Teorija elastičnosti
Svojstvena zadaća za žicu. Evolucijski problem.

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1 Uvod

Inicijalno rubna zadaća napete žice:

$$\begin{aligned}
\rho(x)u_{tt}(x,t) - (a(x)u_x(x,t))_x + b(x)u(x,t) &= f(x,t) \\
u(0,t) &= 0 \\
u_x(l,t) &= 0 \\
u(x,0) &= u_0(x) \\
u_t(x,0) &= u_1(x)
\end{aligned}$$

Tražimo rješenje u separiranim varijablama

$$u(x,t) = X(x)T(t)$$

Slijedi:

$$\begin{aligned}
-(a(x)X'(x))' + b(x)X(x) &= \lambda\rho(x)X(x) \\
X(0) &= 0 \\
X'(l) &= 0
\end{aligned}$$

$$T''(t) + \lambda T(t) = f(t)$$

Trebamo rješiti svojstvenu zadaću te ODJ 2.reda. Svojstvena zadaća se svodi na sljedeći problem: Naći $\lambda \in \mathbb{R}$ i $u \neq 0$ td.

$$\int_0^l (au'v' + buv) = \lambda \int_0^l \rho uv.$$

Stavimo $B(u,v) = \int_0^l (au'v' + buv)$ i $C(u,v) = \int_0^l \rho uv$, te slično kao i na predavanjima raspišemo B i C po baznim funkcijama te dobijemo sljedeću generaliziranu svojstvenu zadaću: Naći $\lambda \in \mathbb{R}$ i $X \neq 0$ td.

$$AX = \lambda MX,$$

gdje je $A = (B(\varphi_i,\varphi_j))_{i,j},\ M = (B(\varphi_i,\varphi_j))_{i,j},\ X = (\alpha_i)_i$.

2 Metoda konačnih elemenata za svojstvenu zadaću

Fiksirajmo $n \in \mathbb{N}$, $h = \frac{l}{n}$. Stavimo $B_1(u,v) = \int_0^l au'v'$, $B_2(u,v) = \int_0^l buv$. Dobivamo sljedeće:

$$
\begin{aligned}
B_1(\varphi_i, \varphi_i) &= \frac{1}{2h}(a(x_{i-1}) + 2a(x_i) + a(x_{i+1})) \\
B_1(\varphi_1, \varphi_1) &= \frac{1}{2h}(a(x_2) + 2a(x_1)) \\
B_1(\varphi_n, \varphi_n) &= \frac{1}{2h}(a(x_n) + 2a(x_{n-1})) \\
B_1(\varphi_i, \varphi_{i+1}) &= -\frac{1}{2h}(a(x_i) + a(x_{i+1})) \\
B_2(\varphi_i, \varphi_i) &= hb(x_i) \\
B_2(\varphi_i, \varphi_{i+1}) &= 0 \\
B_2(\varphi_1, \varphi_1) &= \frac{h}{2}b(x_1) \\
B_2(\varphi_n, \varphi_n) &= \frac{h}{2}b(x_n) \\
C(\varphi_i, \varphi_i) &= h\rho(x_i) \\
C(\varphi_i, \varphi_{i+1}) &= 0 \\
C(\varphi_1, \varphi_1) &= \frac{h}{2}\rho(x_1) \\
C(\varphi_n, \varphi_n) &= \frac{h}{2}\rho(x_n)
\end{aligned}
$$

3 Rezultati

Imamo sljedeću diskretizaciju prostorne i vremenske varijable: uzimamo $t \in [0, T]$, gdje je $T = 1$ sekunda i diskretiziramo sa korakom $\delta t = 0.001$. Prostornu varijablu diskretiziramo na $n + 1$ djelova sa korakom $h = \frac{l}{n}$, gdje je $x \in [0, l]$, i stavljamo $l = 1$. Što je n veći dobivamo "lijepše" rješenje.

3.1 Prvi primjer - usporedba sa analitičkim rješenjem

Rezultate sam testirala na valnoj jednadžbi sa Dirichelovim rubnim uvjetima. Uz $\rho = 1, a = 1, b = 0, f = 0$ dobivam sljedeću jednadžbu:

$$u_{tt} = u_{xx}.$$

Uzimamo sljedeće početne uvjete.

