Finding Maxima and Minima of DiffEq Solutions

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0.0.1 Setup

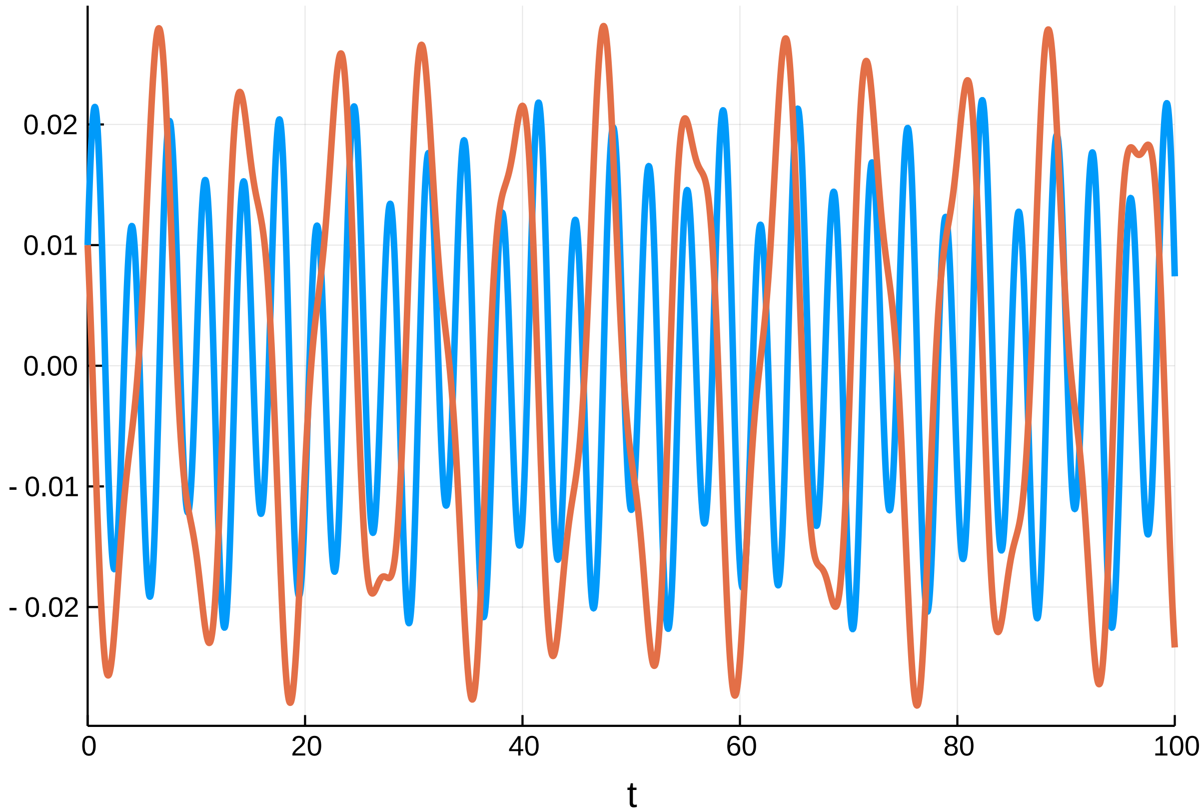
In this tutorial we will show how to use Optim.jl to find the maxima and minima of solutions. Let's take a look at the double pendulum:

```
#Constants and setup
using OrdinaryDiffEq
initial = [0.01, 0.01, 0.01, 0.01]
tspan = (0.,100.)

#Define the problem
function double_pendulum_hamiltonian(udot,u,p,t)
    α  = u[1]
    lα = u[2]
    β  = u[3]
    lβ = u[4]
    udot .=
    [2(lα-(1+cos(β))lβ)/(3-cos(2β)),
    -2sin(α) - sin(α+β),
    2(-(1+cos(β))lα + (3+2cos(β))lβ)/(3-cos(2β)),
    -sin(α+β) - 2sin(β)*(((lα-lβ)lβ)/(3-cos(2β))) + 2sin(2β)*((lα^2 - 2(1+cos(β))lα*lβ
    + (3+2cos(β))lβ^2)/(3-cos(2β))^2)]
end

#Pass to solvers
poincare = ODEProblem(double_pendulum_hamiltonian, initial, tspan)
```

```
sol = solve(poincare, Tsit5())
```

