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The physics of strong magnetic fields in neutron stars

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ABSTRACT

In this paper we present a new result, namely that the primal magnetic field of the collapsed core during a supernova explosion will, as a result of the conservation of magnetic flux, receive a massive boost to more than 90 times its original value by the Pauli paramagnetization of the highly degenerate relativistic electron gas just after the formation of the neutron star. Thus, the observed super-strong magnetic field of neutron stars may originate from the induced Pauli paramagnetization of the highly degenerate relativistic electron gas in the interior of the neutron star. We therefore have an apparently natural explanation for the surface magnetic field of a neutron star.

Key words: magnetic fields – stars: neutron – pulsars: general.

1 INTRODUCTION

It is generally believed that there is a very strong magnetic field, $B > 10^{12}$ G, associated with most neutron stars (e.g. Shapiro & Teukolski 1984). It is also probable that so-called 'magnetars' exist, with an ultra-strong magnetic field strength that exceeds the quantum critical threshold, $H_{\rm cr} = 4.414 \times 10^{13}$ G (Duncan & Thompson 1992; Paczynski 1992; Usov 1992; Thompson & Duncan 1995, 1996). Anomalous X-ray pulsars (AXPs) and soft gamma repeaters (SGRs) are classes of candidates for magnetars (e.g. Kouveliotou et al. 1998, 1999; Hurley et al. 1999; Mereghetti & Stella 1995; Wilson et al. 1999; Kaspi, Chakrabarty & Steinberger 1999). The magnetic field of the magnetars may be so strong that it reaches values two orders of magnitude above the quantum critical threshold, i.e. $B = 10^{14}$–10^{15} G. What is the origin of the strong magnetic field of neutron stars with a magnetic field near the quantum critical threshold $H_{\rm cr}$? What is the origin of the ultra-strong magnetic field of the magnetars? These are indeed very interesting questions.

It is generally believed that the strong magnetic field of neutron stars originates from the collapse of the core of a supernova with the conservation of magnetic flux. The initial magnetic field strength $B^{(0)}$ after the collapse may be estimated as

$$B^{(0)} = \left(\frac{R_{\rm core}}{R_{\rm NS}}\right)^2 B_* \sim (10^8 - 10^{10}) B_* \ {\rm G}, \tag{1}$$

where B_* is the magnetic field strength of the pre-supernova core with radius $R_{\rm core}$. As typical values, we have taken $R_{\rm core} \sim (10^5$–$10^6)$ km and $R_{\rm NS} \sim 10$ km. It is easily seen that the magnetic field of the collapsed supernova core is still much lower than 10^{13} G, even for a very strong magnetic field of the pre-supernova, $B_* \sim 10^3$ G. This means that it is difficult to obtain the strong magnetic fields of $B > 10^{13}$ expected for some pulsars, and especially the ultra-strong field strength of the magnetars, by the collapse process alone.

In this paper we will address in detail the first question given above. We calculate the strong magnetic field produced by the Pauli paramagnetic moment of a highly degenerate Fermi (neutron, proton or electron) system. The second question, concerning the formation of magnetars, will be discussed in detail in another paper (Peng 2006; Peng, Luo & Chou 2006): we have found that the ultra-strong magnetic fields of the magnetars are produced by the induced magnetic moment of the $^3P_2$ neutron pairs in the anisotropic neutron superfluid when the interior temperature of the neutron stars drops to temperatures much lower than 10^7 K.

The treatment of Pauli paramagnetism in terrestrial cases has become a paradigm of Fermi statistics. Employing the grand canonical formalism, starting from the partition function, we can consider the Pauli paramagnetism and Landau diamagnetism in a unified way. We will follow the terrestrial formalism in our celestial case (for the earliest treatment of this problem see Canuto & Chiu 1968). There are, of course, other versions (Mandal & Chakrabarty 2002), and naturally the main results should be and are consistent with each other.

