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Neutrino Cyclotron Radiation from Superfluid Vortexes in Neutron Stars: A New Mechanism for Pulsar Spin Down

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Summary. In this paper the authors discuss a new type of neutrino emission mechanism of neutron stars – the neutrino (cyclotron radiation) emitted by neutron superfluid vortexes in the interior of neutron stars – calculate its power, and derive the increased rate of the pulsar spin down, $\dot{P}=\mathscr{A}e^{-\xi t}P^{-1}+\mathscr{B}G(n)P^2$. For pulsars with short periods ($P<P_m$, $P_m\approx 1$ s), the spin-down of rotation is mainly caused by the dissipation of electromagnetic radiation. For pulsars with long periods ($P>P_m$), it is mainly caused by the dissipation of neutrino radiation. The statistics of 269 pulsars having P, $\dot{P}$(log $\dot{P}=a+b$ log P) show that, when $P<0.5$ s, $b\lesssim -1.2$, when $P>1.25$ s, $b\sim 2.2$. These statistical results do not contradict the theoretical prediction.

Key words: superfluid vortex – neutrino radiation – spin down – neutron star

I. Introduction

In the early stage of pulsar evolution, when the internal temperature is as high as 10^{11}–10^{12} K, various neutrino processes can take place (e.g. URCA process, neutrino bremsstrahlung, photoneutrino process, annihilation process, plasma neutrino process etc.) (e.g., Chiu, 1965). According to Sawyer (1979), however, pulsars will cool rapidly to an internal temperature of 10^8 K. At this temperature only electromagnetic radiation is important, the neutrino radiation mentioned above and gravitational radiation can be neglected because their power is lower than that of the electromagnetic radiation.

In this paper we discuss a new type of neutrino emission mechanism – neutrino cyclotron radiation, and its effect.

According to the Weinberg-Salam theory on elementary particles, when neutrons move in a circle, neutrino pairs (ν, $\bar{\nu}$) can be emitted owing to the weak interaction of the neutral current. This is a cyclotron-synchrotron emission mechanism which is different from those neutrino processes described above. Even at absolute zero this radiation still exists.

Luo et al. (1978) have calculated the power spectrum of this cyclotron radiation through the method of Schwinger et al. (1976). Then they tried to apply their result to neutron stars, but they only thought about the neutrino radiation of the neutron star as a whole, so the power is only 10^{-8} erg/s. It is insignificant indeed.

Considering the neutron superfluid vortexes which exist in the interior of neutron stars, we recalculate the power of the neutrino cyclotron radiation, and find that it plays a very important role at a certain stage.

II. Neutron Superfluid Vortexes in the Interior of Neutron Stars

It is proposed that neutrons in the interior of neutron stars are in a superfluid state. That is to say, when the density is incresed to nuclear density, two neutrons (in a momentum space) will combine to form "bineutron" (Cooper pairs), and a feature of superfluidity will appear in the neutron system. According to the calculation by Yang and Clark (1971), when the neutron density $\varrho=8\ 10^{12}-4\ 10^{13}$ g/cm^3, the energy gap caused by the interaction of attraction of the $^1S_0$ wave between neutrons is very large, $\Delta\geqq 2$ MeV (the corresponding temperature of the normal-superfluid phase transformation $T_\lambda=\dfrac{\Delta}{1.76\ k_B}>10^{10}$ K, where k_B is the Boltzmann constant). When $\varrho\approx 2\ 10^{13}$ g/cm^3, $\Delta(^1S_0)$ reaches the largest value 2.35 MeV, and when $\varrho>1.4\ 10^{14}$ g/cm^3, $\Delta(^1S_0)\sim 0$. But according to the calculation by Hoffberg et al. (1970), when $\varrho>2.5\ 10^{14}$ g/cm^3, a larger anisotropic energy gap $\Delta(^3P_2)$ is produced by the interaction of attraction of the $^3P_2$ wave between neutrons. The neutron superfluid region can extend as far as the core of a neutron star. In the transition layer ($\Delta\sim 0$) between the $^1S_0$ neutron superfluid region and the $^3P_2$ one, however, these may be a neutron fluid layer in the normal phase.

Being analogous to II-type superconductivity, the neutron superfluid in the interior of neutron stars is in a vortex state, i.e., there are plenty of vortex lines (vortex filament). In general, these vortex filaments are arranged in a symmetric lattice, they are parallel to the axis of rotation of the neutron star and as a whole they revolve around the axis of rotation of the neutron star almost rigidly. The circulation of every vortex filament intensity Γ is quantized

$$\Gamma=n\Gamma_0,\qquad \Gamma_0=\frac{2\pi\hbar}{2m_n},\tag{1}$$

where n is a circulation quantum number of vortex, m_n is the mass of neutron, $\hbar$ is Planck's constant divided by 2π, Γ_0 is the intensity of the unit vortex quantum (see, e.g., Feynman, 1955).

It may be supposed that the core of the superfluid vortex is a cylindrical region of normal neutron fluid immersed in the superfluid neutron sea. As usual, the radius of the core of the vortex, a_0, is taken to be the coherent length of neutron fluid.

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Following formula (6) in Ruderman (1976), we have

$$a_0 = E_F/k_F\Delta, \tag{2}$$

where E_F is the Fermi level of neutrons, and k_F is the corresponding Fermi wave number. Then

$$a_0 \simeq (3\pi^2)^{1/3} \frac{\hbar^2}{2m_n^{4/3}} \frac{\varrho^{1/3}}{\Delta}, \tag{3}$$

where ϱ is the total density of neutron fluid. Outside the core of the vortex, neutrons are in a superfluid state. The superfluid neutrons revolve round the vortex line with a velocity (Feynman, 1955)

$$v_s(r) = \frac{n\hbar}{2m_n r}, \tag{4}$$

where r is the distance from the axis of the vortex filament. The distribution of the angular velocity of the neutron superfluid revolving around the vortex filament is

$$\omega_s(r) = \frac{n\hbar}{2m_n r^2}. \tag{5}$$

Therefore the revolution of superfluid neutrons around the vortex filament is placed in a differential state. Near $r \sim a_0$ the angular velocity reaches the largest value

$$\omega_c = \frac{n\hbar}{2m_n a_0^2} \tag{6}$$

and inside the core of the vortex ($r < a_0$) the normal neutron fluid revolves rigidly at angular velocity of ω_c.

