cubic is

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \quad \text{or,} \quad \lambda^3 - 3\lambda - 2 = 0 \quad \text{or,} \quad \lambda = 2, -1, -1.$$

Rotating the axes to the principal directions corresponding to the characteristic roots the equation transforms to $2x^2 - y^2 - z^2 = 1$. It is a hyperboloid of two sheets. It is also a surface of revolution.

3. *Find the nature of the quadric represented by*

$$2y^2 + 4zx + 2x - 4y + 6z + 5 = 0. \qquad \text{[BH 92, 2001; CH 2001]}$$

The coordinates of the centre satisfy the equations

$$\begin{aligned} \frac{\partial F}{\partial x} &= 0, \quad \text{i.e.} \quad 2z + 1 = 0, \\ \frac{\partial F}{\partial y} &= 0, \quad \text{i.e.} \quad y - 1 = 0, \\ \frac{\partial F}{\partial z} &= 0, \quad \text{i.e.} \quad 2x + 3 = 0. \end{aligned}$$

From these equations the centre is $\left(-\frac{3}{2}, 1, -\frac{1}{2}\right)$.

Shifting the origin to the centre the given equation reduces to

$$2y^2 + 4zx + 1 \cdot \left(-\frac{3}{2}\right) - 2 \cdot 1 + 3 \cdot \left(-\frac{1}{2}\right) + 5 = 0$$
$$\text{or,}\quad 2y^2 + 4zx = 0 \quad \text{or,} \quad y^2 + 2zx = 0.$$

The discriminating cubic is

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$
$$\text{or,}\quad \lambda^3 - \lambda^2 - \lambda + 1 = 0 \quad \text{or,} \quad \lambda = 1, 1, -1.$$

Thus the canonical form is $x^2 + y^2 - z^2 = 0$ or, $x^2 + y^2 = z^2$.

It is a right circular cone with the semi-vertical angle $\frac{\pi}{4}$.

It is also a surface of revolution.

The principal directions l, m, n corresponding to the root -1 are obtained from $l + n = 0, 2m = 0, l + n = 0$, i.e. $\frac{l}{1} = \frac{m}{0} = \frac{n}{-1}$. Hence the axis of the cone is

$$\frac{x + \frac{3}{2}}{1} = \frac{y - 1}{0} = \frac{z + \frac{1}{2}}{-1} \quad \text{or,} \quad x + z + 2 = 0, y = 1.$$

4. ***Reduce the equation*** $x^2 + y^2 + z^2 - 2xy - 2yz + 2zx + x - 4y + z + 1 = 0$ ***to its canonical form and determine the type of the quadric represented by it.***

[BH 2007; NH 2004; CH 94]

The given equation can be written as

$$(x - y + z)^2 = -x + 4y - z - 1.$$

Adding a constant λ within the bracket in L.H.S., we have

$$(x - y + z + \lambda)^2 = -x + 4y - z - 1 + \lambda^2 + 2\lambda(x - y + z)$$
$$\text{or,}\quad (x - y + z + \lambda)^2 = (2\lambda - 1)x + (4 - 2\lambda)y + (2\lambda - 1)z + \lambda^2 - 1.$$

Let us choose λ in such a way that the planes

$$x - y + z + \lambda = 0 \quad \text{and} \quad (2\lambda - 1)x + (4 - 2\lambda)y + (2\lambda - 1)z + \lambda^2 - 1 = 0$$

are perpendicular to each other.

For this $(2\lambda - 1) \cdot 1 + (4 - 2\lambda) \cdot (-1) + (2\lambda - 1) \cdot 1 = 0$ or, $\lambda = 1$.

Now the equation is

$$(x - y + z + 1)^2 = x + 2y + z \quad \text{or,} \quad \left(\frac{x - y + z + 1}{\sqrt{1 + 1 + 1}}\right)^2 \cdot 3 = \frac{x + 2y + z}{\sqrt{1 + 4 + 1}} \cdot \sqrt{6}.$$

Taking $\frac{x-y+z+1}{\sqrt{3}} = Y$ and $\frac{x+2y+z}{\sqrt{6}} = X$, the equation takes the form $3Y^2 = \sqrt{6}X$.

Therefore, the given equation represents a parabolic cylinder and the canonical form is $3y^2 = \sqrt{6}x$.

Note. The length of the latus rectum of the normal section $= \frac{\sqrt{6}}{3}$. The vertices lie on the line of intersection of the planes

$$x - y + z + 1 = 0 \quad \text{and} \quad x + 2y + z = 0.$$

The latter plane touches the cylinder along the line of vertices. The foci of the normal sections lie on the common line of the plane $x - y + z + 1 = 0$ and a plane $x + 2y + z = 0$ but at a distance of one-fourth of the latus rectum from this plane. If the plane is $x + 2y + z + k = 0$, then

$$\frac{k}{\sqrt{1+4+1}} = 14 \cdot \frac{\sqrt{6}}{3}.$$

[$\because$ $(0,0,0)$ is a point on the tangent plane.]

$$\therefore\ k = \tfrac{1}{2}.$$

Therefore, the foci lie on the line of intersection of the planes $x-y+z+1 = 0$ and $x + 2y + z + \frac{1}{2} = 0$.

5. *Show that the equation $2y^2 - 2yz + 2zx - 2xy - x - 2y + 3z - 2 = 0$ represents a hyperbolic cylinder and find the equations of its axis.*

The discriminating cubic is

$$\begin{vmatrix} -\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = 0 \quad \text{or,} \quad \lambda^3 - 2\lambda^2 - 3\lambda = 0 \quad \text{or,} \quad \lambda = 0, 3, -1.$$

The d.rs. of the axis corresponding to $\lambda = 0$ are given by $0 \cdot l - m + n = 0, -l + 2m - n = 0, l - m = 0$.

From these $l : m : n = 1 : 1 : 1$.

The line of centres is given by any two of

$$\frac{\partial F}{\partial x} = 0, \quad \text{i.e.} \quad -2y + 2z - 1 = 0,$$
$$\frac{\partial F}{\partial y} = 0, \quad \text{i.e.} \quad -2x + 4y - 2z - 2 = 0,$$
$$\frac{\partial F}{\partial z} = 0, \quad \text{i.e.} \quad 2x - 2y + 3 = 0.$$

Putting $z = 0$ and then solving we have $x = -2, y = -\frac{1}{2}$.

The centre is $\left(-2, -\frac{1}{2}, 0\right)$.

Therefore, the canonical form is

$$3x^2 - y^2 + \left(-\tfrac{1}{2}\right)\cdot(-2) + (-1)\cdot\left(-\tfrac{1}{2}\right) + \tfrac{3}{2}\cdot 0 - 2 = 0$$
$$\text{or,}\quad 3x^2 - y^2 - \tfrac{1}{2} = 0 \quad\text{or,}\quad 6x^2 - 2y^2 = 1.$$

It is a hyperbolic cylinder.

The axis is $\frac{x+2}{1} = \frac{y+1/2}{1} = \frac{z}{1}$.

6. *Determine the nature of the quadric*

$$x^2 + y^2 + z^2 - yz - zx - xy - 3x - 6y - 9z + 21 = 0.$$

The discriminating cubic is

$$\begin{vmatrix} 1-\lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0 \quad\text{or,}\quad 4\lambda^3 - 12\lambda^2 + 9\lambda = 0 \quad\text{or,}\quad \lambda = 0, \frac{3}{2}, \frac{3}{2}.$$

Since one root is zero and the other two roots are equal, the equation represents a paraboloid of revolution.

The direction ratios of the axis corresponding to $\lambda = 0$ are given by

$$l - \frac{m}{2} - \frac{n}{2} = 0, -\frac{l}{2} + m - \frac{n}{2} = 0, -\frac{l}{2} - \frac{m}{2} + n = 0.$$

From these $\frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \frac{1}{\sqrt{3}}$.

The canonical form is

$$\frac{3}{2}x^2 + \frac{3}{2}y^2 + 2\left(-\frac{3}{2}\cdot\frac{1}{\sqrt{3}} - 3\cdot\frac{1}{\sqrt{3}} - \frac{9}{2}\cdot\frac{1}{\sqrt{3}}\right)z = 0$$
$$\text{or,}\quad \frac{3}{2}x^2 + \frac{3}{2}y^2 - 6\sqrt{3}z = 0 \quad\text{or,}\quad x^2 + y^2 - 4\sqrt{3}z = 0.$$

Vertex is obtained from

$$x - \frac{1}{2}y - \frac{1}{2}z - \frac{3}{2} + 3\sqrt{3}\cdot\frac{1}{\sqrt{3}} = 0, \quad\text{i.e.}\quad 2x - y - z + 3 = 0;$$
$$-\frac{1}{2}x + y - \frac{1}{2}z - 3 + 3\sqrt{3}\cdot\frac{1}{\sqrt{3}} = 0, \quad\text{i.e.}\quad x - 2y + z = 0;$$
$$-3\sqrt{3}\left(\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}}\right) - \frac{3x}{2} - 3y - \frac{9z}{2} + 21 = 0,$$

i.e. $9x - 12y - 15z + 42 = 0$.

Solving the equations we get $x = 0, y = 1, z = 2$.

$\therefore$ the vertex is $(0, 1, 2)$ and the axis is $\frac{x}{1} = \frac{y-1}{1} = \frac{z-2}{1}$.

7. *Find the value of d for which the quadric*

$$2x^2 - 2y^2 + 3z^2 + 3xy - yz + 7zx + 3x - 4y - z + d = 0$$

represents a pair of intersecting planes.

The discriminating cubic is

$$\begin{vmatrix} 2-\lambda & \frac{3}{2} & \frac{7}{2} \\ \frac{3}{2} & -2-\lambda & -\frac{1}{2} \\ \frac{7}{2} & -\frac{1}{2} & 3-\lambda \end{vmatrix} = 0 \quad \text{or,} \quad 4\lambda^3 - 12\lambda^2 - 75\lambda = 0 \quad \text{or,} \quad \lambda = 0, \frac{3 \pm 2\sqrt{21}}{2}.$$

The d.cs. of the axis corresponding to $\lambda = 0$ are obtained from

$$2l + \frac{3m}{2} + \frac{7n}{2} = 0, \frac{3l}{2} - 2m - \frac{n}{2} = 0, \frac{7l}{2} - \frac{m}{2} + 3n = 0.$$

From these $\frac{l}{1} = \frac{m}{1} = \frac{n}{-1} = \frac{1}{\sqrt{3}}$.

Since $ul + vm + wn = \frac{3}{2} \cdot \frac{1}{\sqrt{3}} - 2 \cdot \frac{1}{\sqrt{3}} - \frac{1}{2} \cdot \left(-\frac{1}{\sqrt{3}}\right) = 0$ and one characteristic root is zero, the given equation represents either elliptic cylinder or hyperbolic cylinder or pair of planes.

The line of centres is given by

$$4x + 3y + 7z + 3 = 0, 3x - 4y - z - 4 = 0, 7x - y + 6z - 1 = 0.$$

Putting $z = 0$ and solving we get $x = 0$ and $y = -1$. Thus the centre is $(0, -1, 0)$.

The reduced equation is

$$\frac{3 + 2\sqrt{21}}{2}x^2 + \frac{3 - 2\sqrt{21}}{2}y^2 + \left\{\frac{3}{2} \cdot 0 - 2 \cdot (-1) - \frac{1}{2} \cdot 0 + d\right\} = 0.$$

If the equation represents a pair of planes, then $d + 2 = 0$ or, $d = -2$.

8. *Determine the nature of* $x^2 + 4y^2 + z^2 - 4yz + 2zx - 4xy - 2x + 4y - 2z - 3 = 0$ *by reducing to canonical form.*

Comparing the given equation with $F(x, y, z) = 0$, we have $a = 1, b = 4, c = 1, f = -2, g = 1, h = -2, u = -1, v = 2, w = -1, d = -3$.

The discriminating cubic is

$$\begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0 \quad \text{or,} \quad \lambda^2(\lambda - 6) = 0 \quad \text{or,} \quad \lambda = 0, 0, 6.$$

Two roots of the discriminating cubic are zero.

Again

$$\Delta = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{vmatrix} = 0 \quad \text{and} \quad fu = 2 = gv = hw.$$

Therefore, the second degree terms form a square.

The equation can be written as

$$(x - 2y + z)^2 - 2(x - 2y + z) - 3 = 0 \quad \text{or,} \quad (x - 2y + z - 1)^2 - 4 = 0.$$

Let us choose a rectangular system of coordinates (x', y', z') in which $x - 2y + z - 1 = 0$ is the plane $x' = 0$. In this case

$$x'^2 = \frac{(x - 2y + z - 1)^2}{6}.$$

$\therefore$ the equation reduces to $6x'^2 - 4 = 0$ or, $3x'^2 - 2 = 0$.

Thus the canonical form is $3x^2 - 2 = 0$ and the given equation represents two parallel planes.

9. *Show that the equation $x^2 + 2yz = 1$ represents an equation of revolution. Find the axis of revolution.*

The discriminating cubic is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \quad \text{or,} \quad (1-\lambda)(\lambda^2 - 1) = 0 \quad \text{or,} \quad \lambda = 1, 1, -1.$$

Since two roots are equal and non-zero, the equation represents a quadric of revolution.

The canonical form is $x^2 + y^2 - z^2 = 1$. It is a hyperboloid of revolution.

The principal directions l, m, n corresponding to the root -1 are obtained from

$$2l = 0, m + n = 0, \quad \text{i.e.} \quad \frac{l}{0} = \frac{m}{1} = \frac{n}{-1}.$$

Hence the axis of revolution is $\frac{x}{0} = \frac{y}{1} = \frac{z}{-1}$.

EXERCISE X

1. Find the principal plane and directions of the following:

(i) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy + 2x + 12y + 10z + 20 = 0$

(ii) $6x^2 + 3y^2 + 3z^2 - 2yz + 4zx - 4xy - 3y + 5z = 0$

(iii) $x^2 + y^2 + z^2 - 2yz + 2zx - 2xy - 2x + 2y - 2z - 3 = 0$

(iv) $4y^2 - 4yz + 4zx - 4xy - 2x + 2y - 1 = 0$

2. (a) Find the centres of the following conicoids:

   (i) $z^2 - yz + zx + xy - 2y + 2z + 2 = 0$

   (ii) $5x^2 + 26y^2 + 10z^2 + 4yz + 14zx + 6xy - 8x - 18y - 10z + 4 = 0$

   (iii) $4x^2 - 2y^2 - 2z^2 + 5yz + 2zx + 2xy - x + 2y + 2z - 1 = 0$

   (iv) $x^2 + y^2 + z^2 - 2yz + 2zx - 2xy - 2x + 2y - 2z - 3 = 0$

   (b) Find the values of b and v so that the quadric $5x^2 + by^2 + 8z^2 + 8yz + 8zx - 2xy + 12x + 2vy + 6 = 0$ may have a line of centres and hence find its equations.

3. Determine the nature of the following by reducing to canonical form.

   (i) $x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0$

   (ii) $x^2 + y^2 + z^2 + yz + zx + xy = 1$

   (iii) $x^2 + 3y^2 - 4yz - 2x - 2y + 6z + 3 = 0$

   (iv) $2x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 2 = 0$ [CH 92, 2003]

   (v) $2x^2 - y^2 - 10z^2 + 20yz - 8zx - 28xy + 16x + 26y + 16z - 34 = 0$ [NH 2006]

   (vi) $5x^2 - y^2 + z^2 + 6zx + 4xy + 2x + 4y + 6z - 8 = 0$

   (vii) $2x^2 + 5y^2 + 2z^2 - 2yz + 4zx - 2xy + 14x - 16y + 14z + 26 = 0$ [CH 95]

   (viii) $2y^2 + 4zx - 6x - 8y + 2z + 5 = 0$

   (ix) $x^2 + 8z^2 + 2x - 3y + 16z = 0$

   (x) $x^2 + 4y^2 + 9z^2 + 12yz + 6zx + 4xy - 54x - 52y + 62z + 113 = 0$

   (xi) $6x^2 - 2y^2 - 3z^2 + xy + 7yz + 7zx - 9x - 13y + 14z - 15 = 0$

   (xii) $2x^2 + 5y^2 + z^2 - 4xy - 8x + 14y + 3 = 0$

   (xiii) $x^2 + 3y^2 + 3z^2 - 2xy - 2yz - 2zx + 1 = 0$ [BH 2000; CH 91]

   (xiv) $13x^2 + 45y^2 + 40z^2 + 12yz + 36zx - 24xy - 49 = 0$

   (xv) $x^2 + 3y^2 + z^2 + 4zx + 2x + 12y - 2z + 9 = 0$

   (xvi) $4x^2 + 9y^2 + 36z^2 - 36yz + 24zx - 12xy - 10x + 15y - 30z + 6 = 0$

   (xvii) $5x^2 - 4y^2 + 5z^2 + 4xy - 14zx + 4yz + 16x + 16y - 32z + 8 = 0$

   (xviii) $3x^2 + 3y^2 + z^2 - 2xy + 6x - 2y - 2z + 4 = 0$

   (xix) $9x^2 + 16y^2 + z^2 - 24xy - 8yz + 6zx + 12x - 16y + 4z + 4 = 0$

   (xx) $3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 7 = 0$

   (xxi) $4y^2 + 4z^2 + 4yz - 2x - 14y - 22z - 33 = 0$ [CH 2000]

4. (i) Prove that $5x^2 + 5y^2 + 8z^2 + 8yz + 8zx - 2xy + 12x - 12y + 6 = 0$ represents a cylinder whose cross-section is an ellipse of eccentricity $\frac{1}{\sqrt{2}}$ and the axis is $5x - y + 4z + 6 = 0 = x + y + 2z$.

(ii) Show that $\sqrt{x} + \sqrt{y} + \sqrt{z} = 0$ represents a cone of revolution whose canonical form is $2(x^2 + y^2) - z^2 = 0$ and axis is $x = y = z$.
[*Hints.* $\sqrt{x}+\sqrt{y}+\sqrt{z} = 0$ or, $(\sqrt{x}+\sqrt{y})^2 = (-\sqrt{z})^2$ or, $x+y-z = -2\sqrt{xy}$ or, $(x + y - z)^2 = 4xy$ or, $x^2 + y^2 + z^2 - 2xy - 2yz - 2zx = 0$.
Its characteristic roots are $2, 2, -1$, so the canonical form is $2(x^2 + y^2) - z^2 = 0$. The vertex is $(0, 0, 0)$ and the principal directions corresponding to the characteristic root -1 are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. Hence the axis is $x = y = z$.]

5. Prove that $a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 1$ represents a hyperboloid of one sheet and that the sum of the squares on its real axes is equal to the square on the conjugate axis.

6. (i) If $P(x_1, y_1, z_1), Q(x_2, y_2, z_2), R(x_3, y_3, z_3)$ be the extremities of conjugate semi-diameters of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, prove that

$$\frac{x}{a^2}(x_1 + x_2 + x_3) + \frac{y}{b^2}(y_1 + y_2 + y_3) + \frac{z}{c^2}(z_1 + z_2 + z_3) = 1.$$

[*Hints:* Let the equation of the plane PQR be

$$lx + my + nz = p. \tag{1}$$

Here

$$lx_1 + my_1 + nz_1 = p, \tag{2}$$

$$lx_2 + my_2 + nz_2 = p, \tag{3}$$

$$lx_3 + my_3 + nz_3 = p. \tag{4}$$

From $(2) \times x_1 + (3) \times x_2 + (4) \times x_3$, we get

$$l\sum x_1^2 + m\sum x_1y_1 + n\sum z_1x_1 = p\sum x_1.$$

Here

$$\sum x_1^2 = a^2, \sum x_1y_1 = 0 = \sum z_1x_1.$$

$$\therefore\ l = \frac{p}{a^2}\sum x_1.$$

Similarly $m = \frac{p}{b^2}\sum y_1, n = \frac{p}{c^2}\sum z_1$. Hence the result follows.]

(ii) Show that $\frac{x}{1} = \frac{y}{4} = \frac{z}{3}, \frac{x}{4} = \frac{y}{1} = \frac{z}{-9}, \frac{x}{26} = \frac{y}{-28} = \frac{z}{45}$ are three mutually conjugate diameters of the ellipsoid $\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{9} = 1$.

(iii) If λ, μ, ν are the angle between a set of equal conjugate diameters of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, prove that $\cos^2\lambda + \cos^2\mu + \cos^2\nu = \frac{3}{2}\frac{\sum(b^2-c^2)^2}{(a^2+b^2+c^2)^2}$.
[*Hints:* Let r be the length of each semi-diameter and their ends be $P(x_1, y_1, z_1), Q(x_2, y_2, z_2), R(x_3, y_3, z_3)$.
Then

$$\cos\lambda = \frac{x_1x_2 + y_1y_2 + z_1z_2}{r^2}, \cos\mu = \frac{x_2x_3 + y_2y_3 + z_2z_3}{r^2}$$

and

$$\cos\nu = \frac{x_3x_1 + y_3y_1 + z_3z_1}{r^2}.$$

$$\therefore \sum \cos^2\lambda = \frac{1}{r^4}\sum(x_1x_2+y_1y_2+z_1z_2)^2$$
$$= \frac{1}{r^4}\left[\sum(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2) - \sum(x_1y_2-x_2y_1)^2\right]$$
(by Lagrange's identity)
$$= \frac{1}{r^4}\left[3r^4 - \sum\left(\frac{bcx_1}{a}\right)^2\right], \text{\{by Cor.2. of Sec 10.12 (a)\}}$$
$$= 3 - \frac{a^2b^2c^2}{r^4}\sum\frac{x_1^2}{a^4}$$
$$= 3 - \frac{a^2b^2c^2}{r^4}\left[\frac{x_1^2+x_2^2+x_3^2}{a^4} + \frac{y_1^2+y_2^2+y_3^2}{b^4} + \frac{z_1^2+z_2^2+z_3^2}{c^4}\right]$$
$$= 3 - \frac{a^2b^2c^2}{r^4}\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right]$$
{by Cor.1. of Sec 10.12(a)}
$$= 3 - \frac{9a^2b^2c^2}{(a^2+b^2+c^2)^2}\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right], \left\{\because\ r^2 = \frac{a^2+b^2+c}{3}\right.$$
$$= 3\frac{(a^2+b^2+c^2)^2 - 3\sum b^2c^2}{(a^2+b^2+c^2)^2}$$
$$= \frac{3}{2}\frac{\sum(b^2-c^2)^2}{(a^2+b^2+c^2)^2}.]$$

(iv) Show that the diametral planes $x+3y=3, 2x-y=1$ are conjugate for $2x^2+3y^2=4z$.

ANSWERS

1. (i) $\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}; \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}};$
$x-z=2, x+y+z+4=0, x-2y+z=1.$

(ii) $\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}; \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}};$ (not unique)
$4x-2y+2z+1=0, x+y-z=2, 2y+2z+1=0.$

(iii) $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}; \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}};$ (not unique)
$x-y+z=1, x-z=0, x+2y+z=0.$

(iv) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}; \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}};$
Any plane at right angle to $x=y=z-\frac{1}{2}$, $2(x-2y+z)=1, 2(x-z)+1=0.$

2. (a) (i) $(1,1,-1)$ (ii) $\frac{x-5}{-16} = \frac{y}{1} = \frac{z+3}{11}$ is the line of centres. (iii) No centre (iv) $x-y+z=1$ is the plane of centres.

(b) $b=5, v=-6.$

3. (i) $x^2 + y^2 + z^2 = 11$, sphere
(ii) $x^2 + y^2 + 4z^2 = 2$, ellipsoid of revolution
(iii) $x^2 + 4y^2 - z^2 + \dfrac{23}{4} = 0$, hyperboloid of two sheets
(iv) $x^2 + 2y^2 - 4z^2 = 1$, hyperboloid of one sheet
(v) $2x^2 - y^2 - 2z^2 = 1$, hyperboloid of two sheets
(vi) $7x^2 - 2y^2 - \dfrac{8}{\sqrt{14}}z = 0$, hyperbolic paraboloid
(vii) $2x^2 + y^2 = 1$, elliptic cylinder
(viii) $y^2 + z^2 = x^2$, right circular cone
(ix) $x^2 + 8z^2 - 3y = 0$, elliptic paraboloid
(x) $y^2 = 4\sqrt{3}x$, parabolic cylinder
(xi) $15x^2 - 13y^2 = 0$, pair of intersecting planes
(xii) $x^2 + y^2 + 6z^2 = 8$, ellipsoid of revolution
(xiii) $3x^2 + 4y^2 + 1 = 0$, imaginary cylinder
(xiv) $x^2 + y^2 = 1$, right circular cylinder
(xv) $3(x^2 + y^2) - z^2 = 1$, hyperboloid of revolution, axis of revolution $\dfrac{x-1}{1} = \dfrac{y+2}{0} = \dfrac{z+1}{-1}$
(xvi) $196x^2 - 1 = 0$, pair of parallel planes
(xvii) $2x^2 - y^2 = 0$, pair of intersecting planes
(xviii) $x^2 + 2y^2 + 4z^2 = 0$, point ellipsoid
(xix) $x^2 = 0$, coincident planes
(xx) $3x^2 + 2y^2 + 6z^2 + 1 = 0$, imaginary ellipsoid
(xxi) $3x^2 + y^2 = z$, elliptic paraboloid.

Chapter 11

Plane Sections

11.10 All plane sections of a conicoid are conics.

Let the plane

$$lx + my + nz = p \tag{1}$$

intersect the conicoid

$$\begin{aligned} F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \\ + 2ux + 2vy + 2wz + d = 0. \end{aligned} \tag{2}$$

We may assume that $l^2 + m^2 + n^2 = 1$.

By rotation of axes

$$\begin{aligned} x &= \frac{-m}{\sqrt{l^2+m^2}}x' - \frac{ln}{\sqrt{l^2+m^2}}y' + lz', \\ y &= \frac{l}{\sqrt{l^2+m^2}}x' - \frac{mn}{\sqrt{l^2+m^2}}y' + mz', \\ z &= \sqrt{l^2+m^2}y' + nz', \end{aligned}$$

the equation (1) reduces to

$$z' = p. \tag{3}$$

If the equation (2) transforms to

$$\begin{aligned} a_1x'^2 + b_1y'^2 + c_1z'^2 + 2f_1y'z' + 2g_1z'x' + 2h_1x'y' \\ + 2u_1x' + 2v_1y' + 2w_1z' + d_1 = 0 \end{aligned} \tag{4}$$

then the section made by the plane (3) on the surface represented by (4) is

$$\begin{aligned} a_1x'^2 + 2h_1x'y' + b_1y'^2 + 2(g_1p + u_1)x' + 2(f_1p + v_1)y' \\ + c_1p^2 + 2w_1p + d_1 = 0, z' = p. \end{aligned}$$

Obviously it represents a conic and the nature of the conic depends on a_1, b_1 and h_1. Thus all plane sections of a conicoid are conics.

11.11 Similar and similarly situated conics

Definition. *Two conics* $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$ *and* $a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0$ *are said to be similar and similarly situated conics, if and only if* $\frac{a_1}{a_2} = \frac{h_1}{h_2} = \frac{b_1}{b_2}$.

Theorem. *Plane sections made by parallel planes on a conicoid are similar and similarly situated conics.*

The sections made by $lx + my + nz = p_1$ and $lx + my + nz = p_2$, where $l^2 + m^2 + n^2 = 1$, on the conicoid $F(x, y, z) = 0$ are of the forms

$$\begin{aligned} & a_1x'^2 + 2h_1x'y' + b_1y'^2 + 2(g_1p_1 + u_1)x' + 2(f_1p_1 + v_1)y' \\ & \qquad + c_1p_1^2 + 2w_1p_1 + d_1 = 0, z' = p_1 \\ \text{and} \quad & a_1x'^2 + 2h_1x'y' + b_1y'^2 + 2(g_1p_2 + u_1)x' + 2(f_1p_2 + v_1)y' \\ & \qquad + c_1p_2^2 + 2w_1p_2 + d_1 = 0, z' = p_2 \end{aligned}$$

for the transformation of Sec 11.10.

Since the coefficients of $x'^2, x'y', y'^2$ are same, the conics are similar and similarly situated conics.

Note. Two similar and similarly situated conics are of the same species.

11.12 Projection of a plane section (ellipse, parabola, hyperbola) of a conicoid on any plane is a conic of the same species.

Let the given conic be

$$\begin{aligned} & f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0 \\ \text{and} \quad & lx + my + nz = p \text{ be the given plane, where } l^2 + m^2 + n^2 = 1. \end{aligned}$$

Since the section by parallel planes are similar and similarly situated conics, we may consider the projection of the conic on the plane $lx + my + nz = 0$. To obtain the projection, we apply the rotation of axes

$$\begin{aligned} x &= \frac{-m}{\sqrt{l^2 + m^2}}x' - \frac{ln}{\sqrt{l^2 + m^2}}y' + lz', \\ y &= \frac{l}{\sqrt{l^2 + m^2}}x' - \frac{mn}{\sqrt{l^2 + m^2}}y' + mz', \\ z &= \sqrt{l^2 + m^2}y' + nz'. \end{aligned}$$

The transformed equation will be $a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c' +$ (terms of z'^2 and z') $= 0$ and $z' = 0$.

Thus the projection of the given section on the plane $lx + my + nz = 0$ is the same as $a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c' = 0, z' = 0$.

Here $$a' = \frac{am^2 - 2hlm + bl^2}{l^2 + m^2}, h' = \frac{(a-b)lmn + hn(m^2 - l^2)}{l^2 + m^2},$$
$$b' = \frac{(al^2 + 2hlm + bm^2)n^2}{l^2 + m^2}, g' = \frac{fl - gm}{\sqrt{l^2 + m^2}},$$
$$f' = -\frac{(gl + fm)n}{\sqrt{l^2 + m^2}}, c' = c.$$

Now it can be shown that $h'^2 - a'b' = n^2(h^2 - ab)$.

Therefore, $h'^2 - a'b' >=< 0$ according as $h^2 - ab >=< 0$. Hence the projection is of the same species.

Note. In case of a circle $a = b, h = 0$. In this case, $a' = a, h' = 0, b' = n^2b = n^2a$ and $a'b' = n^2a^2 > 0$. Therefore, the projection is an ellipse though the original section is a circle. Thus we cannot say definitely that the original section is an ellipse when the projection is an ellipse. In such a case, it is better to find the equation of the original section.

Example 1. *Show that the curve $x^2 + 7y^2 - 10z^2 + 9 = 0, x + 2y + 3z = 0$ is a hyperbola.*

Eliminating x, we have

$$(2y + 3z)^2 - 10z^2 + 9 = 0 \quad \text{or,} \quad 4y^2 + 12yz - z^2 + 9 = 0.$$

Hence the equation of the projection of the section on the plane $x = 0$ is

$$4y^2 + 12yz - z^2 + 9 = 0, x = 0.$$

$\because\ 6^2 - 4 \cdot (-1) > 0$, the projection is a hyperbola. Hence the section is a hyperbola.

Example 2. *Find the nature of the curve $3y^2 + 12z^2 - x^2 = 75, 2x + 3z = 5$.*

Eliminating z from $2x + 3z = 5$ and $3y^2 + 12z^2 - x^2 = 75$, we have $3y^2 + 12\left(\frac{5-2x}{3}\right)^2 - x^2 = 75$ or, $13x^2 + 9y^2 - 80x - 125 = 0$. The projection of the section on the plane $z = 0$ is $13x^2 + 9y^2 - 80x - 125 = 0$, $z = 0$. It is an ellipse. From this we cannot conclude that the section is an ellipse.

To find the nature, we apply the rotation of axes

$$x = -\frac{3}{\sqrt{13}}y' + \frac{2}{\sqrt{13}}z', y = x', z = \frac{2}{\sqrt{13}}y' + \frac{3}{\sqrt{13}}z'.$$

The plane and the quadric transform to $z' = 5$ and

$$3x'^2 + \frac{12}{13}(2y' + 3z')^2 - \frac{1}{13}(2z' - 3y')^2 = 75$$
$$\text{or,} \quad 3x'^2 + 3y'^2 + 12y'z' = 75 \quad \text{or,} \quad x'^2 + y'^2 + 4y'z' = 25.$$

Thus the section is $x'^2 + y'^2 + 20y' = 25, z' = 5$.

It is the equation of a circle. Hence the required section is a circle.

Example 3. *Show that the section of the surface $yz + zx + xy = a^2$ by the plane $lx + my + nz = p$ will be a parabola if $\sqrt{l} + \sqrt{m} + \sqrt{n} = 0$.*

The projection of the section on the plane $z = 0$ is

$$(y+x)\left(\frac{p - lx - my}{n}\right) + xy = a^2, z = 0$$
$$\text{or,}\quad lx^2 + my^2 + (l + m - n)xy - px - py + na^2 = 0, z = 0.$$

It will be a parabola, if $(l + m - n)^2 = 4lm$

$$\text{or,}\quad l^2 + m^2 + n^2 = 2(lm + mn + nl). \tag{1}$$

If $\sqrt{l} + \sqrt{m} + \sqrt{n} = 0$, then

$$l + m + n = -2\left(\sqrt{lm} + \sqrt{mn} + \sqrt{nl}\right)$$
$$\text{or,}\quad (l + m + n)^2 = 4\left(\sqrt{lm} + \sqrt{mn} + \sqrt{nl}\right)^2$$
$$\text{or,}\quad l^2 + m^2 + n^2 = 2(lm + mn + nl) + 8\sqrt{lmn}\left(\sqrt{l} + \sqrt{m} + \sqrt{n}\right)$$
$$\text{or,}\quad l^2 + m^2 + n^2 = 2(lm + mn + nl).$$

It is the condition (1). Hence the result follows.

11.13 Axes of plane sections

(i) Axes of central sections of central conicoid.
Let the central conicoid be

$$ax^2 + by^2 + cz^2 = 1 \tag{1}$$

and the central plane (the plane through the centre of the conicoid) be

$$lx + my + nz = 0. \tag{2}$$

We consider a concentric sphere

$$x^2 + y^2 + z^2 = r^2. \tag{3}$$

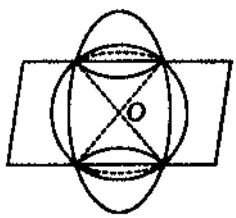

Fig. 44

The equation of the cone with the vertex at the common centre of (1) and (3) and the guiding curve

$$ax^2 + by^2 + cz^2 = 1, x^2 + y^2 + z^2 = r^2 \text{ is}$$
$$ax^2 + by^2 + cz^2 = \frac{x^2 + y^2 + z^2}{r^2}$$
$$\text{or,}\quad (ar^2 - 1)x^2 + (br^2 - 1)y^2 + (cr^2 - 1)z^2 = 0. \tag{4}$$

The plane (2) cuts this cone in two generators. These are equal diameters of length $2r$ of the conic section made by the plane on the conicoid. These diameters are equally inclined to the axes of the section. They will be along the axes if and only if they coincide. In this case, the plane (2) touches the cone (4).

The condition for tangency is $\frac{l^2}{ar^2-1}+\frac{m^2}{br^2-1}+\frac{n^2}{cr^2-1}=0$. It implies that

$$\begin{aligned}(bcl^2+cam^2+abn^2)r^4-\{(b+c)l^2+(c+a)m^2+(a+b)n^2\}r^2&\\ +(l^2+m^2+n^2)=0.&\end{aligned}\tag{5}$$

It is a quadratic equation in r^2. If r_1^2 and r_2^2 are the roots, then they are the squares of the semi-axes of the section.

If λ,μ,ν are d.rs. of the axis of length $2r$, the plane (2) touches the cone along the line $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$.

Therefore, the plane $(ar^2-1)\lambda x+(br^2-1)\mu y+(cr^2-1)\nu z=0$ is identical with $lx+my+nz=0$. Thus we get

$$\frac{\lambda(ar^2-1)}{l}=\frac{\mu(br^2-1)}{m}=\frac{\nu(cr^2-1)}{n}.\tag{6}$$

r is obtained from (5) and the d.rs. of this axis are obtained from (6).

Corollary I. Area of the elliptic section.

Area $=\pi r_1r_2=\pi\sqrt{\frac{l^2+m^2+n^2}{bcl^2+cam^2+abn^2}}$.

If p is the perpendicular distance from the origin to the tangent plane $lx+my+nz=\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$ of the conicoid, then

$$p=\sqrt{\frac{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}{l^2+m^2+n^2}}=\sqrt{\frac{bcl^2+cam^2+abn^2}{l^2+m^2+n^2}}\cdot\frac{1}{\sqrt{abc}}.$$

$\therefore$ area $=\frac{\pi}{p\sqrt{abc}}$.

Corollary II. Condition for rectangular hyperbolic section.

For this $r_1^2+r_2^2=0$.

$\therefore\ (b+c)l^2+(c+a)m^2+(a+b)n^2=0.$

Corollary III. Condition for the lines $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$ and $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$ ***to be the axes of the section.***

The lines are conjugated diameters of the section. The diametral plane of the first line is $al_1x+bm_1y+cn_1z=0$. The second line lies on it.

$\therefore al_1l_2+bm_1m_2+cn_1n_2=0.$

Again the lines must be perpendicular to each other.

$\therefore\ l_1l_2+m_1m_2+n_1n_2=0.$

These are the conditions.

Corollary IV. Cone of axes.

From (6), $$\frac{\lambda(ar^2-1)}{l}=\frac{\mu(br^2-1)}{m}=\frac{\nu(cr^2-1)}{n}=k(\text{say}).$$

Now $$\frac{lk}{\lambda}=ar^2-1,\ \frac{mk}{\mu}=br^2-1,\ \frac{nk}{\nu}=cr^2-1.$$

Multiplying these by $b-c, c-a$ and $a-b$ respectively and adding, we get

$$\frac{l}{\lambda}(b-c)+\frac{m}{\mu}(c-a)+\frac{n}{\nu}(a-b)=0.$$

Hence the axis $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$ lies on the cone

$$\frac{l(b-c)}{x}+\frac{m(c-a)}{y}+\frac{n(a-b)}{z}=0.$$

Example 4. *Find the lengths and equations of the axes of the conic* $9x^2+6y^2+14z^2=3, x+y+z=0$. *Obtain the area of the section.*

Here $a=3, b=2, c=\frac{14}{3}, l=m=n=1$.

Lengths of semi-axes are given by

$$\left(\frac{28}{3}+14+6\right)r^4-\left(\frac{20}{3}+\frac{23}{3}+5\right)r^2+(1+1+1)=0 \quad \text{or,} \quad 88r^4-58r^2+9=0.$$

From this equation

$$r^2=\frac{72}{176},\frac{44}{176}, \quad \text{i.e.} \quad r^2=\frac{9}{22},\frac{1}{4}.$$

Thus the lengths of semi-axes are $\frac{3}{\sqrt{22}}$ and $\frac{1}{2}$.

If λ,μ,ν are d.rs. of the axis of length $2\cdot\frac{3}{\sqrt{22}}$, then

$$\frac{\lambda\left(\frac{27}{22}-1\right)}{1}=\frac{\mu\left(\frac{18}{22}-1\right)}{1}=\frac{\nu\left(\frac{21}{11}-1\right)}{1} \quad \text{or,} \quad \frac{\lambda}{4}=\frac{\mu}{-5}=\frac{\nu}{1}.$$

If λ',μ',ν' are d.rs. of the axis of length $2\cdot\frac{1}{2}$, then

$$\frac{\lambda'\left(\frac{3}{4}-1\right)}{1}=\frac{\mu'\left(\frac{1}{2}-1\right)}{1}=\frac{\nu'\left(\frac{7}{6}-1\right)}{1} \quad \text{or,} \quad \frac{\lambda'}{2}=\frac{\mu'}{1}=\frac{\nu'}{-3}.$$

Hence the equations of the axes are

$$\frac{x}{4}=\frac{y}{-5}=\frac{z}{1} \quad \text{and} \quad \frac{x}{2}=\frac{y}{1}=\frac{z}{-3}.$$

The section is an ellipse.

The area of this section $=\pi\frac{\sqrt{(1+1+1)}}{\sqrt{\left(\frac{28}{3}+14+6\right)}}=\frac{3\pi}{\sqrt{22}}$.

(ii) **Axes of non-central plane section of a central conicoid.**

$$\text{Let} \quad ax^2+by^2+cz^2=1 \tag{7}$$

$$\text{and} \quad lx+my+nz=p \tag{8}$$

be the equations of the central conicoid and the plane respectively.

If (α, β, γ) is the centre of the section, then the equation of the plane of section is

$$a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2. \tag{9}$$

It is identical with (8).

$$\therefore \quad \frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{a\alpha^2+b\beta^2+c\gamma^2}{p}=k(\text{say}).$$

$$\therefore \quad \alpha=\frac{kl}{a}, \beta=\frac{km}{b}, \gamma=\frac{kn}{c}$$

$$\text{and} \quad k=\frac{a\alpha^2+b\beta^2+c\gamma^2}{p}=\frac{k^2}{p}\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)$$

$$\text{or,} \quad k=\frac{p}{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}=\frac{p}{p_\circ^2}, \text{ where } p_\circ^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}.$$

The centre of the section is $\left(\frac{lp}{ap_\circ^2}, \frac{mp}{bp_\circ^2}, \frac{np}{cp_\circ^2}\right)$.

The equation of the conicoid referred to the centre as origin is

$$a\left(x+\frac{lp}{ap_\circ^2}\right)^2+b\left(y+\frac{mp}{bp_\circ^2}\right)^2+c\left(z+\frac{np}{cp_\circ^2}\right)^2=1$$

$$\text{or,} \quad ax^2+by^2+cz^2+\frac{2p}{p_\circ^2}(lx+my+nz)+\frac{p^2}{p_\circ^2}=1 \tag{10}$$

and the equation of the plane is

$$lx+my+nz=0. \tag{11}$$

Now the equation of the conic is

$$ax^2+by^2+cz^2+\frac{p^2}{p_\circ^2}=1, lx+my+nz=0. \tag{12}$$

Putting $1-\frac{p^2}{p_\circ^2}=d^2$, the equation of the conic becomes

$$\frac{a}{d^2}x^2+\frac{b}{d^2}y^2+\frac{c}{d^2}z^2=1, lx+my+nz=0. \tag{13}$$

The squares of the lengths of semi-axes are obtained from the equation

$$\frac{l^2}{ar^2-d^2}+\frac{m^2}{br^2-d^2}+\frac{n^2}{cr^2-d^2}=0. \tag{14}$$

[See the equation (5).]

If λ, μ, ν are d.rs. to the axis, then

$$\frac{\lambda(ar^2 - d^2)}{l} = \frac{\mu(br^2 - d^2)}{m} = \frac{\nu(cr^2 - d^2)}{n} \tag{15}$$

for the value of r^2 obtained from (14).

Corollary V. Area of the elliptic section.

If r_1^2 and r_2^2 are the roots of the equation (14), then the area of the section

$$= \pi r_1 r_2 = \pi d^2 \sqrt{\frac{l^2+m^2+n^2}{bcl^2+cam^2+abn^2}}$$
$$= \pi \left(1 - \frac{p^2abc}{bcl^2+cam^2+abn^2}\right) \sqrt{\frac{l^2+m^2+n^2}{bcl^2+cam^2+abn^2}}.$$

Corollary VI. Parallel plane sections.

The sections made by the planes $lx+my+nz=0$ and $lx+my+nz=p$ on $ax^2+by^2+cz^2=1$ are parallel. If α and β are the lengths of semi-axes of the central section, then those of non-central section are $d\alpha$ and $d\beta$ by (5) and (14).

Example 5. *Find the centre and d.rs. of axes of the section of*

$$11x^2 - 13y^2 - 4z^2 = 5, x+y+z=1.$$

Eliminating z from $11x^2-13y^2-4z^2=5$ by $x+y+z=1$, we have

$$11x^2 - 13y^2 - 4(1-x-y)^2 = 5 \quad \text{or,} \quad 7x^2 - 8xy - 17y^2 + 8x + 8y - 9 = 0.$$

It is the equation of the projection of the section on the plane $z=0$. Since the projection is a hyperbola the given section is a hyperbola.

If (α, β, γ) is the centre of this section, then the equation of it is $11\alpha x - 13\beta y - 4\gamma z = 11\alpha^2 - 13\beta^2 - 4\gamma^2$. It is identical with $x+y+z=1$.

$$\therefore \quad 11\alpha = -13\beta = -4\gamma = 11\alpha^2 - 13\beta^2 - 4\gamma^2 = k \text{ (say).}$$
$$\therefore \quad 11 \cdot \frac{k^2}{11^2} - 13 \cdot \frac{k^2}{13^2} - 4 \cdot \frac{k^2}{4^2} = k \quad \text{or,} \quad k = -\frac{4 \times 143}{135}.$$

Thus the centre is $\left(-\frac{52}{135}, \frac{44}{135}, \frac{143}{135}\right)$.
Here $p=1$ and $p_\circ^2 = \frac{5}{11} - \frac{5}{13} - \frac{5}{4} = -\frac{5\times135}{4\times143}$.

$$\therefore \; d^2 = 1 - \frac{p^2}{p_\circ^2} = 1 + \frac{4\times143}{5\times135} = \frac{1247}{675}.$$

The equation determining the lengths of semi-axes is

$$\frac{1}{\frac{11}{5}\cdot\frac{675}{1247}r^2 - 1} + \frac{1}{-\frac{13}{5}\cdot\frac{675}{1247}r^2 - 1} + \frac{1}{-\frac{4}{5}\cdot\frac{675}{1247}r^2 - 1} = 0$$
$$\text{or,} \quad 135^3r^4 - 12\times135\times1247r^2 - 3\times1247^2 = 0.$$

It gives that $r^2 = \frac{1247\times 27}{135^2}, -\frac{1247\times 15}{135^2}$.

If λ, μ, ν are d.rs. of the axis corresponding to $r^2 = \frac{1247\times 27}{135^2}$, then

$$\lambda\left(\frac{11}{5}\times\frac{675}{1247}\times\frac{1247\times 27}{135^2}-1\right)=\mu\left(-\frac{13}{5}\times\frac{675}{1247}\times\frac{1247\times 27}{135^2}-1\right)$$
$$=\nu\left(-\frac{4}{3}\times\frac{675}{1247}\times\frac{1247\times 27}{135^2}-1\right)$$

or, $\frac{6}{5}\lambda = -\frac{18}{5}\mu = -\frac{9}{5}\nu$ or, $\frac{\lambda}{3} = \frac{\mu}{-1} = \frac{\nu}{-2}$.

If λ', μ', ν' are d.rs. of the axis corresponding to $r^2 = -\frac{1247\times 15}{135^2}$, then

$$\lambda'\left(\frac{11}{5}\times\frac{675}{1247}\times\frac{-1247\times 15}{135^2}-1\right)=\mu'\left(-\frac{13}{5}\times\frac{675}{1247}\times\frac{-1247\times 15}{135^2}-1\right)$$
$$=\nu'\left(-\frac{4}{5}\times\frac{675}{1247}\times\frac{-1247\times 15}{135^2}-1\right)$$

or, $-\frac{20}{9}\lambda' = \frac{4}{9}\mu' = -\frac{5}{9}\nu'$ or, $\frac{\lambda'}{1} = \frac{\mu'}{-5} = \frac{\nu'}{4}$.

Thus the d.rs. of the axes of the section are $3, -1, -2$ and $1, -5, 4$.

(iii) **Axes of a plane section of a paraboloid.**

Let

$$ax^2 + by^2 = 2cz \tag{16}$$
$$\text{and} \quad lx + my + nz = p \tag{17}$$

be the equation of the paraboloid and the plane respectively. If (α, β, γ) is the centre of the section, then the equation (17) is represented by $a\alpha x + b\beta y - cz = a\alpha^2 + b\beta^2 - c\gamma$.

Comparing the coefficients, $\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{-c}{n} = \frac{a\alpha^2+b\beta^2-c\gamma}{p}$.

From these relations

$$\alpha = -\frac{cl}{an}, \beta = -\frac{cm}{bn}, c\gamma = a\alpha^2 + b\beta^2 + \frac{pc}{n} = \frac{c^2}{n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{np}{c}\right).$$

Writing $k = \frac{l^2}{a}+\frac{m^2}{b}+\frac{np}{c}$, the centre is $\left(-\frac{cl}{an}, -\frac{cm}{bn}, \frac{ck}{n^2}\right)$. Shifting the origin to this centre with the axes remaining parallel, the equation (16) transforms to

$$a\left(x-\frac{cl}{an}\right)^2 + b\left(y-\frac{cm}{bn}\right)^2 = 2c\left(z+\frac{ck}{n^2}\right)$$
$$\text{or,} \quad ax^2 + by^2 - \frac{2c}{n}(lx+my+nz) - \frac{c(ck+np)}{n^2} = 0 \tag{18}$$

and the equation (17) becomes

$$lx + my + nz = 0. \tag{19}$$

Now the equation of the conic is

$$ax^2 + by^2 = \frac{c(ck + np)}{n^2}, lx + my + nz = 0. \tag{20}$$

Writing $c(ck + np) = p_o^2$, the semi-diameters of length r of the conicoid $ax^2 + by^2 = \frac{p_o^2}{n^2}$ lie on the cone

$$ax^2 + by^2 = \frac{p_o^2}{n^2}\frac{x^2 + y^2 + z^2}{r^2},$$

$$\text{i.e.} \quad (an^2r^2 - p_o^2)x^2 + (bn^2r^2 - p_o^2)y^2 - p_o^2z^2 = 0. \tag{21}$$

In order that r^2 be the square of the length of semi-axes of the plane section by (19), the plane $lx + my + nz = 0$ must be a tangent plane to the cone (21). For this

$$\frac{l^2}{an^2r^2 - p_o^2} + \frac{m^2}{bn^2r^2 - p_o^2} - \frac{n^2}{p_o^2} = 0$$

$$\text{or,} \quad abn^6r^4 - \{(a + b)n^2 + am^2 + bl^2\}n^2p_o^2r^2 + p_o^4(l^2 + m^2 + n^2) = 0. \tag{22}$$

It is a quadratic equation in r^2. The roots r_1^2 and r_2^2 are the squares of the semi-axes of the section.

If λ, μ, ν are d.rs. of the axis of length $2r$, the plane (19) touches the cone (21) along the line $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$. Therefore, the plane (19) is represented by

$$(an^2r^2 - p_o^2)\lambda x + (bn^2r^2 - p_o^2)\mu y - p_o^2\nu z = 0.$$

$$\text{Consequently} \qquad \frac{(an^2r^2 - p_o^2)\lambda}{l} = \frac{(bn^2r^2 - p_o^2)\mu}{m} = \frac{-p_o^2\nu}{n}. \tag{23}$$

This gives the d.rs. of the axis corresponding to the value of r^2 obtained from the equation (22).

Corollary VII. *The section will be a rectangular hyperbola, if*

$$r_1^2 + r_2^2 = 0, \quad i.e. \quad (a + b)n^2 + am^2 + bl^2 = 0.$$

Corollary VIII. Area of the elliptic section.

$$Area = \pi r_1 r_2 = \frac{\pi p_o^2}{n^3}\sqrt{\frac{l^2 + m^2 + n^2}{ab}}.$$

Note. If $n = 0$, the section will be a parabola.

Example 6. *Show that the lengths of semi-axes of the section of the paraboloid* $2x^2 + y^2 = 12z$ *by the plane* $6x - 3y - 6z = 7$ *are* $\frac{1}{2}\sqrt{78}$ *and* $\frac{1}{2}\sqrt{39}$ *and the d.rs. of the axes are* $2, 2, 1$ *and* $1, -2, 2$.

Here $a = 2, b = 1, c = 6, l = 6, m = -3, n = -6, p = 7, p_o^2 = c^2\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{np}{c}\right) + cnp = 468$.

By the equation (22),

$$2\cdot 1\cdot 6^6 r^4 - \{(2+1)\cdot 36 + 2\cdot 9 + 1\cdot 36\}\cdot 36\cdot 468 r^2 + 468^2(36+9+36) = 0$$

or, $8r^4 - 13\times 18r^2 + 9\times 13^2 = 0$

or, $r^2 = \dfrac{13\times 18 \pm \sqrt{(13^2\times 18^2 - 4\times 8\times 9\times 13^2)}}{2\times 8} = \dfrac{78}{4}, \dfrac{39}{4}.$

Thus the lengths of semi-axes are $\frac{1}{2}\sqrt{78}$ and $\frac{1}{2}\sqrt{39}$.

If λ, μ, ν are the d.rs. corresponding to the length $\frac{1}{2}\sqrt{78}$, then

$$\frac{(2\times 36\times \frac{78}{4} - 468)\lambda}{6} = \frac{(1\times 36\times \frac{78}{4} - 468)\mu}{-3} = \frac{-468}{-6}\nu$$

or, $12\lambda = -6\mu = 6\nu$ or, $\dfrac{\lambda}{1} = \dfrac{\mu}{-2} = \dfrac{\nu}{2}.$

If λ', μ', ν' are the d.rs. corresponding to the length $\frac{1}{2}\sqrt{39}$, then

$$\frac{(2\times 36\times \frac{39}{4} - 468)\lambda'}{6} = \frac{(1\times 36\times \frac{39}{4} - 468)\mu'}{-3} = \frac{-468}{-6}\nu'$$

or, $3\lambda' = 3\mu' = 6\nu'$ or, $\dfrac{\lambda'}{2} = \dfrac{\mu'}{2} = \dfrac{\nu'}{1}.$

Thus the d.rs. of the axes of the section are $1, -2, 2$ and $2, 2, 1$.

WORKED-OUT EXAMPLES

1. *Prove that the section of the conicoid $ax^2+by^2+cz^2=1$ by a tangent plane to the cone $\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$ is a rectangular hyperbola.*

   The lengths of semi-axes of the section $ax^2+by^2+cz^2=1, lx+my+nz=0$ are given by

$$(bcl^2+cam^2+abn^2)r^4 - \{(b+c)l^2+(c+a)m^2+(a+b)n^2\}r^2 + l^2+m^2+n^2 = 0. \quad [(5) \text{ of Sec } 11.13]$$

   The section will be a rectangular hyperbola, if the sum of the squares of its semi-axes is zero.

   For this $(b+c)l^2+(c+a)m^2+(a+b)n^2=0$.

   It shows that the plane $lx+my+nz=0$ touches the cone

$$\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0.$$

2. *If A_1, A_2, A_3 are the areas of the three mutually perpendicular central sections of an ellipsoid, show that $\frac{1}{A_1^2}+\frac{1}{A_2^2}+\frac{1}{A_3^2}$ is constant.*

   Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ be an ellipsoid and $l_r x + m_r y + n_r z = 0 (r=1,2,3)$ be three mutually perpendicular central planes. Here $l_r^2+m_r^2+n_r^2=1$.

If A_1 be the area of the section by the plane $l_1x + m_1y + n_1z = 0$, then

$$A_1 = \frac{\pi abc\sqrt{l_1^2 + m_1^2 + n_1^2}}{\sqrt{a^2l_1^2 + b^2m_1^2 + c^2n_1^2}} = \frac{\pi abc}{\sqrt{a^2l_1^2 + b^2m_1^2 + c^2n_1^2}}.$$

$$\therefore \frac{1}{A_1^2} = \frac{a^2l_1^2 + b^2m_1^2 + c^2n_1^2}{(\pi abc)^2}.$$

Similarly, $$\frac{1}{A_2^2} = \frac{a^2l_2^2 + b^2m_2^2 + c^2n_2^2}{(\pi abc)^2}, \frac{1}{A_3^2} = \frac{a^2l_3^2 + b^2m_3^2 + c^2n_3^2}{(\pi abc)^2}.$$

$$\begin{aligned}\therefore \frac{1}{A_1^2} + \frac{1}{A_2^2} + \frac{1}{A_3^2} &= \frac{a^2(l_1^2 + l_2^2 + l_3^2) + b^2(m_1^2 + m_2^2 + m_3^2) + c^2(n_1^2 + n_2^2 + n_3^2)}{(\pi abc)^2} \\ &= \frac{a^2 + b^2 + c^2}{(\pi abc)^2} \quad \left[\because \sum l_1^2 = 1 = \sum m_1^2 = \sum n_1^2\right] \\ &= \text{constant.}\end{aligned}$$

3. *One axis of a central section of the conicoid $ax^2 + by^2 + cz^2 = 1$ lies in the plane $ux + vy + wz = 0$. Show that the other lies on the cone $(b-c)uyz + (c-a)vzx + (a-b)wxy = 0$.*

Let $\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$ and $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$ be the two axes. If the first lies on the given plane, then

$$ul_1 + vm_1 + wn_1 = 0. \tag{1}$$

Again $$l_1l_2 + m_1m_2 + n_1n_2 = 0 \tag{2}$$

and $$al_1l_2 + bm_1m_2 + cn_1n_2 = 0. \tag{3}$$

[See the Corollary III of Sec 11.13.]

Eliminating l_1, m_1, n_1 from (1), (2) and (3), we have

$$\begin{vmatrix} u & v & w \\ l_2 & m_2 & n_2 \\ al_2 & bm_2 & cn_2 \end{vmatrix} = 0 \quad \text{or,} \quad (b-c)um_2n_2 + (c-a)vn_2l_2 + (a-b)wl_2m_2 = 0.$$

With this condition the locus of $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$ is $(b-c)uyz + (c-a)vzx + (a-b)wxy = 0$.

4. *Prove that the area of the section of an ellipsoid by a plane which passes through the extremities of three conjugate semi-diameters is in a constant ratio to the area of the parallel central section.*

Let $P(x_1, y_1, z_1), Q(x_2, y_2, z_2), R(x_3, y_3, z_3)$ be the coordinates of the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

The equation of the plane PQR is

$$\frac{x_1 + x_2 + x_3}{a^2}x + \frac{y_1 + y_2 + y_3}{b^2}y + \frac{z_1 + z_2 + z_3}{c^2}z = 1. \tag{1}$$

Let this plane be $lx + my + nz = 1$.

The central plane parallel to it is $lx + my + nz = 0$.

If A and $A_\circ$ are areas of non-central and central sections, then

$$\frac{A}{A_\circ} = 1 - \frac{p^2}{p_\circ^2} = 1 - \frac{1}{\sum a^2 l^2}.$$

Here

$$\sum a^2 l^2 = \sum \frac{(x_1 + x_2 + x_3)^2}{a^2} = 3. \;\therefore\; \frac{A}{A_\circ} = 1 - \frac{1}{3} = \frac{2}{3}.$$

5. *Planes are drawn through a fixed point (α, β, γ) so that their sections of the paraboloid $ax^2 + by^2 = 2z$ are rectangular hyperbolas. Prove that they touch the cone*

$$\frac{(x-\alpha)^2}{b} + \frac{(y-\beta)^2}{a} + \frac{(z-\gamma)^2}{a+b} = 0.$$

The equation of a plane through the point (α, β, γ) can be written as

$$l(x-\alpha) + m(y-\beta) + n(z-\gamma) = 0$$
$$\text{or,} \quad lx + my + nz = l\alpha + m\beta + n\gamma = p \text{ (say)}. \tag{1}$$

The lengths of semi-axes are given by the equation

$$abn^6 r^4 - \{(a+b)n^2 + am^2 + bl^2\}n^2 p_\circ^2 r^2 + (l^2 + m^2 + n^2)p_\circ^4 = 0,$$

$$\text{where} \quad p_\circ^2 = \frac{l^2}{a} + \frac{m^2}{b} + 2np = \frac{l^2}{a} + \frac{m^2}{b} + 2n(l\alpha + m\beta + n\gamma).$$

[See equation (22) of Sec 11.13.]

The section will be rectangular hyperbola, if the sum of the squares of semi-axes is zero. In this case

$$(a+b)n^2 + am^2 + bl^2 = 0. \tag{2}$$

The normal to the plane through the point (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}. \tag{3}$$

By (2) and (3) the normals generate the cone

$$(a+b)(z-\gamma)^2 + a(y-\beta)^2 + b(x-\alpha)^2 = 0. \tag{4}$$

The plane (1) touches the reciprocal cone of (4).

Thus the required cone is

$$\frac{(x-\alpha)^2}{b} + \frac{(y-\beta)^2}{a} + \frac{(z-\gamma)^2}{a+b} = 0.$$

EXERCISE XIA

1. Show that the curve

   (i) $3x^2 - 2y^2 + z^2 = 1, x - y + z = 1$ is a hyperbola;

   (ii) $xy + yz + zx + 8 = 0, x + y + z = 0$ is a circle;

   (iii) $2x^2 + y^2 - 2z^2 = 1, x + 2y + 3z - 3 = 0$ is a parabola;

   (iv) $x^2 + 2y^2 + 3z^2 = 1, 2x + 3y + 4z = 1$ is an ellipse.

2. Find the lengths and equations of axis of the following conics.

   (i) $x^2 + y^2 + \frac{z^2}{4} = 1, 2x + 2y + z = 0.$

   (ii) $x^2 + 7y^2 - 10z^2 + 9 = 0, x + 2y + 3z = 0.$

   (iii) $2x^2 - y^2 - z = 0, x + 2y + z = 4.$

3. Find the centre, lengths and equations of the following conics.

   (i) $3x^2 + 3y^2 + 6z^2 = 10, x + y + z = 1,$

   (ii) $14x^2 + 5y^2 - 4z^2 = 72, 2x - y - 2z + 36 = 0.$

   (iii) $3x^2 - 2z^2 = 6y, 3x - 3y + 4z + 2 = 0.$

4. Find the area of the following sections.

   (i) $x^2 - 20y^2 - 6z^2 + 8 = 0, 3x + 6y + 2z = 0.$

   (ii) $x^2 + 2y^2 + 3z^2 = 1, 2x + 3y + 4z = 3.$

   (iii) $2x^2 + y^2 = z, x + 2y + z = 4.$

5. Show that the section $2x^2 + y^2 - 2z^2 = 1, x + 2y + 3z = 4$ is a parabola, the d.rs. of whose axis are $1, 4, -3$.

6. Prove that the equation of the conic $x^2 + 2y^2 - 2z^2 = 1, 3x - 2y - z = 0$, referred to its principal axes, is approximately $1.7x^2 - 1.77y^2 = 1$.

7. Find the lengths and equations of axes of the section of the ellipsoid of revolution $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ by the plane $lx + my + nz = 0$.

8. Planes through the origin cut the quadric $ax^2 + by^2 + cz^2 = 1$ in rectangular hyperbolas. Show that the normals to the planes through the origin lie on a quadric cone.

   [*Hints.* Let the plane $lx + my + nz = 0$ cut the quadric in a rectangular hyperbola.