2 THE MAGNETIC FIELD PRODUCED BY THE PAULI PARAMAGNETIC MOMENT OF A HIGHLY DEGENERATE RELATIVISTIC ELECTRON SYSTEM

The spin of an electron is depicted by $\boldsymbol{S} = \hbar\boldsymbol{\sigma}/2$, $|\sigma| = 1$, $\sigma_z = +1, -1$. The electron possesses a magnetic moment $\boldsymbol{\mu} = -\mu_{\rm e}\boldsymbol{\sigma}$, where $\mu_{\rm e} = 0.927 \times 10^{-20}$ erg G^{-1} is the Bohr magneton. The energy of an electron in an external magnetic field B is $\sigma\mu_{\rm e}B$. The induced magnetic moment (or Pauli paramagnetization) of a Fermi system can be calculated by the standard method of quantum statistics:

$$\mu^{(\rm in)} = k_{\rm B} T \frac{\partial \ln \Xi}{\partial B}, \tag{2}$$

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$$\ln \Xi = \sum_{\sigma=\pm 1} \int_0^{\infty} N(\varepsilon) \ln[1 + \exp\{\beta(\psi - \varepsilon_k - \sigma\mu_e B)\}]\, d\varepsilon, \tag{3}$$

where $\beta = 1/k_B T$, Ξ is the grand partition function of the system, B is the magnitude of the external magnetic field, k_B is Boltzmann's constant, and Ψ is the chemical potential of the Fermi gas (the Fermi energy is E_F). $N(\varepsilon)$ is the density of states: $N(\varepsilon)\, d\varepsilon = (4\pi V/h^3)p^2\, dp$. The Fermi sphere of the electron gas is spherically symmetric when the applied magnetic field is much weaker than the Landau critical value of $B_{cr} = 4.414 \times 10^{13}$ G.

In the regime $\mu_e B(<1\,\mathrm{MeV}) \ll E_F(e)(>60\,\mathrm{MeV})$, the integrand can be expanded as a series in $\sigma\ \mu_e\ B$ as follows.

$$\begin{aligned}\ln[1 + \exp\{\beta(\psi - \varepsilon - \sigma\mu_e B)\}] &= \ln\left[1 + e^{\beta(\psi-\varepsilon)}\right] \\ &- \beta\sigma\mu_e B\bar{n}(\varepsilon) + \frac{1}{2}(\beta\sigma\mu_e B)^2 \bar{n}(\varepsilon)[1 - \bar{n}(\varepsilon)],\end{aligned} \tag{4}$$

where $\bar{n}(\varepsilon)$ is the average occupation number of electrons in the quantum state with energy ε:

$$\bar{n}(\varepsilon) = \left[1 + e^{\beta(\varepsilon-\psi)}\right]^{-1}. \tag{5}$$

The sum of the second term for σ $(= -1, +1)$ is equal to zero, and the sums of both the first term and the third term for σ $(= -1, +1)$ are just doubled when equation (4) is substituted into equation (3):

$$\begin{aligned}\ln \Xi = 2 &\int_0^{\infty} d\varepsilon N(\varepsilon) \ln\left[1 + e^{\beta(\psi-\varepsilon)}\right] \\ &+ (\beta\mu_e B)^2 \int_0^{\infty} d\varepsilon N(\varepsilon)\bar{n}(\varepsilon)[1 - \bar{n}(\varepsilon)].\end{aligned} \tag{6}$$

The first term makes no contribution to the magnetic moment as it is independent of the magnetic field. Using the relationships

$$\bar{n}(\varepsilon)[1 - \bar{n}(\varepsilon)] = -kT \frac{d\bar{n}(\varepsilon)}{d\varepsilon} \tag{7}$$

and

$$-\int_0^{\infty} d\varepsilon N(\varepsilon) \frac{d\bar{n}(\varepsilon)}{d\varepsilon} = N(\psi) + \frac{\pi^2}{6}(kT)^2 \frac{d^2 N(\psi)}{d\psi^2}, \tag{8}$$

we can calculate the induced Pauli paramagnetic magnetic moment for the electron system as follows.

$$\mu^{(in)} = kT \frac{\partial \ln \Xi}{\partial B} = 2\mu_e^2 B^{(0)} N(\psi)\left[1 + \frac{\pi^2}{6}(kT)^2 \frac{1}{N(\psi)} \frac{d^2 N(\psi)}{d\psi^2}\right]. \tag{9}$$

Here, $B^{(0)}$ is the background magnetic field.

The electron gas in the neutron stars is highly relativistic and degenerate. The relationship between the energy and momentum is $\varepsilon = cp$. We then have

$$N(\varepsilon) = \frac{4\pi V}{(hc)^3}\varepsilon^2, \tag{10}$$

$$\frac{d^2 N(\psi)/d\psi^2}{N(\psi)} = \frac{2}{E_F^2}, \tag{11}$$

$$\mu^{(in)} = 2\mu_e^2 B^{(0)} N(\psi)\left\{1 + \frac{\pi^2}{3}\left(\frac{kT}{E_F(e)}\right)^2\right\} \approx 2\mu_e^2 B^{(0)} N(\psi). \tag{12}$$

[This is the same as the non-relativistic case. See also the deduction in Feng & Jin (2005).]