According to Feynman (1955), the number of superfluid vortex filaments per unit area is $2\Omega/\bar{n}\Gamma_0$. Then the order of magnitude of the average separation b between vortex filaments and the total number N of the superfluid vortexes in the interior of a neutron star are respectively:

$$b = \left(\frac{\bar{n}\hbar}{2m_n\Omega}\right)^{1/2}, \tag{7}$$

$$N = \left(\frac{2m_n\Omega}{\bar{n}\hbar}\right) R^2, \tag{8}$$

where R and Ω are respectively the radius of neutron star and the angular velocity of rotation as a whole. $\bar{n}$ is the circulation quantum number of each vortex filament on the average.

The rotation energy of superfluid neutrons in the whole vortex filament, according to Feynman (1955), $E_s^{(v,f)} = \bar{H} \int_a^b \frac{1}{2}\varrho_s(r) v_s^2(r) 2\pi r dr$, where $\varrho_s(r)$ is the density distribution of superfluid neutrons, and $\bar{H}$ is an average length of vortex filaments. If we take it to be $\bar{H} = \frac{\pi}{2} R$, and ϱ_s to be constant, then

$$E_s^{(v,f)} = \frac{\pi^2}{8} \frac{n^2\hbar^2}{m_n^2} \varrho_s R \ln \frac{b}{a_0}, \tag{9}$$

or about $10^{14}\, n^2$ erg.

In the interior of neutron stars, $a_0 \approx 10^{-12}$ cm and $b = 10^{-3}$ cm, therefore, the core of vortex is very tiny, the distribution of vortex filaments is exceedingly sparse (the separation between vortex filaments reaches a macroscopic order of magnitude). It is usual to neglect the effect of the vortex core and the picture of the differential revolution of the superfluid around the vortex filament when one discusses the average motion of superfluid neutrons. But if we consider the new type of neutrino cyclotron radiation pointed out in the introduction of this paper, we cannot neglect the differential rotation of superfluid neutrons and the effect of normal neutrons revolving rigidly at high angular velocity at the core of vortex. On the contrary, they will play a decided role in the neutrino cyclotron radiation. This is the chief problem we want to investigate in this paper.

III. Neutrino Emission by Neutrons Undergoing Cyclotron Movement

According to the Weinberg-Salam theory, considering the weak interaction of neutral current, Luo et al. (1978, see Appendix A) have derived a power spectrum for one neutron in circular motion

$$p(\omega, a) \simeq \frac{G_v^2 \omega^8 a^2}{30\,(2\pi)^3} \qquad (c = \hbar = 1), \tag{10}$$

where G_v is weak interaction constant. In the usual C.G.S. system

$$p(\omega, r) = A\omega^8 r^2, \qquad A = 4.7\ 10^{-159}\ \mathrm{erg\ s^7\ cm^{-2}}. \tag{11}$$

This is the power in neutrino pairs emitted by a neutron revolving around the axis at angular velocity ω and at a distance from the axis r. The frequency of emitted neutrino pairs is equal to the angular velocity of the neutron ω.

IV. A Few Hypotheses

a) Classical Approximation

The discussion in the preceding paragraph is of neutrino emission of one neutron. For superfluid vortexes made of a large quantity of neutrons, which are in a macroscopic quantum state, there are two possible processes in which the cyclotron radiation of these neutrons appears. One is that the superfluid vortex filaments as a whole jump from one quantum state to another lower-lying one. In this case, we ought to use the method of quantum mechanics or quantum statistics to manage all the processes taking place in superfluid vortexes. But there is a second process of "local circulation" thru which neutrino cyclotron radiation can also be emitted without a change in the quantum number, and we will use the classical phenomenological method to explore the radiation problem in this paper (see, e.g. Feynman, 1955; Androdikashvili and Mamaladze, 1966). This process is as follows: the rotational velocities of superfluid neutrons will decrease during the emission of $(\nu, \bar{\nu})$ pairs due to the dissipation of its energy. According to the formula $v_s(r) = n\hbar/2m_n r$, these superfluid neutrons will drift out, then the transverse pressure exerting on the normal neutrons in the vortex cores will decrease. The normal neutrons will move out to $r > a_0$ and become superfluid ones. At the end of the vortexes, other normal neutrons located in the normal neutron layer in the interior of neutron stars will flow into vortex cores along the axes. At the same time the densities of superfluid neutrons at the boundaries of vortex lattices will increase, driving a flow into the normal neutron layer along the direction of axis. A "local circulation" will be formed in the superfluid vortex region in this way. It does not change the superfluid vortex states, n is constant, at the same time the rotational energy of normal neutrons revolving around the axis of neutron star as a whole is transferred to superfluid vortexes, and it leads to the spindown of the neutron star as a whole through the interaction of magnetic moment between normal neutrons and electrons. This process is very much analogous to Ekman pumping (Anderson, 1978; Greenspan, 1968). The condition, which allows it to be accomplished, is that the relaxation time of the interaction of magnetic moments between electrons and normal neutrons is less than the average radiation life of vortex filaments. We will discuss this condition further in Sect. VIII.

b) Non-equilibrium State Hypothesis

It is generally assumed that the superfluid vortex states are all in ground state, $n=1$, because it is the most favourable state from the point of view of energy. We could suppose, however, that if pulsars originate from catastrophic gravitational collapse during the late state of massive stars, resulting in an exceedingly large energy release from the interior of stars, then the interior should be in a chaotic state of turbulence. Because the time scale during which the internal temperature decreases from 10^{11} K to 10^{10} K is very short, the interior of neutron stars is not in an equilibrium thermodynamic state (see, e.g. Bisnovatyi-Kogan and Chechetkin, 1979).

