Math 4990 - Final Exam

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Problem 1: Consider the integral function

$$J[y] = \int_1^2 \frac{y'(x)^2}{x} dx$$

The integrand here is strongly convex (on an appropriately defined set). Find the unique $y \in D$ that minimizes $J[y]$ over D, for the following cases. In each case, is the minimizer (if it exists) unique?

Part (a) $D = \{y \in C^1[1,2] : y(1) = 0, y(2) = 3\}$

Solution
If J is a minimum at $y \in D$ then y must satify the Euler-Lagrange equation $F_y(x,y,y') - \frac{d}{dx}F_{y'}(x,y,y') = 0$ where $F(x,y,y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 &\implies \frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'} \\
&\implies 0 = \frac{d}{dx}\frac{2y'}{x} \implies \int 0 dx = \int \frac{d}{dx}\frac{2y'}{x} dx \\
&\implies \frac{2y'}{x} = C \quad (C \in \mathbb{R}) \\
&\implies y' = Bx \quad (B \in \mathbb{R}) \\
&\implies \int y' dx = \int Bx dx \\
&\implies y = c_1 x^2 + c_2 \quad (c1, c2 \in \mathbb{R})
\end{aligned}$$

y must also satisfy the boundary conditions:

$$\begin{cases} y(1) = c_1 + c_2 = 0 \\ y(2) = 4c_1 + c_2 = 3 \end{cases} \implies c_1 = \frac{3}{5}, c_2 = -\frac{3}{5} \implies \boxed{y = \frac{3}{5}x^2 - \frac{3}{5}}$$

We have shown with Euler-Lagrange equation the necessary conditions for y to minimize $J[y]$ and because we are given the integrand is strongly convex on the defined set D, we have the sufficient conditions to guarantee y is the unique minimizer.

Part (b) $D = \{y \in C^1[1,2] : y(2) = 3\}$

Solution

From the notes given in class, we have the following:

Theorem. If $F(x,y,y')$ is strongly convex in (y,y') then a solution $y(x)$ of the Differential Euler-Lagrange equation uniquely minimizes

$$J[y] = \int_a^b F(x,y,y')dx$$

over $\{y \in C'[a,b] : y(b) = B\}$ if $F_{y'}(a, y(a), y'(a)) = 0$.

We find the solution to the Euler-Lagrange equation $F_y(x,y,y') - \frac{d}{dx}F_{y'}(x,y,y') = 0$ where $F(x,y,y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \Longrightarrow \frac{\partial F}{\partial y} &= \frac{d}{dx}\frac{\partial F}{\partial y'} \\
&\Longrightarrow 0 = \frac{d}{dx}\frac{2y'}{x} \Longrightarrow \int 0dx = \int \frac{d}{dx}\frac{2y'}{x}dx \\
&\Longrightarrow \frac{2y'}{x} = C \quad (C \in \mathbb{R}) \\
&\Longrightarrow y' = Bx \quad (B \in \mathbb{R}) \\
&\Longrightarrow \int y'dx = \int Bxdx \\
&\Longrightarrow y = c_1x^2 + c_2 \quad (c1, c2 \in \mathbb{R})
\end{aligned}$$

y must also satisfy the boundary condition:

$$\begin{aligned}
y(2) = 4c_1 + c_2 = 3 &\Longrightarrow c_2 = 3 - 4c_1 \\
&\Longrightarrow y = c_1x^2 + (3 - 4c_1) \\
&\Longrightarrow y = c_1(x^2 - 4) + 3
\end{aligned}$$

thus

$$\begin{aligned}
F_{y'}(1, y(1), y'(1)) = 0 &\Longrightarrow \frac{2y'(1)}{1} = 0 \\
&\Longrightarrow y'(1) = 0 \\
&\Longrightarrow 2c_1(1) \\
&\Longrightarrow c_1 = 0
\end{aligned}$$

which implies

$$\boxed{y(x) = 3}$$

Given the integrand is strongly convex on the defined set D along with the theorem mentioned above, y uniquely minimizes $J[y]$ over D.

