

# Math 4990 - Final Exam

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**Problem 1:** Consider the integral function

$$J[y] = \int_1^2 \frac{y'(x)^2}{x} dx$$

The integrand here is strongly convex (on an appropriately defined set). Find the unique  $y \in D$  that minimizes  $J[y]$  over  $D$ , for the following cases. In each case, is the minimizer (if it exists) unique?

**Part (a)**  $D = \{y \in C^1[1, 2] : y(1) = 0, y(2) = 3\}$

## Solution

If  $J$  is a minimum at  $y \in D$  then  $y$  must satisfy the Euler-Lagrange equation  $F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0$  where  $F(x, y, y') = \frac{y'(x)^2}{x}$ .

$$\begin{aligned} \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} &= 0 \implies \frac{\partial F}{\partial y} = \frac{d}{dx} \frac{\partial F}{\partial y'} \\ \implies 0 &= \frac{d}{dx} \frac{2y'}{x} \implies \int 0 dx = \int \frac{d}{dx} \frac{2y'}{x} dx \\ \implies \frac{2y'}{x} &= C \quad (C \in \mathbb{R}) \\ \implies y' &= Bx \quad (B \in \mathbb{R}) \\ \implies \int y' dx &= \int Bx dx \\ \implies y &= c_1 x^2 + c_2 \quad (c_1, c_2 \in \mathbb{R}) \end{aligned}$$

$y$  must also satisfy the boundary conditions:

$$\begin{cases} y(1) = c_1 + c_2 = 0 \\ y(2) = 4c_1 + c_2 = 3 \end{cases} \implies c_1 = \frac{3}{5}, c_2 = -\frac{3}{5} \implies \boxed{y = \frac{3}{5}x^2 - \frac{3}{5}}$$

We have shown with Euler-Lagrange equation the necessary conditions for  $y$  to minimize  $J[y]$  and because we are given the integrand is strongly convex on the defined set  $D$ , we have the sufficient conditions to guarantee  $y$  is the unique minimizer.

**Part (b)**  $D = \{y \in C^1[1, 2] : y(2) = 3\}$

**Solution**

From the notes given in class, we have the following:

*Theorem.* If  $F(x, y, y')$  is strongly convex in  $(y, y')$  then a solution  $y(x)$  of the Differential Euler-Lagrange equation uniquely minimizes

$$J[y] = \int_a^b F(x, y, y') dx$$

over  $\{y \in C'[a, b] : y(b) = B\}$  if  $F_{y'}(a, y(a), y'(a)) = 0$ .

We find the solution to the Euler-Lagrange equation  $F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0$  where  $F(x, y, y') = \frac{y'(x)^2}{x}$ .

$$\begin{aligned} \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} &= 0 \implies \frac{\partial F}{\partial y} = \frac{d}{dx} \frac{\partial F}{\partial y'} \\ \implies 0 &= \frac{d}{dx} \frac{2y'}{x} \implies \int 0 dx = \int \frac{d}{dx} \frac{2y'}{x} dx \\ \implies \frac{2y'}{x} &= C \quad (C \in \mathbb{R}) \\ \implies y' &= Bx \quad (B \in \mathbb{R}) \\ \implies \int y' dx &= \int Bx dx \\ \implies y &= c_1 x^2 + c_2 \quad (c_1, c_2 \in \mathbb{R}) \end{aligned}$$

$y$  must also satisfy the boundary condition:

$$\begin{aligned} y(2) = 4c_1 + c_2 &= 3 \implies c_2 = 3 - 4c_1 \\ \implies y &= c_1 x^2 + (3 - 4c_1) \\ \implies y &= c_1(x^2 - 4) + 3 \end{aligned}$$

thus

$$\begin{aligned} F_{y'}(1, y(1), y'(1)) &= 0 \implies \frac{2y'(1)}{1} = 0 \\ \implies y'(1) &= 0 \\ \implies 2c_1(1) & \\ \implies c_1 &= 0 \end{aligned}$$

which implies

$$\boxed{y(x) = 3}$$

Given the integrand is strongly convex on the defined set  $D$  along with the theorem mentioned above,  $y$  uniquely minimizes  $J[y]$  over  $D$ .