   $\therefore (b + c)l^2 + (c + a)m^2 + (a + b)n^2 = 0.$

   The normal to the plane through the origin is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. Thus the normals lie on $(b + c)x^2 + (c + a)y^2 + (a + b)z^2 = 0.$]

9. Prove that the central section of an ellipsoid whose area is constant touches a cone of second degree.

[*Hints.* Let the section be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, lx+my+nz=0$ and the area be $\frac{\pi abc}{k}$.

$$\therefore \quad \frac{\pi abc\sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}=\frac{\pi abc}{k}$$

$$\text{or,}\quad l^2(a^2-k^2)+m^2(b^2-k^2)+n^2(c^2-k^2)=0.$$

Therefore, the normal $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ to the plane is a generator of the cone $x^2(a^2-k^2)+y^2(b^2-k^2)+z^2(c^2-k^2)=0$. Hence the plane touches the reciprocal cone $\frac{x^2}{a^2-k^2}+\frac{y^2}{b^2-k^2}+\frac{z^2}{c^2-k^2}=0$.]

10. Find the angle between the asymptotes of the conic $ax^2+by^2+cz^2=1, lx+my+nz=p$.

[*Hints.* If r_1^2 and r_2^2 are the squares of semi-axes of conic and θ is the angle between the asymptotes, then $\tan\frac{\theta}{2}=\sqrt{\frac{-r_2^2}{r_1^2}}$.

$$\therefore \tan^2\theta=\frac{-4r_1^2r_2^2}{(r_1^2+r_2^2)^2}=\frac{-4(l^2+m^2+n^2)(bcl^2+cam^2+abn^2)}{\{(b+c)l^2+(c+a)m^2+(a+b)n^2\}^2}.]$$

11. Find the locus of the centres of sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose areas are constant $(=\pi k^2)$.

12. Prove that the locus of the centres of sections of the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$, which are of constant area πk^2 is

$$a^2b^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-2z\right)^2=k^4.$$

13. Prove that the axes of the sections of the conicoid $ax^2+by^2+cz^2=1$ which pass through the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ lie on the cone

$$\frac{b-c}{x}(mz-ny)+\frac{c-a}{y}(nx-lz)+\frac{a-b}{z}(ly-mx)=0.$$

[*Hints.* Let the axes of the section through the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}\cdots(1)\text{ be }\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}\cdots(2)\text{ and }\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}.\cdots(3)$$

For the conditions of axes

$l_1l_2+m_1m_2+n_1n_2=0\cdots$ (4) and $al_1l_2+bm_1m_2+cn_1n_2=0.\cdots$ (5)

The lines (1), (2) and (3) are coplanar.

$$\therefore \quad \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$$

$$\text{or,} \quad l_1(m_2n - n_2m) + m_1(n_2l - l_2n) + n_1(l_2m - m_2l) = 0. \tag{6}$$

Eliminating l_1, m_1, n_1 from (4), (5) and (6),

$$\begin{vmatrix} l_2 & m_2 & n_2 \\ al_2 & bm_2 & cn_2 \\ m_2n - n_2m & n_2l - l_2n & l_2m - m_2l \end{vmatrix} = 0.$$

Now eliminating l_2, m_2, n_2 by (3), the required equation is obtained.]

14. Prove that the axes of the section of

$$f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$$

by the plane $lx + my + nz = 0$ are given by

$$r^4(Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm) \\ + r^2\{f(l, m, n) - (a + b + c)(l^2 + m^2 + n^2)\} + l^2 + m^2 + n^2 = 0,$$

where $A = bc - f^2$, etc.

ANSWERS

2. (i) $2, 2\sqrt{3}; \frac{x}{1} = \frac{y}{1} = \frac{z}{-4}, \frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$.

 (ii) transverse axis 6, conjugate axis $6\sqrt{\frac{14}{47}}; \frac{x}{6} = \frac{y}{3} = \frac{z}{-4}, \frac{x}{17} = \frac{y}{-22} = \frac{z}{9}$.

 (iii) $10.56, 3.38; \frac{x}{22.58} = \frac{y}{101.27} = \frac{z}{-226.0544}, \frac{x}{4.51} = \frac{y}{-2.44} = \frac{z}{-.5412}$.

3. (i) $\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right); \sqrt{\frac{44}{15}}, \frac{\sqrt{44}}{5}; \frac{x-2/5}{1} = \frac{y-2/5}{1} = \frac{z-1/5}{0}, \frac{x-2/5}{1} = \frac{y-2/5}{1} = \frac{z-1/5}{-2}$.

 (ii) $(10, -14, 35); 72, 36; \frac{x-10}{1} = \frac{y+14}{-2} = \frac{z-35}{2}, \frac{x-10}{2} = \frac{y+14}{2} = \frac{z-35}{1}$.

 (iii) $(1, -1, -2); \frac{25\pm\sqrt{241}}{36}; \frac{(13+\sqrt{241})(x-1)}{12} = \frac{y+1}{-1} = \frac{(7+\sqrt{241})(z+2)}{-24}$,
 $\frac{(13-\sqrt{241})(x-1)}{12} = \frac{y+1}{-1} = \frac{(7-\sqrt{241})(z+2)}{-24}$.

4. (i) Lengths of semi-axes $2, 1$; area$=2\pi$.

 (ii) Lengths of semi-axes $\frac{\sqrt{29}}{83}\sqrt{52 \pm \sqrt{297}}$; area $= \pi\left(\frac{29}{83}\right)^{3/2}$

 (iii) Lengths of semi-axes $5.28, 1.68$; area $= 8.8704\pi$.

7. $a, ac(l^2 + m^2 + n^2)^{1/2}\{a^2(l^2 + m^2) + c^2n^2\}^{-1/2}$;
 $\frac{x}{m} = \frac{y}{-l} = \frac{z}{0}, \frac{x}{nl} = \frac{y}{mn} = \frac{z}{-(l^2+m^2)}$.

11. $a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)^2 = k^4\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$.

11.20 Circular sections

If $F = 0$ is the equation of a conicoid, then it can be shown into the form $S+\lambda uv = 0$, where $S = 0$ is the equation to a sphere and $u = 0, v = 0$ represent planes. Thus the common points of the conicoid and the planes lie on the sphere $S = 0$. Consequently the sections of the conicoid by the planes $u = 0$ and $v = 0$ are circles.

(i) Circular sections of an ellipsoid.

Let the equation of the ellipsoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. It can be written as in the following forms.

$$\frac{x^2+y^2+z^2}{a^2} - 1 + y^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right) = 0, \tag{1}$$

$$\frac{x^2+y^2+z^2}{b^2} - 1 + z^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right) + x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0, \tag{2}$$

$$\frac{x^2+y^2+z^2}{c^2} - 1 + x^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{c^2}\right) = 0. \tag{3}$$

Hence the planes

$$y^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right) = 0, \tag{4}$$

$$z^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right) + x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0 \tag{5}$$

and $$x^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{c^2}\right) = 0 \tag{6}$$

cut the ellipsoid in circles of radii a, b, c respectively. If $a > b > c$, then $\frac{1}{a^2} < \frac{1}{b^2} < \frac{1}{c^2}$. In this case, the equations (4) and (6) represent imaginary planes and the real circular sections are given by the planes of (5). Hence the planes of circular sections are $\frac{x}{a}\sqrt{a^2-b^2} \pm \frac{z}{c}\sqrt{b^2-c^2} = 0$.

Since parallel plane sections are similar and similarly situated conics, the planes $\frac{x}{a}\sqrt{a^2-b^2} + \frac{z}{c}\sqrt{b^2-c^2} = \lambda$ and $\frac{x}{a}\sqrt{a^2-b^2} - \frac{z}{c}\sqrt{b^2-c^2} = \mu$ intersect the ellipsoid in circles for all values of λ and μ.

Note. If $a > b > c$, the corresponding sphere is $x^2+y^2+z^2 = b^2$.

One important property. *Any two circular sections of an ellipsoid of opposite systems lie on a sphere.*

Let $\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2} = \lambda$ and $\frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2} = \mu$ be the equations of the planes of any two circular sections of opposite systems.

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 + \frac{1}{k}\left(\frac{x}{a}\sqrt{a^2-b^2} + \frac{z}{c}\sqrt{b^2-c^2} - \lambda\right)$$
$$\times\left(\frac{x}{a}\sqrt{a^2-b^2} - \frac{z}{c}\sqrt{b^2-c^2} - \mu\right) = 0 \tag{7}$$

represents a conicoid through the two circular sections for all values of k. It will represent a sphere if the coefficients at x^2, y^2, z^2 in (7) are equal, i.e.

$$\frac{k}{a^2} + \frac{a^2 - b^2}{a^2} = \frac{k}{c^2} - \frac{b^2 - c^2}{c^2} = \frac{k}{b^2}.$$

It implies that

$$\frac{k}{a^2} + 1 - \frac{b^2}{a^2} = \frac{k}{b^2} \quad \text{or,} \quad k\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = b^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \quad \text{or,} \quad k = b^2.$$

Thus the sphere through the circles is

$$x^2 + y^2 + z^2 - \frac{(\lambda + \mu)\sqrt{a^2 - b^2}}{a}x + \frac{(\lambda - \mu)\sqrt{b^2 - c^2}}{c}z + \lambda\mu - b^2 = 0.$$

Hence the two circles lie on a sphere.

Example 7. *Find the real central circular sections of the ellipsoid* $x^2+2y^2+6z^2 = 8$.

$$x^2 + 2y^2 + 6z^2 = 8 \quad \text{or,} \quad \frac{x^2}{8} + \frac{y^2}{4} + \frac{z^2}{4/3} = 1.$$

It can be written as

$$\frac{x^2 + y^2 + z^2}{4} - 1 + x^2\left(\frac{1}{8} - \frac{1}{4}\right) + z^2\left(\frac{3}{4} - \frac{1}{4}\right) = 0$$

$$\text{or,} \quad \frac{x^2 + y^2 + z^2}{4} - 1 - \frac{x^2}{8} + \frac{z^2}{2} = 0.$$

Thus the real sections are given by the planes

$$-\frac{x^2}{8} + \frac{z^2}{2} = 0 \quad \text{or,} \quad x^2 - 4z^2 = 0 \quad \text{or,} \quad x \pm 2z = 0.$$

(ii) **Circular sections of a hyperboloid.**

(a) Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, where $a > b > c$, be the equation of the hyperboloid of one sheet.

This equation can be written as

$$\frac{x^2 + y^2 + z^2}{a^2} - 1 + y^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) - z^2\left(\frac{1}{a^2} + \frac{1}{c^2}\right) = 0$$

$$\text{or,} \quad \frac{x^2 + y^2 + z^2}{b^2} - 1 + x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) - z^2\left(\frac{1}{b^2} + \frac{1}{c^2}\right) = 0$$

$$\text{or,} \quad -\frac{x^2 + y^2 + z^2}{c^2} - 1 + x^2\left(\frac{1}{a^2} + \frac{1}{c^2}\right) + y^2\left(\frac{1}{b^2} + \frac{1}{c^2}\right) = 0.$$

Hence the planes

$$y^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) - z^2\left(\frac{1}{a^2} + \frac{1}{c^2}\right) = 0, \tag{8}$$

$$x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) - z^2\left(\frac{1}{b^2} + \frac{1}{c^2}\right) = 0 \tag{9}$$

$$\text{and} \quad x^2\left(\frac{1}{a^2} + \frac{1}{c^2}\right) + y^2\left(\frac{1}{b^2} + \frac{1}{c^2}\right) = 0 \tag{10}$$

may cut the hyperboloid in circles of radii a, b and c respectively.

Since $a > b > c$, the real sections are only obtained from the planes of (8). Consequently all circular sections are given by

$$y\frac{\sqrt{a^2-b^2}}{b}+z\frac{\sqrt{a^2+c^2}}{c}=\lambda \quad \text{and} \quad y\frac{\sqrt{a^2-b^2}}{b}-z\frac{\sqrt{a^2+c^2}}{c}=\mu$$

for all values of λ and μ.

(b) Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, where $a > b > c$, be the equation of the hyperboloid of two sheets.

This equation can be put in any one of the following forms.

$$\frac{x^2+y^2+z^2}{a^2}-1-y^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-z^2\left(\frac{1}{a^2}+\frac{1}{c^2}\right)=0$$

$$\text{or,}\quad \frac{x^2+y^2+z^2}{b^2}+1-x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)=0$$

$$\text{or,}\quad \frac{x^2+y^2+z^2}{c^2}+1-x^2\left(\frac{1}{a^2}+\frac{1}{c^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)=0.$$

Hence the planes

$$y^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+z^2\left(\frac{1}{a^2}+1c^2\right)=0, \tag{11}$$

$$x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)=0 \tag{12}$$

$$\text{and}\quad x^2\left(\frac{1}{a^2}+\frac{1}{c^2}\right)-y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)=0 \tag{13}$$

may cut the hyperboloid in circular sections.

Since $a > b > c$, the real planes are obtained from (12). These planes are $\frac{x}{a}\sqrt{a^2+b^2}\pm\frac{z}{c}\sqrt{b^2-c^2}=0$ which do not meet the hyperboloid of two sheets in any real point. The real circular sections are given by

$$\frac{x}{a}\sqrt{a^2+b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda \quad \text{and} \quad \frac{x}{a}\sqrt{a^2+b^2}-\frac{z}{c}\sqrt{b^2-c^2}=\mu$$

for some values of λ and μ.

> **Note.** The circular section of the hyperboloid of two sheets will be real, if λ^2 or, $\mu^2 > c^2 + a^2$.
>
> This condition is obtained by considering the points of intersection between the hyperboloid and a plane of λ or, μ system on the plane $y = 0$.

Example 8. *Prove that the planes $2x + 3z - 5 = 0, 2x - 3z + 7 = 0$ meet the hyperboloid $-x^2 + 3y^2 + 12z^2 = 75$ in circles, which lie on the sphere $3x^2 + 3y^2 + 3z^2 + 4x + 36z - 110 = 0$.*

The equation of the hyperboloid can be written as $-\frac{x^2}{75}+\frac{y^2}{25}+\frac{4z^2}{25}=1$.

This can be put in the following forms.

$$\frac{x^2+y^2+z^2}{-75}-1+y^2\left(\frac{1}{25}+\frac{1}{75}\right)+z^2\left(\frac{4}{25}+\frac{1}{75}\right)=0$$

$$\text{or,}\quad \frac{x^2+y^2+z^2}{25}-1+x^2\left(-\frac{1}{75}-\frac{1}{25}\right)+z^2\left(\frac{4}{25}-\frac{1}{25}\right)=0$$

$$\text{or,}\quad \frac{x^2+y^2+z^2}{25/4}-1+x^2\left(-\frac{1}{75}-\frac{4}{25}\right)+y^2\left(\frac{1}{25}-\frac{4}{25}\right)=0.$$

The real circular sections are obtained from

$$x^2\left(-\frac{1}{75}-\frac{1}{25}\right)+z^2\left(\frac{4}{25}-\frac{1}{25}\right)=0,\quad \text{i.e.}\quad 4x^2-9z^2=0.$$

Therefore, the circular sections through the origin are given by $2x+3z=0$ and $2x-3z=0$.

Since $2x+3z-5=0$ and $2x-3z+7=0$ are parallel to $2x+3z=0$ and $2x-3z=0$ respectively, these meet the hyperboloid in circles.

$$-\frac{x^2}{75}+\frac{y^2}{25}+\frac{4z^2}{25}-1+k(2x+3z-5)(2x-3z+7)=0$$

represents a conicoid through the intersection of the hyperboloid and the planes $2x+3z-5=0, 2x-3z+7=0$. This will be a sphere, if the coefficients of x^2, y^2, z^2 are equal, i.e.

$$4k-\frac{1}{75}=\frac{1}{25}=\frac{4}{25}-9k. \text{ It gives that } k=\frac{1}{75}.$$

Thus the sphere through the circle is

$$\left(\frac{4}{75}-\frac{1}{75}\right)x^2+\frac{y^2}{25}+\left(\frac{4}{25}-\frac{9}{75}\right)z^2+\frac{4}{75}x+\frac{36}{75}z-\left(1+\frac{35}{75}\right)=0$$

$$\text{or,}\quad 3x^2+3y^2+3z^2+4x+36z-110=0.$$

(iii) **Circular sections of a paraboloid.**

Let the equation of the paraboloid be

$$ax^2+by^2=2cz. \tag{14}$$

The equation may be written in the forms

$$a\left(x^2+y^2+z^2-\frac{2cz}{a}\right)-(a-b)y^2-az^2=0 \tag{15}$$

$$\text{or,}\quad b\left(x^2+y^2+z^2-\frac{2cz}{b}\right)+(a-b)x^2-bz^2=0 \tag{16}$$

$$\text{or,}\quad ax^2+by^2-(0\cdot x^2+0\cdot y^2+0\cdot z^2+2cz)=0. \tag{17}$$

If $a>b>0$, then $(a-b)x^2-bz^2=0$ represents real planes which meet the paraboloid in circles, where the sphere $x^2+y^2+z^2-\frac{2cz}{b}=0$ intersects the paraboloid. The systems of circular sections are given by

$$\sqrt{a-b}\,x+\sqrt{b}\,z=\lambda,\ \sqrt{a-b}\,x-\sqrt{b}\,z=\mu.$$

If a and b have opposite signs, then the only real planes are given by $ax^2+by^2=0$. Since $0\cdot x^2+0\cdot y^2+0\cdot z^2+2cz=0$ is the limiting form of $kx^2+ky^2+kz^2+k\left(z+\frac{c}{k}\right)^2=\frac{c^2}{k}$ as k tends to zero, the circular sections in this case are circles of infinite radius, i.e. straight lines. They are straight lines in which the plane $z=0$ cuts the surface.

Note. Hyperbolic paraboloids have no real circular sections.

Example 9. *Show that the plane* $7x+2z=5$ *cuts the paraboloid* $53x^2+4y^2=8z$ *in circular section of radius* $\sqrt{\frac{314}{53}}$.

The given equation of the paraboloid can be put in the following forms.

$$53\left(x^2+y^2+z^2-\frac{8}{53}z\right)-49y^2-53z^2=0$$

$$\text{or,}\quad 4(x^2+y^2+z^2-2z)+49x^2-4z^2=0$$

$$\text{or,}\quad 53x^2+4y^2-(0\cdot x^2+0\cdot y^2+0\cdot z^2+8z)=0.$$

Evidently the real circular sections are given by the planes parallel to the planes $49x^2-4z^2=0$, i.e. $7x\pm 2z=0$. Thus $7x+2z=5$ cuts the paraboloid in a circular section.

The area of the section $ax^2+by^2=2cz, lx+my+nz=p$ is

$$\frac{\pi p_o^2}{n^3}\sqrt{\frac{l^2+m^2+n^2}{ab}},\quad \text{where } p_o^2=c^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{2np}{c}\right).$$

If r is the radius of the circular section, then

$$r^2=\frac{p_o^2}{n^3}\sqrt{\frac{l^2+m^2+n^2}{ab}}=\frac{16}{8}\left(\frac{49}{53}+\frac{2\times 2\times 5}{4}\right)\sqrt{\frac{49+4}{53\times 4}}$$

$$=2\times\frac{314}{53}\times\frac{1}{2}=\frac{314}{53}.$$

$$\therefore\ r=\sqrt{\frac{314}{53}}.$$

11.21 Umbilics

Definition. *A point on a quadric such that the planes parallel to the tangent plane at the point determine circular sections is called an umbilic.*

An umbilic is a point circle of a system of parallel circular sections and it is the extremity of the diameter passing through the centres of the system of parallel circular sections. In Fig. 45 P and P' are umbilics, AB is the tangent plane at P and PP' is a diameter.

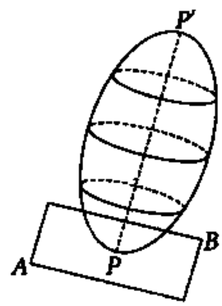

Fig. 45

(i) **Umbilics for an ellipsoid.**

Let $P(\alpha, \beta, \gamma)$ be an umbilic of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, a > b > c.$$

The tangent plane at P is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1. \tag{1}$$

It should be parallel to either of the central circular sections

$$\frac{x}{a}\sqrt{a^2 - b^2} \pm \frac{z}{c}\sqrt{b^2 - c^2} = 0. \tag{2}$$

Comparing (1) and (2),

$$\frac{\alpha/a^2}{\sqrt{a^2 - b^2}/a} = \frac{\beta/b^2}{0} = \frac{\pm\gamma/c^2}{\sqrt{b^2 - c^2}/c} = \pm\frac{\sqrt{\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right)}}{\sqrt{(a^2 - b^2 + b^2 - c^2)}} = \frac{\pm 1}{\sqrt{a^2 - c^2}}$$

as (α, β, γ) lies on the ellipsoid.

$$\therefore \ \alpha = \pm\frac{a\sqrt{a^2 - b^2}}{\sqrt{a^2 - c^2}}, \beta = 0, \gamma = \pm\frac{c\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}}.$$

Since $a > b > c$, the real umbilics are four points, namely

$$\left(\frac{a\sqrt{a^2 - b^2}}{\sqrt{a^2 - c^2}}, 0, \frac{c\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}}\right), \left(\frac{-a\sqrt{a^2 - b^2}}{\sqrt{a^2 - c^2}}, 0, \frac{c\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}}\right),$$
$$\left(\frac{a\sqrt{a^2 - b^2}}{\sqrt{a^2 - c^2}}, 0, \frac{-c\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}}\right), \left(\frac{-a\sqrt{a^2 - b^2}}{\sqrt{a^2 - c^2}}, 0, \frac{-c\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}}\right)$$

Note. Since $\alpha^2 + \beta^2 + \gamma^2 = \frac{a^2(a^2-b^2)}{a^2-c^2} + \frac{c^2(b^2-c^2)}{a^2-c^2} = a^2 + c^2 - b^2$, the umbilics lie on the sphere $x^2 + y^2 + z^2 = a^2 + c^2 - b^2$.

Example 10. *Find the umbilics of* $\frac{x^2}{36} + \frac{y^2}{16} + \frac{z^2}{9} = 1$.

Here $a^2 = 36, b^2 = 16, c^2 = 9$, i.e. $a > b > c$.

The umbilics are

$$\left(\pm\frac{a\sqrt{a^2 - b^2}}{\sqrt{a^2 - c^2}}, 0, \frac{\pm c\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}}\right),$$

$$\text{i.e.} \quad \left(\pm\frac{6\sqrt{36 - 16}}{\sqrt{36 - 9}}, 0, \frac{\pm 3\sqrt{16 - 9}}{\sqrt{36 - 9}}\right), \quad \text{i.e.} \quad \left(\frac{\pm 4\sqrt{5}}{\sqrt{3}}, 0, \frac{\pm\sqrt{7}}{\sqrt{3}}\right).$$

(ii) **Umbilics for a hyperboloid of one sheet.**

Let $P(\alpha, \beta, \gamma)$ be an umbilic of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, (a > b > c)$.

The tangent plane at P is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - \frac{\gamma z}{c^2} = 1. \tag{3}$$

It should be parallel to either of the central circular sections

$$\frac{y}{b}\sqrt{a^2 - b^2} \pm \frac{z}{c}\sqrt{a^2 + c^2} = 0.$$

Comparing it with (1),

$$\frac{\alpha/a^2}{0} = \frac{\beta/b^2}{\frac{1}{b}\sqrt{a^2 - b^2}} = \frac{-\gamma/c^2}{\pm\frac{1}{c}\sqrt{a^2 + c^2}}$$

$$\text{or,}\quad \frac{\alpha/a}{0} = \frac{\beta/b}{\sqrt{a^2 - b^2}} = \frac{-\gamma/c}{\sqrt{a^2 + c^2}} = \pm\frac{\sqrt{\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - \frac{\gamma^2}{c^2}\right)}}{\sqrt{(a^2 - b^2 - a^2 - c^2)}} = \frac{\pm 1}{i\sqrt{b^2 + c^2}}.$$

So (α, β, γ) is an imaginary point, i.e. the hyperboloid of one sheet has no umbilic.

(iii) **Umbilics for a hyperboloid of two sheets.**

Let $P(\alpha, \beta, \gamma)$ be an umbilic of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

It should be parallel to either of the central circular sections

$$\frac{x}{a}\sqrt{a^2 + b^2} \pm \frac{z}{c}\sqrt{b^2 - c^2} = 0.$$

Comparing it with (1),

$$\frac{\alpha/a^2}{\frac{\sqrt{a^2+b^2}}{a}} = \frac{-\beta/b^2}{0} = \frac{-\gamma/c}{\pm\frac{1}{c}\sqrt{b^2 - c^2}}$$

$$\text{or,}\quad \frac{\alpha/a}{\sqrt{a^2 + b^2}} = \frac{-\beta/b}{0} = \frac{-\gamma/c}{\pm\sqrt{b^2 - c^2}} = \frac{\pm\sqrt{\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - \frac{\gamma^2}{c^2}}}{\sqrt{a^2 + b^2 - (b^2 - c^2)}} = \frac{\pm 1}{\sqrt{a^2 + c^2}}.$$

It implies that

$$\alpha = \frac{\pm a\sqrt{a^2 + b^2}}{\sqrt{a^2 + c^2}}, \beta = 0, \gamma = \frac{\pm c\sqrt{b^2 - c^2}}{\sqrt{a^2 + c^2}}.$$

The real umbilics are four points, namely

$$\left(\frac{a\sqrt{a^2 + b^2}}{\sqrt{a^2 + c^2}}, 0, \frac{c\sqrt{b^2 - c^2}}{\sqrt{a^2 + c^2}}\right), \left(\frac{a\sqrt{a^2 + b^2}}{\sqrt{a^2 + c^2}}, 0, \frac{-c\sqrt{b^2 - c^2}}{\sqrt{a^2 + c^2}}\right),$$
$$\left(\frac{-a\sqrt{a^2 + b^2}}{\sqrt{a^2 + c^2}}, 0, \frac{c\sqrt{b^2 - c^2}}{\sqrt{a^2 + c^2}}\right), \left(\frac{-a\sqrt{a^2 + b^2}}{\sqrt{a^2 + c^2}}, 0, \frac{-c\sqrt{b^2 - c^2}}{\sqrt{a^2 + c^2}}\right).$$

(iv) **Umbilics for a paraboloid.**

Let $P(\alpha, \beta, \gamma)$ be an umbilic of the paraboloid $ax^2 + by^2 = 2cz(a > b > 0)$.

The tangent plane at P is

$$a\alpha x + b\beta y = c(z + \gamma). \tag{1}$$

It should be parallel to one of the circular sections

$$x\sqrt{a-b} + z\sqrt{b} = \lambda \quad \text{or,} \quad x\sqrt{a-b} - z\sqrt{b} = \mu.$$

$$\therefore \frac{a\alpha}{\sqrt{a-b}} = \frac{b\beta}{0} = \frac{c}{\pm\sqrt{b}} \Rightarrow \alpha = \pm\frac{c\sqrt{a-b}}{a\sqrt{b}}, \beta = 0.$$

From the equation of the paraboloid

$$2c\gamma = a\alpha^2 + b\beta^2 = \frac{c^2(a-b)}{ab} \quad \text{or,} \quad \gamma = \frac{c(a-b)}{2ab}.$$

Hence $\left[\pm\frac{c\sqrt{a-b}}{a\sqrt{b}}, 0, \frac{c(a-b)}{2ab}\right]$ are two real umbilics of the paraboloid.

Example 11. *Find the umbilics of the paraboloid* $4x^2 + 5y^2 = 40z$.

Here $a = 4, b = 5, c = 20$ and $b > a$.

The umbilics are $\left[0, \pm\frac{c}{b}\sqrt{\frac{b-a}{a}}, \frac{c(b-a)}{2ab}\right]$, i.e. $\left(0, \pm 2, \frac{1}{2}\right)$.

11.30 (i) Circular sections of any central conicoid

$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$ represents a central conicoid. It may be written as

$$(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) - \lambda(x^2 + y^2 + z^2) \\ + \lambda(x^2 + y^2 + z^2 - 1/\lambda) = 0.$$

If $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - \lambda(x^2 + y^2 + z^2) = 0$ represents a pair of planes, they will cut the given conicoid in circles. For a pair of planes

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0.$$

It gives three real values of λ but real planes are obtained only for the mean value.

(ii) **To find the conditions that the equations**

$$\boldsymbol{f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1}$$

and $\boldsymbol{lx + my + nz = 0}$ **should determine a circle.**

The equation of the conicoid can be written as

$$f(x,y,z)-\lambda(x^2+y^2+z^2)+\lambda(x^2+y^2+z^2-1/\lambda)=0.$$

The section $f(x,y,z)=1, lx+my+nz=0$ will be a circle, if $f(x,y,z)-\lambda(x^2+y^2+z^2)=0$ represents a pair of planes one of which is $lx+my+nz=0$.

$$\begin{aligned}\therefore\ f(x,y,z)&-\lambda(x^2+y^2+z^2)\\ &=(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2fyz+2gzx+2hxy\\ &=(lx+my+nz)\left\{\frac{a-\lambda}{l}x+\frac{b-\lambda}{m}y+\frac{c-\lambda}{n}z\right\}.\end{aligned} \tag{1}$$

Here we assume that none of l,m,n is zero.

Equating the coefficients of like powers of x,y,z, we have

$$\left.\begin{aligned}2f&=\frac{n}{m}(b-\lambda)+\frac{m}{n}(c-\lambda), 2g=\frac{l}{n}(c-\lambda)+\frac{n}{l}(a-\lambda)\\ \text{and}\quad 2h&=\frac{l}{m}(b-\lambda)+\frac{m}{l}(a-\lambda).\end{aligned}\right\}. \tag{2}$$

From the first of (2),

$$\lambda\left(\frac{n}{m}+\frac{m}{n}\right)=\frac{bn}{m}+\frac{cm}{n}-2f \quad \text{or,} \quad \lambda=\frac{bn^2+cm^2-2fmn}{m^2+n^2}.$$

Similarly finding the values of λ from the second and third of (2), the required conditions are

$$\frac{bn^2+cm^2-2fmn}{m^2+n^2}=\frac{cl^2+an^2-2gnl}{n^2+l^2}=\frac{am^2+bl^2-2hlm}{l^2+m^2}. \tag{3}$$

If $l=0$, then $\lambda=\frac{cl^2+an^2-2gnl}{n^2+l^2}=\frac{an^2}{n^2}=a$.

Now from (1),

$$(b-a)y^2+(c-a)z^2+2fyz+2gzx+2hxy=(my+nz)\left(\frac{b-a}{m}y+\frac{c-a}{n}z\right).$$

Equating the coefficients of like powers of x,y,z,

$$g=0=h \text{ and } \frac{m}{n}(c-a)+\frac{n}{m}(b-a)=2f \text{ or, } (c-a)m^2+(b-a)n^2=2fmn.$$

Thus the conditions are

$$g=h=0 \quad \text{and} \quad (c-a)m^2+(b-a)n^2-2fmn=0.$$

Example 12. *Show that the plane $x+y-z=0$ cuts the conicoid $4x^2+2y^2+z^2+3yz+zx-1=0$ in a circle. What is the radius of the circle?*

The equation can be written as

$$(4x^2+2y^2+z^2+3yz+zx)-\lambda(x^2+y^2+z^2)+\lambda\left(x^2+y^2+z^2-\frac{1}{\lambda}\right)=0. \quad (1)$$

Here $4x^2+2y^2+z^2+3yz+zx-\lambda(x^2+y^2+z^2)$ will represent a pair of planes,

$$\begin{aligned}
\text{if} \quad & (4-\lambda)(2-\lambda)(1-\lambda)-(4-\lambda)\frac{9}{4}-(2-\lambda)\frac{1}{4}=0\\
\text{or,} \quad & 2\lambda^3-14\lambda^2+23\lambda+3=0\\
\text{or,} \quad & (\lambda-3)(2\lambda^2-8\lambda-1)=0 \quad \text{or,} \quad \lambda=4-3\sqrt{2},3,4+3\sqrt{2}.
\end{aligned}$$

Putting $\lambda=3$ in (1), we have

$$\begin{aligned}
& 3\left(x^2+y^2+z^2-\frac{1}{3}\right)+(x^2-y^2-2z^2+3yz+zx)=0\\
\text{or,} \quad & 3\left(x^2+y^2+z^2-\frac{1}{3}\right)+(x+y-z)(x-y+2z)=0.
\end{aligned}$$

Hence the plane $x+y-z=0$ cuts the given conicoid in a circle. Again $x+y-z=0$ passes through the centre $(0,0,0)$ of the sphere $x^2+y^2+z^2=\frac{1}{3}$. Thus the radius of the circular section is $\frac{1}{\sqrt{3}}$.

WORKED-OUT EXAMPLES

1. *Prove that the radius of the circle in which the plane* $\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda$ *cuts the ellipsoid is* $b\sqrt{\left(1-\frac{\lambda^2}{a^2-c^2}\right)}$.

   Here the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, (a>b>c)$. The plane $\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=0$ cuts the ellipsoid in a circle of radius b. The given plane is parallel to this plane. Parallel plane sections are similar. Thus the radius of the section made by the given plane is

$$\begin{aligned}
& b\sqrt{\left(1-\frac{p^2}{p_o^2}\right)}, \text{ where } p^2=\lambda^2\\
\text{and} \quad & p_o^2=a^2\cdot\frac{a^2-b^2}{a^2}+b^2\cdot 0+c^2\cdot\frac{b^2-c^2}{c^2}=a^2-c^2.
\end{aligned}$$

   Hence the radius of the circle is $b\sqrt{\left(1-\frac{\lambda^2}{a^2-c^2}\right)}$.

2. *Prove that the central circular sections of the conicoid* $(a-b)x^2+ay^2+(a+b)z^2=1$ *are at right angles and that the umbilics are given by*

$$\left[\pm\sqrt{\frac{a+b}{2a(a-b)}},0,\pm\sqrt{\frac{a-b}{2a(a+b)}}\right].$$

The given equation can be written as $[a(x^2+y^2+z^2)-1]+b(z^2-x^2)=0$ and circular sections are given by $z^2-x^2=0$ or, $z=\pm x$. Obviously these are at right angles.

If (α,β,γ) is an umbilic, then the tangent plane at this point is $(a-b)\alpha x+a\beta y+(a+b)\gamma z=1$. It is parallel to either $x+z=0$ or, $x-z=0$.

$$\therefore\ \frac{(a-b)\alpha}{1}=\frac{a\beta}{0}=\frac{(a+b)\gamma}{\pm 1}=k \text{ (say)}.$$

We have

$$(a-b)\alpha^2+a\beta^2+(a+b)\gamma^2=1 \text{ or, } \frac{k^2}{a-b}+\frac{k^2}{a+b}=1 \text{ or, } k=\pm\sqrt{\frac{a^2-b^2}{2a}}.$$

$$\text{Hence the umbilics are}\qquad \left[\pm\sqrt{\frac{a+b}{2a(a+b)}},0,\pm\sqrt{\frac{a-b}{2a(a+b)}}\right].$$

3. *Prove that the sphere* $x^2+y^2+z^2-2\frac{\sqrt{a^2-b^2}\sqrt{a^2-c^2}}{a}x+a^2-b^2-c^2=0$ *meets the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *at its umbilics only.*

The equation of the sphere can be written as

$$b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+\left(1-\frac{b^2}{a^2}\right)x^2+\left(1-\frac{b^2}{c^2}\right)z^2$$
$$-2\sqrt{a^2-b^2}\sqrt{a^2-c^2}\frac{x}{a}+a^2-c^2=0$$

$$\text{or,}\quad b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+\left(\frac{\sqrt{a^2-b^2}}{a}x-\sqrt{a^2-c^2}\right)^2-\frac{b^2-c^2}{c^2}z^2=0.$$

Hence the sphere intersects the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ in two planes

$$\frac{\sqrt{a^2-b^2}}{a}x-\sqrt{a^2-c^2}\pm\frac{\sqrt{b^2-c^2}}{c}z=0,\ \text{i.e.}\ \frac{\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}}\frac{x}{a}\pm\frac{\sqrt{b^2-c^2}}{\sqrt{a^2-c^2}}\frac{z}{c}=1.$$

These are tangent planes to the given sphere and the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at the points

$$\left(a\sqrt{\frac{a^2-b^2}{a^2-c^2}},0,\pm c\sqrt{\frac{b^2-c^2}{a^2-c^2}}\right).$$

Again these points are umbilics to the ellipsoid. Hence the sphere meets the ellipsoid at its umbilics only.

EXERCISE XIB

1. Find the circular sections of the following conicoids.

   (i) $2x^2 + 11y^2 + z^2 = 1$ (ii) $7x^2 + 3y^2 + 5z^2 = 10$

   (iii) $2x^2 + 5y^2 - 10z^2 = 2$ (iv) $15x^2 + y^2 - 10z^2 + 4 = 0$

   (v) $13y^2 + 4z^2 = 2x$ (vi) $x^2 + 5z^2 + 4y = 0$

2. Find the circular sections of the following.

   (i) $5y^2 - 8z^2 + 18yz - 14zx - 10xy + 27 = 0$

   (ii) $3x^2 + 5y^2 + 3z^2 + 2zx = 4$

   (iii) $6x^2 + 13y^2 + 6z^2 - 10yz + 4zx - 10xy = 1$

3. Prove that the planes $\sqrt{3}x + y + 3 = 0$, $\sqrt{3}x - y + 1 = 0$ meet the hyperboloid $10x^2 - 2y^2 + z^2 + 2 = 0$ in circles, which lie on the sphere $x^2 + y^2 + z^2 - 12\sqrt{3}x + 6y - 9 = 0$.

4. Find the umbilics of

   (i) $2x^2 + 3y^2 + 6z^2 = 6$, (ii) $\frac{x^2}{25} - \frac{y^2}{16} - \frac{z^2}{9} = 1$, (iii) $25x^2 + 16y^2 = 2z$.

5. Find the radius of the circle $\frac{x^2}{25} + \frac{y^2}{12} + \frac{z^2}{9} = 1, 3\sqrt{13}x + 5\sqrt{3}z = 10$.

6. Prove that the sphere $x^2 + y^2 + z^2 - 2\sqrt{15}x + 11 = 0$ meets the conicoid $\frac{x^2}{36} + \frac{y^2}{16} + \frac{z^2}{9} = 1$ at umbilics only.

7. Find the equations to the circular sections of the conicoid

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) + 1 = 0.$$

   [*Hints.* The given equation can be written as

$$\sum yz\left(\frac{b}{c} + \frac{c}{b}\right) - \lambda(x^2 + y^2 + z^2) + \lambda\left(x^2 + y^2 + z^2 + \frac{1}{\lambda}\right) = 0.$$

   Choose λ in such a way that $\sum yz\left(\frac{b}{c} + \frac{c}{b}\right) - \lambda(x^2 + y^2 + z^2) = 0$ represents a pair of planes. For this λ is 1.]

8. Show that the circular sections of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ passing through one extremity of x- axis are both of radius r, where $\frac{r^2}{b^2} = \frac{b^2 - c^2}{a^2 - c^2}$.

9. Show that the locus of the centres of the spheres which pass through the origin and cut the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in a pair of real circles is the hyperbola $\frac{a^2x^2}{a^2 - b^2} - \frac{c^2z^2}{b^2 - c^2} = b^2, y = 0$.

   [*Hints.* The circles lie on the sphere

$$x^2 + y^2 + z^2 - \frac{(\lambda + \mu)\sqrt{a^2 - b^2}}{a}x + \frac{(\lambda - \mu)\sqrt{b^2 - c^2}}{c}z + \lambda\mu - b^2 = 0.$$

   {See the note of (i) of Sec 11.20}.

If the centre is (α, β, γ), then

$$2\alpha = \frac{(\lambda+\mu)\sqrt{a^2-b^2}}{a}, \beta = 0, 2\gamma = -\frac{(\lambda-\mu)\sqrt{b^2-c^2}}{c}.$$

Now $$\frac{4a^2\alpha^2}{a^2-b^2} - \frac{4c^2\gamma^2}{b^2-c^2} = (\lambda+\mu)^2 - (\lambda-\mu)^2 = 4\lambda\mu.$$

If the sphere passes through the origin, then $\lambda\mu = b^2$.

$$\therefore \frac{a^2\alpha^2}{a^2-b^2} - \frac{c^2\gamma^2}{b^2-c^2} = b^2.$$

Hence the result follows.]

10. Prove that the umbilics of the conicoid $\frac{x^2}{a+b} + \frac{y^2}{a} + \frac{z^2}{a-b} = 1$ are the extremities of the equal conjugate diameters of the ellipse $y = 0, \frac{x^2}{a+b} + \frac{z^2}{a-b} = 1$.

[*Hints.* By (1) of Sec 11.21, the umbilics are $\left(\pm\sqrt{\frac{a+b}{2}}, 0, \pm\sqrt{\frac{a-b}{2}}\right)$.

These are the extremities of equiconjugate diameters of the given ellipse.]

11. Prove that the normals to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ at all points of a central circular section are parallel to the plane that makes an angle $\cos^{-1}\frac{ac}{b\sqrt{a^2-b^2+c^2}}$ with the section.

[*Hints.* One real central circular section is

$$\frac{x}{a}\sqrt{a^2-b^2} + \frac{z}{c}\sqrt{b^2-c^2} = 0, \text{ if } a > b > c.$$

Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be a generator of the enveloping cylinder which touches the ellipsoid along the section.

The diametral plane which bisects chords parallel to the generator is $\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} = 0$. It is identical with the circular section.

$$\therefore \frac{l}{a\sqrt{a^2-b^2}} = \frac{m}{0} = \frac{n}{c\sqrt{b^2-c^2}}.$$

Again the normals to the ellipsoid will be parallel to the normal section of the cylinder, i.e $lx + my + nz = 0$ or, $a\sqrt{a^2-b^2}x + c\sqrt{b^2-c^2}z = 0$.

If θ is the angle between the circular section and the normal section of the cylinder, then

$$\cos\theta = \frac{(a^2-b^2)+(b^2-c^2)}{\left[\left(\frac{a^2-b^2}{a^2} + \frac{b^2-c^2}{c^2}\right)\{a^2(a^2-b^2)+c^2(b^2-c^2)\}\right]^{1/2}}$$
$$= \frac{ac}{b\sqrt{a^2-b^2+c^2}}.]$$

ANSWERS

1. (i) $3y + z = \lambda, 3y - z = \mu$

   (ii) $x + y = \lambda, x - y = \mu$

   (iii) $y + 2z = \lambda, y - 2z = \mu$

   (iv) $4x + 3z = \lambda, 4x - 3z = \mu$

   (v) $2x + 3y = \lambda, 2x - 3y = \mu$

   (vi) $y + 2z = \lambda, y - 2z = \mu$.

2. (i) $(x - 2y - 5z)(3x - 4y + z) = 0$

   (ii) $(x + y - z)(x - y - z) = 0$

   (iii) $2(x + z)^2 - 10y(x + z) + 9y^2 = 0$.

4. (i) $\left(\pm\frac{\sqrt{3}}{2}, 0, \pm\frac{1}{\sqrt{2}}\right)$

   (ii) $\left(\pm 5\sqrt{\frac{41}{34}}, 0, \pm 3\sqrt{\frac{7}{34}}\right)$

   (iii) $\left(\pm\frac{3}{100}, 0, \frac{9}{800}\right)$.

5. 3.

Chapter 12

Cylindrical and Spherical Coordinates

12.10 Different system of coordinates

To deal the problems of analytical coordinate geometry we have so far used the rectangular cartesian coordinate system. Sometimes other system of coordinates are useful to define the position of a point in space. In this chapter we introduce two new systems, namely cylindrical and spherical (polar) systems. These are convenient to use in situation, where there is an axis of symmetry and a centre of symmetry respectively.

12.11 Cylindrical coordinates

Let P be a point in space in the rectangular cartesian frame $OXYZ$. PN is perpendicular to XOY-plane and MN is parallel to OY.

We write that $ON = r, \angle XON = \theta$ and $NP = z$. (r, θ, z) are called the cylindrical coordinates of P w.r.t. the fixed point O, the fixed line OX and the fixed plane XOY. These are *reference elements*. r, θ and z are called the *radius vector*, the *vectorial angle* and the *applicate* of P respectively.

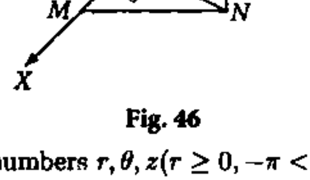

Fig. 46

For a point P in space there is an ordered triple of numbers $r, \theta, z (r \geq 0, -\pi < \theta \leq \pi)$.

Conversely there exists a unique point P in space for a given ordered triple of numbers $r, \theta, z (r \geq 0, -\pi < \theta \leq \pi)$.

Transformation formulae. If (x, y, z) are the cartesian coordinates of P w.r.t. three rectangular axes OX, OY, OZ and (r, θ, z) are cylindrical coordinates of P w.r.t. the point O, the line OX and the plane XOY as reference elements, then

$$x = r\cos\theta, y = r\sin\theta, z = z \text{ and } r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}, z = z.$$

Loci

(i) *The locus of $r = a$, where a is a constant.*

From $r = a$ we have $x^2 + y^2 = a^2$. It represents a right circular cylinder with the z-axis as the axis of it. The coordinates of a point on the surface of the cylinder is of the form (a, θ, z). The term 'cylindrical coordinates' arises from this fact. (See Fig. 47).

(ii) *The locus of θ = constant.*

$\theta = \alpha$ (constant) represents a plane making an angle α with the plane ZOX and passing through the z-axis. (See Fig. 48).

(iii) *The locus of z = constant.*

$z = k$ (constant) is a plane parallel to the XOY-plane. It is at a distance k from the XOY-plane. (See Fig. 49).

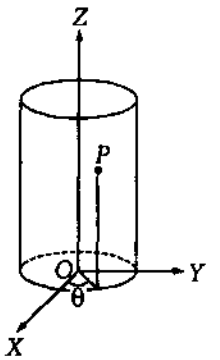

Fig. 47

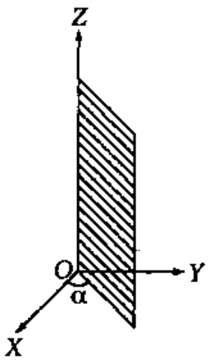

Fig. 48

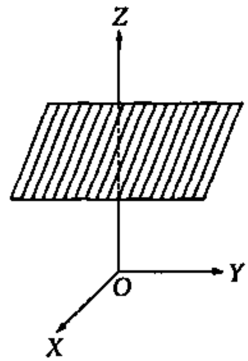

Fig. 49

12.12 Distance between two points in cylindrical coordinates

Let P and Q be two pints in space and (r_1, θ_1, z_1) and (r_2, θ_2, z_2) be their cylindrical coordinates.

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are their respective cartesian coordinates, then

$$\begin{aligned} PQ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \\ &= \sqrt{(r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2 + (z_1 - z_2)^2} \\ &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + (z_1 - z_2)^2}. \end{aligned}$$

12.20 Spherical coordinates

Let P be a point in space in the rectangular-cartesian frame $OXYZ$. PN is perpendicular to the XOY-plane and MN is parallel to OY.

We write that $OP = r, \angle ZOP = \theta$ and $\angle XON = \phi$. (r, θ, ϕ) are called the spherical coordinates of P w.r.t. the fixed point O and two perpendicular lines OX and OZ. These are *reference elements.* r, θ and ϕ are called the *radius vector, the colatitude* and the *latitude* of P respectively.

For a point P in space there is an ordered triple of numbers $r, \theta, \phi (r \geq 0, 0 \leq \theta \leq \pi, -\pi < \phi \leq \pi)$. Conversely a given ordered triple of numbers $r, \theta, \phi (r \geq 0, 0 \leq \theta \leq \pi, -\pi < \phi \leq \pi)$ corresponds to a unique point P in space.

Transformation formulae. If (x, y, z) are rectangular cartesian coordinates of P with respective references allied with the spherical coordinates of P, then

$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$$

and
$$r = \sqrt{x^2+y^2+z^2},$$
$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}, \phi = \tan^{-1}\frac{y}{x}.$$

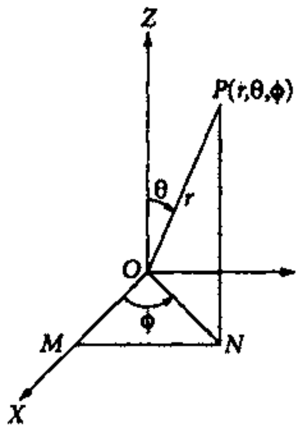

Fig. 50

If (ρ, ψ, z) are the cylindrical coordinates of P in the allied frame, then

$$\rho = r\sin\theta, \psi = \phi, z = r\cos\theta \text{ and}$$
$$r = \sqrt{\rho^2+z^2}, \theta = \cos^{-1}\frac{z}{\sqrt{\rho^2+z^2}}, \phi = \psi.$$

Loci

(i) *The locus of* r = *constant.*

$r = a$ (constant) gives $x^2+y^2+z^2 = a^2$. It is a sphere of radius a with the centre at the origin. The spherical coordinates of a point P on the sphere are (a, θ, ϕ) and its cartesian coordinates are $(a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)$. The term 'spherical coordinates' comes from the fact of relation with a sphere. (See Fig. 51).

Fig. 51

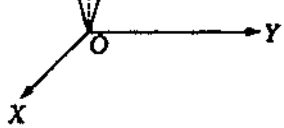

Fig. 52

(ii) *The locus of* θ = *constant.*

From θ = constant, we get

$$\cos\theta = \text{constant or, } \frac{x^2+y^2+z^2}{z^2} = \text{constant} = \lambda \text{ (say) or, } x^2+y^2 = (\lambda-1)z^2.$$

It is a right circular cone with the z-axis as the axis of the cone and the semi-vertical angle θ. Thus θ (= constant) represents a right circular cone. (See Fig. 52).

(iii) *The locus of ϕ = constant.*

$\phi = \alpha$ (constant) represents a plane passing through the z-axis and making an angle α with the ZOX-plane.

12.21 Distance between two points in spherical coordinates

Let the spherical coordinates of two points P and Q be (r_1, θ_1, ϕ_1) and (r_2, θ_2, ϕ_2). If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the cartesian coordinates of P and Q in the allied frame of references, then

$$\begin{aligned}
PQ &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2} \\
&= \sqrt{(r_1\sin\theta_1\cos\phi_1 - r_2\sin\theta_2\cos\phi_2)^2 + (r_1\sin\theta_1\sin\phi_1 - r_2\sin\theta_2\sin\phi_2)^2 + (r_1\cos\theta_1 - r_2\cos\theta_2)^2} \\
&= \sqrt{r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) + (r_1\cos\theta_1 - r_2\cos\theta_2)^2} \\
&= \sqrt{r_1^2+r_2^2-2r_1r_2\{\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\}+\cos\theta_1\cos\theta_2}.
\end{aligned}$$

WORKED-OUT EXAMPLES

1. *Find the cylindrical coordinates of the point whose cartesian coordinates are* $(1, \sqrt{3}, 2)$.

   If (r, θ, z) are cylindrical coordinates of this point, then $r\cos\theta = 1, r\sin\theta = \sqrt{3}, z = 2$.

   $\therefore\ r = \sqrt{1+3} = 2, \theta = \tan^{-1}\sqrt{3} = \pi/3$.

   Thus the required coordinates are $(2, \pi/3, 2)$.

2. *Find the cartesian coordinates of the point whose spherical coordinates are* $(3, 2\pi/3, -\pi/6)$.

   Here $r = 3, \theta = \frac{2\pi}{3}, \phi = -\frac{\pi}{6}$.

   If (x, y, z) are the cartesian coordinates of the point, then

$$\begin{aligned}
x &= r\sin\theta\cos\phi = 3\sin\frac{2\pi}{3}\cos\left(-\frac{\pi}{6}\right) = 3\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{9}{4}, \\
y &= r\sin\theta\sin\phi = 3\sin\frac{2\pi}{3}\sin\left(-\frac{\pi}{6}\right) = 3\cdot\frac{\sqrt{3}}{2}\cdot\left(-\frac{1}{2}\right) = -\frac{3\sqrt{3}}{4}, \\
z &= r\cos\theta = 3\cos\frac{2\pi}{3} = -\frac{3}{2}.
\end{aligned}$$

   Thus the required coordinates are $\left(\frac{9}{4}, -\frac{3\sqrt{3}}{4}, -\frac{3}{2}\right)$.

3. *Find the spherical coordinates of the point whose cylindrical coordinates are* $(2, \pi/4, -1)$.

   If (r, θ, ϕ) are the spherical coordinates, then

$$r = \sqrt{2^2+(-1)^2} = \sqrt{5}, \theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right), \phi = \pi/4.$$

Thus the required coordinates are $\left(\sqrt{5}, \cos^{-1}\frac{-1}{\sqrt{5}}, \frac{\pi}{4}\right)$.

4. *Show that the equation $r = 2\cos\theta$ in spherical coordinates represents a sphere.*

$r = 2\cos\theta$.

Transforming into cartesian coordinates,

$$\sqrt{x^2+y^2+z^2} = 2\cdot\frac{z}{\sqrt{x^2+y^2+z^2}}$$
$$\text{or,}\quad x^2+y^2+z^2-2z=0 \quad \text{or,}\quad x^2+y^2+(z-1)^2=1.$$

It is a sphere whose centre is $(0,0,1)$ and radius is 1.

5. *If ds represents the distance between two points with spherical coordinates (r,θ,ϕ) and $(r+dr,\theta+d\theta,\phi+d\phi)$, where $dr, d\theta, d\phi$ are small increments of r,θ,ϕ respectively show that $(ds)^2=(dr)^2+r^2(d\theta)^2+r^2\sin^2\theta(d\phi)^2$.*

Let (x,y,z) and $(x+dx, y+dy, z+dz)$ be the cartesian coordinates of the points whose spherical coordinates are (r,θ,ϕ) and $(r+dr,\theta+d\theta,\phi+d\phi)$.

We have $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$.

Considering differentials we have

$$\begin{aligned}
dx &= dr\sin\theta\cos\phi + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi,\\
dy &= dr\sin\theta\sin\phi + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi,\\
dz &= dr\cos\theta - r\sin\theta d\theta.\\
(ds)^2 &= \{x-(x+dx)\}^2 + \{y-(y+dy)\}^2 + \{z-(z+dz)\}^2\\
&= (dx)^2+(dy)^2+(dz)^2\\
&= (dr\sin\theta\cos\phi + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi)^2 + (dr\sin\theta\sin\phi\\
&\qquad + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi)^2 + (dr\cos\theta - r\sin\theta d\theta)^2\\
&= (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2.
\end{aligned}$$

EXERCISE XII

1. Find the cylindrical and spherical coordinates of the points whose cartesian coordinates are $(3,4,5), (-2,1,-2)$ and $(2,-3,2)$.

2. Find the cartesian and cylindrical coordinates of the points whose spherical coordinates are

$$\left(8,\frac{2\pi}{3},\frac{\pi}{3}\right), \left(-3,\frac{\pi}{4},-\frac{\pi}{3}\right), \left(2,\frac{5\pi}{6},-\frac{3\pi}{4}\right) \quad\text{and}\quad \left(3,\cos^{-1}\frac{1}{3},\frac{\pi}{4}\right).$$

3. Find the cartesian and spherical coordinates of the points whose cylindrical coordinates are $\left(4,\frac{\pi}{2},5\right), \left(3,-\frac{\pi}{4},2\right)$ and $\left(1,\frac{5\pi}{6},-3\right)$.

4. Show that the equation $r^2 - z^2 = 1$ in cylindrical coordinates represents a hyperboloid of one sheet.

5. Find the distance between the points whose spherical coordinates are $\left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{6}\right)$ and $\left(2, \frac{\pi}{3}, \frac{\pi}{3}\right)$.

6. Show that $r \sin\theta = 2$ in spherical coordinates represents a right circular cylinder.

7. Find the distance between two points whose cylindrical coordinates are $\left(6, \frac{\pi}{6}, -3\right)$ and $\left(8, \frac{2\pi}{3}, 3\right)$.

8. If ds represents the distance between two points having cylindrical coordinates (r, θ, z) and $(r + dr, \theta + d\theta, z + dz)$, where $dr, d\theta, dz$ are small increments of r, θ and z respectively show that $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2$.

ANSWERS

1. $\left(5, \tan^{-1}\frac{4}{3}, 5\right), \left(5\sqrt{2}, \frac{\pi}{4}, \tan^{-1}\frac{4}{3}\right); \left(\sqrt{5}, \pi - \tan^{-1}\frac{1}{2}, -2\right),$
$\left(3, \pi - \cos^{-1}\frac{2}{\sqrt{5}}, \pi - \tan^{-1}\frac{1}{2}\right); \left(\sqrt{13}, -\tan^{-1}\frac{3}{2}, 2\right), \left(\sqrt{17}, \cos^{-1}\frac{2}{\sqrt{17}}, -\tan^{-1}\frac{3}{2}\right)$

2. $(2\sqrt{3}, 6, -4), (4\sqrt{3}, \pi/3, -4); \left(-\dfrac{3\sqrt{2}}{4}, \dfrac{3\sqrt{6}}{4}, -\dfrac{3\sqrt{2}}{2}\right), \left(-\dfrac{3\sqrt{2}}{2}, -\dfrac{\pi}{3}, -\dfrac{3\sqrt{2}}{2}\right);$
$(2, 2, 1), \left(2\sqrt{2}, \frac{\pi}{4}, 1\right).$

3. $(0, 4, 5), \left(\sqrt{41}, \cos^{-1}\frac{5}{\sqrt{41}}, \frac{\pi}{2}\right); \left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}, 2\right), \left(\sqrt{13}, \cos^{-1}\frac{2}{\sqrt{13}}, -\frac{\pi}{4}\right);$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, -3\right), \left(\sqrt{10}, \cos^{-1}\frac{-3}{\sqrt{10}}, \frac{5\pi}{6}\right).$

4. $x^2 + y^2 - z^2 = 1.$

5. 1.

6. $x^2 + y^2 = 4.$

7. $\sqrt{136}.$

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Part III

Vector Analysis

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Chapter 1

Definition and Elementary Operations

1.10 Definitions

Scalars. Quantities which have magnitude only are known as *scalars*. Mass, area, volume, temperature, work, etc. are scalar quantities. A scalar can be represented by a number with the proper sign.

Vectors. Quantities which have magnitude as well as direction and which also obey certain law of addition are known as *vectors*. Force, displacement, velocity, moment, electric current, etc. are vector quantities.

Representation of a vector. A vector can be represented by a directed line segment AB. The length AB represents the magnitude of the vector to some convenient scale and the direction of it is indicated by an arrow as shown in Fig. 1.

A vector from the point A to the point B is generally denoted by $\overrightarrow{AB}$. Thus one can say that a vector is a *directed line segment*. For simple notation and complete distinction from others we denote the vectors by small letters with bars on them as $\overrightarrow{AB} = \bar{a}$. The magnitude or the absolute value or module of $\overrightarrow{AB}$ is written as $|\overrightarrow{AB}|$ or, $|\bar{a}|$. It is always positive. [Generally vectors are denoted by bold face letters as $\mathbf{a}$, $\mathbf{b}$, ... It is difficult to write these in practice.]

Fig. 1

Like vectors. Vectors which have the same directions are said to be *like vectors*.

Position vectors. Let O be an arbitrary point. O is called the *vector origin* or *the initial point*. The position of a point P w.r.t. O is represented by $\overrightarrow{OP}$. The vector $\overrightarrow{OP}$ is known as the position vector of P with the initial point O.

Fig. 2

Coinitial vectors. Vectors having the same initial point are called *co-initial vectors*.

Unit vector. A vector whose length is unity is called a *unit vector*.

Null vector (*Zero vector*). It is a vector whose absolute value is zero.

Proper vector. A vector whose length is not zero is called a *proper vector*.

Equal vectors. Two vectors are said to be equal, if they have the same magnitudes and directions. If $\bar{a}$ and $\bar{b}$ are equal, we will write $\bar{a} = \bar{b}$. Thus a vector $\bar{a} = \overrightarrow{AB}$ is equal to all vectors obtained from $\overrightarrow{AB}$ by parallel displacement or translation. For example, the vectors ($\overrightarrow{AB}$ and $\overrightarrow{DC}$) forming the opposite sides of a parallelogram $ABCD$ are equal.

Vectors that conform to the above definition of equality are said to be *free*. But forces acting on rigid bodies are not free vectors. They may only be shifted along their lines of action since other translations alter their dynamic effect. A vector confined to a definite line of action is called a *line vector*. Two line vectors are equal when they have the same length and the same direction and lie on the same line.

Collinear vectors. Any set of vectors is said to be collinear if each member of the set is parallel to the same line.

If $\bar{a}$ is collinear with $\bar{b}$, then we can write $\bar{a} = x\bar{b}$ where x is a scalar.

Coplanar vectors. A system of vectors is said to be coplanar if a plane can be drawn which is parallel to each of the vectors. Otherwise, they are called non-coplanar.

1.20 Addition of vectors (Triangle law)

If $\overrightarrow{AB} = \bar{a}$, $\overrightarrow{BC} = \bar{b}$ and $\overrightarrow{AC} = \bar{c}$, then $\bar{a}+\bar{b} = \bar{c}$, i.e. $\overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$. It is the *triangle law* of addition.

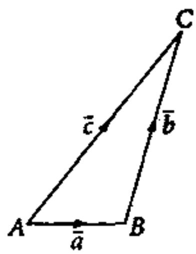


Fig. 3

Theorem 1. *Vector addition is commutative.* ($\bar{a} + \bar{b} = \bar{b} + \bar{a}$).

Let $\overrightarrow{AB} = \bar{a}$, $\overrightarrow{BC} = \bar{b}$ and $\overrightarrow{AC} = \bar{c}$ in Fig. 4.

The parallelogram $ABCD$ is completed.

Now $\overrightarrow{AD} = \overrightarrow{BC} = \bar{b}, \overrightarrow{DC} = \overrightarrow{AB} = \bar{a}$.

From $\triangle ABC, \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \bar{a} + \bar{b}$.

From $\triangle ADC, \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \bar{b} + \bar{a}$.

$\therefore\ \bar{a} + \bar{b} = \bar{b} + \bar{a} (= \bar{c})$.

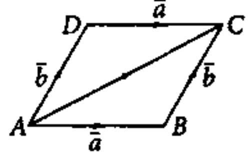


Fig. 4

Note 1. Vector addition can also be made by using parallelogram law (see Fig. 4). The addition of vectors represented by two adjacent sides of a parallelogram is represented by the diagonal of the parallelogram passing through the common point.

Theorem 2. *Vector addition is associative.* $(\bar{a}+\bar{b})+\bar{c}=\bar{a}+(\bar{b}+\bar{c})$.

Let $\overrightarrow{AB}=\bar{a}, \overrightarrow{BC}=\bar{b}, \overrightarrow{CD}=\bar{c}$ in Fig. 5.

From $\triangle ABC, \overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\bar{a}+\bar{b}$.

From $\triangle ACD, \overrightarrow{AD}=\overrightarrow{AC}+\overrightarrow{CD}=(\bar{a}+\bar{b})+\bar{c}$. (1)

From $\triangle BCD, \overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}=\bar{b}+\bar{c}$.

