Week 4

Perfect Information Extensive form

time plays important role
steps

games in extensive forms (game trees)
take into account time

normal form doesn't reflect time

2 variants

perfect information	imperfect information
opponents know what others do	cannot observe others actions

A (finite) perfect-information game (in extensive form)

is defined by $(N, A, \underbrace{H, Z, \chi, \rho, \sigma, u}_{\text{for utility}})$
where

N a set of n players

A a set of actions (for all actions

Choices

Choice nodes – where players make choices

H – a set of non-terminal choice nodes

Action function $\chi : H \to 2^A$
set of actions available for players in h

Player function $\rho : H \to N$

assigns to each non-terminal node h a player $i \in N$
who chooses an action at h

Terminal nodes

where game ends

Z – terminal nodes, disjoint from H

Successor function

(edges in a game tree)

$$\sigma : \underbrace{H \times A}_{\text{node}} \to \underbrace{H \cup Z}_{\text{next node}}$$ ← defines a tree

maps a choice node and an action to a new choice node or terminal node

such that

$\forall h_1, h_2 \in H$ and $a_1, a_2 \in A$

if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then
$h_1 = h_2$ and $a_1 = a_2$ ← forms a tree

Choice nodes form a tree: nodes encode history

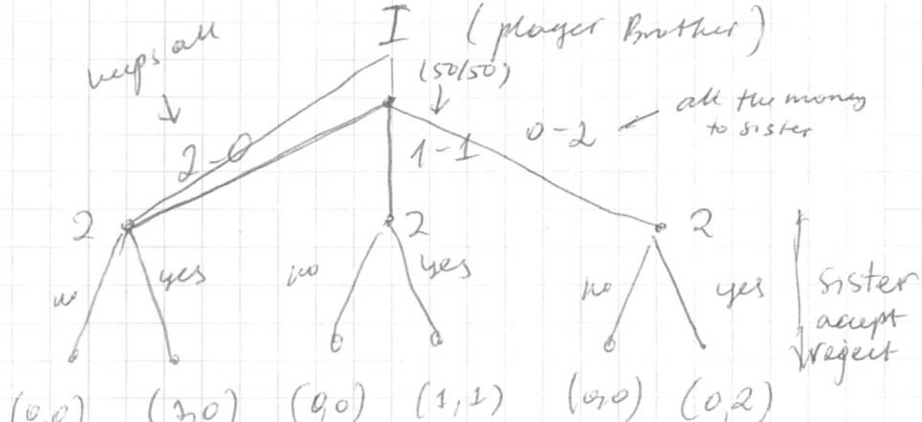
Utility function

$u = (u_1 .. u_n)$; $u_i : Z \to \mathbb{R}$

for player i on terminal nodes Z

the sharing game

(a brother and a sister decide how they're going to share 2 dollars)



Strategies, BR, NE

Pure Strategies

Sharing Game

how many pure strategies?

player I : 3

player II : 8

for player II

if 1 choose no
if 2 choose no
if 3 choose yes

if 1 choose no
if 2 choose yes
3 choose yes

8 combinations → 2 choices for 1
2 cases for 2, 3 ⇒
$2 \times 2 \times 2 = 8$

A pure strategy for a player in a perfect-information game is a
complete specification
of which actions to take at each node belonging to that player

$G = (N, A, H, Z, \chi, \rho, \sigma, u)$ -

perfect information extensive form game

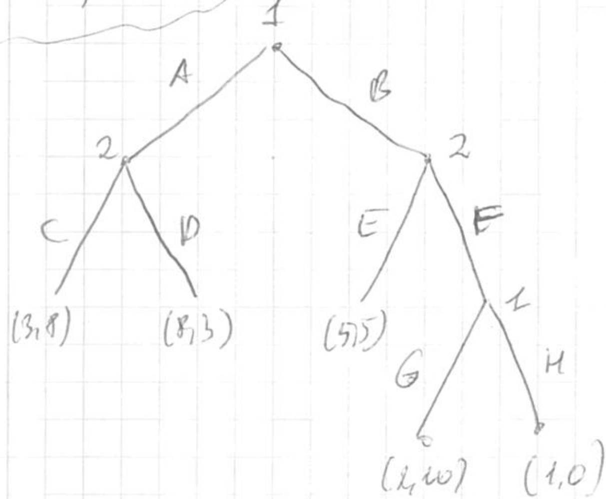
The pure strategies of player i consist of the cross product

$$\prod_{h \in H, \rho(h) = i} \chi(h)$$

for every choice you encounter you should know what to do -
It's a strategy

(policies what to do on every choice of a game)

