Math 4990 - Final Exam

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Problem 1: Consider the integral function

$$J[y] = \int_1^2 \frac{y'(x)^2}{x} dx$$

The integrand here is strongly convex (on an appropriately defined set). Find the unique $y \in D$ that minimizes $J[y]$ over D, for the following cases. In each case, is the minimizer (if it exists) unique?

Part (a) $D = \{y \in C^1[1,2] : y(1) = 0, y(2) = 3\}$

Solution

If J is a minimum at $y \in D$ then y must satify the Euler-Lagrange equation $F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0$ where $F(x, y, y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 &\implies \frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'} \\
&\implies 0 = \frac{d}{dx}\frac{2y'}{x} \implies \int 0 dx = \int \frac{d}{dx}\frac{2y'}{x} dx \\
&\implies \frac{2y'}{x} = C \quad (C \in \mathbb{R}) \\
&\implies y' = Bx \quad (B \in \mathbb{R}) \\
&\implies \int y' dx = \int Bx dx \\
&\implies y = c_1 x^2 + c_2 \quad (c1, c2 \in \mathbb{R})
\end{aligned}$$

y must also satisfy the boundary conditions:

$$\begin{cases} y(1) = c_1 + c_2 = 0 \\ y(2) = 4c_1 + c_2 = 3 \end{cases} \implies c_1 = \frac{3}{5}, c_2 = -\frac{3}{5} \implies \boxed{y = \frac{3}{5}x^2 - \frac{3}{5}}$$

We have shown with Euler-Lagrange equation the necessary conditions for y to minimize $J[y]$ and because we are given the integrand is strongly convex on the defined set D, we have the sufficient conditions to guarantee y is the unique minimizer.