It would be expected to nucleate the superfluid vortexes in $n>1$ state. As the neutron star evolves further, these superfluid vortexes will jump from high-lying excited states to low-lying ones. But we have already seen that the total rotational energy of superfluid neutrons in a whole vortex filament $E_s(n) \approx 10^{14}\, n^2$ erg, so $\Delta E_{21} = E_s(2) - 2E_s(1) \approx 10^{14}$ erg. This "difference of energy level" is very large. The larger n is, the larger ΔE will be. Therefore, it is necessary that the decay from a high-lying state to a low-lying one be accompanied by some drastic change. The pattern of vortexes will be rearranged also. This is not so easy to accomplish.

In conclusion, we would expect that the interior of relatively young neutron stars, such as pulsars, are not already in thermodynamic equilibrium state, but are still far from thermodynamic equilibrium. The observed period discontinuities and the "restless" behavior in pulsar periods might be the very results of non-equilibrium processes.

If we accept the Ekman mechanism thru which neutrino cyclotron radiation can be emitted without a change in the circulation quantum number, n, and if we accept that interior of relative young pulsars are not already in thermodynamic equilibrium state, so we could expect to have the superfluid vortexes in $n>1$ state for younger pulsars, then it is very reasonable to assume $n>1$ holds for the old neutron stars. So the vortex quantum number n will be reserved in all the formulae of this paper (see, e.g. Khalatnikov, 1971). The question of what quantum lavel the vortexes are in is a crucial one for the process that we are considering, and much more detailed study of this question is needed.

c) Magnetic Decay Model

In this paper we suppose that all of the electromagnetic radiation is converted from magnetic dipole radiation and adopt the magnetic decay model of pulsars (Ostriker and Gunn, 1969), then the total power of electromagnetic radiation W_{em} is

$$W_{em} = \frac{2\mu_0^2 e^{-\xi t}}{3c^3}\,\Omega^4.$$

For typical pulsars, we adopt the statistical value from Qu et al. (1976)

$$\langle \xi \rangle \simeq 1.3\ 10^{-6}/\mathrm{yr}.$$

In Sect. VIII we will show that the results we get in this paper are still available if the life time of magnetic field is much larger (say 10^{10} yr) or even no decay at all instead of 10^6 yr.

V. Neutrino Cyclotron Radiation of Superfluid Vortexes in the Interior of Neutron Stars

a) The Neutrino Cyclotron Radiation Power of Superfluid Neutrons Inside One Vortex

With a large quantity of neutrons, it is necessary to consider the coherence of their radiation. To discuss this problem in the configuration of "local circulation" is analogous to discussing the electromagnetic radiation of a charged particle system. If this system is small with a scale $a \ll \lambda$ the emitted wavelength, then the electromagnetic waves emitted by this system are strongly coherent, or vice versa. Because there is a different law of rotation between neutrons in the superfluid phase near a vortex filament and that in normal phase inside the core of the vortex filament, it is necessary, therefore, to take a different treatment for them, respectively.

The angular velocity of the superfluid neutrons at a distance r from the axis of the vortex filament is $\omega_s(r)$, and the corresponding wavelength is

$$\lambda_s(r) = \frac{2\pi c}{\omega_s(r)} = \frac{4\pi c m_n r^2}{n\hbar} \qquad (r > a_0). \tag{12}$$

Owing to $\frac{\lambda_s(\mathrm{r})}{\mathrm{r}} = \frac{4\pi c m_n}{n\hbar} r (r > a_0) > \frac{6\ 10^{14}}{n} a_0$ (C.G.S. system), when $n \ll 100$, $\lambda_s(r)/r \gg 1$, i.e., the wavelength of the neutrinos emitted by superfluid neutrons at a distance r from the axis of vortex filament is much larger than r. For that reason, it may be assumed that for neutrinos emitted by all the superfluid neutrons inside a ring of $r \to r + dr$ on a plane perpendicular to the vortex filament, not only are of equal wavelengths, but their phases are almost strictly coherent. The total power of the neutrino radiation within the coherent extent is equal to the radiation power of a single neutron multiplied by the square of the number of neutrons in this volume. The vortex filaments can be roughly divided into $\bar{H}/\lambda_s(r)$ pieces of cylindrical ring with a height $\lambda_s(\mathrm{r})$ in the axial direction. The neutrino radiation inside the cylinder is coherent, but that from different cylinders is non-coherent. The number of superfluid neutrons inside the cylindrical ring at a distance $r \to r + dr$ from the axis of vortex filament is

$$dN_1 = \frac{\varrho_s(r)}{m_n}\,\lambda_s(r) 2\pi r dr. \tag{13}$$

For convenience, introducing non-dimensional quantities

$$\omega' = \omega_s(\mathrm{r})/\omega_c \quad \text{and} \quad r' = r/a_0, \tag{14}$$

the distribution of angular velocity of rotation of superfluid neutrons (5) becomes

$$\omega'(r') = 1/(r')^2. \tag{15}$$

Following the description given above, the neutrino radiation power of all the superfluid neutrons inside the cylindrical ring at a distance $r \to r + dr$ from the axis of the vortex is

$$\frac{\bar{H}}{\lambda_s(r)}\, dN_1^2 p(\omega, r) = 4\pi^4 cA\, [\varrho_s(r)/m_n]^2 R\omega_c^7 a_0^6 \omega'^7(r) r'^4 (dr)^2.$$

The relation (15) between $\omega'(r')$ and r' shows that a δ-function: $\delta\left(\omega'(r') - \frac{1}{r'^2}\right)$ ought to be introduced in the above formula. After integrating through once the quadric differential in the formula above becomes a single differential. Thereupon, the neutrino radiation power of superfluid neutrons inside the cylindrical ring is

$$dw_1^{(s)}(\omega) = \pi^4 cAR \frac{\omega_c^7 a_0^6}{m_n^2} \varrho_s^2 \omega'^2 d\omega'. \tag{16}$$

After integrating this formula over the frequency of the neutrinos emitted by the superfluid neutrons, we obtain the total neutrino radiation power of all the superfluid neutrons in one vortex.

$$w_1^{(s)} = \int_{\omega'=0}^{\omega'=1} dw_1^{(s)}(\omega') = \frac{1}{3}\pi^4 cA \frac{\omega_c^7 a_0^6}{m_n^2} \varrho_s^2 R. \tag{17}$$

b) The Neutrino Radiation Power of Neutrons in Normal Phase Inside the Core of Every Vortex