Employing the dipolar model of neutron stars, it can be estimated that the induced magnetic field arising from the Pauli paramagnetism of the electron system is

$$B_e^{(in)} = \frac{2\mu^{(in)}}{R_{NS}^3} \approx AB^{(0)}, \tag{13}$$

where

$$A = \frac{4\mu_e^2}{R_{NS}^3} N(E_F(e)). \tag{14}$$

Alternatively, we can express the amplification factor of the magnetic field by combining equation (14) with equation (10) for the highly relativistic degenerate electron gas in the form

$$A \approx \frac{64\pi^2}{3} \frac{\mu_e^2}{(hc)^3} E_F^2(e). \tag{15}$$

Using the relationships $n_e = (8\pi/3h^3)p_F^3$, $p_F = E_F/c$, $n_e = Y_e N_A \rho$, we finally obtain

$$\begin{aligned}A &= \frac{64\pi^2}{3}\left(\frac{3}{8\pi}\right)^{2/3} \frac{\mu_e^2}{hc} N_A^{2/3}(Y_e\rho)^{2/3} \\ &\approx 0.91 \times 10^2 \left[\frac{Y_e}{0.05}\frac{\rho}{\rho_{nuc}}\right]^{2/3},\end{aligned} \tag{16}$$

where Y_e is the mass fraction of electrons. In addition, an electron gas also has a diamagnetic susceptibility originating from changes in the orbital states caused by the applied magnetic field, the so-called Landau diamagnetic susceptibility. Besides the Pauli paramagnetization, therefore, the Landau diamagnetization of the electron gas must also be included.

For a non-relativistic but highly degenerate electron gas (terrestrial laboratory case), the Landau diamagnetic susceptibility is $-1/3$ of the Pauli paramagnetic susceptibility [Feng & Jin 2005; for detailed calculations see Pathria (2003)]. The Landau diamagnetic susceptibility is much smaller than the Pauli paramagnetic susceptibility for a relativistic highly degenerate electron gas by a factor of about -10^{-4}; see the Appendix for a detailed discussion.

It can be deduced that the induced magnetic field produced by the Pauli paramagnetic magnetization for the highly degenerate relativistic electron gas in neutron stars is 90 times stronger than the applied magnetic field, which is just the initial magnetic field $B^{(0)}$ arising from the gravitational collapse of the pre-supernova core

As regards the neutron and proton systems, their Pauli paramagnetic magnetization is much weaker than that of the electron system in neutron stars. The main reason for this is the inherent magnetic moment of a neutron or a proton is only one-thousandth of the electron magnetic moment.

The Pauli paramagnetization of the neutron system can be obtained simply by calculating the amplification factor, A, of the magnetic field by equation (14) with the substitution $\mu_e \to \mu_n$ and using the expression for the energy-level density at the Fermi surface of the non-relativistic degenerate neutron (or proton) system, namely $N(\varepsilon) = (V/2\pi^2\hbar^3)(2m_n)^{1/2} m_n \varepsilon^{1/2}$. Taking $V = (4\pi/3)R^3$, $R \approx R_{NS}$, we obtain $A \sim 2 \times 10^{-3}$. The Pauli paramagnetic magnetization of the neutron and of the proton systems can therefore be neglected.

According to Mandal & Chakrabarty (2002), under the magnetic field of neutron stars the neutron and the proton systems never become fully polarized, while the electron system is very close to being fully polarized. This is qualitatively consistent with our result here. The difference is that we have made quantitative calculations.

We therefore conclude that the observed strong magnetic field for neutron stars may originate from the induced magnetic field produced by the Pauli paramagnetization of the highly degenerate relativistic electron gas in the interior of the neutron star.

3 INDUCED PAULI PARAMAGNETIC MAGNETIZATION OF A HIGHLY DEGENERATE RELATIVISTIC ELECTRON GAS UNDER A LANDAU CRITICAL MAGNETIC FIELD

We now consider the Landau quantum effect on the motion of electrons in the direction perpendicular to the applied magnetic field. The energy of an electron in a magnetic field $\boldsymbol{B}$ is

$$E^2(p_z, B, n, \sigma) = m^2c^4 + p_z^2c^2 + (2n+1+\sigma)2mc^2\mu_e B. \tag{17}$$

Comparing this with the relativistic energy expression

$$E^2 = m^2c^4 + p_z^2c^2 + p_\perp^2c^2, \tag{18}$$

we obtain

$$p_\perp^2(B, n, \sigma) = 2m(2n+1+\sigma)\mu_e B, \tag{19}$$

where σ is the projection of the spin quantum number, $\sigma = \pm 1$, and n is the quantum number of the Landau energy level, $n = 0, 1, 2, 3\ldots$.