From $\triangle ABD, \overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}=\bar{a}+(\bar{b}+\bar{c})$. (2)

By (1) and (2) $(\bar{a}+\bar{b})+\bar{c}=\bar{a}+(\bar{b}+\bar{c})$.

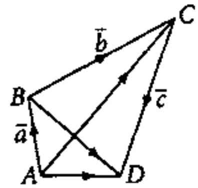


Fig. 5

Hence the sum of the vectors is independent of the order in which they are added. Thus we write the sum as $\bar{a}+\bar{b}+\bar{c}$.

1.21 Subtraction of vectors

The sum of two vectors is zero when they are equal in magnitude but opposite in directions. $\overrightarrow{AB}+\overrightarrow{BA}=\bar{0}$.

If $\overrightarrow{AB}=\bar{a}$, then $\overrightarrow{BA}=-\bar{a}$.

From Fig. 3, we have $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.

It can be written as $\overrightarrow{AB}=\overrightarrow{AC}-\overrightarrow{BC}$, i.e. $\bar{a}=\bar{c}-\bar{b}$.

Subtraction can be conceived as addition of $\bar{c}$ with $-\bar{b}$.

1.22 Multiplication of a vector by a scalar

If $\bar{a}$ is a vector and m is a scalar, then their product is written as $m\bar{a}$ or, $\bar{a}m$. It is also a vector in the same direction of $\bar{a}$ when m is positive and in the opposite direction of $\bar{a}$ when m is negative but its magnitude is $|m|$ times that of $\bar{a}$.

Vectors obey the rules of ordinary algebra for addition, subtraction and multiplication by scalars.

So (i) $(m+n)\bar{a}=m\bar{a}+n\bar{a}$, (ii) $m(\bar{a}+\bar{b})=m\bar{a}+m\bar{b}$, (iii) $m(n\bar{a})=n(m\bar{a})=mn\bar{a}$.

1.30 Point of division (Section ratio)

Let the point C divide AB in the ratio $m:n$. Here n is non-zero positive number and m is positive or negative according as C lies within or outside the segment AB.

Let the position vectors of A and B w.r.t. the vector origin O be $\bar{a}$ and $\bar{b}$, i.e. $\overrightarrow{OA}=\bar{a}$ and $\overrightarrow{OB}=\bar{b}$.

From $\triangle OAC, \overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}$ or, $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$.

From $\triangle OCB, \overrightarrow{OC}+\overrightarrow{CB}=\overrightarrow{OB}$ or, $\overrightarrow{CB}=\overrightarrow{OB}-\overrightarrow{OC}$.

Since $\frac{AC}{CB}=\frac{m}{n}, \overrightarrow{AC}=\frac{m}{n}\overrightarrow{CB}$.

$$\therefore \quad n(\overrightarrow{OC}-\overrightarrow{OA})=m(\overrightarrow{OB}-\overrightarrow{OC})$$

$$\text{or,} \quad (m+n)\overrightarrow{OC}=m\overrightarrow{OB}+n\overrightarrow{OA}$$

$$\text{or,} \quad \overrightarrow{OC}=\frac{m\overrightarrow{OB}+n\overrightarrow{OA}}{m+n}=\frac{m\bar{b}+n\bar{a}}{m+n}.$$

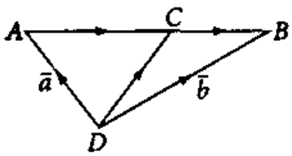


Fig. 6

It gives the position vector of C w.r.t. the same initial point O.

Corollary. *If C is the midpoint of AB, then $m = n$. In this case,* $\overrightarrow{OC} = \frac{\bar{a}+\bar{b}}{2}$.

Note. *Harmonic conjugates.* If the point C divides AB in the ratio $m : n$ and the point D divides AB in the ratio $-m : n$, then C and D are called harmonic conjugates of A and B. Conversely A and B are harmonic conjugates of C and D.

1.31 Theorem 1. Collinearity of three points.

The necessary and sufficient condition for the three distinct points A, B, C to lie on a straight line (i.e. collinear) is that there exist three scalars α, β, γ (not all zero) such that $\alpha\bar{a} + \beta\bar{b} + \gamma\bar{c} = \bar{0}$ and $\alpha + \beta + \gamma = 0$ where $\bar{a}, \bar{b}, \bar{c}$ are the position vectors of A, B, C respectively w.r.t. a certain vector origin.

Proof. (i) *The condition is necessary.*

Let A, B, C lie on the same line. Then C must divide AB in some ratio $m : n$ which is not zero, ∞ or, -1, so that $m, n, m+n$ are not zero.

Thus we write $\bar{c} = \frac{m\bar{b}+n\bar{a}}{m+n}$ or, $n\bar{a} + m\bar{b} - (m+n)\bar{c} = 0$.

Putting $n = \alpha, m = \beta$ and $-(m+n) = \gamma$, we have

$$\alpha\bar{a} + \beta\bar{b} + \gamma\bar{c} = \bar{0} \quad \text{and} \quad \alpha + \beta + \gamma = 0.$$

(ii) *The condition is sufficient.*

Let the relations $\alpha\bar{a} + \beta\bar{b} + \gamma\bar{c} = \bar{0}$ and $\alpha + \beta + \gamma = 0$ hold.

Then $\alpha + \beta = -\gamma$ and $-\gamma\bar{c} = \alpha\bar{a} + \beta\bar{b}$ or, $\bar{c} = \frac{\alpha\bar{a}+\beta\bar{b}}{-\gamma} = \frac{\alpha\bar{a}+\beta\bar{b}}{\alpha+\beta}$.

This suggests that $\bar{c}$ lies on the line of join of $\bar{a}$ and $\bar{b}$ and also it divides AB in the ratio $\beta : \alpha$. Hence the points are collinear.

Theorem 2. Coplanarity of four points.

The necessary and sufficient condition for four points defined by the position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ (no three of them are collinear) to lie on the same plane (i.e. coplanar) is that there exist four scalars $\alpha, \beta, \gamma, \delta$ (not all zero) such that

$$\alpha\bar{a} + \beta\bar{b} + \gamma\bar{c} + \delta\bar{d} = \bar{0}, \alpha + \beta + \gamma + \delta = 0.$$

Proof. (i) *The condition is necessary.*

Let A, B, C, D be the four points given by the position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ lie on the same plane where no three of them are collinear. In this case, we can choose that AB and CD meet at a point P (say). If $\bar{p}$ is the position vector of P w.r.t. the same vector origin and it divides AB and CD in the ratio $m : n$ and $r : s$ respectively, then

$$\bar{p} = \frac{n\bar{a} + m\bar{b}}{m+n} \quad \text{and} \quad \bar{p} = \frac{s\bar{c} + r\bar{d}}{r+s}.$$

$$\therefore \quad \frac{n\bar{a} + m\bar{b}}{m+n} = \frac{s\bar{c} + r\bar{d}}{r+s} \quad \text{or,} \quad \frac{n}{m+n}\bar{a} + \frac{m}{m+n}\bar{b} - \frac{s}{r+s}\bar{c} - \frac{r}{r+s}\bar{d} = \bar{0}.$$

Putting $\alpha = \frac{n}{m+n}, \beta = \frac{m}{m+n}, \gamma = -\frac{s}{r+s}, \delta = -\frac{r}{r+s}$, we have

$$\alpha\bar{a} + \beta\bar{b} + \gamma\bar{c} + \delta\bar{d} = \bar{0} \quad \text{and} \quad \alpha + \beta + \gamma + \delta = 0.$$

(ii) *The condition is sufficient.*

Let the relations $\alpha\bar{a} + \beta\bar{b} + \gamma\bar{c} + \delta\bar{d} = \bar{0}$ and $\alpha + \beta + \gamma + \delta = 0$ hold.

$\therefore\ \alpha\bar{a} + \beta\bar{b} = -(\gamma\bar{c} + \delta\bar{d})$ and $\alpha + \beta = -(\gamma + \delta)$.

Dividing one by the other, $\frac{\alpha\bar{a}+\beta\bar{b}}{\alpha+\beta} = \frac{\gamma\bar{c}+\delta\bar{d}}{\gamma+\delta}$.

It suggests that the line joining the points defined by the position vectors $\bar{a}$ and $\bar{b}$ is divided by a point in the ratio $\beta : \alpha$ and the line joining the points defined by the position vectors $\bar{c}$ and $\bar{d}$ is divided by the same point in the ratio $\delta : \gamma$. As these four points lie on two intersecting lines, they are coplanar.

1.40 Rectangular components (resolution of a vector)

Let $\bar{i}, \bar{j}, \bar{k}$ be the unit vectors along the three rectangular axes OX, OY and OZ respectively and P be a point whose coordinates are (x, y, z) w.r.t. OX, OY and OZ.

In Fig. 7, PN is perpendicular to XOY-plane and MN is parallel to OY.

Therefore, $OM = x, MN = y, NP = z$ and $\overrightarrow{OM} = x\bar{i}$, $MN = y\bar{j}$, $\overrightarrow{NP} = z\bar{k}$.

From $\triangle OMN$, $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN} = x\bar{i} + y\bar{j}$.

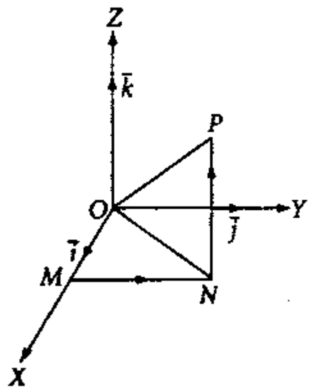


Fig. 7

From $\triangle ONP$, $\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP} = x\bar{i} + y\bar{j} + z\bar{k}$. $x\bar{i}, y\bar{j}, z\bar{k}$ are called the vector components of $\overrightarrow{OP}$ in the directions of OX, OY and OZ. Thus a vector can be resolved into three orthogonal components.

Here $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$ and the d.cs. of OP are $\frac{x}{|\overline{OP}|}, \frac{y}{|\overline{OP}|}, \frac{z}{|\overline{OP}|}$.

Sometimes the vector, $\overrightarrow{OP}$ is denoted by (x, y, z) when the cartesian coordinates of P are (x, y, z).

Note 1. If $\overrightarrow{OP} = x\bar{i} + y\bar{j} + z\bar{k}$, (x, y, z) are the coordinates of P.

Note 2. If the coordinates of A and B are (x_1, y_1, z_1) and (x_2, y_2, z_2), then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1)\bar{i} + (y_2 - y_1)\bar{j} + (z_2 - z_1)\bar{k}$.

Note 3. It is customary to use the notations $\bar{i}, \bar{j}, \bar{k}$ for the unit vectors along the three rectangular axes.

1.50 Theorem 1. *If $\bar{a}$ and $\bar{b}$ are two non-zero, non-collinear vectors and x, y are two scalars, then it is not possible to express $x\bar{a} + y\bar{b} = \bar{0}$ with $x \neq 0, y \neq 0$.*

Proof. Suppose $x \neq 0$. Then it can be written as $\bar{a} = -\frac{y}{x}\bar{b}$. Since $-\frac{y}{x}$ is a scalar, $\bar{a}$ and $\bar{b}$ are collinear. It is a contradiction. Therefore, $x = 0$. Similarly $y = 0$.

Theorem 2. *If $\bar{r}, \bar{a}, \bar{b}$ are coplanar vectors and $\bar{a}, \bar{b}$ are non-collinear, then it can be uniquely expressed as $\bar{r} = x\bar{a} + y\bar{b}$ where x and y are scalars.*

Proof. (i) Let $\bar{r} = \overrightarrow{OC}$ be in the plane of $\bar{a}$ and $\bar{b}$ or in the plane parallel to $\bar{a}$ and $\bar{b}$.

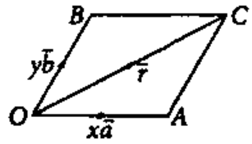


Fig. 8

Through O and C lines are drawn to $\bar{a}$ and $\bar{b}$ to form the parallelogram $OACB$. Since $\overrightarrow{OA}$ is parallel to $\bar{a}, \overrightarrow{OA} = x\bar{a}$ where x is a scalar. Similarly $\overrightarrow{OB} = y\bar{b}$ where y is a scalar.

Now $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + \overrightarrow{OB}$.

$\therefore\ \bar{r} = x\bar{a} + y\bar{b}$.

(ii) *The expression is unique.*

Suppose $\bar{r} = x_1\bar{a} + y_1\bar{b}$ where x_1 and y_1 are scalars.

$\therefore\ x\bar{a} + y\bar{b} = x_1\bar{a} + y_1\bar{b}$ or, $(x - x_1)\bar{a} + (y - y_1)\bar{b} = \bar{0}$.

Since $\bar{a}$ and $\bar{b}$ are non-collinear, $x - x_1 = 0$ and $y - y_1 = 0$, i.e. $x = x_1$ and $y = y_1$.

Hence the theorem is proved.

Theorem 3. *If $\bar{a}, \bar{b}, \bar{c}$ are three non-coplanar vectors and x, y, z are three scalars, then we cannot express $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ with $x \neq 0, y \neq 0, z \neq 0$.*

Proof. Suppose $x \neq 0$. Then it can be written as $\bar{a} = -\frac{y}{x}\bar{b} - \frac{z}{x}\bar{c}$. Thus $\bar{a}$ is coplanar with $\bar{b}$ and $\bar{c}$. It leads to a contradiction. Hence $x = 0$. Similarly $y = 0$ and $z = 0$.

Theorem 4. *Any vector, $\bar{r}$ in space can be uniquely expressed as the linear combination of three non-coplanar vectors $\bar{a}, \bar{b}$ and $\bar{c}$ i.e. $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$ where x, y, z are scalars.*

Proof. (i) Let $\bar{r} = \overrightarrow{OG}$ and OA, OC, OE be parallel to $\bar{a}, \bar{b}, \bar{c}$ respectively.

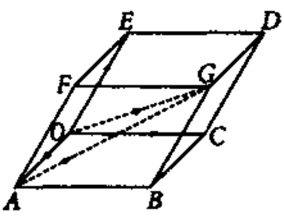


Fig. 9

Since $\bar{a}, \bar{b}, \bar{c}$ are non-coplanars, OA, OC and OE are non-coplanar. Through G three planes parallel to the planes OAC, COE and EOA are drawn to meet OA, OC and OE at A, C, E respectively. Now the parallelopiped $OABCDEFG$ is obtained. OG is a diagonal of this parallelopiped. AG is joined. Again there exist three scalars x, y, z such that $\overrightarrow{OA} = x\bar{a}, \overrightarrow{OC} = y\bar{b}, \overrightarrow{OE} = z\bar{c}$.

From Fig. 9, $\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BG} = \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OE}$.

$\therefore\ \bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$.

(ii) *The expression is unique.*

Suppose $\bar{r} = x_1\bar{a} + y_1\bar{b} + z_1\bar{c}$ where x_1, y_1, z_1 are scalars.

Then $x\bar{a} + y\bar{b} + z\bar{c} = x_1\bar{a} + y_1\bar{b} + z_1\bar{c}$ or, $(x - x_1)\bar{a} + (y - y_1)\bar{b} + (z - z_1)\bar{c} = \bar{0}$.

Since $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar

$$x - x_1 = 0, y - y_1 = 0, z - z_1 = 0, \quad \text{i.e.} \quad x = x_1, y = y_1, z = z_1.$$

Thus the result follows.

Note 1. If $\bar{r} = x\bar{a} + y\bar{b}$, then $x\bar{a}$ and $y\bar{b}$ called components of $\bar{r}$ and the scalars x and y are called the coordinates of $\bar{r}$ relative to the vectors $\bar{a}$ and $\bar{b}$.

Similarly in case of $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}, x\bar{a}, y\bar{b}, z\bar{c}$ are called components of $\bar{r}$ and the scalars x, y, z are called the coordinates of $\bar{r}$ relative to the vectors $\bar{a}, \bar{b}$ and $\bar{c}$.

Note 2. If $x\bar{a} + y\bar{b} = \bar{0}$ and x, y are not zero, then $\bar{a}$ and $\bar{b}$ are linearly dependent and collinear.

Similarly if $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ and x, y, z are not all zero, then the vectors $\bar{a}, \bar{b}, \bar{c}$ are linearly dependent and coplanar.

WORKED-OUT EXAMPLES

1. *If $\bar{a} = (1, 2, 3)$ and $\bar{b} = (2, -3, 4)$, find $\bar{a} + \bar{b}$ and $2\bar{b} - \bar{a}$. Calculate the module and d.cs. of $\bar{a} + \bar{b}$.*

   Here $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}, \bar{b} = 2\bar{i} - 3\bar{j} + 4\bar{k}$ where $\bar{i}, \bar{j}, \bar{k}$ are unit vectors along the three rectangular axes.

   Now $\bar{a} + \bar{b} = (\bar{i} + 2\bar{j} + 3\bar{k}) + (2\bar{i} - 3\bar{j} + 4\bar{k}) = 3\bar{i} - \bar{j} + 7\bar{k} = (3, -1, 7)$, $2\bar{b} - \bar{a} = 2(2\bar{i} - 3\bar{j} + 4\bar{k}) - (\bar{i} + 2\bar{j} + 3\bar{k}) = 3\bar{i} - 8\bar{j} + 5\bar{k} = (3, -8, 5)$.

   $|\bar{a} + \bar{b}| = \sqrt{3^2 + 1^2 + 7^2} = \sqrt{59}$.

   The d.cs. of $\bar{a} + \bar{b}$ are $\frac{3}{\sqrt{59}}, \frac{-1}{\sqrt{59}}, \frac{7}{\sqrt{59}}$.

2. *If the position vectors of A, B, C are $2\bar{i} + 4\bar{j} - \bar{k}, 4\bar{i} + 5\bar{j} + \bar{k}$ and $3\bar{i} + 6\bar{j} - 3\bar{k}$ respectively show that $\triangle ABC$ is right-angled.*

   With O as vector origin $\overrightarrow{OA} = 2\bar{i} + 4\bar{j} - \bar{k}$, $\overrightarrow{OB} = 4\bar{i} + 5\bar{j} + \bar{k}$ and $\overrightarrow{OC} = 3\bar{i} + 6\bar{j} - 3\bar{k}$.

   $$\begin{aligned}
   \text{Now } \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = (4\bar{i} + 5\bar{j} + \bar{k}) - (2\bar{i} + 4\bar{j} - \bar{k}) = 2\bar{i} + \bar{j} + 2\bar{k},\\
   \overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB} = (3\bar{i} + 6\bar{j} - 3\bar{k}) - (4\bar{i} + 5\bar{j} + \bar{k}) = -\bar{i} + \bar{j} - 4\bar{k},\\
   \overrightarrow{CA} &= \overrightarrow{OA} - \overrightarrow{OC} = (2\bar{i} + 4\bar{j} - \bar{k}) - (3\bar{i} + 6\bar{j} - 3\bar{k}) = -\bar{i} - 2\bar{j} + 2\bar{k}.
   \end{aligned}$$

   $$|\overrightarrow{AB}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3, |\overrightarrow{BC}| = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18} = 3\sqrt{2},$$
   $$|\overrightarrow{CA}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3.$$
   $$\therefore \quad AB^2 + CA^2 = 9 + 9 = 18 = BC^2.$$

   Thus $\triangle ABC$ is right-angled.

3. *Show by vector method that the medians of a triangle are concurrent.*

   [BH 96, 2003, 09; CH 2002, NH 2008]

Let ABC be a triangle and $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of A, B, C respectively.
If D is the midpoint of BC, then the position vector of D is $\frac{\bar{b}+\bar{c}}{2}$. If the point G divides AD in the ratio $2:1$, then the position vector of G is

$$\frac{2\frac{\bar{b}+\bar{c}}{2} + 1 \cdot \bar{a}}{2+1}, \quad \text{i.e.} \quad \frac{\bar{a}+\bar{b}+\bar{c}}{3}.$$

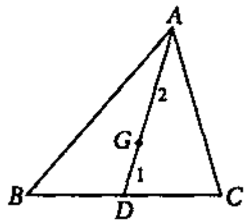


Fig. 10

The symmetry of this expression suggests that the other two medians will be divided in the ratio $2:1$ by the same point G.

Hence they are concurrent.

4. *ABCD is a parallelogram. P, Q are the midpoints of the sides AB and CD respectively. Show that DP and BQ trisect AC and are trisected by AC.*

[CH 2004; BH 2007]

Let the position vectors of A, B, C, D, P and Q be $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{p}$ and $\bar{q}$ respectively.
Since P and Q are the midpoints of AB and CD, $\bar{p} = \frac{\bar{a}+\bar{b}}{2}, \bar{q} = \frac{\bar{c}+\bar{d}}{2}$.
$ABCD$ is a parallelogram.

$\therefore\ \overrightarrow{AB} = \overrightarrow{DC}$ or, $\bar{b} - \bar{a} = \bar{c} - \bar{d}$. (1)

Now $\bar{p} = \frac{\bar{a}+\bar{a}+\bar{c}-\bar{d}}{2}$ [by (1)] or, $2\bar{a} + \bar{c} = 2\bar{p} + \bar{d}$ or, $\frac{2\bar{a}+\bar{c}}{3} = \frac{2\bar{p}+\bar{d}}{3}$.
This results indicate that the common point of AC and PD intersects them in the ratio $1:2$.
Again

$$\bar{q} = \frac{\bar{c}+\bar{d}}{2} = \frac{\bar{c}+\bar{c}+\bar{a}-\bar{b}}{2} \text{ [by (1)]}$$

$$\text{or,} \quad \frac{2\bar{c}+\bar{a}}{3} = \frac{2\bar{q}+\bar{b}}{3}.$$

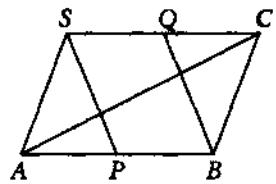


Fig. 11

Therefore, CA and QB are intersected by their common points in the ratio $1:2$.

Hence DP and BQ trisect AC and are trisected by AC.

5. *Show that the lines joining the vertices to the centroids of the opposite faces of a tetrahedron are concurrent.*

Let $ABCD$ be a tetrahedron and $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the position vectors of its vertices.
If X is the centroid of $\triangle BCD$, the position vector of X is $\frac{\bar{b}+\bar{c}+\bar{d}}{3}$.
Let the point G divide AX in the ratio $3:1$. Then the position vector of G is $\frac{\bar{a}+\bar{b}+\bar{c}+\bar{d}}{4}$.
The symmetry of this expression suggests that the other three lines joining the other vertices to the centroids of the opposite faces will be divided in the ratio $3:1$ by the same point G.

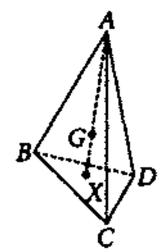


Fig. 12

Hence they are concurrent.

6. *P is a point on BC of $\triangle ABC$. If $\overrightarrow{PQ}$ is the resultant of $\overrightarrow{AP}, \overrightarrow{PB}, \overrightarrow{PC}$, show that $ABQC$ is a parallelogram and Q therefore, is a fixed point.*

Here $\overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PC} = (\overrightarrow{AP} + \overrightarrow{PB}) + \overrightarrow{PC} = \overrightarrow{AB} + \overrightarrow{PC}$. CD is drawn parallel and equal to AB. Therefore, $ABDC$ is a parallelogram.
Now $\overrightarrow{AB} + \overrightarrow{PC} = \overrightarrow{CD} + \overrightarrow{PC} = \overrightarrow{PC} + \overrightarrow{CD} = \overrightarrow{PD}$.
Since $\overrightarrow{PQ}$ is the resultant, D coincides with Q.
Therefore, $ABQC$ is a parallelogram and Q is fixed w.r.t. A, B and C.

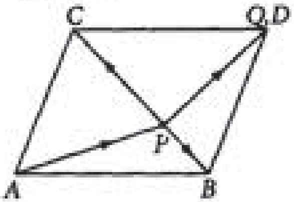


Fig. 13

7. *$\bar{a}$ and $\bar{b}$ are vectors representing consecutive sides of a regular hexagon. Find the vectors forming the other four sides.*

Let $ABCDEF$ be a regular hexagon and $\overrightarrow{AB} = \bar{a}, \overrightarrow{BC} = \bar{b}$. Now $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \bar{a} + \bar{b}$.
In the regular hexagon AD is parallel to BC and $AD = 2BC$.
$\therefore\ \overrightarrow{AD} = 2\overrightarrow{BC} = 2\bar{b}$.
Again $AB \| ED$ and $AB = ED$.
$\therefore\ \overrightarrow{DE} = -\overrightarrow{AB} = -\bar{a}$.
$\overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC} = 2\bar{b} - \bar{a} - \bar{b} = \bar{b} - \bar{a}$.
$\overrightarrow{EF} = -\overrightarrow{BC} = -\bar{b}$.
$\overrightarrow{FA} = -\overrightarrow{CD} = \bar{a} - \bar{b}$.

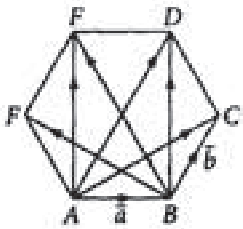


Fig. 14

8. *Show that the points $A(\bar{i} - 2\bar{j} + 3\bar{k})$, $B(2\bar{i} - 3\bar{j} + 4\bar{k})$ and $C(-2\bar{i} + \bar{j})$ are collinear.*

Let $x(\bar{i} - 2\bar{j} + 3\bar{k}) + y(2\bar{i} - 3\bar{j} + 4\bar{k}) + z(-2\bar{i} + \bar{j}) = \bar{0}$ where x, y, z are scalars.

It implies that $x + 2y - 2z = 0$, $-2x - 3y + z = 0$, $3x + 4y = 0$.

The coefficient matrix of these equations and $x + y + z = 0$ is

$$M = \begin{bmatrix} 1 & 2 & -2 \\ -2 & -3 & 1 \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\ \rho(M) = 2 < \text{number of unknowns.}$$

Therefore, the system of homogeneous equations has non-trivial solutions. Hence the points are collinear.

9. *Show by vector method that the points $P(1, 5, -1)$, $Q(0, 4, 5)$, $R(-1, 5, 1)$ and $S(2, 4, 3)$ are coplanar.*

Let O be the vector origin and $\bar{i}, \bar{j}, \bar{k}$ the unit vectors along the three rectangular axes through O. Then

$\overrightarrow{OP} = \bar{r}_1 = \bar{i} + 5\bar{j} - \bar{k}$, $\overrightarrow{OQ} = \bar{r}_2 = 4\bar{j} + 5\bar{k}$, $\overrightarrow{OR} = \bar{r}_3 = -\bar{i} + 5\bar{j} + \bar{k}$, $\overrightarrow{OS} = \bar{r}_4 = 2\bar{i} + 4\bar{j} + 3\bar{k}$.

Let $x\bar{r}_1 + y\bar{r}_2 + z\bar{r}_3 + t\bar{r}_4 = \bar{0}$ where x, y, z, t are scalars. It implies that $x - z + 2t = 0$, $5x + 4y + 5z + 4t = 0$, $-x + 5y + z + 3t = 0$.

The coefficient matrix of these equations and $x + y + z + t = 0$ is

$$M = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 5 & 4 & 5 & 4 \\ -1 & 5 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \ \rho(M) = 3 < \text{number of unknowns.}$$

$\therefore$ the system has non-trivial solution. Hence the points are coplanar.

10. *Show that the vectors $\bar{a} - 2\bar{b} + 3\bar{c}$, $-2\bar{a} + 3\bar{b} - 4\bar{c}$, $-\bar{b} + 2\bar{c}$ are coplanar; $\bar{a}, \bar{b}, \bar{c}$ being any vectors.*

Let $x(\bar{a} - 2\bar{b} + 3\bar{c}) + y(-2\bar{a} + 3\bar{b} - 4\bar{c}) + z(-\bar{b} + 2\bar{c}) = \bar{0}$ (1)

where x, y, z are scalars.

Then $(x - 2y)\bar{a} + (-2x + 3y - z)\bar{b} + (3x - 4y + 2z)\bar{c} = \bar{0}$.

Since $\bar{a}, \bar{b}, \bar{c}$ are any vectors, $x - 2y = 0$, $-2x + 3y - z = 0$, $3x - 4y + 2z = 0$.

These equations are satisfied by $x = 2, y = 1, z = -1$. The relation (1) holds for these non-zero values of x, y, z. Hence the given vectors are coplanar.

EXERCISE I

1. If $\bar{a} = (2, 3, 4)$ and $\bar{b} = (-2, 1, 5)$, find $\bar{a} + \bar{b}$ and $\bar{a} - 2\bar{b}$. Calculate the module of $\bar{a} - 2\bar{b}$ and the d.cs. of it.

2. Determine the values of λ and μ for which the vectors $-3\bar{i} + 4\bar{j} + \lambda\bar{k}$ and $\mu\bar{i} + 8\bar{j} + 6\bar{k}$ are collinear. [NH 2005]

3. If the diagonals of a quadrilateral bisect one another, then prove by vector method that the figure is a parallogram. [BH 2006, CH 2006]

4. If the vertices of a triangle are the points $\bar{i} - \bar{j} + 2\bar{k}$, $2\bar{i} + 3\bar{j} + 4\bar{k}$, $3\bar{i} + 3\bar{j} - 4\bar{k}$, what are the vectors determined by the sides ?

5. If the position vectors of A, B, C are $3\bar{i} + 4\bar{j} - 6\bar{k}$, $4\bar{i} - 6\bar{j} + 3\bar{k}$ and $-6\bar{i} + 3\bar{j} + 4\bar{k}$, show that ABC is an equilateral triangle.

6. The vertices of a triangle are $(2, 4, -1)$, $(4, 5, 1)$ and $(3, 6, -3)$. Show that the triangle is an isosceles right-angled triangle.

7. Show by the method of vectors that the join of the middle points of two sides of a triangle is parallel to the third side and is half of its length. [BH 2009]

8. $ABCD$ is a parallelogram whose diagonals are AC and BD. Prove that $\overrightarrow{AC} + \overrightarrow{BD} = 2\overrightarrow{BC}$, $\overrightarrow{AC} - \overrightarrow{BD} = 2\overrightarrow{AB}$. Also prove that AC and BD bisect each other.

9. If D, E, F are the midpoints of the sides BC, CA, AB of $\triangle ABC$, prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \bar{0}$.

10. $ABCD$ is a quadrilateral and P is a variable point such that $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD}$ is constant. Prove that the locus of P is a circle. [BH 2009]

11. Show that the lines joining the midpoints of the opposite sides of a tetrahedron are concurrent.

12. D and E are any two points on the sides BC and CA of $\triangle ABC$. Prove that the segments AD and BE cannot bisect each other.

13. Show that (i) the points $(1,2,1), (2,3,4)$ and $(4,5,10)$ are collinear; (ii) the points $(\bar{a} - 2\bar{b} + 3\bar{c}), (-2\bar{a} + 3\bar{b} + 2\bar{c}), (-8\bar{a} + 13\bar{b})$ are linearly dependent.

14. Show that the points (i) $6\bar{a} - 4\bar{b} + 10\bar{c}, -5\bar{a} + 3\bar{b} - 10\bar{c}, 4\bar{a} - 6\bar{b} - 10\bar{c}, 2\bar{b} + 10\bar{c}$ and (ii) $(-1,4,-3), (3,2,-5), (-3,8,-5), (-3,2,1)$ are coplanar.

15. If $\bar{r}_1 = \bar{a} + \bar{b} + \bar{c}, \bar{r}_2 = 7\bar{a} + 6\bar{c}, \bar{r}_3 = 2\bar{a} - \bar{b} + \bar{c}, \bar{r}_4 = \bar{a} - \bar{b} - \bar{c}$ where $\bar{a}, \bar{b}, \bar{c}$ are non-zero, non-coplanar vectors, show that $2\bar{r}_1 = \bar{r}_2 - 3\bar{r}_3 + \bar{r}_4$.

16. Show that the following vectors are coplanar.

    (i) $4\bar{a} + 5\bar{b} + \bar{c}, 5\bar{a} + 9\bar{b} + 4\bar{c}, -\bar{b} - \bar{c}$.

    (ii) $\bar{a} - 3\bar{b} + 5\bar{c}, \bar{a} - 2\bar{b} + 3\bar{c}, -2\bar{a} + 3\bar{b} - 4\bar{c}$.

17. Show that the vectors $\bar{i} - 3\bar{j} + 2\bar{k}, 2\bar{i} - 4\bar{j} - \bar{k}, 3\bar{i} + 2\bar{j} - \bar{k}$ are linearly independent.

18. If $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ are vectors from the origin to the points A, B, C, D respectively such that $\bar{\beta} - \bar{\alpha} = \bar{\gamma} - \bar{\delta}$, prove that $ABCD$ is a parallelogram.

19. If $\bar{i}, \bar{j}$ denote unit vectors having the directions of positive x, y-axes respectively of a two-dimensional rectangular coordinate system and $\bar{\alpha} = 2\bar{i} + 3\bar{j}, \bar{\beta} = 4\bar{i} + \bar{j}, \bar{\gamma} = 5\bar{i} + d\bar{j}$ are three vectors having their initial points at the origin, find the value of d so that their extremities may be collinear.

ANSWERS

1. $\bar{a} + \bar{b} = (0,4,9), \bar{a} - 2\bar{b} = (6,1,-6), |\bar{a} - 2\bar{b}| = \sqrt{73}$, d.cs. $\frac{6}{\sqrt{73}}, \frac{1}{\sqrt{73}}, \frac{-6}{\sqrt{73}}$.

2. $\lambda = 3, \mu = -6$.

3. $\bar{i} + 4\bar{j} + 2\bar{k}, \bar{i} - 8\bar{k}, -2\bar{i} - 4\bar{j} + 6\bar{k}$.

19. $d = 0$.

Chapter 2

Products of Vectors

2.10 Introduction

Multiplication between vectors and numbers obeys the rules of simple algebra. Now we define two operations between vectors, which are known as products. These have some properties in common with the products of numbers and some properties which are in striking disagreement with the usual products of numbers. The result of one of these products is scalar and of the other is vector, so one will be called scalar product and the other will be vector product.

2.11 Scalar or dot product

Let $\bar{a}$ and $\bar{b}$ be two vectors and θ the smallest non-negative angle between them. The scalar product of these vectors is written as $\bar{a}\cdot\bar{b}$ and is defined as $|\bar{a}||\bar{b}|\cos\theta$.

$\therefore\ \bar{a}\cdot\bar{b}=|\bar{a}||\bar{b}|\cos\theta.$

It is a scalar quantity.

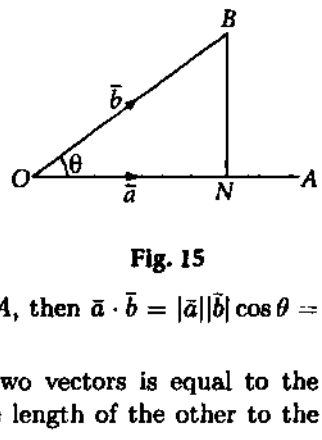


Fig. 15

If $\bar{a}=\overrightarrow{OA}, \bar{b}=\overrightarrow{OB}$ and BN is perpendicular to OA, then $\bar{a}\cdot\bar{b}=|\bar{a}||\bar{b}|\cos\theta= OA\cdot OB\cos\theta=OA\cdot ON$.

Thus geometrically the scalar or dot product of two vectors is equal to the product of the length of one and the projection of the length of the other to the former.

Properties.

(i) *Scalar product is commutative.* $(\bar{a}\cdot\bar{b}=\bar{b}\cdot\bar{a})$.

Since $\bar{a}\cdot\bar{b}=|\bar{a}||\bar{b}|\cos\theta$ and $\bar{b}\cdot\bar{a}=|\bar{b}||\bar{a}|\cos\theta=|\bar{a}||\bar{b}|\cos\theta, \bar{a}\cdot\bar{b}=\bar{b}\cdot\bar{a}$.

(ii) $\bar{a}\cdot\bar{b}$ is positive, zero or negative according as θ is acute, right or obtuse.

If $\bar{a}$ and $\bar{b}$ are proper vectors, i.e. non-zero vectors, then they will be at right angle when $\bar{a}\cdot\bar{b}=0$.

(iii) $\bar{a}\cdot\bar{a}=|\bar{a}||\bar{a}|\cos 0=|\bar{a}|^2=\bar{a}^2$.

If $\bar{i}, \bar{j}, \bar{k}$ are the unit vectors along three rectangular axes, then $\bar{i} \cdot \bar{i} = \bar{i}^2 = 1 = \bar{j}^2 = \bar{k}^2$. Again $\bar{i} \cdot \bar{j} = 0 = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{i}$.

(iv) **Distributive law.** $\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$.

Let $\overrightarrow{OA} = \bar{a}, \overrightarrow{OB} = \bar{b}$ and $\overrightarrow{BC} = \bar{c}$, then $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \bar{b} + \bar{c}$.

$$\begin{aligned} \text{Now } \bar{a} \cdot (\bar{b} + \bar{c}) &= OA \cdot \text{proj. of } OC \text{ on } OA \\ &= OA \cdot ON = OA \cdot (OM + MN) \\ &= OA \cdot OM + OA \cdot MN. \\ &= OA \cdot \text{proj. of } OB \text{ on } OA + OA \cdot \\ &\quad\ \text{proj. of } BC \text{ on } OA \\ &= \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}. \quad \text{(see Fig. 16).} \end{aligned}$$

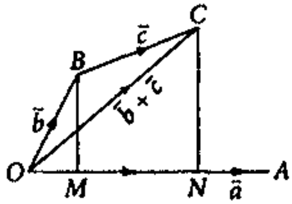


Fig. 16

(v) If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$,
then $\bar{a} \cdot \bar{b} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}) \cdot (b_1\bar{i} + b_2\bar{j} + b_3\bar{k}) = a_1b_1 + a_2b_2 + a_3b_3$.

Again $\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos\theta$, where θ is the acute angle between $\bar{a}$ and $\bar{b}$.

$$\therefore \cos\theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

(vi) *If $\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}$, then $\bar{b}$ may not be equal to $\bar{c}$.*

$\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}$ or, $\bar{a} \cdot (\bar{b} - \bar{c}) = 0$.

Here $\bar{a} = \bar{0}$ or, $\bar{b} - \bar{c} = \bar{0}$, i.e. $\bar{b} = \bar{c}$ or, $\bar{a}$ is perpendicular to $\bar{b} - \bar{c}$.

2.12 Vector or Cross Product

Let $\bar{a}$ and $\bar{b}$ be two vectors and θ the smallest non-negative acute angle between them. The vector product between them is written as $\bar{a} \times \bar{b}$ and is defined as $|\bar{a}||\bar{b}| \sin\theta\, \bar{n}$, where $\bar{n}$ is a unit vector perpendicular to $\bar{a}$ and $\bar{b}$ and its direction coincides with the direction of the motion of a right-handed screw rotating from a to $\bar{b}$.

In Fig. 17, $\overrightarrow{OA} = \bar{a}, \overrightarrow{OB} = \bar{b}, \angle AOB = \theta$ and $\bar{n}$ is a unit vector perpendicular to the plane of OA and OB.

$\therefore\ \bar{a} \times \bar{b} = OA \cdot OB \sin\theta\, \bar{n}$.

It is a vector quantity.

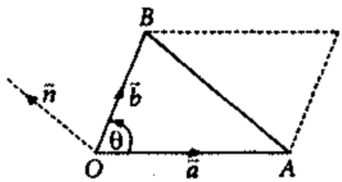


Fig. 17

Properties.

(i) *Vector product is non-commutative*, i.e. $\bar{a} \times \bar{b} \neq \bar{b} \times \bar{a}$.

The directions of $\bar{a} \times \bar{b}$ and $\bar{b} \times \bar{a}$ are opposite to each other.

$\therefore\ \bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$.

(ii) $|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}| \sin\theta = OA \cdot OB \sin\theta = 2\triangle OAB =$ *area of the parallelogram whose adjacent sides are OA and OB.*

Thus $\bar{a}\times\bar{b}$ is called a vector area of a parallelogram whose adjacent sides represent $\bar{a}$ and $\bar{b}$.

(iii) *If $\bar{a}$ and $\bar{b}$ are both proper vectors and $\bar{a}\times\bar{b}=\bar{0}$, then $\bar{a}$ and $\bar{b}$ are parallel.*

(iv) $\bar{a}\times\bar{a}=\bar{0}$, since $\theta=0$.

If $\bar{i},\bar{j},\bar{k}$ are the unit vectors along three rectangular axes, then $\bar{i}\times\bar{i}=\bar{0}=\bar{j}\times\bar{j}=\bar{k}\times\bar{k}$.

Again $\bar{i}\times\bar{j}=\bar{k},\bar{j}\times\bar{k}=\bar{i},\bar{k}\times\bar{i}=\bar{j}$ and $\bar{j}\times\bar{i}=-\bar{k},\bar{k}\times\bar{j}=-\bar{i},\bar{i}\times\bar{k}=-\bar{j}$.

(v) $(\bar{a}\times\bar{b})^2=(\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{b})=|\bar{a}\times\bar{b}|^2=|\bar{a}|^2|\bar{b}|^2\sin^2\theta=|\bar{a}|^2|\bar{b}|^2(1-\cos^2\theta)$
$$=|\bar{a}|^2|\bar{b}|^2-|\bar{a}|^2|\bar{b}|^2\cos^2\theta=\bar{a}^2\bar{b}^2-(\bar{a}\cdot\bar{b})^2.$$

(vi) *If $\bar{a}\times\bar{b}=\bar{c}$, then $\bar{c}$ is perpendicular to $\bar{a}$ and $\bar{b}$.*

Consequently $\bar{a}\cdot\bar{c}=0$ and $\bar{b}\cdot\bar{c}=0$.

If $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k},\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$, then $\bar{c}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$.

Let $\bar{c}=c_1\bar{i}+c_2\bar{j}+c_3\bar{k}$. Then $a_1c_1+a_2c_2+a_3c_3=0$ and $b_1c_1+b_2c_2+b_3c_3=0$.

By cross-multiplication,

$$\frac{c_1}{a_2b_3-a_3b_2}=\frac{c_2}{a_3b_1-a_1b_3}=\frac{c_3}{a_1b_2-a_2b_1}\ (=\lambda \text{ say}).$$

$\therefore\ c_1=\lambda(a_2b_3-a_3b_2),c_2=\lambda(a_3b_1-a_1b_3),c_3=\lambda(a_1b_2-a_2b_1)$.

$$\begin{aligned}\text{Now } \bar{c}^2&=c_1^2+c_2^2+c_3^2\\&=\lambda^2[(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2]\\&=\lambda^2[a_1^2(b_2^2+b_3^2)+a_2^2(b_3^2+b_1^2)+a_3^2(b_1^2+b_2^2)\\&\qquad-2(a_2b_2a_3b_3+a_3b_3a_1b_1+a_1b_1a_2b_2)]\\&=\lambda^2[(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2]\\&=\lambda^2[\bar{a}^2\bar{b}^2-(\bar{a}\cdot\bar{b})^2]=\lambda^2\{\bar{a}^2\bar{b}^2-\bar{a}^2\bar{b}^2\cos^2\theta\}\\&=\lambda^2\bar{a}^2\bar{b}^2(1-\cos^2\theta)=\lambda^2\bar{a}^2\bar{b}^2\sin^2\theta.\end{aligned}$$

But $\bar{c}^2=\bar{a}^2\bar{b}^2\sin^2\theta$. $\therefore\ \lambda^2=1$ or, $\lambda=\pm1$.

If we take $\bar{a}=\bar{i},\bar{b}=\bar{j}$, then $\bar{c}=\bar{i}\times\bar{j}=\bar{k}$.

In this case, $a_1=b_2=c_3=1,a_2=a_3=b_1=b_3=c_1=c_2=0$.

We have $c_3=\lambda(a_1b_2-a_2b_1)$ or, $1=\lambda\cdot1$ or, $\lambda=1$.

Thus taking $\lambda=1$,

$$\begin{aligned}\bar{a}\times\bar{b}&=c_1\bar{i}+c_2\bar{j}+c_3\bar{k}\\&=(a_2b_3-a_3b_2)\bar{i}+(a_3b_1-a_1b_3)\bar{j}+(a_1b_2-a_2b_1)\bar{k}\\&=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}.\end{aligned}$$

(vii) Distributive law. $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$.

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}, \bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}, \bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$.

Then $\bar{b} + \bar{c} = (b_1 + c_1)\bar{i} + (b_2 + c_2)\bar{j} + (b_3 + c_3)\bar{k}$.

By (vi) $\bar{a} \times (\bar{b} + \bar{c})$

$$\begin{aligned}
&= \{a_2(b_3 + c_3) - a_3(b_2 + c_2)\}\bar{i} + \{a_3(b_1 + c_1) - a_1(b_3 + c_3)\}\bar{j} \\
&\quad + \{a_1(b_2 + c_2) - a_2(b_1 + c_1)\}\bar{k} \\
&= \{(a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2)\}\bar{i} + \{(a_3b_1 - a_1b_3) + (a_3c_1 - a_1c_3)\}\bar{j} \\
&\quad + \{(a_1b_2 - a_2b_1) + (a_1c_2 - a_2c_1)\}\bar{k} \\
&= \{(a_2b_3 - a_3b_2)\bar{i} + (a_3b_1 - a_1b_3)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}\} \\
&\quad + \{(a_2c_3 - a_3c_2)\bar{i} + (a_3c_1 - a_1c_3)\bar{j} + (a_1c_2 - a_2c_1)\bar{k}\} \\
&= \bar{a} \times \bar{b} + \bar{a} \times \bar{c}.
\end{aligned}$$

Similarly $(\bar{b} + \bar{c}) \times \bar{a} = \bar{b} \times \bar{a} + \bar{c} \times \bar{a}$.

(viii) *If* $\bar{a} \times \bar{b} = \bar{a} \times \bar{c}, \bar{b}$ *may not be equal to* $\bar{c}$.

$\bar{a} \times \bar{b} = \bar{a} \times \bar{c}$ *or,* $\bar{a} \times (\bar{b} - \bar{c}) = \bar{0}$.

It shows that either $\bar{a} = \bar{0}$ or, $\bar{b} - \bar{c} = \bar{0}$, i.e. $\bar{b} = \bar{c}$ or, $\bar{a}$ is parallel to $\bar{b} - \bar{c}$.

(ix) If $\bar{a} = (a_1, a_2, a_3), \bar{b} = (b_1, b_2, b_3)$ and θ be the angle between them, then

$$|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}| \sin\theta$$

$$\text{or,}\quad \sin\theta = \frac{\sqrt{\{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2\}}}{\sqrt{(a_1^2 + a_2^2 + a_3^2)} \cdot \sqrt{(b_1^2 + b_2^2 + b_3^2)}}.$$

2.20 Triple product

(a) Scalar triple product.

Let $\bar{a}, \bar{b}$ and $\bar{c}$ be any three vectors. The expression $\bar{a} \cdot (\bar{b} \times \bar{c})$ is known as a scalar triple product of $\bar{a}, \bar{b}, \bar{c}$. It is a scalar and is numerically equal to the volume v of the parallelopiped whose coterminus edges represent $\bar{a}, \bar{b}$ and $\bar{c}$.

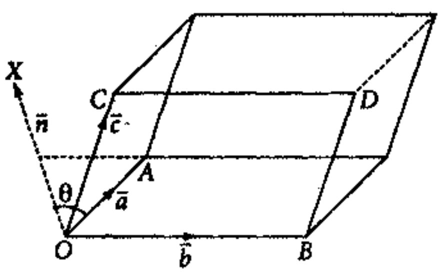


Fig. 18

Let $\bar{a}, \bar{b}$ and $\bar{c}$ be the concurrent edges OA, OB, OC of the parallelopipied as shown in Fig. 18.

$\bar{b} \times \bar{c} = OB \cdot OC \sin BOC \bar{n}$, where $\bar{n}$ is the unit vector perpendicular to the plane $OBDC$ in the direction of OX.

Let θ be the angle between OX and OA.

Now $\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{n})\cdot$ (area of parallelogram $OBDC$)

$= (OA\cos\theta)\cdot$ (area of $OBDC$).

$OA\cos\theta =$ projection of OA on $OX =$ distance of A from the plane $OBDC$.

$\therefore\ \bar{a}\cdot(\bar{b}\times\bar{c}) =$ (area of $OBDC$) $\cdot$ (distance of A from the plane $OBDC$)

$=$ (area of the base) $\cdot$ (altitude of the parallelopiped)

$=$ volume of the parallelopiped.

Properties.

(i) $\bar{a}\cdot(\bar{b}\times\bar{c})$ is denoted by $[\bar{a}\bar{b}\bar{c}]$. (box notation)

$[\bar{a}\bar{b}\bar{c}] = [\bar{b}\bar{c}\bar{a}] = [\bar{c}\bar{a}\bar{b}] = -[\bar{b}\bar{a}\bar{c}] = -[\bar{a}\bar{c}\bar{b}] = -[\bar{c}\bar{b}\bar{a}]$.

(ii) If $\bar{i},\bar{j},\bar{k}$ are the three unit vectors along the rectangular axes, then $[\bar{i}\bar{j}\bar{k}] = [\bar{j}\bar{k}\bar{i}] = [\bar{k}\bar{i}\bar{j}] = 1$, $[\bar{i}\bar{k}\bar{j}] = [\bar{k}\bar{j}\bar{i}] = [\bar{j}\bar{i}\bar{k}] = -1$.

(iii) If $\bar{a} = (a_1,a_2,a_3), \bar{b} = (b_1,b_2,b_3)$ and $\bar{c} = (c_1,c_2,c_3)$, then $\bar{b}\times\bar{c} = (b_2c_3 - b_3c_2)\bar{i} + (b_3c_1 - b_1c_3)\bar{j} + (b_1c_2 - b_2c_1)\bar{k}$.

$$\text{Hence } \bar{a}\cdot(\bar{b}\times\bar{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

(iv) **$\bar{a}\cdot(\bar{b}\times\bar{c}) = 0$ is the necessary and sufficient condition for the coplanarity of the three proper vectors $\bar{a},\bar{b},\bar{c}$.**

Proof. The condition is necessary.

Let the vectors $\bar{a},\bar{b},\bar{c}$ lie in the same plane.

Since $\bar{b}\times\bar{c}$ is a vector perpendicular to the plane of $\bar{b}$ and $\bar{c}$, it is perpendicular to $\bar{a}$. Hence $\bar{a}\cdot(\bar{b}\times\bar{c}) = 0$.

The condition is sufficient.

Let $\bar{a}\cdot(\bar{b}\times\bar{c}) = 0$. $\bar{a}$ is perpendicular to $\bar{b}\times\bar{c}$. $\bar{b}\times\bar{c}$ is a vector perpendicular to the plane of $\bar{b}$ and $\bar{c}$. Therefore, $\bar{a}$ is perpendicular to the normal to the plane of $\bar{b}$ and $\bar{c}$. Hence $\bar{a}$ lies in the plane of $\bar{b}$ and $\bar{c}$, i.e. $\bar{a},\bar{b},\bar{c}$ lie in the same plane.

(v) **Proof of the distributive law of the cross product by scalar triple product.**

Let $\bar{p} = \bar{a}\times(\bar{b}+\bar{c}) - \bar{a}\times\bar{b} - \bar{a}\times\bar{c}$. Considering the dot product with a proper vector $\bar{q}$ which is not perpendicular to $\bar{p}$,

$$\begin{aligned} \bar{q}\cdot\bar{p} &= \bar{q}\cdot\{\bar{a}\times(\bar{b}+\bar{c})\} - \bar{q}\cdot(\bar{a}\times\bar{b}) - \bar{q}\cdot(\bar{a}\times\bar{c}) \\ &= (\bar{q}\times\bar{a})\cdot(\bar{b}+\bar{c}) - \bar{q}\cdot(\bar{a}\times\bar{b}) - \bar{q}\cdot(\bar{a}\times\bar{c}) \\ &= (\bar{q}\times\bar{a})\cdot\bar{b} + (\bar{q}\times\bar{a})\cdot\bar{c} - (\bar{q}\times\bar{a})\cdot\bar{b} - (\bar{q}\times\bar{a})\cdot\bar{c} = 0. \end{aligned}$$

$$\therefore\ \bar{p} = \bar{0}. \qquad \therefore\quad \bar{a}\times(\bar{b}+\bar{c}) = \bar{a}\times\bar{b} + \bar{a}\times\bar{c}.$$

(b) **Vector triple product.**

Let $\bar{a}, \bar{b}, \bar{c}$ be any three vectors. The expression $\bar{a} \times (\bar{b} \times \bar{c})$ is a vector and is called a vector triple product of $\bar{a}, \bar{b}, \bar{c}$. This vector lies in the plane of $\bar{b}$ and $\bar{c}$.

Now we shall prove that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$.

Let $\bar{i}, \bar{j}, \bar{k}$ be the unit vectors along the three rectangular axes. Without any loss of generality we may choose $\bar{a}$ along $\bar{i}$, $\bar{b}$ in the plane of $\bar{i}$ and $\bar{j}$. Thus we can express $\bar{a}, \bar{b}$ and $\bar{c}$ in the following form.

$$\bar{a} = a_1\bar{i}, \bar{b} = b_1\bar{i} + b_2\bar{j}, \bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}.$$

$$\text{Now} \qquad \bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = b_2c_3\bar{i} - b_1c_3\bar{j} + (b_1c_2 - b_2c_1)\bar{k}.$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & 0 & 0 \\ b_2c_3 & -b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix} = -a_1(b_1c_2 - b_2c_1)\bar{j} - a_1b_1c_3\bar{k}.$$

$$\begin{aligned} \bar{a} \cdot \bar{c} &= a_1\bar{i} \cdot (c_1\bar{i} + c_2\bar{j} + c_3\bar{k}) = a_1c_1, \\ (\bar{a} \cdot \bar{c})\bar{b} &= a_1b_1c_1\bar{i} + a_1b_2c_1\bar{j}, \\ \bar{a} \cdot \bar{b} &= a_1\bar{i} \cdot (b_1\bar{i} + b_2\bar{j}) = a_1b_1, \\ (\bar{a} \cdot \bar{b})\bar{c} &= a_1b_1c_1\bar{i} + a_1b_1c_2\bar{j} + a_1b_1c_3\bar{k}. \end{aligned}$$

$$\therefore \ (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} = -a_1(b_1c_2 - b_2c_1)\bar{j} - a_1b_1c_3\bar{k} = \bar{a} \times (\bar{b} \times \bar{c}).$$

Note 1. $\bar{a} \times (\bar{b} \times \bar{c}) = -(\bar{b} \times \bar{c}) \times \bar{a}$.

$\therefore \ (\bar{b} \times \bar{c}) \times \bar{a} = (\bar{a} \cdot \bar{b})\bar{c} - (\bar{a} \cdot \bar{c})\bar{b}$.

Note 2. If $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \times \bar{c}$, then $(\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{b} \cdot \bar{c})\bar{a}$.

$\therefore \ (\bar{b} \cdot \bar{c})\bar{a} = (\bar{a} \cdot \bar{b})\bar{c}$ or, $(\bar{c} \times \bar{a}) \times \bar{b} = \bar{0}$.

For this $\bar{c}$ is parallel to $\bar{a}$ or, $\bar{b}$ is perpendicular to $\bar{a}$ and $\bar{c}$.

2.21 Product of four vectors

(a) **Scalar product of four vectors.** $(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d})$.

We shall prove that $(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = \begin{vmatrix} \bar{a} \cdot \bar{c} & \bar{a} \cdot \bar{d} \\ \bar{b} \cdot \bar{c} & \bar{b} \cdot \bar{d} \end{vmatrix}$.

Let $\bar{a} \times \bar{b} = \bar{p}$.

$$\begin{aligned} (\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) &= \bar{p} \cdot (\bar{c} \times \bar{d}) = (\bar{p} \times \bar{c}) \cdot \bar{d} = \{(\bar{a} \times \bar{b}) \times \bar{c}\} \cdot \bar{d} \\ &= \{(\bar{a} \cdot \bar{c})\bar{b} - (\bar{b} \cdot \bar{c})\bar{a}\} \cdot \bar{d} = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{a} \cdot \bar{d})(\bar{b} \cdot \bar{c}) \\ &= \begin{vmatrix} \bar{a} \cdot \bar{c} & \bar{a} \cdot \bar{d} \\ \bar{b} \cdot \bar{c} & \bar{b} \cdot \bar{d} \end{vmatrix}. \end{aligned}$$

(b) **Vector product of four vectors.** $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})$.

We shall prove that

$$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = [\bar{a}\bar{b}\bar{d}]\bar{c} - [\bar{a}\bar{b}\bar{c}]\bar{d} = [\bar{a}\bar{c}\bar{d}]\bar{b} - [\bar{b}\bar{c}\bar{d}]\bar{a}.$$

Let $\bar{a} \times \bar{b} = \bar{p}$.

Now
$$\begin{aligned}(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) &= \bar{p} \times (\bar{c} \times \bar{d}) = (\bar{p} \cdot \bar{d})\bar{c} - (\bar{p} \cdot \bar{c})\bar{d} \\ &= \{(\bar{a} \times \bar{b}) \cdot \bar{d}\}\bar{c} - \{(\bar{a} \times \bar{b}) \cdot \bar{c}\}\bar{d} = [\bar{a}\bar{b}\bar{d}]\bar{c} - [\bar{a}\bar{b}\bar{c}]\bar{d}.\end{aligned}$$

Let $\bar{c} \times \bar{d} = \bar{q}$.

Then
$$\begin{aligned}(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) &= (\bar{a} \times \bar{b}) \times \bar{q} = (\bar{q} \cdot \bar{a})\bar{b} - (\bar{q} \cdot \bar{b})\bar{a} \\ &= \{(\bar{c} \times \bar{d}) \cdot \bar{a}\}\bar{b} - \{(\bar{c} \times \bar{d}) \cdot \bar{b}\}\bar{a} \\ &= [\bar{c}\bar{d}\bar{a}]\bar{b} - [\bar{c}\bar{d}\bar{b}]\bar{a} = [\bar{a}\bar{c}\bar{d}]\bar{b} - [\bar{b}\bar{c}\bar{d}]\bar{a}.\end{aligned}$$

Hence the result follows.

Corollary I.

$$[\bar{a}\bar{b}\bar{d}]\bar{c} - [\bar{a}\bar{b}\bar{c}]\bar{d} = [\bar{a}\bar{c}\bar{d}]\bar{b} - [\bar{b}\bar{c}\bar{d}]\bar{a}$$

$$\text{or, } [\bar{a}\bar{b}\bar{c}]\bar{d} = [\bar{b}\bar{c}\bar{d}]\bar{a} - [\bar{a}\bar{c}\bar{d}]\bar{b} + [\bar{a}\bar{b}\bar{d}]\bar{c} = [\bar{d}\bar{b}\bar{c}]\bar{a} + [\bar{d}\bar{c}\bar{a}]\bar{b} + [\bar{d}\bar{a}\bar{b}]\bar{c}$$

$$\text{or, } \bar{d} = \frac{[\bar{d}\bar{b}\bar{c}]}{[\bar{a}\bar{b}\bar{c}]}\bar{a} + \frac{[\bar{d}\bar{c}\bar{a}]}{[\bar{a}\bar{b}\bar{c}]}\bar{b} + \frac{[\bar{d}\bar{a}\bar{b}]}{[\bar{a}\bar{b}\bar{c}]}\bar{c}.$$

It shows that $\bar{d}$ can be expressed as a linear combination of three non-coplanar vectors $\bar{a}, \bar{b}, \bar{c}$. Thus any vector $\bar{r}$ can be expressed as a linear combination of three non-coplanar vectors $\bar{a}, \bar{b}, \bar{c}$ in the form

$$\bar{r} = \frac{[\bar{r}\bar{b}\bar{c}]}{[\bar{a}\bar{b}\bar{c}]}\bar{a} + \frac{[\bar{r}\bar{c}\bar{a}]}{[\bar{a}\bar{b}\bar{c}]}\bar{b} + \frac{[\bar{r}\bar{a}\bar{b}]}{[\bar{a}\bar{b}\bar{c}]}\bar{c}.$$

Corollary II. *If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar, then* $[\bar{a}\bar{b}\bar{d}] = 0 = [\bar{a}\bar{b}\bar{c}]$.

$\therefore\ (\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = \bar{0}$.

Corollary III. Reciprocal vectors.

If $\bar{a}, \bar{b}, \bar{c}$ form a set of three non-coplanar vectors and

$$\bar{a}' = \frac{\bar{b} \times \bar{c}}{[\bar{a}\bar{b}\bar{c}]}, \bar{b}' = \frac{\bar{c} \times \bar{a}}{[\bar{c}\bar{a}\bar{b}]} \text{ and } \bar{c}' = \frac{\bar{a} \times \bar{b}}{[\bar{a}\bar{b}\bar{c}]},$$

then $\bar{a}', \bar{b}', \bar{c}'$ are called reciprocal to the set $\bar{a}, \bar{b}, \bar{c}$.

Now any vector $\bar{r}$ can be expressed as

$$\bar{r} = (\bar{r} \cdot \bar{a}')\bar{a} + (\bar{r} \cdot \bar{b}')\bar{b} + (\bar{r} \cdot \bar{c}')\bar{c} \text{ or, } \bar{r} = (\bar{r} \cdot \bar{a})\bar{a}' + (\bar{r} \cdot \bar{b})\bar{b}' + (\bar{r} \cdot \bar{c})\bar{c}'.$$

WORKED-OUT EXAMPLES

1. *If $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ be three vectors defined by $\bar{\alpha} = \lambda(2\bar{i} + 3\bar{j} + 6\bar{k}), \bar{\beta} = \lambda(3\bar{i} - 6\bar{j} + 2\bar{k})$ and $\bar{\gamma} = \lambda(6\bar{i} + 2\bar{j} - 3\bar{k})$, where λ is a scalar and $\bar{i}, \bar{j}, \bar{k}$ are three orthogonal unit vectors, determine λ for which $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ may be each of length unity. Also prove that $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$ are mutually perpendicular.*

   Here $|\bar{\alpha}| = \lambda\sqrt{2^2 + 3^2 + 6^2} = 7\lambda, |\beta| = \lambda\sqrt{3^2 + 6^2 + 2^2} = 7\lambda$, $|\bar{\gamma}| = \lambda\sqrt{6^2 + 2^2 + 3^2} = 7\lambda$.

