Math 4990 - Final Exam

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Problem 1: Consider the integral function

$$J[y] = \int_1^2 \frac{y'(x)^2}{x} dx$$

The integrand here is strongly convex (on an appropriately defined set). Find the unique $y \in D$ that minimizes $J[y]$ over D, for the following cases. In each case, is the minimizer (if it exists) unique?

Part (a) $D = \{y \in C^1[1,2] : y(1) = 0, y(2) = 3\}$

Solution
If J is a minimum at $y \in D$ then y must satify the Euler-Lagrange equation
$F_y(x, y, y') - \frac{d}{dx}F_{y'}(x, y, y') = 0$ where $F(x, y, y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \implies \frac{\partial F}{\partial y} &= \frac{d}{dx}\frac{\partial F}{\partial y'} \\
&\implies 0 = \frac{d}{dx}\frac{2y'}{x} \implies \int 0dx = \int \frac{d}{dx}\frac{2y'}{x}dx \\
&\implies \frac{2y'}{x} = C \quad (C \in \mathbb{R}) \\
&\implies y' = Bx \quad (B \in \mathbb{R}) \\
&\implies \int y'dx = \int Bxdx \\
&\implies y = c_1x^2 + c_2 \quad (c1, c2 \in \mathbb{R})
\end{aligned}$$

y must also satisfy the boundary conditions:

$$\begin{cases} y(1) = c_1 + c_2 = 0 \\ y(2) = 4c_1 + c_2 = 3 \end{cases} \implies c_1 = \frac{3}{5}, c_2 = -\frac{3}{5} \implies \boxed{y = \frac{3}{5}x^2 - \frac{3}{5}}$$

We have shown with Euler-Lagrange equation the necessary conditions for y to minimize $J[y]$ and because we are given the integrand is strongly convex on the defined set D, we have the sufficient conditions to guarantee y is the unique minimizer.

Part (b) $D = \{y \in C^1[1,2] : y(2) = 3\}$

Solution
From the notes given in class, we have the following:
Theorem. If $F(x,y,y')$ is strongly convex in (y,y') then a solution $y(x)$ of the Differential Euler-Lagrange equation uniquely minimizes

$$J[y] = \int_a^b F(x,y,y')dx$$

over $\{y \in C'[a,b] : y(b) = B\}$ if $F_{y'}(a,y(a),y'(a)) = 0$.

We find the solution to the Euler-Lagrange equation $F_y(x,y,y') - \frac{d}{dx}F_{y'}(x,y,y') = 0$ where $F(x,y,y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \Longrightarrow \frac{\partial F}{\partial y} &= \frac{d}{dx}\frac{\partial F}{\partial y'} \\
&\Longrightarrow 0 = \frac{d}{dx}\frac{2y'}{x} \Longrightarrow \int 0dx = \int \frac{d}{dx}\frac{2y'}{x}dx \\
&\Longrightarrow \frac{2y'}{x} = C \quad (C \in \mathbb{R}) \\
&\Longrightarrow y' = Bx \quad (B \in \mathbb{R}) \\
&\Longrightarrow \int y'dx = \int Bxdx \\
&\Longrightarrow y = c_1x^2 + c_2 \quad (c1, c2 \in \mathbb{R})
\end{aligned}$$

y must also satisfy the boundary condition:

$$\begin{aligned}
y(2) = 4c_1 + c_2 = 3 &\Longrightarrow c_2 = 3 - 4c_1 \\
&\Longrightarrow y = c_1x^2 + (3 - 4c_1) \\
&\Longrightarrow y = c_1(x^2 - 4) + 3
\end{aligned}$$

thus

$$\begin{aligned}
F_{y'}(1,y(1),y'(1)) = 0 &\Longrightarrow \frac{2y'(1)}{1} = 0 \\
&\Longrightarrow y'(1) = 0 \\
&\Longrightarrow 2c_1(1) \\
&\Longrightarrow c_1 = 0
\end{aligned}$$

which implies

$$\boxed{y(x) = 3}$$

Given the integrand is strongly convex on the defined set D along with the theorem mentioned above, y uniquely minimizes $J[y]$ over D.

Part (c) $D = C^1[1,2]$

Solution
If J is a minimum at $y \in D$ then y must satify the Euler-Lagrange equation $F_y(x,y,y') - \frac{d}{dx}F_{y'}(x,y,y') = 0$ where $F(x,y,y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \implies \frac{\partial F}{\partial y} &= \frac{d}{dx}\frac{\partial F}{\partial y'} \\
&\implies 0 = \frac{d}{dx}\frac{2y'}{x} \implies \int 0dx = \int \frac{d}{dx}\frac{2y'}{x}dx \\
&\implies \frac{2y'}{x} = C \quad (C \in \mathbb{R}) \\
&\implies y' = Bx \quad (B \in \mathbb{R}) \\
&\implies \int y'dx = \int Bxdx \\
&\implies y = c_1x^2 + c_2 \quad (c1, c2 \in \mathbb{R})
\end{aligned}$$

From strong convexity and observation we see if $y = c_2$, we achieve a minimum for $J[y] = \int_1^2 \frac{y'(x)^2}{x}dx$ since $y'(x)^2 \geq 0$ and $x \in [1,2]$ which implies $J[y] \geq 0$. However y is not a unique minimizer since $c2 \in \mathbb{R}$.

