I. Objective

The objective of this project be to price a European up-and-out call option held with a risky counterparty. This is a call option whose payoff becomes 0 if the share price gets too high over the lifetime of the option. Note that this limits the final payoff of the option, which subsequently makes it cheaper that a vanilla call option.

II. Introduction

Options are financial instruments that are derivatives or based on underlying securities such as stocks. An options contract offers the buyer the opportunity to buy or sell—depending on the type of contract they hold—the underlying asset.

Option are of two types:

1) European Option - European-style options can be exercised only at expiration.
2) American Option - American-style options can be exercise at any time before the option expires.

In this project, we are going to work on special category of options called Barrier Options. Let's first understand the dynamics of barrier option.

A **barrier option** is a type of derivative where the payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price. A barrier option can be a knock-out, meaning it expires worthless if the underlying exceeds a certain price, limiting profits for the holder and limiting losses for the writer. It can also be a knock-in, meaning it has no value until the underlying reaches a certain price.

Knock-in options may be classified as up-and-in or down-and-in.

In an up-and-in barrier option, the option only comes into existence if the price of the underlying asset rises above the pre-specified barrier, which is set above the underlying's initial price.
In a down-and-in barrier option, the option only comes into existence when the underlying asset price moves below a pre-determined barrier that is set below the underlying's initial price.

Knock-out barrier options cease to exist if the underlying asset reaches a barrier during the life of the option. If an underlying asset reaches the barrier at any time during the option's life, the option is knocked out, or terminated.
Knock-out barrier options may be classified as up-and-out or down-and-out.

An up-and-out barrier option ceases to exist when the underlying security moves above a barrier that is set above the underlying's initial price. In this submission we are going to work on estimating the price of such an option held with a risky counterparty using Monte Carlo simulations.

A down-and-out option ceases to exist when the underlying asset moves below a barrier that is set below the underlying's initial price.

In the next section we are going to understand and implement dynamics and pricing of *up-and-out call option*.

III. Algorithm

Pricing of European up-and-out call option depends on the following parameters:

- Maturity – For how long option is held.
- Spot Price – Current price of underlying asset. Here it is a share of company.
- Strike Price – Price of underlying asset, at which owner of option can buy it.
- Risk Free Interest Rate – This is the opportunity cost - it is the return that is missed out or given up when investor choose one investment over another.
- Volatility – Standard deviation of underlying asset, it shows fluctuation in price of the underlying asset.
- Up and Out Barrier – Value of option will become zero when price of underlying asset crosses this barrier.

Payoff of this option at maturity is derived by following formula:

Payoff = max (S_T – K, 0) given S_t < L where $t \in [0, T]$

S_t is the share price at time t, K is the strike price of the option and L is the barrier level.

Payoff of the option is dependent on the value of the share price between the inception of the option and maturity. This means that the option payoff is dependent on the history of the share price, and not just its terminal value on maturity.

We monitor the barrier, that is, check if the share price has crossed the barrier, only once a month. Due to this discrete barrier monitoring, the estimated option price deviates from the analytical option price. To account for this, we used barrier adjustment[1] according to the following formula.

$$V_m(H) = V\left(He^{\pm\beta\sigma\sqrt{\frac{T}{m}}}\right)$$

Where $V_m(H)$ is the price of a discretely monitored barrier option with barrier H

V(H) is the price of the corresponding continuously monitored barrier option

+ applies if $H > S_0$, - applies if $H < S_0$

$$\beta = -\frac{£\left(\frac{1}{2}\right)}{\sqrt{2Pi}} \sim 0.5826$$

[1] http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.434.9647&rep=rep1&type=pdf
Broadie, Glasserman, and Kou suggest that "to use the continuous price as an approximation to the discrete price, we should first shift the barrier away from S0 by a factor of exp(βσ√dT)".

TASK 1: Simulate paths for the underlying share and for the counterparty's firm value using sample sizes of 1000, 2000, .., 50000 and monthly simulations for the lifetime of the option.

