Math 4990 - Final Exam

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Problem 1: Consider the integral function

$$J[y] = \int_1^2 \frac{y'(x)^2}{x} dx$$

The integrand here is strongly convex (on an appropriately defined set). Find the unique $y \in D$ that minimizes $J[y]$ over D, for the following cases. In each case, is the minimizer (if it exists) unique?

Part (a) $D = \{y \in C^1[1,2] : y(1) = 0, y(2) = 3\}$

Solution
If J is a minimum at $y \in D$ then y must satify the Euler-Lagrange equation $F_y(x,y,y') - \frac{d}{dx}F_{y'}(x,y,y') = 0$ where $F(x,y,y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 &\Longrightarrow \frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'} \\
&\Longrightarrow 0 = \frac{d}{dx}\frac{2y'}{x} \Longrightarrow \int 0 dx = \int \frac{d}{dx}\frac{2y'}{x} dx \\
&\Longrightarrow \frac{2y'}{x} = C \quad (C \in \mathbb{R}) \\
&\Longrightarrow y' = Bx \quad (B \in \mathbb{R}) \\
&\Longrightarrow \int y' dx = \int Bx dx \\
&\Longrightarrow y = c_1 x^2 + c_2 \quad (c1, c2 \in \mathbb{R})
\end{aligned}$$

y must also satisfy the boundary conditions:

$$\begin{cases} y(1) = c_1 + c_2 = 0 \\ y(2) = 4c_1 + c_2 = 3 \end{cases} \Longrightarrow c_1 = \frac{3}{5}, c_2 = -\frac{3}{5} \Longrightarrow \boxed{y = \frac{3}{5}x^2 - \frac{3}{5}}$$

We have shown with Euler-Lagrange equation the necessary conditions for y to minimize $J[y]$ and because we are given the integrand is strongly convex on the defined set D, we have the sufficient conditions to guarantee y is the unique minimizer.

Part (b) $D = \{y \in C^1[1,2] : y(2) = 3\}$

Solution
From the notes given in class, we have the following:
Theorem. If $F(x,y,y')$ is strongly convex in (y,y') then a solution $y(x)$ of the Differential Euler-Lagrange equation uniquely minimizes

$$J[y] = \int_a^b F(x,y,y')dx$$

over $\{y \in C'[a,b] : y(b) = B\}$ if $F_{y'}(a, y(a), y'(a)) = 0$.

We find the solution to the Euler-Lagrange equation $F_y(x,y,y') - \frac{d}{dx}F_{y'}(x,y,y') = 0$ where $F(x,y,y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \implies \frac{\partial F}{\partial y} &= \frac{d}{dx}\frac{\partial F}{\partial y'} \\
&\implies 0 = \frac{d}{dx}\frac{2y'}{x} \implies \int 0dx = \int \frac{d}{dx}\frac{2y'}{x}dx \\
&\implies \frac{2y'}{x} = C \quad (C \in \mathbb{R}) \\
&\implies y' = Bx \quad (B \in \mathbb{R}) \\
&\implies \int y'dx = \int Bxdx \\
&\implies y = c_1x^2 + c_2 \quad (c1, c2 \in \mathbb{R})
\end{aligned}$$

y must also satisfy the boundary condition:

$$\begin{aligned}
y(2) = 4c_1 + c_2 = 3 &\implies c_2 = 3 - 4c_1 \\
&\implies y = c_1x^2 + (3 - 4c_1) \\
&\implies y = c_1(x^2 - 4) + 3
\end{aligned}$$

thus

$$\begin{aligned}
F_{y'}(1, y(1), y'(1)) = 0 &\implies \frac{2y'(1)}{1} = 0 \\
&\implies y'(1) = 0 \\
&\implies 2c_1(1) \\
&\implies c_1 = 0
\end{aligned}$$

which implies

$$\boxed{y(x) = 3}$$

Given the integrand is strongly convex on the defined set D along with the theorem mentioned above, y uniquely minimizes $J[y]$ over D.