The neutrons in normal phase inside the core of the vortex revolve rigidly at an unitary angular velocity ω_c. Therefore, the average radiation power of every neutron in the normal phase is $p(\omega) = A\omega_c^8 r^2 (0 < r < a_0)$. We can take $r^2 = \frac{1}{3}a_0^2$. Analogous to the preceding discussion, we obtain the neutrino radiation power from all of the neutrons in the normal phase inside the core of one vortex

$$w_1^{(n)}(\omega_c) = \frac{\bar{H}}{\lambda_n}\left(\lambda_n \pi a_0^2 \frac{\varrho_n}{m_n}\right)^2 \frac{1}{3} A\omega_c^8 a_0^2 = \frac{\pi^4}{3} cA \frac{\omega_c^7 a_0^6}{m_n^2} \varrho_n^2 R, \tag{18}$$

where ϱ_n is the (mass) density of the neutron fluid at the core of vortex. If we assume it is equal to the average superfluid density in the interior of neutron stars, then we have $w_1^{(n)} = w_1^{(s)}$.

c) The Total Neutrino Radiation Power of the Whole Neutron Star

For every vortex, the total power of neutrino cyclotron radiation from neutrons in the superfluid phase and neutrons in the normal phase ought to be

$$w_1 = w_1^{(s)} + w_1^{(n)}$$

$$w_1 = \frac{4}{3^{11/3}\pi^{4/3}} cA \frac{m_n^{5/3}}{\hbar^9} \frac{\Delta^8}{\varrho^{2/3}} n^7 R \tag{19}$$

so that the total power of neutrino cyclotron radiation from the whole neutron star is

$$W_\nu = N\bar{w}_1 = 8(3^{11}\pi^4)^{-1/3} cAm_n^{8/3}\hbar^{-10}\overline{\left(\frac{\Delta^8}{\varrho^{2/3}} n^7\right)}(\bar{n})^{-1}\Omega R^3, \tag{20}$$

where the bar is used to represent an average of some physical quantity for all the vortex lines. We assume

$$\overline{(\Delta^8 \varrho^{-2/3} n^7)} \simeq \overline{(\Delta^8/\varrho^{2/3})}\, \overline{n^7}$$

then

$$W_\nu = BG(n)\Omega, \tag{21}$$

where

$$B = 1.0\ 10^{57}\overline{(\Delta^8/\varrho^{2/3})} R^3, \tag{22}$$

$$G(n) = \overline{n^7}/\bar{n}. \tag{23}$$

It is evident that the value of B is very sensitive to the value of the energy gap Δ. When we take an average over a particular vortex filament, that region in which the energy gap Δ reaches the largest value (when $\varrho = 2\ 10^{13}$ g/cm^3, $\Delta \simeq \Delta_M = 2.35$ MeV) has the predominant weight in contributing to B's value. For this reason, we can approximately take $\Delta = 2.3$ MeV, $\varrho \simeq 2\ 10^{13}$ g/cm^3, hence, $B \simeq 1.1\ 10^5 R^3$ (C.G.S. system). If we take the radius of the neutron star to be 10–20 km, then $B = 10^{23}$–10^{24} erg. If $n = 5$, then the maximum neutrino luminosity of this neutron star is only of the order of 10^{30} erg/s. This seems to be much less than the luminosity of the electromagnetic radiation. Under two conditions given below, however, the importance of this neutrino emission mechanism will be shown:

1. For old pulsars, the electromagnetic radiation should be greatly decreased, so that neutrino radiation becomes more important relatively, and there is a possibility of seeing some observational effect. We shall go into this problem in detail in the next section.

2. For very young pulsars, the interior is probably in a more turbulent condition, and the circulation quantum number of the vortex n is likely to be very high, which makes the neutrino radiation very strong. It may thus reach the same order of magnitude as the luminosity of electromagnetic radiation. We shall briefly discuss this problem at the end.

VI. The Effect of Neutrino Cyclotron Radiation on the Spin-down of Pulsars

It is usually suggested that vortexes in a superfluid are conservative (Anderson, 1966). According to Feynman (1955), the sum of the vortex intensity $\Gamma_i\ (= n\Gamma_0)$ of every vortex line is determined by the circulation speed of the whole neutron star

$$\Sigma\Gamma_i = 2\pi R^2 \Omega. \tag{24}$$

It can be assumed that during a relatively stable stage, i.e. non-glitch, the picture of superfluid vortex motion in the interior of neutron stars is quasi-stationary. But the neutrino pairs emitted by neutrons in gyroscopic motion must still carry some energy and momentum. It can be supposed then through some mechanism (e.g. the local circulation process), that the rotational energy of the neutron star as a whole can be gradually converted to the interior vortex energy, and it can make the superfluid vortex motion stabilized. In this way, the neutrino cyclotron radiation described above provides a new mechanism for slowing down the rotation of neutron stars.

The kinetic energy of the neutron star as a whole is $E_{\mathrm{rot}} = \frac{1}{2} I\Omega^2$, where I is the rotational inertial of the whole neutron star. If we suggest that the variation of I with time be very slow, so that dI/dt can be neglected. If we do not consider the gravitational radiation at the earliest stage of neutron star evolution or those known neutrino radiations mentioned in the introduction of this paper, but think only about electromagnetic radiation and neutrino cyclotron radiation discussed in this paper, then we have

$$\dot{P} = \mathscr{A} e^{-\xi t} P^{-1} + \mathscr{B} G(n) P^2, \tag{25}$$

where

$$\mathscr{A} = \frac{8\pi^2}{3c^3} \frac{\mu_0^2}{I}, \qquad \mathscr{B} = \frac{B}{2\pi I}. \tag{26}$$

For typical pulsars, we adopt the statistical values from Qu et al. (1976)

$$\left\langle \frac{I}{\mu_0^2} \right\rangle \simeq 7.8\ 10^{-17}, \qquad \langle \xi \rangle \simeq 1.3\ 10^{-6}/\mathrm{yr}.$$

If we investigate pulsars of older age, we can take $I \sim 10^{44}$ gcm^2, then

$$\mathscr{A} \simeq 1.0\ 10^{-14}\,\mathrm{s}, \qquad \mathscr{B} \simeq 10^{-22} \to 10^{-21}\,\mathrm{s}^{-2}. \tag{27}$$

The variation of the period P with the age of pulsars, t, can be derived from Eq. (25). But because the variation of $G(n)$ with t is not clear, and the variation of the parameters I and R (contained in the coefficients $\mathscr{A}$ and $\mathscr{B}$) are not clear either, to solve Eq. (25) is difficult. Nevertheless, we can discuss it qualitatively.