For the electron gas in the neutron stars, $E_F(e) \sim 60$–100 MeV, $mc^2 = 0.511$ MeV, $\mu_e B \sim 0.3(B/B_{cr})$ MeV ($B_{cr} = 4.414 \times 10^{13}$ G), and we have

$$E_F(e) \approx p_z c + \varepsilon, \varepsilon \ll p_z c, \tag{20}$$

$$\varepsilon = \frac{1}{2p_z c}\left[m^2c^4 + (2n+1+\sigma)2mc^2\mu_e B\right]. \tag{21}$$

The population of electrons near the Fermi surface with Landau energy level n is

$$\bar{n}(\varepsilon \sim E_F) = \left\{1 + \exp\left[\left(E_F - \psi + \frac{m^2c^4}{2E_F} + \frac{(2n+1+\sigma)mc\mu_e B}{2E_F}\right)/kT\right]\right\}^{-1}. \tag{22}$$

The Fermi sphere with spherical symmetry for the electron system will be distorted by the strong applied magnetic field into a Landau cylinder. The (level) density of state for the Landau cylinder is

$$N(\varepsilon)\,d\varepsilon = \frac{V}{h^3}\pi p_\perp^2\,dp_z = \frac{2\pi V}{h^3}(2n+1+\sigma)m\mu_e B\,dp_z \tag{23}$$

or

$$N(B, n, \sigma) = \frac{2\pi V}{h^3 c}(2n+1+\sigma)m\mu_e B, \tag{24}$$

because $d\varepsilon = c\,dp_z$. The total level density of states is

$$N(\varepsilon, B, T) = \sum_{n,\sigma=\pm1} N(B, n, \sigma)\bar{n}(p_z, B, n, \sigma). \tag{25}$$

Near the Fermi surface,

$$N(E_F, B, T) = \frac{2\pi V m\mu_e B}{h^3 c}G, \tag{26}$$

where

$$G = \sum_{n,\sigma=\pm1}(2n+1+\sigma)\bar{n}(\varepsilon). \tag{27}$$

The great majority of the electrons are in the lower energy states $n = 0, 1, 2\ldots$, and G is generally less than 10. We can estimate the value of the amplification factor of the magnetic field, A, in equation (13) using equations (14) and (26):

$$A = \frac{32\pi^2}{3}\frac{m\mu_e^3 B^{(0)}}{h^3 c}G = 3.78 \times 10^{-4}\left(\frac{B^{(0)}}{B_{cr}}\right)G. \tag{28}$$

We can therefore conclude that the induced Pauli paramagnetic magnetization of the highly degenerate relativistic electron gas under the influence of the Landau critical magnetic field is very weak and may be neglected. This is because the level density of states of the Landau cylinder is much lower (by a factor of about 10^{-6}) than that of the Fermi surface with spherical symmetry in a weak magnetic field [see the discussion in the foundational work of Canuto & Chiu (1968)].

Consequently, the ultra-strong magnetic field, $B \sim 10^{14}$–10^{15} G, of the magnetars cannot originate from the induced Pauli paramagnetic magnetization of the highly degenerate relativistic electron gas under the Landau critical magnetic field. We thus have to find an alternative novel model for the origin of the ultra-strong magnetic field of the magnetars (Peng 2006; Peng et al. 2006).

4 CONCLUSIONS

The primal magnetic field is rapidly boosted to about 90 times its initial value by the Pauli paramagnetic magnetization of the highly degenerate relativistic electron gas in the interior of neutron stars. This mechanism naturally explains the strong surface magnetic field of the neutron star. In the case of magnetars, however, other models should be employed.

For main-sequence stars, the A_p (peculiar A) stars have the strongest surface magnetic field, of about 1000 G. We thus obtain an upper limit for the magnetic field of a neutron star of about 10^{15} G. It should be borne in mind, however, that A_p stars are not the progenitors of supernovae. Our upper limit here can be compared with that of Mandal & Chakrabarty (2002): they obtained an upper limit of 10^{16} G, which is the saturation value for a magnetic dipole system.