Part (c) $D = C^1[1,2]$

Solution
If J is a minimum at $y \in D$ then y must satify the Euler-Lagrange equation $F_y(x,y,y') - \frac{d}{dx}F_{y'}(x,y,y') = 0$ where $F(x,y,y') = \frac{y'(x)^2}{x}$.

$$\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \implies \frac{\partial F}{\partial y} &= \frac{d}{dx}\frac{\partial F}{\partial y'} \\
&\implies 0 = \frac{d}{dx}\frac{2y'}{x} \implies \int 0dx = \int \frac{d}{dx}\frac{2y'}{x}dx \\
&\implies \frac{2y'}{x} = C \quad (C \in \mathbb{R}) \\
&\implies y' = Bx \quad (B \in \mathbb{R}) \\
&\implies \int y'dx = \int Bxdx \\
&\implies y = c_1x^2 + c_2 \quad (c1, c2 \in \mathbb{R})
\end{aligned}$$

From strong convexity and observation we see if $y = c_2$, we achieve a minimum for $J[y] = \int_1^2 \frac{y'(x)^2}{x}dx$ since $y'(x)^2 \geq 0$ and $x \in [1,2]$ which implies $J[y] \geq 0$. However y is not a unique minimizer since $c2 \in \mathbb{R}$.

Problem 2: Let $\delta(x)$ be the Dirac delta function. Justify the identity

$$\delta(1-x^2) = \frac{\delta(x-1)+\delta(x+1)}{2}$$

.
Hint: consider the integral $\int_{-\infty}^{\infty} \delta(1-x^2)dx$

Solution For any continuous function f(x), we have

$$\int_{\infty}^{\infty} f(x)\delta(1-x^2)dx = \underbrace{\int_{0}^{\infty} f(x)\delta(1-x^2)dx}_{(1)} + \underbrace{\int_{-\infty}^{0} f(x)\delta(1-x^2)dx}_{(2)}$$

If we let $\sqrt{u} = x$ and $\frac{1}{2\sqrt{u}}du = dx$

$$(1) = \int_{0}^{\infty} \frac{f(\sqrt{u})\delta(1-u)}{2\sqrt{u}}du = \frac{f(1)}{2}$$

If we let $-\sqrt{u} = x$ and $-\frac{1}{2\sqrt{u}}du = dx$

$$(2) = -\int_{-\infty}^{0} \frac{f(-\sqrt{u})\delta(1-u)}{2\sqrt{u}}du = \int_{0}^{\infty} \frac{f(-\sqrt{u})\delta(1-u)}{2\sqrt{u}}du = \frac{f(-1)}{2}$$

thus

$$\begin{aligned}
\int_{\infty}^{\infty} f(x)\delta(1-x^2)dx &= \frac{1}{2}[f(1)+f(-1)] \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)\delta(x-1)dx + \int_{-\infty}^{\infty} f(x)\delta(x+1)dx\right] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} f(x)(\delta(x-1)+\delta(x+1))dx \\
&= \int_{-\infty}^{\infty} f(x)\frac{\delta(x-1)+\delta(x+1)}{2}dx
\end{aligned}$$

Since δ functions are defined only by how they behave in integrals we conclude

$$\boxed{\delta(1-x^2) = \frac{\delta(x-1)+\delta(x+1)}{2}}$$

Problem 3: Let $f : [0,1] \to \mathbb{R}$ be a given continuous function and consider the following differential equation for the function y:

$$\begin{cases} y'''(x) = f(x) & 0 \leq x \leq 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases} \tag{1}$$

Intuitively, finding $y(x)$ requires three integrations. This problem illustrates that by employing a Green's function we can actually find y via a *single* integral.

Part (a) Show that if the Green's function $u = G(x; s)$ satisfies

$$\begin{cases} u'''(x) = \delta(x-s) & 0 \leq x \leq 1 \\ u(0) = u'(0) = u''(0) = 0 \end{cases}$$

then the function

$$y(x) = \int_0^1 f(s)G(x;s)ds \tag{2}$$

satisfies (1).