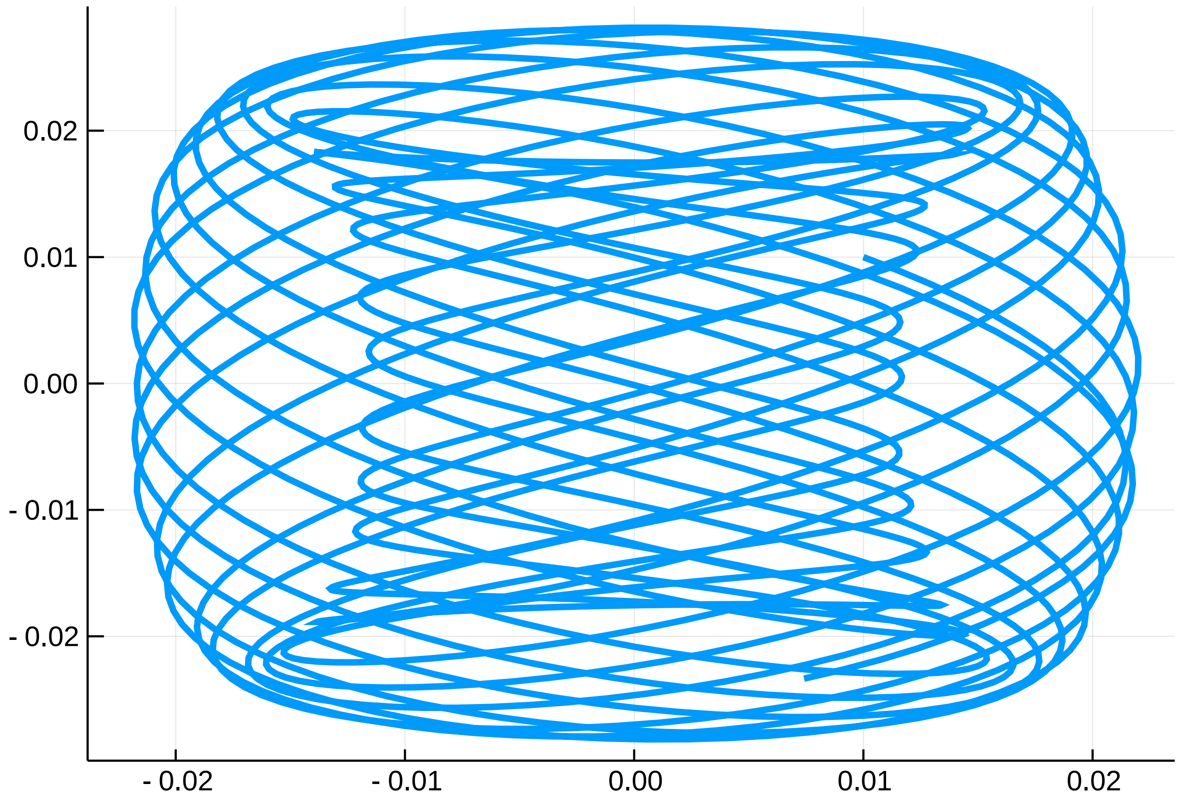
In time, the solution looks like:

```
using Plots; gr()
plot(sol, vars=[(0,3),(0,4)], leg=false, plotdensity=10000)
```



while it has the well-known phase-space plot:

```
plot(sol, vars=(3,4), leg=false)
```



0.0.2 Local Optimization

Let's fine out what some of the local maxima and minima are. Optim.jl can be used to minimize functions, and the solution type has a continuous interpolation which can be used. Let's look for the local optima for the 4th variable around `t=20`. Thus our optimization function is:

```
f = (t) -> sol(t,idxs=4)
```

`first(t)` is the same as `t[1]` which transforms the array of size 1 into a number. `idxs=4` is the same as `sol(first(t))[4]` but does the calculation without a temporary array and thus is faster. To find a local minima, we can simply call Optim on this function. Let's find a local minimum:

```
using Optim
opt = optimize(f,18.0,22.0)
```

From this printout we see that the minimum is at `t=18.63` and the value is `-2.79e-2`. We can get these in code-form via:

```
println(opt.minimizer)
```

```
18.632126799604933
```

```
println(opt.minimum)
```