   $|\bar{\alpha}|, |\bar{\beta}|, |\bar{\gamma}|$ will be unity, if $\lambda = \frac{1}{7}$.

$$\bar{\alpha} \cdot \bar{\beta} = \lambda^2(2\bar{i} + 3\bar{j} + 6\bar{k}) \cdot (3\bar{i} - 6\bar{j} + 2\bar{k}) = \lambda^2(6 - 18 + 12) = 0.$$
$$\bar{\beta} \cdot \bar{\gamma} = \lambda^2(3\bar{i} - 6\bar{j} + 2\bar{k}) \cdot (6\bar{i} + 2\bar{j} - 3\bar{k}) = \lambda^2(18 - 12 - 6) = 0.$$
$$\bar{\gamma} \cdot \bar{\alpha} = \lambda^2(6\bar{i} + 2\bar{j} - 3\bar{k}) \cdot (2\bar{i} + 3\bar{j} + 6\bar{k}) = \lambda^2(12 + 6 - 18) = 0.$$

   $\therefore$ $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are mutually orthogonal.

2. *The position vectors of two points P and Q are respectively $3\bar{i} + 7\bar{j} - 4\bar{k}$ and $6\bar{i} - 2\bar{j} + 12\bar{k}$, where $\bar{i}, \bar{j}, \bar{k}$ are unit vectors parallel to the axes of co rdinates. Calculate the angle between OP and OQ, where O is the origin.*

   Here $\overrightarrow{OP} = 3\bar{i} + 7\bar{j} - 4\bar{k}, \overrightarrow{OQ} = 6\bar{i} - 2\bar{j} + 12\bar{k}$.

   If θ is the angle between them

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = |\overrightarrow{OP}||\overrightarrow{OQ}| \cos\theta$$
$$\text{or,} \quad (3\bar{i} + 7\bar{j} - 4\bar{k}) \cdot (6\bar{i} - 2\bar{j} + 12\bar{k}) = |3\bar{i} + 7\bar{j} - 4\bar{k}||6\bar{i} - 2\bar{j} + 12\bar{k}| \cos\theta$$
$$\text{or,} \quad 18 - 14 - 48 = \sqrt{3^2 + 7^2 + 4^2}\sqrt{6^2 + 2^2 + 12^2} \cos\theta$$
$$\text{or,} \quad -44 = \sqrt{74}\sqrt{184} \cos\theta$$
$$\text{or,} \quad \cos\theta = \frac{-44}{\sqrt{74} \cdot \sqrt{184}} = \frac{-11}{\sqrt{37} \cdot \sqrt{23}} = \frac{-11}{\sqrt{851}}.$$
$$\therefore \quad \theta = \cos^{-1}\left(\frac{-11}{\sqrt{851}}\right).$$

3. *Given $\bar{a} = \bar{i} - 2\bar{j}, \bar{b} = \bar{j} + \bar{k}$, find the component of $\bar{a}$ along $\bar{b}$. Also find $\bar{a} \times \bar{b}$.*

   [BH 2007]

   The component of $\bar{a}$ along $\bar{b} = |\bar{a}| \cos\theta \frac{\bar{b}}{|\bar{b}|}$, where θ is the angle between $\bar{a}$ and $\bar{b}$

$$= |\bar{a}| \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} \frac{\bar{b}}{|\bar{b}|} = \frac{\bar{a} \cdot \bar{b}}{|\bar{b}|^2} \bar{b}$$
$$= \frac{(\bar{i} - 2\bar{j}) \cdot (\bar{j} + \bar{k})}{\left(\sqrt{1^2 + 1^2}\right)^2} (\bar{j} + \bar{k}) = \frac{-2}{2}(\bar{j} + \bar{k}) = -(\bar{j} + \bar{k}).$$

$$\bar{a} \times \bar{b} = (\bar{i} - 2\bar{j}) \times (\bar{j} + \bar{k}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -2\bar{i} - \bar{j} + \bar{k}.$$

4. *Find the unit vector perpendicular to each of*

$$\bar{a} = 6\bar{i} + 2\bar{j} + 3\bar{k} \text{ and } \bar{b} = 3\bar{i} - 6\bar{j} - 2\bar{k}.$$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 6 & 2 & 3 \\ 3 & -6 & -2 \end{vmatrix} = 14\bar{i} + 21\bar{j} - 42\bar{k}.$$

It is a vector perpendicular to $\bar{a}$ and $\bar{b}$. Therefore, the required unit vector $= \frac{14\bar{i}+21\bar{j}-42\bar{k}}{\sqrt{14^2+21^2+42^2}} = \frac{1}{7}(2\bar{i} + 3\bar{j} - 6\bar{k})$.

5. *Prove, by vector method, the trigonometrical formula* $\cos C = \frac{a^2+b^2-c^2}{2ab}$.

[BH 91]

Let ABC be a triangle.

Now $\quad \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

or, $\quad \overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC} = \overrightarrow{CB} - \overrightarrow{CA}$

or, $\quad \overrightarrow{AB}^2 = \overrightarrow{CB}^2 + \overrightarrow{CA}^2 - 2\overrightarrow{CA} \cdot \overrightarrow{CB}$

or, $\quad c^2 = a^2 + b^2 - 2ba\cos C$

or, $\quad \cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

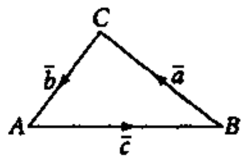


Fig. 19

6. *Prove the formula* $c = a\cos b + b\cos A$ *by vector method.*

In $\triangle ABC$ (Fig. 19) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \bar{0}$.

$\therefore\ \overrightarrow{AB} \cdot (\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) = 0$ or, $\overrightarrow{AB}^2 + \overrightarrow{AB} \cdot \overrightarrow{BC} + \overrightarrow{AB} \cdot \overrightarrow{CA} = 0$

or, $\quad c^2 + ca\cos(\pi - B) + cb\cos(\pi - A) = 0$

or, $\quad c - a\cos B - b\cos A = 0$ or, $c = a\cos B + b\cos A$.

7. *Prove by vector method* $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ *in a triangle ABC.*

[NH 2004, 05, 07; BH 99; CH 94, '96, '99]

From $\triangle ABC$ (Fig. 19), $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \bar{0}$.

$\therefore\ \overrightarrow{AB} \times (\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) = \bar{0}$ or, $\overrightarrow{AB} \times \overrightarrow{BC} + \overrightarrow{AB} \times \overrightarrow{CA} = \bar{0}$

or, $\quad AB \cdot BC \sin(\pi - B)\bar{n} - AB \cdot CA \sin(\pi - A)\bar{n} = \bar{0}$

(where $\bar{n}$ is the unit vector perpendicular to the plane of $\triangle ABC$)

or, $\quad a\sin B - b\sin A = 0 \quad$ or, $\quad \frac{a}{\sin A} = \frac{b}{\sin B}$.

Similarly considering cross-multiplication by $\overrightarrow{BC}$, we get $\frac{b}{\sin B} = \frac{c}{\sin C}$.

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

8. Show by vector method, that the angle in a semi-circle is a right angle.

[NH 2003; BH 2001]

Let P be a point on the semi-circle APB, whose centre is O. Considering O as vector origin,

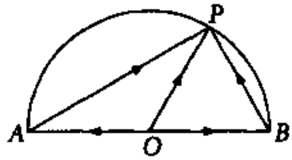


Fig. 20

$$\begin{aligned}\overrightarrow{AP}\cdot\overrightarrow{BP} &= (\overrightarrow{OP}-\overrightarrow{OA})\cdot(\overrightarrow{OP}-\overrightarrow{OB})\\ &= (\overrightarrow{OP}-\overrightarrow{OA})\cdot(\overrightarrow{OP}+\overrightarrow{OA})\\ &\qquad (\because\ \overrightarrow{OB}=-\overrightarrow{OA})\\ &= \overrightarrow{OP}^2-\overrightarrow{OA}^2=0\ [\because\ OP=OA=\text{radius}].\end{aligned}$$

$\therefore$ PA and PB are at right angle. Hence the result follows.

9. *Show by vector method, that the perpendicular from the vertices of a triangle to the opposite sides are concurrent.* **[BH 2002, 07; CH 94, 2000, 07]**

Let ABC be a triangle and AD, BE perpendiculars to BC and CA from A and B respectively.

Let them meet at O. CO is joined and produced to meet at F.

With O as vector origin

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}, \overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}$.

$\because$ AD is perpendicular to BC,

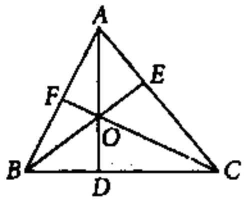


Fig. 21

$$\overrightarrow{BC}\cdot\overrightarrow{OA}=0 \quad \text{or,} \quad (\overrightarrow{OC}-\overrightarrow{OB})\cdot\overrightarrow{OA}=0. \quad (1)$$

Similarly

$$\overrightarrow{CA}\cdot\overrightarrow{OB}=0 \quad \text{or,} \quad (\overrightarrow{OA}-\overrightarrow{OC})\cdot\overrightarrow{OB}=0. \quad (2)$$

From (1) and (2), $\overrightarrow{OC}\cdot\overrightarrow{OA}-\overrightarrow{OC}\cdot\overrightarrow{OB}=0$

or, $(\overrightarrow{OA}-\overrightarrow{OB})\cdot\overrightarrow{OC}=0$ or, $\overrightarrow{BA}\cdot\overrightarrow{OC}=0$.

$\therefore$ CF is perpendicular to AB.

Hence the result follows.

10. *If the two medians of a triangle are equal, show that the triangle is isosceles.* **[BH 2000]**

Let BE and CF be two medians of $\triangle ABC$ (Fig. 22).

Let $\overrightarrow{AB}=\bar{b}$ and $\overrightarrow{AC}=\bar{c}$ with respect to the vector origin A.

Now $\overrightarrow{CF}=\overrightarrow{AF}-\overrightarrow{AC}=\frac{\bar{b}}{2}-\bar{c}$ and $\overrightarrow{BE}=\overrightarrow{AE}-\overrightarrow{AB}=\frac{\bar{c}}{2}-\bar{b}$.

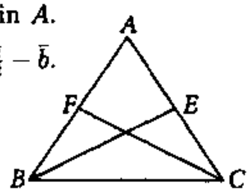


Fig. 22

$$\because\ |\overrightarrow{BE}|=|\overrightarrow{CF}|, \left(\frac{\bar{c}}{2}-\bar{b}\right)^2=\left(\frac{\bar{b}}{2}-\bar{c}\right)^2$$

$$\text{or,}\quad \frac{\bar{c}^2}{4}+\bar{b}^2-2\left(\bar{b}\cdot\frac{\bar{c}}{2}\right)=\frac{\bar{b}^2}{4}+\bar{c}^2-2\left(\frac{\bar{b}}{2}\cdot\bar{c}\right)$$

$$\text{or,}\quad \frac{3}{4}\bar{b}^2=\frac{3}{4}\bar{c}^2 \quad \text{or,}\quad \bar{b}^2=\bar{c}^2 \quad \text{or,}\quad AB^2=AC^2 \quad \text{or,}\quad AB=AC.$$

Hence the triangle is isosceles.

11. *Show that* $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ *by vector method.*

Let $\bar{i}$ and $\bar{j}$ be two unit vectors in two rectangular directions OX and OY.
Let the two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ make angles α and β with OX as shown in Fig. 23.
Taking $OA = r_1, OB = r_2$.

$\overrightarrow{OA} = \bar{i}r_1\cos\alpha - \bar{j}r_1\sin\alpha, \overrightarrow{OB} = \bar{i}r_2\cos\beta + \bar{j}r_2\sin\beta.$

The angle between OA and OB is $\alpha+\beta$.

$$\therefore \quad \overrightarrow{OA}\cdot\overrightarrow{OB} = (\bar{i}r_1\cos\alpha - \bar{j}r_1\sin\alpha)\cdot(\bar{i}r_2\cos\beta + \bar{j}r_2\sin\beta)$$

or, $r_1r_2\cos(\alpha+\beta) = r_1r_2(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$

or, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

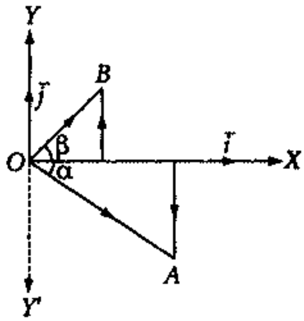


Fig. 23

12. *Show that* $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ *by vector method.*

[CH 2000; NH 2001]

Let $\bar{i}$ and $\bar{j}$ be the unit vectors in the two rectangular axes OX and OY. Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ make angles α and β with OX as shown in Fig. 24.
If $OA = r_1$ and $OB = r_2$, then
$\overrightarrow{OA} = \bar{i}r_1\cos\alpha + \bar{j}r_1\sin\alpha$ and
$\overrightarrow{OB} = \bar{i}r_2\cos\beta + \bar{j}r_2\sin\beta.$
The angle between OA and OB is $\alpha-\beta$.

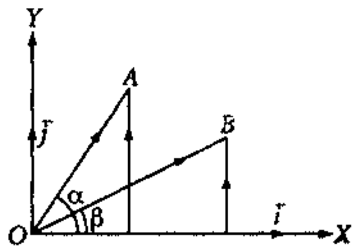


Fig. 24

Now $\quad |\overrightarrow{OB}\times\overrightarrow{OA}| = |r_1r_2(\bar{i}\cos\beta + \bar{j}\sin\beta)\times(\bar{i}\cos\alpha + \bar{j}\sin\alpha)|$

or, $\quad r_1r_2\sin(\alpha-\beta) = r_1r_2|\bar{k}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)|$

(where $\bar{k}$ is a unit vector perpendicular to $\bar{i}$ and $\bar{j}$)

$= \pm r_1r_2(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

or, $\quad \sin(\alpha-\beta) = \pm(\sin\alpha\cos\beta - \cos\alpha\sin\beta).$

To determine the exact sign, we choose $\beta = 0$. Then left-hand side $= \sin\alpha$ and the right-hand side $= \pm\sin\alpha$.

Thus the negative sign is discarded.

Hence $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

13. *Show that the vectors* $\bar{A} = 2\bar{i} - \bar{j} + \bar{k}, \bar{B} = \bar{i} - 3\bar{j} - 5\bar{k}, \bar{C} = 3\bar{i} - 4\bar{j} - 4\bar{k}$, *form the sides of a right-angled triangle.*

To form a triangle the vectors must be coplanar. To prove the coplanarity of the vectors we consider $\bar{A}\cdot(\bar{B}\times\bar{C})$.

$$\bar{B} \times \bar{C} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -3 & -5 \\ 3 & -4 & -4 \end{vmatrix} = -8\bar{i} - 11\bar{j} + 5\bar{k}.$$

$\bar{A} \cdot (\bar{B} \times \bar{C}) = (2\bar{i} - \bar{j} + \bar{k}) \cdot (-8\bar{i} - 11\bar{j} + 5\bar{k}) = -16 + 11 + 5 = 0.$

$\therefore$ the vectors are coplanar.

Again $\bar{A} \cdot \bar{B} = 2 + 3 - 5 = 0$. Therefore, $\bar{A}$ is perpendicular to $\bar{B}$. Consequently the vectors form the sides of a right-angled triangle.

14. *If $\bar{\alpha} = (2, -10, 2), \bar{\beta} = (3, 1, 2)$ and $\bar{\gamma} = (2, 1, 3)$, find the vector $\bar{\alpha} \times (\bar{\beta} \times \bar{\gamma})$ and interpret the result geometrically.*

$\bar{\alpha} \times (\bar{\beta} \times \bar{\gamma}) = (\bar{\alpha} \cdot \bar{\gamma})\bar{\beta} - (\bar{\alpha} \cdot \bar{\beta})\bar{\gamma}.$

Here $\bar{\alpha} \cdot \bar{\gamma} = 2 \cdot 2 - 10 \cdot 1 + 2 \cdot 3 = 0, \bar{\alpha} \cdot \bar{\beta} = 2 \cdot 3 - 10 \cdot 1 + 2 \cdot 2 = 0.$

$\therefore\ \bar{\alpha} \times (\bar{\beta} \times \bar{\gamma}) = \bar{0}$ and $\bar{\alpha}$ is perpendicular to $\bar{\beta}$ and $\bar{\gamma}$.

15. *Find the projection of the join of two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line whose direction cosines are l, m, n.*

Let OS be the line with direction consines l, m, n. If OS be of unit length, then $\overrightarrow{OS} = l\bar{i} + m\bar{j} + n\bar{k}$.

Again $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (\bar{i}x_2 + \bar{j}y_2 + \bar{k}z_2) - (\bar{i}x_1 + \bar{j}y_1 + \bar{k}z_1)$

$= \bar{i}(x_2 - x_1) + \bar{j}(y_2 - y_1) + \bar{k}(z_2 - z_1).$

The required projection $= \overrightarrow{OS} \cdot \overrightarrow{PQ} = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$

16. *Find a vector $\bar{\delta}$ which is perpendicular to both $\bar{\alpha} = 4\bar{i} + 5\bar{j} - \bar{k}$ and $\bar{\beta} = \bar{i} - 4\bar{j} + 5\bar{k}$ and $\bar{\delta} \cdot \bar{\gamma} = 21$, where $\bar{\gamma} = 3\bar{i} + \bar{j} - \bar{k}$.*

Let $\bar{\delta} = x\bar{i} + y\bar{j} + z\bar{k}$.

Here $\bar{\alpha} \cdot \bar{\delta} = 0$ and $\bar{\beta} \cdot \bar{\delta} = 0$.

$\therefore\ 4x + 5y - z = 0$ and $x - 4y + 5z = 0$.

From these two equations $x = -y = -z = \lambda$ (say).

Again $\bar{\delta} \cdot \bar{\gamma} = 21$ or, $3x + y - z = 21$ or, $3\lambda = 21$ or, $\lambda = 7$.

$\therefore\ \bar{\delta} = 7\bar{i} - 7\bar{j} - 7\bar{k}.$

17. *Solve $k\bar{r} + \bar{r} \times \bar{a} = \bar{b}$, where k is a non-zero scalar and $\bar{a}, b$ are two given vectors.* **[CH 2004, 08]**

$$k\bar{r} + \bar{r} \times \bar{a} = \bar{b}. \tag{1}$$

Considering dot product with $\bar{a}$,

$$k(\bar{a} \cdot \bar{r}) = \bar{a} \cdot \bar{b} \quad [\because\ \bar{a} \cdot (\bar{r} \times \bar{a}) = 0]. \tag{2}$$

Considering cross-product with $\bar{a}$,

$$k(\bar{a}\times\bar{r})+\bar{a}^2\bar{r}-(\bar{a}\cdot\bar{r})\bar{a}=\bar{a}\times\bar{b}$$

$$\text{or,}\quad k(k\bar{r}-\bar{b})+\bar{a}^2\bar{r}-\frac{\bar{a}\cdot\bar{b}}{k}\bar{a}=\bar{a}\times\bar{b},\ \text{[by (1) and (2)]}$$

$$\text{or,}\quad (k^2+\bar{a}^2)\bar{r}=k\bar{b}+\frac{\bar{a}\cdot\bar{b}}{k}\bar{a}+\bar{a}\times\bar{b}$$

$$\text{or,}\quad \bar{r}=\frac{1}{k^2+\bar{a}^2}\left(k\bar{b}+\frac{\bar{a}\cdot\bar{b}}{k}\bar{a}+\bar{a}\times\bar{b}\right).$$

18. If $\bar{\alpha},\bar{\beta},\bar{\gamma}$ are non-coplanar, prove that $\bar{\alpha}+\bar{\beta},\bar{\beta}+\bar{\gamma},\bar{\gamma}+\bar{\alpha}$ are non-coplanar.

[NH 2007, 08]

$$\begin{aligned}[\bar{\alpha}+\bar{\beta},\bar{\beta}+\bar{\gamma},\bar{\gamma}+\bar{\alpha}] &= (\bar{\alpha}+\bar{\beta})\cdot\{(\bar{\beta}+\bar{\gamma})\times(\bar{\gamma}+\bar{\alpha})\}\\ &= (\bar{\alpha}+\bar{\beta})\cdot\{\bar{\beta}\times\bar{\gamma}-\bar{\alpha}\times\bar{\beta}+\bar{\gamma}\times\bar{\alpha}\}\\ &= \bar{\alpha}\cdot(\bar{\beta}\times\bar{\gamma})-\bar{\alpha}\cdot(\bar{\alpha}\times\bar{\beta})+\bar{\alpha}\cdot(\bar{\gamma}\times\bar{\alpha})\\ &\qquad+\bar{\beta}\cdot(\bar{\beta}\times\bar{\gamma})-\bar{\beta}\cdot(\bar{\alpha}\times\bar{\beta})+\bar{\beta}\cdot(\bar{\gamma}\times\bar{\alpha})\\ &= [\bar{\alpha}\bar{\beta}\bar{\gamma}]+[\bar{\beta}\bar{\gamma}\bar{\alpha}]=[\bar{\alpha}\bar{\beta}\bar{\gamma}]+[\bar{\alpha}\bar{\beta}\bar{\gamma}]\\ &= 2[\bar{\alpha}\bar{\beta}\bar{\gamma}]\neq 0,\ \text{since } \bar{\alpha},\bar{\beta},\bar{\gamma} \text{ are non-coplanar.}\end{aligned}$$

Hence $\bar{\alpha}+\bar{\beta},\bar{\beta}+\bar{\gamma},\bar{\gamma}+\bar{\alpha}$ are non-coplanar.

EXERCISE II

1. Find the dot product between $2\bar{i}-3\bar{j}+\bar{k}$ and $7\bar{i}+5\bar{j}+\bar{k}$ and hence find the angle between the vectors.

2. Show that the vectors $\bar{\alpha}=\bar{i}+2\bar{j}+\bar{k},\bar{\beta}=\bar{i}+\bar{j}-3\bar{k},\bar{\gamma}=7\bar{i}-4\bar{j}+\bar{k}$ are perpendicular to each other.

3. Find the value of $\bar{a}\times\bar{b}$, where $\bar{a}=2\bar{i}+3\bar{j}+4\bar{k}$ and $\bar{b}=4\bar{i}+3\bar{j}+\bar{k}$.

4. (i) Find the unit vector perpendicular to each of the vector $2\bar{i}+\bar{j}+\bar{k}$ and $\bar{i}-\bar{j}+2\bar{k}$.

   (ii) Prove that the points $\bar{i}-2\bar{j}+3\bar{k},2\bar{j}+4\bar{k}$ and $5\bar{i}+3\bar{j}+5\bar{k}$ form the vertices of a right-angled triangle (without using Pythagorian theorem).

   (iii) Find a unit vector in the plane of the vectors $\bar{\alpha}$ and $\bar{\beta}$ and is perpendicular to the vector $\bar{\gamma}$, where $\bar{\alpha}=\bar{i}+2\bar{j}-\bar{k},\bar{\beta}=\bar{i}+\bar{j}-2\bar{k},\bar{\gamma}=2\bar{i}-\bar{j}+\bar{k}$.

   [NH 2006]

5. Show, by vector method, that the perpendicular bisectors of the sides of a triangle are concurrent. [NH 2002, 06]

6. Show, by vector method, that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides. [NH 2001]

7. (i) Show, by vector method, that a parallelogram will be rectangle, if the diagonals are equal.

   (ii) Show that the diagonals of a rhombus are at right-angles. [BH 93]

8. (i) If $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are three non-coplanar vectors and $\bar{\delta} \cdot \bar{\alpha} = \bar{\delta} \cdot \bar{\beta} = \bar{\delta} \cdot \bar{\gamma} = 0$, show that $\bar{\delta}$ is a zero vector. [BH 2003]

   [*Hints.* $\because$ $\bar{\delta}$ is perpendicular to three non-coplanar vectors, $\bar{\delta}$ is zero.]

   (ii) If $\bar{\alpha} \neq \bar{0}$ but $\bar{\alpha} \cdot \bar{\beta} = \bar{\alpha} \cdot \bar{\gamma}$ and $\bar{\alpha} \times \bar{\beta} = \bar{\alpha} \times \bar{\gamma}$ hold simultaneously show that $\bar{\beta} = \bar{\gamma}$.

   (iii) If a straight line is equally inclined to three coplanar straight lines, show by vector methods, that it is perpendicular to their plane.

   [*Hints.* Let the coplanar lines be represented by unit vectors $\bar{\beta}, \bar{\gamma}, \bar{\delta}$ and the vector $\bar{\alpha}$ be equally inclined to them. Then $\bar{\alpha} \cdot \bar{\beta} = \bar{\alpha} \cdot \bar{\gamma} = \bar{\alpha} \cdot \bar{\delta}$.
   $\therefore$ $\bar{\alpha} \cdot (\bar{\beta} - \bar{\gamma}) = 0$ and $\bar{\alpha} \cdot (\bar{\beta} - \bar{\delta}) = 0$. Thus $\bar{\alpha}$ is perpendicular to the coplanar lines $\bar{\beta} - \bar{\gamma}$ and $\bar{\beta} - \bar{\delta}$. Hence the result follows.]

   (iv) If a line g makes angles α, β, γ with the coordinate axes, prove that the projection of a vector (X, Y, Z) on g is given by $X \cos\alpha + Y \cos\beta + Z \cos\gamma$.

9. $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are three vectors satisfying the condition $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = \bar{0}$. If $|\bar{\alpha}| = 3$, $|\bar{\beta}| = 4$ and $|\bar{\gamma}| = 5$, show that $\bar{\alpha} \cdot \bar{\beta} + \bar{\beta} \cdot \bar{\gamma} + \bar{\gamma} \cdot \bar{\alpha} = -25$. [BH 2003, 07]

10. (i) Use vector method to show that $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$.

    (ii) Use vector method to show that $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$.

11. Find the area of the triangle ABC whose vertices are $A(1,1,2), B(2,-1,1)$ and $C(3,2,-1)$. Also find the angles of $\triangle ABC$.

12. If $\bar{e}_1$ and $\bar{e}_2$ are the two unit vectors inclined at angle θ, then show that $2\sin\frac{\theta}{2} = |\bar{e}_1 - \bar{e}_2|$. [BH 2001, 07]

13. (i) Find the component of $\bar{a}(= 3\bar{i} - \bar{j})$ along $\bar{b}(= \bar{j} - 3\bar{k})$.

    (ii) Show that the component of $\bar{a}$ perpendicular to $\bar{b}$ is $\frac{\bar{b} \times (\bar{a} \times \bar{b})}{|\bar{b}|^2}$.

    [*Hints.* $(\bar{b} \times \bar{a}) \times \bar{b}$ is coplanar with $\bar{b}$ and $\bar{a}$ and is perpendicular to $\bar{b}$.
    Let $\bar{a} = x\bar{b} + y(\bar{b} \times \bar{a}) \times \bar{b}$, where x and y are scalars.
    Now $\bar{a} \cdot \bar{b} = x\bar{b}^2$ or, $x = \frac{\bar{a} \cdot \bar{b}}{\bar{b}^2}$ and $\bar{a} \times \bar{b} = y(\bar{b} \cdot \bar{b})(\bar{a} \times \bar{b})$ or, $y = \frac{1}{|\bar{b}|^2}$.
    Hence the component perpendicular to $\bar{b}$ is $\frac{(\bar{b} \times \bar{a}) \times \bar{b}}{|\bar{b}|^2}$, i.e. $\frac{\bar{b} \times (\bar{a} \times \bar{b})}{|\bar{b}|^2}$.]

14. Find, by vector method, the area of the triangle whose vertices are $A(1,2,3)$, $B(2,5,-1), C(-1,1,2)$.

    [*Hints.* Find $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.]

15. (i) If $\bar{\alpha} = (-2,-2,4), \bar{\beta} = (-2,4,-2)$ and $\bar{\gamma} = (4,-2,-2)$, calculate the value of $\bar{\alpha} \cdot (\bar{\beta} \times \bar{\gamma})$ in its simplest form and interpret your result geometrically.

    (ii) Find the value of the constant d, such that the vectors $2\bar{i} - \bar{j} + \bar{k}, \bar{i} + 2\bar{j} - 3\bar{k}$ and $3\bar{i} + d\bar{j} + 5\bar{k}$ are coplanar. [NH 2008]

(iii) Find a vector of magnitude $\sqrt{51}$ which makes equal analges with the vectors $\bar{a} = \frac{1}{3}(\bar{i} - 2\bar{j} + 2\bar{k}), \bar{b} = \frac{1}{5}(-4\bar{i} - 3\bar{k}), \bar{c} = \bar{j}$.

16. (i) If $\bar{\alpha} = 3\bar{i} - \bar{j} + 2\bar{k}, \bar{\beta} = 2\bar{i} + \bar{j} - \bar{k}$ and $\bar{\gamma} = \bar{i} - 2\bar{j} + 2\bar{k}$, show that $\bar{\alpha} \times (\bar{\beta} \times \bar{\gamma}) \neq (\bar{\alpha} \times \bar{\beta}) \times \bar{\gamma}$.

(ii) $\bar{\beta}, \bar{\gamma}, \bar{\alpha}$ are three vectors in the plane of $\bar{i}$ and $\bar{j}$. If $\bar{\beta} = 4\bar{i} + 3\bar{j}, \bar{\gamma}$ is perpendicular to $\bar{\beta}$ and the projections of $\bar{\alpha}$ on $\bar{\beta}$ and $\bar{\gamma}$ are 1 and 2 respectively find $\bar{\alpha}$.

17. (i) Show that $\frac{1}{2}(\bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b})$ is the vector area of the triangle ABC, where $\bar{a}, \bar{b}, \bar{c}$ are the position vectors of A, B, C respectively. Interpret your result, if this is zero. **[BH 93, 2000]**

(ii) Prove that $(\bar{a} - \bar{b}) \times (\bar{a} + \bar{b}) = 2(\bar{a} \times \bar{b})$ and interpret it.

(iii) Show that the three vectors $\bar{\alpha} \times (\bar{\beta} \times \bar{\gamma}), \bar{\beta} \times (\bar{\gamma} \times \bar{\alpha})$ and $\bar{\gamma} \times (\bar{\alpha} \times \bar{\beta})$ are coplanar and their mutual perpendicular vector is $\frac{\bar{\alpha} \times \bar{\beta}}{\bar{\alpha} \cdot \bar{\beta}} + \frac{\bar{\beta} \times \bar{\gamma}}{\bar{\beta} \cdot \bar{\gamma}} + \frac{\bar{\gamma} \times \bar{\alpha}}{\bar{\gamma} \cdot \bar{\alpha}}$.

(iv) Express $\bar{a}$ as a linear combination of $\bar{b} \times \bar{c}, \bar{c} \times \bar{a}$ and $\bar{a} \times \bar{b}$.

(v) Show that four points whose position vectors are $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ are coplanars iff $[\bar{\alpha}\bar{\beta}\bar{\gamma}] = [\bar{\beta}\bar{\gamma}\bar{\delta}] + [\bar{\gamma}\bar{\alpha}\bar{\delta}] + [\bar{\alpha}\bar{\beta}\bar{\delta}]$. **[BH 2005; CH 2001, 05]**

18. Show that

(i) $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = \bar{0}$, **[BH 2001; NH 2005; CH 2005]**

(ii) $(\bar{b} \times \bar{c}) \cdot (\bar{a} \times \bar{d}) + (\bar{c} \times \bar{a}) \cdot (\bar{b} \times \bar{d}) + (\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = 0$, **[BH 2007; CH 2007]**

(iii) $[\bar{a} \times \bar{b}, \bar{b} \times \bar{c}, \bar{c} \times \bar{a}] = [\bar{a}\bar{b}\bar{c}]^2$, **[NH 2004, 2006; CH 2003]**

(iv) $|\bar{a} \times \bar{b}|^2 |\bar{a} \times \bar{c}|^2 - \{(\bar{a} \times \bar{b}) \cdot (\bar{a} \times \bar{c})\}^2 = |\bar{a}|^2 [\bar{a}\bar{b}\bar{c}]^2$, **[NH 03]**

(v) $\bar{a} \times [\bar{a} \times \{\bar{a} \times (\bar{a} \times \bar{b})\}] = |\bar{a}|^4 \bar{b}$, where $\bar{a} \cdot \bar{b} = 0$,

(vi) $[\bar{a}'\bar{b}'\bar{c}'][\bar{a}\bar{b}\bar{c}] = 1$, where $\bar{a}', \bar{b}', \bar{c}'$ and $\bar{a}, \bar{b}, \bar{c}$ are reciprocal systems.

(vii) $\left[\bar{a} \quad \bar{b} \quad \bar{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \left[\bar{l} \quad \bar{m} \quad \bar{n}\right]$, where $\bar{a} = a_1\bar{l} + a_2\bar{m} + a_3\bar{n}$, $\bar{b} = b_1\bar{l} + b_2\bar{m} + b_3\bar{n}$, $\bar{c} = c_1\bar{l} + c_2\bar{m} + c_3\bar{n}$ and $\bar{l}, \bar{m}, \bar{n}$ are non-coplanar. **[CH 96]**

[*Hints.* $\bar{b} \times \bar{c} = (b_2c_3 - b_3c_2)(\bar{m} \times \bar{n}) + (b_3c_1 - b_1c_3)(\bar{n} \times \bar{l}) + (b_1c_2 - b_2c_1)(\bar{l} \times \bar{m})$. $\bar{a} \cdot (\bar{b} \times \bar{c}) = \{a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + {}_3(b_1c_2 - b_2c_1)\}\{\bar{l} \cdot (\bar{m} \times \bar{n})\}$. Hence the result follows.]

19. (i) Solve $\bar{r} \times \bar{a} = \bar{b}$, where $\bar{a}$ and $\bar{b}$ are given vectors and $\bar{b}$ is $\perp$ to $\bar{a}$. **[CH 2003, 09]**

[*Hints.* Let $\bar{r} = x\bar{a} + y\bar{b} + z(\bar{a} \times \bar{b})$. Then $\bar{r} = x\bar{a} + \frac{1}{a^2}\bar{a} \times \bar{b}, x$ is arbitrary.]

(ii) Solve $\bar{r} \times \bar{b} = \bar{a} \times \bar{b}$, where $\bar{a}, \bar{b}$ are two given vectors.
[*Hints.* Here $(\bar{r} - \bar{a}) \times \bar{b} = \bar{0}$. $\therefore$ $\bar{r} - \bar{a}$ is parallel to $\bar{b}$.
Thus $\bar{r} - \bar{a} = t\bar{b}$ or, $\bar{r} = \bar{a} + t\bar{b}$, where t is a scalar.]

20. Prove that $\sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta$ and $\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta$ by vector method. **[BH 2002, 07; CH 91, 96]**

[*Hints.* We have $(\bar{b}\times\bar{c})\cdot(\bar{a}\times\bar{d})+(\bar{c}\times\bar{a})\cdot(\bar{b}\times\bar{d})+(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})=0$. Let $\angle AOC=\alpha, \angle AOB=\beta, \angle COD=\beta$, and $\overrightarrow{OA}=\bar{a}, \overrightarrow{OB}=\bar{b}, \overrightarrow{OC}=\bar{c}, \overrightarrow{OD}=\bar{d}$.
We assume that $\bar{a},\bar{b},\bar{c},\bar{d}$ are coplanar and $\bar{n}$ is a unit vector perpendicular to this plane.
If $OA=a, OB=b, OC=c, OD=d$, then $(bc\sin BOC)\bar{n}\cdot(ad\sin AOD)\bar{n}$
$-(ac\sin AOC)\bar{n}\cdot(bd\sin BOD)\bar{n}+$
$(ab\sin AOB)\bar{n}\cdot(cd\sin COD)\ \bar{n}=0$ or,
$\sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta$.

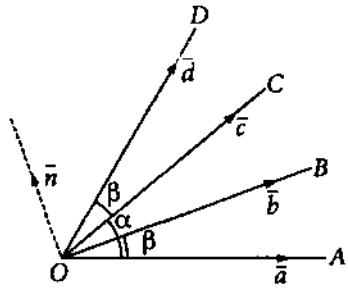


Fig. 25

Now use $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})=(\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d})-(\bar{b}\cdot\bar{c})(\bar{a}\cdot\bar{d})$ to get the other result.]

21. Prove that $[\bar{l}\bar{m}\bar{n}][\bar{a}\bar{b}\bar{c}]=\begin{vmatrix}\bar{l}\cdot\bar{a} & \bar{l}\cdot\bar{b} & \bar{l}\cdot\bar{c}\\ \bar{m}\cdot\bar{a} & \bar{m}\cdot\bar{b} & \bar{m}\cdot\bar{c}\\ \bar{n}\cdot\bar{a} & \bar{n}\cdot\bar{b} & \bar{n}\cdot\bar{c}\end{vmatrix}$.

[*Hints.* Let $\bar{a}',\bar{b}',\bar{c}'$ be the reciprocal system to $\bar{a},\bar{b},\bar{c}$.

Then $[\bar{a}'\bar{b}'\bar{c}'][\bar{a}\bar{b}\bar{c}]=1$ and

$$\bar{l}=(\bar{l}\cdot\bar{a})\bar{a}'+(\bar{l}\cdot\bar{b})\bar{b}'+(\bar{l}\cdot\bar{c})\bar{c}',$$
$$\bar{m}=(\bar{m}\cdot\bar{a})\bar{a}'+(\bar{m}\cdot\bar{b})\bar{b}'+(\bar{m}\cdot\bar{c})\bar{c}',$$
$$\bar{n}=(\bar{n}\cdot\bar{a})\bar{a}'+(\bar{n}\cdot\bar{b})\bar{b}'+(\bar{n}\cdot\bar{c})\bar{c}'.$$
$$\therefore\ [\bar{l}\bar{m}\bar{n}]=\begin{vmatrix}\bar{l}\cdot\bar{a} & \bar{l}\cdot\bar{b} & \bar{l}\cdot\bar{c}\\ \bar{m}\cdot\bar{a} & \bar{m}\cdot\bar{b} & \bar{m}\cdot\bar{c}\\ \bar{n}\cdot\bar{a} & \bar{n}\cdot\bar{b} & \bar{n}\cdot\bar{c}\end{vmatrix}[\bar{a}'\bar{b}'\bar{c}'].$$

Hence the result follows.]

ANSWERS

1. $0, \frac{\pi}{2}$.

3. $-9\bar{i}+14\bar{j}-6\bar{k}$.

4. (i) $\frac{\bar{i}-\bar{j}-\bar{k}}{\sqrt{3}}$, (iii) $\frac{\bar{j}+\bar{k}}{\sqrt{2}}$.

11. $\frac{5\sqrt{5}}{2}$; $\cos^{-1}\frac{3}{\sqrt{84}}, \cos^{-1}\frac{3}{\sqrt{84}}, \cos^{-1}\frac{11}{14}$.

13. (i) $-\frac{1}{10}(\bar{j}-3\bar{k})$.

14. $\frac{\sqrt{155}}{2}$.

15. (i) 0, coplanar, (ii) -4, (iii) $\pm(5\bar{i}-\bar{j}-5\bar{k})$.

16. (ii) $\bar{i}-\bar{j}$.

17. (iv) $\bar{a}=\frac{1}{[\bar{a}\bar{b}\bar{c}]}\{(\bar{a}\cdot\bar{a})(\bar{b}\times\bar{c})+(\bar{a}\cdot\bar{b})(\bar{c}\times\bar{a})+(\bar{a}\cdot\bar{c})(\bar{a}\times\bar{b})\}$.

Chapter 3

Simple Applications in Geometry and Mechanics

3.10 Straight line

(i) **Equation of a line passing through a given point and parallel to a given vector.** (*Parametric form*)

Let A be a given point whose position vector is $\bar{a}$ w.r.t. the origin O and $\bar{b}$ the given vector. Let P be a point on the line and its position vector be $\bar{r}$.

Now $\overrightarrow{AP}$ is parallel to $\bar{b}$ and so we can write $\overrightarrow{AP} = t\bar{b}$, where t is a scalar.

From Fig. 26,

$$\bar{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \bar{a} + t\bar{b}. \qquad (1)$$

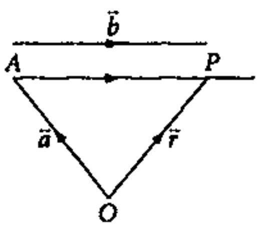


Fig. 26

This relation gives the position vector of the moving point P at any position on the line for different values of t.

Hence it is the vector equation of the line.

Corollary I. *The equation of a line through the origin is* $\bar{r} = t\bar{b}$. $[\bar{a} = \bar{0}]$

Note. The equation (1) can be written as $\bar{r} \times \bar{b} = \bar{a} \times \bar{b}$.

(ii) **Equation of a line passing through two points.**

Let A and B be two given points with the position vectors $\bar{a}$ and $\bar{b}$ referred to the vector origin O. Let P be a point on the line and $\bar{r}$ be its position vector.

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \bar{b} - \bar{a}$.

$\overrightarrow{AP}$ is collinear with $\overrightarrow{AB}$. Therefore, we can write $\overrightarrow{AP} = t(\bar{b} - \bar{a})$, where t is a scalar.

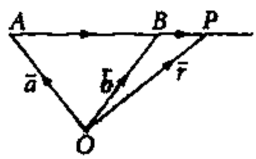


Fig. 27

From $\triangle OAP, \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$\text{or,} \quad \bar{r} = \bar{a} + t(\bar{b} - \bar{a}) = (1-t)\bar{a} + t\bar{b} \tag{2}$$

$$\text{or,} \quad \bar{r} = s\bar{a} + (1-s)\bar{b}, \tag{3}$$

where $1 - t = s$.

Anyone of (2) and (3) is the equation of the required line.

Note 2. The equation (3) can be written as $\bar{r} - s\bar{a} - (1-s)\bar{b} = \bar{0}$.

The sum of scalar coefficients$=1 - s - (1-s) = 0$.

The condition of collinearity of three points is satisfied.

Note 3. Let the coordinates of P, B, A be $(x, y, z), (x_1, y_1, z_1)$ and (x_2, y_2, z_2) w.r.t. three rectangular axes through the vector origin O.

From the equation (3), $(x, y, z) = s(x_2, y_2, z_2) + (1-s)(x_1, y_1, z_1)$.

It gives $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = s$.

These are the equations of a line through two given points in the cartesian system.

3.11 Equations of the bisectors of angles between two lines

Let OP be the internal bisector of the angle AOB.

Let $\bar{a}$ and $\bar{b}$ be unit vectors along OA and OB.

We consider that O is the vector origin, P is any point on the bisector and MP is parallel to OB.

Here $OM = MP = t$ (say).

$\therefore\ \overrightarrow{OM} = t\bar{a}, \overrightarrow{MP} = t\bar{b}$.

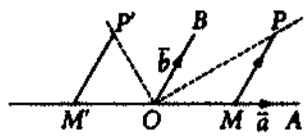


Fig. 28

Let $\overrightarrow{OP} = \bar{r}$. From $\triangle OMP, \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$ or, $\bar{r} = t(\bar{a} + \bar{b})$. This relation gives the position vector of P at any position on the bisector for different values of t. Hence it is the equation of the internal bisector.

Let OP' be the external bisector and $\overrightarrow{OP'} = \bar{r}$. $M'P'$ is parallel to OB and it meets AO (produced) at M'. If $OM' = M'P' = t$, then $\overrightarrow{OM'} = -t\bar{a}$ and $\overrightarrow{M'P'} = t\bar{b}$.

$\therefore\ \bar{r} = t(\bar{b} - \bar{a})$. It is the equation of the external bisector.

Corollary. If $\overrightarrow{OA} = \bar{a}, \overrightarrow{OB} = \bar{b}$, then unit vectors along OA and OB are $\frac{\bar{a}}{|\bar{a}|}$ and $\frac{\bar{b}}{|\bar{b}|}$. In this case, the equations are $\bar{r} = t\left(\frac{\bar{b}}{|\bar{b}|} \pm \frac{\bar{a}}{|\bar{a}|}\right)$.

3.12 Theorems on bisectors

Theorem 1. *The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the other two sides.*

Let AD be the internal bisector of the angle A of $\triangle ABC$ and $\overrightarrow{AB} = \bar{c}, \overrightarrow{AC} = \bar{b}$ with A as vector origin.

Now the equation of AD is

$$\bar{r} = t\left(\frac{\bar{c}}{|\bar{c}|} + \frac{\bar{b}}{|\bar{b}|}\right) = t\frac{|\bar{b}|\bar{c} + |\bar{c}|\bar{b}}{|\bar{b}||\bar{c}|},$$

where t is a scalar.

Let us choose $t = \dfrac{|\bar{b}||\bar{c}|}{|\bar{b}| + |\bar{c}|}$.

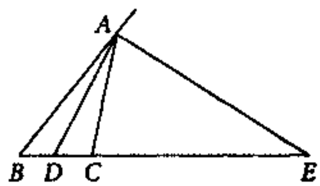


Fig. 29

Then $\bar{r} = \frac{|\bar{b}|\bar{c}+|\bar{c}|\bar{b}}{|\bar{b}|+|\bar{c}|}$ is the position vector of a point on the line BC, which divides BC in the ratio $|\bar{c}| : |\bar{b}|$. So it is the common point of BC and AD. Hence the internal bisector divides BC in the ratio of $AB : AC$. This proves the theorem.

Theorem 2. *The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the other two sides.*

Let AE be the external bisector of the angle A of $\triangle ABC$ and $\overrightarrow{AB} = \bar{c}, \overrightarrow{AC} = \bar{b}$ w.r.t. the vector origin A (see Fig. 29). Now the equation of AE is

$$\bar{r} = t\left(\frac{\bar{b}}{|\bar{b}|} - \frac{\bar{c}}{|\bar{c}|}\right) = t\frac{|\bar{c}|\bar{b} - |\bar{b}|\bar{c}}{|\bar{b}||\bar{c}|}, \text{ where } t \text{ is a scalar.}$$

Let us choose $t = \frac{|\bar{b}||\bar{c}|}{|\bar{c}|-|\bar{b}|}$. Then $\bar{r} = \frac{|\bar{c}|\bar{b}-|\bar{b}|\bar{c}}{|\bar{c}|-|\bar{b}|}$, $\frac{|\bar{c}|\bar{b}-|\bar{b}|\bar{c}}{|\bar{c}|-|\bar{b}|}$ is the position vector of a point on BC, which divides BC externally in the ratio $|\bar{c}| : |\bar{b}|$. So it is the common point of BC and AE. Thus the external bisector divides the opposite side externally in the ratio of the other two sides.

Note. C and D are harmonic conjugates to A and B.

Theorem 3. *The internal bisectors of the angles of a triangle are concurrent.*

Let $\bar{r}_1, \bar{r}_2, \bar{r}_3$ be the position vectors of the vertices A, B, C of ΔABC w.r.t. a certain vector origin.

Let $BC = a, CA = b, AB = c$.

If the internal bisector of the angle A meets the side BC at D, then D divides BC in the ratio $c : b$.

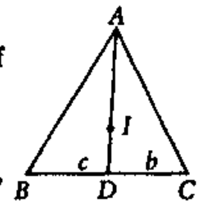


Fig. 30

Therefore, the position vector of D is $\frac{b\bar{r}_2+c\bar{r}_3}{b+c}$.

Let the point I divide AD in the ratio $b + c : a$. Then the position vector of I is $\frac{a\bar{r}_1+b\bar{r}_2+c\bar{r}_3}{a+b+c}$. The symmetry of this expression suggests that this point must also lie on the other bisectors. Hence the bisectors are concurrent.

Theorem 4. *The internal bisector of one angle and the external bisectors of the other two angles of a triangle are concurrent.*

Let $\bar{r}_1, \bar{r}_2, \bar{r}_3$ be the position vectors of the vertices A, B, C of $\triangle ABC$ w.r.t. a certain vector origin.

If the external bisector of the angle B meets AC at E, then the point E divides AC externally in the ratio $c : a$.

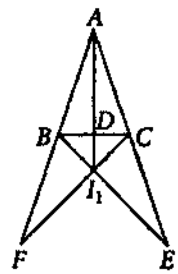


Fig. 31

Therefore, the position of E is $\frac{c\bar{r}_3 - a\bar{r}_1}{c-a}$. Let the point I_1 divide BE in the ratio $c - a : b$. Then the position vector of I_1 is $\frac{b\bar{r}_2 + c\bar{r}_3 - a\bar{r}_1}{b+c-a}$.

If the external bisector of the angle C meets AB externally at F, then the position vector of F is $\frac{b\bar{r}_2 - a\bar{r}_1}{b-a}$. Now CF is divided internally in the ratio $b - a : c$ by a point whose position is that of I_1. Similarly the internal bisector AD of the angle A is divided externally in the ratio $b + c : a$ by the point whose position vector is alike to that of I_1. Hence the result follows.

3.20 Theorems on complete quadrangle

Definition. Four coplanar points A, B, C, D, no three of which are collinear, form a figure called a *complete quadrangle*. The four points are called the *vertices* and the six straight lines joining them in pairs are called the *sides* of the quadrangle. Two sides which do not meet in a vertex are termed *opposite*. There are three pairs of opposite sides, namely $(AB, CD), (BC, AD)$ and (CA, BD). These pairs meet at P, Q, R. These are termed *diagonal points* and $\triangle PQR$ is called the *diagonal triangle*.

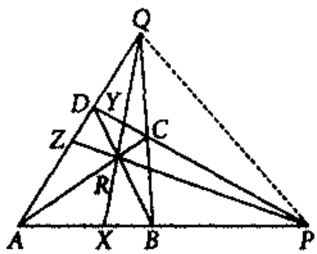


Fig. 32

Position vectors of P, Q, R. If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are the position vectors of A, B, C, D then

$$\alpha\bar{a} + \beta\bar{b} + \gamma\bar{c} + \delta\bar{d} = \bar{0} \tag{1}$$

$$\text{and} \quad \alpha + \beta + \gamma + \delta = 0. \tag{2}$$

$\alpha, \beta, \gamma, \delta$ are four scalars and they are not all zero.

Let $\bar{p}, \bar{q}, \bar{r}$ be the position vectors of P, Q, R.

By (1) and (2), $\frac{\alpha\bar{a}+\beta\bar{b}}{\alpha+\beta} = \frac{\gamma\bar{c}+\delta\bar{d}}{\gamma+\delta}$.

It represents the position vector of the common of AB and CD.

Thus

$$\bar{p} = \frac{\alpha\bar{a} + \beta\bar{b}}{\alpha + \beta} = \frac{\gamma\bar{c} + \delta\bar{d}}{\gamma + \delta}. \tag{3}$$

It shows that P divides AB and CD in the ratio $\beta : \alpha$ and $\delta : \gamma$ respectively.

$$\text{Similarly} \quad \bar{q} = \frac{\beta\bar{b} + \gamma\bar{c}}{\beta + \gamma} = \frac{\alpha\bar{a} + \delta\bar{d}}{\alpha + \delta} \tag{4}$$

$$\text{and} \quad \bar{r} = \frac{\gamma\bar{c} + \alpha\bar{a}}{\gamma + \alpha} = \frac{\beta\bar{b} + \delta\bar{d}}{\beta + \delta}. \tag{5}$$

Theorem 1. *If QR cuts AB and CD at X and Y, then X, Y are harmonic conjugates to Q, R.*

Proof. From (5), $(\gamma + \alpha)\bar{r} = \gamma\bar{c} + \alpha\bar{a}$.

From (4), $(\beta + \gamma)\bar{q} = \beta\bar{b} + \gamma\bar{c}$.

From these it follows that

$$\frac{(\gamma+\alpha)\bar{r}-(\beta+\gamma)\bar{q}}{(\gamma+\alpha)-(\beta+\gamma)}=\frac{\alpha\bar{a}-\beta\bar{b}}{\alpha-\beta}=\bar{x}, \tag{6}$$

where $\bar{x}$ is the position vector of X.

Similarly $$\frac{(\alpha+\gamma)\bar{r}-(\alpha+\delta)\bar{q}}{(\alpha+\gamma)-(\alpha+\delta)}=\frac{\gamma\bar{c}-\delta\bar{d}}{\gamma-\delta}=\bar{y}, \tag{7}$$

where $\bar{y}$ is the position vector of Y.

The result (6) shows that X divides QR in the ratio $-(\gamma+\alpha):(\beta+\gamma)$.

The result (7) shows that Y divides QR in the ratio $-(\alpha+\gamma):(\alpha+\delta)$, i.e. $(\alpha+\gamma):(\beta+\gamma)$. $[\because\ \alpha+\beta+\gamma+\delta=0]$

Hence X, Y are harmonic conjugates to Q, R.

Note. X, P are harmonic conjugates to $A, B; Y, P$ are harmonic conjugates to $D, C; Z, Q$ are harmonic conjugates to A, D.

Theorem 2. *The midpoints of AC, BD and PQ of the complete quadrangle are collinear.*

Proof. From the results of (1), (2), (3) and (4),

$$\alpha\bar{a}+\beta\bar{b}+\gamma\bar{c}+\delta\bar{d}=\bar{0},\quad \alpha+\beta+\gamma+\delta=0;$$

$$(\alpha+\beta)\bar{p}=\alpha\bar{a}+\beta\bar{b},\quad (\beta+\gamma)\bar{q}=\beta\bar{b}+\gamma\bar{c}.$$

Now $$\begin{aligned}(\alpha+\beta)(\beta+\gamma)(\bar{p}+\bar{q}) &= (\beta+\gamma)(\alpha\bar{a}+\beta\bar{b})+(\alpha+\beta)(\beta\bar{b}+\gamma\bar{c})\\ &= \gamma\alpha(\bar{a}+\bar{c})+\beta(\alpha+2p\beta+\gamma)\bar{b}+\beta(\alpha\bar{a}+\gamma\bar{c})\\ &= \gamma\alpha(\bar{a}+\bar{c})+\beta(\beta-\delta)\bar{b}+\beta(-\beta\bar{b}-\delta\bar{d})\\ &= \gamma\alpha(\bar{a}+\bar{c})-\beta\delta(\bar{b}+\bar{d})\end{aligned}$$

or, $$(\alpha+\beta)(\beta+\gamma)\tfrac{\bar{p}+\bar{q}}{2}-\gamma\alpha\tfrac{\bar{a}+\bar{c}}{2}+\beta\delta\tfrac{\bar{b}+\bar{d}}{2}=\bar{0}.$$

It shows that the position vectors of the midpoints of AC, BD and PQ are linearly dependent.

Again $(\alpha+\beta)(\beta+\gamma)-\gamma\alpha+\beta\delta=\beta(\alpha+\beta+\gamma)+\beta\delta=\beta(\alpha+\beta+\gamma+\delta)=0$.

Hence the result follows.

3.21 Theorems on complete quadrilateral

Definition. Four coplanar straight lines AB, BC, CD, DA, no three of which are concurrent, form a figure called a *complete quadrilateral*. The four straight lines are the *sides* of the quadrilateral. The six points in which the sides cut one another two and two are called the *vertices*. Two vertices which do not lie o a side are called *opposite vertices* and the line joining them is known as a diagonal. In Fig. 33, (A, C), (B, D) and (E, F) are three pairs of opposite vertices and AC, BD, EF are three diagonals.

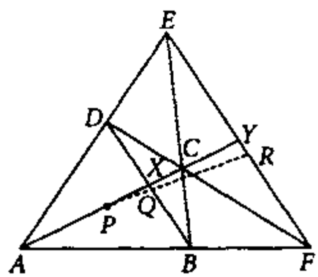


Fig. 33

Theorem 1. *The midpoints of the diagonals of a complete quadrilateral are collinear.*

Let P, Q, R be the midpoints of the diagonals AC, BD, EF of the complete quadrilateral $ABCD$.

If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are the position vectors of A, B, C, D, then

$$\alpha\bar{a} + \beta\bar{b} + \gamma\bar{c} + \delta\bar{d} = \bar{0} \tag{1}$$

$$\text{and} \quad \alpha + \beta + \gamma + \delta = 0. \tag{2}$$

$\alpha, \beta, \gamma, \delta$ are scalars and they are not all zero.

E and F are the common points of AB, DC and AD, BC respectively.

Thus

$$\bar{e} = \frac{\alpha\bar{a} + \beta\bar{b}}{\alpha + \beta} = \frac{\gamma\bar{c} + \delta\bar{d}}{\gamma + \delta} \tag{3}$$

$$\text{and} \quad \bar{f} = \frac{\alpha\bar{a} + \delta\bar{d}}{\alpha + \delta} = \frac{\beta\bar{b} + \gamma\bar{c}}{\beta + \gamma}, \tag{4}$$

where $\bar{e}$ and $\bar{f}$ are the position vectors of E and F.

From (3) and (4),

$$(\alpha + \beta)\bar{e} = \alpha\bar{a} + \beta\bar{b} \quad \text{and} \quad (\alpha + \delta)\bar{f} = \alpha\bar{a} + \delta\bar{d}.$$

$$\begin{aligned}
\text{Now} \qquad (\alpha + \beta)(\alpha + \delta)(\bar{e} + \bar{f}) &= (\alpha + \delta)(\alpha\bar{a} + \beta\bar{b}) + (\alpha + \beta)(\alpha\bar{a} + \delta\bar{d}) \\
&= \alpha(2\alpha + \beta + \delta)\bar{a} + \beta\delta(\bar{b} + \bar{d}) + \alpha(\beta\bar{b} + \delta\bar{d}) \\
&= \alpha(\alpha - \gamma)\bar{a} + \beta\delta(\bar{b} + \bar{d}) - \alpha(\alpha\bar{a} + \gamma\bar{c}) \\
&= -\alpha\gamma(\bar{a} + \bar{c}) + \beta\delta(\bar{b} + \bar{d})
\end{aligned}$$

$$\text{or,} \quad \alpha\gamma\frac{\bar{a} + \bar{c}}{2} - \beta\delta\frac{\bar{b} + \bar{d}}{2} + (\alpha + \beta)(\alpha + \delta)\frac{\bar{e} + \bar{f}}{2} = \bar{0}. \tag{5}$$

It shows that the position vectors of the midpoints of AC, BD and EF are linearly dependent.

Again $\alpha\gamma - \beta\delta + (\alpha + \beta)(\alpha + \delta) = \alpha\gamma + \alpha(\alpha + \beta + \delta) = \alpha(\alpha + \beta + \gamma + \delta) = 0.$

Hence the result follows.

Theorem 2. *Each diagonal of a complete quadrilateral is cut harmonically by the other two.*

Let the diagonal AC be cut by the diagonals BD and EF at the points X and Y.

X is the common point of AC and BD. By (1) and (2)

$$\bar{x} = \frac{\alpha\bar{a} + \gamma\bar{c}}{\alpha + \gamma} = \frac{\beta\bar{b} + \delta\bar{d}}{\beta + \delta}, \tag{6}$$

where $\bar{x}$ is the position vector of X.

From (3) and (4)

$$(\alpha+\beta)\bar{e}-(\beta+\gamma)\bar{f}=(\alpha\bar{a}+\beta\bar{b})-(\beta\bar{b}+\gamma\bar{c})=\alpha\bar{a}-\gamma\bar{c}$$

$$\text{or,}\quad \frac{(\alpha+\beta)\bar{e}-(\beta+\gamma)\bar{f}}{(\alpha+\beta)-(\beta+\gamma)}=\frac{\alpha\bar{a}-\gamma\bar{c}}{\alpha-\gamma}=\bar{y}, \tag{7}$$

where $\bar{y}$ is the position vector of Y which is the common point of EF and AC.

The relation (6) shows that X divides AC in the ratio $\gamma:\alpha$.

The relation (7) shows that Y divides AC in the ratio $-\gamma:\alpha$.

Hence X and Y are harmonic conjugates to A and C.

3.30 Equation of a plane

(i) **Normal form.** *To find the equation of a plane perpendicular to the unit vector $\bar{n}$ and passing through a point whose position vector is $\bar{a}$.*

Let O be the vector origin, ON perpendicular to the plane, A the given point on the plane and P any point on the plane. Here $\overrightarrow{OA}=\bar{a}, \overrightarrow{ON}=p\bar{n}$, where $p=|\overrightarrow{ON}|$. Let $\overrightarrow{OP}=\bar{r}$.

From Fig. 34, $\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\bar{r}-\bar{a}$. AP lies on the plane. Therefore, it is perpendicular to $\bar{n}$.

Thus $(\bar{r}-\bar{a})\cdot\bar{n}=0$. This relation holds for any position of P on the plane. Hence it is the equation of the plane.

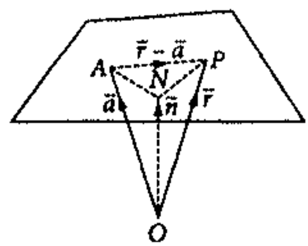


Fig. 34

Again $\bar{a}\cdot\bar{n}$=projection of OA on $ON=p$.

$\therefore\ \bar{r}\cdot\bar{n}=p$.

It is known as the normal form of the equation of the plane.

Corollary. The equation of the plane passing through the origin is $\bar{r}\cdot\bar{n}=0$.

(ii) *To find the equation of the plane passing through a given point and parallel to two given vectors.*

Let the position vector of the given point be $\bar{a}$ and the given vectors be $\bar{b}$ and $\bar{c}$. Since $\bar{b}$ and $\bar{c}$ are parallel to the plane, $\bar{b}\times\bar{c}$ is perpendicular to the plane. Moreover, the plane passes through the point whose position vector is $\bar{a}$. If $\bar{r}$ is a point on the plane, $\bar{r}-\bar{a}$ is perpendicular to $\bar{b}\times\bar{c}$.

Thus $(\bar{r}-\bar{a})\cdot(\bar{b}\times\bar{c})=0$ or, $\bar{r}\cdot(\bar{b}\times\bar{c})=\bar{a}\cdot(\bar{b}\times\bar{c})$ or, $[\bar{r}\bar{b}\bar{c}]=[\bar{a}\bar{b}\bar{c}]$. It is the equation of the plane.

Note 1. The plane passing through two points with position vectors $\bar{a}$ and $\bar{b}$ and parallel to $\bar{c}$ is identical with the plane passing through the point with position vector $\bar{a}$ and parallel to the vectors $\bar{b}-\bar{a}$ and $\bar{c}$.

Note 2. The plane containing the line $\bar{r}=\bar{\alpha}+t\bar{\beta}$, where t is a scalar and perpendicular to the plane $\bar{r}\cdot\bar{\delta}=q$ is identical with the plane passing through the point having the position vector $\bar{\alpha}$ and parallel to the vectors $\bar{\beta}$ and $\bar{\delta}$.

(iii) *To find the equation of the plane passing through three given points.*

Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of the three given points. Then $\bar{b} - \bar{a}$ and $\bar{c} - \bar{a}$ lie in the same plane. Therefore, $(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})$ is perpendicular to the plane. If $\bar{r}$ is a point on the plane, then $\bar{r} - \bar{a}$ is perpendicular to $(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})$.

$\therefore \quad (\bar{r} - \bar{a}) \cdot \{(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})\} = 0$

or, $\quad \bar{r} \cdot \{(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})\} = \bar{a} \cdot \{(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})\}$

or, $\quad \bar{r} \cdot (\bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b}) = \bar{a} \cdot (\bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b})$

or, $\quad \bar{r} \cdot \bar{m} = [\bar{a}\bar{b}\bar{c}]$.

It is the equation of the plane.