With the help of python code we will run Monte Carlo simulation for underlying share price and counterparty's firm in sample size of multiple of 1000 for next 12 months (i.e. 1 year).

We will start by first importing libraries in our python code. Four important libraries are used in this project are as follows:

- Numpy
- Matplotlib
- Scipy
- Random

1st Step: We define and declare all the variables that will be used in all three task, these are information about share, option and firm. Value of all these variables are already given in problem statement of task 1.
We also allocate empty list of 50 numpy matrices with sample size increasing in multiple of 1000 for storing the Share price, Firm Value, Option Payoff, and Losses (CVA).

2nd Step: We will define two function by the name of *"share_path"* and *"upandout_call_payoff_disc".*

"share_path" – This function simulates the share price at time T according to Black Scholes equation. It takes in an initial share price, a risk-free rate, and volatility, an array of standard normal random variables, and a terminal time. It returns an array of terminal stock prices. We also use this function simulates the share price.

Black Scholes equation used in above function is as follow:

$S_T = S_0 * \exp((r - \sigma^2/2) * T + \sigma * \sqrt{T} * Z)$

This equation comes from by solving the Black Scholes's stochastic differential equation. This model relies on a number of (unrealistic) assumptions:

- There is a constant, continuously compounded, risk-free rate of r.

- There is no default risk, no transaction costs, no spreads, no tax, and no dividends.
- Markets are perfectly liquid with unlimited short selling permitted.
- No arbitrage is possible.

"upandout_call_payoff_disc" – This function implements the formula for the value of a call option as defined in Payoff of Up and Out call option above. It takes in an array of terminal stock values, a strike price, Barrier of option, a risk-free rate, and a terminal time. It returns an array which has the value of the call for each terminal stock price. Logic used in this option is if barrier is not crossed during life of the option, payoff is same as that of a regular European call option else payoff becomes 0

3rd Step: We had fixed our time T as 1/12 year. We run two loop for generating our samples in size of multiple of 1000, so that will get 50 samples, size of each sample will be increasing till 50000.
According to the problem statement, correlation between the share and counterparty firm is 0.2, we will generate correlated standard normal numbers using a Cholesky decomposition. To get this, first we have define and declare correlation matrix by the name of *corr_matrix*, and then generate standard normal variables according to size of our sample. We then take the vector of these standard normals and multiply them by the correlation, and combine these two vectors into a matrix with each array making up a row with the help of *matmul* function.

Once we have our matrix of standard normal random numbers, we can generate our terminal stock and firm values. Calling *"share_path" function in loop* creates an array of stock values using the first row of the matrix of correlated standard normals. We perform similar task for our firm value.Simultaneously with above task we also generated the option pay off and amount of loss in option pay off due to risk of default in counterparty firm.

After running the loop successfully, we get our 50 samples of different sizes (in multiple of 1000) containing share path and firm value in two list of numpy matrices.

Finally we plot our share path and Firm value in two different graphs by randomly choosing samples from our generated matrices.

TASK 2: Determine Monte Carlo estimates of both, the default-free value of the option and the Credit Valuation Adjustment (CVA).

We accomplishes given task by performing in two steps.

1st step: First we perform the Monte Carlo estimation of default free value of option.

We calculated the mean of every samples of different sizes for Option Payoff (calculated/generated in previous task). These 50 mean are the option price estimate for different sizes. Simultaneously we also calculated the standard deviation in same way for 50 samples. We now focus on calculating analytical call price of up and out option or closed form solution. With help of below formula we are successful in calculating the closed form solutions of given option in our python code.

Price of up-and-out European call option = Cbs(S_0, k) - Cbs(S_0, L) - (L-k)e^(-rt)*phi(dBS(S_0,L)) - (L/ S_0)^(2v/sigma^2) { Cbs(L^2/ S_0,k) - Cbs(L^2/ S_0, L) - (L-k)e^(-rt)*phi(dBS(L, S_0)}

where

Cbs(S_0,K) = price of vanilla European call option with initial stock price s0 and strike price k.
dBS = (ln (S_0/k)+v*sqrt(t))/ (sigma*sqrt(T)).
v = r - (sigma^2/2)

We then plot these estimated mean for 50 samples with closed form solutions.