The age and period for which the two terms are equal can be represented by t_m and P_m, respectively, and

$$P_m^3=\frac{\mathscr{A}}{\mathscr{B}G(n)}\,e^{-\xi t_m}. \tag{28}$$

When $t<t_m(P<P_m)$, the contribution of neutrino cyclotron radiation [the second term in formula (25)] can be neglected. That is, for pulsars with short period, the period growth is mainly caused by dissipation of electromagnetic radiation. On the contrary, when $t>t_m(P>P_m)$, the contribution of neutrino cyclotron radiation will surpass that of electromagnetic radiation, and the spin-down is mainly caused by the neutrino radiation. In order to find P_m, we must integrate formula (25) over t from $t=0$ to $t=t_m$. We have

$$P_m^2=\frac{2\mathscr{A}}{\xi}\,(1-e^{-\xi t_m})+2\mathscr{B}\int_0^{t_m} G(n)\,P^3\,dP. \tag{29}$$

It is difficult to solve the integral of second term, but it may be supposed that its contribution is less than the first term. If we neglect the second term, P_m and t_m can be obtained from formulas (29) and (28) by the iterative method. Generally speaking, if $\xi t_m>5$ (see below), then the term of $e^{-\xi t_m}$ in formula (29) can be neglected, and we can directly calculate $P_m\simeq(2\mathscr{A}/\xi)^{1/2}\simeq 0.78$ s. The P_m obtained like this, however, is only a lower limit. The reason is as follows:

1. The statistical average value, $\langle I/\mu_0^2\rangle$, adopted from Qu et al. (1976) is obtained for pulsars with short periods. For pulsars with long periods, the value of I will probably be smaller. Then the value of $\mathscr{A}$ will become larger, and the value of P_m will probably be larger than that calculated here.

2. If we consider the second term of formula (29), it also makes the value of P_m grow. But the value of P_m can not exceed $\sqrt{2}$ times the above calculated one because the second term is less than the first one in formula (29). Owing to these indefinite factors, we can only roughly estimate that $P_m\lesssim 1$ s, and the value of t_m can be evaluated from formula (28)

$$t_m=\frac{2.3}{\xi}\,\{\log\mathscr{A}-\log\mathscr{B}-\log G(n)-3\log P_m\}$$
$$=1.77\,\{\genfrac{}{}{0pt}{}{7}{8}-\log G(n)\}\ (10^6\,\mathrm{yr}), \tag{30}$$

where the upper value is corresponding to $\mathscr{B}=10^{-21}$, and the lower is corresponding to $\mathscr{B}=10^{-22}$.

It is evident from formula (23), that $n_{\max}$ will have the largest weight when taking an average of n^7. If there is an $\alpha\leq 1$ fraction of the superfluid vortex filaments having the highest circulation quantum number $n_{\max}$, and a $(1-\alpha)$ fraction having quantum number n', the corresponding value of $G(n)$ and t_m are then listed in Table 1.

Table 1. The value of $G(n)$ *and* t_m, corresponding to different $n_{\max}$ and α

$n_{\max}$	α	n'	$\bar{n}$	$G(n)$	$t\,(10^6\,\mathrm{yr})$[a]	$\mathscr{B}G(n)$
3	0.5	2	2.5	$4.6\ 10^2$	7.7 9.4	$\sim 10^{-19}$
3	1	3	3	$7.3\ 10^2$	7.3 9.1	
4	0.5	3	3.5	$2.7\ 10^3$	6.3 8.1	$\sim 10^{-18}$
4	1	4	4	$4.1\ 10^3$	6.0 7.8	
5	0.5	4	4.5	$1.1\ 10^4$	5.3 7.0	
5	1	5	5	$1.6\ 10^4$	5.0 6.7	
6	0.5	5	5.5	$1.3\ 10^4$	4.4 6.2	
6	1	6	6	$4.7\ 10^4$	4.1 5.9	$\sim 10^{-17}$
7	0.5	6	6.5	$8.5\ 10^4$	3.7 5.4	
7	1	7	7	$1.2\ 10^5$	3.4 5.2	
8	0.5	7	7.5	$2.0\ 10^5$	3.0 4.8	
8	1	8	8	$2.6\ 10^5$	2.8 4.6	$\sim 10^{-16}$
9	0.5	8	8.5	$4.1\ 10^5$	2.5 4.2	
9	1	9	9	$5.3\ 10^5$	2.3 4.0	
10	0.5	9	9.5	$7.8\ 10^5$	2.0 3.7	
10	1	10	10	$1.0\ 10^6$	1.8 3.5	$\sim 10^{-15}$

[a] The upper value is corresponding to $\mathscr{B}=10^{-21}$, and the lower one $\mathscr{B}=10^{-22}$

VII. Observational Effects

The Statistical Relation Between the Spindown Rate of Pulsars $\dot{P}$ and the Period P

We can draw a conclusion from formula (25): when $P^3\ll P_m^3$ (the magnetic dipole radiation dominates), it ought to have $\log\dot{P}\simeq a_1+b_1\log P$, and $b_1<-1$. When $P^3\gg P_m^3$ (the neutrino cyclotron radiation dominates), the $\log\dot{P}=a_2+b_2\log P$, and $b_2=2$. In order to test whether this theoretical prediction is correct or not, we have performed a statistical analysis of the relation between P and $\dot{P}$ for pulsars having measured P and $\dot{P}$. The regression equation adopted is $\log\dot{P}=a+b\log P$, the statistical results are tabulated in Table 2.

It can be seen from Table 2, the observed value of correlation coefficient for $P<0.5$ s is just significant at the 1% level of significance, and obviously significant for $P>1.1$ s, $P>1.2$ s, and $P>1.25$ s at the same level. The regression coefficients are about -1 and $+2$, respectively.