$[\bar{m} = \bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b} =$ twice the vector area of the triangle whose vertices are $\bar{a}, \bar{b}, \bar{c}]$

Corollary. If p is the perpendicular distance of the plane from the origin, then $p = \frac{[\bar{a}\bar{b}\bar{c}]}{|\bar{m}|}$.

(iv) *To find the equation of the plane through two given lines.*

Let the given lines be $\bar{r} = \bar{a}_1 + t\bar{b}_1$ and $\bar{r} = \bar{a}_2 + s\bar{b}_2$, where t and s are scalars.

Obviously the plane through these lines passes through the point $\bar{a}_1$ and it is parallel to $\bar{b}_1$ and $\bar{b}_2$. Therefore, the equation of the plane is $[\bar{r}\bar{b}_1\bar{b}_2] = [\bar{a}_1\bar{b}_1\bar{b}_2]$ by (ii).

Corollary. Condition of the coplanarity of two straight lines.

The plane passes through the point $\bar{a}_2$. Hence the condition of the coplanarity of the two given lines is $[\bar{a}_2\bar{b}_1\bar{b}_2] = [\bar{a}_1\bar{b}_1\bar{b}_2]$.

(v) *To find the equation of the plane through the intersection of two planes.*

Let the given planes be $\bar{r} \cdot \bar{n}_1 = p_1$ and $\bar{r} \cdot \bar{n}_2 = p_2$. (1)

Obviously all those points which satisfy the equations (1) also satisfy the equation

$$(\bar{r} \cdot \bar{n}_1 - p_1) - \lambda(\bar{r} \cdot \bar{n}_2 - p_2) = 0, \tag{2}$$

whatever be the value of λ.

Thus the equation (2) represents the general equation of the plane passing through the line of intersection of the planes (1).

The equation (2) can be written as $\bar{r} \cdot (\bar{n}_1 - \lambda\bar{n}_2) = p_1 - \lambda p_2$.

If the plane passes through a given point $\bar{a}$, then

$$\bar{a} \cdot (\bar{n}_1 - \lambda\bar{n}_2) = p_1 - \lambda p_2 \quad \text{or,} \quad \lambda = \frac{\bar{a} \cdot \bar{n}_1 - p_1}{\bar{a} \cdot \bar{n}_2 - p_2}.$$

Thus the equation, $(\bar{r} \cdot \bar{n}_1 - p_1)(\bar{a} \cdot \bar{n}_2 - p_2) - (\bar{r} \cdot \bar{n}_2 - p_2)(\bar{a} \cdot \bar{n}_1 - p_1) = 0$ represents the equation of the plane passing through the line of intersection of the planes (1) and the point $\bar{a}$.

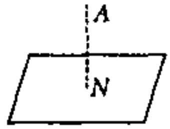
3.31 (i) To find the distance of a point from a given plane

Let the position vector of the given point A be $\bar{a}$ and the equation of the given plane be $\bar{r}\cdot\bar{n}=p$, where $\bar{n}$ is normal to the plane.

If AN is perpendicular to the plane, then this line passes through the point $\bar{a}$ and it is parallel to $\bar{n}$. Therefore, the equation of AN is $\bar{r}=\bar{a}+t\bar{n}$, where t is a scalar. At the point of intersection of the line AN and the plane $(\bar{a}+t\bar{n})\cdot\bar{n}=p$ or, $t=\frac{p-\bar{a}\cdot\bar{n}}{\bar{n}^2}$.

$\therefore$ the position vector of the point N is

$$\bar{a}+t\bar{n}=\bar{a}+\frac{p-\bar{a}\cdot\bar{n}}{\bar{n}^2}\bar{n}.$$



Hence $\overrightarrow{AN}=\bar{a}+\frac{p-\bar{a}\cdot\bar{n}}{\bar{n}^2}\bar{n}-\bar{a}=\frac{p-\bar{a}\cdot\bar{n}}{\bar{n}^2}\bar{n}$.

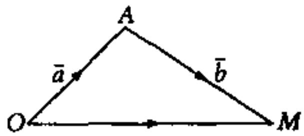
$\therefore$ the required distance $=|\overrightarrow{AN}|=\left|\frac{p-\bar{a}\cdot\bar{n}}{\bar{n}^2}\right||\bar{n}|=\frac{|p-\bar{a}\cdot\bar{n}|}{|\bar{n}|}$.

Fig. 35

(ii) *To find the distance of a point from a plane measured in a given direction.*

Let the equation of the plane be

$$\bar{r}\cdot\bar{n}=p \qquad (1)$$



and the given point be A with the position vector $\bar{a}$.

Fig. 36

Let the given direction be that of unit vector $\bar{b}$ and the line through A and parallel to $\bar{b}$ meet the plane at M.

If $AM=d$, then $\overrightarrow{AM}=d\bar{b}$.

Again $\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}=\bar{a}+d\bar{b}$.

Since M lies on the plane, $(\bar{a}+d\bar{b})\cdot\bar{n}=p$ or, $d=\frac{p-\bar{a}\cdot\bar{n}}{\bar{b}\cdot\bar{n}}$.

(iii) *Bisector planes of the angles between two planes.*

Let the equations of the planes be

$$\bar{r}\cdot\bar{n}_1=p_1 \quad \text{and} \quad \bar{r}\cdot\bar{n}_2=p_2.$$

If $\bar{r}$ be any point on the bisecting plane, then the perpendicular distances of the planes of this point are equal in magnitude.

$$\therefore \quad \frac{p_1-\bar{r}\cdot\bar{n}_1}{|\bar{n}_1|}=\pm\frac{p_2-\bar{r}\cdot\bar{n}_2}{|\bar{n}_2|}.$$

$$\therefore \quad \bar{r}\cdot\left(\frac{\bar{n}_1}{|\bar{n}_1|}+\frac{\bar{n}_2}{|\bar{n}_2|}\right)=\frac{p_1}{|\bar{n}_1|}+\frac{p_2}{|\bar{n}_2|}$$

$$\text{and} \quad \bar{r}\cdot\left(\frac{\bar{n}_1}{|\bar{n}_1|}-\frac{\bar{n}_2}{|\bar{n}_2|}\right)=\frac{p_1}{|\bar{n}_1|}-\frac{p_2}{|\bar{n}_2|}$$

are the equations of the bisector planes.

3.40 Equations of lines relating with planes

(i) *To find the equation of the line passing through the point $\bar{a}$ and parallel to the line of intersection of the planes $\bar{r}\cdot\bar{n}_1 = p_1$ and $\bar{r}\cdot\bar{n}_2 = p_2$.*

The line of intersection of the planes is parallel to $\bar{n}_1 \times \bar{n}_2$. Thus the required line is parallel to $\bar{n}_1 \times \bar{n}_2$ and it passes through the point $\bar{a}$.

Hence the equation is $(\bar{r} - \bar{a}) \times (\bar{n}_1 \times \bar{n}_2) = \bar{0}$.

(ii) *To find the equation of the line passing through the point $\bar{a}$, parallel to $\bar{r}\cdot\bar{n} = p$ and intersecting the line $\bar{r} = \bar{\alpha} + t\bar{\beta}$.*

Let us consider a plane through the required line and the given line. The plane is parallel to $\bar{\beta}$ and it contains the vector $\bar{a} - \bar{\alpha}$. Since $(\bar{a} - \bar{\alpha}) \times \bar{\beta}$ is normal to the plane, it is perpendicular to the required line.

The required line is parallel to $\bar{r}\cdot\bar{n} = p$, so it is perpendicular to $\bar{n}$. Consequently the line is parallel to $\{(\bar{a} - \bar{\alpha}) \times \bar{\beta}\} \times \bar{n}$.

Thus the equation of the line is $(\bar{r} - \bar{a}) \times \{(\bar{a} - \bar{\alpha}) \times \bar{\beta} \times \bar{n}\} = \bar{0}$.

(iv) *To find the equation of the line of intersection of two planes $\bar{r}\cdot\bar{n}_1 = p_1$ and $\bar{r}\cdot\bar{n}_2 = p_2$.*

The line of intersection lies on both the planes, so it is perpendicular to $\bar{n}_1$ and $\bar{n}_2$. Hence it is parallel to $\bar{n}_1 \times \bar{n}_2$.

If ON is perpendicular to the line, then ON lies on the plane containing $\bar{n}_1$ and $\bar{n}_2$. Here O is the vector origin and N is the foot of the perpendicular on the line. $\overrightarrow{ON}$ can be expressed as a linear combination of $\bar{n}_1$ and $\bar{n}_2$.

Let $\overrightarrow{ON} = l_1\bar{n}_1 + l_2\bar{n}_2$, where l_1 and l_2 are scalars. The point N lies on both the planes. Therefore

$$(l_1\bar{n}_1 + l_2\bar{n}_2)\cdot\bar{n}_1 = p_1 \quad \text{or,} \quad l_1\bar{n}_1^2 + l_2\bar{n}_1\cdot\bar{n}_2 = p_1$$
$$\text{and} \quad (l_1\bar{n}_1 + l_2\bar{n}_2)\cdot\bar{n}_2 = p_2 \quad \text{or,} \quad l_1\bar{n}_1\cdot\bar{n}_2 + l_2\bar{n}_2^2 = p_2.$$

From these two relations,

$$l_1 = \frac{p_1\bar{n}_2^2 - p_2\bar{n}_1\cdot\bar{n}_2}{\bar{n}_1^2\bar{n}_2^2 - (\bar{n}_1\cdot\bar{n}_2)^2}, \quad l_2 = \frac{p_2\bar{n}_1^2 - p_1\bar{n}_1\cdot\bar{n}_2}{\bar{n}_1^2\bar{n}_2^2 - (\bar{n}_1\cdot\bar{n}_2)^2}.$$

Hence the required equation is $\bar{r} = l_1\bar{n}_1 + l_2\bar{n}_2 + t(\bar{n}_1 \times \bar{n}_2)$, where t is a parameter.

3.41 (i) To find the distance of a point from a given line.

Let the equation of the line BN be $\bar{r} = \bar{b} + t\bar{n}$, where t is a scalar, $\bar{b}$ the position vector of the point B and $\bar{n}$ the unit vector parallel to BN.

Let $\bar{a}$ be the position vector of the point A. Now $\overrightarrow{BA} = \bar{a} - \bar{b}$. $\therefore\ \overrightarrow{BA}^2 = (\bar{a} - \bar{b})^2$. AN is perpendicular to BN. BN is the projection of BA along the unit vector $\bar{n}$.

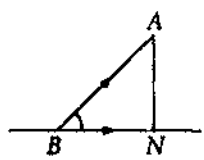


Fig. 37

This projection $= BA\cos\theta = 1.\ BA\cos\theta = \bar{n}\cdot(\bar{a} - \bar{b})$. From Fig. 37, $AN^2 = BA^2 - BN^2 = (\bar{a} - \bar{b})^2 - \{\bar{n}\cdot(\bar{a} - \bar{b})\}^2$. It gives the required distance.

Note 1. $\overrightarrow{NA} = \overrightarrow{BA} - \overrightarrow{BN} = (\bar{a} - \bar{b}) - \bar{n} \cdot (\bar{a} - \bar{b})\bar{n}$.

Note 2. If $\bar{n}$ is not the unit vector, then $\bar{n}$ is replaced by $\frac{\bar{n}}{|\bar{n}|}$.

(ii) **Shortest distance between two skew lines.**

Let LA and MC be the two skew lines and their equations be $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ respectively.

Here t and s are scalars, $\bar{a}$ and $\bar{c}$ are the position vectors of A and C. LA and MC are parallel to $\bar{b}$ and $\bar{d}$.

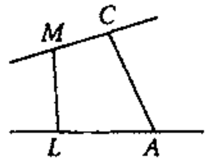


Fig. 38

Let LM be the shortest distance between the lines. Then it is perpendicular to both the lines, so it is parallel to $\bar{b} \times \bar{d}$. The shortest distance is the projection of AC on LM, i.e. the projection of $\bar{c} - \bar{a}$ on $(\bar{b} \times \bar{d})/|\bar{b} \times \bar{d}|$.
Hence

$$LM = (\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d})/|\bar{b} \times \bar{d}|. \tag{1}$$

The equation of the shortest distance. The shortest distance is the line of intersection of the two planes drawn through the two skew lines and the line of s.d.

The equation of the plane through $\bar{r} = \bar{a} + t\bar{b}$ and LM is

$$(\bar{r} - \bar{a}) \cdot \{\bar{b} \times (\bar{b} \times \bar{d})\} = 0. \tag{2}$$

The equation of the plane through $\bar{r} = \bar{c} + s\bar{d}$ and LM is

$$(\bar{r} - \bar{c}) \cdot \{\bar{d} \times (\bar{b} \times \bar{d})\} = 0. \tag{3}$$

The planes (2) and (3) determine the equation of the s.d.

Note 1. cartesian form of s.d.

$$(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d})/|\bar{b} \times \bar{d}| = \frac{[(\bar{c} - \bar{a})\bar{b}\bar{d}]}{|\bar{b} \times \bar{d}|}$$

$$= \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\sum(m_1 n_2 - m_2 n_1)^2}.$$

Here

$$\bar{a} = \bar{i}x_1 + \bar{j}y_1 + \bar{k}z_1, \quad \bar{c} = \bar{i}x_2 + \bar{j}y_2 + \bar{k}z_2,$$
$$\bar{b} = \bar{i}l_1 + \bar{j}m_1 + \bar{k}n_1, \quad \bar{d} = \bar{i}l_2 + \bar{j}m_2 + \bar{k}n_2.$$

Note 2. Condition for intersection of two lines.

If the lines $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ intersect, then the s.d. between them is zero. In this case, $(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d}) = 0$ or, $[(\bar{c} - \bar{a})\bar{b}\bar{d}] = 0$.

3.50 Volume of a tetrahedron

Let $ABCD$ be a tetrahedron and the position vectors of A, B, C, D be $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ w.r.t. certain vector origin O.

DL is perpendicular to the plane of $\triangle ABC$.

Now $\overrightarrow{AB} \times \overrightarrow{AC} = (\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})$.

It represents a vector parallel to LD and its magnitude $= 2\triangle ABC$.

Again LD is the projection of AD on LD.

$\therefore$ volume of the tetrahedron $= \frac{1}{3}\triangle ABC \cdot LD$.

$$= \tfrac{1}{6}(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}$$

$$= \tfrac{1}{6}\{(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})\} \cdot (\bar{d} - \bar{a})$$

$$= \tfrac{1}{6}[\bar{b} - \bar{a}, \bar{c} - \bar{a}, \bar{d} - \bar{a}]$$

$$= \tfrac{1}{6}\{[\bar{b}\bar{c}\bar{d}] + [\bar{c}\bar{a}\bar{d}] + [\bar{d}\bar{a}\bar{b}] - [\bar{a}\bar{b}\bar{c}]\}.$$

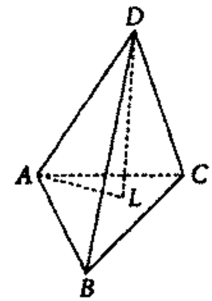


Fig. 39

Note 1. cartesian equivalence.

$$\text{Let } \bar{a} = \bar{i}x_1 + \bar{j}y_1 + \bar{k}z_1, \bar{b} = \bar{i}x_2 + \bar{j}y_2 + \bar{k}z_2,$$
$$\bar{c} = \bar{i}x_3 + \bar{j}y_3 + \bar{k}z_3, \bar{d} = \bar{i}x_4 + \bar{j}y_4 + \bar{k}z_4.$$
$$\text{Then } \bar{b} - \bar{a} = \bar{i}(x_2 - x_1) + \bar{j}(y_2 - y_1) + \bar{k}(z_2 - z_1),$$
$$\bar{c} - \bar{a} = \bar{i}(x_3 - x_1) + \bar{j}(y_3 - y_1) + \bar{k}(z_3 - z_1),$$
$$\bar{d} - \bar{a} = \bar{i}(x_4 - x_1) + \bar{j}(y_4 - y_1) + \bar{k}(z_4 - z_1).$$

$\therefore$ volume of the tetrahedron $= \frac{1}{6}[\bar{b} - \bar{a}, \bar{c} - \bar{a}, \bar{d} - \bar{a}]$

$$= \frac{1}{6}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix}.$$

Note 2. Condition for the coplanarity of four points.

Here the volume of the tetrahedron is zero.

$\therefore$ the condition is $[\bar{b}\bar{c}\bar{d}] + [\bar{c}\bar{a}\bar{d}] + [\bar{d}\bar{a}\bar{b}] - [\bar{a}\bar{b}\bar{c}] = 0$.

Note 3. Shortest distance between two opposite edges.

The s.d. between the opposite edges AC and BD is the numerical value of $(\bar{d} - \bar{a}) \cdot \{(\bar{c} - \bar{a}) \times (\bar{d} - \bar{b})\}/|(\bar{c} - \bar{a}) \times (\bar{d} - \bar{b})|$.

Note 4. If the vertex A is taken as vector origin, then the volume of the tetrahedron is $\frac{1}{6}[\bar{b}\bar{c}\bar{d}]$.

3.60 Sphere

(i) General equation.

Let $\bar{c}$ be the position vector of the centre C of the sphere of radius a w.r.t. the vector origin O. If $\bar{r}$ is the position vector of any point P on the sphere, then $\overrightarrow{CP} = \bar{r} - \bar{c}$.

Now $|\overrightarrow{CP}|^2 = a^2$

$$\text{or, } \quad (\bar{r} - \bar{c})^2 = a^2$$
$$\text{or, } \quad \bar{r}^2 - 2\bar{r} \cdot \bar{c} + \bar{c}^2 - a^2 = 0. \qquad (1)$$

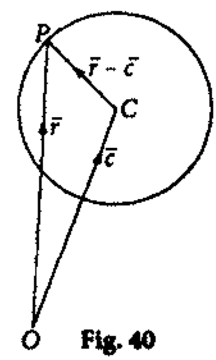


Fig. 40

It is the equation of the sphere.

If $k = \bar{c}^2 - a^2$, then the equation is

$$\bar{r}^2 - 2\bar{r} \cdot \bar{c} + k = 0. \tag{2}$$

Corollary I. *If the centre is the origin, then $\bar{c} = 0$ and the equation of the sphere is*

$$\bar{r}^2 = a^2. \tag{3}$$

Corollary II. *If the origin lies on the surface, then $\bar{c}^2 = a^2$ and the equation of the sphere is*

$$\bar{r}^2 - 2\bar{r} \cdot \bar{c} = 0. \tag{4}$$

(ii) **Equation of sphere with diameter ends as $\bar{a}$ and $\bar{b}$.**

Here $\bar{a}$ and $\bar{b}$ are the position vectors of A and B, the ends of a diameter of the sphere. Let P be any point on the sphere whose position vector is $\bar{r}$.

Then $\overrightarrow{AP} = \bar{r} - \bar{a}$ and $\overrightarrow{BP} = \bar{r} - \bar{b}$.

Since PA is perpendicular to PB

$$(\bar{r} - \bar{a}) \cdot (\bar{r} - \bar{b}) = 0. \tag{5}$$

It is the equation of the sphere.

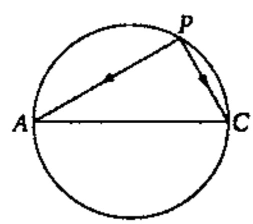


Fig. 41

(iii) **Intersection of a line and a sphere.**

Let the sphere and the line be

$$\bar{r}^2 - 2\bar{r} \cdot \bar{c} + k = 0 \quad \text{and} \quad \bar{r} = \bar{a} + t\bar{b}.$$

The values of t, where the line cuts the sphere are given by

$$(\bar{a} + t\bar{b})^2 - 2(\bar{a} + t\bar{b}) \cdot \bar{c} + k = 0$$

$$\text{or,} \quad \bar{b}^2 t^2 + 2\bar{b} \cdot (\bar{a} - \bar{c})t + \bar{a}^2 - 2\bar{a} \cdot \bar{c} + k = 0. \tag{6}$$

The two values of t obtained from (6) correspond to two points of intersection between the line and the sphere.

Note 1. The line would touch the sphere if two values of t are equal.

Note 2. Let the line cut the sphere at P and Q and $\bar{a}$ be the position vector of A. If t_1 and t_2 are the roots of (6), then the position vectors of P and Q are $\bar{a} + t_1\bar{b}$ and $\bar{a} + t_2\bar{b}$.

Hence

$$AP \cdot AQ = \bar{a}^2 - 2\bar{a} \cdot \bar{c} + k. \tag{7}$$

If the line touches the sphere at T, then P and Q coincide with T and

$$AT^2 = \bar{a}^2 - 2\bar{a} \cdot \bar{c} + k. \tag{8}$$

It is the *power of the point* $\bar{a}$ w.r.t. the sphere.

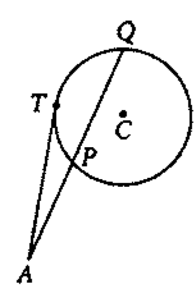


Fig. 42

(iv) **Tangent plane at the point $\bar{a}$.**

If the line is the tangent at A (see Fig. 42), then A, P, Q coincide, i.e. $t_1 = 0 = t_2$.

In this case, the sum of the roots and the product of the roots of (6) are zero. Consequently

$$\bar{b} \cdot (\bar{a} - \bar{c}) = 0 \tag{9}$$

$$\text{and} \quad \bar{a}^2 - 2\bar{a} \cdot \bar{c} + k = 0. \tag{10}$$

The condition suggests that the line $\bar{r} = \bar{a} + t\bar{b}$ is perpendicular to $\bar{a} - \bar{c}$.

$$\therefore \ (\bar{r} - \bar{a}) \cdot (\bar{a} - \bar{c}) = 0. \tag{11}$$

It shows that all tangents through the point $\bar{a}$ lie on the plane (11).

From (11),

$$\bar{r} \cdot \bar{a} - \bar{r} \cdot \bar{c} - \bar{a}^2 + \bar{a} \cdot \bar{c} = 0$$

$$\text{or,} \quad \bar{r} \cdot \bar{a} - \bar{c} \cdot (\bar{r} + \bar{a}) + k = 0 \ [\text{by (10)}]. \tag{12}$$

It is the *standard equation of the tangent plane* at $\bar{a}$.

(v) **Condition of tangency.**

A plane $\bar{r} \cdot \bar{n} = p$ would be tangent to the sphere $\bar{r}^2 - 2\bar{r} \cdot \bar{c} + k = 0$ if the perpendicular distance of the plane from the centre $\bar{c}$ = radius, i.e.

$$\frac{(p - \bar{r} \cdot \bar{n})^2}{\bar{n}^2} = a^2. \tag{13}$$

3.70 Centroids and centre of mass

Centroid. *If $\bar{a}, \bar{b}, \bar{c}, \cdots, \bar{l}$ be position vectors of n given points and $\bar{g} = \overrightarrow{OG} = \frac{1}{n}(\bar{a} + \bar{b} + \bar{c} + \cdots + \bar{l})$, then G is called the centroid (mean centre, centre of mean position) of the given points.* [O is the vector origin.]

If $\bar{g} = \overrightarrow{OG} = \frac{m_1\bar{a} + m_2\bar{b} + m_3\bar{c} + \cdots + m_n\bar{l}}{m_1 + m_2 + m_3 + \cdots + m_n}$, then G is called the centroid of n weighted points.

Here $m_1 + m_2 + m_3 + \cdots + m_n \neq 0$ and $m_1, m_2, \cdots, m_n$ are the weights of the points $\bar{a}, \bar{b}, \cdots \bar{l}$ respectively.

Centre of mass. *If $m_1, m_2, \cdots, m_n$ masses are situated at the points $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n$, then the point*

$$\bar{a} = \frac{m_1\bar{a}_1 + m_2\bar{a}_2 + \cdots + m_n\bar{a}_n}{m_1 + m_2 + \cdots + m_n}$$

is called the centre of mass.

Centre of gravity. *If n parallel forces whose magnitudes are proportional to $m_1, m_2, \cdots, m_n$ act at the points $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n$, then the point*

$$\bar{a} = \frac{m_1\bar{a}_1 + m_2\bar{a}_2 + \cdots + m_n\bar{a}_n}{m_1 + m_2 + \cdots + m_n}$$

is called the centre of gravity of these points. It is equivalent to the centre of mass.

Note. cartesian equivalence.

I' .. $= (x_t, y_t, z_t)$ and $\bar{a} = (x, y, z)$, then

$$x\bar{i} + y\bar{j} + z\bar{k} = \frac{\sum m_t(x_t\bar{i} + y_t\bar{j} + z_t\bar{k})}{\sum m_t}, \ t = 1, 2, \cdots, n,$$

$$= \frac{\sum m_t x_t\bar{i} + \sum m_t y_t\bar{j} + \sum m_t z_t\bar{k}}{\sum m_t}$$

$$\therefore \ x = \frac{\sum m_t x_t}{\sum m_t}, y = \frac{\sum m_t y_t}{\sum m_t}, z = \frac{\sum m_t z_t}{\sum m_t}.$$

3.71 Forces

(i) **Resultant of concurrent forces.** If $\bar{F}_1, \bar{F}_2, \cdots, \bar{F}_n$ forces act at O, then the resultant of these forces is $\bar{F} = \bar{F}_1 + \bar{F}_2 + \cdots + \bar{F}_n$. It acts at O.

If $\bar{F} = 0$, then the forces are in equilibrium.

(ii) **Lami's theorem.** *If three coplanar forces acting at a point be in equilibrium, then magnitude of each force is proportional to the sine of the angle between the other two.*

Let P, Q, R be the magnitudes of the forces and $\bar{a}, \bar{b}, \bar{c}$ be the unit vectors along them.

Since the forces are in equilibrium,

$$P\bar{a} + Q\bar{b} + R\bar{c} = \bar{0}. \tag{1}$$

Taking vector product with $\bar{a}$,

$$P(\bar{a} \times \bar{a}) + Q(\bar{a} \times \bar{b}) + R(\bar{a} \times \bar{c}) = \bar{0}$$

$$\text{or,} \quad Q(\bar{a} \times \bar{b}) = R(\bar{c} \times \bar{a}) \text{ or, } \frac{Q}{|\bar{c} \times \bar{a}|} = \frac{R}{|\bar{a} \times \bar{b}|}.$$

Taking vector product with $\bar{b}$, $\frac{P}{|\bar{b} \times \bar{c}|} = \frac{R}{|\bar{a} \times \bar{b}|}$.

$$\therefore \ \frac{P}{|\bar{b} \times \bar{c}|} = \frac{Q}{|\bar{c} \times \bar{a}|} = \frac{R}{|\bar{a} \times \bar{b}|} \text{ or, } \frac{P}{\sin \widehat{bc}} = \frac{Q}{\sin \widehat{ca}} = \frac{R}{\sin \widehat{ab}}.$$

(iii) **Rankine's theorem.** *If four forces acting at a point are in equilibrium, then each force is proportional to the volume of the parallelopiped whose coterminus edges represent the unit vectors in the directions of the other three.*

Let P_1, P_2, P_3, P_4 be the magnitudes of the forces and $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ unit vectors in the direction of the forces.

Since the forces are in equilibrium,

$$P_1\bar{a} + P_2\bar{b} + P_3\bar{c} + P_4\bar{d} = \bar{0}. \tag{1}$$

Multiplying by $\bar{c} \times \bar{d}$,

$$P_1[\bar{a}\bar{c}\bar{d}] + P_2[\bar{b}\bar{c}\bar{d}] = 0. \tag{2}$$

Similarly multiplying (1) by $\bar{a} \times \bar{b}$ and $\bar{a} \times \bar{c}$, we get

$$P_3[\bar{a}\bar{b}\bar{c}] + P_4[\bar{a}\bar{b}\bar{d}] = 0 \tag{3}$$

$$\text{and} \quad P_2[\bar{b}\bar{a}\bar{c}] + P_4[\bar{a}\bar{c}\bar{d}] = 0. \tag{4}$$

From (2), (3) and (4),

$$\frac{P_1}{[\bar{b}\bar{c}\bar{d}]} = \frac{P_2}{[\bar{c}\bar{a}\bar{d}]} = \frac{P_3}{[\bar{a}\bar{b}\bar{d}]} = \frac{-P_4}{[\bar{a}\bar{b}\bar{c}]}.$$

Hence the result follows.

(iv) **Work done by a force.**

If a force $\bar{F}$ experiences a displacement $\bar{d}$, then the work done by the force is the product of the magnitudes of the displacement and the component of the force in the direction of the displacement.

$\therefore$ work done $= |\bar{d}||\bar{F}|\cos\theta = \bar{F} \cdot \bar{d}$.

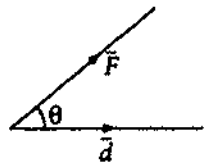


Fig. 43

Here θ is the angle between $\bar{F}$ and $\bar{d}$. Hence the dot product between a force and its displacement represent the work done by the force for this displacement.

If there is a number of forces $\bar{F}_1, \bar{F}_2, \cdots, \bar{F}_n$ acting on the particle and the particle experiences a displacement $\bar{d}$, then the work done

$$= \bar{F}_1 \cdot \bar{d} + \bar{F}_2 \cdot \bar{d} + \cdots + \bar{F}_n \cdot \bar{d} = \bar{d} \cdot (\bar{F}_1 + \bar{F}_2 + \cdots + \bar{F}_n) = \bar{d} \cdot \bar{F},$$

where $\bar{F}$ is the resultant of the forces.

(v) **Moment of a force about a point.**

Let a force $\bar{F}$ be applied on a body and O be the given point.

If P is any point on the line of $\bar{F}$ and $\overrightarrow{OP} = \bar{r}$, then $\bar{m} = \bar{r} \times \bar{F}$ is called the *moment vector* of $\bar{F}$ or *torque* of $\bar{F}$ about the point O. It is perpendicular to the plane through the point O and the line of $\bar{F}$.

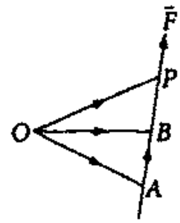


Fig. 44

Note 1. The moment vector does not depend on the position of P.

Let A and B be any other points on the line of action of $\bar{F}$. Then $\overrightarrow{OB} \times \bar{F} = (\overrightarrow{OA} + \overrightarrow{AB}) \times \bar{F} = \overrightarrow{OA} \times \bar{F} + \overrightarrow{AB} \times \bar{F}$.

But $\overrightarrow{AB} \times \bar{F} = \bar{0}$ because $\overrightarrow{AB}$ and $\bar{F}$ are in the same line.

$\therefore$ $\overrightarrow{OB} \times \bar{F} = \overrightarrow{OA} \times \bar{F}$.

Here A and B are arbitrary points, hence the moment vector about O is same for all points on the line of action of $\bar{F}$.

Note 2. If $\bar{F} = \bar{F}_1 + \bar{F}_2 + \cdots + \bar{F}_n$, then

$$\begin{aligned} \bar{m} &= \bar{r} \times \bar{F} = \bar{r} \times (\bar{F}_1 + \bar{F}_2 + \cdots + \bar{F}_n) \\ &= \bar{r} \times \bar{F}_1 + \bar{r} \times \bar{F}_2 + \cdots + \bar{r} \times \bar{F}_n. \end{aligned}$$

It shows that *the moment of the resultant of any number of forces about a point is equal to the sum of separate moments.* (Varignon's theorem)

(vi) **Moment of a force about an axis**

Let $\bar{b}$ be the unit vector in the direction of the axis or the line and O be any point on the axis. If $\bar{m} = \bar{r} \times \bar{F}$ is the moment of the force $\bar{F}$ about O, then the moment of $\bar{F}$ about the axis is $\bar{m} \cdot \bar{b} = (\bar{r} \times \bar{F}) \cdot \bar{b}$. It is the component of $\bar{m}$ on the axis.

3.72 Rotation about a fixed axis

Let a rigid body rotate about a fixed axis ON. The position of the body at any instant may be specified by the angle ϕ (radians) between an axial plane fixed in our frame of reference and an axial plane fixed in the body. The angular speed of the body at this instant is defined as $\frac{d\phi}{dt}$.

If $\bar{k}$ is the unit vector along the axis ON, then the angular velocity vector $\bar{\omega} = \bar{k}\frac{d\phi}{dt}$. $\bar{\omega}$ is related to the instantaneous sense of rotation by the rule of the right-hand screw.

Let $\bar{r}$ be the position vector of a point P of the body w.r.t. the chosen vector origin O on the axis. PM is perpendicular to ON. For the rigid body the position of P relative to the fixed axis ON remains fixed. Thus the point P is moving in a circle with centre M and radius MP. The linear velocity of P is perpendicular to the plane OPM. If $\bar{v}$ is the velocity of P at time t, then $\bar{v} = \bar{\omega} \times \bar{r}$ and $|\bar{v}| = \frac{d\phi}{dt} \cdot OP \sin\theta = MP \cdot \frac{d\phi}{dt}$.

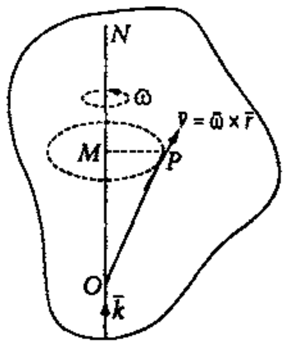


Fig. 45

WORKED-OUT EXAMPLES

1. *Find the equation of the plane through the points $A(-1,1,2), B(1,-2,1)$ and $C(2,2,4)$.* [CH 2003]

   Here $\overrightarrow{AB} = (2,-3,-1)$ and $\overrightarrow{AC} = (3,1,2)$.

   Let $P(x,y,z)$ be any point on the plane.

   $\overrightarrow{AB} \times \overrightarrow{AC} = (-5,-7,11)$ is perpendicular to the plane.

   $\overrightarrow{AP} = (x+1, y-1, z-2)$. It lies in the plane. Therefore, it is perpendicular to $\overrightarrow{AB} \times \overrightarrow{AC}$.

   $$\text{Thus} \quad \overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$$
   $$\text{or,} \quad -5(x+1) - 7(y-1) + 11(z-2) = 0 \text{ or, } 5x + 7y - 11z + 20 = 0.$$

   It is the equation of the plane.

2. *If $\bar{a}$ is the position vector of a given point A and $\bar{r}$ is the position vector of any point P in space, find the locus of P in each of the following cases.*

   (i) $(\bar{r} - \bar{a}) \cdot \bar{a} = 0$, (ii) $(\bar{r} - \bar{a}) \cdot \bar{r} = 0$.

(i) Let $\bar{a}$ and $\bar{r}$ be the vector positions of points A and P w.r.t. the vector origin O.

Then $\overrightarrow{AP} = \bar{r} - \bar{a}$.

Since $(\bar{r} - \bar{a}) \cdot \bar{a} = 0, \overrightarrow{AP} \cdot \overrightarrow{OA} = 0$.

Therefore, AP is perpendicular to OA.

Hence the locus of P is a plane passing through the point A and perpendicular to OA.

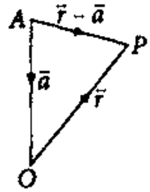


Fig. 46

(ii) Here also $\bar{a}$ and $\bar{r}$ are position vectors of A and P w.r.t. the vector origin O.

Since $(\bar{r} - \bar{a}) \cdot \bar{r} = 0, \overrightarrow{AP} \cdot \overrightarrow{OP} = 0$, i.e. AP is perpendicular to OP. Hence the locus of P is a sphere with OA as diameter.

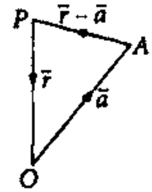


Fig. 47

3. *If the vector equations of lines are $\bar{r} = \bar{r}_1 + t\bar{\alpha}, \bar{r} = \bar{r}_2 + t\bar{\beta}$ where t is a scalar and $\bar{r}_1, \bar{\alpha}, \bar{r}_2, \bar{\beta}$ are vectors with coordinates $(1,4,5), (2,1,2), (2,8,11)$ and $(-1,3,4)$ respectively, show that the lines are coplanar.*

The lines will be coplanar, if $[\bar{r}_2\bar{\alpha}\bar{\beta}] = [\bar{r}_1\bar{\alpha}\bar{\beta}]$.

$\bar{\alpha} \times \bar{\beta} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & 2 \\ -1 & 3 & 4 \end{vmatrix} = -2\bar{i} - 10\bar{j} + 7\bar{k}$, where $\bar{i}, \bar{j}, \bar{k}$ are unit vectors parallel to x, y, z-axes.

$$\begin{aligned} \bar{r}_2 \cdot (\bar{\alpha} \times \bar{\beta}) &= (2\bar{i} + 8\bar{j} + 11\bar{k}) \cdot (-2\bar{i} - 10\bar{j} + 7\bar{k}) \\ &= -4 - 80 + 77 = -7. \\ \bar{r}_1 \cdot (\bar{\alpha} \times \bar{\beta}) &= (\bar{i} + 4\bar{j} + 5\bar{k}) \cdot (-2\bar{i} - 10\bar{j} + 7\bar{k}) \\ &= -2 - 40 + 35 = -7. \end{aligned}$$

Hence the lines are coplanar.

4. *Find the perpendicular distance of a corner of a unit cube from a diagonal not passing through it.* **[CH 2006; BH 2000]**

Let $OABCDEFG$ be a unit cube and $\overrightarrow{OA} = \bar{i}, \overrightarrow{OC} = \bar{j}, \overrightarrow{OE} = \bar{k} \cdot \bar{i}, \bar{j}, \bar{k}$ are three unit vectors.

Now the diagonal $\overrightarrow{OG} = \bar{i} + \bar{j} + \bar{k}$.

A unit vector along OG is $\frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}$.

Let EM be perpendicular to OG.

$\therefore\ OM = \bar{k} \cdot \frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$.

Again $OE = |\overrightarrow{OE}| = |\bar{k}| = 1$.

$\therefore\ EM = \sqrt{(OE^2 - OM^2)} = \sqrt{(1 - \frac{1}{3})} = \sqrt{\frac{2}{3}}$.

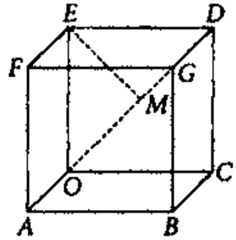


Fig. 48

5. *Prove that* $\bar{r} \times \bar{a} = \bar{b} \times \bar{a}$ *and* $\bar{r} \times \bar{b} = \bar{a} \times \bar{b}$ *intersect and find their point of intersection.*

The first equation can be written as $(\bar{r} - \bar{b}) \times \bar{a} = \bar{0}$. It implies that the line passes through the point of $\bar{b}$ and it is parallel to $\bar{a}$. Hence it can be written as

$$\bar{r} = \bar{b} + t\bar{a}, \tag{1}$$

where t is a scalar.

Similarly the second equation can be written as

$$\bar{r} = \bar{a} + s\bar{b}, \tag{2}$$

where s is a scalar.

If these lines intersect, they lie in the plane which is parallel to $\bar{b} - \bar{a}, \bar{a}$ and $\bar{b}$. For this $\bar{b} - \bar{a}, \bar{a}$ and $\bar{b}$ must be coplanar.

Since $(\bar{b} - \bar{a}) \cdot (\bar{a} \times \bar{b}) = \bar{b} \cdot (\bar{a} \times \bar{b}) - \bar{a} \cdot (\bar{a} \times \bar{b}) = 0$, the vectors are coplanar.

For the point of intersection we should have identical values of $\bar{r}$.

It happens for $t = s = 1$.

$\therefore$ the required point is $\bar{a} + \bar{b}$.

6. *$A(4, d, -1), B(4, 2, -2)$ and $C(6, 4, -1)$ are three non-collinear points. Use vector method to find d such that the perpendicular distance of A from the line joining B and C is 1.*

$\overrightarrow{BA} = (0, d-2, 1)$ and $\overrightarrow{BC} = (2, 2, 1)$.

By the given condition

$$\frac{|\overrightarrow{BC} \times \overrightarrow{BA}|}{|\overrightarrow{BC}|} = 1. \tag{1}$$

$$\overrightarrow{BC} \times \overrightarrow{BA} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 2 & 1 \\ 0 & d-2 & 1 \end{vmatrix} = \bar{i}(4-d) - \bar{j}2 + \bar{k}(2d-4).$$

$$\begin{aligned} \therefore \quad & (4-d)^2 + 4 + (2d-4)^2 = 9, \quad \text{[by (1)]} \\ \text{or,} \quad & 5d^2 - 24d + 27 = 0 \\ \text{or,} \quad & (d-3)(5d-9) = 0 \quad \text{or,} \quad d = 3, 9/5. \end{aligned}$$

7. *Find the equation of the plane containing the line $\bar{r} = t\bar{\alpha}$ and perpendicular to the plane containing the lines $\bar{r} = t_1\bar{\beta}$ and $\bar{r} = t_2\bar{\gamma}$.*

$\bar{\beta} \times \bar{\gamma}$ is perpendicular to the plane containing the lines $\bar{r} = t_1\bar{\beta}$ and $\bar{r} = t_2\bar{\gamma}$. Therefore, the required plane is parallel to the vector $\bar{\beta} \times \bar{\gamma}$.

The line $\bar{r} = t\bar{\alpha}$ passes through the origin and it is parallel to $\bar{\alpha}$. Since the required plane contains this line, the origin lies on the plane and $\bar{\alpha}$ is parallel to the plane. Hence $\bar{\alpha} \times (\bar{\beta} \times \bar{\gamma})$ is normal to the plane.

Thus the required equation is $\bar{r} \cdot \{\bar{\alpha} \times (\bar{\beta} \times \bar{\gamma})\} = 0$.

8. *Find the point of intersection of the planes $\bar{r}\cdot\bar{n}_1 = p_1, \bar{r}\cdot\bar{n}_2 = p_2, \bar{r}\cdot\bar{n}_3 = p_3$, where $\bar{n}_1, \bar{n}_2, \bar{n}_3$ are three non-coplanar vectors.*

   Since $\bar{n}_1, \bar{n}_2, \bar{n}_3$ are three given non-coplanar vectors, $\bar{n}_1 \times \bar{n}_2, \bar{n}_2 \times \bar{n}_3, \bar{n}_3 \times \bar{n}_1$ are also non-coplanar.

   Let the planes meet at a point whose vector position is $\bar{\alpha}$ (say). Any vector can be expressed as a linear combination of three non-coplanar vectors. Thus we can write

$$\bar{\alpha} = l(\bar{n}_2 \times \bar{n}_3) + m(\bar{n}_3 \times \bar{n}_1) + n(\bar{n}_1 \times \bar{n}_2), \tag{1}$$

   where l, m, n are constants.

   $\bar{\alpha}$ satisfies the equations of the given planes. From the first equation

$$\begin{aligned} &\bar{\alpha}\cdot\bar{n}_1 = p_1 \\ \text{or,}\quad &\{l(\bar{n}_2 \times \bar{n}_3) + m(\bar{n}_3 \times \bar{n}_1) + n(\bar{n}_1 \times \bar{n}_2)\}\cdot\bar{n}_1 = p_1 \\ \text{or,}\quad &l[\bar{n}_1\bar{n}_2\bar{n}_3] = p_1 \quad \text{or,} \quad l = \frac{p_1}{[\bar{n}_1\bar{n}_2\bar{n}_3]}. \end{aligned}$$

   Similarly,

$$m = \frac{p_2}{[\bar{n}_1\bar{n}_2\bar{n}_3]} \quad \text{and} \quad n = \frac{p_3}{[\bar{n}_1\bar{n}_2\bar{n}_3]}.$$

   Putting these values of l, m, n in (1), the position vector of the common point is obtained.

9. *Find the s.d. between two skew-lines $\bar{r} = \bar{r}_1 + t\bar{\alpha}, \bar{r} = \bar{r}_2 + t\bar{\beta}$, where t is a scalar and $\bar{r}_1, \bar{\alpha}, \bar{r}_2, \bar{\beta}$ are vectors with coordinates* $(1,-2,3)$, $(2,1,1)$, $(-2,2,-1)$, $(-3,1,2)$ *respectively.*

   Here s.d. $= |(\bar{r}_2 - \bar{r}_1)\cdot(\bar{\alpha}\times\bar{\beta})|/|\bar{\alpha}\times\bar{\beta}|$.

$$\bar{\alpha}\times\bar{\beta} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & 1 \\ -3 & 1 & 2 \end{vmatrix} = \bar{i} - 7\bar{j} + 5\bar{k}.$$

$$\bar{r}_2 - \bar{r}_1 = (-2\bar{i} + 2\bar{j} - \bar{k}) - (\bar{i} - 2\bar{j} + 3\bar{k}) = -3\bar{i} + 4\bar{j} - 4\bar{k}.$$

$$\begin{aligned} \text{Now s.d.} &= \frac{|(-3\bar{i} + 4\bar{j} - 4\bar{k})\cdot(\bar{i} - 7\bar{j} + 5\bar{k})|}{|\bar{i} - 7\bar{j} + 5\bar{k}|} \\ &= \frac{|-3-28-20|}{\sqrt{1+49+25}} = \frac{51}{5\sqrt{3}} = \frac{17\sqrt{3}}{5}. \end{aligned}$$

10. *If the volume of a tetrahedron is 2 and three of its vertices have position vectors* $(1,1,0), (1,0,1), (2,-1,1)$, *find the locus of the fourth vertex.* [NH 99]

    Let $ABCD$ be the tetrahedron and the position vectors of A, B, C be $(1,1,0)$, $(1,0,1)$, $(2,-1,1)$ respectively. Let the position vector of D be (α, β, γ).

Volume of the tetrahedron $= \frac{1}{6}(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}$.

Here $\overrightarrow{AB} = -\bar{j} + \bar{k}, \overrightarrow{AC} = \bar{i} - 2\bar{j} + \bar{k}, \overrightarrow{AD} = (\alpha - 1)\bar{i} + (\beta - 1)\bar{j} + \gamma\bar{k}$.

$\therefore \quad \frac{1}{6}\{(-\bar{j} + \bar{k}) \times (\bar{i} - 2\bar{j} + \bar{k})\} \cdot \{(\alpha - 1)\bar{i} + (\beta - 1)\bar{j} + \gamma\bar{k}\} = 2$

or, $\quad (\bar{i} + \bar{j} + \bar{k}) \cdot \{(\alpha - 1)\bar{i} + (\beta - 1)\bar{j} + \gamma\bar{k}\} = 12$

or, $\quad (\alpha - 1) + (\beta - 1) + \gamma = 12 \quad$ or, $\quad \alpha + \beta + \gamma = 14$.

Hence the locus of D is $x + y + z = 14$.

11. *Find the coordinates of the centre of the sphere inscribed in the tetrahedron bounded by the planes*

$\bar{r} \cdot \bar{i} = 0, \bar{r} \cdot \bar{j} = 0, \bar{r} \cdot \bar{k} = 0 \quad$ and $\quad \bar{r} \cdot (\bar{i} + \bar{j} + \bar{k}) = a$.

Also write down the equation of the sphere.

Let the centre of the sphere be $\bar{\alpha}$.

The given planes are tangent planes. The perpendiculars from the centre to these planes are equal to the radius of the sphere.

$\therefore \quad \bar{\alpha} \cdot \bar{i} = \bar{\alpha} \cdot \bar{j} = \bar{\alpha} \cdot \bar{k} = \frac{\bar{\alpha} \cdot (\bar{i} + \bar{j} + \bar{k}) - a}{\sqrt{3}}$.

If $\bar{\alpha} = \bar{i}\alpha_1 + \bar{j}\alpha_2 + \bar{k}\alpha_3$, then

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{3\alpha_1 - a}{\sqrt{3}}.$$

From this $\alpha_1 = \frac{a}{3 - \sqrt{3}} = \frac{a(3 + \sqrt{3})}{6}$.

Thus the centre is $\frac{a}{6}(3 + \sqrt{3})(\bar{i} + \bar{j} + \bar{k})$.

The equation of the sphere is

$$(\bar{r} - \bar{\alpha})^2 = \left\{\frac{a}{6}(3 + \sqrt{3})\right\}^2.$$

12. *A particle, acted on by constant forces $4\bar{i} + \bar{j} - 3\bar{k}$ and $3\bar{i} + \bar{j} - \bar{k}$, is displaced from the point $\bar{i} + 2\bar{j} + 3\bar{k}$ to the point $5\bar{i} + 4\bar{j} + \bar{k}$. Find the work done by the force on the particle.* **[CH 95, 98, 2002, 03]**

Let $\bar{F}_1 = 4\bar{i} + \bar{j} - 3\bar{k}, \bar{F}_2 = 3\bar{i} + \bar{j} - \bar{k}$ and $\bar{F}$ be the resultant of these forces. Then $\bar{F} = \bar{F}_1 + \bar{F}_2 = 7\bar{i} + 2\bar{j} - 4\bar{k}$.

The displacement vector

$$\bar{d} = (5\bar{i} + 4\bar{j} + \bar{k}) - (\bar{i} + 2\bar{j} + 3\bar{k}) = 4\bar{i} + 2\bar{j} - 2\bar{k}.$$

Work done $= \bar{F} \cdot \bar{d} = (7\bar{i} + 2\bar{j} - 4\bar{k}) \cdot (4\bar{i} + 2\bar{j} - 2\bar{k}) = 28 + 4 + 8 = 40$ units.

13. *A force of 15 units acts in the direction of the vector $\bar{i} - 2\bar{j} + \bar{k}$ and passes through a point $\bar{i} + \bar{j} + \bar{k}$. Find the moment of the force about the point $2\bar{i} - 2\bar{j} + 2\bar{k}$.* **[CH 2009]**

Let $2\bar{i} - 2\bar{j} + 2\bar{k}$ and $\bar{i} + \bar{j} + \bar{k}$ be the position vectors of the points A and P.

The unit vector in the direction of the force is $\frac{1}{\sqrt{6}}(\bar{i} - 2\bar{j} + \bar{k})$.

$$\text{Required moment} = \overrightarrow{AP} \times \frac{15}{\sqrt{6}}(\bar{i} - 2\bar{j} + \bar{k})$$

$$= \frac{15}{\sqrt{6}}(-\bar{i} + 3\bar{j} - \bar{k}) \times (\bar{i} - 2\bar{j} + \bar{k})$$

$$= \frac{15}{\sqrt{6}}\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 3 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \frac{15}{\sqrt{6}}(\bar{i} - \bar{k}).$$

14. *A rigid body is spinning with an angular velocity 4 radians per second about an axis parallel to $3\bar{j} - \bar{k}$ passing through the point $\bar{i} + 3\bar{j} - \bar{k}$. Find the velocity of the particle at the point $4\bar{i} - 2\bar{j} + \bar{k}$.*

Let $\bar{i} + 3\bar{j} - \bar{k}$ and $4\bar{i} - 2\bar{j} + \bar{k}$ be the position vectors of the points A and P.

$\overrightarrow{AP} = (4\bar{i} - 2\bar{j} + \bar{k}) - (\bar{i} + 3\bar{j} - \bar{k}) = 3\bar{i} - 5\bar{j} - 2\bar{k}$.

The unit vector parallel to the axis of rotation is $\frac{3\bar{j} - \bar{k}}{\sqrt{3^2 + 1^2}} = \frac{3\bar{j} - \bar{k}}{\sqrt{10}}$.

The vector specifying angular velocity $\bar{\omega} = \frac{4}{\sqrt{10}}(3\bar{j} - \bar{k})$.

Now the required velocity $\bar{v} = \bar{\omega} \times \overrightarrow{AP}$

$= \frac{4}{\sqrt{10}}(3\bar{j} - \bar{k}) \times (3\bar{i} - 5\bar{j} - 2\bar{k}) = \frac{4}{\sqrt{10}}(\bar{i} - 3\bar{j} - 9\bar{k})$.

It represents a speed of $4\sqrt{9.1}$ units in the direction of $\bar{i} - 3\bar{j} - 9\bar{k}$.

EXERCISE III

1. Find the equation of the line through the points $(2, 3, 4)$ and $(3, 4, 5)$.

2. Find the equation of a line through the point $(-2, 1, 0)$ and parallel to the vector $5\bar{i} - 3\bar{j} + 4\bar{k}$.

3. Find the equation of the plane passing through the point $2\bar{i} - 3\bar{j} + \bar{k}$ and perpendicular to the line joining the points $3\bar{i} + 4\bar{j} - \bar{k}$ and $2\bar{i} - \bar{j} + 5\bar{k}$.

4. Find the equation of the plane through the points $(2, -1, 4), (3, 4, 7)$ and $(-2, 3, -1)$.

5. Show that $\bar{r} = \bar{a} + t\bar{b} + s\bar{c}$ represents a plane through the point $\bar{a}$ and parallel to $\bar{b}$ and $\bar{c}$. t and s are scalars.

6. Prove that the lines $\bar{r} = \bar{a} + t(\bar{b} \times \bar{c})$ and $\bar{r} = \bar{b} + s(\bar{c} \times \bar{a})$ will intersect if $\bar{a} \cdot \bar{c} = \bar{b} \cdot \bar{c}$.

7. Find the equation of the plane passing through the point $(2, 3, -1)$ and perpendicular to the vector $(3, -4, 7)$. Find the length of the perpendicular from the origin to the plane.

8. Find the equation of the plane which contains two parallel lines $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + t\bar{b}$.

   [*Hints.* The plane passes through $\bar{a}$ and is parallel to $\bar{a} - \bar{c}$ and $\bar{b}$.]

9. Referred to O as origin the coordinates of A, B, C, D are $(1,1,0), (1,\frac{1}{2},1), (0,1,1)$ and $(1,1,1)$ respectively. Find by vector methods the ratio $OP : PD$, P being the point, where OD meets the plane ABC. [CH 2001]

10. Find the equation of the plane which passes through the line of intersection of the planes $\bar{r}\cdot\bar{n}_1 = p_1, \bar{r}\cdot\bar{n}_2 = p_2$ and is parallel to the line of intersection of the planes $\bar{r}\cdot\bar{n}_3 = p_3, \bar{r}\cdot\bar{n}_4 = p_4$.

    [*Hint.* The equation of the plane is of the form $\bar{r}\cdot(\bar{n}_1 - \lambda\bar{n}_2) = p_1 - \lambda p_2$, where λ is a parameter. The plane is parallel to $\bar{n}_3 \times \bar{n}_4$, so it is perpendicular to $\bar{n}_1 - \lambda\bar{n}_2$. Therefore, $(\bar{n}_1 - \lambda\bar{n}_2)\cdot(\bar{n}_3 \times \bar{n}_4) = 0$, $\lambda = [\bar{n}_1\bar{n}_3\bar{n}_4]/[\bar{n}_2\bar{n}_3\bar{n}_4]$. Hence the equation is

    $$[\bar{n}_2\bar{n}_3\bar{n}_4](\bar{r}\cdot\bar{n}_1 - p_1) = [\bar{n}_1\bar{n}_3\bar{n}_4](\bar{r}\cdot\bar{n}_2 - p_2).]$$

11. Find the locus of a point which is equidistant from the three planes $\bar{r}\cdot\bar{n}_1 = q_1, \bar{r}\cdot\bar{n}_2 = q_2, \bar{r}\cdot\bar{n}_3 = q_3$.

12. Show that the lines $\bar{r} = \bar{r}_1 + t\bar{\alpha}, \bar{r} = \bar{r}_2 + t\bar{\beta}$, where t is a scalar and $\bar{r}_1, \bar{\alpha}, \bar{r}_2, \bar{\beta}$ are vectors with coordinates $(4,5,1), (-4,-6,-2), (3,9,4), (-7,-5,0)$ respectively are coplanar.

13. Show that the equation of the line which passes through the point $\bar{c}$, is parallel to the plane $\bar{r}\cdot\bar{a} = 0$ and intersects the line $\bar{r} - \bar{b} = t\bar{d}$ is $\bar{r} = \bar{c} + k[\bar{a}\times\{\bar{d}\times(\bar{b}-\bar{c})\}]$, where k is a scalar.

14. The vector equations of two lines are $\bar{r} = \bar{r}_1 + t\bar{\alpha}, \bar{r} = \bar{r}_2 + t\bar{\beta}$, where t is a scalar and $\bar{r}_1, \bar{\alpha}, \bar{r}_2, \bar{\beta}$ are vectors with coordinates $(-14,8,6)$, $(25,-4,-5)$, $(3,5,5)$, $(3,6,3)$ respectively. Find the s.d. between the lines.

15. O, Q, R, S are four points no three of which are collinear. If $\bar{\beta}, \bar{\gamma}, \bar{\delta}$ be the position vectors of Q, R, S respectively w.r.t. O as origin, find the s.d. between the lines OQ and RS.

16. Find the s.d. between two opposite edges of a regular tetrahedron.

    [The faces of a regular tetrahedron are equilateral triangles.]

17. If $\bar{\alpha}$ be the position vector of a point P, find by vector method the distance of P from the line $\bar{r} = \bar{\beta} + t\bar{\gamma}$, where the vectors $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ have coordinates $(5,-6,2)$, $(1,-1,2)$ and $(0,-4,-3)$ respectively.

18. Find, by vector method, the position vector of the point of intersection of the line joining the points $(1,1,2)$ and $(5,-1,0)$ and the plane $\bar{r}\cdot\bar{n} = 5\sqrt{2}$, where $\bar{n}$ is the vector $\left(\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

19. $PQRS$ is a tetrahedron with $(-5,-4,8),(2,3,1),(4,1,2),(6,3,7)$ as the coordinates of P, Q, R, S respectively. Find by vector method the distance of the point P from the plane determined by Q, R and S.

20. $P(1,3,-1), Q(0,1,6), R(-1,3,1)$ are three points in space. Find the coordinates of a point S on the y-axis such that the volume of the tetrahedron $PQRS$ is 10.

21. A line through the vector origin O meets a sphere at P. A point Q divides OP in such a way that $OP : OQ$ is constant. Prove that the focus of Q is a sphere.

22. If a tangent plane to the sphere $x^2+y^2+z^2=d^2$ makes intercepts a, b, c on the axes, prove by vector method that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{d^2}$.

    [The plane $\bar{r}\cdot\bar{n}=p$ touches the sphere if the perpendicular distance from the centre = radius, i.e. $\frac{p}{|\bar{n}|}=d$.

    The intercepts made by the plane on the axes are

    $$\frac{p}{\bar{i}\cdot\bar{n}}=a, \frac{p}{\bar{j}\cdot\bar{n}}=b, \frac{p}{\bar{k}\cdot\bar{n}}=c.$$

    Now $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{n_1^2+n_2^2+n_3^2}{p^2}=\frac{1}{d^2}$.]

23. Show that the work done by a particle acted on by a force $5\bar{i}+10\bar{j}+15\bar{k}$ for the displacement from the point $\bar{i}+3\bar{k}$ to $3\bar{i}-\bar{j}-6\bar{k}$ is 135 units.

24. If a force given by $\bar{F}=2\bar{i}+4\bar{j}-\bar{k}$ displaces a particle from the position A to B, where position vectors of A and B are given by $\bar{i}+\bar{j}+2\bar{k}$ and $3\bar{i}-\bar{j}-\bar{k}$ respectively, find the work done by the force.

25. Show that the moment of a force $4\bar{i}+2\bar{j}+\bar{k}$ through the point $5\bar{i}+2\bar{j}+4\bar{k}$ about the point $3\bar{i}-\bar{j}+3\bar{k}$ is $\bar{i}+2\bar{j}-8\bar{k}$.

26. Consider the line vector $\bar{F}=(1,-1,2)$ acting through the point $A(2,4,-1)$. Find the moment of $\bar{F}$ about the point $P(3,-1,2)$ and about an axis through P in the direction $(2,-1,2)$.

27. Particles of equal masses are placed at $(n-2)$ of the vertices of a regular polygon of n sides. Find their centre of mass.

28. A rigid body is spinning about the fixed point $3\bar{i}-\bar{j}-2\bar{k}$ with angular velocity 5 radians/sec., the axis of rotation being in the direction of the vector $2\bar{i}+\bar{j}-2\bar{k}$. Find the velocity of the particle at the point $4\bar{i}+\bar{j}$.

29. Forces $\bar{P}, \bar{Q}$ act at a point O and have a resultant $\bar{R}$. If any transversal cuts their lines of actions at A, B, C respectively, show that $\frac{P}{OA}+\frac{Q}{OB}=\frac{R}{OC}$.

    [BH 94, 96; CH 93, 2003]

    [*Hints.* Let $\bar{P}, \bar{Q}$ and $\bar{R}$ be represented by the sides OL, OM and the diagonal ON of the parallelogram $OLNM$. $\therefore\ \bar{P}+\bar{Q}=\bar{R}$. (1)

The transversal cuts OL, OM and ON at A, B, C respectively.

Suppose that $\bar{P} = p\overrightarrow{OA}, \bar{Q} = q\overrightarrow{OB}, \bar{R} = r\overrightarrow{OC}$.

By (1), $p\overrightarrow{OA} + q\overrightarrow{OB} = r\overrightarrow{OC}$. The points A, B, C are collinear. $\therefore\ p + q = r$. Thus $\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$.]

30. P is any point in the plane of ΔABC and I is the incentre of ΔABC. Prove that $a\,\overrightarrow{AP} + b\overrightarrow{BP} + c\overrightarrow{CP} = (a+b+c)\overrightarrow{IP}$, where a, b, c are the lengths of the sides of ΔABC.

ANSWERS

1. $x - 2 = y - 3 = z - 4$.
2. $\frac{x+2}{5} = \frac{y-1}{-3} = \frac{z}{4}$.
3. $x + 5y - 6z + 19 = 0$.
4. $37x + 7y - 24z + 29 = 0$.
7. $3x - 4y + 7z + 13 = 0, \frac{13}{\sqrt{74}}$.
8. $\bar{r} \cdot \{(\bar{a} - \bar{c}) \times \bar{b}\} + [\bar{a}\bar{c}\bar{b}] = 0$.
9. $3 : 1$.
11. $\frac{q_1 - \bar{r}\cdot\bar{n}_1}{|\bar{n}_1|} = \frac{q_2 - \bar{r}\cdot\bar{n}_2}{|\bar{n}_2|} = \frac{q_3 - \bar{r}\cdot\bar{n}_3}{|\bar{n}_3|}$.
14. $\frac{23}{\sqrt{107}}$.
15. $\frac{[\bar{\gamma}\bar{\beta}\bar{\delta}]}{|\bar{\beta}\times\bar{\delta} - \bar{\beta}\times\bar{\gamma}|}$
16. $\frac{a}{\sqrt{2}}$, where a is the length of each edge.
17. 5.
18. $(-21, 12, 13)$.
19. $\frac{49}{\sqrt{17}}$.
20. $(0, 8, 0), (0, -2, 0)$.
24. 1.
26. $(7, -1, -4), \frac{7}{3}$.

27. (i) The centre of mass divides the join of the middle point of the segment joining the points of no particles and the centre of the polygon externally in the ratio $n : n - 2$.