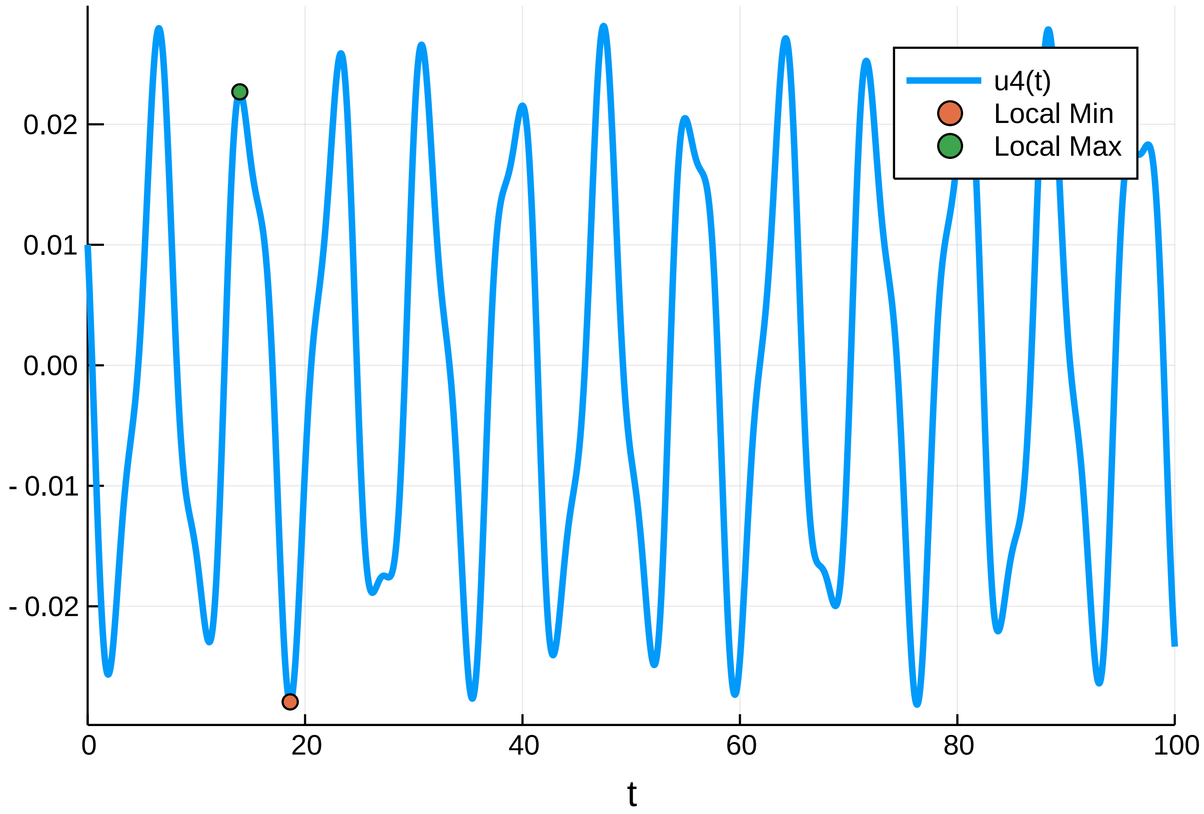
```
-0.027931635264245896
```

To get the maximum, we just minimize the negative of the function:

```
f = (t) -> -sol(first(t),idxs=4)
opt2 = optimize(f,0.0,22.0)
```

Let's add the maxima and minima to the plots:

```
plot(sol, vars=(0,4), plotdensity=10000)
scatter!([opt.minimizer],[opt.minimum],label="Local Min")
scatter!([opt2.minimizer],[-opt2.minimum],label="Local Max")
```



Brent's method will locally minimize over the full interval. If we instead want a local maxima nearest to a point, we can use `BFGS()`. In this case, we need to optimize a vector `[t]`, and thus dereference it to a number using `first(t)`.

```
f = (t) -> -sol(first(t),idxs=4)
opt = optimize(f,[20.0],BFGS())
```