Solution
From the notes given in class, we have the following:

Theorem.
The function $y(x) = \int_0^1 f(s)G(x;s)ds$ satisfies the differential equation

$$\begin{cases} L[y] = y'''(x) = f(x) & 0 \leq x \leq 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

Let us first observe

$$\begin{aligned} L[y] = y''' &= \frac{d^3}{dx^3}\Big(\int_0^1 f(s)G(x;s)ds\Big) \\ &= \int_0^1 \frac{d^3}{dx^3}\big(f(s)G(x;s)\big)ds \\ &= \int_0^1 f(s)\frac{d^3}{dx^3}\big(G(x;s)\big)ds \\ &= \int_0^1 f(s)G'''(x;s)ds \\ &= \int_0^1 f(s)L[G]ds \end{aligned}$$

and

$$L[G] = G'''(x;s) = \delta(x-s), \qquad 0 \leq x \leq 1$$

thus

$$\begin{aligned}
L[y] &= L\left[\int_0^1 f(s)G(x;s)ds\right] \\
&= \int_0^1 L\Big[f(s)G(x;s)\Big]ds \\
&= \int_0^1 f(s)L[G]ds \\
&= \int_0^1 f(s)\delta(x-s)ds \\
&= f(x)
\end{aligned}$$

along with the boundary conditions given $G(0;s) = 0$

$$\begin{aligned}
y(0) &= \int_0^1 f(s)\underbrace{G(0;s)}_{0}ds = 0 \\
y'(0) &= \int_0^1 f'(s)\underbrace{G(0;s)}_{0}ds = 0 \\
y''(0) &= \int_0^1 f''(s)\underbrace{G(0;s)}_{0}ds = 0
\end{aligned}$$

We have showed (2) satisfies (1).

Part (b) Find the Green's function $G(x:s)$ for this problem

$$\begin{cases} y'''(x) = f(x) & 0 \le x \le 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

Solution To find Green's function we solve for

$$\begin{cases} u'''(x) = \delta(x-s) & 0 \le x \le 1 \\ u(0) = u'(0) = u''(0) = 0 \end{cases}$$

where $u(x) = G(x;s)$.

For $x \neq s$ the differential equation is

$$u'''(x) = 0 \qquad\qquad 0 \le x \le 1$$

whose general solution is found to be

$$u(x) = Ax^2 + Bx + C \qquad (A, B, C \in \mathbb{R})$$

thus

$$u(x) = \begin{cases} Ax^2 + Bx + C & x < s \\ Ex^2 + Dx + F & x > s \end{cases}$$

The boundary conditions require

$$\begin{aligned} u(0) = 0 &\implies Ax^2 + Bx + C = 0 &&\implies C = 0 \\ u'(0) = 0 &\implies 2Ax + B = 0 &&\implies B = 0 \\ u''(0) = 0 &\implies 2A = 0 &&\implies A = 0. \end{aligned}$$

The Green function now has the form

$$u(x) = \begin{cases} 0 & x < s \\ Ex^2 + Dx + F & x > s. \end{cases}$$

We also need to satisfy the second derivative discontinuity condition since $u''' = \delta(x-s) \Rightarrow u'' = H(x-s)$

$$\begin{aligned} u''(s^+) - u''(s^-) = 1 &\implies 2E - 0 = 1 \\ &\implies 2E = 1 \\ &\implies E = \frac{1}{2} \end{aligned}$$

The Green function now has the form

$$u(x) = \begin{cases} 0 & x < s \\ \frac{1}{2}x^2 + Dx + F & x > s. \end{cases}$$

Together with the need to satisfy the first derivative continuity condition since $u' = \int H(x-s)dx$ is continuous.

$$\begin{aligned} u'(s^+) = u''(s^-) &\implies s + D = 0 \\ &\implies D = -s \end{aligned}$$