(ii) If n is even and if any two diametrically opposite corners have no particles, then the centre of the polynomial is the centre of mass.

28. $5(2\bar{i} - 2\bar{j} + \bar{k})$.

Chapter 4

Differentiation, Integration and Operators

4.10 Vector function: Limit and continuity

Vector function. A vector $\bar{F}$ is said to be a function of a scalar variable t if to each value of t in some domain there corresponds a value of the vector $\bar{F}$. It is usually denoted by $\bar{F}(t)$. $\bar{F}(t)$ can be decomposed as $\bar{F}(t) = F_1(t)\bar{i} + F_2(t)\bar{j} + F_3(t)\bar{k}$, where $\bar{i}, \bar{j}, \bar{k}$ are three unit vectors along three fixed directions and $F_1(t), F_2(t), F_3(t)$ are scalar functions of t. This relation may be indicated by $\bar{F} = (F_1, F_2, F_3)$.

Limit of a vector function. A vector function $\bar{F}(t)$ is said to tend a limit $\bar{A}$ as t tends to t_0, if to any preassigned positive number ϵ, however small, there corresponds a positive number δ such that

$$|\bar{F}(t) - \bar{A}| < \epsilon \text{ when } 0 < |t - t_o| \leq \delta.$$

It is expressed by writing $\lim_{t \to t_0} \bar{F}(t) = \bar{A}$.

In the neighbourhood of t_0 the magnitude and direction of $\bar{F}(t)$ are almost same as those $\bar{A}$ when $\bar{A} \neq \bar{0}$, but if $\bar{A} = \bar{0}$, then the direction of $\bar{F}(t)$ may be arbitrary.

If $\bar{F}(t) = F_1(t)\bar{i} + F_2(t)\bar{j} + F_3(t)\bar{k}$ and $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$, then $\lim_{t \to t_0} \bar{F}(t) = \bar{A}$ implies that

$\lim_{t \to t_0} F_n(t) = A_n (n = 1, 2, 3)$ and conversely.

Continuity. A vector function $\bar{F}(t) = F_1(t)\bar{i} + F_2(t)\bar{j} + F_3(t)\bar{k}$ is said to be continuous at $t = t_0$ if and only if the three scalar functions $F_1(t), F_2(t)$ and $F_3(t)$ are continuous at $t = t_0$.

4.11 Differentiation of a vector function

Let $\bar{F}(t)$ be a vector function of the scalar parameter t and $\Delta\bar{F}$ represent the increment of $\bar{F}$ corresponding to the increment Δt of the parameter t.

Thus $\Delta\bar{F} = \bar{F}(t+\Delta t) - \bar{F}(t)$

$$\begin{aligned} \text{or,} \quad \Delta\bar{F} &= F_1(t+\Delta t)\bar{i} + F_2(t+\Delta t)\bar{j} + F_3(t+\Delta t)\bar{k} \\ &\quad - [F_1(t)\bar{i} + F_2(t)\bar{j} + F_3(t)\bar{k}] \\ &= \Delta F_1\bar{i} + \Delta F_2\bar{j} + \Delta F_3\bar{k}. \end{aligned} \tag{1}$$

Now the derivative of $\bar{F}(t)$ w.r.t. the scalar t is defined as

$$\begin{aligned} \frac{d\bar{F}}{dt} &= \lim_{\Delta t\to 0} \frac{\bar{F}(t+\Delta t) - \bar{F}(t)}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Delta\bar{F}}{\Delta t} \\ &= \lim_{\Delta t\to 0} \frac{\Delta F_1}{\Delta t}\bar{i} + \lim_{\Delta t\to o} \frac{\Delta F_2}{\Delta t}\bar{j} + \lim_{\Delta t\to o} \frac{\Delta F_3}{\Delta t}\bar{k} \\ &= \frac{dF_1}{dt}\bar{i} + \frac{dF_2}{dt}\bar{j} + \frac{dF_3}{dt}\bar{k}. \end{aligned}$$

The derivative of $\bar{F}(t)$ w.r.t. the scalar t exists if and only if the derivatives of the scalar functions $F_1(t), F_2(t)$ and $F_3(t)$ exist w.r.t. scalar t.

Illustration. Let P be the position of a moving particle at time t and the cartesian coordinates of P be (x, y, z).

If $\bar{r}(t)$ is the position vector of P, then

$$\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}.$$

Here $\frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$.

It represents the velocity vector of the particle with components $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$.

Note. $\frac{d\bar{F}}{dt}$ is also denoted by $\bar{F}'(t)$.

Differentials. If $\bar{F}'(t)$ exists, then $\bar{F}'(t)dt$ is called the differential of $\bar{F}(t)$ and it is denoted by $d\bar{F}$, i.e. $d\bar{F} = \bar{F}'(t)dt$. If $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$, then $d\bar{F} = dF_1\bar{i} + dF_2\bar{j} + dF_3\bar{k}$.

Higher derivatives. If $F_1(t), F_2(t), F_3(t)$ are differentiable up to nth order, then by successive differentiation

$$\begin{aligned} \frac{d\bar{F}}{dt} &= \frac{dF_1}{dt}\bar{i} + \frac{dF_2}{dt}\bar{j} + \frac{dF_3}{dt}\bar{k}, \\ \frac{d^2\bar{F}}{dt^2} &= \frac{d^2F_1}{dt^2}\bar{i} + \frac{d^2F_2}{dt^2}\bar{j} + \frac{d^2F_3}{dt^2}\bar{k}, \\ &\cdots \quad \cdots \quad \cdots \\ \frac{d^n\bar{F}}{dt^n} &= \frac{d^nF_1}{dt^n}\bar{i} + \frac{d^nF_2}{dt^n}\bar{j} + \frac{d^nF_3}{dt^n}\bar{k}. \end{aligned}$$

Partial derivatives. If $\bar{F}$ is a continuous vector function and depends on more than one independent scalar variables, say x, y, z, then the vector function is written as $\bar{F}(x, y, z)$.

Now the partial derivatives of $\bar{F}$ w.r.t. x is defined as

$$\frac{\partial \bar{F}}{\partial x} = \lim_{\Delta x \to o} \frac{\bar{F}(x + \Delta x, y, z) - \bar{F}(x, y, z)}{\Delta x}$$

provided the limit exists.

Similarly,
$$\frac{\partial \bar{F}}{\partial y} = \lim_{\Delta y \to o} \frac{\bar{F}(x, y + \Delta y, z) - \bar{F}(x, y, z)}{\Delta y},$$
$$\frac{\partial \bar{F}}{\partial z} = \lim_{\Delta z \to o} \frac{\bar{F}(x, y, z + \Delta z) - \bar{F}(x, y, z)}{\Delta z}$$

are the partial derivatives of $\bar{F}(x, y, z)$ w.r.t. y and z respectively provided these limits exist.

Example. $\bar{r} = t^2\bar{i} - t^3\bar{j} + t^4\bar{k}$ represents the position vector of a moving particle, find its velocity at time $t = 1$.

The velocity vector at $t = 1$ is

$$\left[\frac{d\bar{r}}{dt}\right]_{t=1} = [2t\bar{i} - 3t^2\bar{j} + 4t^3\bar{k}]_{t=1} = 2\bar{i} - 3\bar{j} + 4\bar{k}.$$

The resultant velocity $= \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$.

4.12 Derivatives of sums and products

If $\bar{u}(t), \bar{v}(t)$ are differentiable vector functions, $f(t)$ a differentiable scalar function and $\bar{c}$ a constant vector, then we can prove that

(i) $\dfrac{d\bar{c}}{dt} = \bar{0}$, (ii) $\dfrac{d}{dt}(\bar{u} + \bar{v}) = \dfrac{d\bar{u}}{dt} + \dfrac{d\bar{v}}{dt}$,

(iii) $\dfrac{d(f\bar{u})}{dt} = f\dfrac{d\bar{u}}{dt} + \dfrac{df}{dt}\bar{u}$, (iv) $\dfrac{d}{dt}(\bar{u} \cdot \bar{v}) = \bar{u} \cdot \dfrac{d\bar{v}}{dt} + \dfrac{d\bar{u}}{dt} \cdot \bar{v}$,

(v) $\dfrac{d}{dt}(\bar{u} \times \bar{v}) = \bar{u} \times \dfrac{d\bar{v}}{dt} + \dfrac{d\bar{u}}{dt} \times \bar{v}$.

(i) An increment Δt in the scalar variable t produces no change in $\bar{c}$, i.e. $\Delta\bar{c} = \bar{0}$.

$\therefore\ \dfrac{d\bar{c}}{dt} = \lim_{\Delta t \to 0} \dfrac{\Delta \bar{c}}{\Delta t} = \bar{0}$.

The converse is also true.

Let $\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$.

Then $\dfrac{d\bar{c}}{dt} = \dfrac{dc_1}{dt}\bar{i} + \dfrac{dc_2}{dt}\bar{j} + \dfrac{dc_3}{dt}\bar{k}$.

$\dfrac{d\bar{c}}{dt} = \bar{0}$ implies that $\dfrac{dc_1}{dt} = 0 = \dfrac{dc_2}{dt} = \dfrac{dc_3}{dt}$.

$\therefore\ c_1, c_2$ and c_3 are constants. Consequently $\bar{c}$ is a constant vector.

(ii) $\Delta(\bar{u} + \bar{v}) = (\bar{u} + \Delta\bar{u} + \bar{v} + \Delta\bar{v}) - (\bar{u} + \bar{v}) = \Delta\bar{u} + \Delta\bar{v}$.

$$\therefore \quad \frac{\Delta(\bar{u}+\bar{v})}{\Delta t} = \frac{\Delta \bar{u}}{\Delta t} + \frac{\Delta \bar{v}}{\Delta t}.$$

Taking the limit as $\Delta t \to 0$, $\dfrac{d}{dt}(\bar{u}+\bar{v}) = \dfrac{d\bar{u}}{dt} + \dfrac{d\bar{v}}{dt}$.

(iii) $\Delta(f\bar{u}) = (f+\Delta f)(\bar{u}+\Delta\bar{u}) - f\bar{u} = \Delta f\bar{u} + f\Delta\bar{u} + \Delta f\Delta\bar{u}$.

$$\therefore \lim_{\Delta t\to 0}\frac{\Delta(f\bar{u})}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta f}{\Delta t}\bar{u} + \lim_{\Delta t\to 0} f\frac{\Delta\bar{u}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\Delta f}{\Delta t}\Delta\bar{u}$$

or, $\quad \dfrac{d}{dt}(f\bar{u}) = \dfrac{df}{dt}\bar{u} + f\dfrac{d\bar{u}}{dt} \quad$ since $\Delta\bar{u} \to \bar{0}$ as $\Delta t \to 0$.

(iv) $\Delta(\bar{u}\cdot\bar{v}) = (\bar{u}+\Delta\bar{u})\cdot(\bar{v}+\Delta\bar{v}) - \bar{u}\cdot\bar{v} = \Delta\bar{u}\cdot\bar{v} + \bar{u}\cdot\Delta\bar{v} + \Delta\bar{u}\cdot\Delta\bar{v}$.

$$\therefore \quad \lim_{\Delta t\to 0}\frac{\Delta(\bar{u}\cdot\bar{v})}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\bar{u}}{\Delta t}\cdot\bar{v} + \lim_{\Delta t\to 0}\bar{u}\cdot\frac{\Delta\bar{v}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\Delta\bar{u}}{\Delta t}\cdot\Delta\bar{v}$$

or, $\quad \dfrac{d}{dt}(\bar{u}\cdot\bar{v}) = \dfrac{d\bar{u}}{dt}\cdot\bar{v} + \bar{u}\cdot\dfrac{d\bar{v}}{dt}$.

(v) $\Delta(\bar{u}\times\bar{v}) = (\bar{u}+\Delta\bar{u})\times(\bar{v}+\Delta\bar{v}) - \bar{u}\times\bar{v} = \bar{u}\times\Delta\bar{v} + \Delta\bar{u}\times\bar{v} + \Delta\bar{u}\times\Delta\bar{v}$.

$$\therefore \quad \lim_{\Delta t\to 0}\frac{\Delta(\bar{u}\times\bar{v})}{\Delta t} = \lim_{\Delta t\to 0}\bar{u}\times\frac{\Delta\bar{v}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\Delta\bar{u}}{\Delta t}\times\bar{v} + \lim_{\Delta t\to 0}\frac{\Delta\bar{u}}{\Delta t}\times\Delta\bar{v}$$

or, $\quad \dfrac{d}{dt}(\bar{u}\times\bar{v}) = \bar{u}\times\dfrac{d\bar{v}}{dt} + \dfrac{d\bar{u}}{dt}\times\bar{v}$.

Note 1. If $\bar{u}$ is a function of s and s is a function of t, then $\frac{d\bar{u}}{dt} = \frac{d\bar{u}}{ds}\frac{ds}{dt}$.

Note 2. Derivatives of triple product.

(a) Let $r = [\bar{u}\bar{v}\bar{\omega}]$,

then $\quad \dfrac{d\bar{r}}{dt} = \dfrac{d}{dt}[\bar{u}\bar{v}\bar{\omega}] = \dfrac{d}{dt}\{\bar{u}\cdot(\bar{v}\times\bar{\omega})\}$

$= \bar{u}\cdot\left(\dfrac{d\bar{v}}{dt}\times\bar{\omega}\right) + \bar{u}\cdot\left(\bar{v}\times\dfrac{d\bar{\omega}}{dt}\right) + \dfrac{d\bar{u}}{dt}\cdot(\bar{v}\times\bar{\omega})$ by (iv) and (v).

(b) Let $\bar{r} = \bar{u}\times(\bar{v}\times\bar{\omega})$.

Then $\dfrac{d\bar{r}}{dt} = \dfrac{d\bar{u}}{dt}\times(\bar{v}\times\bar{\omega}) + \bar{u}\times\left(\dfrac{d\bar{v}}{dt}\times\bar{\omega}\right) + \bar{u}\times\left(\bar{v}\times\dfrac{d\bar{\omega}}{dt}\right)$.

4.20 Two theorems

Theorem 1. *A necessary and sufficient condition that a proper vector $\bar{u}$ has a constant length is that* $\bar{u}\cdot\dfrac{d\bar{u}}{dt} = 0$.

Proof. Since $|\bar{u}|^2 = \bar{u}^2 = \bar{u}\cdot\bar{u}$,

$$\frac{d}{dt}|\bar{u}|^2 = \frac{d}{dt}\bar{u}^2 = 2\bar{u}\cdot\frac{d\bar{u}}{dt}.$$

Now $|\bar{u}| =$ constant. It implies that $\bar{u}\cdot\dfrac{d\bar{u}}{dt} = 0$ and conversely.

> **Note.** The derivative of a vector of constant length is perpendicular to that vector.

Theorem 2. *A necessary and sufficient condition that a proper vector* $\bar{u}$ *always remains parallel to a fixed line is that* $\bar{u} \times \dfrac{d\bar{u}}{dt} = \bar{0}$.

Proof. Let $\bar{u} = u(t)\bar{e}$, where $\bar{e}$ is a unit vector.

$$\bar{u} \times \frac{d\bar{u}}{dt} = u(t)\bar{e} \times \left(\frac{du}{dt}\bar{e} + u\frac{d\bar{e}}{dt}\right) = u^2\bar{e} \times \frac{d\bar{e}}{dt}, \text{ since } \bar{e} \times \bar{e} = \bar{0}.$$

If $\bar{u}$ remains parallel to a fixed line, then $\bar{e}$ is a constant vector and hence $\frac{d\bar{e}}{dt} = \bar{0}$. Therefore, the condition is necessary.

Here $\bar{u} \neq \bar{0}$. Therefore, the condition implies that $\bar{e} \times \frac{d\bar{e}}{dt} = \bar{0}$. $\bar{e}$ is a vector of constant length. By theorem 1, $\bar{e} \cdot \frac{d\bar{e}}{dt} = 0$. These two equations are contradictory unless $\frac{d\bar{e}}{dt} = \bar{0}$; that is, $\bar{e}$ is a constant vector. Consequently $\bar{u}$ is parallel to a fixed line and the condition is sufficient.

4.21 Geometrical interpretation of $\frac{d\bar{F}}{dt}$.

If the values of a continuous vector function $\bar{F}(t)$ are drawn from a common origin, then their end points will define a curve C whose points will be in correspondence with the values of the scalar variable t. In this representation the direction of $\Delta\bar{F}$ will be that of an infinitesimal chord of the curve C. Since Δt is a scalar, the direction of $\frac{\Delta\bar{F}}{\Delta t}$ is the same as that of $\Delta\bar{F}$. Now as Δt approaches zero, the direction of $\Delta\bar{F}$ approaches the direction of the tangent at a point on C corresponding to the scalar t.

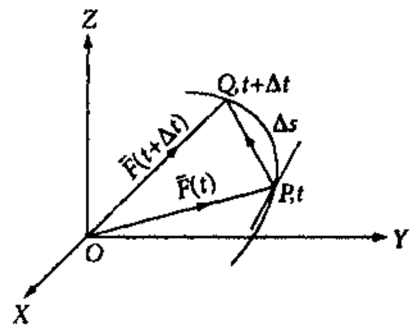


Fig. 49

Thus $\dfrac{d\bar{F}}{dt}$ is a vector along the tangent to the curve C.

If the scalar variable t is taken as the arc length s of the curve C defined by the end points of the vector function $\bar{F}(s)$, then

$$\left|\frac{d\bar{F}}{ds}\right| = \lim_{\Delta s \to o}\left|\frac{\Delta\bar{F}}{\Delta s}\right| = \lim_{\Delta s \to o} \frac{\text{infinitesimal chord of } C}{\text{infinitesimal arc of } C} = 1. \qquad (1)$$

Thus $\dfrac{d\bar{F}}{ds}$ represents the unit vector along a tangent to C. It is generally denoted by $\bar{T}$ and we shall call it as the *unit tangent*.

Corollary I. Tangent at a point.

$\bar{r} = \bar{F}(t)$ represents the equation of a curve in space. If $\bar{r}$ is the position vector of the point P on the curve, then $\frac{d\bar{F}}{dt}$, i.e. $\frac{d\bar{r}}{dt}$ is parallel to the tangent at P. Hence the equation of the tangent at P is

$$(\bar{R} - \bar{r}) \times \frac{d\bar{r}}{dt} = \bar{0}. \tag{2}$$

Here $\bar{R}$ is the position vector of any point on the tangent.

For the equation of the curve $\bar{r} = \bar{F}(s)$ the equation of the tangent is

$$(\bar{R} - \bar{r}) \times \frac{d\bar{r}}{ds} = \bar{0}. \tag{3}$$

If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then $\dfrac{d\bar{r}}{ds} = \dfrac{dx}{ds}\bar{i} + \dfrac{dy}{ds}\bar{j} + \dfrac{dz}{ds}\bar{k}$. $\dfrac{dx}{ds}, \dfrac{dy}{ds}, \dfrac{dz}{ds}$ are the d.cs. of the tangent at P.

Corollary II. Normal plane at a point.

The plane through a point P, perpendicular to the tangent at P, is called the *normal plane* at that point. If $\bar{R}$ is the position vector of a point on the normal plane and $\bar{r}$ is the position vector of P on the curve, then the equation of the normal plane is

$$(\bar{R} - \bar{r}) \cdot \frac{d\bar{r}}{dt} = 0. \tag{4}$$

4.22 Principal normal, curvature, binormal, torsion

$\bar{T}$ is the unit tangent. It is a vector of unit length. Therefore, $\frac{d\bar{T}}{ds}$ is perpendicular to $\bar{T}$ and normal to the curve at the point corresponding to $\bar{F}(s)$, i.e. at P. If $\bar{N}$ is the *unit normal* in this normal direction, then we may write $\frac{d\bar{T}}{ds} = k\bar{N}$, where k is a positive scalar.

Any straight line in the normal plane through P is a normal to the curve at P. The normal which is in the direction of $\bar{N}$ is called the *principal normal* and $\bar{N}$ is known as the *unit principal normal*.

Let $\bar{T}$ and $\bar{T} + \Delta\bar{T}$ be the unit tangents at P and Q on the curve of $\bar{r} = \bar{F}(s)$ (see Fig. 49) and meet at A at an angle $\Delta\psi$. $\bar{T}$ and $\bar{T} + \Delta\bar{T}$ are represented by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively.

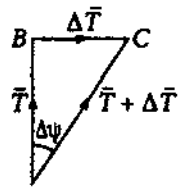


Fig. 50

Now

$$k = \left|\frac{d\bar{T}}{ds}\right| = \lim_{\Delta s \to 0}\left|\frac{\Delta\bar{T}}{\Delta s}\right| = \lim_{\Delta s \to 0}\left(\frac{BC}{\Delta\psi} \cdot \frac{\Delta\psi}{\Delta s}\right)$$
$$= \lim_{\Delta s \to 0}\frac{\Delta\psi}{\Delta s}\left(\because \lim_{\Delta s \to 0}\frac{BC}{\Delta\psi} = 1\right) = \frac{d\psi}{ds} = \textit{curvature of the curve at } P.$$

$\frac{1}{k} = \rho =$ ***radius of curvature*** **at** P.

The unit vector perpendicular to both $\bar{T}$ and $\bar{N}$ is called the *binormal* to the curve at the point P. If it is $\bar{B}, \bar{B} = \bar{T} \times \bar{N}$. $\bar{B}$ is a unit vector of constant length. Therefore, $\frac{d\bar{B}}{ds}$ is perpendicular to $\bar{B}$. Again $\bar{T} \cdot \bar{B} = 0$. Differentiating w.r.t.s

$$\frac{d\bar{T}}{ds} \cdot \bar{B} + \bar{T} \cdot \frac{d\bar{B}}{ds} = 0 \text{ or, } k\bar{N} \cdot \bar{B} + \bar{T} \cdot \frac{d\bar{B}}{ds} = 0 \text{ or, } \bar{T} \cdot \frac{d\bar{B}}{ds} = 0 \ (\because \ \bar{N} \cdot \bar{B} = 0).$$

Thus $\frac{d\bar{B}}{ds}$ is perpendicular to $\bar{T}$ and $\bar{B}$. Therefore, $\frac{d\bar{B}}{ds}$ is parallel to $\bar{N}$ and we may write $\frac{d\bar{B}}{ds} = -\tau\bar{N}$. The scalar τ measures the arc-rate of turning of the binormal and it is called the *torsion* of the curve at P. The reciprocal of τ is known as the *radius of torsion* which is denoted by δ.

Note 1. The Serret-Frenet formulae are (Fig. 51)

(i) $\frac{d\bar{T}}{ds} = k\bar{N}$, (ii) $\frac{d\bar{B}}{ds} = -\tau\bar{N}$ and (iii) $\frac{d\bar{N}}{ds} = \tau\bar{B} - k\bar{T}$.

The formulae (i) and (ii) have been proved. The three unit vectors $\bar{T}, \bar{N}$ and $\bar{B}$ form a right-handed system of mutually orthogonal vectors.

Hence $\bar{N}, \bar{B}, \bar{T}$ also form a right-handed system. Thus we can write $\bar{N} = \bar{B} \times \bar{T}$.

$$\therefore \ \frac{d\bar{N}}{ds} = \frac{d\bar{B}}{ds} \times \bar{T} + \bar{B} \times \frac{d\bar{T}}{ds} = -\tau\bar{N} \times \bar{T} + \bar{B} \times k\bar{N} = \tau\bar{B} - k\bar{T}$$
$$(\because \ \bar{N} \times \bar{T} = -\bar{B}, \bar{B} \times \bar{N} = -\bar{T}).$$

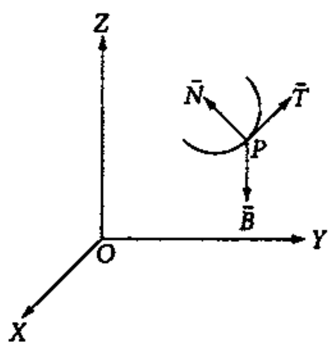


Fig. 51

Note 2. The system of unit vectors $\bar{T}, \bar{N}, \bar{B}$ is the localised unit vectors since the system moves with the change of s.

Note 3. Osculating plane. The plane containing $\bar{T}$ and $\bar{N}$ at a point P on the curve C is called the *osculating plane* to $\bar{C}$ at the point P.

Normal plane. The plane containing $\bar{N}$ and $\bar{B}$ is known as the *normal plane* to the curve at the point of $\bar{N}$ and $\bar{B}$.

Rectifying plane. The plane containing $\bar{B}$ and $\bar{T}$ is called the *rectifying plane.*

WORKED-OUT EXAMPLES

1. *If* $\bar{u} = 2t^2\bar{i} + t\bar{j} - 3t^3\bar{k}$ *and* $\bar{v} = \cos t\bar{i} - \sin t\bar{j} - \bar{k}$, *find the values of* $\frac{d}{dt}(\bar{u} \cdot \bar{v})$, $\frac{d}{dt}(\bar{u} \times \bar{v})$ *and* $\frac{d}{dt}\bar{v}^2$.

$\bar{u} \cdot \bar{v} = 2t^2 \cos t - t \sin t + 3t^3.$

$$\therefore \ \frac{d}{dt}(\bar{u} \cdot \bar{v}) = 4t \cos t - 2t^2 \sin t - \sin t - t \cos t + 9t^2$$
$$= (9 - 2 \sin t)t^2 + 3t \cos t - \sin t.$$

$\bar{u} \times \bar{v} = -(t + 3t^3 \sin t)\bar{i} + (2t^2 - 3t^3 \cos t)\bar{j} - (2t^2 \sin t + t \cos t)\bar{k}.$

$$\therefore \ \frac{d}{dt}(\bar{u} \times \bar{v}) = -(1 + 9t^2 \sin t + 3t^3 \cos t)\bar{i} + (4t - 9t^2 \cos t + 3t^3 \sin t)\bar{j}$$
$$- (4t \sin t + 2t^2 \cos t + \cos t - t \sin t)\bar{k}$$
$$= -(1 + 9t^2 \sin t + 3t^2 \cos t)\bar{i} + (4t - 9t^2 \cos t + 3t^3 \sin t)\bar{j}$$
$$- (3t \sin t + 2t^2 \cos t + \cos t)\bar{k}.$$

$\bar{v}^2 = \bar{v} \cdot \bar{v} = \cos^2 t + \sin^2 t + 1 = 2.$

$$\therefore \ \frac{d\bar{v}^2}{dt} = 0.$$

2. *If* $\bar{r} = \bar{i} \cos \omega t + \bar{j} \sin \omega t$, *show that*

$$\bar{r} \times \frac{d\bar{r}}{dt} = \omega(\bar{i} \times \bar{j}) \quad \text{and} \quad \frac{d^2\bar{r}}{dt^2} + \omega^2 \bar{r} = \bar{0}.$$

$$\bar{r} = \bar{i} \cos \omega t + \bar{j} \sin \omega t. \tag{1}$$
$$\frac{d\bar{r}}{dt} = -\bar{i}\omega \sin \omega t + \bar{j}\omega \cos \omega t. \tag{2}$$

$$\therefore \ \bar{r} \times \frac{d\bar{r}}{dt} = (\bar{i} \cos \omega t + \bar{j} \sin \omega t) \times (-\bar{i}\omega \sin \omega t + \bar{j}\omega \cos \omega t)$$
$$= \omega(\cos^2 \omega t + \sin^2 \omega t)(\bar{i} \times \bar{j}) = \omega(\bar{i} \times \bar{j}).$$

From (2),

$$\frac{d^2\bar{r}}{dt^2} = -\bar{i}\omega^2 \cos \omega t - \bar{j}\omega^2 \sin \omega t = -\omega^2 \bar{r} \quad \text{or,} \quad \frac{d^2\bar{r}}{dt^2} + \omega^2 \bar{r} = \bar{0}.$$

3. *Find the angle between the tangents to the curve* $x = t, y = t^2, z = t^3$ *at the points* $t = 1$ *and* $t = -1$.

Let $\bar{F}(t) = t\bar{i} + t^2\bar{j} + t^3\bar{k}$ be the vector function whose end points define the curve.

Now $\dfrac{d\bar{F}}{dt} = \bar{i} + 2t\bar{j} + 3t^2\bar{k}.$

Thus the tangents at $t = 1$ and $t = -1$ are along the vectors

$$\bar{T}_1 = \left.\frac{d\bar{F}}{dt}\right|_{t=1} = \bar{i} + 2\bar{j} + 3\bar{k} \quad \text{and} \quad \bar{T}_2 = \left.\frac{d\bar{F}}{dt}\right|_{t=-1} = \bar{i} - 2\bar{j} + 3\bar{k}.$$

If θ is the angle between them,

$$\cos\theta = \frac{\bar{T}_1 \cdot \bar{T}_2}{|\bar{T}_1||\bar{T}_2|} = \frac{1-4+9}{\sqrt{14}\sqrt{14}} = \frac{6}{14} = \frac{3}{7}. \ \therefore\ \theta = \cos^{-1}\frac{3}{7}.$$

4. ***Show that the curve*** $x = 3\cos t, y = 3\sin t, z = 4t$ ***lies on a cylinder and find*** $\bar{T}, \bar{N}, \bar{B}, k, \tau$ ***at any point on the curve.***

Since $t = z/4$, the equations of the curve are $x = 3\cos\frac{z}{4}, y = 3\sin\frac{z}{4}$. Thus the curve lies on the cylinder $x^2 + y^2 = 9$.

The position vector of any point on the curve is $\bar{r} = \bar{i}3\cos t + \bar{j}3\sin t + \bar{k}4t$.

$$\therefore\ \frac{d\bar{r}}{dt} = -\bar{i}3\sin t + \bar{j}3\cos t + \bar{k}4$$

$$\text{and}\quad \frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5.$$

(i) $\bar{T} = \dfrac{d\bar{r}}{ds} = \dfrac{d\bar{r}}{dt}\Big/\dfrac{ds}{dt} = -\dfrac{3}{5}\sin t\bar{i} + \dfrac{3}{5}\cos t\bar{j} + \dfrac{4}{5}\bar{k}.$

(ii) $\dfrac{d\bar{T}}{dt} = -\dfrac{3}{5}\cos t\bar{i} - \dfrac{3}{5}\sin t\bar{j}$

$\dfrac{d\bar{T}}{ds} = \dfrac{d\bar{T}}{dt}\Big/\dfrac{ds}{dt} = -\dfrac{3}{25}\cos t\bar{i} - \dfrac{3}{25}\sin t\bar{j}.$

We have $\dfrac{d\bar{T}}{ds} = k\bar{N}$ and $|k\bar{N}| = k$.

$\therefore\ k = \sqrt{\dfrac{9}{625}(\cos^2 t + \sin^2 t)} = \dfrac{3}{25}$ and $\bar{N} = -\cos t\bar{i} - \sin t\bar{j}.$

(iii) $\bar{B} = \bar{T} \times \bar{N} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{4}{5}\sin t\bar{i} - \frac{4}{5}\cos t\bar{j} + \frac{3}{5}\bar{k}.$

$$\frac{d\bar{B}}{dt} = \frac{4}{5}\cos t\bar{i} + \frac{4}{5}\sin t\bar{j}.$$

$$\therefore\ \frac{d\bar{B}}{ds} = \frac{d\bar{B}}{dt}\Big/\frac{ds}{dt} = \frac{4}{25}\cos t\bar{i} + \frac{4}{25}\sin t\bar{j}.$$

We have $-\tau\bar{N} = \frac{d\bar{B}}{ds}$.

$$\therefore\ -\tau(-\cos t\bar{i} - \sin t\bar{j}) = \frac{4}{25}(\cos t\bar{i} + \sin t\bar{j}).$$

$$\therefore\ \tau = \frac{4}{25}.$$

5. *Find the two points on the curve*

$$x = 3t^4 - 6t^2 + 12, y = 4t^3 - 6t^2, z = 12t$$

at which the tangents are parallel.

Defining $\bar{F}(t) = (3t^4 - 6t^2 + 12)\bar{i} + (4t^3 - 6t^2)\bar{j} + 12t\bar{k}$,

$$\frac{d\bar{F}}{dt} = 12(t^3 - t)\bar{i} + 12(t^2 - t)\bar{j} + 12\bar{k}.$$

Let the tangents at t_1 and t_2 be parallel.

Then

$$\frac{t_1^3 - t_1}{t_2^3 - t_2} = \frac{t_1^2 - t_1}{t_2^2 - t_2} = \frac{12}{12} = 1.$$

These give

$$(t_1^2 - t_2^2) - (t_1 - t_2) = 0, \quad \text{i.e.} \quad t_1 + t_2 = 1 \quad (\because\ t_1 \neq t_2),$$

$$\text{and} \quad (t_1^3 - t_2^3) - (t_1 - t_2) = 0, \quad \text{i.e.} \quad t_1^2 + t_1 t_2 + t_2^2 = 1.$$

Now using $t_1 + t_2 = 1, (t_1 + t_2)^2 - t_1 t_2 = 1$ gives $t_1 t_2 = 0$.

Taking $t_1 = 0, t_2 = 1$.

Thus the required points are $(12, 0, 0)$ and $(9, -2, 12)$.

6. *A particle $P(r, \theta)$ moves in a plane and its position vector at time t is $\bar{r}$ referred to some fixed point in the plane. If $\bar{R}$ and $\bar{S}$ be unit vectors in radial and cross-radial directions, show that*

$$\frac{d\bar{R}}{dt} = \bar{S}\frac{d\theta}{dt} \quad \text{and} \quad \frac{d\bar{S}}{dt} = -\bar{R}\frac{d\theta}{dt}.$$

Hence deduce that

$$\frac{d\bar{r}}{dt} = \bar{R}\frac{dr}{dt} + \bar{S}r\frac{d\theta}{dt} \text{ and } \frac{d^2\bar{r}}{dt^2} = \bar{R}\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} + \bar{S}\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right).$$

Let $\bar{i}$ and $\bar{j}$ be the two unit vectors along OX and OY and $\overrightarrow{OP} = \bar{r}$.

From Fig. 52,

$$\bar{R} = \bar{i}\cos\theta + \bar{j}\sin\theta \qquad (1)$$

$$\text{and } \bar{S} = -\bar{i}\sin\theta + \bar{j}\cos\theta. \qquad (2)$$

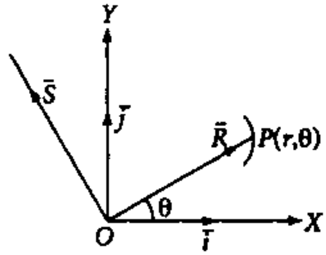


Fig. 52

Now

$$\frac{d\bar{R}}{dt} = (-\bar{i}\sin\theta + \bar{j}\cos\theta)\frac{d\theta}{dt} = \bar{S}\frac{d\theta}{dt} \qquad (3)$$

$$\text{and } \frac{d\bar{S}}{dt} = -(\bar{i}\cos\theta + \bar{j}\sin\theta)\frac{d\theta}{dt} = -\bar{R}\frac{d\theta}{dt}. \qquad (4)$$

$\bar{r} = \bar{R}r$, where $|\bar{r}| = r$.

$$\frac{d\bar{r}}{dt} = \frac{d\bar{R}}{dt}r + \bar{R}\frac{dr}{dt} = \bar{R}\frac{dr}{dt} + \bar{S}r\frac{d\theta}{dt} \quad \text{[by (3)]}, \tag{5}$$

$$\frac{d^2\bar{r}}{dt^2} = \frac{d}{dt}\left(\bar{R}\frac{dr}{dt} + \bar{S}r\frac{d\theta}{dt}\right)$$

$$= \bar{R}\frac{d^2r}{dt^2} + \frac{d\bar{R}}{dt}\frac{dr}{dt} + \bar{S}r\frac{d^2\theta}{dt^2} + \bar{S}\frac{dr}{dt}\frac{d\theta}{dt} + \frac{d\bar{S}}{dt}r\frac{d\theta}{dt}$$

$$= \bar{R}\frac{d^2r}{dt^2} + \bar{S}\frac{dr}{dt}\frac{d\theta}{dt} + \bar{S}r\frac{d^2\theta}{dt^2} + \bar{S}\frac{dr}{dt}\frac{d\theta}{dt} - \bar{R}r\left(\frac{d\theta}{dt}\right)^2 \quad \text{[by (3) and (4)]}$$

$$= \bar{R}\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} + \bar{S}\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right).$$

7. *A particle moves along a curve in three-dimensional space with velocity $\bar{v}$ and acceleration $\bar{a}$. Show that*

$$\bar{v} = v\bar{T}, \bar{a} = \frac{dv}{dt}\bar{T} + \frac{v^2}{\rho}\bar{N}.$$

($\bar{T}$ = unit tangent vector, $\bar{N}$ = unit principal normal, ρ = radius of curvature and $v = |\bar{v}|$.)

Let $\bar{r}$ be the position vector of the particle at time t on the curve.

Now the velocity vector $\bar{v} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds} \cdot \frac{ds}{dt} = \bar{T}v, v = \frac{ds}{dt}$ and s is the arc length from a specified point of the point $\bar{r}$ on the curve.

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}(\bar{T}v) = \bar{T}\frac{dv}{dt} + \frac{d\bar{T}}{dt}v = \bar{T}\frac{dv}{dt} + \frac{d\bar{T}}{ds}\frac{ds}{dt}v$$

$$= \bar{T}\frac{dv}{dt} + \frac{\bar{N}}{\rho}v^2 = \frac{dv}{dt}\bar{T} + \frac{v^2}{\rho}\bar{N}.$$

(The acceleration of a particle is a vector in the plane of the tangent and the principal normal to the path at the point of the particle. Its tangential and normal components are $\frac{dv}{dt}$ and $\frac{v^2}{\rho}$.)

8. *Show that the Serret-Frenet formulae can be written in the form*

$$\frac{d\bar{T}}{ds} = \bar{\omega} \times \bar{T}, \frac{d\bar{N}}{ds} = \bar{\omega} \times \bar{N}, \frac{d\bar{B}}{ds} = \bar{\omega} \times \bar{B}.$$

Determine $\bar{\omega}$.

We have $\bar{T} \times \bar{N} = \bar{B}, \bar{N} \times \bar{B} = \bar{T}, \bar{B} \times \bar{T} = \bar{N}$.

The first two Serret-Frenet formulae are

$$\frac{d\bar{T}}{ds} = k\bar{N} = k\bar{B} \times \bar{T} \quad \text{and} \quad \frac{d\bar{B}}{ds} = -\tau\bar{N} = \tau\bar{T} \times \bar{B}.$$

Thus we set $\bar{\omega} = k\bar{B} + \tau\bar{T}$.

Now

$$\frac{d\bar{T}}{ds} = \bar{\omega} \times \bar{T} \quad (\because \ \bar{T} \times \bar{T} = \bar{0})$$

$$\text{and} \quad \frac{d\bar{B}}{ds} = \bar{\omega} \times \bar{B} \quad (\because \ \bar{B} \times \bar{B} = \bar{0}).$$

The third formula is $\frac{d\bar{N}}{ds} = \tau\bar{B} - k\bar{T}$.

Since $\bar{\omega} \times \bar{N} = (k\bar{B} + \tau\bar{T}) \times \bar{N} = -k\bar{T} + \tau\bar{B}$, $\frac{d\bar{N}}{ds} = \bar{\omega} \times \bar{N}$.

Note. $\bar{\omega}$ is known as *Darboux's vector.*

EXERCISE IV

1. If $\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$, find the values of

$$\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \left|\frac{d\bar{r}}{dt}\right| \quad \text{and} \quad \bar{r} \times \frac{d^2\bar{r}}{dt^2} \text{ at } t = 0.$$

2. If $\bar{u} = 2t\bar{i} + \bar{j} + t\bar{k}$ and $\bar{v} = t^2\bar{i} + t\bar{j} + 2t\bar{k}$, find

   (i) $\frac{d}{dt}(\bar{u} \cdot \bar{v})$, (ii) $\frac{d}{dt}(\bar{u} \times \bar{v})$ and (iii) $\frac{d}{dt}\left(\bar{u} \times \frac{d\bar{v}}{dt}\right)$.

3. Find the derivatives of the following w.r.t. t.

   (i) $\bar{r}^2 + \frac{1}{\bar{r}^2}$ (ii) $\frac{\bar{r} \times \bar{a}}{\bar{r} \cdot \bar{a}}$ (iii) $\bar{u} \times \left(\frac{d\bar{u}}{dt} \times \frac{d^2\bar{u}}{dt^2}\right)$

   (iv) $\left[\bar{u} \frac{d\bar{u}}{dt} \frac{d^2\bar{u}}{dt^2}\right]$ (v) $\bar{u} \times \frac{d\bar{v}}{dt} - \frac{d\bar{u}}{dt} \times \bar{v}$.

4. For $\bar{r} = (3t, 3t^2, 2t^3)$, find $\left[\frac{d\bar{r}}{dt} \frac{d^2\bar{r}}{dt^2} \frac{d^3\bar{r}}{dt^3}\right]$. [CH 2002]

5. A particle moves along a curve $x = e^{-t}, y = 2\cos 3t, z = 2\sin 3t$, where t is the time. Determine its velocity and acceleration at $t = \pi$.

6. The radius vector of a moving point is, at any instant of time t, given by $\bar{r} = \bar{i} - 4t^2\bar{j} + 3t^2\bar{k}$.

   Determine the trajectory of motion and the magnitude and directions of the velocity and acceleration at any time t.

7. (i) Show that

$$k = \frac{\left|\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right|}{\left|\frac{d\bar{r}}{dt}\right|^3} \quad \text{and} \quad \tau = \frac{\left(\frac{d\bar{r}}{dt} \quad \frac{d^2\bar{r}}{dt^2}\right) \cdot \frac{d^3\bar{r}}{dt^3}}{\left|\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right|^2}$$

   for the coordinate t.

[*Hints.* $\dfrac{d\bar{r}}{dt} = \dfrac{d\bar{r}}{ds}\cdot\dfrac{ds}{dt} = \bar{T}\dfrac{ds}{dt}, \dfrac{d^2\bar{r}}{dt^2} = \dfrac{d^2s}{dt^2}\bar{T} + \left(\dfrac{ds}{dt}\right)^2 k\bar{N},$

$$\frac{d^3\bar{r}}{dt^3} = \left\{\frac{d^3s}{dt^3} - \left(\frac{ds}{dt}\right)^3 k^2\right\}\bar{T} + \left\{3\frac{ds}{dt}\frac{d^2s}{dt^2}k + \left(\frac{ds}{dt}\right)^2\frac{dk}{dt}\right\}\bar{N} + \left(\frac{ds}{dt}\right)^3 k\tau\bar{B}.]$$

(ii) Show that for the curve $\bar{r} = \bar{r}(s)$, $\dfrac{d\bar{r}}{ds}\cdot\dfrac{d^2\bar{r}}{ds^2}\times\dfrac{d^3\bar{r}}{ds^3} = \dfrac{\tau}{\rho^2}$, where ρ is the radius of curvature, τ the torsion and s the arc length of the curve.

[BH 2006]

[*Hints.*

$$\begin{aligned}\frac{d\bar{r}}{ds} = \bar{T}, \frac{d^2\bar{r}}{ds^2} &= \frac{d\bar{T}}{ds} = k\bar{N}, \frac{d^3\bar{r}}{ds^3} = k\frac{d\bar{N}}{ds} + \frac{dk}{ds}\bar{N}\\ &= k(\tau\bar{B} - k\bar{T}) + \frac{dk}{ds}\bar{N} = k\tau\bar{B} - k^2\bar{T} + \frac{dk}{ds}\bar{N}.\\ \therefore\ \frac{d\bar{r}}{ds}\cdot\frac{d^2\bar{r}}{ds^2}\times\frac{d^3\bar{r}}{ds^3} &= \bar{T}\cdot k\bar{N}\times\left(k\tau\bar{B} - k^2\bar{T} + \frac{dk}{ds}\bar{N}\right)\\ &= \bar{T}\cdot k(k\tau\bar{T} + k^2\bar{B}) = k^2\tau = \frac{\tau}{\rho^2}.]\end{aligned}$$

8. (i) At the point $t = 1$ of the twisted cubic (which does not lie in a plane) $x = 2t, y = t^2, z = \dfrac{t^3}{3}$, find $\bar{T}, \bar{N}, \bar{B}, k$ and τ.

[*Hints.* $\bar{T}, \bar{B}, \bar{N}$ have the directions of $\dfrac{d\bar{r}}{dt}, \dfrac{d\bar{r}}{dt}\times\dfrac{d^2\bar{r}}{dt^2}$ & $\left(\dfrac{d\bar{r}}{dt}\times\dfrac{d^2\bar{r}}{dt^2}\right)\times\dfrac{d\bar{r}}{dt}$.]

(ii) In E_3 the vector equation of a curve is $\bar{r} = (4\cos t)\bar{i} + (4\sin t)\bar{j} + (2t)\bar{k}$. Find $\bar{T}, \bar{N}, \bar{B}, \rho$ and the radius of torsion δ at the point t.

9. A space curve is defined by the equations $x = \arctan s, y = \dfrac{1}{\sqrt{2}}\log(s^2+1), z = s -\arctan s$, s being the arc length. Find $\bar{T}, \bar{N}, \bar{B}, k$ and τ.

[*Hints.* Here $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

$$\bar{T} = \frac{d\bar{r}}{ds} = \frac{dx}{ds}\bar{i} + \frac{dy}{ds}\bar{j} + \frac{dz}{ds}\bar{k} = \frac{1}{1+s^2}\bar{i} + \frac{\sqrt{2}s}{1+s^2}\bar{j} + \left(1 - \frac{1}{1+s^2}\right)\bar{k},$$

i.e. $\bar{T} = (\bar{i} + \sqrt{2}s\bar{j} + s^2\bar{k})/(1+s^2)$.

$$\frac{d\bar{T}}{ds} = k\bar{N}, k = \left|\frac{d\bar{T}}{ds}\right| = \frac{\sqrt{4s^2 + 2(1-s^2)^2 + 4s^2}}{(1+s^2)^2} = \frac{\sqrt{2}}{1+s^2}.$$

Hence $\bar{N} = \{-\sqrt{2}s\bar{i} + (1-s^2)\bar{j} + \sqrt{2}s\bar{k}\}/(1+s^2)$.

$$\bar{B} = \bar{T}\times\bar{N} = (s^2\bar{i} - \sqrt{2}s\bar{j} + s^2)/(1+s^2)\cdot\frac{d\bar{B}}{ds} = -\tau\bar{N} \text{ gives } \tau = \frac{\sqrt{2}}{1+s^2}.]$$

10. A helix is a twisted curve whose tangent makes a constant angle with a fixed direction ($\bar{e} \cdot \bar{T} = \cos\alpha, 0 < \alpha < \pi/2$, where $\bar{e}$ is the unit vector in the fixed direction known as axis). Prove that (i) the principal normal is perpendicular to the axis, (ii) the Darboux's vector is parallel to the axis, (iii) $\frac{k}{\tau} = \pm\tan\alpha$.

    [*Hints.* We have $\bar{e}\cdot\bar{T} = \cos\alpha$. Differentiating w.r.t. s, $\bar{e}\cdot\frac{d\bar{T}}{ds} = 0$ or, $\bar{e}\cdot k\bar{N} = 0$, i.e. $\bar{N}$ is perpendicular to $\bar{e}$. Consequently $\bar{e}, \bar{B}, \bar{T}$ are coplanar and $\bar{e}\cdot\bar{B} = \pm\sin\alpha$. From $\bar{e}\cdot\bar{N} = 0, \bar{e}\cdot\frac{d\bar{N}}{ds} = 0$ or, $\bar{e}\cdot(\tau\bar{B} - k\bar{T}) = 0$ or, $\pm\tau\sin\alpha - k\cos\alpha = 0$ or, $\frac{k}{\tau} = \pm\tan\alpha$.

    Now $\bar{e}\times\bar{\omega} = \bar{e}\times(k\bar{B} + \tau\bar{T}) = -k\cos\alpha\bar{N} + \tau\sin\alpha\bar{N}$ when $k = \tau\tan\alpha$ and $\bar{e}\times(k\bar{B} + \tau\bar{T}) = -k\cos\alpha\bar{N} - \tau\sin\alpha\bar{N}$ when $k = -\tau\tan\alpha$.

    $\therefore$ $\bar{e}\times\bar{\omega} = 0$ and $\bar{\omega}$ is parallel to $\bar{e}$.]

11. Show that $k = \frac{a}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$ for the circular helix $x = a\cos t, y = a\sin t, z = bt$. (In case of circular helix, k and τ are constants.)

12. Prove that a necessary and sufficient condition that a curve, $\bar{r} = \bar{F}(s)$ be a helix is that $\bar{r}''\cdot\bar{r}'''\times\bar{r}^{\text{iv}} = 0$.

13. Prove that the radius of curvature of the curve with parametric equations $x = x(s), y = y(s), z = z(s)$ is given by $\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2$, s being arc-length.

14. Find the equations of the tangent and the principal normal of the circular helix $x = 3\cos t, y = 3\sin t, z = 4t$ at $t = \frac{\pi}{4}$.

ANSWERS

1. $\bar{i}, 2\bar{j}, 1, 0$. 2. (i) $6t^2 + 4t + 1$ (ii) $(2-2t)\bar{i} + (3t^2 - 8t)\bar{j} + 2t\bar{k}$ (iii) $-\bar{i} + 4(t-1)\bar{j}$.

3. (i) $2\bar{r}\cdot\frac{d\bar{r}}{dt}\left(1 - \frac{1}{\bar{r}^4}\right)$ (ii) $\frac{\frac{d\bar{r}}{dt}\times\bar{a}}{\bar{r}\cdot\bar{a}} - \frac{(\frac{d\bar{r}}{dt}\cdot\bar{a})(\bar{r}\times\bar{a})}{(\bar{r}\cdot\bar{a})^2}$

   (iii) $\frac{d\bar{u}}{dt}\times\left(\frac{d\bar{u}}{dt}\times\frac{d^2\bar{u}}{dt^2}\right) + \bar{u}\times\left(\frac{d\bar{u}}{dt}\times\frac{d^3\bar{u}}{dt^3}\right)$

   (iv) $\left[\bar{u}\frac{d\bar{u}}{dt}\frac{d^3\bar{u}}{dt^3}\right]$ (v) $\bar{u}\times\frac{d^2\bar{v}}{dt^2} - \frac{d^2\bar{u}}{dt^2}\times\bar{v}$

4. 216. 5. $\bar{v} = -e^{-\pi}\bar{i} - 6\bar{k}, \bar{a} = e^{-\pi}\bar{i} + 18\bar{j}$.

6. $x = 1, 3y + 4z = 0; \bar{v} = -8t\bar{j} + 6t\bar{k}, \bar{a} = -8\bar{j} + 6\bar{k}$.

8. (i) $\bar{T} = \frac{1}{3}(2,2,1), \bar{B} = \frac{1}{3}(1,-2,2), \bar{N} = \frac{1}{3}(-2,1,2), k = \frac{2}{9}, \tau = \frac{2}{9}$.

   (ii) $\bar{T} = \frac{1}{\sqrt{5}}(-2\sin t\bar{i} + 2\cos t\bar{j} + \bar{k})$,

   $\bar{N} = -(\cos t\bar{i} + \sin t\bar{j})$,

   $\bar{B} = \frac{1}{\sqrt{5}}(\sin t\bar{i} - \cos t\bar{j} + 2\bar{k}), \rho = 5, \delta = 10$.

14. Tangent: $\frac{\sqrt{2x-3}}{-3} = \frac{\sqrt{2y-3}}{3} = \frac{z-\pi}{4}$,
    Normal: $\frac{\sqrt{2x-3}}{1} = \frac{\sqrt{2y-3}}{1} = \frac{z-\pi}{0}$.

4.30 Scalar and vector field

Field. A field is a set of functions of coordinates of a point in space. Modern physics deals with various types of fields as potential fields, probability fields, electromagnetic fields, tensor fields, etc. The fields characterised by a single magnitude at each point are called *scalar fields*. Many other fields characterised by magnitude and direction at each point are known as *vector fields*.

Mathematical concept of scalar field. Let $\phi(x, y, z)$ be a single valued function defined at certain points $P(x, y, z)$ in space whose values are real numbers depending only on the points in space but not on the particular choice of the coordinate system. Such a function may be defined in a domain D consisting of points on a curve, a surface or a three-dimensional region of space. The function $\phi(x, y, z)$ is said to define a scalar field in the domain D when for each point P in D we get a scalar real number given by $\phi(x, y, z)$. It is independent of the coordinates system but depends on the position of P. All scalar fields have the property of invariance under a transformation of space coordinates. The function ϕ may also depend on some particular such as time. It is then written as $\phi(P, t)$.

The temperature T within a body B is a scalar field, namely the temperature field. When a bar is heated at one end, the temperature at various points will attain a steady state and will be independent of time so that it will depend on position only.

Vector field. Let a vector $\bar{F}(P)$ be assigned to each point P of a set of points in space either lying on a curve, a surface or a three-dimensional region. Then $\bar{F}(P)$ is called a vector function and we can say, a vector field is defined at those set of points.

Examples of vector fields are velocity field of a moving particle, force field defined by forces acting on a body, gravitational field defined by a system of particles under the action of gravity, etc.

Note. Point function. A variable quantity which depends upon the coordinates of the point of a region of space is called a *point function*. $\phi(P)$ is a point function.

4.31 Gradient of a scalar field

Let a scalar field be defined by the scalar function $\phi(x, y, z)$ of the coordinates x, y, z which is also defined and differentiable at each point (x, y, z) in some region of space. The vector function

$\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is called the *gradient of the scalar function* ϕ.

This gradient is frequently written in operational notation as

$$\operatorname{grad}\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi. \tag{1}$$

Using the differential operator

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} \tag{2}$$

we may write

$$\text{grad}\,\phi = \nabla\phi. \tag{3}$$

The operator ∇ is read as *nabla* or *del* and it is called *Hamiltonian operator*. Hamilton first introduced this operator to develop the theory of quaternions. Now it is used to derive many complicated results.

Significance of grad ϕ. Let $P(x, y, z)$ and $Q(x + \Delta x, y + \Delta y, z + \Delta z)$ be two neighbouring points and let $\phi(x, y, z)$ change by an amount $\Delta\phi$ which can be written as

$$\Delta\phi = \frac{\partial\phi}{\partial x}\Delta x + \frac{\partial\phi}{\partial y}\Delta y + \frac{\partial\phi}{\partial z}\Delta z + \epsilon_1\Delta x + \epsilon_2\Delta y + \epsilon_3\Delta z,$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ are quantities which approach zero as $Q \to P$, i.e. $\Delta x, \Delta y, \Delta z$ approach zero. If $\Delta\phi$ is divided by the distance Δs between P and Q, a measure of the rate of change of ϕ is obtained when P moves to Q along the arc PQ. Thus

$$\frac{\partial\phi}{\partial s} = \lim_{\Delta s \to o}\frac{\Delta\phi}{\Delta s} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}. \tag{4}$$

$\dfrac{\partial\phi}{\partial s}$ is the directional derivative of the function ϕ at P.

Again

$$\begin{aligned}\frac{\partial\phi}{\partial s} &= \left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\bar{i}\frac{dx}{ds} + \bar{j}\frac{dy}{ds} + \bar{k}\frac{dz}{ds}\right) \\ &= (\text{grad}\,\phi)\cdot\frac{d\bar{r}}{ds}. \end{aligned} \tag{5}$$

Here the first vector depends on the scalar ϕ and the coordinates of P and the second depends on the direction of ds.

The equation (5) can be used to determine the significance of grad ϕ. $\frac{d\bar{r}}{ds}$ is a unit vector along the tangent at P and grad $\phi \cdot \frac{d\bar{r}}{ds}$ is the projection of grad ϕ in the direction of $\frac{d\bar{r}}{ds}$. Thus grad ϕ has the property that its projection in any direction is equal to the derivative of ϕ in that direction. In particular, if $\frac{d\bar{r}}{ds}$ has the direction of positive x-axis, then

$$\frac{\partial\phi}{\partial s} = \text{grad}\,\phi\cdot\bar{i} = \frac{\partial\phi}{\partial x}. \tag{6}$$

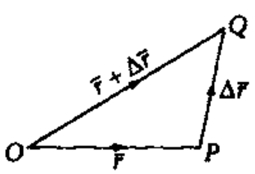


Fig. 53

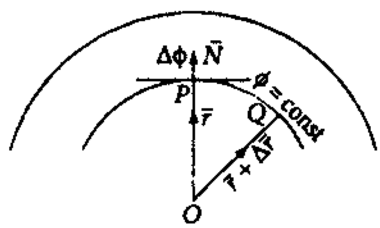


Fig. 54

Geometrical interpretation of $\nabla\phi$. (see Fig. 54)

The equation $\phi(x,y,z)=c$ represents a family of surfaces for various values of c. These are called *level surfaces* of ϕ. Since ϕ is a single valued function, one and only one level surface passes through any given point P. Now we consider a neighbouring point Q on this surface. We have $\frac{\Delta\phi}{\Delta s}=0$ since ϕ has the same value at all points on the level surface. Hence by (5),

$$\nabla\phi\cdot\frac{d\bar{r}}{ds}=0. \tag{7}$$

This asserts that grad ϕ is perpendicular to every tangent to the level surface at P. Thus geometrical interpretation of $\nabla\phi$ can be stated as if $\phi(x,y,z)$ is a scalar function such that through any point P in space, there passes precisely one level surface S of ϕ and grad $\phi\neq 0$ at P, then grad ϕ has the direction of the normal of S at P.

Note. Maximum value of $\frac{\partial\phi}{\partial s}$.

$\nabla\phi$ is a vector. Its magnitude and direction are independent of the choice of coordinate system.

From (5),

$$\frac{\partial\phi}{\partial s}=|\text{grad}\,\phi|\left|\frac{d\bar{r}}{ds}\right|\cos\gamma=|\text{grad}\,\phi|\cos,\gamma, \tag{8}$$

where γ is the angle between grad ϕ and $\frac{d\bar{r}}{ds}$. It shows that $\frac{\partial\phi}{\partial s}$ is maximum when $\cos\gamma=1$, i.e. $\gamma=0$ and then $\frac{\partial\phi}{\partial s}=|\text{grad}\,\phi|$.

Example. Show that the unit normal of the cone of revolution $z^2=2(x^2+y^2)$ at the point $P(1,0,2)$ is $\frac{1}{\sqrt{2}}(\bar{i}-\bar{k})$.

The cone may be considered as the level surface $\phi=0$ of the function $\phi(x,y,z)=2(x^2+y^2)-z^2$.

Now grad $\phi=\bar{i}4x+\bar{j}4y-\bar{k}2z$.

At $P(1,0,2)$ grad $\phi=\bar{i}4-\bar{k}4$.

Hence the unit normal $\bar{N}=\frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}=\frac{1}{\sqrt{2}}(\bar{i}-\bar{k})$.

4.32 Divergence of a vector field

Let $\bar{F}(x,y,z)$ be a differentiable vector function of the cartesian coordinates (x,y,z) in space and $\bar{F}=\bar{i}F_1+\bar{j}F_2+\bar{k}F_3$. Then the scalar function given by

$$\text{div}\,\bar{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} \tag{1}$$

is called the *divergence* of $\bar{F}$ or the divergence of the vector field defined by $\bar{F}$. In the operator notation

$$\begin{aligned}\operatorname{div}\bar{F} = \nabla\cdot\bar{F} &= \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot(\bar{i}F_1+\bar{j}F_2+\bar{k}F_3)\\ &= \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}.\end{aligned}$$

Note. A vector field $\bar{F}$ is called *solenoidal,* if div $\bar{F}=0$ everywhere in the space considered.

Physical concept of the divergence of vector function.

To obtain the physical meaning of the divergence of a vector function we consider the case of a fluid flow in a stream and compute the net out-flow over the volume element $\Delta x\Delta y\Delta z$ at the point $A(x,y,z)$. For this we consider a small parallelopiped in a mass of fluid with edges Δx, Δy, Δz and a corner at A. Let $\bar{V}=\bar{i}V_x+\bar{j}V_y+\bar{k}V_z$ be the velocity of fluid at the point A. First we compute the flow of fluid parallel to y-axis, i.e. across the faces $ADHE$ and $BCGF$. Since V_y is the component of $\bar{V}$ along y-axis and perpendicular to the face $ADHE$, the flow per unit time through the face

$$ADHE = V_y\Delta z\Delta x. \tag{2}$$

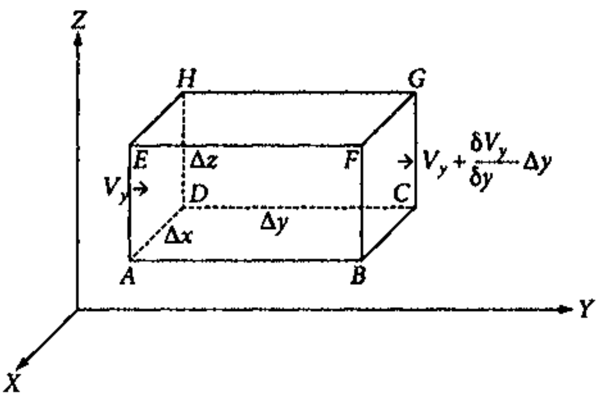


Fig. 55

If V_y is a function of y only, then the velocity of fluid across the face $BCGF$ is $V_y+\frac{\partial V_y}{\partial y}\Delta y$ ($\because$ Δy is very small). Hence the flow per unit time through the face $BCGF$

$$=\left(V_y+\frac{\partial V_y}{\partial y}\Delta y\right)\Delta z\Delta x. \tag{3}$$

By (2) and (3) the total rate of flow outward from the elementary volume $\Delta x\Delta y\Delta z$ along y-axis

$$=\left(V_y+\frac{\partial V_y}{\partial y}\Delta y\right)\Delta z\Delta x - V_y\Delta z\Delta x = \frac{\partial V_y}{\partial y}\Delta x\Delta y\Delta z. \tag{4}$$

Similarly, the total rate of outward flow along the x-axis and z-axis are $\frac{\partial V_x}{\partial x}\Delta x\Delta y\Delta z$ and $\frac{\partial V_z}{\partial z}\Delta x\Delta y\Delta z$ respectively. Hence the net outflow through the elementary volume is

$$\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)\Delta x\Delta y\Delta z.$$

Therefore, $\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}$ is the rate of outward flow through unit volume.

Note. If div $\bar{V}=0$ everywhere, the fluid is, then said to be *incompressible*.

4.33 Curl of a vector field

The curl of a vector function $\bar{F}$ or the curl of a vector field by $\bar{F}$ is defined as

$$\text{curl } \bar{F}=\nabla\times\bar{F}=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\times(\bar{i}F_1+\bar{j}F_2+\bar{k}F_3).$$

Thus

$$\text{curl } \bar{F}=\bar{i}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+\bar{j}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+\bar{k}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right). \qquad (1)$$

Written in determinant,

$$\text{curl } \bar{F}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}. \qquad (2)$$

Note 1. curl $\bar{F}$ is also called rot $\bar{F}$.