In order to realize that this statistical results are not due to fluctuation, we use v^2-test and t-test, respectively, to check if there are significant differences in standard deviations and regression coefficients for $P<0.5$ s, $P>1.1$ s, $P>1.2$ s, and $P>1.25$ s at the 1% level. The results are as follows. The standard deviations for $P>1.1$ s, $P>1.2$ s, and $P>1.25$ s do not differ significantly each other at the 1% level of significance, but only the standard deviation for $P>1.25$ s does not differ significantly from that for P

Table 2

Statistical region P(s)	Star number	Correlation initial value ($\alpha=0.01$)	Coefficient observed value R	Standard deviation s	b	Standard deviation for b S_b
<0.5	90	0.270	0.270	0.869	-1.228	0.466
<0.6	121	0.240	0.137	0.853	-0.569	0.377
<0.8	162	0.210	0.040	0.823	-0.147	0.293
<1.0	190	0.190	0.023	0.812	$+0.080$	0.250
>0.9	90	0.270	0.262	0.633	$+1.253$	0.493
>1.1	69	0.309	0.343	0.654	$+1.904$	0.638
>1.20	60	0.331	0.432	0.612	$+2.367$	0.650
>1.25	52	0.354	0.386	0.644	$+2.190$	0.740

<0.5 s at the same level. On the other hand, the regression coefficients for $P>1.1$ s, $P>1.2$ s, and $P>1.25$ s do not differ significantly each other at the 1% level. The regression coefficient for $P>1.25$ s, however, do differ significantly from that for $P<0.5$ s at the same level. Therefore, we can say that for $P<0.5$ s and $P>1.25$ s the relation between log $\dot{P}$ and log P is significantly different which is just what is predicted from our model.

Figure 1 is a diagram of log $\dot{P}$ vs. log P for 269 pulsars with measured P and $\dot{P}$ (see, Backus et al., 1980; Newton et al., 1980). This relation changes from an inverse proportionality to a direct one as the pulsars evolve. The correlation is rather weak around log $P\sim -0.1$ (corresponding $P\sim 0.8$ s, the shadow region in diagram), where we expect the two emission mechanisms ($\propto P^{-1}$ and $\propto P^2$) is to be equally effective. The dashed lines in diagram are theoretical values for $\log \dot{P} = \log \mathscr{A} e^{-\xi t} P^{-1}$ and $\log \dot{P} = \log \mathscr{B} G(n) P^2$, respectively. The solid lines are the best fit to the data from the statistical results.

VIII. Discussion

a) On the Transfer Problem of Rotational Energy of Neutron Stars as a Whole to the Energy of Vortex Motion

If there is no energy compensation, the characteristic time (or "average life") in which every superfluid vortex line of circulation quantum number n can exist is about

$$\tau \simeq \frac{E_s^{(v,f)}}{w_1} = K \frac{\varrho^{5/3}}{\Delta^8} \frac{1}{n^5} \ln \frac{b}{a_0} \text{ (s)}$$

$$K = 6.1\ 10^{-61} \qquad \text{(C.G.S. system).} \tag{31}$$

Substituting $\varrho \sim 2\ 10^{13}$ g/cm^3, $\Delta \simeq 2.3$ MeV, we have

$$\tau \simeq \frac{6\ 10^6}{n^5} \{1 + 0.05 \log n - 0.05 \log \Omega\} \text{ (s).} \tag{32}$$

Hence a superfluid vortex filament will exhaust its entire energy in a time of the order of months divided by n^5. The higher the quantum number of the vortex state n is, the shorter the "average life" will be.

However, in Sect. IV we propose a process named "local circulation" which could continuously transfer some of the rotation energy of the neutron star as a whole to the internal vortexes and thus maintain the superfluid vortexes. One factor in this process is the interaction between the magnetic moments of electrons and normal neutrons at the core of the vortexes. Feibelman (1971) has calculated the rotational energy of the neutron star as a whole may be transferred to the internal vortexes. Feibelman (1971) has calculated the relaxation time of this interaction to be about one day (for $\Delta \sim 1.7$ MeV) to a few years (for $\Delta \sim 2.4$ MeV). As is pointed out by Ruderman (1976), however, if the region between the $^1S_0$ neutron superfluid region and the $^3P_2$ one in the interior of neutron stars is a layer made of normal neutron fluid, the time of interaction between the magnetic moments of electrons and neutrons is only of a few seconds. Thereupon, if n is not too high ($n \sim 10$), the relaxation time of this interaction is less than the "average life" of vortex filaments described above. The time scale is thus sufficient. Further investigation of this process, of course, is required.

b) On the Synchrotron Emission (Neutrino) in the Early Stage of Pulsar Evolution

In this paper we have discussed only cases of relatively low quantum numbers for the neutron superfluid vortex state, i.e., $n \ll 100$. When pulsars are just born, however, the interior may be in a highly turbulent condition due to the catastrophic gravitational collapse, and the superfluid vortexes are likely to be in much higher excited states where n can be very large ($n \sim 100$). At that time the rotation speed of superfluid neutrons around the axis of vortex filaments near the cores of vortexes will approach the speed of light. We then have to treat synchrotron radiation (neutrino) of relativistic neutrons. And some new phenomena might appear. This problem is under study now.

c) On the Transfer of Neutrinos

Above $\varrho \sim 10^{11}$ g/cm^3 there is absorption of neutrinos due to neutral currents interactions and hence there is a problem of transfer neutrinos. However, in our case it may not be considered. First of all, one important factor is the effect of degeneracy of electrons in the neutron stars. The electron degeneracy causes inhibition in all reactions that have electrons in the final state, such as the neutrino-electron scattering and the neutrino absorption by neutrons going to a proton plus an electron (see, e.g. Freedman et al., 1977; Lichtenstadt et al., 1978).