2nd Step: We now focusing on calculation of Monte Carlo estimation of CVA.
The way we calculated mean of option payoff in previous step, same way we have calculated the mean of every samples of different sizes for amount of loss in option payoff (called estimated CVA), calculated in previous task.
Similarly we have calculated the standard deviation of amount of loss in option payoff.
In this step also we first going to calculate the closed form solution for CVA when correlation between share and firm is zero (uncorrelated), same will be calculated by below formula:

Uncorrelated CVA = (1 - recovery rate) * price of asset * default probability

Where

Default probability is phi(-d2).
Price of asset is closed form solution for up and out call option.
Recovery rate is the proportion of our portfolio that we recover if there is a default.

We now plot estimated CVA with closed form solution of CVA (Uncorrelated CVA).

TASK 3: Monte Carlo estimates for the price of the option incorporating counterparty risk

This task is much easy than other task, Here we have to find the market value of option taking consideration of default risk of counterparty firm.

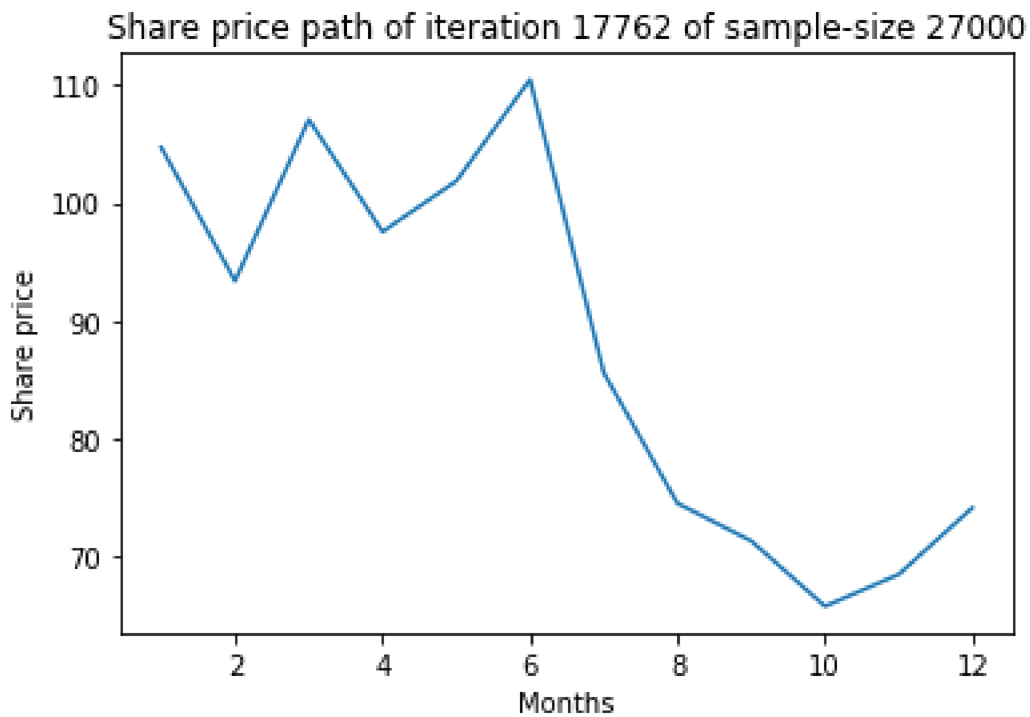
We will calculate the same by subtracting estimates of option price (Payoff) from CVA estimates, we have calculated the both in task 2.
For closed form solution, we apply same logic as above. We have subtracted closed form solution of call option from closed form solution of CVA.
Now we plot estimates of market price of call option with closed form solution of market price of call option.