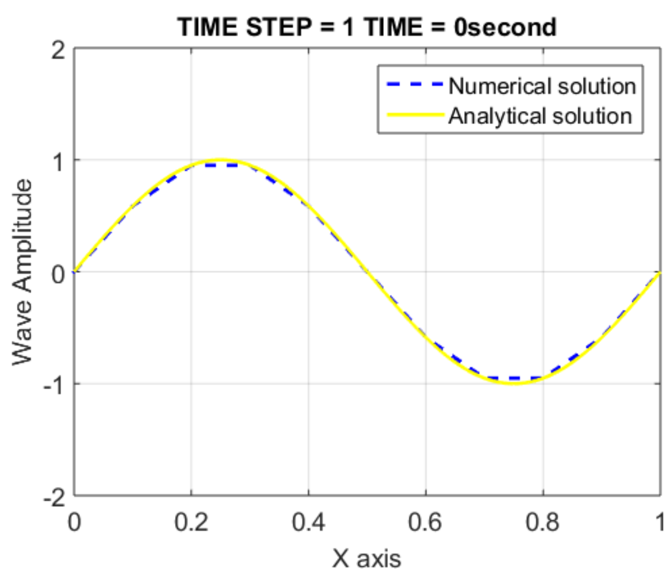
$$u(x, 0) = \sin(2\pi x)$$
$$u_t(x, 0) = 0.$$

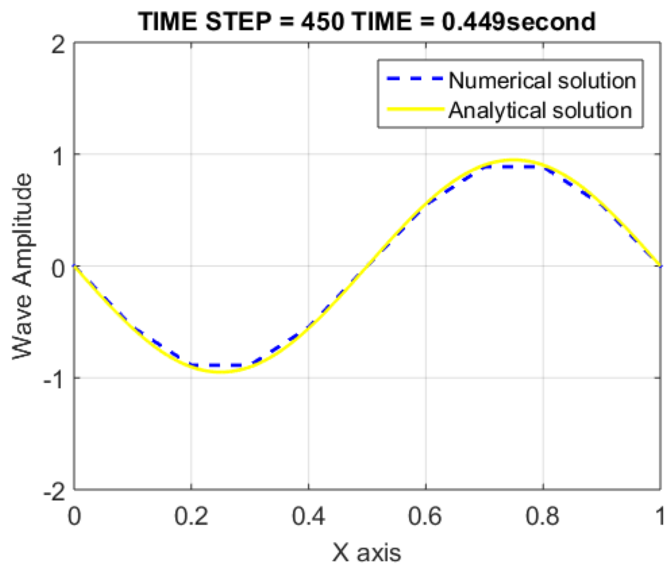
Analitičko rješenje valne jednadžbe u tom slučaju je:

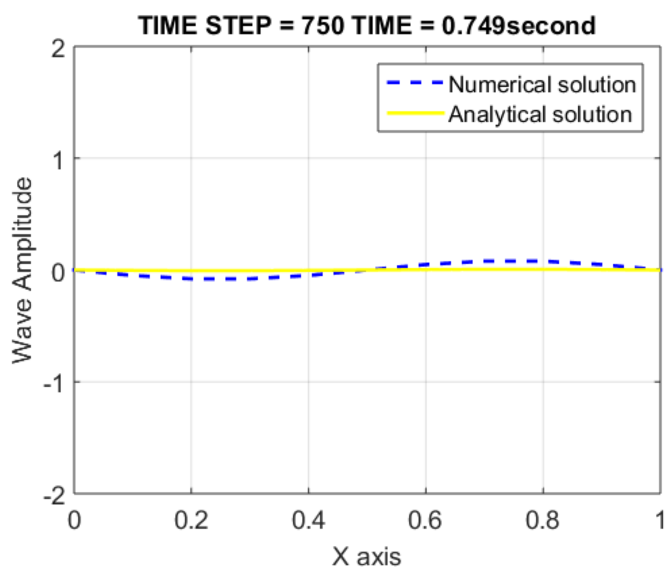
$$\frac{1}{2}(\sin(2\pi(x - t)) + \sin(2\pi(x + t))).$$