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APPENDIX A: LANDAU DIAMAGNETISM IN THE RELATIVISTIC CASE

The energy–momentum relationship is given by the Landau solution of the Dirac equation in the presence of a homogeneous magnetic field:

$$E^2 = p_z^2c^2 + m^2c^4 + (2n + 1 + \sigma)\hbar ceB, \tag{A1}$$

where n is orbital quantum number having values 0, 1, 2, and so on, and σ is the polarization quantum number, as stated at the beginning of Section 2. The above equation can be modified by employing the Bohr magneton $\mu_{\mathrm{B}} = e\hbar/(2mc)$, and thus we can rewrite the energy–momentum relationship as follows:

$$E^2 = p_z^2c^2 + m^2c^4 + (2n + 1 + \sigma)2mc^2\mu_{\mathrm{B}}B. \tag{A2}$$

Following the traditional treatment of Landau diamagnetism, we neglect the Zeeman energy term. Thus

$$E^2 = p_z^2c^2 + m^2c^4 + (2n + 1)2mc^2\mu_{\mathrm{B}}B, \tag{A3}$$

and henceforth the energy can be treated as a function of $n + \frac{1}{2}$.

After the energy–momentum relationship has been obtained, we can calculate the multiplicity factor

$$\frac{1}{h^2}\int \mathrm{d}p_x\,\mathrm{d}p_y = \frac{1}{h^2}\pi p_\perp^2\Big|_n^{n+1} = \frac{4\pi m\mu_{\mathrm{B}}B}{h^2}, \tag{A4}$$

which is the same as in the non-relativistic case (Pathria 2003). This will be useful in calculating the grand partition function.

In our case, the grand partition function is

$$\begin{aligned}\Xi &= \sum_\varepsilon \log\left(1 + ze^{-\beta\varepsilon}\right)\\ &= 2\int \frac{\mathrm{d}p_z}{h}\sum_{n=0}^{\infty}\frac{4\pi m\mu_{\mathrm{B}}B}{h^2}\log\left(1 + ze^{-\beta\varepsilon(n+\frac{1}{2})}\right)\\ &= \frac{8\pi m\mu_{\mathrm{B}}B}{h^3}\int_{-\infty}^{\infty}\mathrm{d}p_z\sum_{n=0}^{\infty}\log\left(1 + ze^{-\beta\varepsilon(n+\frac{1}{2})}\right),\end{aligned} \tag{A5}$$

where z is the fugacity of the system. As pointed out above, ε is an even function of the z-direction momentum, and, with the help of the Euler summation formula,

$$\sum_{j=0}^{\infty} f\left(j + \frac{1}{2}\right) \cong \int_0^{\infty} f(x)\,\mathrm{d}x + \frac{1}{24}f'(0), \tag{A6}$$

the grand partition function becomes

$$\Xi = \frac{8\pi m\mu_{\mathrm{B}}B}{h^3}2\int_0^{+\infty}\mathrm{d}p_z\frac{1}{24}\log\left(1 + ze^{-\beta\varepsilon(n+\frac{1}{2})}\right)', \tag{A7}$$

where a prime denotes differentiation with respect to the variable $n + \frac{1}{2}$ in the expression for ε, and we have drop terms independent of B.

In a magnetized relativistic gas, the magnetic moment is $\mu = (1/\beta)(\partial\Xi/\partial B)$, which has an analytical expression if we adopt the zero-temperature approximation

$$\mu = -\frac{1}{3}\frac{1}{x_{\mathrm{F}}^3}\sinh^{-1}(x_{\mathrm{F}})\,2\mu_{\mathrm{B}}^2 g(\varepsilon_{\mathrm{F}})B, \tag{A8}$$

where x_{F} is the dimensionless Fermi momentum $x_{\mathrm{F}} = p_{\mathrm{F}}/(mc)$, and $g(\varepsilon_{\mathrm{F}})$ is the density of states at the Fermi surface. Comparing this with the Pauli paramagnetism (equation 12), we obtain

$$\begin{aligned}\mu &= -\frac{1}{3}\frac{1}{x_{\mathrm{F}}^2}\sinh^{-1}(x_{\mathrm{F}})\mu_{\mathrm{Pauli}}\\ &= -10^{-4}\mu_{\mathrm{Pauli}},\end{aligned} \tag{A9}$$

where $x_{\mathrm{F}} \sim 200$, as in the case of neutron stars (Shapiro & Teukolski 1984, p. 310). That is, in the relativistic case, the Landau diamagnetism is only -10^{-4} of the Pauli paramagnetism, and is thus insignificant.

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