**Part (c)**  $D = C^1[1, 2]$

**Solution**

If  $J$  is a minimum at  $y \in D$  then  $y$  must satisfy the Euler-Lagrange equation

$F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0$  where  $F(x, y, y') = \frac{y'(x)^2}{x}$ .

$$\begin{aligned} \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} &= 0 \implies \frac{\partial F}{\partial y} = \frac{d}{dx} \frac{\partial F}{\partial y'} \\ \implies 0 &= \frac{d}{dx} \frac{2y'}{x} \implies \int 0 dx = \int \frac{d}{dx} \frac{2y'}{x} dx \\ \implies \frac{2y'}{x} &= C \quad (C \in \mathbb{R}) \\ \implies y' &= Bx \quad (B \in \mathbb{R}) \\ \implies \int y' dx &= \int Bx dx \\ \implies y &= c_1 x^2 + c_2 \quad (c_1, c_2 \in \mathbb{R}) \end{aligned}$$

From strong convexity and observation we see if  $y = c_2$ , we achieve a minimum for  $J[y] = \int_1^2 \frac{y'(x)^2}{x} dx$  since  $y'(x)^2 \geq 0$  and  $x \in [1, 2]$  which implies  $J[y] \geq 0$ . However  $y$  is not a unique minimizer since  $c_2 \in \mathbb{R}$ .

**Problem 2:** Let  $\delta(x)$  be the Dirac delta function. Justify the identity

$$\delta(1 - x^2) = \frac{\delta(x - 1) + \delta(x + 1)}{2}$$

Hint: consider the integral  $\int_{-\infty}^{\infty} \delta(1 - x^2) dx$

**Solution** For any continuous function  $f(x)$ , we have

$$\int_{-\infty}^{\infty} f(x) \delta(1 - x^2) dx = \underbrace{\int_0^{\infty} f(x) \delta(1 - x^2) dx}_{(1)} + \underbrace{\int_{-\infty}^0 f(x) \delta(1 - x^2) dx}_{(2)}$$

If we let  $\sqrt{u} = x$  and  $\frac{1}{2\sqrt{u}} du = dx$

$$(1) = \int_0^{\infty} \frac{f(\sqrt{u}) \delta(1 - u)}{2\sqrt{u}} du = \frac{f(1)}{2}$$

If we let  $-\sqrt{u} = x$  and  $-\frac{1}{2\sqrt{u}} du = dx$

$$(2) = - \int_{-\infty}^0 \frac{f(-\sqrt{u}) \delta(1 - u)}{2\sqrt{u}} du = \int_0^{\infty} \frac{f(-\sqrt{u}) \delta(1 - u)}{2\sqrt{u}} du = \frac{f(-1)}{2}$$

thus

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \delta(1 - x^2) dx &= \frac{1}{2} [f(1) + f(-1)] \\ &= \frac{1}{2} \left[ \int_{-\infty}^{\infty} f(x) \delta(x - 1) dx + \int_{-\infty}^{\infty} f(x) \delta(x + 1) dx \right] \\ &= \frac{1}{2} \int_{-\infty}^{\infty} f(x) (\delta(x - 1) + \delta(x + 1)) dx \\ &= \int_{-\infty}^{\infty} f(x) \frac{\delta(x - 1) + \delta(x + 1)}{2} dx \end{aligned}$$

Since  $\delta$  functions are defined only by how they behave in integrals we conclude

$$\boxed{\delta(1 - x^2) = \frac{\delta(x - 1) + \delta(x + 1)}{2}}$$

**Problem 3:** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a given continuous function and consider the following differential equation for the function  $y$ :

$$\begin{cases} y'''(x) = f(x) & 0 \leq x \leq 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases} \quad (1)$$

Intuitively, finding  $y(x)$  requires three integrations. This problem illustrates that by employing a Green's function we can actually find  $y$  via a *single* integral.