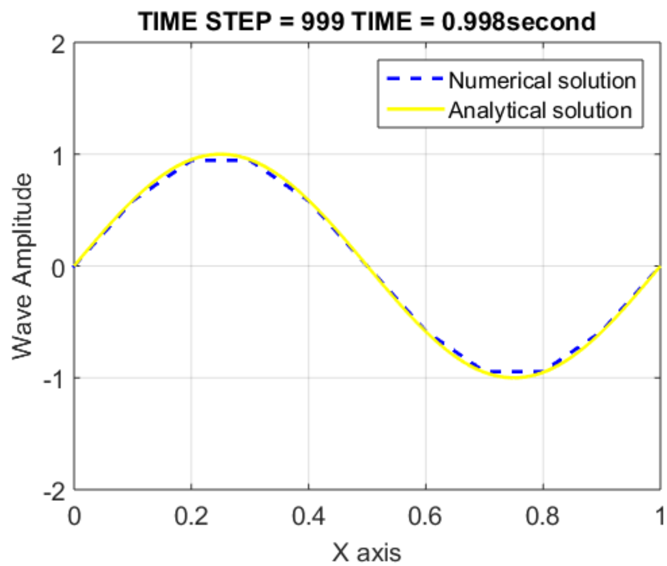
Usporedit ćemo numeričko i analitičko rješenje za $n = 10$ i $n = 100$.

Za $n = 10$ imamo:

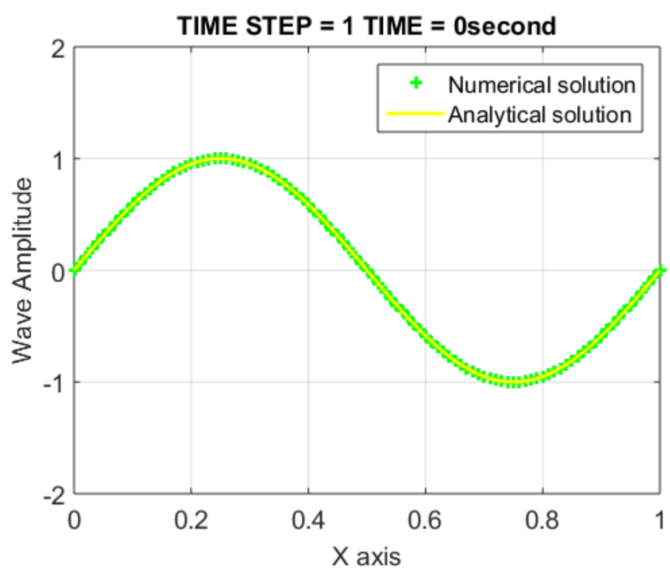


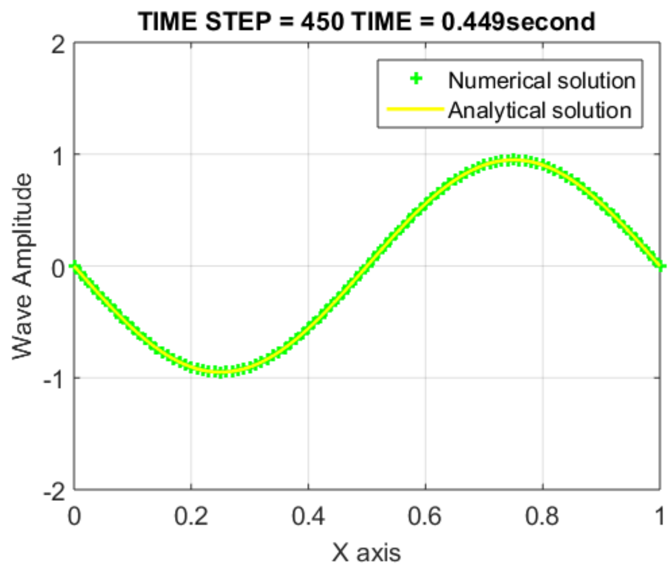

TIME STEP = 450 TIME = 0.449second
Numerical solution
Analytical solution
Wave Amplitude
X axis
2
1
0
-1
-2
0
0.2
0.4
0.6
0.8
1