Problem 2: Let $\delta(x)$ be the Dirac delta function. Justify the identity

$$\delta(1-x^2) = \frac{\delta(x-1)+\delta(x+1)}{2}$$

.

Hint: consider the integral $\int_{-\infty}^{\infty} \delta(1-x^2)dx$

Solution For any continuous function f(x), we have

$$\int_{\infty}^{\infty} f(x)\delta(1-x^2)dx = \underbrace{\int_{0}^{\infty} f(x)\delta(1-x^2)dx}_{(1)} + \underbrace{\int_{-\infty}^{0} f(x)\delta(1-x^2)dx}_{(2)}$$

If we let $\sqrt{u} = x$ and $\frac{1}{2\sqrt{u}}du = dx$

$$(1) = \int_{0}^{\infty} \frac{f(\sqrt{u})\delta(1-u)}{2\sqrt{u}}du = \frac{f(1)}{2}$$

If we let $-\sqrt{u} = x$ and $-\frac{1}{2\sqrt{u}}du = dx$

$$(2) = -\int_{-\infty}^{0} \frac{f(-\sqrt{u})\delta(1-u)}{2\sqrt{u}}du = \int_{0}^{\infty} \frac{f(-\sqrt{u})\delta(1-u)}{2\sqrt{u}}du = \frac{f(-1)}{2}$$

thus

$$\begin{aligned}
\int_{\infty}^{\infty} f(x)\delta(1-x^2)dx &= \frac{1}{2}[f(1)+f(-1)] \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)\delta(x-1)dx + \int_{-\infty}^{\infty} f(x)\delta(x+1)dx\right] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} f(x)(\delta(x-1)+\delta(x+1))dx \\
&= \int_{-\infty}^{\infty} f(x)\frac{\delta(x-1)+\delta(x+1)}{2}dx
\end{aligned}$$

Since δ functions are defined only by how they behave in integrals we conclude

$$\boxed{\delta(1-x^2) = \frac{\delta(x-1)+\delta(x+1)}{2}}$$

Problem 3: Let $f : [0,1] \to \mathbb{R}$ be a given continuous function and consider the following differential equation for the function y:

$$\begin{cases} y'''(x) = f(x) & 0 \le x \le 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases} \tag{1}$$

Intuitively, finding $y(x)$ requires three integrations. This problem illustrates that by employing a Green's function we can actually find y via a *single* integral.

Part (a) Show that if the Green's function $u = G(x; s)$ satisfies

$$\begin{cases} u'''(x) = \delta(x-s) & 0 \le x \le 1 \\ u(0) = u'(0) = u''(0) = 0 \end{cases}$$

then the function

$$y(x) = \int_0^1 f(s)G(x;s)ds \tag{2}$$

satisfies (1).

Solution
From the notes given in class, we have the following:

Theorem.
The function $y(x) = \int_0^1 f(s)G(x;s)ds$ satisfies the differential equation

$$\begin{cases} L[y] = y'''(x) = f(x) & 0 \le x \le 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

Let us first observe

$$\begin{aligned} L[y] = y''' &= \frac{d^3}{dx^3}\big(\int_0^1 f(s)G(x;s)ds\big) \\ &= \int_0^1 \frac{d^3}{dx^3}\big(f(s)G(x;s)\big)ds \\ &= \int_0^1 f(s)\frac{d^3}{dx^3}\big(G(x;s)\big)ds \\ &= \int_0^1 f(s)G'''(x;s)ds \\ &= \int_0^1 f(s)L[G]ds \end{aligned}$$

and

$$L[G] = G'''(x;s) = \delta(x-s), \qquad 0 \le x \le 1$$

thus

$$\begin{aligned} L[y] &= L\left[\int_0^1 f(s)G(x;s)ds\right] \\ &= \int_0^1 L\left[f(s)G(x;s)\right]ds \\ &= \int_0^1 f(s)L[G]ds \\ &= \int_0^1 f(s)\delta(x-s)ds \\ &= f(x) \end{aligned}$$

along with the boundary conditions given $G(0;s) = 0$

$$\begin{aligned} y(0) &= \int_0^1 f(s)\underbrace{G(0;s)}_{0}ds = 0 \\ y'(0) &= \int_0^1 f'(s)\underbrace{G(0;s)}_{0}ds = 0 \\ y''(0) &= \int_0^1 f''(s)\underbrace{G(0;s)}_{0}ds = 0 \end{aligned}$$

We have showed (2) satisfies (1).