0.0.3 Global Optimization

If we instead want to find global maxima and minima, we need to look somewhere else. For this there are many choices. A pure Julia option is BlackBoxOptim.jl, but I will use NLopt.jl. Following the NLopt.jl tutorial but replacing their function with out own:

```
import NLopt, ForwardDiff

count = 0 # keep track of # function evaluations

function g(t::Vector, grad::Vector)
  if length(grad) > 0
    #use ForwardDiff for the gradients
    grad[1] = ForwardDiff.derivative((t)->sol(first(t),idxs=4),t)
  end
  sol(first(t),idxs=4)
end
opt = NLopt.Opt(:GN_ORIG_DIRECT_L, 1)
NLopt.lower_bounds!(opt, [0.0])
NLopt.upper_bounds!(opt, [40.0])
NLopt.xtol_rel!(opt,1e-8)
```

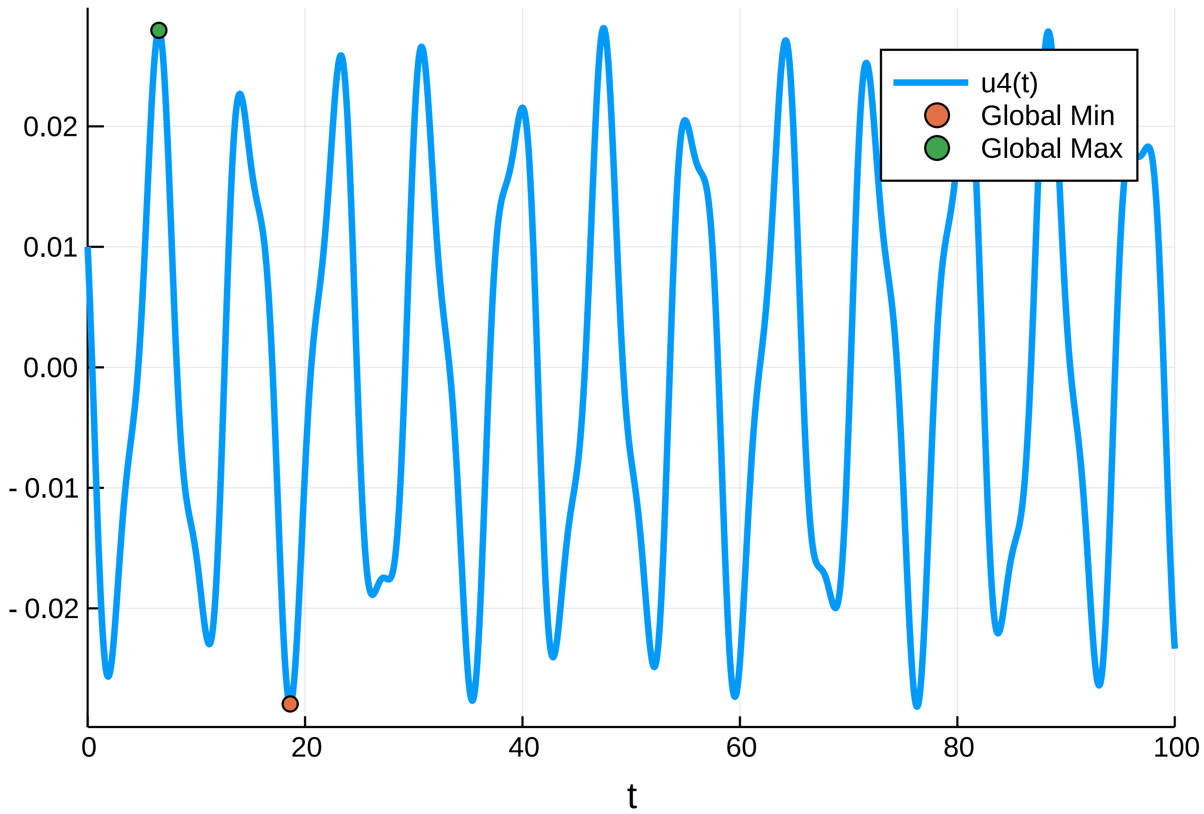
```
NLopt.min_objective!(opt, g)
(minf,minx,ret) = NLopt.optimize(opt,[20.0])
println(minf," ",minx," ",ret)
```

```
-0.027931635264245837 [18.6321] XTOL_REACHED
```

```
NLopt.max_objective!(opt, g)
(maxf,maxx,ret) = NLopt.optimize(opt,[20.0])
println(maxf," ",maxx," ",ret)
```

```
0.027968571933041954 [6.5537] XTOL_REACHED
```

```
plot(sol, vars=(0,4), plotdensity=10000)
scatter!([minx],[minf],label="Global Min")
scatter!([maxx],[maxf],label="Global Max")
```