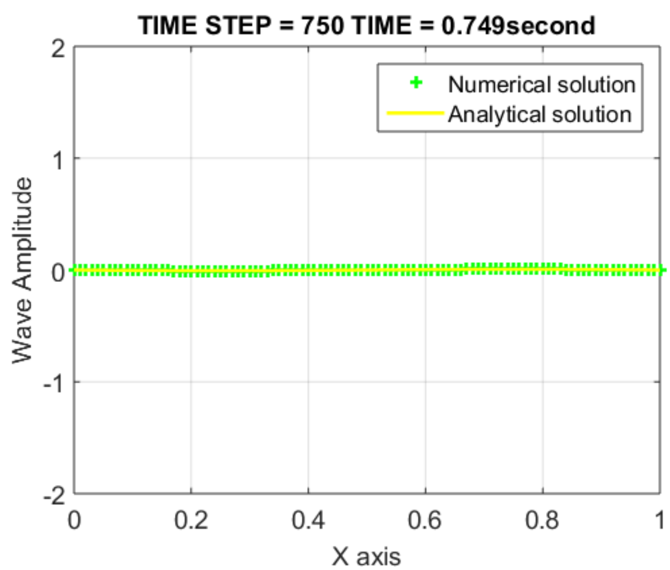

TIME STEP = 750 TIME = 0.749second
Numerical solution
Analytical solution
Wave Amplitude
X axis
2
1
0
-1
-2
0
0.2
0.4
0.6
0.8
1

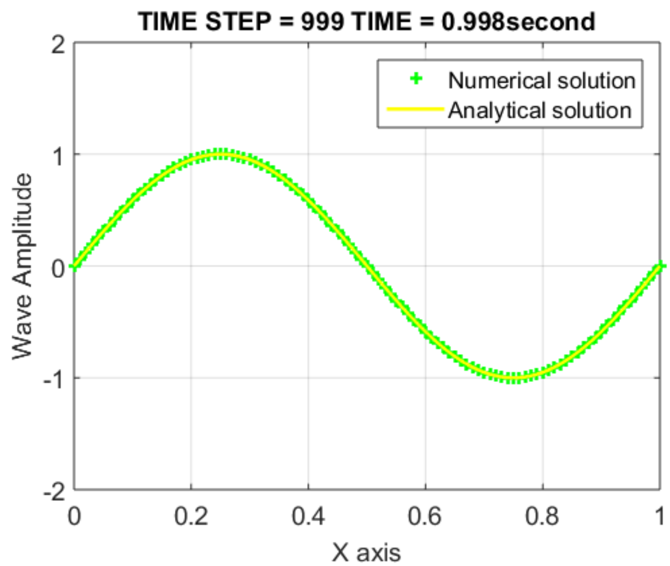


Za $n = 100$ imamo:



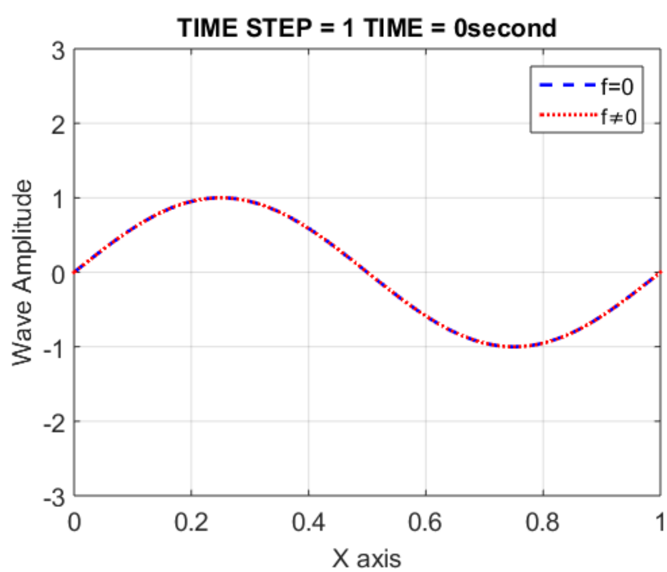

TIME STEP = 450 TIME = 0.449second
Numerical solution
Analytical solution
Wave Amplitude
X axis
2
1
0
-1
-2
0
0.2
0.4
0.6
0.8
1