**Part (a)** Show that if the Green's function  $u = G(x; s)$  satisfies

$$\begin{cases} u'''(x) = \delta(x - s) & 0 \leq x \leq 1 \\ u(0) = u'(0) = u''(0) = 0 \end{cases}$$

then the function

$$y(x) = \int_0^1 f(s)G(x; s)ds \quad (2)$$

satisfies (1).

### Solution

From the notes given in class, we have the following:

*Theorem.*

The function  $y(x) = \int_0^1 f(s)G(x; s)ds$  satisfies the differential equation

$$\begin{cases} L[y] = y'''(x) = f(x) & 0 \leq x \leq 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

Let us first observe

$$\begin{aligned} L[y] = y''' &= \frac{d^3}{dx^3} \left( \int_0^1 f(s)G(x; s)ds \right) \\ &= \int_0^1 \frac{d^3}{dx^3} (f(s)G(x; s))ds \\ &= \int_0^1 f(s) \frac{d^3}{dx^3} (G(x; s))ds \\ &= \int_0^1 f(s)G'''(x; s)ds \\ &= \int_0^1 f(s)L[G]ds \end{aligned}$$

and

$$L[G] = G'''(x; s) = \delta(x - s), \quad 0 \leq x \leq 1$$

thus

$$\begin{aligned}
L[y] &= L \left[ \int_0^1 f(s) G(x; s) ds \right] \\
&= \int_0^1 L \left[ f(s) G(x; s) \right] ds \\
&= \int_0^1 f(s) L[G] ds \\
&= \int_0^1 f(s) \delta(x - s) ds \\
&= f(x)
\end{aligned}$$

along with the boundary conditions given  $G(0; s) = 0$

$$\begin{aligned}
y(0) &= \int_0^1 f(s) \underbrace{G(0; s)}_0 ds = 0 \\
y'(0) &= \int_0^1 f'(s) \underbrace{G(0; s)}_0 ds = 0 \\
y''(0) &= \int_0^1 f''(s) \underbrace{G(0; s)}_0 ds = 0
\end{aligned}$$

We have showed (2) satisfies (1).

**Part (b)** Find the Green's function  $G(x : s)$  for this problem

$$\begin{cases} y'''(x) = f(x) & 0 \leq x \leq 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

**Solution** To find Green's function we solve for

$$\begin{cases} u'''(x) = \delta(x - s) & 0 \leq x \leq 1 \\ u(0) = u'(0) = u''(0) = 0 \end{cases}$$

where  $u(x) = G(x; s)$ .

For  $x \neq s$  the differential equation is

$$u'''(x) = 0 \quad 0 \leq x \leq 1$$

whose general solution is found to be

$$u(x) = Ax^2 + Bx + C \quad (A, B, C \in \mathbb{R})$$

thus

$$u(x) = \begin{cases} Ax^2 + Bx + C & x < s \\ Ex^2 + Dx + F & x > s \end{cases}$$

The boundary conditions require

$$\begin{aligned} u(0) = 0 & \implies Ax^2 + Bx + C = 0 & \implies C = 0 \\ u'(0) = 0 & \implies 2Ax + B = 0 & \implies B = 0 \\ u''(0) = 0 & \implies 2A = 0 & \implies A = 0. \end{aligned}$$

The Green function now has the form

$$u(x) = \begin{cases} 0 & x < s \\ Ex^2 + Dx + F & x > s. \end{cases}$$

We also need to satisfy the second derivative discontinuity condition since

$$u''' = \delta(x - s) \implies u'' = H(x - s)$$

$$\begin{aligned} u''(s^+) - u''(s^-) = 1 & \implies 2E - 0 = 1 \\ & \implies 2E = 1 \\ & \implies E = \frac{1}{2} \end{aligned}$$

The Green function now has the form

$$u(x) = \begin{cases} 0 & x < s \\ \frac{1}{2}x^2 + Dx + F & x > s. \end{cases}$$

Together with the need to satisfy the first derivative continuity condition since  $u' = \int H(x-s)dx$  is continuous.