Note 2. If $\bar{V}$ is the velocity vector at the point (x,y,z), then $\frac{1}{2}$ curl $\bar{V}$ represents the circulation of fluid around the point (x,y,z). It is called the *vorticity vector* of the fluid. If curl $\bar{V}=\bar{0}$ everywhere, the flow of fluid is said to be *irrotational*.

4.34 Some deduction

(i) **grad $(\phi\pm\psi)$ = grad $\phi\pm$ grad ψ**, i.e. $\nabla(\phi\pm\psi)=\nabla\phi\pm\nabla\psi$.

Proof.

$$\begin{aligned}\nabla(\phi\pm\psi)&=\bar{i}\frac{\partial}{\partial x}(\phi\pm\psi)+\bar{j}\frac{\partial}{\partial y}(\phi\pm\psi)+\bar{k}\frac{\partial}{\partial z}(\phi\pm\psi)\\&=\left(\bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}\right)\pm\left(\bar{i}\frac{\partial\psi}{\partial x}+\bar{j}\frac{\partial\psi}{\partial y}+\bar{k}\frac{\partial\psi}{\partial z}\right)\\&=\nabla\phi\pm\nabla\psi.\end{aligned}$$

(ii) **div $(\bar{E}\pm\bar{F})$ = div $\bar{E}\pm$ div $\bar{F}$**, i.e. $\nabla\cdot(\bar{E}\pm\bar{F})=\nabla\cdot\bar{E}\pm\nabla\cdot\bar{F}$.

(iii) **curl $(\bar{E}\pm\bar{F})$ = curl $\bar{E}\pm$ curl $\bar{F}$**, i.e. $\nabla\times(\bar{E}\pm\bar{F})=\nabla\times\bar{E}\pm\nabla\times\bar{F}$.

(iv) **grad** $(\phi\psi) = \phi$ **grad** $\psi + \psi$ **grad** ϕ, i.e. $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$.

Proof.
$$\nabla(\phi\psi) = \bar{i}\frac{\partial}{\partial x}(\phi\psi) + \bar{j}\frac{\partial}{\partial y}(\phi\psi) + \bar{k}\frac{\partial}{\partial z}(\phi\psi) = \sum \bar{i}\left(\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x}\right)$$
$$= \phi\sum\bar{i}\frac{\partial\psi}{\partial x} + \psi\sum\bar{i}\frac{\partial\phi}{\partial x} = \phi\nabla\psi + \psi\nabla\phi.$$

(v) **div** $(\phi\bar{F}) = \phi$ **div** $\bar{F} + \bar{F}\cdot$ **grad** ϕ, i.e. $\nabla\cdot(\phi\bar{F}) = \phi\nabla\cdot\bar{F} + \bar{F}\cdot\nabla\phi$.

Proof.
$$\nabla\cdot(\phi\bar{F}) = \sum\bar{i}\cdot\left(\frac{\partial\phi}{\partial x}\bar{F} + \phi\frac{\partial\bar{F}}{\partial x}\right)$$
$$= \left(\sum\bar{i}\frac{\partial\phi}{\partial x}\right)\cdot\bar{F} + \phi\sum\bar{i}\cdot\frac{\partial\bar{F}}{\partial x} = \nabla\phi\cdot\bar{F} + \phi\nabla\cdot\bar{F}.$$

(vi) **curl** $(\phi\bar{F}) = \phi$ **curl** $\bar{F} +$ **grad**$\phi\times\bar{F}$, i.e. $\nabla\times(\phi\bar{F}) = \phi\nabla\times\bar{F} + \nabla\phi\times\bar{F}$.

Proof.
$$\nabla\times(\phi\bar{F}) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\times(\phi\bar{F})$$
$$= \sum\bar{i}\times\frac{\partial}{\partial x}(\phi\bar{F}) = \sum\bar{i}\times\left(\frac{\partial\phi}{\partial x}\bar{F} + \phi\frac{\partial\bar{F}}{\partial x}\right)$$
$$= \sum\bar{i}\frac{\partial\phi}{\partial x}\times\bar{F} + \phi\sum\bar{i}\frac{\partial}{\partial x}\times\bar{F}$$
$$= \nabla\phi\times\bar{F} + \phi\nabla\times\bar{F}.$$

(vii) **div** $(\bar{E}\times\bar{F}) = \bar{F}\cdot$ **curl** $\bar{E} - \bar{E}\cdot$ **curl** $\bar{F}$, i.e. $\nabla\cdot(\bar{E}\times\bar{F}) = \bar{F}.\nabla\times\bar{E} - \bar{E}\cdot\nabla\times\bar{F}$.

Proof.

$$\nabla\cdot(\bar{E}\times\bar{F}) = \sum\bar{i}\cdot\frac{\partial}{\partial x}(\bar{E}\times\bar{F}) = \sum\bar{i}\cdot\left(\frac{\partial\bar{E}}{\partial x}\times\bar{F} + \bar{E}\times\frac{\partial\bar{F}}{\partial x}\right)$$
$$= \sum\bar{i}\cdot\left(\frac{\partial\bar{E}}{\partial x}\times\bar{F}\right) + \sum\bar{i}\cdot\left(\bar{E}\times\frac{\partial\bar{F}}{\partial x}\right)$$
$$= \bar{F}\cdot\left(\sum\bar{i}\times\frac{\partial\bar{E}}{\partial x}\right) - \bar{E}\cdot\left(\sum\bar{i}\times\frac{\partial\bar{F}}{\partial x}\right) = \bar{F}\cdot\nabla\times\bar{E} - \bar{E}\cdot\nabla\times\bar{F}.$$

(viii) **curl** $(\bar{E}\times\bar{F}) = \nabla\times(\bar{E}\times\bar{F}) = (\bar{F}\cdot\nabla)\bar{E} - (\bar{E}\cdot\nabla)\bar{F} + \bar{E}(\nabla\cdot\bar{F}) - \bar{F}(\nabla\cdot\bar{E})$.

Proof.
$$\begin{aligned}\nabla\times(\bar{E}\times\bar{F}) &= \sum \bar{i}\times\frac{\partial}{\partial x}(\bar{E}\times\bar{F}) = \sum\bar{i}\times\left(\bar{E}\times\frac{\partial\bar{F}}{\partial x}+\frac{\partial\bar{E}}{\partial x}\times\bar{F}\right)\\
&= \sum\bar{i}\times\left(\bar{E}\times\frac{\partial\bar{F}}{\partial x}\right)+\sum\bar{i}\times\left(\frac{\partial\bar{E}}{\partial x}\times\bar{F}\right)\\
&= \sum\left\{\left(\bar{i}\cdot\frac{\partial\bar{F}}{\partial x}\right)\bar{E}-(\bar{i}\cdot\bar{E})\frac{\partial\bar{F}}{\partial x}\right\}\\
&\quad+\sum\left\{(\bar{i}\cdot\bar{F})\frac{\partial\bar{E}}{\partial x}-\left(\bar{i}\cdot\frac{\partial\bar{E}}{\partial x}\right)\bar{F}\right\}\\
&= \bar{E}\sum\left(\bar{i}\frac{\partial}{\partial x}\right)\cdot\bar{F}-\sum\left(\bar{E}\cdot\bar{i}\frac{\partial}{\partial x}\right)\bar{F}\\
&\quad+\sum\left(\bar{F}\cdot i\frac{\partial}{\partial x}\right)\bar{E}-\bar{F}\sum\left(\bar{i}\frac{\partial}{\partial x}\right)\bar{E}\\
&= \bar{E}(\nabla\cdot\bar{F})-(\bar{E}\cdot\nabla)\bar{F}+(\bar{F}\cdot\nabla)\bar{E}-\bar{F}(\nabla\cdot\bar{E})\\
&= (\bar{F}\cdot\nabla)\bar{E}-(\bar{E}\cdot\nabla)\bar{F}+\bar{E}(\nabla\cdot\bar{F})-\bar{F}(\nabla\cdot\bar{E}).\end{aligned}$$

(ix) **grad** $(\bar{E}\cdot\bar{F}) = \bar{E}\times$ **curl** $\bar{F}+\bar{F}\times$ **curl** $\bar{E}+(\bar{E}\cdot\nabla)\bar{F}+(\bar{F}\cdot\nabla)\bar{E}$,
i.e. $\nabla(\bar{E}\cdot\bar{F}) = \bar{E}\times(\nabla\times\bar{F})+\bar{F}\times(\nabla\times\bar{E})+(\bar{E}\cdot\nabla)\bar{F}+(\bar{F}\cdot\nabla)\bar{E}$.

Proof.
$$\begin{aligned}\nabla(\bar{E}\cdot\bar{F}) &= \sum\bar{i}\frac{\partial}{\partial x}(\bar{E}\cdot\bar{F}) = \sum\bar{i}\left(\frac{\partial\bar{E}}{\partial x}\cdot\bar{F}+\bar{E}\cdot\frac{\partial\bar{F}}{\partial x}\right)\\
&= \sum\left(\bar{F}\cdot\frac{\partial\bar{E}}{\partial x}\right)\bar{i}+\sum\left(\bar{E}\cdot\frac{\partial\bar{F}}{\partial x}\right)\bar{i}.\end{aligned}\tag{1}$$

It can be proved that

$$\bar{E}\times(\nabla\times\bar{F}) = \sum\left(\bar{E}\cdot\frac{\partial\bar{F}}{\partial x}\right)\bar{i}-(\bar{E}\cdot\nabla)\bar{F}$$
$$\text{and}\quad \bar{F}\times(\nabla\times\bar{E}) = \sum\left(\bar{F}\cdot\frac{\partial\bar{E}}{\partial x}\right)\bar{i}-(\bar{F}\cdot\nabla)\bar{E}.$$

Now the result follows.

(x) **div grad** $\phi = \nabla\cdot\nabla\phi = \nabla^2\phi$, where $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

Proof.
$$\begin{aligned}\nabla\cdot\nabla\phi &= \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot\left(\bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}\right)\\
&= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi.\end{aligned}$$

(xi) **div** $(\phi$ **grad** $\psi) = \nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi$.

Proof.
$$\begin{aligned}\nabla\cdot(\phi\nabla\psi) &= \frac{\partial}{\partial x}\left(\phi\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\phi\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\phi\frac{\partial\psi}{\partial z}\right)\\
&= \phi\sum\frac{\partial^2\psi}{\partial x^2}+\sum\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} = \phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi.\end{aligned}$$

(xii) **curl grad** $\phi = \bar{0}$, i.e. $\nabla \times (\nabla\phi) = \bar{0}$.

Proof. $\nabla \times (\nabla\phi) = \nabla \times \sum \bar{i}\dfrac{\partial\phi}{\partial x} = \sum \bar{i}\left(\dfrac{\partial^2\phi}{\partial y\partial z} - \dfrac{\partial^2\phi}{\partial z\partial y}\right) = \bar{0}$.

(xiii) **div curl** $\bar{F} = 0$, i.e. $\nabla \cdot (\nabla \times \bar{F}) = 0$.

(xiv) **curl curl** $\bar{F} =$ **grad div** $\bar{F} - \sum \dfrac{\partial^2 \bar{F}}{\partial x^2}$,

i.e. $\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$.

Proof.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \bar{i}L_1 + \bar{j}L_2 + \bar{k}L_3,$$

where $L_1 = \dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}, L_2 = \dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}, L_3 = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$.

$$\nabla \times (\nabla \times \bar{F}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ L_1 & L_2 & L_3 \end{vmatrix} = \bar{i}G_1 + \bar{j}G_2 + \bar{k}G_3 \quad \text{(say)}.$$

Here

$$\begin{aligned} G_1 = \frac{\partial L_3}{\partial y} - \frac{\partial L_2}{\partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \\ &= -\frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \\ &= -\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right) + \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \\ &= -\nabla^2 F_1 + \frac{\partial}{\partial x}\nabla \cdot \bar{F}. \end{aligned}$$

Similarly, $G_2 = -\nabla^2 F_2 + \dfrac{\partial}{\partial y}\nabla \cdot \bar{F}$, $G_3 = -\nabla^2 F_3 + \dfrac{\partial}{\partial z}\nabla \cdot \bar{F}$

$\therefore\ \nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$.

WORKED-OUT EXAMPLES

1. *Find the directional derivative of* $\phi(x, y, z) = 2xy + z^2$ *at the point* $(1, 2, 3)$ *in the direction of the vector* $\bar{i} + 2\bar{j} + 2\bar{k}$.

   We have $\dfrac{\partial\phi}{\partial s} = \nabla\phi \cdot \dfrac{\bar{F}}{|\bar{F}|}$, where $\bar{F} = \bar{i} + 2\bar{j} + 2\bar{k}$.

Here $\nabla\phi = \bar{i}2y + \bar{j}2x + \bar{k}2z$; $\nabla\phi|_{(1,2,3)} = 4\bar{i} + 2\bar{j} + 6\bar{k}$.

$$\frac{\bar{F}}{|\bar{F}|} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{1+4+4}} = \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k}).$$

Thus the directional derivative

$$= (4\bar{i}+2\bar{j}+6\bar{k})\cdot\frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k}) = \frac{1}{3}(4+4+12) = \frac{20}{3}.$$

2. *Prove that* $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$, *where* $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

$$\frac{\partial f(r)}{\partial x} = f'(r)\frac{x}{r}, \text{ since } r^2 = x^2+y^2+z^2.$$

Similarly, $\dfrac{\partial f(r)}{\partial y} = f'(r)\dfrac{y}{r}$ and $\dfrac{\partial f(r)}{\partial z} = f'(r)\dfrac{z}{r}$.

$$\therefore\ \nabla f(r) = \frac{f'(r)}{r}(x\bar{i}+y\bar{j}+z\bar{k}) = f'(r)\frac{\bar{r}}{r}.$$

3. *Find grad* $(\log|r|)$, *where* $\bar{r} = \bar{i}x + \bar{j}y + \bar{k}z$.

Here $|\bar{r}| = \sqrt{x^2+y^2+z^2}$ and $\log|\bar{r}| = \frac{1}{2}\log(x^2+y^2+z^2)$.

Thus

$$\begin{aligned}\text{grad }(\log|\bar{r}|) &= \frac{1}{2}\left(\bar{i}\frac{2x}{x^2+y^2+z^2} + \bar{j}\frac{2y}{x^2+y^2+z^2} + \bar{k}\frac{2z}{x^2+y^2+z^2}\right)\\ &= \frac{1}{r^2}(\bar{i}x+\bar{j}y+\bar{k}z) = \frac{\bar{r}}{r^2} = \frac{\bar{r}}{\bar{r}\cdot\bar{r}}.\end{aligned}$$

4. *If* $\bar{A} = zx\bar{i} + yz\bar{j} - 3xy\bar{k}$ *and* $\bar{B} = y^2\bar{i} - x^2\bar{j} + z\bar{k}$, *find* $(\bar{A}\cdot\nabla)\bar{B}$.

$$\begin{aligned}\bar{A}\cdot\nabla &= (zx\bar{i}+yz\bar{j}-3xy\bar{k})\cdot\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\\ &= zx\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} - 3xy\frac{\partial}{\partial z}.\\ (\bar{A}\cdot\nabla)\bar{B} &= \left(zx\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} - 3xy\frac{\partial}{\partial z}\right)(y^2\bar{i} - x^2\bar{j} + z\bar{k})\\ &= yz2y\bar{i} - zx\cdot 2x\bar{j} - 3xy\bar{k} = 2y^2z\bar{i} - 2x^2z\bar{j} - 3xy\bar{k}.\end{aligned}$$

5. *If* $\bar{A} = 2xz^2\bar{i} - yz\bar{j} + 3xz^3\bar{k}$ *and* $\phi = x^2yz$, *find* (i) *curl* $(\phi\bar{A})$ *and* (ii) *curl curl* $\bar{A}$.

(i) $$\operatorname{curl}(\phi\bar{A}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^3yz^3 & -x^2y^2z^2 & 3x^3yz^4 \end{vmatrix}$$

$$= \bar{i}\left\{\frac{\partial}{\partial y}(3x^3yz^4) - \frac{\partial}{\partial z}(-x^2y^2z^2)\right\} + \bar{j}\left\{\frac{\partial}{\partial z}(2x^3yz^3) - \frac{\partial}{\partial x}(3x^3yz^4)\right\}$$

$$+ \bar{k}\left\{\frac{\partial}{\partial x}(-x^2y^2z^2) - \frac{\partial}{\partial y}(2x^3yz^3)\right\}$$

$$= \bar{i}(3x^3z^4 + 2x^2y^2z) + \bar{j}(6x^3yz^2 - 9x^2yz^4) + \bar{k}(-2xy^2z^2 - 2x^3z^3).$$

(ii) $$\operatorname{curl}\bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix} = y\bar{i} + (4zx - 3z^3)\bar{j}.$$

$$\operatorname{curl}\operatorname{curl}\bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 4zx - 3z^3 & 0 \end{vmatrix} = (-4x + 9z^2)\bar{i} + (4z - 1)\bar{k}.$$

6. *For what value of the constant α will the vector $\bar{A} = (\alpha xy - z^3)\bar{i} + (\alpha - 2)x^2\bar{j} + (1 - \alpha)xz^2\bar{k}$ have its curl identically equal to zero?*

$$\operatorname{curl}\bar{A} = \bar{i}\left\{\frac{\partial}{\partial y}(1-\alpha)xz^2 - \frac{\partial}{\partial z}(\alpha - 2)x^2\right\}$$

$$+ \bar{j}\left\{\frac{\partial}{\partial z}(\alpha xy - z^3) - \frac{\partial}{\partial x}(1-\alpha)xz^2\right\}$$

$$+ \bar{k}\left\{\frac{\partial}{\partial x}(\alpha - 2)x^2 - \frac{\partial}{\partial y}(\alpha xy - z^3)\right\}$$

$$= -\bar{j}z^2(4 - \alpha) + \bar{k}x(\alpha - 4).$$

For curl $\bar{A} = \bar{0}, \alpha - 4 = 0$ or, $\alpha = 4$.

7. *Prove that curl curl curl curl $\bar{F} = \nabla^4\bar{F}$, where $\bar{F}$ is a vector and div $\bar{F} = 0$.*

We have

$$\text{curl curl}\,\bar{F} = \nabla(\nabla\cdot\bar{F}) - \nabla^2\bar{F} = -\nabla^2\bar{F}\ (\because\ \nabla\cdot\bar{F}=0)$$
$$= \bar{H}\ \text{(say)}.$$

Again curl curl $\bar{H} = \nabla(\nabla\cdot\bar{H}) - \nabla^2\bar{H}$.

But $\nabla(\nabla\cdot\bar{H}) = -\nabla(\nabla\cdot\nabla^2\bar{F}) = -\nabla(\nabla^2\nabla\cdot\bar{F}) = 0$.

$\therefore$ curl curl curl curl $\bar{F} = -\nabla^2\bar{H} = \nabla^4\bar{F}$.

8. *Given that* (i) $\nabla\times\bar{D} = -\dfrac{1}{c}\dfrac{\partial\bar{H}}{\partial t}$, (ii) $\nabla\times\bar{H} = \dfrac{1}{c}\left(\dfrac{\partial\bar{D}}{\partial t} + \rho\bar{v}\right)$,

(iii) $\nabla\cdot\bar{D} = \rho$, (iv) $\nabla\cdot\bar{H} = 0$. *t is the time variable and c is a constant. Prove that*

$$\nabla^2\bar{D} - \frac{1}{c^2}\frac{\partial^2\bar{D}}{\partial t^2} = \nabla\rho + \frac{1}{c^2}\frac{\partial}{\partial t}(\rho\bar{v})\ \textit{and}\ \nabla^2\bar{H} - \frac{1}{c^2}\frac{\partial^2\bar{H}}{\partial t^2} = -\frac{1}{c}\nabla\times(\rho\bar{v}).$$

From (ii) $\rho\bar{v} = c(\nabla\times\bar{H}) - \dfrac{\partial\bar{D}}{\partial t}$

$$\therefore\ \frac{\partial(\rho\bar{v})}{\partial t} = c\frac{\partial}{\partial t}(\nabla\times\bar{H}) - \frac{\partial^2\bar{D}}{\partial t^2}$$
$$= -c^2(\nabla\times\nabla\times\bar{D}) - \frac{\partial^2\bar{D}}{\partial t^2}\quad \text{[by (i)]}$$
$$= -c^2\nabla(\nabla\cdot\bar{D}) + c^2\nabla^2\bar{D} - \frac{\partial^2\bar{D}}{\partial t^2}$$
$$= -c^2\nabla\rho + c^2\nabla^2\bar{D} - \frac{\partial^2\bar{D}}{\partial t^2}\quad \text{[by (iii)]}.$$

Thus $\nabla^2\bar{D} - \dfrac{1}{c^2}\dfrac{\partial^2\bar{D}}{\partial t^2} = \nabla\rho + \dfrac{1}{c^2}\dfrac{\partial}{\partial t}(\rho\bar{v})$.

Again

$$\nabla\times(\rho\bar{v}) = c(\nabla\times\nabla\times\bar{H}) - \frac{\partial}{\partial t}\nabla\times\bar{D}$$
$$= c\nabla(\nabla\cdot\bar{H}) - c\nabla^2\bar{H} + \frac{1}{c}\frac{\partial^2\bar{H}}{\partial t^2}\quad \text{[by (i)]}$$
$$= -c\nabla^2\bar{H} + \frac{1}{c}\frac{\partial^2\bar{H}}{\partial t^2}\quad \text{[by (iv)]}.$$

$$\therefore\ \nabla^2\bar{H} - \frac{1}{c^2}\frac{\partial^2\bar{H}}{\partial t^2} = -\frac{1}{c}\nabla\times(\rho\bar{v}).$$

EXERCISE V

1. Find $\nabla\phi$, if

(i) $\phi(x,y) = x^2 - 2xy + y^2$ at $(2,3)$, (ii) $\phi(x,y,z) = xy + yz + zx$,

(iii) $\phi(r) = r^2e^{-r}$.

2. If $\bar{A} = 2xz^2\bar{i} - yz\bar{j} + 3xz^3\bar{k}$, find curl $\bar{A}$.

3. If $r = \sqrt{x^2 + y^2 + z^2}$, show that

   (i) $\nabla r^2 = 2\bar{r}$, (ii) $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\bar{r}}{r^3}$, (iii) $\nabla \log r = \dfrac{\bar{r}}{r^2}$,

   (iv) $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$, (v) $\nabla \times \bar{r} = \bar{0}$, (vi) $\nabla\cdot\bar{r} = 3$,

   (vii) $\nabla r^m = mr^{m-2}\bar{r}$.

4. Show that (i) $\nabla^2 r^m = m(m+1)r^{m-2}$, (ii) $\nabla \times (r^n\bar{r}) = \bar{0}$,

   (iii) $\nabla\left(\nabla\cdot\dfrac{\bar{r}}{r}\right) = -\dfrac{2\bar{r}}{r^3}$.

5. (a) If $\bar{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$, show that

   (i) $\nabla\cdot\bar{F} = 6(x+y+z)$ and (ii) $\nabla\times\bar{F} = \bar{0}$.

   (b) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $\bar{c}$ (a constant vector) $= c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$, prove that curl $(\bar{r}\times\bar{c}) = -2\bar{c}$. **[BH 2006]**

6. Find a unit normal to the surface $x^2y - 2xz = 4$ at the point $(2, 2, -3)$.

7. Find the equation of the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$.

8. (i) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, 1)$ in the direction $(2\bar{i} - \bar{j} - 2\bar{k})$. **[NH 2005]**

   (ii) In what direction from the point $(2, 1, -1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum ?. What is the magnitude of this maximum?

   [*Hints.* $\nabla\phi = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$, $\nabla\phi|_{(2,1,-1)} = -4\bar{i} - 4\bar{j} + 12\bar{k}$.

   The directional derivative is maximum in the direction of $-4\bar{i} - 4\bar{j} + 12\bar{k}$. Maximum value$=|\nabla\phi| = \sqrt{16+16+144} = 4\sqrt{11}$.]

9. Find the angle between the surfaces $x^2 + y^2 + z^2 = 14$ and $x^2 - y^2 = z$ at $(1, 2, -3)$.

   [*Hints.* The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

   A normal to $x^2 + y^2 + z^2 = 14$ at $(1, 2, -3)$ is

   $$\nabla(x^2 + y^2 + z^2)|_{(1,2,-3)} = 2\bar{i} + 4\bar{j} - 6\bar{k}.$$

   A normal to $x^2 - y^2 - z = 0$ at $(1, 2, -3)$ is

   $$\nabla(x^2 - y^2 - z)|_{(1,2,-3)} = 2\bar{i} - 4\bar{j} - \bar{k}.$$

   If θ is the angle between the surfaces, then

   $$\cos\theta = \frac{(2\bar{i} + 4\bar{j} - 6\bar{k})\cdot(2\bar{i} - 4\bar{j} - \bar{k})}{|2\bar{i} + 4\bar{j} - 6\bar{k})||2\bar{i} - 4\bar{j} - \bar{k}|} = \frac{-\sqrt{6}}{14}.$$

   Thus the acute angle between the normals is $\cos^{-1}\dfrac{\sqrt{6}}{14}$.]

10. If $\bar{A} = x^2z\bar{i} + yz^3\bar{j} - 3xy\bar{k}$ and $\bar{B} = y^2\bar{i} - yz\bar{j} + 2x\bar{k}$, find $(\nabla \cdot \bar{A})\bar{B}$.

11. If $\bar{A} = x^2\bar{i} + yz\bar{j} + xy\bar{k}$ and $\phi = 3x^2yz$, find $\bar{A} \cdot \nabla\phi$ and $(\bar{A} \cdot \nabla)\phi$ and show that they give the same scalar quantity. Also show that $(\bar{A} \times \nabla)\phi = \bar{A} \times \nabla\phi$.

12. (i) Show that the gravitational field $\bar{F} = \frac{g}{r^2}\bar{r}$ is irrotational, g being a constant.

    (ii) For what value of a the vector $\bar{v} = (x + 3y)\bar{i} + (y - 2z)\bar{j} + (x + az)\bar{k}$ is solenoidal. [NH 2005]

13. Find the values of constants a, b, c so that the vector field $\bar{V} = (3y + az)\bar{i} + (bx - 4y)\bar{j} + (8x + cy)\bar{k}$ is irrotational.

14. Prove that (i) $\nabla \times [(\bar{r} \times \bar{a}) \times \bar{b}] = \bar{b} \times \bar{a}$, (ii) $\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = -\frac{2}{r^3}\bar{r}$,

    (iii) $\nabla \times \frac{\bar{a} \times \bar{r}}{r^n} = \frac{2 - n}{r^n}\bar{a} + \frac{n}{r^{n+2}}(\bar{a} \cdot \bar{r})\bar{r}$.

15. If a rigid body moves about a fixed axis through O with the angular velocity $\bar{\omega}$ and $\bar{v}$ is the velocity of any particle $P(\bar{r})$, show that curl $\bar{v} = 2\bar{\omega}$.

    [*Hints.* $\bar{v} = \bar{\omega} \times \bar{r}$.]

16. If $\bar{F}(x, y, z, t)$ is a differentiable function, prove that $\frac{d\bar{F}}{dt} = \frac{\partial \bar{F}}{\partial t} + (\bar{q} \cdot \nabla)\bar{F}$, where $\bar{q} = \frac{d\bar{r}}{dt}$.

17. If $\nabla \times \bar{A} = \bar{0}$ and $\nabla \times \bar{B} = \bar{0}$, prove that $\nabla \cdot (\bar{A} \times \bar{B}) = 0$. [NH 2007]

18. Prove that (i) $\nabla \times (\phi\nabla\psi) + \nabla \times (\psi\nabla\phi) = \bar{0}$,

    (ii) $(\bar{v} \cdot \nabla)\bar{v} = \frac{1}{2}\nabla(|\nabla\phi|^2)$ when $\bar{v} = \nabla\phi$.

ANSWERS

1. (i) $-2\bar{i} + 2\bar{j}$ (ii) $(y + z)\bar{i} + (z + x)\bar{j} + (x + y)\bar{k}$ (iii) $(2r - r^2)e^{-r}\frac{\bar{r}}{r}$.

2. $y\bar{i} + (4xz - 3z^3)\bar{j}$ 6. $\frac{1}{\sqrt{57}}(7\bar{i} + 2\bar{j} - 2\bar{k})$.

7. $7x - 3y + 8z = 26$. 10. $(2xz + z^3)(y^2\bar{i} - yz\bar{j} + 2x\bar{k})$.

11. $6x^3yz + 3x^2yz^2 + 3x^3y^2$. 12. (ii) -2. 13. $a = 8, b = 3, c = 0$.

4.40 Vector Integration

Ordinary vector integration can be defined as

$$\int \bar{F}(u)du = \bar{i}\int F_1(u)du + \bar{j}\int F_2(u)du + \bar{k}\int F_3(u)du, \tag{1}$$

where the vector $\bar{F}(u)$ with components $F_1(u), F_2(u), F_3(u)$ depend on a single scalar variable u. This integral is called the *indefinite integral* of $\bar{F}(u)$. If there exists a vector $\bar{G}(u)$ such that $\bar{F}(u) = \frac{d}{du}\bar{G}(u)$, then

$$\int \bar{F}(u)du = \bar{G}(u) + \bar{c}, \tag{2}$$

where $\bar{c}$ is an arbitrary constant vector independent of u. Just like scalar calculus $\bar{F}(u)$ and $\bar{G}(u)$ are called *integrand* and *integral* respectively.

The *definite integral* between limits $u = a$ and $u = b$ can in such case be written as

$$\int_a^b \bar{F}(u)du = [\bar{G}(u)]_a^b = \bar{G}(b) - \bar{G}(a). \tag{3}$$

This definite integral can also be conceived as the limit of a sum in a manner similar to that of elementary integral calculus. The integrals which have been developed in vector analysis are in most cases scalar quantities which have physical interpretations. The integration may be taken along a curve C or over a regular surface S or over a closed volume V.

Line integrals. Let $\bar{F} = \bar{i}F_1(x,y,z) + \bar{j}F_2(x,y,z) + \bar{k}F_3(x,y,z)$ be a vector function of position $P(x,y,z)$, defined and continuous along a curve C. Then the line integral of the tangential component of $\bar{F}(x,y,z)$ along C is defined by

$$\int_C \bar{F}\cdot\bar{T}ds, \tag{4}$$

$\bar{T}$ being the unit vector along the tangent to C at $P(x,y,z)$.

Since

$$\bar{T} = \frac{d\bar{r}}{ds}, \int_C \bar{F}\cdot\bar{T}ds = \int_C \bar{F}\cdot\frac{d\bar{r}}{ds}ds = \int_C \bar{F}\cdot d\bar{r}. \tag{5}$$

Now $\bar{F}\cdot d\bar{r} = F_1dx + F_2dy + F_3dz$.

Thus

$$\int_C \bar{F}\cdot\bar{T}ds = \int_C (F_1dx + F_2dy + F_3dz). \tag{6}$$

If $\bar{F}$ is the force on a particle moving along the curve C, then the above line integral represents the work done by the force.

Note. The line integral may or may not depend upon the path of integration.

Surface integral. Let $\bar{n}$ be a unit normal (outward) to a regular surface S. Then the surface integral

$$\iint_S (\bar{F}\cdot\bar{n})d\bar{S} \tag{7}$$

is an example of surface integral across the surface S. Another example of surface integral is

$$\iint_S \bar{F}\times d\bar{S}, \tag{8}$$

where the vector $d\bar{S}$ has magnitude as dS and the direction normal to the surface, i.e. along $\bar{n}$.

Note. Surface integrals are conveniently evaluated by expressing them as double integrals taken over the projected area of the surface S on one of the coordinate planes.

Volume integral. The volume integral of the divergence of a vector $\bar{F}$ over a closed volume V is defined as

$$\iiint_V \nabla \cdot \bar{F}\, dV. \tag{9}$$

Other examples of volume integrals are

$$\iiint_V \bar{F}\, dV \quad \text{and} \quad \iiint_V \phi\, dV,$$

where ϕ is a scalar function.

Note. Sometimes surface integrals are converted into volume integrals by using some integral theorems.

Example 1. *If $\bar{F} = (2xy + z^3)\bar{i} + x^2\bar{j} + 3xz^2\bar{k}$, find the work done in moving an object in this field from $P(1,-2,1)$ to $Q(3,1,4)$.*

$$\begin{aligned}\text{Work done} &= \int_P^Q \bar{F} \cdot d\bar{r} = \int_P^Q \{(2xy + z^3)dx + x^2 dy + 3xz^2 dz\} \\ &= \int_P^Q d(x^2y + xz^3) = [x^2y + xz^3]_P^Q = \Big[x^2y + xz^3\Big]_{(1,-2,1)}^{(3,1,4)} = 202.\end{aligned}$$

Note. Here $\bar{F} \cdot d\bar{r}$ is a total differential. In this case, the line integral is independent of the path C.

Example 2. *Integrate $\bar{F} = x^2y\bar{i} - y^3\bar{j}$ from the origin O to $P(1,1)$ in each of the following cases:*

(i) along the line OP; (ii) along the parabola $y^2 = x$ and

(iii) along the y-axis from $y = 0$ to $y = 1$ and then along the line $y = 1$ from $x = 0$ to $x = 1$.

(i) The equation of the line OP is $y = x$.

Its parametric equation is $x = t, y = t$. Now the line integral along OP

$$= \int_{OP} \bar{F} \cdot d\bar{r} = \int_{OP} (x^2 y\bar{i} - y^3 \bar{j}) \cdot (\bar{i} dx + \bar{j} dy)$$

$$= \int_{OP} (x^2 y\, dx - y^3 dy) = \int_{t=0}^{1} (t^3 - t^3) dt = 0.$$

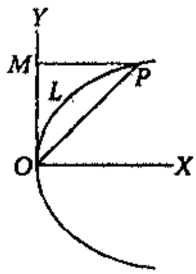


Fig. 56

(ii) The parametric equation of the parabola $y^2 = x$ is $x = t^2, y = t$. Now the line integral along the parabola is

$$\int_{OLP} \bar{F} \cdot d\bar{r} = \int_{OLP} (x^2 y\, dx - y^3 dy) = \int_{t=0}^{1} (t^5 \cdot 2t - t^3) dt = \left[\frac{2t^7}{7} - \frac{t^4}{4}\right]_0^1 = \frac{1}{28}.$$

(iii) Here the computation is first done from $y = 0$ to $y = 1$ and then along the line $y = 1$, i.e. MP.

Now

$$\int_{OMP} \bar{F} \cdot d\bar{r} = \int_{OM} \bar{F} \cdot d\bar{r} + \int_{MP} \bar{F} \cdot d\bar{r} = I_1 + I_2 \text{ (say)}.$$

For $I_1, x = 0$.

$$\therefore\ I_1 = \int_{y=0}^{1} (-y^3 \bar{j}) \cdot (\bar{j} dy) = \int_{y=0}^{1} -y^3 dy = \left[-\frac{y^4}{4}\right]_0^1 = -\frac{1}{4}.$$

For $I_2, y = 1.\ \therefore\ dy = 0$.

$\therefore\ \bar{F} = x^2 \bar{i} - \bar{j}$ and $d\bar{r} = \bar{i} dx$.

Thus

$$I_2 = \int_{x=0}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

Hence

$$\int_{OMP} \bar{F} \cdot d\bar{r} = -\frac{1}{4} + \frac{1}{3} = \frac{1}{12}.$$

Example 3. *Evaluate* $\iint_S (\bar{F} \cdot \bar{n})\, dS$, *where* $\bar{F} = yz\bar{i} + x\bar{j} + z\bar{k}$ *and* S *is the surface of the cylinder* $x^2 + y^2 = 4$ *included in the first octant between* $z = 0$ *and* $z = 2$.

Projection of dS on yz-plane $= \dfrac{dy\, dz}{|\bar{n} \cdot \bar{i}|}$.

If R is the projection of S on the yz-plane, then

$$\iint_S (\bar{F} \cdot \bar{n})\, dS = \iint_R (\bar{F} \cdot \bar{n}) \frac{dy\, dz}{|\bar{n} \cdot \bar{i}|}.$$

$$\bar{n} = \frac{\nabla(x^2+y^2-4)}{|\nabla(x^2+y^2-4)|} = \frac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}} = \frac{x\bar{i}+y\bar{j}}{\sqrt{x^2+y^2}} = \frac{x\bar{i}+y\bar{j}}{2}.$$

$\therefore$ surface integral

$$= \iint_R (yz\bar{i}+x\bar{j}+z\bar{k}) \cdot \left(\frac{x\bar{i}+y\bar{j}}{2}\right) \frac{dy\,dz}{x/2}$$

$$= \int_{z=0}^{2} \int_{y=-2}^{2} \frac{xyz+xy}{x} dy\,dz$$

$$= \int_{z=0}^{2} \int_{y=-2}^{2} (yz+y) dy\,dz = \int_{z=0}^{2} (z+1) \left[\frac{y^2}{2}\right]_{-2}^{2} dz$$

$$= \int_{z=0}^{2} (z+1) \cdot 0 dz = 0.$$

Example 4. *Evaluate* $\iiint_V \bar{F}\,dV$, *where* V *is the region bounded by the faces* $x=0$, $x=2$, $y=0$, $y=2$, $z=0$, $z=2$ *and* $\bar{F} = 2zx\bar{i} - x\bar{j} + y\bar{k}$.

$$\iiint_V \bar{F}\,dV = \int_0^2 \int_0^2 \int_0^2 (2zx\bar{i} - x\bar{j} + y\bar{k}) dx\,dy\,dz = 16\bar{i} - 8\bar{j} + 8\bar{k}.$$

Some definitions and theorems

(1) *Curve in space.* Let $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ be the position vector of (x, y, z). It is the equation of a curve C in space and t is the parameter. For a value of t in $a \le t \le b$, a definite point is obtained on the curve.

(2) *Regular curve.* If A and B are two points of the curve corresponding two values of t in $a \le t \le b$ and $\frac{d\bar{r}}{dt}$ is continuous in this interval, then the arc AB is called a regular arc. A curve is called regular, if it contains finite number of regular arcs without double points.

(3) *Connected region.* A region R is called a connected region, if any two points of R can be joined by arcs lying completely in R. The interior of a sphere is a connected region R. The region consisting of two non-intersecting circles is not a connected region.

Simply connected region. It is a region, if any closed curve within it can be contracted or shrunk continuously to C point without crossing its boundary. A non-simply connected region is a *multiple connected* region.

(4) *Circulation.* The line integral $\int_C \bar{F} \cdot d\bar{r}$, where C is a simply closed curve is called the circulation of $\bar{F}$ about C. It is denoted by $\oint \bar{F} \cdot d\bar{r}$. If the vector $\bar{F}$ is irrotational, the circulation is zero.

(5) *Conservative field.* If the line integral of a vector field $\bar{F}$ depends on $\bar{F}$ and end points of the path but not on the nature of the path, then the vector field $\bar{F}$ is called the conservative vector field.

Theorem 1. *$\bar{F}$ is the gradient of some scalar point function ϕ, if the line integral of $\bar{F}$ is independent of the path.*

Proof. By hypothesis $\oint_C \bar{F}\cdot d\bar{r}$ is independent of the path C joining any two points (x_1, y_1, z_1) and (x, y, z). Here (x_1, y_1, z_1) is a fixed point.

Then
$$\phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \bar{F}\cdot d\bar{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \bar{F}\cdot \frac{d\bar{r}}{ds}ds,$$
where s is the length of the arc of the path from the point (x_1, y_1, z_1).

Now
$$\frac{d\phi}{ds} = \bar{F}\cdot\frac{d\bar{r}}{ds}.$$
But
$$\frac{d\phi}{ds} = \nabla\phi\cdot\frac{d\bar{r}}{ds}.$$
$$\therefore\ \left(\bar{F} - \nabla\phi\right)\cdot\frac{d\bar{r}}{ds} = 0.$$

It is independent of $\frac{d\bar{r}}{ds}$.
$$\therefore\ \bar{F} - \nabla\phi = \bar{0} \quad \text{or,} \quad \bar{F} = \nabla\phi.$$

Theorem 2. *The necessary and sufficient condition that $F_1dx + F_2dy + F_3dz$ will be exact differential is that $\nabla\times\bar{F} = \bar{0}$, where $\bar{F} = \bar{i}F_1 + \bar{j}F_2 + \bar{k}F_3$.*

Proof. The condition is necessary.

Let
$$F_1dx + F_2dy + F_3dz = d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \text{ (an exact differential).}$$

Since x, y and z are independent variables, $F_1 = \frac{\partial\phi}{\partial x}, F_2 = \frac{\partial\phi}{\partial y}, F_3 = \frac{\partial\phi}{\partial z}$.
$$\therefore\ \bar{F} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = \nabla\phi.$$

Thus $\nabla\times\bar{F} = \nabla\times\nabla\phi = \bar{0}$.

The condition is sufficient. If $\nabla\times\bar{F} = \bar{0}$, then $\bar{F} = \nabla\phi$.

Now $\bar{F}\cdot d\bar{r} = \nabla\phi\cdot d\bar{r} = d\phi$, i.e. $F_1dx + F_2dy + F_3dz = d\phi$.

$\therefore\ F_1dx + F_2dy + F_3dz$ is an exact differential.

Theorem 3. *$\bar{F}$ is a conservative field iff $\nabla\times\bar{F} = \bar{0}$.*

Proof. The condition is necessary. Let $\bar{F} = \nabla\phi$, where ϕ is a single valued and has continuous partial derivatives.

Work done by $\bar{F}$ for the displacement from the point $P_1\,(x_1, y_1, z_1)$ to the point $P_2(x_2, y_2, z_2)$ in the field of $\bar{F}$

$$\begin{aligned}
= \int_{P_1}^{P_2} \bar{F}\cdot d\bar{r} = \int_{P_1}^{P_2} \nabla\phi\cdot d\bar{r} &= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial j}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}\right)\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k}) \\
&= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) \\
&= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1).
\end{aligned}$$

The integral depends on the points P_1 and P_2 and not on the path joining the points.

Therefore, $\bar{F}$ is a conservative field.

Here $\nabla\times\bar{F} = \nabla\times\nabla\phi = \bar{0}$.

The condition is sufficient. Let $\nabla\times\bar{F} = \bar{0}$. It holds, if $\bar{F} = \nabla\phi$.

Consequently, $\bar{F}$ is a conservative field.

4.41 Divergence theorem of Gauss or Green's First Theorem

Let $\bar{F}(x, y, z)$ and $\nabla\cdot\bar{F}$ are continuous over the closed regular surface and its interior V. Then the divergence theorem asserts that

$$\iiint\limits_V \nabla\cdot\bar{F}\,dV = \iint\limits_S \bar{F}\cdot\bar{n}\,dS, \tag{1}$$

where $\bar{n}$ is the unit normal to the surface in the outward direction.

Proof. Let the closed surface S be such that a line parallel to any coordinate axis cuts it in not more than two points.

Now if $\bar{F} = \bar{i}F_1 + \bar{j}F_2 + \bar{k}F_3$ and $\cos\alpha, \cos\beta, \cos\gamma$ are the d.cs. of $\bar{n}$, then the integral relation may be written as

$$\iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV = \iint\limits_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dS. \tag{2}$$

Let us first consider the integral

$$\iiint\limits_V \frac{\partial F_3}{\partial z}\,dV = \iiint\limits_V \frac{\partial F_3}{\partial z}dx\,dy\,dz.$$

The closed surface S when projected on xy-plane gives a regular two dimensional closed region R. We assume that the surface may be divided into lower and upper portions S_1 and S_2 whose equations are of the form $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively.

Now consider the integral

$$\iiint\limits_V \frac{\partial F_3}{\partial z}\, dV = I.$$

$$I = \iint\limits_R \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z} dz\right] dy\, dx$$

$$= \int_R F_3\{x, y, f_2(x, y)\} dy\, dx - \iint\limits_R F_3\{x, y, f_1(x, y)\} dy\, dx.$$

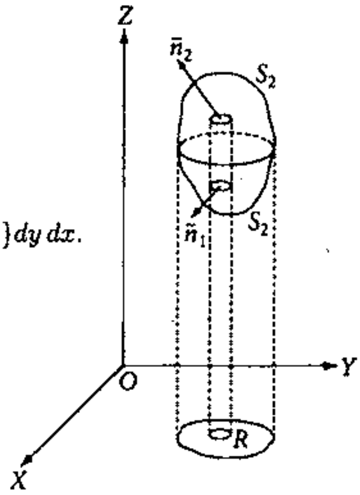


Fig. 57

For points on S_2,

$$dy\, dx = dS_2 \cos\gamma_2 = \bar{k}\cdot\bar{n}_2\, dS_2$$

since the normal $\bar{n}_2$ to dS_2 makes an acute angle γ_2 with $\bar{k}$.

Similarly, for $S_1, dy\, dx = -dS_1 \cos\gamma_1$ $= -\bar{k}\cdot\bar{n}_1\, dS_1$.

The normal $\bar{n}_1$, to dS_1 makes an obtuse angle γ_1 with $\bar{k}$.

Thus

$$\iint\limits_R F_3(x, y, f_2) dy\, dx = \iint\limits_{S_2} F_3\bar{k}\cdot\bar{n}_2\, dS_2$$

and $$\iint\limits_R F_3(x, y, f_1) dy\, dx = -\iint\limits_{S_1} F_3\bar{k}\cdot\bar{n}_1\, dS_1.$$

$$\therefore \iiint\limits_V \frac{\partial F_3}{\partial z}\, dV = \iint\limits_{S_2} F_3\bar{k}\cdot\bar{n}_2\, dS_2 + \iint\limits_{S_1} F_3\bar{k}\cdot\bar{n}_1\, dS_1$$

$$= \iint\limits_S F_3\bar{k}\cdot\bar{n}\, dS.$$

Similarly, by projecting S on the yz- and zx-planes respectively, we have

$$\iiint\limits_V \frac{\partial F_1}{\partial x}\, dV = \iint\limits_S F_1\bar{i}\cdot\bar{n}\, dS \quad \text{and} \quad \iiint\limits_V \frac{\partial F_2}{\partial y}\, dV = \iint\limits_S F_2\bar{j}\cdot\bar{n}\, dS.$$

Hence

$$\iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV = \iint\limits_S (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\cdot\bar{n}\, dS$$

$$\text{or,} \quad \iiint\limits_V \nabla\cdot\bar{F}\, dV = \iint\limits_S \bar{F}\cdot\bar{n}\, dS.$$

Example 1. *Transform the following surface integrals into volume integrals by the help of divergence theorem.*

(i) $\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,dS$ (ii) $\iint_S (xy\,dx\,dy + yz\,dy\,dz + zx\,dz\,dx)$.

(i) Let $\bar{F} = \bar{i}x + \bar{j}y + \bar{k}z, \bar{n} = \bar{i}\cos\alpha + \bar{j}\cos\beta + \bar{k}\cos\gamma$.

So that

$$\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,dS = \iint_S \bar{F}\cdot\bar{n}\,dS.$$

Now

$$div\bar{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

$\therefore$ by divergence theorem,

$$\iint_S \bar{F}\cdot\bar{n}\,dS = \iiint_V div\bar{F}\,dV = \iiint_V 3dx\,dy\,dz.$$

(ii) Taking $\bar{F} = \bar{i}yz + \bar{j}zx + \bar{k}xy$, we have

$$\iint_S (xy\,dx\,dy + yz\,dy\,dz + zx\,dz\,dx) = \iint_S \bar{F}\cdot\bar{n}\,dS.$$

By divergence theorem,

$$\iint_S \bar{F}\cdot\bar{n}\,dS = \iiint_V div\bar{F}\,dV = 0 \ (\because\ div\bar{F} = 0).$$

Example 2. *Verify the divergence theorem for the function $\bar{F} = \bar{i}x^2 + \bar{j}z + \bar{k}yz$ over the unit cube.*

We may assume that the unit cube is bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

Here $\nabla\cdot\bar{F} = 2x + y$.

$$\therefore\ \iiint_V \nabla\cdot\bar{F}\,dV = \int_0^1\int_0^1\int_0^1 (2x+y)dx\,dy\,dz = \frac{3}{2}.$$

To evaluate $\iint_S \bar{F}\cdot\bar{n}\,dS$, we compute it over the surfaces of the cube separately.

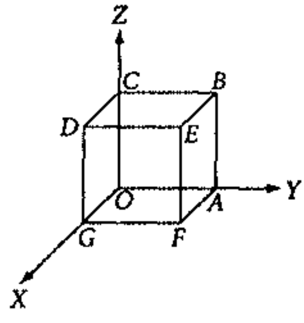


Fig. 58

On the face $OABC, \bar{n} = -\bar{i}, x = 0, \bar{F}\cdot\bar{n} = -x^2 = 0$.

$$\therefore\ \iint_{OABC} \bar{F}\cdot\bar{n}\,dS = \int_0^1\int_0^1 0\cdot dy\,dz = 0.$$

On the face $DEFG, \bar{n} = \bar{i}, x = 1, \bar{F} \cdot \bar{n} = x^2 = 1.$

$$\therefore \iint_{DEFG} \bar{F} \cdot \bar{n}\, dS = \int_0^1 \int_0^1 1 \cdot dy\, dz = 1.$$

On the face $OCDG, \bar{n} = -\bar{j}, y = 0, \bar{F} \cdot \bar{n} = -z.$

$$\therefore \iint_{OCDG} \bar{F} \cdot \bar{n}\, dS = \int_0^1 \int_0^1 -z\, dz\, dx = -\frac{1}{2}.$$

On the face $BEFA, \bar{n} = \bar{j}, y = 1, \bar{F} \cdot \bar{n} = z.$

$$\therefore \iint_{BEFA} \bar{F} \cdot \bar{n}\, dz\, dx = \int_0^1 \int_0^1 z\, dz\, dx = \frac{1}{2}.$$

On the face $DAFG, \bar{n} = -\bar{k}, z = 0, \bar{F} \cdot \bar{n} = -yz = 0.$

$$\therefore \iint_{DAFG} \bar{F} \cdot \bar{n}\, dS = \int_0^1 \int_0^1 0 \cdot dx\, dy = 0.$$

On the face $CBED, \bar{n} = \bar{k}, z = 1, \bar{F} \cdot \bar{n} = yz = y.$

$$\therefore \iint_{CBED} \bar{F} \cdot \bar{n}\, dS = \int_0^1 \int_0^1 y\, dx\, dy = \frac{1}{2}.$$

Adding these six integrals

$$\iint_S \bar{F} \cdot \bar{n}\, dS = \frac{3}{2} \text{ which is equal to } \iiint_V \nabla \cdot \bar{F}\, dV.$$

Thus the divergence theorem is verified.

4.42 Some important deductions from divergence theorem

(i) $\displaystyle\iiint_V \nabla\phi\, dV = \iint_S \phi\bar{n}\, dS,$ (ii) $\displaystyle\iiint_V (\nabla \times \bar{F})\, dV = -\iint_S (\bar{F} \times \bar{n})\, dS,$

(iii) $\displaystyle\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi) \cdot d\bar{S}$ (Green's identity).

(i) Let $\bar{F} = \phi\bar{c}$, where ϕ is a scalar function and $\bar{c}$ is a constant vector. Putting **this** value of $\bar{F}$ in the divergence theorem

$$\iiint_V div(\phi\bar{c}) dv = \iint_S \phi\bar{c} \cdot \bar{n}\, dS.$$

Now $div(\phi\bar{c}) = grad\,\phi \cdot \bar{c} = \bar{c} \cdot grad\,\phi$ and $\phi\bar{c} \cdot \bar{n} = \bar{c} \cdot \phi\bar{n}.$

Thus

$$\iiint_V \bar{c}\cdot grad\,\phi\, dV = \iint_S \bar{c}\cdot \phi\bar{n}\, dS.$$

Since $\bar{c}$ is a constant vector we can take it outside the integrals to give

$$\bar{c}\cdot \iiint_V grad\,\phi\, dV = \bar{c}\cdot \iint_S \phi\bar{n}\, dS.$$

$\bar{c}$ can be any arbitrary vector.

$$\therefore\ \iiint_V grad\,\phi\, dV = \iint_S \phi\bar{n}\, dS.$$

(ii) Replacing $\bar{F}$ by $\bar{A}\times\bar{F}$ in divergence theorem, where $\bar{A}$ is any constant vector, we get

$$\iiint_V \nabla\cdot(\bar{A}\times\bar{F})\, dV = \iint_S (\bar{A}\times\bar{F})\cdot\bar{n}\, dS$$

or, $$\iiint_V \{\bar{F}\cdot(\nabla\times\bar{A}) - \bar{A}\cdot(\nabla\times\bar{F})\}\, dV = \iint_S \bar{A}\cdot(\bar{F}\times\bar{n})\, dS$$

or, $$-\iiint_V \bar{A}\cdot(\nabla\times\bar{F})\, dV = \iint_S \bar{A}\cdot(\bar{F}\times\bar{n})\, dS$$

[$\because$ $\bar{A}$ is a constant vector.]

or, $$-\bar{A}\cdot\iiint_V \nabla\times\bar{F}\, dV = \bar{A}\cdot\iint_S \bar{F}\times\bar{n}\, dS$$

or, $$\iiint_V \nabla\times\bar{F}\, dV = -\iint_S \bar{F}\times\bar{n}\, dS.\ (\because\ \bar{A} \text{ is arbitrary}).$$

(iii) Let $\bar{F} = \phi\nabla\psi - \psi\nabla\phi$.

Then $\nabla\cdot\bar{F} = \nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$.

$$\bar{F}\cdot\bar{n} = (\phi\nabla\psi - \psi\nabla\phi)\cdot\bar{n}.$$

Thus

$$\iiint_V \nabla\cdot\bar{F}\, dV = \iint_S \bar{F}\cdot\bar{n}\, dS$$

gives

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\bar{n}\, dS$$

$$= \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\bar{S}.$$

4.43 Green's theorem in the plane

If R is a closed region in the xy-plane bounded by a simple closed curve C and if X and Y are continuous functions of x and y having continuous partial derivatives in R, then

$$\int_C (X\,dx + Y\,dy) = \iint_R \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dx\,dy.$$

Proof. Let the closed curve C have the property that any straight line parallel to a coordinate axis meets the curve C in at most two points. Let the equations of the curves AEB and AFB be $y = f_1(x)$ and $y = f_2(x)$ respectively.

Now

$$\iint_R \frac{\partial X}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=f_1(x)}^{f_2(x)} \frac{\partial X}{\partial y}\,dy\right] dx$$

$$= \int_{x=a}^{b} [X\{x, f_2(x)\} - X\{x, f_1(x)\}]\,dx$$

$$= -\int_a^b X\{x, f_1(x)\}dx - \int_b^a X\{x, f_2(x)\}dx$$

$$= -\int_C X\,dx.$$

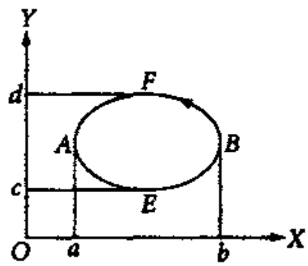


Fig. 59

Thus

$$\int_C X\,dx = -\iint_R \frac{\partial X}{\partial y}\,dx\,dy.$$

Similarly, let the equation of the curves EAF and EBF be $x = g_1(y)$ and $x = g_2(y)$ respectively.

Then

$$\iint_R \frac{\partial Y}{\partial x}\,dx\,dy = \int_{y=c}^{d}\left[\int_{x=g_1(x)}^{g_2(x)} \frac{\partial Y}{\partial x}\,dx\right] dy = \int_c^d [Y(y, g_2) - Y(y, g_1)]dy$$

$$= \int_d^c Y(y, g_1)dy + \int_c^d Y(y, g_2)dy = \int_c^d Y\,dy.$$

Thus

$$\int_C Y\,dy = \iint_R \frac{\partial Y}{\partial x}\,dx\,dy.$$

Adding these two line integrals we get

$$\int_C (X\,dx + Y\,dy) = \iint_R \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dx\,dy. \tag{1}$$

In vector notation this can be expressed as

$$\int_C \bar{F}\cdot d\bar{r} = \iint_R (\nabla \times \bar{F})\cdot \bar{k}\,dR. \tag{2}$$

Example. Show that $\frac{1}{2}\int_C (X\,dy - Y\,dx)$ is equal to the area bounded by the curve C, where C is a simple closed curve on a plane. **[BH 2007]**

Putting $X = -y$ and $Y = x$ in Green's theorem,

$$\int_C (x\,dy - y\,dx) = \iint_R \left\{\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right\} dx\,dy = 2\iint dx\,dy$$

$$= 2 \times \text{area of the region } R \text{ bounded by the closed curve } C.$$

4.44 Stokes's theorem

The line integral of the tangential component of a vector $\bar{F}$ taken along a simple closed curve C is equal to the surface integral of the normal component of curl $\bar{F}$ taken over any surface S having C as its boundary.

Written in vector notation

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (curl\bar{F})\cdot \bar{n}\,dS,$$

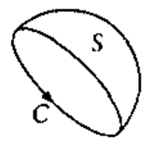


Fig. 60

where C is traversed in the positive direction and $\bar{F}$ has continuous derivatives.

This theorem can be expressed in cartesian form in terms of $\bar{F}$ and curl $\bar{F}$.

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ and $\bar{n} = \bar{i}\cos\alpha + \bar{j}\cos\beta + \bar{k}\cos\gamma$.

Then in cartesian form Stoke's theorem is

$$\int_C (F_1dx + F_2dy + F_3dz)$$
$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS.$$

Proof. Let the surface S be such that its projection on the coordinate planes are regions bounded by simple closed curves and any straight line parallel to the z-axis cuts S in one point.

Let the equation of the surface S be $z = f(x,y)$. The direction cosines of the normal can be expressed by

$$\cos\alpha = \frac{-\frac{\partial f}{\partial x}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}},$$

$$\cos\beta = \frac{-\frac{\partial f}{\partial y}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}},$$

$$\cos\gamma = \frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}. \qquad (1)$$

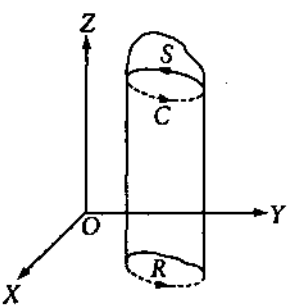


Fig. 61

Now we consider the line integral

$$\int_C F_1(x,y,z)dx.$$

On $C, z = f(x,y)$, where (x,y) are the coordinates of the points on the curve L which is the projection of C on the xy-plane.

Thus

$$\int_C F_1(x,y,z)dx = \int_L F_1(x,y,f)dx. \tag{2}$$

Substituting $Y = 0$ and $X = F_1(x,y,f)$ in Green's formula

$$\int_L (X\,dx + Y\,dy) = \iint\limits_R \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dx\,dy.$$

we obtain

$$\int_L F_1(x,y,f)dx = -\iint\limits_R \frac{\partial F_1}{\partial y} dx\,dy, \tag{3}$$

where the region R is bounded by the curve L. Since $F_1(x,y,z)$ is a composite function the derivative $\frac{\partial F_1}{\partial y}$ can be written as $\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f(x,y)}{\partial y}$.

Substituting this in the above double integral, we have

$$\int_C F_1(x,y,z)dx = -\iint\limits_R \left[\frac{\partial F_1}{\partial y}(x,y,z) + \frac{\partial F_1}{\partial z}(x,y,z)\frac{\partial f}{\partial y}(x,y)\right] dx\,dy. \tag{4}$$

The integral over R can be transformed into surface integral by using the formula

$$\iint\limits_S A(x,y,z)\cos\gamma\,dS = \iint\limits_R A\,dx\,dy.$$

Then

$$\iint\limits_R \frac{\partial F_1}{\partial y} dx\,dy = \iint\limits_S \frac{\partial F_1}{\partial y}\cos\gamma\,dS$$

$$\text{and}\quad \iint\limits_R \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y} dx\,dy = \iint\limits_S \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\cos\gamma\,dS.$$

From (i)

$$\frac{\cos\beta}{\cos\gamma} = -\frac{\partial f}{\partial y} \quad \text{or,} \quad -\frac{\partial f}{\partial y}\cos\gamma = \cos\beta.$$

Now the relation (4) takes the form

$$\int_C F_1(x,y,z)dx = -\iint\limits_S \frac{\partial F_1}{\partial y}\cos\gamma\,dS + \iint\limits_S \frac{\partial F_1}{\partial z}\cos\beta\,dS. \tag{5}$$

The direction of integral along the contour C must agree with positive normal $\bar{n}$. Namely, for an observer looking from the end of the normal the direction of integration will be anti-clockwise.

By similar considerations

$$\int_C F_2(x,y,z)dy = \iint_S \left[-\frac{\partial F_2}{\partial z}\cos\alpha + \frac{\partial F_2}{\partial x}\cos\gamma\right] dS \tag{6}$$

and $$\int_C F_3(x,y,z)dz = \iint_S \left[-\frac{\partial F_3}{\partial x}\cos\beta + \frac{\partial F_3}{\partial y}\cos\alpha\right] dS. \tag{7}$$

Adding (5), (6) and (7),

$$\int_C (F_1dx + F_2dy + F_3dz) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha \right.$$
$$\left. + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS.$$

Note 1. The integral $\int_C (F_1dx + F_2dy + F_3dz)$ along a closed curve is called the circulation of the vector $\bar{F}$ along the contour C. If the vector $\bar{F}$ is irrotational, the circulation is zero.

Note 2. If $\bar{F}$ is a vector force on the particle, the $\int_C \bar{F}\cdot d\bar{r}$ represents the work done in moving the particle around a closed path C. For $\int \bar{F}\cdot d\bar{r} = 0$, the field is said to be *conservative*. Hence an irrotation field is always conservative.

If $\bar{F} = \nabla\phi$, then $\nabla\times\bar{F} = 0$. ϕ is called *potential*.

$$\bar{F} = \bar{i}F_1 + \bar{j}F_2 + \bar{k}F_3 = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}.$$

$$\therefore\ F_1 = \frac{\partial\phi}{\partial x}, F_2 = \frac{\partial\phi}{\partial y}, F_3 = \frac{\partial\phi}{\partial z}.$$

Again

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = F_1dx + F_2dy + F_3dz.$$

On integration,

$$\phi_B - \phi_A = \int_A^B (F_1dx + F_2dy + F_3dz) = \int_A^B \bar{F}\cdot d\bar{r}.$$

It shows that in an irrotational field that potential difference between two points A and B is equal to the line integral of a function from A to B. Obviously this value is independent of the path of integration.

