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Sicp 1.13

No

Prove that Fib(n) is the closest integer to $\varphi^n/\sqrt{5}$ where $\varphi = (1+\sqrt{5})/2$

Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction and definition of the Fibonacci numbers to prove that $Fib(n) = (\varphi^n - \psi^n)/\sqrt{5}$

<u>Proof</u>

We will show by induction $Fib(n) = (\varphi^n - \psi^n)/\sqrt{5}$

Base case:

$$Fib(0) = (\varphi^0 - \psi^0)/\sqrt{5} = 0/\sqrt{5} = 0$$

$$Fib(1) = \left(\frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}}\right) = 1$$

Suppose

$$Fib(k) = (\varphi^k - \psi^k)/\sqrt{5}$$

then

$$Fib(k+1) = Fib(k) + Fib(k-1)$$

$$= \frac{\varphi^k - \psi^k}{\sqrt{5}} + \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}} = \frac{\varphi^k - \psi^k + \varphi^{k-1} - \psi^{k-1}}{\sqrt{5}}$$

$$= \frac{\varphi^{k-1}(\varphi+1) - \psi^{k-1}(\psi+1)}{\sqrt{5}} = \frac{\varphi^{k-1}\left(\frac{3+\sqrt{5}}{2}\right) - \psi^{k-1}\left(\frac{3-\sqrt{5}}{2}\right)}{\sqrt{5}}$$

$$= \frac{\varphi^{k-1}(\varphi^2) - \psi^{k-1}(\psi^2)}{\sqrt{5}} = \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}}$$

$$= (\varphi^k - \psi^k)/\sqrt{5}$$

To show Fib(n) is the closest integer to $\varphi^n/\sqrt{5}$, notice that

$$-\frac{1}{2} < \frac{-1}{\sqrt{5}} < -\frac{\psi^n}{\sqrt{5}} < \frac{1}{\sqrt{5}} < \frac{1}{2}$$

$$Fib(n) = \frac{\varphi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}} \Rightarrow Fib(n) - \frac{\varphi^n}{\sqrt{5}} = \frac{-\psi^n}{\sqrt{5}} \Rightarrow Fib(n) - \frac{\varphi^n}{\sqrt{5}} < 1/2$$

and $Fib(n) - \frac{\varphi^n}{\sqrt{5}} > -\frac{1}{2}$

thus

$$\left| Fib(n) - \frac{\varphi^n}{\sqrt{5}} \right| < \frac{1}{2} \qquad \text{Q.E.D}$$