Reference Weiqun Zhang's blog ~ SICP 1.14 excercise

SICP - Exercise 1.14

space and time complexity of (count-change)

Space complexity of count-change is $O(n)$ because the depth of the tree grows linearly with n. The time complexity is $O(n^M)$ where M is the number of denominations of the coin.

Let $N(n,M)$ be the number of steps required.

For $M=1$, we have
$$\begin{aligned} N(n,1) &= 1 + N(n-d,1) \\ &= 1+1+N(n-2d,1) \\ &= 2+N(n-2d,1) \\ &\approx n/d \sim O(n) \end{aligned}$$

For arbitrary n and M, we have
$$\begin{aligned} cc(n,M) &= cc(n,M-1) + cc(n-d_M,M) \\ &= cc(n,M-1) + cc(n-d_M,M-1) + cc(n-2d_M,M) \\ &= cc(n,M-1) + cc(n-d_M,M-1) + \dots + cc(n-i_M d_M,M-1) + cc(n-(i_M+1)d_M,M) \end{aligned}$$

here d_M is the denominator of coin number M, and i_M is floor of n/d.

Note that $n-(i_M+1)d_M < 0$. Hereafter drop the subscript M for simplicity.

Assuming $N(M,n-1) \sim O(n^{M-1})$ we have

$$\begin{aligned} N(M,n) &= N(M,n-1) + N(M-d,n-1) + \dots + N(M-id,n-1) + \underbrace{N(M-(i+1)d,n)}_{=1} \\ &\sim O(n^{M-1}) + O((n-d)^{M-1}) + \dots + O((n-id)^{M-1}) \\ &\sim O\left(\Sigma_{j=0}^{i} (n-jd)^{M-1}\right) \\ &\sim O\left(\Sigma_{j=0}^{i} (r+jd)^{M-1}\right) \quad \text{where } r \text{ is remainder from } n/d \end{aligned}$$

Binomial theorem
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Which by the binomial theorem, can be written as
$$\begin{aligned} &\sim O\left(\Sigma_{j=0}^{i} \Sigma_{k=0}^{M-1} \binom{M-1}{k} r^{M-1-k} (jd)^k\right) \\ &\sim O\left(\Sigma_{k=0}^{M-1} \binom{M-1}{k} d^k r^{M-1-k} \Sigma_{j=0}^{i} j^k\right) \end{aligned}$$

Since $\Sigma_{j=0}^{i} j^k$ dominating term is proportional to i^{k+1}, we have

$$\begin{aligned} &\sim O\left(\Sigma_{k=0}^{M-1} \binom{M-1}{k} (id)^k r^{M-1-k} i\right) \\ &\sim O\left((r+id)^{M-1} i\right) \\ &\sim O\left(n^{M-1} (n/d)\right) \\ &\sim O(n^M) \end{aligned}$$

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