(proxy instruction)
(to smb)



pure strategies for player 2: $(C, D) \underset{4}{\times} (E, F)$

player 1: $(A, B)\ (G, H) \rightarrow 4!$ (not 3)

3

$(A\ G)$ is different from (A, H)

mixed strategies, best response, NE

$\underbrace{\text{mixed strategies, best response, NE}}$

the same, ~~but adapted~~
(not even adapted)

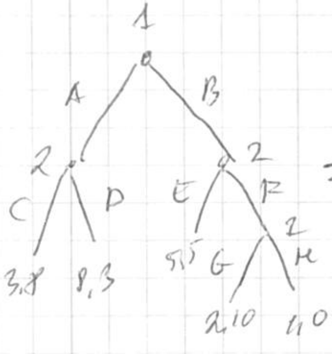
definitions

mixed strategy – probability distribution over pure strategies

best response – mixed strategy that maximizes utility, given profile of utility of other agents

NE – a strategy profile when every agent best-responds to every agent

We can convert an extensive form game into normal form



⟹

	CE	CF	DE	DF
AG	3,8	3,8	8,3	8,3
AH	3,8	3,8	8,3	8,3
BG	5,5	2,10	5,5	2,10
BH	5,5	1,0	5,5	1,0

pure strategies of each agents

induced normal form game

play the game using the strategy

- normal form is not compact
  (grows exponentially!)

- we can't always perform the reverse transformation

<u>Theorem</u> Every perfect information game in extensive form has a PSNE

(you can see what others do, you don't have to randomize)

What are the pure-strategy equilibria?

(A G) (C F)

(A H) (C F)

(B H) (C E)

is there any deviation that would give a player better utility?

Subgame perfection

Subgame of G rooted at H

restriction of G to the descendents of H

Subgames of G

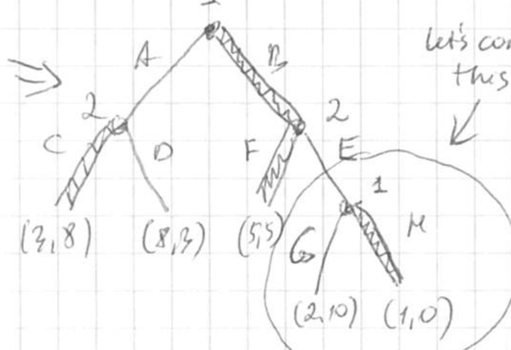
defined by the subgames of G rooted at each node in G

S is a *subgame perfect equilibrium* of G iff for any subgame G' of G, the restriction of S to G' is a NE of G'

Notes:

- since G its own subgame ⟹ every SPE is a NE
- rules out „non-credible theats"

(BH)(CE) equilibrium ⟹



Let's consider this subgame

1 wants to deviate to G

⇓

not a subgame perfect equilibrium

(BH) (CE) – equilibrium

- Why would player I choose H?
  G dominates it

- He does it to threaten player II to
  prevent him from choosing F,
  and gets 5.

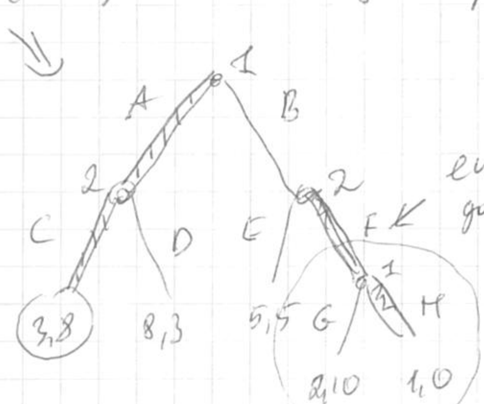
  - Non-credible threat
  - if player I reached his second
    decision, he would play H?

What are subgame perfect equilibria?