Graphs are plotted with the help of Matplotlib library.

IV. Results

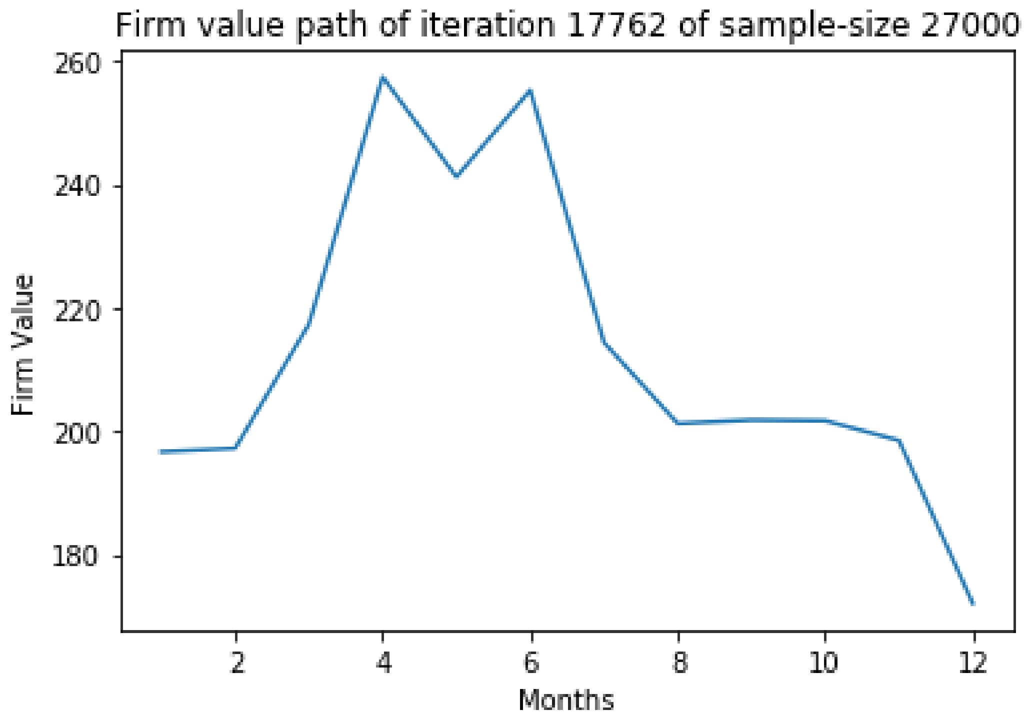
(I) We have simulated the price path of the stock price at monthly time-steps over the life of the option (i.e., 1 year) for 50 different samples – with sample sizes of 1000, 2000… 50000.
One of these samples has been chosen randomly. From that sample, one price path has been selected randomly, and plotted below in Figure 1.

Figure 1



(II) We have simulated the path of firm value at monthly time-steps over the life of the option (i.e., 1 year) for 50 different samples – with sample sizes of 1000, 2000… 50000.
One of these samples has been chosen randomly. From that sample, one firm value path has been selected randomly, and plotted below in Figure 2.