The Green function now has the form

$$u(x) = \begin{cases} 0 & x < s \\ \frac{1}{2}x^2 - sx + F & x > s. \end{cases}$$

Along with the continuity condition of u since u' is continuous which implies u is continuous.

$$\begin{aligned} u(s^+) = u(s^-) &\implies \frac{1}{2}s^2 - s^2 + F = 0 \\ &\implies -\frac{1}{2}s^2 + F = 0 \\ &\implies F = \frac{1}{2}s^2. \end{aligned}$$

Thus the Green function is as follows

$$\boxed{u(x) = \begin{cases} 0 & x < s \\ \frac{1}{2}(x^2 + s^2) - sx & x > s \end{cases}}$$

Part (c) For the case $f(x) = x$, solve (1) directly by integrating thrice. Verify that (2) gives the same result.

Solution We are given

$$\begin{cases} y'''(x) = x & 0 \leq x \leq 1 \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

By integrating $f(x) = x$ thrice,

$$\begin{aligned} \iiint x dx &\Longrightarrow \iint \frac{x^2}{2} + c_1 dx \\ &\Longrightarrow \int \frac{x^3}{6} + c_1 x + c_2 dx \\ &\Longrightarrow \frac{x^4}{24} + \frac{c_1 x^2}{2} + c_2 x + c_3 \qquad (c_1, c_2, c_3 \in R) \end{aligned}$$

and in order to satisfy the boundary conditions, $c_1 = c_2 = c_3 = 0$ since

$$\begin{cases} y(0) &= \frac{0^4}{24} = 0 \\ y'(0) &= \frac{0^3}{6} = 0 \\ y''(0) &= \frac{0^2}{3} = 0. \end{cases}$$

Thus

$$y = \frac{x^4}{24}.$$

Now to verify green's function with $y(x) = \int_0^1 f(s)G(x;s)ds$, we have

$$G(x;s) = \begin{cases} 0 & x < s \\ \frac{1}{2}(x^2 + s^2) - sx & x > s \end{cases}$$

therefore

$$\begin{aligned} y(x) &= \int_0^x s\Big(\frac{1}{2}(x^2 + s^2) - sx\Big) ds + \int_x^1 0 ds \\ &= \int_0^x \frac{sx^2}{2} + \frac{s^3}{2} - s^2 x ds \\ &= \left[\frac{s^2 x^2}{4} + \frac{s^4}{8} - \frac{s^3 x}{3}\right]_0^x \\ &= \frac{x^4}{4} + \frac{x^4}{8} - \frac{x^4}{3} \\ &= \frac{x^4}{24}. \end{aligned}$$

We have showed both methods give the same results.

Problem 4: The Fourier transform is often useful for finding Green's functions. Consider the following partial differential equation for the function $u(x,t)$:

$$\begin{cases} u_{tt} - u_{xx} + u = 0 & -\infty < x < \infty \\ u(x,0) = \delta(x-s) & s \in \mathbb{R} \\ u_t(x,0) = 0. \end{cases}$$

Part (a) Apply a Fourier transform with respect to x to obtain a 2nd-order ordinary differential equation for the function $\hat{u}(\omega, t)$.

Solution
By applying the Fourier transform with respect to x such that $\mathcal{F}\{u(x,t)\} = \hat{u}(\omega,t)$ we get

$$\begin{cases} u_{tt} - u_{xx} + u &= 0 \\ u(x,0) &= \delta(x-s) \\ u_t(x,0) &= 0 \end{cases} \quad \xrightarrow{\mathcal{F}} \quad \begin{cases} \hat{u}_{tt} - (i\omega)^2\hat{u} + \hat{u} &= 0 \quad ^{(1)} \\ \hat{u}(\omega,0) &= \frac{1}{2\pi}e^{-i\omega s} \quad ^{(2)} \\ \hat{u}_t(\omega,0) &= 0. \end{cases}$$

which is a 2nd order ordinary differential equation. Second order due to $\hat{u}_{tt}$ and ordinary due to derivatives only involving t.