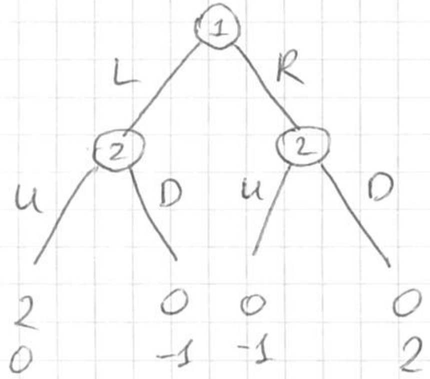
(AG) (CF) : subgame perfect
(noone would deviate)

(BH) (CE) : not subgame perfect

(AH) (CF) : not subgame perfect



even if player II got a change to choose, he would pick up G, not H.



How many subgames are in this game?

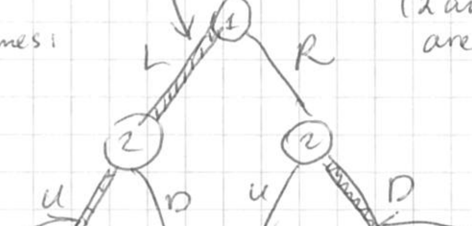
3, original game and 2 subgames.

Which is a subgame perfect equilibrium?

(L)(U,D)

player I wouldn't want to deviate from L (2 and 0 are better than 0 and 0)

3 subgames:



player II wouldn't want to deviate from U (0 is better than -1)

II wouldn't want to deviate from D (2 is better than -1)

Backward Induction

a way of computing a subgame perfect equilibrium

Idea: Identify the equilibria in the bottom-most trees, and adopt these as one moves up the tree

function BackwardInduction (node h)
    returns $u(h)$

    if $h \in Z$ (leaf node)
        return $u(h)$

    best_util $= -\infty$ all actions available at h (all one for each agent)

    forall $a \in \mathcal{X}(h)$ do

        util_at_child $=$
            BackwardInduction $(\sigma(h,a))$ take action at ~~all children~~ node h (where you'll arrive)

        if util_at_child$_{\rho(h)}$ $>$ best_util$_{\rho(h)}$ then
            best_util $=$ util_at_child

    return best_util

util_at_child - vector which denotes the utility for each player

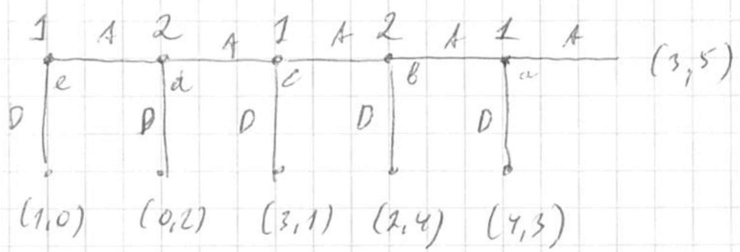
This procedure doesn't return an equilibrium strategy, but rather labels each node with a vector of real numbers

- This labeling may be seen as extension of the games utility function to the non-terminal nodes
- Eq. strategies take a best action at each node

For zero-sum games BI has another name: minimax

- enough to store one number per node
- you may speed up by pruning nodes that will never be visited

сороконожка
Centipede Game (centipede - многоножка)



The only equilibrium - player one goes down on the first move

Backward induction:

on $\underline{\underline{a}}$ player 1 would go D rather than A (4 vs 3)

knowing that, on $\underline{\underline{b}}$ player 2 go D (4 vs 3)

on $\underline{\underline{c}}$, player 1 would go D (3 vs 2)

on $\underline{\underline{d}}$, player 2 then would go D (2 vs 1)

on $\underline{\underline{e}}$, player 1 would go D (1 vs 0)

Two considerations

- practical: human subjects don't go down right away, they cooperate for a while until one defects

- theoretical
  What player 2 would do if pl 1 doesn't go down?

  why did he go A if there is a 0 prob for this? Should I also go A?

Ultimatum Bargaining

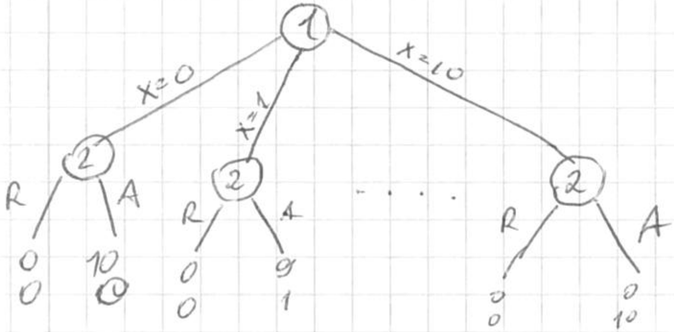
10 units are to be split between pl 1 and pl2

pl 1 offers $x \in \{0, \dots, 10\}$ to pl 2

p2 accepts or rejects

I gets $10-x$ and II gets x
if accepted

both get 0 if rejected



for player II: you should accept any offer
$x > 0$.