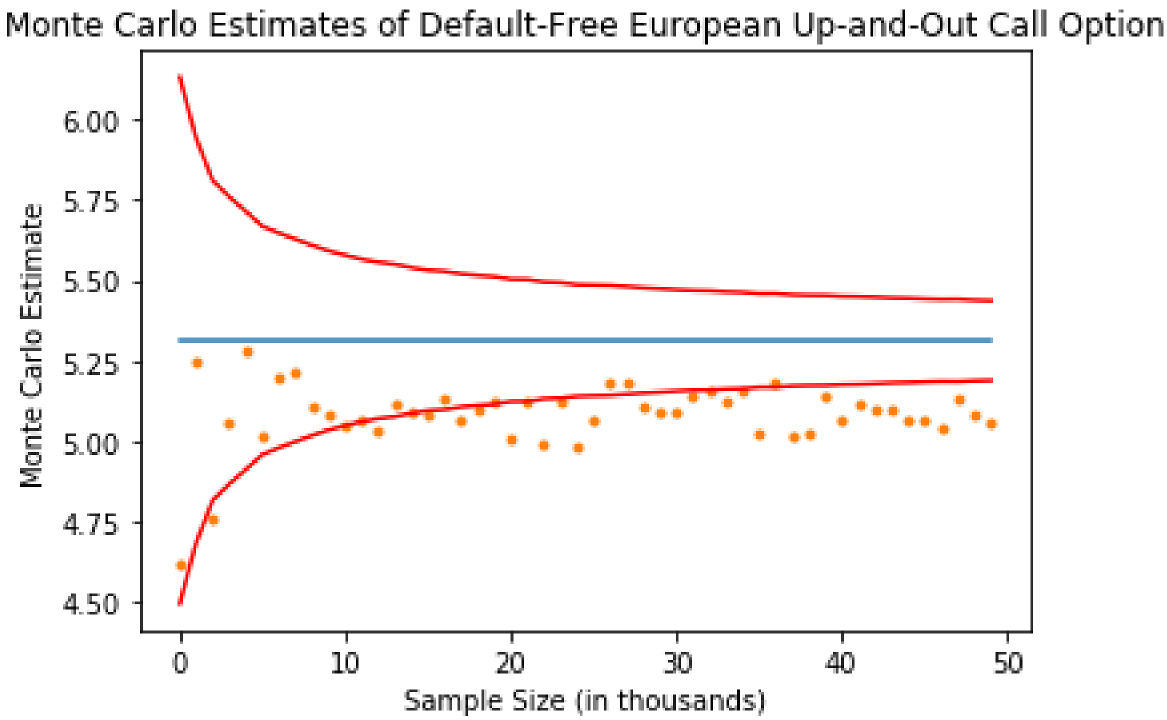
Figure 2



(III) Monte Carlo estimates of default-free European up-and-out call option have been plotted below (in Figure 3) with increasing sample size – from 1000 to 50000. For comparison, we have also plotted the analytical price of the option as a horizontal line and error bounds (in red) at 3 standard deviations from the analytical value.

- The analytical price of the option is approximately 5.3, and the estimated price converges to approximately 5.0, which deviates from the analytical value by 5%
- This difference exists despite using barrier adjustment to account for discrete barrier monitoring (as explained in the previous section).

Figure 3



(IV) Monte Carlo estimates of Credit Valuation Adjustment (CVA) have been plotted below (in Figure 4) with increasing sample size – from 1000 to 50000. For comparison, we have also plotted the analytical value of uncorrelated CVA as a horizontal line and error bounds (in red) at 3 standard deviations from the analytical value.

- It should be noted that this comparison is not ideal because the analytical value is for correlation of 0, and the estimate is for correlation of 0.2.
- The analytical value of uncorrelated CVA is approximately 0.93, and the estimated correlated CVA converges to approximately 0.74.
- For the sake of an ideal comparison, we have also estimated CVA for correlation of 0, and plotted it in Figure 5.