(1)By derivative property of Fourier Transforms.

(2)By shift property and Fourier Transform of the Dirac Delta function.

Part (b) Find $\hat{u}(\omega,t)$ by solving the differential equation from part (a).

Solution From part a we have the following:

$$\begin{cases} \hat{u}_{tt} - (i\omega)^2\hat{u} + \hat{u} &= 0 \\ \hat{u}(\omega,0) &= \frac{1}{2\pi}e^{-i\omega s} \\ \hat{u}_t(\omega,0) &= 0. \end{cases}$$

For $\hat{u}_{tt} - (i\omega)^2\hat{u} + \hat{u} = 0$,

$$\begin{aligned} \hat{u}_{tt} = (i\omega)^2\hat{u} - \hat{u} &\Longrightarrow \hat{u}_{tt} = (-\omega^2 - 1)\hat{u} \\ &\Longrightarrow \hat{u} = c_1 e^{\sqrt{-\omega^2-1}t} + c_2 e^{-\sqrt{-\omega^2-1}t} \quad (c_1, c_2 \in \mathbb{R}) \qquad ^{(1)} \end{aligned}$$

(1)By solving for second-order linearly ordinary differential equation on Wolfram Alpha

together with $\hat{u}_t(\omega,0) = 0$

$$\begin{aligned} \hat{u}_t(\omega,0) = 0 &\Longrightarrow \frac{d}{dt}(c_1 e^{\sqrt{-\omega^2-1}t} + c_2 e^{-\sqrt{-\omega^2-1}t}) = 0 \\ &\Longrightarrow c_1\sqrt{-\omega^2-1}e^{\sqrt{-\omega^2-1}(0)} - c_2\sqrt{-\omega^2-1}e^{-\sqrt{-\omega^2-1}(0)} = 0 \\ &\Longrightarrow c_1\sqrt{-\omega^2-1} = c_2\sqrt{-\omega^2-1} \\ &\Longrightarrow c_1 = c_2 \end{aligned}$$

along with $\hat{u}(\omega, 0) = \frac{1}{2\pi}e^{-i\omega s}$

$$\implies \frac{1}{2\pi}e^{-i\omega s}\underbrace{\left(c_1 e^{\sqrt{-\omega^2-1}(0)} + c_1 e^{-\sqrt{-\omega^2-1}(0)}\right)}_{=1}$$

$$\implies c_1 = \frac{1}{2}$$

therefore

$$\begin{aligned}
\hat{u}(\omega, t) &= \frac{1}{4\pi}e^{-i\omega s}\left(e^{\sqrt{-\omega^2-1}t} + e^{-\sqrt{-\omega^2-1}t}\right) \\
&= \boxed{\frac{1}{2\pi}e^{-i\omega s}cosh\big(t\sqrt{-\omega^2-1}\big)} \qquad \text{since } e^x + e^-x = 2\cosh(x)
\end{aligned}$$

Part (c) Invert the Fourier transform to obtain $u(x, t)$

Solution

$$\begin{aligned}
u(x, t) &= \int_{-\infty}^{\infty} \hat{u}(\omega, t)e^{i\omega t}d\omega \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\cosh\big(t\sqrt{-\omega^2-1}\big)\right]e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cosh\big(t\sqrt{-\omega^2-1}\big)e^{i\omega(t-s)}d\omega
\end{aligned}$$

Problem 5: Consider the cubic equation

$$x^3 - x + \varepsilon = 0 \tag{3}$$

with $|\varepsilon| \ll 1$. Assume a series solution of the form

$$x(\varepsilon) = x_0 + x_1\varepsilon + x_2\varepsilon^2 + ...$$

Substitute this onto (3) and consider the limit $\varepsilon \to 0$ to determine the coefficients x_0, x_1, x_2 and thereby obtain a three-term asymptotic approximation for each of the three roots. Verify that your series gives reasonably accurate results for the cases $\varepsilon = 0.1$ and $\varepsilon = 0.01$.