$\Downarrow$

for player I: don't make offer more than 1.
(II will accept any possible amount)

but in fact: players don't act this way

(pl 2 ~~ave~~ expected payoff is about 5)

So ⇓

Subgame Perfection doesn't always match data

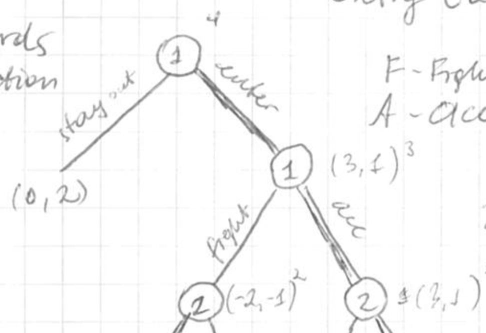
Rejections violate „rationality"

People value equity? (Behavioral Game Theory)

Examples

Entry Game

Backwards Induction



F - Fight
A - Accomodate

I: (Enter, Acc)

II: (F, A)

① 2 prefeers A over F since it gives 1 vs -1 => (3,1) goes to node ④ (A)

② 2 prefers F over A, (-2,-1) goes to node ③ (F)

③ 1 prefers 3 over -2, (3,1) goes to ④ (A)

④ 1 prefers 3 over 0, (3,1) goes to ④. (E)

Ultimate Bargaining

Consider the following:

A offers $x \in \{0, 1, \dots, 10\}$ to B

B accepts or rejects

A gets $10 - x$, B gets x if accepted

A gets 0, B gets -1 if rejected

What is a possible outcome from Backwards Induction?

B always accepts offers - he doesn't want punishment of -1

$\Downarrow$

Therefore A chooses max of 10

$\Downarrow$

Outcome is $(10, 0)$

Imperfect Information Extensive Form

Poker

Moves are sequential but there might be some uncertainty about moves of others

Hidden Information: Poker

- it's sequential (betting calling, folding)
- some cards, but not all
- see bets and react to them
- you have beliefs about rationality and motivation of other players

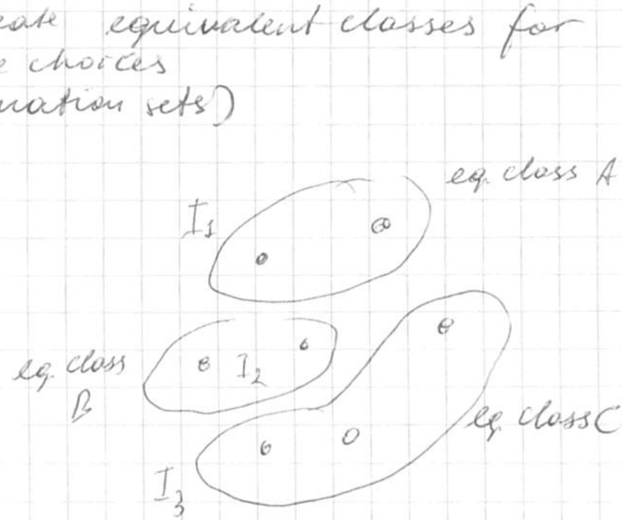
- Many hands!
- Many betting strategies
- Impossible to draw the tree

Formal Definition

in perfect inf. game you
- can see what opponent does
- know the whole history

in imp. you sometimes don't see what others are doing, but it affects your payoff

So we create equivalent classes for some choices (information sets)



A player may not be sure at which of exact nodes he ~~are~~ is, but knows about eq. class of them

Imperfect-information game (in extensive form) is a tuple

$(N, A, H, Z, \chi, \rho, \sigma, u, I)$ where

- $(N, A, H, Z, \chi, \rho, \sigma, u)$ - perfect-inf game
- $I = (I_1, \dots, I_n)$ where

for each player $I_i = (I_{i,1}, \dots, I_{i,k_i})$ - equivalence relation (a partition of)