Figure 4

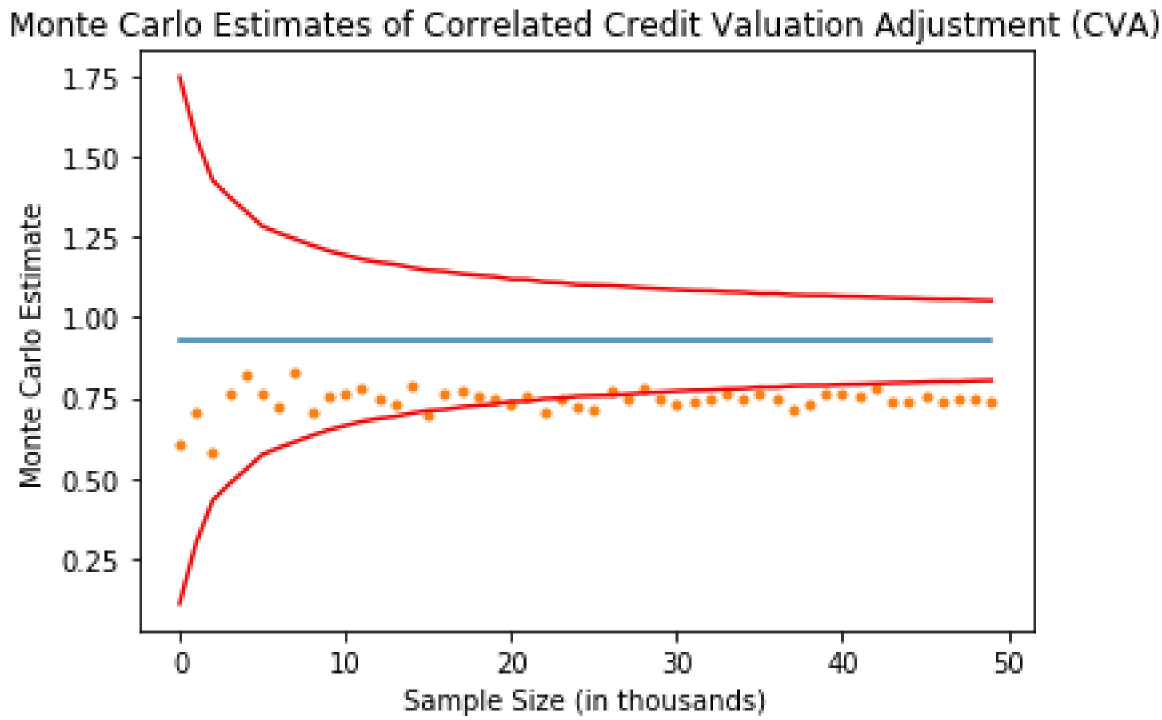
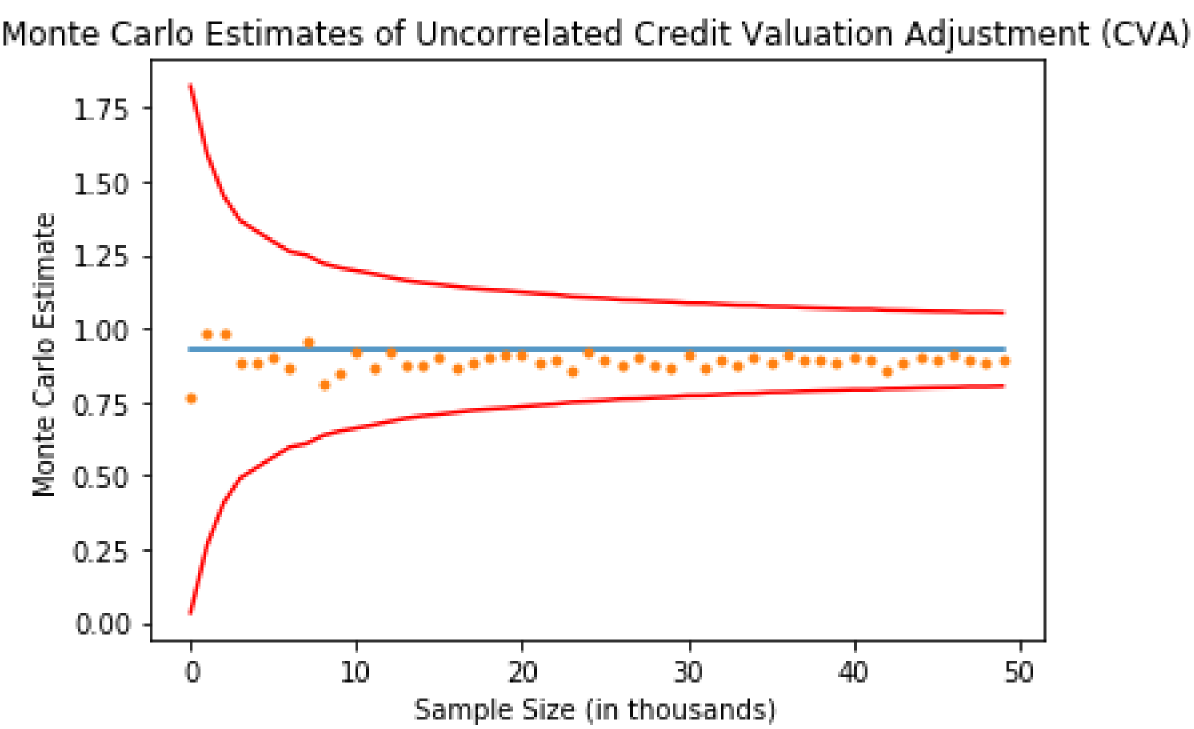