The main contribution to the interaction of neutrinos with matter comes from absorption by free nucleons and from elastic scattering by protons and neutrons as well as coherent scattering

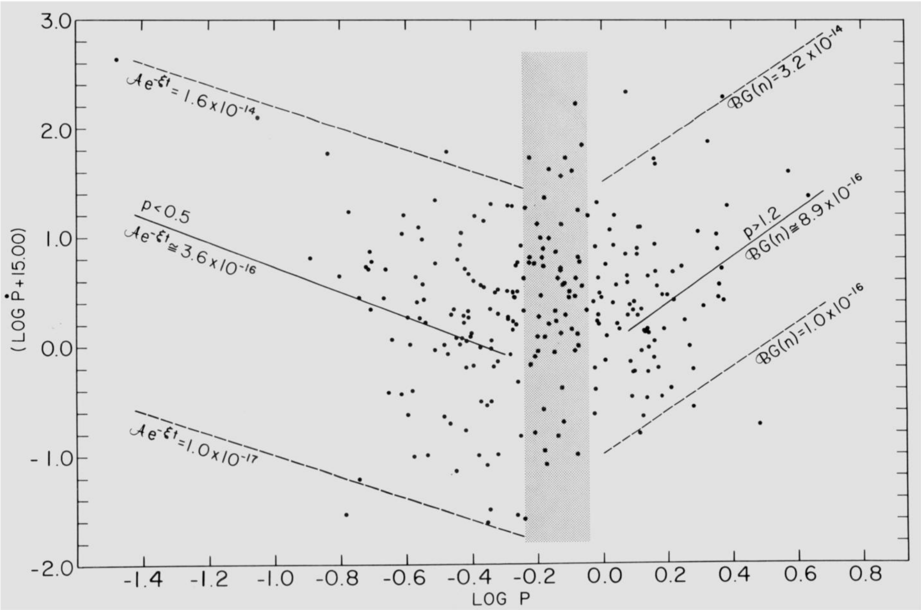

Fig. 1. It is a diagram of log $\dot{P}$ vs. log P for 255 pulsars with measured P and $\dot{P}$. It is obvious that for $P<1$ s and $P>1$ s the relation between log $\dot{P}$ and log P is very different. When $P<1$ s the magnetic dipole radiation dominates, when $P>1$ s the neutrino radiation dominates

by nuclei (see, e.g. Nadëzhin and Otroshchenko, 1980; Lamb and Pethick, 1976). The contribution from neutrino-proton scattering could be omitted since it is expected to be always negligible compared with the other two scattering contributions as pointed out by Lamb and Pethick (1976).

As for the absorption process, because of the effect of electron degeneracy we just consider the absorption process in the reaction $\bar{\nu}+p\rightarrow n+e^+$, the mean free path of antineutrinos is given by (Nadëzhin and Otroshchenko, 1980)

$$l_\nu^{-1}=\frac{\sigma_0}{m_p}\frac{\varrho}{1+\theta}\left[\frac{Q_{\bar{\nu}}}{1-\exp\left(-\dfrac{E_{e+}+\psi_e}{kT}\right)}\right]\cdot\left[1+\exp\left(-\frac{E_{\bar{\nu}}+\psi_\nu}{kT}\right)\right], \tag{33}$$

where m_p is the proton mass; $\sigma_0=1.63\ 10^{-44}\ \mathrm{cm}^2$; $\theta=n_n/n_p$ is the ratio of the neutron and proton number densities; ψ_ν, ψ_e are the chemical potential of neutrinos and electrons, respectively; and Q_ν are expressed by

$$Q_\nu=\begin{cases}0 & (E_{\bar{\nu}}\leqq 3.53\,m_ec^2)\\ 1.488\left(\dfrac{E_{\bar{\nu}}}{m_ec^2}-2.53\right)\left[\left(\dfrac{E_{\bar{\nu}}}{m_ec^2}-2.53\right)^2-1\right]^{1/2} & (E_{\bar{\nu}}>3.53\,m_ec^2).\end{cases} \tag{34}$$

The antineutrino energies emitted in our case are $E_{\bar{\nu}}\lesssim\hbar\omega_c\sim1$ MeV, it is less than $3.53\,m_ec^2$, then $Q_\nu=0$ and $l_\nu\rightarrow\infty$.

As for the elastic scattering transport processes, to a good approximation, the neutrino mean free path is given by (Lamb and Pethick, 1976)

$$\lambda_{e1}\approx1.0\ 10^6\left(\frac{\varrho}{10^{12}\,\mathrm{g\,cm}^3}\right)^{-1}\left(\frac{1}{12}X_h\bar{A}+X_n\right)^{-1}\left(\frac{\varepsilon_\nu}{10\,\mathrm{MeV}}\right)^{-2}. \tag{35}$$

Here ϱ is mass density, ε_ν is the neutrino energy, X_h and X_n are the mass fractions of heavy nuclei ($A>1$) and free neutrons and

$\bar{A}=\Sigma_{A>1}X_AA/\Sigma_{A>1}X_A$

is the mass fraction average of A (atomic mass number) for all $A>1$.

In our case, $\varepsilon_\nu\lesssim1$ MeV, $\varrho\sim10^{13}$, so the $\lambda_{e1}\sim10^2$ km if we take $X_h\bar{A}+12\,X_n\sim12$ following Lamb and Pethick (1976). If we consider the majority of neutrons and protons in neutron stars are at superfluid state and superconductivity state, respectively, the scattering probability would be much smaller than that given by Lamb and Pethick, and the mean free path λ_{e1} would be much larger than that given above.

So, roughly speaking, in our case the emitted neutrinos will leave the star, carrying angular momentum without significant absorption.

e) On the Frequency Distribution of Period of Pulsars

The bulk of the pulsars have periods less than 1 s, and the number of pulsars with $P>1$ s decreases drastically.

Gunn and Ostriker (1970) tried to explain this frequency distribution. They assumed that the distribution of the initial pulsar magnetic moment (or surface magnetic field) was normal. And they then assumed that the magnetic moment decayed with time in order to account for the lack of pulsars with $P\gg1$ s. But this would also predict that those pulsars with $P>1$ s would have decreasingly smaller $\dot{P}$'s, not increasingly larger $\dot{P}$'s as is observed (Fig. 1).