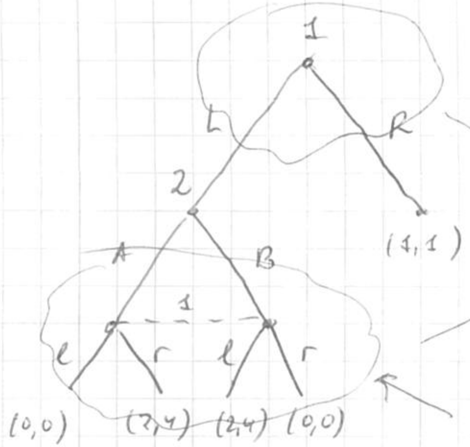
$\{h \in H : \rho(h) = i\}$

If every I contains one el, we'll back to perfect form

with property that

same set of available actions $\rightarrow$ $\chi(h) = \chi(h')$ and $\overbrace{\rho(h) = \rho(h')}^{\text{belongs to the same player}}$

whenever there exists a j which $h \in I_{i,j}$ and $h' \in I_{i,j}$



2 eq. classes:

I_1

I_2

pl 1 doesn't know what action pl 2 took

How we should define pure strategies?

pure strategies of player i consist of the cross product

$$\prod_{I_{i,j} \in I_i} \chi(I_{i,j})$$

of all action sets of different eq. classes

For Pl 1: 4 pure strategies

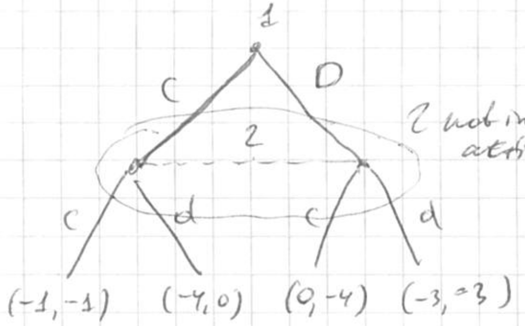
Ll Rl

Lr Rr

We can represent any normal form game

(Prisoners' Dilemma)

would be the same if we put pl 1 at the root →

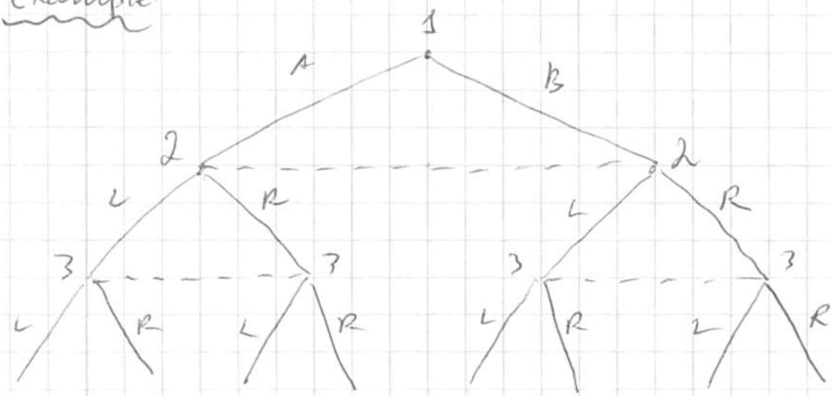


2 not inform about action of 1.

Produced Normal Form:

same as before:
enumerate pure strategies for all agents

Example



What does pl 3 know about pl 1's choice?

PL 3 knows whether PL1 has choosen A or B.
(but doesn't know about PL2's moves)

Mixed And Behavioral Strategies

2 meaningfully different randomized strategies

Mixed

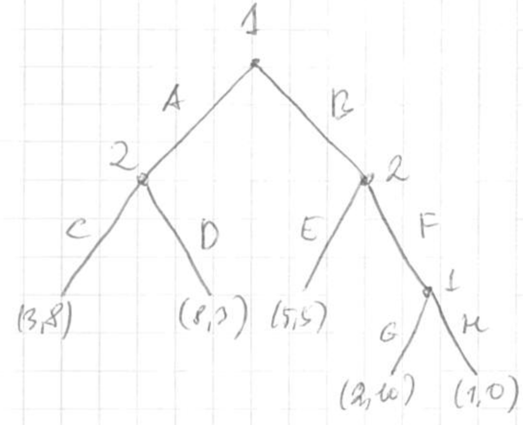
randomize over pure strategies

(distribution over such pure strategies)

Behavioral

independent coin toss every time

(in each set how you should randomize)



Example of beh. strategy:

PI: A with prob = 0.5
G with prob = 0.3

In this game behaviour strategy corresponds to a mixed strategy

Example of mixed strategy:

PI (A, G) – one p.s. (B, H) – another

[0.6 × (A, G), 0.4 × (B, H)]

~~Def~~ Perfect Recall - players know what they already visited and all the action they have taken

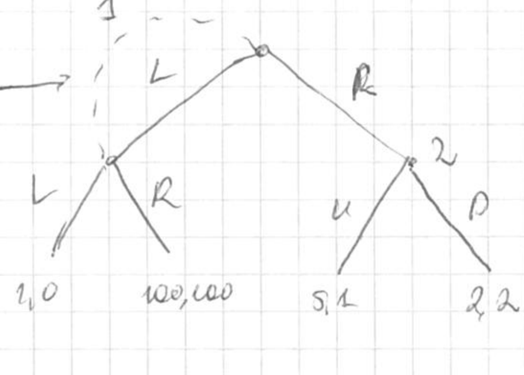
Imperfect Recall:

(landing on an island, and you don't know on which one)

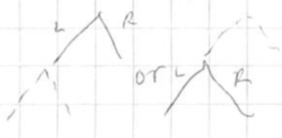
Imagine that pl I sends 2 proxies to the game with the same strategies

When he arrives, he doesn't know if the other has arrived before him, or he's the first one

pl I cannot tell these nodes apart



Pure Strategies: I (L, R) II (U, D)



Mixed Strategy Equilibrium.

For PII: play D - dominant strategy

$\Downarrow$

for pI - play R (2 vs 1)

(RD is better than LD for I)

$\Rightarrow$ R, D is equilibrium

100, 100 - not accessible in mixed strategy

What is an equilibrium in behavioral strategies?

- D strongly dominant for pII
- pI randomizes

goes Left with p, Right with 1-p

(p, 1-p) - his behavioural strategy

$$p^2 + 100p(1-p) + 1\cdot 2\cdot(1-p)$$

$$-99p^2 + 98p + 2$$

$$p = \frac{98}{198}$$

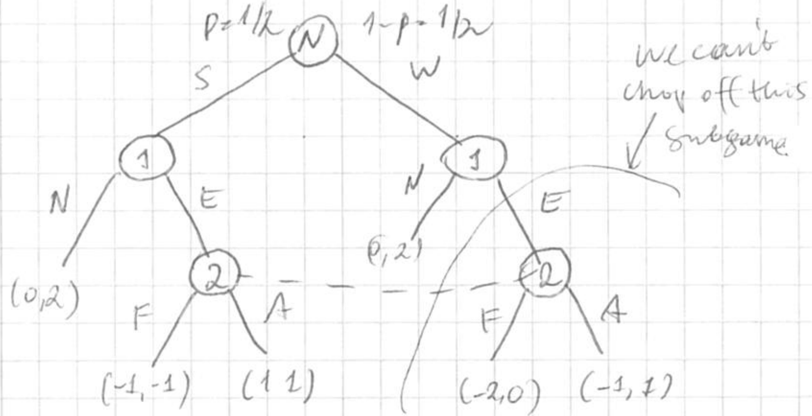
Thus equilibrium is

$$\left(\underset{L}{\underset{\downarrow}{98/198}},\ \underset{R}{\underset{\downarrow}{100/198}}\right)\ (0, 1)$$

Thus, when we have imperfect recall, equilibria in behavioral strategies are different from equilibria in different strategies

Beyond Subgame Perfection

no many proper subgames



Should I enter this market?

N - nature

N decides if player I strong or weak

player II doesn't know if I is S or W.

We cannot chop any subgame in this game. In fact, there is only one subgame - the whole game.

NE: 2 fights, 1 doesn't enter

(that's strange, but they won't since I doesn't enter) → (it hurts)

not really credible

NE: Q A; ~~A~~ I enters if strong doesn't if week

⇓

more credible (following best response)

There are more NE (mixing, etc)

Sequential Equilibrium, Perfect Bayesian Eq.

- Beliefs are not contradicted by the actual play of the game

- players best respond to their beliefs

(i.e. a belief for p2 - may be a probability that p1 is weak = p and strong 1-p.)

$\Downarrow$

(they always should A).

$$\boxed{\begin{array}{l} 1: \begin{array}{l} S \to E \\ W \to N \end{array} \\ \\ 2 \text{ always } A \end{array}}$$

unique prediction