Solution By substituting $x(\varepsilon) = x_0 + x_1\varepsilon + x_2\varepsilon^2 + ...$ into $x^3 - x + \varepsilon = 0$ we get

$$\begin{aligned}
x(\varepsilon)^3 - x(\varepsilon) + \varepsilon \Longrightarrow & (x_0 + x_1\varepsilon + x_2\varepsilon^2 + ...)^3 - (x_0 + x_1\varepsilon + x_2\varepsilon^2 + ...) + \varepsilon \\
\Longrightarrow & ({x_0}^3 + 3{x_0}^2x_1\varepsilon + 3{x_0}^2x_2e^2 + 3x_0{x_1}^2\varepsilon^2 + 6x_0x_1x_2e^3 + \\
& 3x_0{x_2}^2\varepsilon^4 + {x_1}^3\varepsilon^3 + 3{x_1}^2x_2\varepsilon^4 + 3x_1{x_2}^2\varepsilon^5 + {x_2}^3\varepsilon^6) \\
& - (x_0 + x_1\varepsilon + x_2\varepsilon^2) + \varepsilon = 0
\end{aligned}$$

For ε^0 terms:

$${x_0}^3 - x_0 = 0 \Longrightarrow \begin{cases} x_0 = -1 \\ x_0 = 0 \\ x_0 = 1 \end{cases}$$

For ε^1 terms:

$$3{x_0}^2x_1 - x_1 + 1 = 0 \Longrightarrow \begin{cases} x_0 = \pm 1 \Longrightarrow 3x_1 - x_1 + 1 = 0 & \Longrightarrow x_1 = -\frac{1}{2} \\ x_0 = 0 \Longrightarrow -x_1 + 1 = 0 & \Longrightarrow x_1 = 1 \end{cases}$$

For ε^2 terms:

$$3{x_0}^2x_2 + 3x_0{x_1}^2 - x_2 = 0 \Longrightarrow \begin{cases} x_0 = 1, x_1 = -\frac{1}{2} \Longrightarrow 3x_2 + \frac{3}{4} - x_2 = 0 & \Longrightarrow x_2 = -\frac{3}{8} \\ x_0 = -1, x_1 = -\frac{1}{2} \Longrightarrow 3x_2 - \frac{3}{4} - x_2 = 0 & \Longrightarrow x_2 = \frac{3}{8} \\ x_0 = 0, x_1 = 1 & \Longrightarrow x_2 = 0 \end{cases}$$

Thus the three-term asymptotic approximation for each of the three roots are:

$$x = \begin{cases} 1 - \frac{1}{2}\varepsilon - \frac{3}{8}\varepsilon^2 \\ -1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 \\ \varepsilon \end{cases}$$

Verification
For $\varepsilon = 0.1$, $x^3 - x + 0.1 = 0$,

$$x = 0.945649, \quad x = -1.04668, \quad x = 0.101031,$$

$$\begin{cases} 1 - \frac{1}{2}(0.1) - \frac{3}{8}(0.1)^2 & = 0.94625 \\ -1 - \frac{1}{2}(0.1) + \frac{3}{8}(0.1)^2 & = -1.04625 \\ 0.1 & = 0.1 \end{cases}$$

For $\varepsilon = 0.01$, $x^3 - x + 0.01 = 0$,

$$x = 0.994962, \quad x = -1.00496, \quad x = 0.010001,$$

$$\begin{cases} 1 - \frac{1}{2}(0.1) - \frac{3}{8}(0.1)^2 & = 0.9949625 \\ -1 - \frac{1}{2}(0.1) + \frac{3}{8}(0.1)^2 & = -1.004962 \\ 0.01 & = 0.01 \end{cases}$$

So we get reasonably accurate results for cases $\varepsilon = 0.1$ and $\varepsilon = 0.01$