0.1 Appendix

```
using DiffEqTutorials
DiffEqTutorials.tutorial_footer(WEAVE_ARGS[:folder],WEAVE_ARGS[:file])
```

These benchmarks are part of the DiffEqTutorials.jl repository, found at:

https://github.com/JuliaDiffEq/DiffEqTutorials.jl

To locally run this tutorial, do the following commands:

```
using DiffEqTutorials
DiffEqTutorials.weave_file("ode_extras","ode_minmax.jmd")
```

Computer Information:

```
Julia Version 1.1.0
Commit 80516ca202 (2019-01-21 21:24 UTC)
Platform Info:
  OS: Windows (x86_64-w64-mingw32)
  CPU: Intel(R) Core(TM) i7-8700 CPU @ 3.20GHz
  WORD_SIZE: 64
  LIBM: libopenlibm
  LLVM: libLLVM-6.0.1 (ORCJIT, skylake)
Environment:
  JULIA_EDITOR = "C:\Users\accou\AppData\Local\atom\app-1.34.0\atom.exe" -a
  JULIA_NUM_THREADS = 6
```

Package Information:

```
    Status `C:\Users\accou\.julia\environments\v1.1\Project.toml`
  [7e558dbc] ArbNumerics v0.3.6
  [c52e3926] Atom v0.7.14
  [6e4b80f9] BenchmarkTools v0.4.2
  [336ed68f] CSV v0.4.3
  [3895d2a7] CUDAapi v0.5.4
  [be33ccc6] CUDAnative v1.0.1
  [3a865a2d] CuArrays v0.9.1
  [a93c6f00] DataFrames v0.17.1
  [55939f99] DecFP v0.4.8
  [abce61dc] Decimals v0.4.0
  [39dd38d3] Dierckx v0.4.1
  [459566f4] DiffEqCallbacks v2.5.2
  [f3b72e0c] DiffEqDevTools v2.6.1
  [aae7a2af] DiffEqFlux v0.2.0
  [c894b116] DiffEqJump v6.1.0+ [`C:\Users\accou\.julia\dev\DiffEqJump`]
  [1130ab10] DiffEqParamEstim v1.6.0+ [`C:\Users\accou\.julia\dev\DiffEqPar
amEstim`]
  [055956cb] DiffEqPhysics v3.1.0
  [225cb15b] DiffEqTutorials v0.0.0 [`C:\Users\accou\.julia\external\DiffEq
Tutorials.jl`]
  [0c46a032] DifferentialEquations v6.3.0
  [497a8b3b] DoubleFloats v0.7.5
  [587475ba] Flux v0.7.3
  [f6369f11] ForwardDiff v0.10.3+ [`C:\Users\accou\.julia\dev\ForwardDiff`]
  [28b8d3ca] GR v0.38.1
  [7073ff75] IJulia v1.17.0
  [c601a237] Interact v0.9.1
  [b6b21f68] Ipopt v0.5.4
  [4076af6c] JuMP v0.19.0
  [e5e0dc1b] Juno v0.5.4
  [eff96d63] Measurements v2.0.0
  [76087f3c] NLopt v0.5.1
  [429524aa] Optim v0.17.2
```

```
  [1dea7af3] OrdinaryDiffEq v5.2.1+ [`C:\Users\accou\.julia\dev\OrdinaryDif
fEq`]
  [65888b18] ParameterizedFunctions v4.1.1
  [91a5bcdd] Plots v0.23.0
  [71ad9d73] PuMaS v0.0.0 [`C:\Users\accou\.julia\dev\PuMaS`]
  [d330b81b] PyPlot v2.7.0
  [731186ca] RecursiveArrayTools v0.20.0
  [90137ffa] StaticArrays v0.10.2
  [789caeaf] StochasticDiffEq v6.1.1+ [`C:\Users\accou\.julia\dev\Stochasti
cDiffEq`]
  [c3572dad] Sundials v3.0.0
  [1986cc42] Unitful v0.14.0
  [2a06ce6d] UnitfulPlots v0.0.0 #master (https://github.com/ajkeller34/Uni
tfulPlots.jl)
  [44d3d7a6] Weave v0.7.1 [`C:\Users\accou\.julia\dev\Weave`]
```