Part (b) Find the Green's function $G(x:s)$ for this problem

$$\begin{cases} y'''(x) = f(x) & 0 \le x \le 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

Solution To find Green's function we solve for

$$\begin{cases} u'''(x) = \delta(x-s) & 0 \le x \le 1 \\ u(0) = u'(0) = u''(0) = 0 \end{cases}$$

where $u(x) = G(x;s)$.

For $x \neq s$ the differential equation is

$$u'''(x) = 0 \qquad\qquad 0 \le x \le 1$$

whose general solution is found to be

$$u(x) = Ax^2 + Bx + C \qquad (A, B, C \in \mathbb{R})$$

thus

$$u(x) = \begin{cases} Ax^2 + Bx + C & x < s \\ Ex^2 + Dx + F & x > s \end{cases}$$

The boundary conditions require

$$\begin{aligned} u(0) = 0 &\implies Ax^2 + Bx + C = 0 &&\implies C = 0 \\ u'(0) = 0 &\implies 2Ax + B = 0 &&\implies B = 0 \\ u''(0) = 0 &\implies 2A = 0 &&\implies A = 0. \end{aligned}$$

The Green function now has the form

$$u(x) = \begin{cases} 0 & x < s \\ Ex^2 + Dx + F & x > s. \end{cases}$$

We also need to satisfy the second derivative discontinuity condition since $u''' = \delta(x-s) \Rightarrow u'' = H(x-s)$

$$\begin{aligned} u''(s^+) - u''(s^-) = 1 &\implies 2E - 0 = 1 \\ &\implies 2E = 1 \\ &\implies E = \frac{1}{2} \end{aligned}$$

The Green function now has the form

$$u(x) = \begin{cases} 0 & x < s \\ \frac{1}{2}x^2 + Dx + F & x > s. \end{cases}$$

Together with the need to satisfy the first derivative continuity condition since $u' = \int H(x-s)dx$ is continuous.

$$\begin{aligned} u'(s^+) = u''(s^-) &\Longrightarrow s + D = 0 \\ &\Longrightarrow D = -s \end{aligned}$$

The Green function now has the form

$$u(x) = \begin{cases} 0 & x < s \\ \frac{1}{2}x^2 - sx + F & x > s. \end{cases}$$

Along with the continuity condition of u since u' is continuous which implies u is continuous.

$$\begin{aligned} u(s^+) = u(s^-) &\Longrightarrow \frac{1}{2}s^2 - s^2 + F = 0 \\ &\Longrightarrow -\frac{1}{2}s^2 + F = 0 \\ &\Longrightarrow F = \frac{1}{2}s^2. \end{aligned}$$

Thus the Green function is as follows

$$\boxed{u(x) = \begin{cases} 0 & x < s \\ \frac{1}{2}(x^2 + s^2) - sx & x > s \end{cases}}$$

Part (c) For the case $f(x) = x$, solve (1) directly by integrating thrice. Verify that (2) gives the same result.

Solution We are given

$$\begin{cases} y'''(x) = x & 0 \leq x \leq 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

By integrating $f(x) = x$ thrice,

$$\begin{aligned} \iiint x dx &\Longrightarrow \iint \frac{x^2}{2} + c_1 dx \\ &\Longrightarrow \int \frac{x^3}{6} + c_1 x + c_2 dx \\ &\Longrightarrow \frac{x^4}{24} + \frac{c_1 x^2}{2} + c_2 x + c_3 \qquad (c_1, c_2, c_3 \in R) \end{aligned}$$

and in order to satisfy the boundary conditions, $c_1 = c_2 = c_3 = 0$ since

$$\begin{cases} y(0) & = \frac{0^4}{24} = 0 \\ y'(0) & = \frac{0^3}{6} = 0 \\ y''(0) & = \frac{0^2}{3} = 0. \end{cases}$$

Thus

$$y = \frac{x^4}{24}.$$

Now to verify green's function with $y(x) = \int_0^1 f(s)G(x;s)ds$, we have

$$G(x;s) = \begin{cases} 0 & x < s \\ \frac{1}{2}(x^2 + s^2) - sx & x > s \end{cases}$$

therefore

$$\begin{aligned} y(x) &= \int_0^x s\Big(\frac{1}{2}(x^2 + s^2) - sx\Big)ds + \int_x^1 0 ds \\ &= \int_0^x \frac{sx^2}{2} + \frac{s^3}{2} - s^2 x ds \\ &= \left[\frac{s^2 x^2}{4} + \frac{s^4}{8} - \frac{s^3 x}{3}\right]_0^x \\ &= \frac{x^4}{4} + \frac{x^4}{8} - \frac{x^4}{3} \\ &= \frac{x^4}{24}. \end{aligned}$$

We have showed both methods give the same results.