Part (c) $D = C^1[1,2]$

Solution
If J is a minimum at $y \in D$ then y must satify the Euler-Lagrange equation $F_y(x,y,y') - \frac{d}{dx}F_{y'}(x,y,y') = 0$ where $F(x,y,y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \implies \frac{\partial F}{\partial y} &= \frac{d}{dx}\frac{\partial F}{\partial y'} \\
&\implies 0 = \frac{d}{dx}\frac{2y'}{x} \implies \int 0dx = \int \frac{d}{dx}\frac{2y'}{x}dx \\
&\implies \frac{2y'}{x} = C \quad (C \in \mathbb{R}) \\
&\implies y' = Bx \quad (B \in \mathbb{R}) \\
&\implies \int y'dx = \int Bxdx \\
&\implies y = c_1x^2 + c_2 \quad (c1, c2 \in \mathbb{R})
\end{aligned}$$

From strong convexity and observation we see if $y = c_2$, we achieve a minimum for $J[y] = \int_1^2 \frac{y'(x)^2}{x}dx$ since $y'(x)^2 \geq 0$ and $x \in [1,2]$ which implies $J[y] \geq 0$. However y is not a unique minimizer since $c2 \in \mathbb{R}$.

Problem 2: Let $\delta(x)$ be the Dirac delta function. Justify the identity

$$\delta(1-x^2) = \frac{\delta(x-1)+\delta(x+1)}{2}$$

.

Hint: consider the integral $\int_{-\infty}^{\infty} \delta(1-x^2)dx$

Solution For any continuous function f(x), we have

$$\int_{\infty}^{\infty} f(x)\delta(1-x^2)dx = \underbrace{\int_{0}^{\infty} f(x)\delta(1-x^2)dx}_{(1)} + \underbrace{\int_{-\infty}^{0} f(x)\delta(1-x^2)dx}_{(2)}$$

If we let $\sqrt{u} = x$ and $\frac{1}{2\sqrt{u}}du = dx$

$$(1) = \int_{0}^{\infty} \frac{f(\sqrt{u})\delta(1-u)}{2\sqrt{u}}du = \frac{f(1)}{2}$$

If we let $-\sqrt{u} = x$ and $-\frac{1}{2\sqrt{u}}du = dx$

$$(2) = -\int_{-\infty}^{0} \frac{f(-\sqrt{u})\delta(1-u)}{2\sqrt{u}}du = \int_{0}^{\infty} \frac{f(-\sqrt{u})\delta(1-u)}{2\sqrt{u}}du = \frac{f(-1)}{2}$$

thus

$$\begin{aligned}
\int_{\infty}^{\infty} f(x)\delta(1-x^2)dx &= \frac{1}{2}[f(1)+f(-1)] \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)\delta(x-1)dx + \int_{-\infty}^{\infty} f(x)\delta(x+1)dx\right] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} f(x)(\delta(x-1)+\delta(x+1))dx \\
&= \int_{-\infty}^{\infty} f(x)\frac{\delta(x-1)+\delta(x+1)}{2}dx
\end{aligned}$$

Since δ functions are defined only by how they behave in integrals we conclude

$$\boxed{\delta(1-x^2) = \frac{\delta(x-1)+\delta(x+1)}{2}}$$

Problem 3: Let $f : [0,1] \to \mathbb{R}$ be a given continuous function, and consider the following differential equation for the function y:

$$\begin{cases} y'''(x) = f(x) & 0 \leq x \leq 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases} \tag{1}$$

Intuitively, finding $y(x)$ requires three integrations. This problem illustrates that by employing a Green's function we can actually find y via a *single* integral.

Part (a) Show that if the Green's function $u = G(x : s)$ satisfies

$$\begin{cases} u'''(x) = \delta(x - s) & 0 \leq x \leq 1 \\ u(0) = u'(0) = u''(0) = 0 \end{cases}$$

then the function

$$y(x) = \int_0^1 f(s)G(x : s)ds \tag{2}$$

satisfies (1).