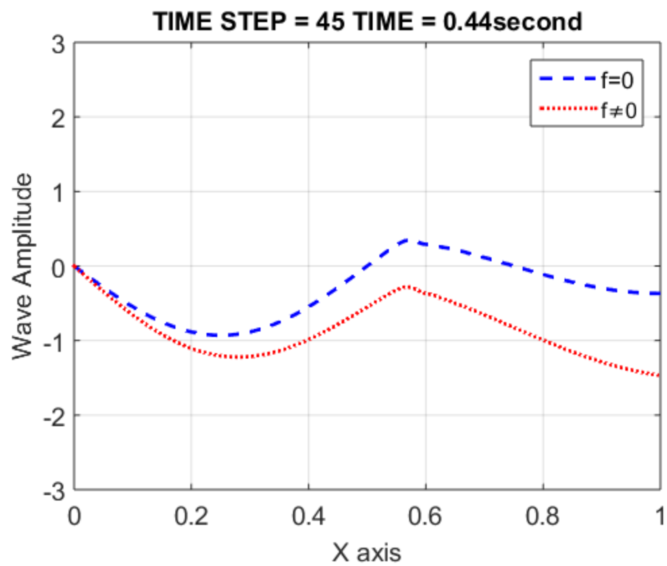

TIME STEP = 750 TIME = 0.749second
Numerical solution
Analytical solution
Wave Amplitude
X axis
2
1
0
-1
-2
0
0.2
0.4
0.6
0.8
1

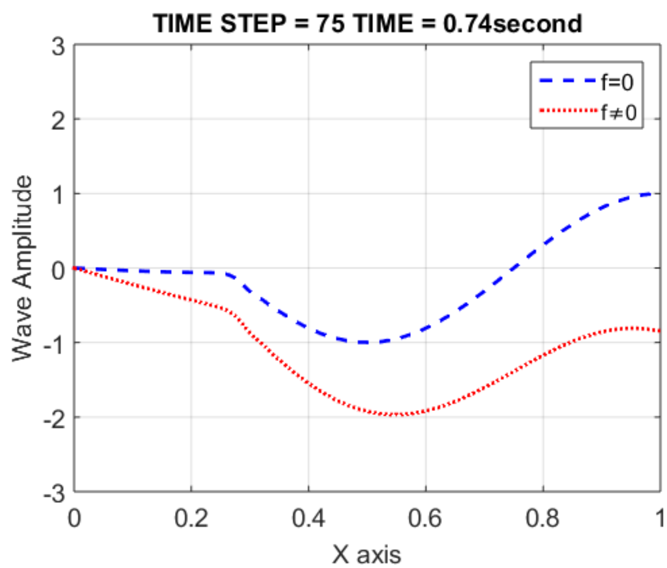


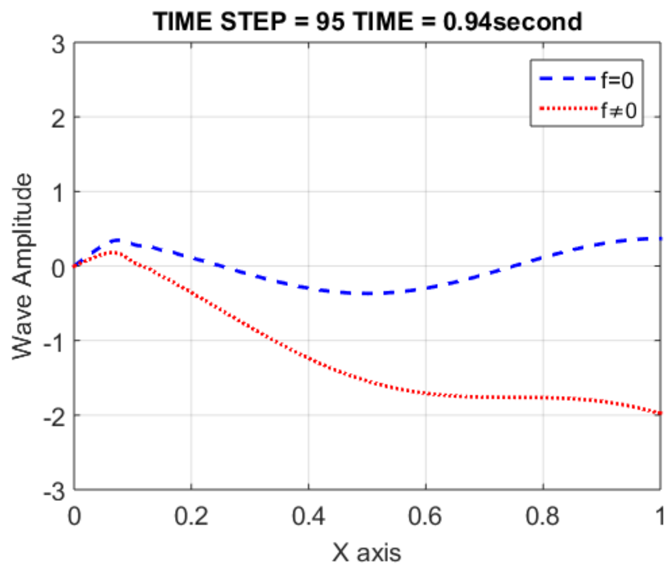
3.2 Drugi primjer - Neumannov rubni uvjet u $x = l$

Stavimo $b = 0$, nema elastičnog sredstva, $a = 1$, $\rho = 1$. Promatrati ćemo primjer za $f = 0$ i $f \neq 0$, tj u ovom slučaju sam uzela $f = 1000x$. Uzmimo $n = 100, t \in [0, T]$, gdje je $T = 1$, sa korakom $\delta t = 0.01$.




TIME STEP = 45 TIME = 0.44second
Wave Amplitude
X axis
f=0
f≠0
3
2
1
0
-1
-2
-3
0
0.2
0.4
0.6
0.8
1


TIME STEP = 75 TIME = 0.74second
Wave Amplitude
X axis
f=0
f≠0
3
2
1
0
-1
-2
-3
0
0.2
0.4
0.6
0.8
1


TIME STEP = 95 TIME = 0.94second
f=0
f≠0
Wave Amplitude
X axis
3
2
1
0
-1
-2
-3
0
0.2
0.4
0.6
0.8
1