Example 1. *Using Stoke's theorem transform the integral* $\int_C (y\,dx + z\,dy + x\,dz)$ *into a surface integral.*

Here $\bar{F} = \bar{i}y + \bar{j}z + \bar{k}x$.

$$\begin{aligned}\nabla \times \bar{F} &= \left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}\right)\bar{i} + \left(\frac{\partial y}{\partial z} - \frac{\partial x}{\partial x}\right)\bar{j} + \left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial y}\right)\bar{k} \\ &= -\bar{i} - \bar{j} - \bar{k}.\end{aligned}$$

Thus by Stoke's theorem,

$$\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, dS = -\iint (\cos\alpha + \cos\beta + \cos\gamma)\, dS,$$

where $\bar{n} = \bar{i}\cos\alpha + \bar{j}\cos\beta + \bar{k}\cos\gamma$.

Example 2. *Verify Stoke's theorem for $\bar{F} = (x+y)\bar{i} + yz^2\bar{j} + y^2z\bar{k}$, where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.*

Here

$$curl\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & yz^2 & y^2z \end{vmatrix} = -\bar{k}.$$

Let the boundary curve C be xy-plane and the projection of the surface S on the xy-plane is a circle of unit area. This region is R.

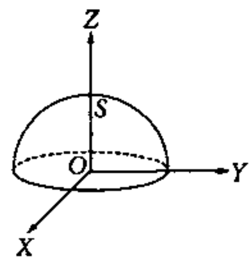


Fig. 62

$$\begin{aligned}\iint_S curl\bar{F} \cdot \bar{n}\, dS &= \iint_S (-\bar{k} \cdot \bar{n})\, dS \\ &= -\iint_R dx\, dy = -\pi.\end{aligned}$$

Here $\bar{n} = \bar{k}$.

$$\begin{aligned}\int_C \bar{F} \cdot d\bar{r} &= \int_C \left\{(x+y)dx + yz^2dy + y^2z\, dz\right\} \\ &= \int_0^{2\pi} (\cos t + \sin t)(-\sin t)dt = -\pi.\end{aligned}$$

Since on $C, x = \cos t, y = \sin t, z = 0$.

Thus Stoke's theorem is verified.

Note. From Stokes theorem it follows that the surface integral of $\nabla \times \bar{F}$ taken over a surface S is equal to the line integral of $\bar{F}$ taken around C, where C is the bounding curve of S. Thus the surface integral $\nabla \times \bar{F}$ taken over a surface S will be the same as the surface integral of $\nabla \times \bar{F}$ taken over the plane area made by the section of C.

WORKED-OUT EXAMPLES

1. *Prove the identity* $\iiint\limits_V \nabla^2 u\, dx\, dy\, dz = \iint\limits_C \frac{\partial u}{\partial n}\, dS.$

Substituting $\bar{F} = \nabla u$ in the divergence theorem we get

$$\iiint\limits_V \nabla \cdot \nabla u\, dV = \iint\limits_S \nabla u \cdot \bar{n}\, dS,$$

$$\text{i.e.} \quad \iiint\limits_V \nabla^2 u\, dx\, dy\, dz = \iint\limits_S \frac{\partial u}{\partial n}\, dS.$$

2. *Prove that*

$$\text{(i) } \nabla\phi = \lim_{\Delta V \to 0} \frac{\iint\limits_S \bar{n}\phi\, dS}{\Delta V},$$

$$\text{(ii) } \nabla \cdot \bar{F} = \lim_{\Delta V \to 0} \frac{\iint\limits_S \bar{n} \cdot \bar{F}\, dS}{\Delta V}, \quad \text{(iii) } \nabla \times \bar{F} = \lim_{\Delta V \to 0} \frac{\iint\limits_S \bar{n} \times \bar{F}\, dS}{\Delta V}.$$

(i) We have

$$\iint\limits_S \bar{n}\phi\, dS = \iiint\limits_V (\nabla\phi)\, dV.$$

From this it follows that

$$\iint\limits_{\Delta S} \bar{n}\phi\, dS = \iiint\limits_{\Delta V} (\nabla\phi)\, dV.$$

By the mean value theorem,

$$\iiint\limits_{\Delta V} \nabla\phi\, dV = \Delta V\{(\nabla\phi)_P + \epsilon\},$$

where $(\nabla\phi)_P$ is the value of $\nabla\phi$ at the point P and $\epsilon \to 0$ as $\Delta V \to 0$.

$$\therefore\ \Delta V\{(\nabla\phi)_P + \epsilon\} = \iint\limits_{\Delta S} \bar{n}\phi\, dS$$

$$\text{or,} \quad \{(\nabla\phi)_P + \epsilon\} = \frac{\iint\limits_{\Delta S} \bar{n}\phi\, dS}{\Delta V}$$

$$\text{or,} \quad (\nabla\phi)_P = \lim_{\Delta V \to 0} \frac{\iint\limits_{\nabla S} \bar{n}\phi\, dS}{\Delta V}.$$

(ii) We have

$$\iiint_V \nabla \cdot \bar{F}\, dV = \iint_S \bar{n} \cdot \bar{F}\, dS.$$

It follows that

$$\iiint_{\Delta V} \nabla \cdot \bar{F}\, dV = \iint_{\Delta S} \bar{n} \cdot \bar{F}\, dS.$$

By the mean value theorem,

$$\iiint_{\Delta V} \nabla \cdot \bar{F}\, dV = \Delta V[(\nabla \cdot F)_P + \epsilon],$$

where $(\nabla \cdot \bar{F})_P$ is the value of $\nabla \cdot \bar{F}$ at the point P and $\epsilon \to 0$ as $\Delta V \to 0$.

$$\therefore \quad \Delta V\Big[(\nabla \cdot \bar{F})_P + \epsilon\Big] = \iint_{\Delta S} \bar{n} \cdot \bar{F}\, dS$$

$$\text{or,} \quad \Big[(\nabla \cdot \bar{F})_P + \epsilon\Big] = \frac{\displaystyle\iint_{\Delta S} \bar{n} \cdot \bar{F}\, dS}{\Delta V}$$

$$\text{or,} \quad (\nabla \cdot \bar{F})_P = \lim_{\Delta V \to 0} \frac{\displaystyle\iint_{\Delta S} \bar{n} \cdot \bar{F}\, dS}{\Delta V}.$$

(iii) The result is followed from

$$\iiint_V (\nabla \times \bar{F})\, dV = -\iint_S (\bar{F} \times \bar{n})\, dS$$

in a similar way.

3. *Prove that $\bar{F} = \bar{i}(x^2 - yz) + \bar{j}(y^2 - zx) + \bar{k}(z^2 - xy)$ is an irrotational vector and find ϕ, where $\bar{F} = -\nabla\phi$.*

Here

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \bar{i}\left\{\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right\} + \bar{j}\left\{\frac{\partial}{\partial z}(x^2 - yz) - \frac{\partial}{\partial x}(z^2 - xy)\right\}$$

$$+ \bar{k}\left\{\frac{\partial}{\partial x}(y^2 - zx) - \frac{\partial}{\partial y}(x^2 - yz)\right\}$$

$$= \bar{i}(-x + x) + \bar{j}(-z + z) + \bar{k}(-y + y) = \bar{0}.$$

Hence $\bar{F}$ is an irrotational vector.

If $\bar{F} = -\nabla\phi$, then

$$d\phi = -(F_1 dx + F_2 dy + F_3 dz).$$
$$\therefore\ \phi = -\int_A^B \left\{(x^2 - yz)dx + (y^2 - zx)dy + (z^2 - xy)dz\right\},$$

where A and B are (x_1, y_1, z_1) and (x, y, z). Here A is fixed.

ϕ can be obtained in the following way.

$$\phi = -\left[\frac{x^3}{3} - xyz - \frac{x_1^3}{3} + x_1y_1z_1 + \frac{y^3}{3} - xyz - \frac{y_1^3}{3} + x_1y_1z_1 + \frac{z^3}{3} - xyz - \frac{z_1^3}{3} + x_1y_1z_1\right] = -\frac{1}{3}\left(x^3 + y^3 + z^3 - 3xyz\right) + \text{constant}.$$

If A is $(0,0,0)$, then

$$\phi = -\tfrac{1}{3}\left(x^3 + y^3 + z^3 - 3xyz\right).$$

4. *Evaluate* $\iint_S \bar{A}\cdot\bar{n}\,dS$, *where S is the surface of the parabolic cylinder $y^2 = 8x$ in the positive octant bounded by the planes $y = 4$ and $z = 5$ and $\bar{A} = 3y\bar{i} + 2z\bar{j} + x^3\bar{k}$.*

Here
$$\bar{n} = \frac{\nabla(y^2 - 8x)}{|\nabla(y^2 - 8x)|} = \frac{-8\bar{i} + 2y\bar{j}}{\sqrt{8^2 + 4y^2}}.$$

Projection of dS on yz-plane $= \dfrac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \dfrac{\sqrt{8^2 + 4y^2}}{8}dy\,dz.$

If R is the projection of S on the yz-plane, then

$$\iint_S \bar{A}\cdot\bar{n}\,dS = \iint_R \bar{A}\cdot\bar{n}\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \int_{y=0}^{4}\int_{z=0}^{5} \frac{-24y + 4yz}{\sqrt{8^2 + 4y^2}}\cdot\frac{\sqrt{8^2 + 4y^2}}{8}dy\,dz$$
$$= \int_{y=0}^{4}\int_{z=0}^{5} \frac{-6 + z}{2}\cdot y\,dy\,dz = \int_{z=0}^{5} \frac{-6 + z}{2}\left[\frac{y^2}{2}\right]_0^4 dz$$
$$= 4\left[-6z + \frac{z^2}{2}\right]_0^5 = -70.$$

5. *Find the work done in moving a particle in a force field given by $\bar{F} = (2x - y + z)\bar{i} + (x + y - z)\bar{j} + (3x - 2y - 5z)\bar{k}$ along an ellipse in the $z = 0$-plane*

given by $\frac{x^2}{9}+\frac{y^2}{16}=1$.

$$\begin{aligned}
\text{Work done} &= \int_C \bar{F}\cdot d\bar{r}\\
&= \int_C \{(2x-y+z)dx+(x+y-z)dy+(3x-2y-5z)dz\}\\
&= \int_0^{2\pi} \{6\cos\theta-4\sin\theta)(-3\sin\theta)d\theta+(3\cos\theta+4\sin\theta)4\cos\theta\,d\theta\}\\
&\qquad (\because \text{ for the curve } C, x=3\cos\theta, y=4\sin\theta,\\
&\qquad z=0, dz=0, 0\le\theta\le 2\pi.)\\
&= \int_0^{2\pi}(12-\sin 2\theta)d\theta = \left[12\theta+\frac{\cos 2\theta}{2}\right]_0^{2\pi} = 24\pi \text{ units.}
\end{aligned}$$

6. *Find the area of a loop of the four-leafed rose* $\rho = 3\sin 2\phi$.

$$\text{Area} = \frac{1}{2}\oint(x\,dy-y\,dx) = \frac{1}{2}\int \rho^2 d\phi,$$

where $x=\rho\cos\phi$ and $y=\rho\sin\phi$.

$$\begin{aligned}
\text{Thus the required area} &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} 9\sin^2 2\phi d\phi = \frac{9}{4}\int_{-\pi/4}^{\pi/4}(1-\cos 4\phi)d\phi\\
&= \frac{9}{4}\left[\phi-\frac{\sin 4\phi}{4}\right]_{-\pi/4}^{\pi/4} = \frac{9\pi}{8}.
\end{aligned}$$

7. *Show that* $\bar{F}=(y+\sin z)\bar{i}+x\bar{j}+x\cos z\,\bar{k}$ *is a conservative vector field. For this find the scalar function* $\phi(x,y,z)$ *such that grad* $\phi=\bar{F}$.

Here

$$\begin{aligned}
\nabla\times\bar{F} &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y+\sin z & x & x\cos z\end{vmatrix}\\
&= \bar{i}\left\{\frac{\partial}{\partial y}(x\cos z)-\frac{\partial}{\partial z}x\right\}+\bar{j}\left\{\frac{\partial}{\partial z}(y+\sin z)-\frac{\partial}{\partial x}(x\cos z)\right\}\\
&\quad +\bar{k}\left\{\frac{\partial x}{\partial x}-\frac{\partial}{\partial y}(y+\sin z)\right\}\\
&= \bar{i}0+\bar{j}(\cos z-\cos z)+\bar{k}(1-1)=\bar{0}.
\end{aligned}$$

Thus $\bar{F}$ is a conservative vector field.

Let $\bar{F} = \nabla\phi$.

$$\therefore \frac{\partial\phi}{\partial x} = y + \sin z, \quad \frac{\partial\phi}{\partial y} = x, \quad \frac{\partial\phi}{\partial z} = x\cos z.$$

$$\phi = \int_{(0,0,0)}^{(x,y,z)} \bar{F}\cdot d\bar{r} = \int_{(0,0,0)}^{(x,y,z)} \{(y+\sin z)dx + x\,dy + x\cos z\,dz\}$$
$$= (y+\sin z)x + xy + x\sin z.$$

8. *Show that the work done by a force $\bar{F}$ to displace a particle of mass m from A to B is equal to the change in kinetic energy $\frac{1}{2}mv^2$, where $\bar{v}$ is the velocity of the particle.*

$$\text{Work done} = \int_A^B \bar{F}\cdot d\bar{r} = \int_A^B m\frac{d\bar{v}}{dt}\cdot d\bar{r} = \int_A^B m\frac{d\bar{v}}{dt}\cdot\frac{d\bar{r}}{dt}dt = \int_A^B m\bar{v}\cdot d\bar{v}$$
$$= \frac{1}{2}\int_A^B m\,d\left(\bar{v}^2\right) = \frac{1}{2}m\left(v_B^2 - v_A^2\right) = \text{change in KE.}$$

Note. If $\bar{F} = \nabla\phi$ and the path C is closed, then work done

$$\oint_C \bar{F}\cdot d\bar{r} = 0,$$

i.e. the change in KE $= 0$.

9. ***Evaluate*** $\oint_C \{(y-\sin x)dx + \cos x\,dy\}$ ***directly and by Green's theorem. Here C is the triangle as shown in the figure.*** **[BH 2006]**

(a) Along OA,

$$I_1 = \int_0^{\pi/2} \{(0-\sin x)dx + \cos x\cdot 0\} = \Big[\cos x\Big]_0^{\pi/2} = -1.$$

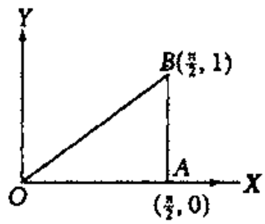


Fig. 63

Along $AB, x = \frac{\pi}{2}$ and $dx = 0$.

$$I_2 = \int_0^1 \{(y-1)0 + 0dy\} = 0.$$

Along $BO, y = \frac{2x}{\pi}, dy = \frac{2dx}{\pi}$.

$$I_3 = \int_{\pi/2}^0 \left\{\left(\frac{2x}{\pi} - \sin x\right)dx + \cos x\frac{2dx}{\pi}\right\} = \left[\frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x\right]_{\pi/2}^0$$
$$= 1 - \frac{\pi}{4} - \frac{2}{\pi}.$$

Thus

$$\oint_C \{(y-\sin x)dx + \cos x\,dy\} = -1 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}.$$

(b) By Green's theorem

$$\oint_C (X\,dx + Y\,dy) = \iint_R \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dx\,dy,$$

where X and Y are continuous functions of x and y, R is a closed region in the xy-plane bounded by a closed curve C.

Here $X = y - \sin x, Y = \cos x$.

$$\begin{aligned}
\therefore \quad \oint_C (X\,dx + y\,dy) &= \iint_R (-\sin x - 1)dx\,dy \\
&= \int_{x=0}^{\pi/2} \left[\int_{y=0}^{2x/\pi} (-\sin x - 1)dy\right] dx \\
&= \int_{x=0}^{\pi/2} (-\sin x - 1)\frac{2x}{\pi} dx \\
&= \left[-\frac{2}{\pi}(-x\cos x + \sin x) - \frac{x^2}{\pi}\right]_0^{\pi/2} \\
&= -\frac{2}{\pi} - \frac{\pi}{4}.
\end{aligned}$$

10. *Verify the divergence theorem for $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ taken over the region bounded by $x^2 + y^2 = a^2, z = 0$ and $z = h$.*

$$\begin{aligned}
\text{Volume integral} &= \iiint_V \nabla \cdot \bar{F}\,dV = \iiint_V \left\{\frac{\partial}{\partial x}4x + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}z^2\right\} dV \\
&= \iiint_V (4 - 4y + 2z)\,dV \\
&= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{h} (4 - 4y + 2z)dx\,dy\,dz \\
&= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (4h - 4hy + h^2)dx\,dy \\
&= \int_{-a}^{a} (8h + 2h^2)\sqrt{a^2 - x^2}dx \\
&= (8h + 2h^2)\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a} \\
&= (4h + h^2)a^2\pi.
\end{aligned}$$

The surface S of the cylinder consists of a base $S_1(z = 0)$, the top base $S_2(z = h)$ and the convex portion $S_3(x^2 + y^2 = a^2)$.

Surface integral

$$= \iint_{S_1} \bar{F} \cdot \bar{n}\,dS_1 + \iint_{S_2} \bar{F} \cdot \bar{n}\,dS_2 + \iint_{S_3} \bar{F} \cdot \bar{n}\,dS_3.$$

On $S_1, \bar{n} = -\bar{k}$,

$$\therefore \ \bar{F}\cdot\bar{n} = 0 \quad \text{and} \quad \iint_{S_1} \bar{F}\cdot\bar{n}\, dS_1 = 0.$$

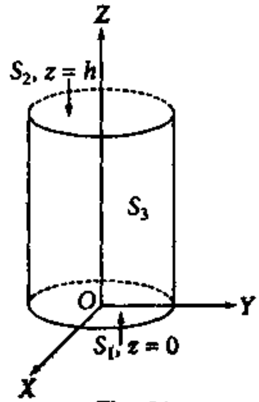


Fig. 64

On $S_2, \bar{n} = \bar{k}$ and $\bar{F}\cdot\bar{n} = h^2$, so

$$\iint_{S_2} h^2\, dS_2 = \pi a^2 h^2 \ \ (\because \text{ area of } S_2 = \pi a^2)\,.$$

On S_3,

$$\bar{n} = \frac{\nabla(x^2+y^2)}{|\nabla(x^2+y^2)|} = \frac{2x\bar{i}+2y\bar{j}}{|2x\bar{i}+2y\bar{j}|} = \frac{x\bar{i}+y\bar{j}}{a},$$

$$\bar{F}\cdot\bar{n} = (4x\bar{i} - 2y^2\bar{j} + z^2\bar{k})\cdot\frac{x\bar{i}+y\bar{j}}{a} = \frac{1}{a}(4x^2 - 2y^3).$$

$$\therefore \ \iint_{S_3} (\bar{F}\cdot\bar{n})\, dS_3 = \iint_{S_3} \frac{1}{a}(4x^2 - 2y^3)\, dS_3.$$

Taking $x = a\cos\theta, y = a\sin\theta$, $dS_3 = a d\theta\, dz$.

$$\therefore \ \iint_{S_3} (\bar{F}\cdot\bar{n})\, dS_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{h} \frac{1}{a}(4a^2\cos^2\theta - 2a^3\sin^3\theta) a d\theta\, dz$$

$$= 2a^2 h\int_{\theta=0}^{2\pi} (2\cos^2\theta - a\sin^3\theta) d\theta = 4ha^2\pi.$$

$\therefore$ surface integral $= (4h + h^2)a^2\pi$.

It is equal to the volume integral.

11. *Verify Stoke's theorem for the vector field* $\bar{F} = (y-z)^2\bar{i} + (z-x)^2\bar{j} + (x-y)^2\bar{k}$ *taken over the portion of the surface* $x^2+y^2+z^2-2ax+az=0$, *where the boundary curve is in the plane* $z=0$.

We are to verify

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\, dS.$$

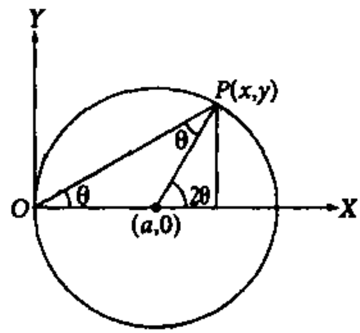


Fig. 65

Here C is $x^2+y^2-2ax=0$, the surface S is $x^2+y^2+z^2-2ax+az=0$,

$$\bar{F} = (y-z)^2\bar{i} + (z-x)^2\bar{j} + (x-y)^2\bar{k}, \quad d\bar{r} = \bar{i}dx + \bar{j}dy + \bar{k}dz,$$

$\bar{n}$ is the unit normal.

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C (y^2dx + x^2dy), \text{ since } z = 0,$$

$$= \int_{-\pi/2}^{\pi/2} \{a^2\sin^2 2\theta(-a\sin 2\theta)2d\theta + (a + a\cos 2\theta)^2(a\cos 2\theta)\cdot 2d\theta\}$$

$$(\because\ x = a + a\cos 2\theta,\ y = a\sin 2\theta)$$

$$= 2a^3 \int_{-\pi/2}^{\pi/2} \{-\sin^3 2\theta + (1+\cos 2\theta)^2\cos 2\theta\}d\theta$$

$$= a^3 \int_{-\pi/2}^{\pi/2} \left\{(1-\cos 2\theta)\frac{d\cos 2\theta}{2} + (2\cos 2\theta + 2 + 2\cos 4\theta)d\theta + 2(1-\sin^2 2\theta)\frac{d\sin 2\theta}{2}\right\}$$

$$= a^3\left[\frac{\cos 2\theta}{2} - \frac{1}{4}\cos^2 2\theta + \sin 2\theta + 2\theta + \frac{1}{2}\sin 4\theta + \sin 2\theta - \frac{\sin^3 2\theta}{3}\right]_{-\pi/2}^{\pi/2}.$$

$$= 2\pi a^3.$$

$$\iint_S (\nabla\times\bar{F})\cdot\bar{n}ds = \iint\{2(y-z)\bar{i} + 2(z-x)\bar{j} + 2(x-y)\bar{k}\}\cdot\bar{n}\,dS$$

$$= \iint 2(x-y)dx\,dy$$

(considering projection on $z = 0$ and $\bar{n} = \bar{k}$)

$$= 2\int_{x=0}^{2a}\left\{\int_{y=-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}}(x-y)dy\right\}dx$$

$$= 2\int_{x=0}^{2a} 2x\sqrt{2ax-x^2}dx$$

$$= 64a^3\int_{\theta=0}^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta \text{ (putting } x = 2a\sin^2\theta)$$

$$= 64a^3\cdot\frac{1.3.1}{2.4.6}\cdot\frac{\pi}{2} = 2\pi a^3.$$

Hence the theorem is verified.

12. *Prove that* $\oint_C (Mdx + Ndy) = 0$ *around every closed curve* C *in a simply connected region, if and only if* $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ *everywhere in the region.*

[*Simply connected region.* A region R for which any closed curve lying in R can be continuously shrunk to a point without leaving R is called a simply connected region.]

Let M and N be continuous and have continuous partial derivatives in the region R bounded by C. In this region Green's theorem is applicable.

Thus

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

(i) If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ in R, then clearly $\oint_c (M\,dx + N\,dy) = 0$.

(ii) Let us suppose that $\oint_C (M\,dx + N\,dy) = 0$.

If $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} > 0$ at a point P, then by the property of continuous functions it follows that $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} > 0$ in some region A surrounding P. If Γ is the boundary of that region, then

$$\oint_L (M\,dx + N\,dy) = \iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy > 0.$$

It contradicts the assumption.

Similarly, the assumption $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} < 0$ leads to a contradiction. Thus $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 0$ at all points in R.

13. *Prove that a necessary and sufficient condition that* $\oint_C \bar{F} \cdot d\bar{r} = 0$ *for every closed curve* C *is that* $\nabla \times \bar{F} = \bar{0}$ *identically.*

(i) *The condition is sufficient.*

Let R be a simply connected region and let $\nabla \times \bar{F} = \bar{0}$ everywhere in R. Since R is simply connected, a surface S can be described on R having C as its boundary. Now by Stoke's theorem

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\,dS = 0.$$

(ii) *The condition is necessary.*

Suppose $\oint_C \bar{F} \cdot d\bar{r} = 0$ around every closed path C and assume $\nabla \times \bar{F} \neq \bar{0}$ at some point P. Assuming $\nabla \times \bar{F}$ is continuous at P there will be a region with P as an interior point, where $\nabla \times \bar{F} \neq \bar{0}$. Let S be a surface contained in this region whose outward unit normal $\bar{n}$ at each point has the same direction as $\nabla \times \bar{F}$, i.e. $\nabla \times \bar{F} = \lambda \bar{n}$, where λ is a positive constant.

Let C be the boundary of S. Then by Stoke's theorem

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\,dS = \lambda \iint_S \bar{n} \cdot \bar{n}\,dS > 0.$$

It contradicts the hypothesis that $\oint_C \bar{F} \cdot d\bar{r} = 0$.

Therefore, $\nabla \times \bar{F} = 0$ everywhere in R.

14. *Define* $\iint_S \bar{F} \cdot \bar{n}\, dS$ *over a surface* S *as the limit of a sum.*

Let the surface area S be subdivided into a finite number of elements of area ΔS_p, where $p = 1, 2, 3, \cdots, m$. A point P_p with coordinates (x_p, y_p, z_p) is taken within the sub-surface ΔS_p.

Let the position vector of the point P be defined as $\bar{F}_p$ and the unit normal to the sub-surface ΔS_p at P be $\bar{n}_p$.

Now we form the sum $\sum_{p=1}^{m} (\bar{F}_p \cdot \bar{n}_p)\Delta S_p$, where $\bar{F}_p \cdot \bar{n}_p$ is the normal component of $\bar{F}_p$ at P_p and consider the limit of this sum as $m \to \infty$ in such a way that the largest dimension of each $\Delta S_p \to 0$. This limit, if it exists, is called the surface integral of $\bar{F}$ over S and is denoted by $\iint_S \bar{F} \cdot \bar{n}\, dS$.

15. *If the surface* S *has projection* R *on the* xy*-plane, show that* $\iint_S \bar{F} \cdot \bar{n}\, dS =$

$\iint_R \bar{F} \cdot \bar{n} \dfrac{dx\, dy}{|\bar{n} \cdot \bar{k}|}$ *provided* $|\bar{n} \cdot \bar{k}| \neq 0$.

By the problem 14,

$$\iint_S \bar{F} \cdot \bar{n}\, dS = \lim_{m \to \infty} \sum_{p=1}^{m} (\bar{F}_p \cdot \bar{n}_p)\Delta S_p. \tag{1}$$

The projection of ΔS_p on the xy-plane

$= \Delta S_p$ multiplied by the cosine of the angle between the plane ΔS_p and the xy-plane

$= \Delta S_p$ multiplied by the cosine of the angle between the normals to the plane ΔS_p and the xy-plane.

$= \Delta S_p |\bar{n}_p \cdot \bar{k}| = \Delta x_p \cdot \Delta y_p$.

Hence from (1),

$$\iint_S \bar{F} \cdot \bar{n}\, dS = \lim_{m \to \infty} \sum (\bar{F}_p \cdot \bar{n}_p) \frac{\Delta x_p \Delta y_p}{|\bar{n}_p \cdot \bar{k}|}$$

$$= \iint_R (\bar{F} \cdot \bar{n}) \frac{dx\, dy}{|\bar{n} \cdot \bar{k}|} \text{ provided } |\bar{n} \cdot \bar{k}| \neq 0.$$

EXERCISE VI

1. Evaluate the following.

(i) $\int_1^2 \bar{r}\,dt$, where $\bar{r} = \bar{i}(3t^2 - 1) + \bar{j}(2 - 6t) - \bar{k}4t$.

(ii) $\int_1^2 \left(\bar{r} \times \frac{d^2\bar{r}}{dt^2}\right) dt$, where $\bar{r} = 2t^2\bar{i} + t\bar{j} - 3t^2\bar{k}$.

(iii) $\int_C \bar{F} \cdot d\bar{r}$, where $\bar{F} = (x^2 + y^2)\bar{i} + (x^2 - y^2)\bar{j}$ and C denotes the curve $\bar{r} = \bar{i}t + \bar{j}t^2, t$ varying from 0 to 1.

(iv) $\int_C \phi\, d\bar{r}$, where $\phi = 2xyz^2$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$.

2. Find the work done by a particle moving under a force field $\bar{F} = 2xy\bar{i} - 3x\bar{j} - 5z\bar{k}$ along the curve $x = t, y = t^2 + 1, z = 2t^2$ from $t = 0$ to 1.

3. Integrate $\bar{F} = x^2\bar{i} - xy\bar{j}$ from the origin O to the point $P(1,1)$ along (i) the line OP, (ii) the parabola $y^2 = x$, (iii) x-axis from $x = 0$ to 1 and then along the line $x = 1$ from $y = 0$ to 1.

4. Evaluate $\iint_S \bar{F} \cdot \bar{n}\, dS$, where S is the surface of a unit cube bounded by planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ and $\bar{F} = 2x^2\bar{i} - 4yz\bar{j} + zx\bar{k}$.

5. If $\bar{F} = x\bar{i} + y\bar{j} + (z^2 - 1)\bar{k}$, prove that $\iint_S \bar{F} \cdot \bar{n}\, dS = 4\pi$, where S is the closed surface bounded by the planes $z = 0, z = 1$ and the cylinder $x^2 + y^2 = 4$.

6. Verify the divergence theorem for $\bar{F} = 2zx\bar{i} + yz\bar{j} + z^2\bar{k}$ over the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

7. If $\phi(x, y, z)$ is a solution of the Laplace's equation, then

$$\iiint_V \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] dV = \iint_S \phi \frac{\partial\phi}{\partial n}\, dS.$$

[*Hints.* Substitute $\bar{F} = \phi\nabla\phi$ in the divergence theorem. Here $\nabla^2\phi = 0$. In this case, ϕ is called a *harmonic function.*]

8. Use the divergence theorem to transform the following surface integral into a volume integral.

$$\iint_S \left(\frac{\partial u}{\partial x} dy\, dz + \frac{\partial u}{\partial y} dz\, dx + \frac{\partial u}{\partial z} dx\, dy\right).$$

9. Find the area

   (i) bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}, a > 0$;

   (ii) of both loops of the lemniscate $\rho^2 = a^2 \cos^2 \phi$.

10. Verify Green's theorem in the plane for $\int_C \{(xy + y^2)dx + x^2dy\}$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

11. Prove that $\frac{1}{3}\iint \bar{r} \cdot d\bar{S}$ is the volume enclosed by the surface S.

    [*Hints.* $\frac{1}{3}\iint\limits_S \bar{r} \cdot d\bar{S} = \frac{1}{3}\iiint\limits_V (\nabla \cdot \bar{r})\, dV = \frac{1}{3}\iiint\limits_V 3dV = V.$]

12. Show that the work done on a particle in moving it from A to B equals its change in kinetic energies at these points whether the force field is conservative or not.

13. Verify Stoke's theorem for $\bar{F} = (2x - y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$, where S is the hemisphere with the boundary circle $x^2 + y^2 + z^2 = a^2, z = 0$.

14. Use Stoke's theorem to transform the integral

$$\int_C \{(y + z)dx + (z + x)dy + (x + y)dz\}$$

    into a surface integral. Here C is the circle $x^2 + y^2 + z^2 = a^2, x + y + z = 0$.

15. Verify the divergence theorem for $\bar{F} = y\bar{i} + x\bar{j} + z^2\bar{k}$ over the cylinder $x^2 + y^2 = a^2, z = 0$ and $z = h$.

16. Verify Stoke's theorem for $\bar{F} = (y - z + 2)\bar{i} + (yz + 4)\bar{j} - zx\bar{k}$, where S is the surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the xy-plane.

17. A vector field is given by $\bar{F} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}$. Show that the field is irrotational and find the scalar potential. Hence evaluate the line integral from $(1, 2)$ to $(2, 1)$.

18. Show that $\bar{B} = \bar{i}(y^2 + z^3) + \bar{j}(2xy - 5z) + \bar{k}(3xz^2 - 5y)$ satisfies $\nabla \times \bar{B} = \bar{0}$. Find a scalar function $\phi(x, y, z)$ such that $\bar{B} = grad\, \phi$.

19. Prove that $(\nabla \times \bar{F}) \cdot \bar{n} = \lim\limits_{\Delta S \to 0} \dfrac{\int_C \bar{F} \cdot d\bar{r}}{\Delta S}$, where ΔS is a surface bounded by a closed curve C, P is any point of ΔS not on C and $\bar{n}$ is a unit normal vector to ΔS at P, ΔS shrinks to P in the limit.

    [*Hints.* By Stoke's theorem $\iint\limits_{\Delta S} (\nabla \times \bar{F}) \cdot \bar{n}\, dS = \int_p \bar{F} \cdot d\bar{r}$.

    By mean value theorem

$$\iint\limits_{\Delta S} (\nabla \times \bar{F}) \cdot \bar{n}\, dS = (\nabla \times \bar{F}) \cdot \bar{n} \iint\limits_{\Delta S} dS = (\nabla \times \bar{F}) \cdot \bar{n}\Delta S,$$

where $(\nabla \times \bar{F}) \cdot \bar{n}$ lies between the maximum and minimum of $(\nabla \times \bar{F}) \cdot \bar{n}$ in ΔS.]

20. A fluid of density $\rho(x, y, z, t)$ moves with velocity $\bar{v}(x, y, z, t)$. If there are no sources or sinks, prove that $\nabla \cdot \bar{J} + \dfrac{\partial \rho}{\partial t} = 0$, where $\bar{J} = \rho \bar{v}$.

[*Hints.* Let M be the mass enclosed in a volume V with surface area S. $M = \iiint\limits_V \rho\, dV$. Time rate of increase of mass is $\dfrac{\partial M}{\partial t} = \iiint\limits_V \dfrac{\partial \rho}{\partial t}\, dV$. Mass of fluid leaving V per unit time $= \iint\limits_S \rho \bar{v} \cdot \bar{n}\, dS$.

$\therefore$ the increase of mass per unit time $= -\iint\limits_S \rho \bar{v} \cdot \bar{n}\, dS$.

$= -\iiint\limits_V \nabla \cdot (\rho \bar{v})\, dV$ by the divergence theorem.

$$\therefore \quad \iiint\limits_V \frac{\partial \rho}{\partial t}\, dV = -\iiint\limits_V \nabla \cdot (\rho \bar{v})\, dV$$

$$\text{or,} \quad \iiint\limits_V \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) \right\} dV = 0.$$

This is an identity.

$$\therefore \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0.]$$

ANSWERS

1. (i) $\frac{11}{2}\bar{i} - 7\bar{j} - 6\bar{k}$ (ii) $-9\bar{i} - 6\bar{k}$ (iii) $\frac{7}{10}$ (iv) $\frac{8}{11}\bar{i} + \frac{4}{5}\bar{j} + \bar{k}$.

2. $10\frac{1}{2}$ 3. (i) 0 (ii) $\frac{1}{12}$ (iii) $-\frac{1}{6}$. 4. $\frac{1}{2}$.

9. (i) $\frac{3}{8}\pi a^2$ (ii) a^2.

17. $-\frac{1}{6}(2x^3 + 3x^2 - 6xy^2 - 3y^2)$. 18. $xy^2 + xz^3 - 5yz$.

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SHORT QUESTIONS

1. The point $(2, 3)$ is transformed to $(3, 2)$ under a reflection about a line, find the equation of the line. [*Ans.* $y = x$]
2. Find the equations of the rigid motion that transforms $x^2 + 2x + y^2 - 10y + 25 = 0$ to a circle about the new origin as centre. $\left[\textit{Ans. } x'^2 + y'^2 = 26\right]$
3. Show that $x^2 + y^2$ remains unchanged under rotation of the xy- plane.
4. The origin is shifted to the point $(3, -1)$ and the axes are turned through an angle $\tan^{-1}\frac{3}{4}$. If the coordinates of a point are $(5, 10)$ in the new system, find its old coordinates. [*Ans.* $(1, 10)$]
5. Rotating the rectangular axes through the angle $\frac{\pi}{4}$, determine the eccentricity of the conic $xy = c^2$. $\left[\textit{Ans. } e = \sqrt{2}\right]$
6. Find the equation of the line $\frac{x}{a} + \frac{y}{b} = 2$, when the origin is transferred to the point (a, b). $\left[\textit{Ans. } \frac{x'}{a} + \frac{y'}{b} = 0\right]$
7. Find the equation of the curve $3x^2 + 3y^2 + 6x - 18y = 14$ referred to parallel axes through the point $(-1, 3)$. [*Ans.* $3x^2 + 3y^2 + 4x - 46 = 0$]
8. Show that the equation $2x^2 + 3xy + y^2 = 0$ represents a pair of lines. Find the angle between them. [*Ans.* $x + y = 0, 2x + y = 0$, angle $= \tan^{-1}\frac{1}{3}$]
9. Find the form of equation of the straight line $x\cos\alpha + y\sin\alpha = p$, when the axes are rotated through an angle α in the anticlockwise sense. [*Ans.* $x' = p$]
10. For what value of a, the transformation $x' = -x + 2, y' = ay + 3$ is a translation. [*Ans.* $a = \pm 1$]
11. The coordinates of two points are $(1, -2)$ and $\left(1 + 3\sqrt{3}, 1\right)$. The origin is shifted to $(1, -2)$ and the new x-axis is the line joining the given points. Find the formula for the rigid motion. $\left[\textit{Ans. } x = x'\frac{\sqrt{3}}{2} - y'\frac{1}{2} + 1 \text{ and } y = x'\frac{1}{2} + y'\frac{\sqrt{3}}{2} - 2\right]$
12. Is the transformation $x^4 = x, y^4 = y$ a rigid motion? [*Ans.* No. It is not one-one mapping]
13. Find the transformed equation of the curve $(4x + 3y + 1)(3x - 4y + 2) = 75$, when the axes are $4x + 3y + 1 = 0$ and $3x - 4y + 2 = 0$. [*Ans.* $x'y' = 3$]
14. Find the translation which transforms the equation $x^2 + y^2 - 2x + 14y + 20 = 0$ into $x'^2 + y'^2 - 30 = 0$. [*Ans.* $x = x' + 1, y = y' - 7$]
15. Find the angle of rotation of the coordinate axes about the origin which will transform the equation $x^2 - y^2 = 4$ to $x'y' = 2$. [*Ans.* $\theta = \frac{3\pi}{4}$]
16. The coordinate axes are rotated through an angle 45° if the transformed coordinates of a point are $\left(0, \sqrt{2}\right)$, find its original coordinates. [*Ans.* $(1, 1)$]
17. Find the equation of the conic with focus at the pole, eccentricity $e = 2$, directrix $r\cos\theta + 5 = 0$. $\left[\textit{Ans. } \frac{10}{r} = 1 - 2\cos\theta\right]$
18. Write down the polar equation of the straight line $x = 0$. $\left[\textit{Ans. } \theta = \frac{\pi}{2}\right]$
19. The three vertices of a rhombus, taken in order are $(2, -1), (3, 4)$ and $(-2, 3)$. Find the fourth vertex. [*Ans.* $(-3, -2)$]
20. The centroid of a triangle is $(1, 4)$ and the coordinates of its two vertices are $(4, -3)$ and $(-9, 7)$. Find the area of the triangle. [*Ans.* $\frac{183}{2}$ sq units]
21. The point A divides the join of $P(-5, 1)$ and $Q(3, 5)$ in the ratio $k : 1$. Find the two values of k for which the area of ΔABC, where B is $(1, 5)$ and C is $(4, -5)$ is equal to 2 units. $\left[\textit{Ans. } \frac{19}{4}, \frac{17}{6}\right]$
22. The equation of the sides of a triangle are $y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$. Prove that the area is $\frac{1}{2}\frac{(c_1 - c_2)^2}{m_1 - m_2}$.
23. A line L intersects the three sides BC, CA and AB of a triangle ABC at P, Q, R respectively. Show that $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1$.

24. Prove that that points $(a, 0)$, $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are collinear, if $t_1t_2 + 1 = 0$.

25. The vertices of a triangle are $\{at_1t_2, a(t_1 + t_2)\}$, $\{at_2t_3, a(t_2 + t_3)\}$, $\{at_3t_1, a(t_3 + t_1)\}$. Find the coordinates of the orthocentre.
[*Ans.* $\{-a, a(t_1 + t_2 + t_3 + t_1t_2t_3)\}$]

26. For what value of λ does the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ represent a pair of lines. [*Ans.* $\lambda = 2$]

27. Prove that $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ represents two parallel lines and find the distance between them. [*Ans.* $\frac{7}{2\sqrt{5}}$]

28. Prove that the lines $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$ and $ax + by + c = 0$ form an equilateral triangle whose area is $c^2/(a^2 + b^2)\sqrt{3}$.

29. Prove that the lines $2x^2 + 6xy + y^2 = 0$ are inclined at the same angle to the lines $4x^2 + 18xy + y^2 = 0$.

30. The straight lines passing through the point $(1, 0)$ intersect the curve $2x^2 + 5y^2 - 7x = 0$ at two points. Prove that the portion of the curve between these two points subtend a right angle at the origin.

31. Prove that the two lines $(x_1y - y_1x)^2 = a^2(x^2 + y^2)$ are at a distance a from the point (x_1, y_1).

32. What is meant by rigid motion?

33. What is the effect of an orthogonal transformation on the degree of an equation?
[*Ans.* Degree remains the same]

34. Can you give the axes a rotation by means of which $lx + my + n = 0$ can be made to pass through the origin? [*Ans.* It is not possible]

35. What is the rank of a second degree curve? How are they classified?

36. Write down the polar equation of a line perpendicular to the initial line and at a distance of 5 units from the pole. [*Ans.* $r\cos\theta = 5$]

37. Find the equation of one directrix of the ellipse $4x^2 + 3y^2 = 24$. [*Ans.* $y = 4\sqrt{2}$]

38. Find the polar equation to the line parallel to the initial line at at a distance of 2 units from the pole. [*Ans.* $r\sin\theta = 2$]

39. Write down the polar equation of the circle of radius 2 units with centre on the initial line at a distance of 2 units from the pole on the positive size.[*Ans.* $r = 4\cos\theta$]

40. Find the coordinates of the focus of the parabola $x^2 + y + 1 = 0$. [*Ans.* $(0, -\frac{5}{2})$]

41. Find the parametric equation of the parabola $(y - 1)^2 = 4(x + 2)$.
[*Ans.* $x = t^2 - 2, y = 2t + 1$]

42. Find the point on the conic $\frac{8}{r} = 3 - \sqrt{2}\cos\theta$ whose radius vector is 4.[*Ans.* $(4, \pm\frac{\pi}{4})$]

43. Find the pole of the line $x + y + 3 = 0$ w.r.t. the circle $x^2 + y^2 - 2x + 5 = 0$.[*Ans.* $(2, 1)$]

44. Find the polar equation of the circle which passes through the pole and two points $(d, 0)$ and $(2d, \frac{\pi}{2})$. [*Ans.* $r = d(\cos\theta + 2\sin\theta)$]

45. Find the polar coordinates of the centre of the circle $r = 12\cos\theta + 5\sin\theta$.
[*Ans.* $(\frac{13}{2}, \tan^{-1}\frac{5}{12})$]

46. Find the polar equation of the left branch of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
[*Ans.* $\frac{9}{r} = 4 + 5 + \cos\theta$]

47. In which quadrant does the focus of each of the parabolas $(y - 2)^2 = 4(x + 1)$ and $(x + 3)^2 = 4(y - 1)$ lie? [*Ans.* foci $(0, 2)$ and $(-3, 2)$]

48. Determine the nature of the conic $r = \frac{1}{4 - 5\cos\theta}$ and the length of the latus rectum.
[*Ans.* hyperbola, length of latus rectum $= \frac{1}{2}$]

49. Test whether the circles $x^2 + y^2 - 2bx + c = 0$ and $x^2 + y^2 + 2ay - c = 0$ cut orthogonally. [*Ans.* Since $2b \cdot 0 + 2a \cdot 0 = 0$ and $c - c = 0$, the circles cut orthogonally]

50. Find the eccentricity and the vertex of $r = \frac{6}{1 - \cos\theta}$. [*Ans.* $e = 1, (3, \pi)$]

51. If $\left(5, \frac{\pi}{4}\right)$ and $\left(8, -\frac{\pi}{12}\right)$ are the polar coordinates of two adjacent vertices of a square, find its area. [*Ans.* 49 square units]

52. Find the coordinates of one end of latus rectum for $\frac{l}{r} = 1 + \cos\theta$ and $\frac{l}{r} = 1 - e\cos\theta$. [*Ans.* $\left(l, \frac{5\pi}{4}\right)$ and $\left(l, \frac{\pi}{4}\right)$]

53. Find the area of the triangle formed by pair of tangents from $(5, 0)$ to $x^2 + y^2 = 4^2$ and the chord of contact. [*Ans.* $\frac{9}{2}$ square units]

54. What is the polar of a central conic for the pole as centre. [*Ans.* No real existence]

55. The forms of equations of tangent, chord of contact and polar w.r.t. a point (x_1, y_1) are alike. Are they identical? Justify your answer.

[*Ans.* For the tangent the point is on the conic. Chord of contact and polar are identical, when the point is outside the conic. If the point is inside the conic, there is no existence of the chord of contact.]

56. What is the polar equation of a circle when the pole is at the centre. [*Ans.* $r = a$, where a is the radius]

57. Show that the diameter of a parabola is parallel to its axis.

58. What do you mean by co-normal points?

59. Show that the feet of the normals to a parabola drawn from a point outside the conic lie on a hyperbola.

60. Show that the points of intersection of perpendicular tangents to a parabola lie on the directrix.

61. What is the director circle? Find the director circle for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [*Ans.* $x^2 + y^2 = a^2 + b^2$]

62. What are the limiting points of $x^2 + y^2 + 2fy - c = 0$? [*Ans.* $(0, \sqrt{-c}), (0, -\sqrt{-c})$]

63. Show that the sum of the squares of two conjugate semi-diameters of an ellipse is constant.

64. Find the projection of the line segment joining the points $(2, 3, 4)$ and $(4, 4, 2)$ on the straight line joining the points $(2, 1, 2)$ and $(4, 3, 1)$. [*Ans.* $\frac{8}{5}$]

65. Find the area of the triangle whose vertices are the points $(1, 2, 3), (-2, 1, 5), (3, 4, 2)$. [*Ans.* $\frac{\sqrt{13}}{2}$ sq units]

66. Do the planes $x = a, y = b$ and $z = c$ meet at a point? [*Ans.* yes, (a, b, c)]

67. Find the ratio and the coordinates of the points $(2, 4, 5)$ and $(3, 5, -4)$ divided by the xy-plane. [*Ans.* $5 : 4, \left(\frac{23}{9}, \frac{41}{9}, 0\right)$]

68. Show that the planes given by $2x + 3y - z = 0, 3x + 3y + z = 2, x - y + 2z = 5$ intersect at a point. [*Ans.* $(4, -3, -1)$]

69. Find the intercepts made on the coordinate axes by the plane $2x + 2y - z = 12$ and the direction cosines of the normal to the plane. [*Ans.* $6, 6, 12$; d.cs. are $\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$]

70. Find the direction cosines of the line which is equally inclined to the axes. [*Ans.* $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$]

71. If α, β, γ be the angles which a line makes with the coordinate axes, then show that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

72. If the three concurrent lines whose d.cs. are $l_1, m_1, n_1; l_2, m_2, n_2$ and l_3, m_3, n_3 are coplanar, prove that $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$.

73. A line makes angle $45°$ with the positive direction of x- and y-axes. Show that it makes angle $90°$ with the positive direction of z-axis.

74. The foot of the perpendicular from the origin to a plane is $(1, -3, 1)$. Find the equation of the plane.

75. Whether the straight lines $\frac{x-1}{3} = \frac{y-2}{3} = \frac{z-3}{4}$ and $4x - 3y + 1 = 0 = 5x - 3z + 2$ are skew or coplanar? [*Ans.* skew lines]

76. Determine the value of k such that the lines $\frac{x-1}{2} = \frac{y-4}{1} = \frac{z-k}{2}$ and $\frac{x-2}{-1} = \frac{y-8}{k} = \frac{z-11}{4}$ may intersect. $\left[Ans.\ \frac{19 \pm \sqrt{161}}{4}\right]$

77. Show that $\frac{x+2}{2} = \frac{y}{3} = \frac{z-1}{-2}$ is a generator of $\frac{x^2}{4} - \frac{y^2}{9} = z$.

78. Find the coordinates of the point equidistant from the four points $(a,0,0), (0,b,0)$, $(0,0,c)$ and $(0,0,0)$. $\left[Ans.\ \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)\right]$

79. Verify that the plane $2x - 2y + 3z = 9$ touches the ellipsoid $x^2 + 2y^2 + 3z^2 = 9$.

80. Find the enveloping cone of the sphere $x^2 + y^2 + z^2 = 25$ with the vertex at $(10,0,0)$. $[Ans.\ 3(x^2 + y^2 + z^2 - 25) = (2x - 5)^2]$

81. Find the equation of the cone whose vertex is origin and base is the circle $x = a, y^2 + z^2 = b^2$. $[Ans.\ a^2(y^2 + z^2) = b^2x^2]$

82. Find the value of k for which the plane $x + kz - 2 = 0$ intersects $x^2 + y^2 - z^2 = -1$ is a hyperbola. $[Ans.\ -1 < k < 1]$

83. Explain that a right circular cone is a surface generated by a line under certain conditions.

84. Write down the equation of the plane containing the lines $x + 2 = 0 = z$ and $z = 0 = y$. $[Ans.\ z = 0]$

85. What type of surface does the following equation $x^2 + 3y^2 - 2z^2 = 0$ represent? [*Ans.* Cone with vertex at the origin.]

86. Find the centre and radius of the circle represented by $x^2 + y^2 + z^2 = 1, x + y = z$. [*Ans.* great circle, radius = 1, centre $(0,0,0)$]

87. Find the points in which the $\frac{x-1}{-1} = \frac{y+1}{2} = \frac{z+2}{3}$ intersect the coordinate axes. [*Ans.* The given line is not coplanar with any coordinate axes]

88. Find the shortest distance between the lines $x = 0, 2y - 3z = 0$ and $x = 3, y = 0$ in space. [*Ans.* s.d. = 3 units]

89. Obtain the equation of the cylinder whose generators intersect the plane curve $ax^2 + by^2 = 1, z = 0$ and are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. $[Ans.\ a(nx - lz)^2 + b(ny - mz)^2 = n^2]$

90. Find the equation of the surface of revolution whose generatix is $x^2 - 4z = 0, y = 0$ and the axis is the z-axis. $[Ans.\ x^2 + y^2 - 4z = 0]$

91. What is a ruled surface? Prove that hyperboloid of one sheet is a ruled surface.

92. What surface is represented by $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$. [*Ans.* sphere]

93. Find the cylindrical coordinates of the point whose cartesian coordinates are $(1, \sqrt{3}, 2)$. $\left[Ans.\ (2, \frac{\pi}{3}, 2)\right]$

94. What is the difference between the equations of circles in two dimension and in three dimensions?

95. What relation exists between a vector $\bar{r}$ and two other non-collinear vectors $\bar{\alpha}, \bar{\beta}$ lying on the same plane? Is the relation unique? What is the name given to this type of relation?

[*Ans.* The relation is $\bar{r} = x\bar{\alpha} + y\bar{\beta}$, where x and y are scalars. Yes. Linear combination.]

96. Show that the sum of three vectors determined by the medians of a triangle directed from the vertices is a zero vector.

97. Show that the figure formed by joining the middle points of the sides of quadrilateral taken in order is a parallelogram.

98. Determine the values of λ and μ for which the vectors $-3\bar{i} + 4\bar{j} + \lambda\bar{k}$ and $\mu\bar{i} + 8\bar{j} + 6\bar{k}$ are collinear. $[Ans.\ \lambda = 3, \mu = -6]$

99. The sides of a parallelogram are $2\bar{i} + 4\bar{j} - 5\bar{k}$ and $\bar{i} + 2\bar{j} + 3\bar{k}$. Find the unit vectors parallel to the diagonals. $\left[Ans.\ \frac{3\bar{i} + 6\bar{j} - 2\bar{k}}{7} \text{ and } \frac{-\bar{i} - 2\bar{j} + 8\bar{k}}{\sqrt{69}}\right]$

100. What is the unit vector parallel to $\bar{a} = 3\bar{i} + 4\bar{j} - 2\bar{k}$? What vector should be added to $\bar{a}$, so that the resultant is unit vector $\bar{i}$? $\left[Ans.\ \frac{3\bar{i}+4\bar{j}-2\bar{k}}{\sqrt{29}}, -2\bar{i} - 4\bar{j} + 2\bar{k}\right]$

101. A particle, in equilibrium, is subjected to four forces $\bar{F}_1 = -10\bar{k}$, $\bar{F}_2 = U\left(\frac{4}{13}\bar{i} - \frac{12}{13}\bar{j} + \frac{3}{13}\bar{k}\right), \bar{F}_3 = V\left(-\frac{4}{13}\bar{i} - \frac{12}{13}\bar{j} + \frac{3}{13}\bar{k}\right), \bar{F}_4 = W(\cos\theta\bar{i} + \sin\theta\bar{j})$. Solve for U, V and W as function of θ.

$\left[Ans.\ U = \frac{65}{3}(1 - 3\cot\theta), V = \frac{65}{3}(1 + 3\cot\theta), W = 40\operatorname{cosec}\theta\right]$

102. If $l_1\bar{a} + m_1\bar{b} = l_2\bar{a} + m_2\bar{b}$, where $\bar{a}$ and $\bar{b}$ are non-collinear vectors, then prove that $l_1 = l_2$ and $m_1 = m_2$.

103. Find the scalar product and the cosine of the angle between the vectors $\bar{i} + 3\bar{j} + 2\bar{k}$ and $2\bar{i} - 4\bar{j} + \bar{k}$. $\left[Ans.\ -8, \cos^{-1}\frac{-8}{7\sqrt{6}}\right]$

104. Prove that $\bar{\alpha}\cdot\bar{\beta} \le |\bar{\alpha}|\,|\bar{\beta}|$.

105. Is vector product commutative? What is the geometrical significance of $|\bar{a} \times \bar{\beta}|$?

106. If $|\bar{a} + \bar{b}| = |\bar{a} - \bar{b}|$, show that $\bar{a}$ is perpendicular to $\bar{b}$.

107. Prove that $\left(\frac{\bar{a}}{\bar{a}^2} - \frac{\bar{b}}{\bar{b}^2}\right) = \left(\frac{\bar{a}-\bar{b}}{|\bar{a}||\bar{b}|}\right)^2$.

108. Find the unit vector perpendicular to each of the vectors $\bar{i} + \bar{j} + \bar{k}$ and $2\bar{i} + 3\bar{j} - \bar{k}$. Find the sine of the angle between the vectors. $\left[Ans.\ \frac{-4\bar{i}+3\bar{j}+\bar{k}}{\sqrt{26}}, \sin\theta = \sqrt{\frac{13}{21}}\right]$

109. Is the vector product commutative? What is the geometrical significance of $|\bar{\alpha} \times \bar{\beta}|$?

110. What is the value of $(\bar{\alpha}\cdot\bar{\beta}) \times \bar{\gamma}^2$. [*Ans.* It is not defined]

111. Show that $|\bar{\alpha} \times \bar{\beta}|^2 + |\bar{\alpha}\cdot\bar{\beta}|^2 = |\bar{\alpha}|^2|\bar{\beta}|^2$.

112. If $\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ and $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$, show that $\bar{a} - \bar{d}$ is parallel to $\bar{b} - \bar{c}$.

113. If $A_1, A_2, \ldots, A_n$ are the vertices of a regular polygon with n sides and O is the centre, show that

$$\sum_{i=1}^{n-1}\left(\overrightarrow{OA}_i \times \overrightarrow{OA}_{i-n}\right) = (1-n)\left(\overrightarrow{OA}_2 \times \overrightarrow{OA}_1\right).$$

114. $[\bar{i} + \bar{j}, \bar{j} + \bar{k}, \bar{k} + \bar{i}] = 2$.

115. If $\bar{a}, \bar{b}, \bar{c}$ are position vectors of A, B, C, prove that $\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}$ is perpendicular to the plane of $\triangle ABC$.

116. If $\bar{a}, \bar{b}, \bar{c}$ and $\bar{a}', \bar{b}', \bar{c}'$ are reciprocal system of vectors, then prove that $\bar{a}' \times \bar{b}' + \bar{b}' \times \bar{c}' + \bar{c}' \times \bar{a}' = \frac{\bar{a}+\bar{b}+\bar{c}}{[\bar{a}\bar{b}\bar{c}]}$.

117. The vertices of a triangle are $A(4,2,3)$, $B(1,3,1)$ and $C(-5,5,-2)$. Find the length of the perpendicular drawn from the origin to the plane of $\triangle ABC$. $\left[Ans.\ \sqrt{10}\text{ units.}\right]$

118. If C is the middle point of AB and P is any point outside AB, then show that $\overrightarrow{PA} + \overrightarrow{PB} = 2\overrightarrow{PC}$.

119. Let $\bar{a} = 2\bar{i}+\bar{j}+\bar{k}, \bar{b} = \bar{i}+2\bar{j}+\bar{k}$ and $\bar{c} = 2\bar{i}-3\bar{j}+4\bar{k}$. A vector $\bar{r}$ satisfying $\bar{r}\times\bar{b} = \bar{c}\times\bar{b}$ and $\bar{r}\cdot\bar{a} = 0$. Show that $\bar{r} = \bar{i} - 5\bar{j} + 3\bar{k}$.

120. If $\bar{A} = t^2\bar{i} - t\bar{j} + (2t+1)\bar{k}$ and $\bar{B} = (2t-3)\bar{i} + \bar{j} - t\bar{k}$, find $\frac{d}{dt}\left(\bar{A} \times \frac{d\bar{B}}{dt}\right)$.

$\left[Ans.\ \bar{i} + 6\bar{j} + 2\bar{k}\right]$

121. If $\bar{A} = \cos(xy)\bar{i} + (3xy - 2x^2)\bar{j} - (3x+2y)\bar{k}$, show that $\frac{\partial^2\bar{A}}{\partial x\partial y} = \frac{\partial^2\bar{A}}{\partial y\partial x} = -(xy\cos xy + \sin xy)\bar{i} + 3\bar{j}$.

122. Find the equations for the tangent plane and normal line to the surface $4z = x^2 - y^2$ at the point $(3,1,1)$. $[Ans.\ 3x - y - 2z = 4; x = 3t + 3, y = 1 - t, z = 2 - 2t]$

123. Find $\nabla\phi$ if $\phi = \log|\bar{r}|$. $\left[Ans.\ \frac{\bar{r}}{r^2}\right]$

124. If $\bar{A} = xz^3\bar{i} - 2x^2yz\bar{j} + 2yz^4\bar{k}$, find $\nabla \times \bar{A}$ at the point $(1,-1,1)$. $[Ans.\ 3\bar{j} + 4\bar{k}]$

125. Show that $\nabla\cdot(\nabla \times \bar{A}) = 0$.

126. If $\bar{v} = \bar{w} \times \bar{r}$, prove that $\bar{w} = \frac{1}{2}\nabla \times \bar{v}$, where $\bar{w}$ is a constant vector.

127. Find the directional derivative of $P = 4e^{2x-y+z}$ at the point $(1,1,-1)$ in a direction toward the point $(-3,5,6)$. [*Ans.* $-\frac{20}{9}$]

128. If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$, where C is the curve in the xy-plane $y = 2x^2$ from $(0,0)$ to $(1,2)$. [*Ans.* $-\frac{7}{6}$]

129. Show that $\bar{F} = (2xy + z^3)\,\bar{i} + x^2\bar{j} + 3xz^2\bar{k}$ is a conservative force field and find the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

[*Ans.* $\nabla \times \bar{F} = \bar{0}$ and 202.]

130. If the section of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ by $z = k$ is real, find the range of k. [*Ans.* $-c < k < c$]

131. Does the plane $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$, pass through a fixed point, the plane passes through a fixed point $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{k}$?

[*Ans.* (k,k,k)]

132. Find the shortest distance between the lines $x + y = 1, z = 0$ and $x = 0 = y$.

[*Ans.* $\frac{1}{\sqrt{2}}$]

133. What type of surface does the following equation $y^2 + x^2 - 2y = 0$ represent?

[*Ans.* circle, centre $(0,1)$, radius $= 1$]

134. Find the common point of the planes $x = 1, y = 2$ and $z = -1$. [*Ans.* $(1,2,-1)$]

135. Write down the equation of the plane containing the lines $x + 2 = 0 = z$ and $z = 0 = y$. [*Ans.* $z = 0$]

136. What is represented by $r^2 - 2(a\cos\theta + b\sin\theta) + c = 0$, where a, b, c are constants such that $a^2 + b^2 - c > 0$. [*Ans.* circle]

137. Determine whether the equation $y^2+z^2-2y = 0$ represents a right circular cylinder?

[*Ans.* $(y-1)^2 + z^2 = 1$]

138. Justify $(\bar{r} \times \bar{c}) + 2\bar{c} = \bar{0}$ where $\bar{r} = \bar{i}x + \bar{j}y + \bar{k}z$ and $\bar{c} = \bar{i}c_1 + \bar{j}c_2 + \bar{k}c_3$.

139. $f = yz + zx + xy$. Show that the normal derivative at $(1,1,3)$ is 6.

140. Find $\int \bar{F}\cdot d\bar{r}$ along the line joining $(0,0,0)$ to $(1,1,1)$, where $\bar{F} = \bar{i}(y^2 + z^2) + \bar{j}(z^2 + x^2) + \bar{k}(z^2 + x^2)$. [*Ans.* 2]

141. Prove that $\nabla\cdot(\bar{a} \times \bar{u}) = -\bar{a}\cdot\nabla\times\bar{u}$.

142. Through what angle is the frame to be rotated about the origin so that $3x^2 - 5xy + 3y^2 = 1$ may not contain xy term. [*Ans.* $\frac{\pi}{2}$ or $-\frac{\pi}{2}$]

143. If $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors and $\bar{d}$ is a vector with $\bar{d}\cdot\bar{a} = \bar{d}\cdot\bar{b} = \bar{d}\cdot\bar{c} = 0$, then prove that $\bar{d} = \bar{0}$.

144. Find the angle between $\bar{r}\cdot(2\bar{i} + 3\bar{j} + \bar{k}) = 7$ and $\bar{r}\cdot(3\bar{i} - 2\bar{j} + 5\bar{k}) = 5$.

[*Ans.* $\cos^{-1}\frac{5}{\sqrt{14}\sqrt{38}}$]

145. Find the value of k for which the plane $x + kz = 2$ intersects $x^2 + y^2 - z^2 + 1 = 0$ in a hyperbola. [*Ans.* $-1 < k < 1$]

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