$$\begin{aligned} u'(s^+) = u''(s^-) &\implies s + D = 0 \\ &\implies D = -s \end{aligned}$$

The Green function now has the form

$$u(x) = \begin{cases} 0 & x < s \\ \frac{1}{2}x^2 - sx + F & x > s. \end{cases}$$

Along with the continuity condition of  $u$  since  $u'$  is continuous which implies  $u$  is continuous.

$$\begin{aligned} u(s^+) = u(s^-) &\implies \frac{1}{2}s^2 - s^2 + F = 0 \\ &\implies -\frac{1}{2}s^2 + F = 0 \\ &\implies F = \frac{1}{2}s^2. \end{aligned}$$

Thus the Green function is as follows

$$u(x) = \begin{cases} 0 & x < s \\ \frac{1}{2}(x^2 + s^2) - sx & x > s \end{cases}$$



**Part (c)** For the case  $f(x) = x$ , solve (1) directly by integrating thrice. Verify that (2) gives the same result.

**Solution** We are given

$$\begin{cases} y'''(x) = x & 0 \leq x \leq 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

By integrating  $f(x) = x$  thrice,

$$\begin{aligned} \iiint x dx &\Rightarrow \iint \frac{x^2}{2} + c_1 dx \\ &\Rightarrow \int \frac{x^3}{6} + c_1 x + c_2 dx \\ &\Rightarrow \frac{x^4}{24} + \frac{c_1 x^2}{2} + c_2 x + c_3 \quad (c_1, c_2, c_3 \in R) \end{aligned}$$

and in order to satisfy the boundary conditions,  $c_1 = c_2 = c_3 = 0$  since

$$\begin{cases} y(0) &= \frac{0^4}{24} = 0 \\ y'(0) &= \frac{0^3}{6} = 0 \\ y''(0) &= \frac{0^2}{2} = 0. \end{cases}$$

Thus

$$y = \frac{x^4}{24}.$$

Now to verify green's function with  $y(x) = \int_0^1 f(s)G(x;s)ds$ , we have

$$G(x;s) = \begin{cases} 0 & x < s \\ \frac{1}{2}(x^2 + s^2) - sx & x > s \end{cases}$$

therefore

$$\begin{aligned} y(x) &= \int_0^x s \left( \frac{1}{2}(x^2 + s^2) - sx \right) ds + \int_x^1 0 ds \\ &= \int_0^x \frac{sx^2}{2} + \frac{s^3}{2} - s^2 x ds \\ &= \left[ \frac{s^2 x^2}{4} + \frac{s^4}{8} - \frac{s^3 x}{3} \right]_0^x \\ &= \frac{x^4}{4} + \frac{x^4}{8} - \frac{x^4}{3} \\ &= \frac{x^4}{24}. \end{aligned}$$

We have showed both methods give the same results.

**Problem 4:** The Fourier transform is often useful for finding Green's functions. Consider the following partial differential equation for the function  $u(x, t)$ :

$$\begin{cases} u_{tt} - u_{xx} + u = 0 & -\infty < x < \infty \\ u(x, 0) = \delta(x - s) & s \in \mathbb{R} \\ u_t(x, 0) = 0. \end{cases}$$

**Part (a)** Apply a Fourier transform with respect to  $x$  to obtain a 2nd-order ordinary differential equation for the function  $\hat{u}(\omega, t)$ .

**Solution**

By applying the Fourier transform with respect to  $x$  such that  $\mathcal{F}\{u(x, t)\} = \hat{u}(\omega, t)$  we get

$$\begin{cases} u_{tt} - u_{xx} + u = 0 \\ u(x, 0) = \delta(x - s) \\ u_t(x, 0) = 0 \end{cases} \xrightarrow{\mathcal{F}} \begin{cases} \hat{u}_{tt} - (i\omega)^2 \hat{u} + \hat{u} = 0 & (1) \\ \hat{u}(\omega, 0) = \frac{1}{2\pi} e^{-i\omega s} & (2) \\ \hat{u}_t(\omega, 0) = 0. \end{cases}$$

which is a 2nd order ordinary differential equation. Second order due to  $\hat{u}_{tt}$  and ordinary due to derivatives only involving  $t$ .