According to our model, when the pulsar period becomes longer than $P_m\sim1$ s, the effect of neutrino cyclotron radiation on the spin-down of pulsars will surpass that of electromagnetic radiation. Moreover the longer the period is, the faster the spin-down will be (proportional to P^2), and this will make the angular velocity of rotation Ω very small. Because the power of electromagnetic radiation is proportional to Ω^4, it will rapidly decrease to below the observational flux threshold. In addition, the time spent by pulsars in the region with $P>1$ s rapidly decreases, so that the probability to find a pulsar with such periods is much smaller.

Owing to these two factors, the observed number of pulsars with longer periods will be greatly decreased. In summary, we see that the neutrino cyclotron radiation appears to be better explain the observed frequency distribution. We shall discuss this problem elsewhere.

Conclusion

We have proposed a new type of pulsar emission and spindown mechanism by neutrino cyclotron emission from neutron superfluid vortexes in the interior of the star. Statistical studies of pulsars P and $\dot{P}$ do not contradict the prediction of this emission mechanism. Although the results obtained are interesting, the discussion of it is rather rough. As usual much more work remains to be done than has been done, especially in the following aspects: the details of the local circulation process, the transfer of the rotational energy of the neutron star as a whole to the vortexes, the quantum level of vortexes and the transfer of neutrinos emitted.

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Appendix

Following Schwinger et al. (Schwinger et al., 1976) Luo et al. adopt a vacuum persistent amplitude

$$\langle 0_+|0_-\rangle^J = e^{iW} \tag{A1}$$

in which the background filed of neutrons has already been included in the vacuum state, the superscript J indicates the situation in the external field (cyclotron movement). W is complex, exp $(-\mathrm{Im}\ W)$ represents cyclotron radiation of neutrons, i.e. the radiation power is related to Im W (Schwinger et al., 1976). The Hamiltonian density of weak interaction due to the neutral current is

$$\hat{H}_I = \frac{1}{\sqrt{2}} G_v J_\mu \bar{\Psi}_v \gamma_\mu (1+\gamma_5)\Psi_v, \tag{A2}$$

where G_v is the weak interaction constant, J_μ is the four vector of neutral current density, Ψ_v is the field operator of neutrino, γ_μ and γ_5 are Dirac matrices. If we only consider the process in which neutrinos and antineutrinos are emitted by neutrons in cyclotron motion through weak interaction of the neutral current. In that case the only part of field operators we must consider is that for creation of neutrinos and antineutrinos

$$\begin{aligned}\bar{\Psi}_v^{(+)}(x) &= \frac{1}{(2\pi)^{3/2}} \sum_{t=\pm 1} \int d^3\vec{p}_v a_t^*(\vec{p}_v)\bar{u}_{vt}(\vec{p}_v)e^{-i(p_v x)} \\ \Psi_v^{(+)}(x) &= \frac{1}{(2\pi)^{3/2}} \sum_{t=\pm 1} \int d^3\vec{p}_{\bar{v}} b_t^*(\vec{p}_{\bar{v}})v_{vt}(\vec{p}_{\bar{v}})e^{-i(p_{\bar{v}} x)}\end{aligned} \tag{A3}$$

here $u_{vt}(\vec{p}_v)$ and $v_{vt}(\vec{p}_{\bar{v}})$ are the Dirac wave functions of neutrinos and antineutrinos, respectively. $a_t^*(\vec{p}_v)$ and $b_t^*(\vec{p}_{\bar{v}})$ are the corresponding creation operators of neutrinos and antineutrinos, respectively.

According to the usual approach, we can get

$$\begin{aligned} iW = \frac{G_v}{(2\pi)^8} \int d^4x \int d^4x' Sp \Big\{ \gamma_\mu \int d^4p \frac{i\hat{p}}{p^2} e^{ip(x-x')}\gamma_v \\ \cdot \int d^4q \frac{i\hat{q}}{q^2} e^{iq(x-x')} \Big\} J_\mu(x) J_v(x') \end{aligned} \tag{A4}$$

when the rotation velocities of neutrons are less than the speed of light, the four vector current can be treated classically (Schwinger et al., 1976)

$$\varrho(\boldsymbol{r},t) = \sum_i \delta(\boldsymbol{r}-\boldsymbol{a}_i(t)), \quad \boldsymbol{J}(\boldsymbol{r},t) = \sum_i \boldsymbol{\omega}\times\boldsymbol{a}_i(t)\,\delta(\boldsymbol{r}-\boldsymbol{a}_i(t)), \tag{A5}$$

where $\boldsymbol{a}_i$ is the vector radius of the ith neutron, $\boldsymbol{\omega}$ is the corresponding vector angular velocity. When $v=|\boldsymbol{\omega}\times\boldsymbol{a}|\ll 1$ (in the unit system $c=\hbar=1$), Luo et al. have derived a power spectrum for one neutron in circular motion

$$p(\omega,t) = \sum_{l=1}^{\infty} \delta(\omega - l\omega')p_l(\omega',a)$$

$$\begin{aligned} p_l(\omega',a) = \frac{G_v^2 l\omega'}{(2\pi)^3 a^2} \Big\{ \frac{4}{3}\omega^3 \sum_{k=0}^{\infty} (k+1)J_{2(l+k)+2}(z) \\ - \frac{2\omega^2}{a} \sum_{k=0}^{\infty} (k+1)(k+2)J_{2(l+k)+3}(z) \\ + \frac{4}{3}\frac{\omega}{a^2} \sum_{k=0}^{\infty} (k+1)(k+2)(k+3)J_{2(l+k)+4}(z) \\ - \frac{1}{3a^3} \sum_{k=0}^{\infty} (k+1)(k+2)(k+3)(k+4)J_{2(l+k)+5}(z) \Big\}, \end{aligned} \tag{A6}$$

where $J_m(z)$ is the mth order Bessel function, the argument $z=2lv=2l|\boldsymbol{\omega}\times\boldsymbol{a}|$. Obviously, the radiation is dominated by $p_{l=1}$. Considering that the maximum velocity of neutrons revolving around the filament is $v_{\max}=n\hbar/(2m_n a_0)$, $a_0\approx 10^{-12}$ cm, if $n\ll 100$, then $z\ll 1$, Luo et al., therefore, derived the formula (10) in this paper.

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