

Math 4990 - Final Exam

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Problem 1: Consider the integral function

$$J[y] = \int_1^2 \frac{y'(x)^2}{x} dx$$

The integrand here is strongly convex (on an appropriately defined set). Find the unique $y \in D$ that minimizes $J[y]$ over D , for the following cases. In each case, is the minimizer (if it exists) unique?

Part (a) $D = \{y \in C^1[1, 2] : y(1) = 0, y(2) = 3\}$

Solution

If J is a minimum at $y \in D$ then y must satisfy the Euler-Lagrange equation $F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0$ where $F(x, y, y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned} \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} &= 0 \implies \frac{\partial F}{\partial y} = \frac{d}{dx} \frac{\partial F}{\partial y'} \\ \implies 0 &= \frac{d}{dx} \frac{2y'}{x} \implies \int 0 dx = \int \frac{d}{dx} \frac{2y'}{x} dx \\ \implies \frac{2y'}{x} &= C \quad (C \in \mathbb{R}) \\ \implies y' &= Bx \quad (B \in \mathbb{R}) \\ \implies \int y' dx &= \int Bx dx \\ \implies y &= c_1 x^2 + c_2 \quad (c_1, c_2 \in \mathbb{R}) \end{aligned}$$

y must also satisfy the boundary conditions:

$$\begin{cases} y(1) = c_1 + c_2 = 0 \\ y(2) = 4c_1 + c_2 = 3 \end{cases} \implies c_1 = \frac{3}{5}, c_2 = -\frac{3}{5} \implies \boxed{y = \frac{3}{5}x^2 - \frac{3}{5}}$$

We have shown with Euler-Lagrange equation the necessary conditions for y to minimize $J[y]$ and because we are given the integrand is strongly convex on the defined set D , we have the sufficient conditions to guarantee y is the unique minimizer.

Part (b) $D = \{y \in C^1[1, 2] : y(2) = 3\}$

Solution

From the notes given in class, we have the following:

Theorem. If $F(x, y, y')$ is strongly convex in (y, y') then a solution $y(x)$ of the Differential Euler-Lagrange equation uniquely minimizes

$$J[y] = \int_a^b F(x, y, y') dx$$

over $\{y \in C'[a, b] : y(b) = B\}$ if $F_{y'}(a, y(a), y'(a)) = 0$.

We find the solution to the Euler-Lagrange equation $F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0$ where $F(x, y, y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned} \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} &= 0 \implies \frac{\partial F}{\partial y} = \frac{d}{dx} \frac{\partial F}{\partial y'} \\ \implies 0 &= \frac{d}{dx} \frac{2y'}{x} \implies \int 0 dx = \int \frac{d}{dx} \frac{2y'}{x} dx \\ \implies \frac{2y'}{x} &= C \quad (C \in \mathbb{R}) \\ \implies y' &= Bx \quad (B \in \mathbb{R}) \\ \implies \int y' dx &= \int Bx dx \\ \implies y &= c_1 x^2 + c_2 \quad (c_1, c_2 \in \mathbb{R}) \end{aligned}$$

y must also satisfy the boundary condition:

$$\begin{aligned} y(2) = 4c_1 + c_2 &= 3 \implies c_2 = 3 - 4c_1 \\ \implies y &= c_1 x^2 + (3 - 4c_1) \\ \implies y &= c_1(x^2 - 4) + 3 \end{aligned}$$

thus

$$\begin{aligned} F_{y'}(1, y(1), y'(1)) &= 0 \implies \frac{2y'(1)}{1} = 0 \\ \implies y'(1) &= 0 \\ \implies 2c_1(1) & \\ \implies c_1 &= 0 \end{aligned}$$

which implies

$$\boxed{y(x) = 3}$$

Given the integrand is strongly convex on the defined set D along with the theorem mentioned, $y \in D$ uniquely minimizes $J[y]$ over D .