<sup>(1)</sup>By derivative property of Fourier Transforms.

<sup>(2)</sup>By shift property and Fourier Transform of the Dirac Delta function.

**Part (b)** Find  $\hat{u}(\omega, t)$  by solving the differential equation from part (a).

**Solution** From part a we have the following:

$$\begin{cases} \hat{u}_{tt} - (i\omega)^2 \hat{u} + \hat{u} = 0 \\ \hat{u}(\omega, 0) = \frac{1}{2\pi} e^{-i\omega s} \\ \hat{u}_t(\omega, 0) = 0. \end{cases}$$

For  $\hat{u}_{tt} - (i\omega)^2 \hat{u} + \hat{u} = 0$ ,

$$\begin{aligned} \hat{u}_{tt} &= (i\omega)^2 \hat{u} - \hat{u} \implies \hat{u}_{tt} = (\omega^2 - 1) \hat{u} \\ \implies \hat{u} &= c_1 e^{\sqrt{\omega^2 - 1}t} + c_2 e^{-\sqrt{\omega^2 - 1}t} \quad (c_1, c_2 \in \mathbb{R}) \end{aligned} \quad (1)$$

<sup>(1)</sup>By solving for second-order linearly ordinary differential equation on Wolfram Alpha

together with  $\hat{u}_t(\omega, 0) = 0$

$$\begin{aligned} \hat{u}_t(\omega, 0) &= 0 \implies \frac{d}{dt}(c_1 e^{\sqrt{\omega^2 - 1}t} + c_2 e^{-\sqrt{\omega^2 - 1}t}) = 0 \\ \implies c_1 \sqrt{\omega^2 - 1} e^{\sqrt{\omega^2 - 1}(0)} - c_2 \sqrt{\omega^2 - 1} e^{-\sqrt{\omega^2 - 1}(0)} &= 0 \\ \implies c_1 \sqrt{\omega^2 - 1} &= c_2 \sqrt{\omega^2 - 1} \\ \implies c_1 &= c_2 \end{aligned}$$

along with  $\hat{u}(\omega, 0) = \frac{1}{2\pi}e^{-i\omega s}$

$$\begin{aligned} &\Rightarrow \frac{1}{2\pi}e^{-i\omega s} \underbrace{\left( c_1 e^{\sqrt{\omega^2-1}(0)} + c_1 e^{-\sqrt{\omega^2-1}(0)} \right)}_{=1} \\ &\Rightarrow c_1 = \frac{1}{2} \end{aligned}$$

therefore

$$\begin{aligned} \hat{u}(\omega, t) &= \frac{1}{4\pi}e^{-i\omega s} \left( e^{\sqrt{\omega^2-1}t} + e^{-\sqrt{\omega^2-1}t} \right) \\ &= \boxed{\frac{1}{2\pi}e^{-i\omega s} \cosh(t\sqrt{\omega^2-1})} \quad \text{since } e^x + e^{-x} = 2\cosh(x) \end{aligned}$$

**Part (c)** Invert the Fourier transform to obtain  $u(x, t)$

**Solution**

$$\begin{aligned} u(x, t) &= \int_{-\infty}^{\infty} \hat{u}(\omega, t) e^{i\omega t} d\omega \\ &= \int_{-\infty}^{\infty} \left[ \frac{1}{4\pi} e^{-i\omega s} \left( e^{\sqrt{\omega^2-1}t} + e^{-\sqrt{\omega^2-1}t} \right) \right] e^{i\omega t} d\omega \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[ e^{-i\omega s} \left( e^{\sqrt{\omega^2-1}t} + e^{-\sqrt{\omega^2-1}t} \right) \right] e^{i\omega t} d\omega \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( e^{\sqrt{\omega^2-1}t} + e^{-\sqrt{\omega^2-1}t} \right) e^{i\omega(t-s)} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \cosh(t\sqrt{\omega^2-1}) e^{i\omega(t-s)} d\omega \end{aligned}$$