Problem 4: The Fourier transform is often useful for finding Green's functions. Consider the following partial differential equation for the function $u(x,t)$:

$$\begin{cases} u_{tt} - u_{xx} + u = 0 & -\infty < x < \infty \\ u(x,0) = \delta(x-s) & s \in \mathbb{R} \\ u_t(x,0) = 0. \end{cases}$$

Part (a) Apply a Fourier transform with respect to x to obtain a 2nd-order ordinary differential equation for the function $\hat{u}(\omega,t)$.

Solution
By applying the Fourier transform with respect to x such that $\mathcal{F}\{u(x,t)\} = \hat{u}(\omega,t)$ we get

$$\begin{cases} u_{tt} - u_{xx} + u & = 0 \\ u(x,0) & = \delta(x-s) \\ u_t(x,0) & = 0 \end{cases} \xrightarrow{\mathcal{F}} \begin{cases} \hat{u}_{tt} - (i\omega)^2\hat{u} + \hat{u} & = 0 & ^{(1)} \\ \hat{u}(\omega,0) & = \frac{1}{2\pi}e^{-i\omega s} & ^{(2)} \\ \hat{u}_t(\omega,0) & = 0. \end{cases}$$

which is a 2nd order ordinary differential equation. Second order due to $\hat{u}_{tt}$ and ordinary due to derivatives only involving t.

(1) By derivative property of Fourier Transforms.
(2) By shift property and Fourier Transform of the Dirac Delta function.

Part (b) Find $\hat{u}(\omega,t)$ by solving the differential equation from part (a).

Solution From part a we have the following:

$$\begin{cases} \hat{u}_{tt} - (i\omega)^2\hat{u} + \hat{u} & = 0 \\ \hat{u}(\omega,0) & = \frac{1}{2\pi}e^{-i\omega s} \\ \hat{u}_t(\omega,0) & = 0. \end{cases}$$

For $\hat{u}_{tt} - (i\omega)^2\hat{u} + \hat{u} = 0$,

$$\begin{aligned} \hat{u}_{tt} = (i\omega)^2\hat{u} - \hat{u} &\implies \hat{u}_{tt} = (-\omega^2 - 1)\hat{u} \\ &\implies \hat{u} = c_1 e^{\sqrt{-\omega^2-1}t} + c_2 e^{-\sqrt{-\omega^2-1}t} \quad (c_1, c_2 \in \mathbb{R}) \qquad ^{(1)} \end{aligned}$$

(1) By solving for second-order linearly ordinary differential equation on Wolfram Alpha

together with $\hat{u}_t(\omega,0) = 0$

$$\begin{aligned} \hat{u}_t(\omega,0) = 0 &\implies \frac{d}{dt}(c_1 e^{\sqrt{-\omega^2-1}t} + c_2 e^{-\sqrt{-\omega^2-1}t}) = 0 \\ &\implies c_1\sqrt{-\omega^2-1}e^{\sqrt{-\omega^2-1}(0)} - c_2\sqrt{-\omega^2-1}e^{-\sqrt{-\omega^2-1}(0)} = 0 \\ &\implies c_1\sqrt{-\omega^2-1} = c_2\sqrt{-\omega^2-1} \\ &\implies c_1 = c_2 \end{aligned}$$

along with $\hat{u}(\omega,0) = \frac{1}{2\pi}e^{-i\omega s}$

$$\implies \frac{1}{2\pi}e^{-i\omega s}\underbrace{\left(c_1 e^{\sqrt{-\omega^2-1}(0)} + c_1 e^{-\sqrt{-\omega^2-1}(0)}\right)}_{=1}$$

$$\implies c_1 = \frac{1}{2}$$

therefore

$$\begin{aligned}\hat{u}(\omega,t) &= \frac{1}{4\pi}e^{-i\omega s}\left(e^{\sqrt{-\omega^2-1}t} + e^{-\sqrt{-\omega^2-1}t}\right)\\ &= \boxed{\frac{1}{2\pi}e^{-i\omega s}cosh\left(t\sqrt{-\omega^2-1}\right)} \qquad \text{since } e^x + e^-x = 2\cosh(x)\end{aligned}$$

Part (c) Invert the Fourier transform to obtain $u(x,t)$

Solution

$$\begin{aligned}u(x,t) &= \int_{-\infty}^{\infty}\hat{u}(\omega,t)e^{i\omega t}d\omega\\ &= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\cosh\left(t\sqrt{-\omega^2-1}\right)\right]e^{i\omega t}d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cosh\left(t\sqrt{-\omega^2-1}\right)e^{i\omega(t-s)}d\omega\end{aligned}$$