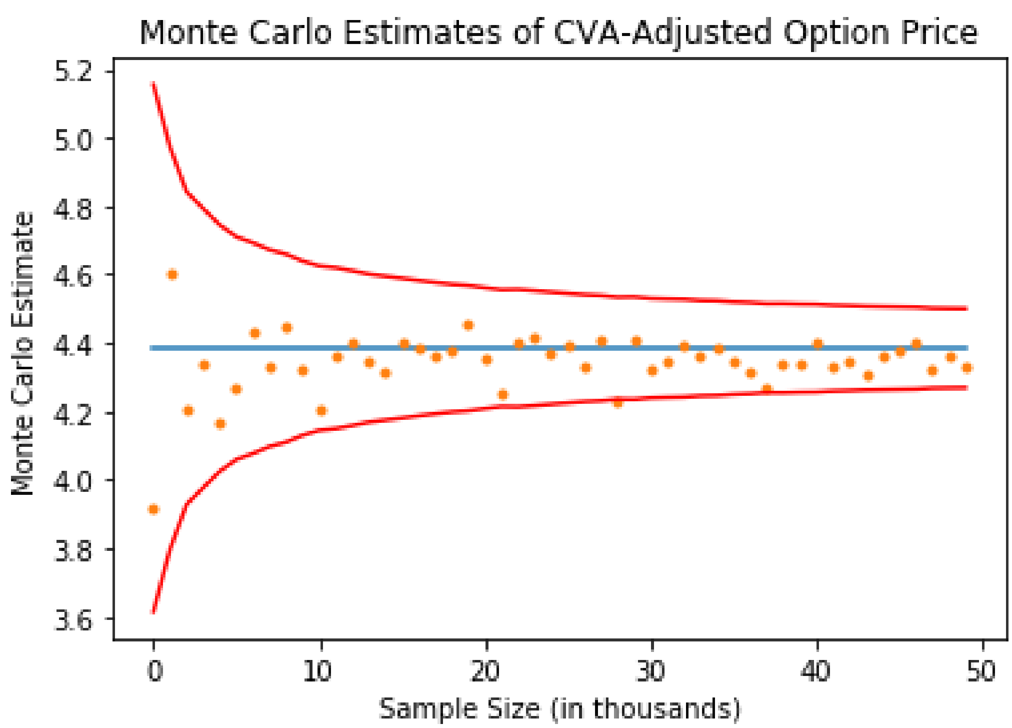
Figure 5



(V) Monte Carlo estimates of CVA-adjusted option price have been plotted below (in Figure 6) with increasing sample size – from 1000 to 50000. For comparison, we have also plotted the analytical value of CVA-adjusted option price as a horizontal line and error bounds (in red) at 3 standard deviations from the analytical value.

- It should be noted that this comparison is not ideal because the analytical value corresponds to correlation of 0, and the estimate is for correlation of 0.2.
- The analytical value of CVA-adjusted option price is approximately 4.4, and the estimated value converges to approximately 4.4 as well.

Figure 6



V. Bibliography

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