Week 7

Bayesian Games

Auctions – I'm not quite sure ~~th~~ what are utilities of other players

Usually everyone knows

- the number of players
- the actions available to each player
- payoff associated with each action vector

Here we will assume

only utility functions differ

1. There are some games (not one).
   All possible games have the same number of agents and the same strategy space for each agent
   (differing only in payoffs)

2. Agents' beliefs are posteriors ([illegible]) obtained by conditioning a common prior on individual private signals

what's possible

updating

beliefs agents have

1) Definition (based on Information sets)

Bayesian game - a set of games that differ only in their payoffs

a common ~~payoff~~ prior defined over them

and a partition structure over the games for each agent

A Bayesian game is a tuple (N, G, P, I) where

- N - is a set of agents
- G is a set of games with N agents each such that if

$$g, g' \in G$$

then for each agent $i \in N$

the strategy space in g is identical to the strategy space in g'

so games differ only in utility function

- $P \in \Pi(G)$ a common prior over games, where

prior distribution (how likely each of these games is) $\Pi(G)$ - set of probability distribution over G

- $I = (I_1, \dots, I_N)$ is a set of partitions of G, one for each agent

(set of eq. classes: for some agents some games are indistinguishable)

Example : 4 games are played: • Matching Pennies
• Prisoners Dilemma
• Coordination
• Battle of the Sexes

$I_{2,1}$ $I_{2,2}$

	MP		PD	
$I_{1,1}$	2,0	0,2	2,2	0,3
	0,2	2,0	3,0	1,1
	$p=0.3$		$p=0.1$	

	Coord		BoS	
$I_{1,2}$	2,2	0,0	2,1	0,0
	0,0	1,1	0,0	1,2
	$p=0.2$		$p=0.4$	

player I has the same set of actions for every game:

Top or Bottom

player II can choose

Left or Right

Equivalence classes for players:

players I can't distinguish between

(MP and PD) and (Coord, BoS)

eq class eq. class

pl II,

(MP, Coord)

(PD, BoS)

~~Why~~ when playing, players won't know for sure what game they're playing,

only the eq. class

② Definition (alternative)

based on epistemic types

Directly represents uncertainty over utility function using the notion of epistemic type (private information of the agents)

A bayesian game is a ~~tuple~~ tuple (N, A, Θ, p, u) where

- N - a set of agents
- $A = (A_1 \dots A_n)$

  A_i - set of actions available to player i
- $\Theta = (\Theta_1 \dots \Theta_n)$

  Θ_i - type space of player i

- $p: \Theta \mapsto [0,1]$

  the common prior over types

- $u = (u_1 \dots u_n)$

  where $u_i = A \times \Theta \mapsto \mathbb{R}$

  utility function for player i

Consider the same game (MP, Coord, PD, BoS)

what if p_1 plays U, p_2 plays L?

depends on the type. Information available for players

for $\theta_{1,1}$ and $\theta_{2,1}$ $(I_{1,1} + I_{2,1})$

payoffs are $(2, 0)$

for D and L, $\theta_{1,2}$, $\theta_{2,2} \Rightarrow 0, 0$

///

So we can write everything in a table with

$\underbrace{a_1 \ a_2}_{\text{actions}} \quad \underbrace{\theta_1 \ \theta_2}_{\text{role}} \quad \underbrace{u_1 \ u_2}_{\text{resulted utility}}$ columns

+ 16 rows

By fixing a type, you end up with a specific game.

Analysing Bayesian Games

Bayesian (Nash) Equilibrium

it a plan of actions for each player as a function that maximizes each type's expected utility

(should be a best reply)

if I observe a certain type, what am I going to do?

- expecting over the actions of other players (what are the expected action distributions we're going to face)

- expecting over the types of other players

Strategies

Given a Bayesian game (N, A, Θ, p, u) (finite)

- Pure strategy: $s_i : \Theta_i \mapsto A_i$ for a type, what action you'll take

a choice of a pure action for player i as a function of his/her type

- mixed strategy

$$s_i : \Theta_i \longmapsto \underbrace{\Pi(A_i)}$$

probability distribution over actions of your ~~type~~ type

a choice of a mixed action for player i as a function of his type

- $s_i(a_i \mid \Theta_i)$ – distribution over actions

(what's the probability that action a_i will be chosen if they happen to be of the type Θ_i)

the probability under mixed strategy s_i that agent i plays action a_i, given that is type is Θ_i

Expected Utility

- ex-ante

the agents knows nothing about anyone's actual type

- interim

agents know their own types, but don't know the types of each other

- ex-post

everybody knows everything

Interim expected utility

for player i with respect to type θ_i

and mixed strategy profile s

← probability of others' type

$$EU_i(s \mid \theta_i) = \underbrace{\sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} \mid \theta_i)}_{\text{sum across all probabilities of types for others}} \cdot$$

what can i expect if he of type θ_i and follows s

$$\cdot \sum_{a \in A} \Big(\prod_{j \in N} \underbrace{s_i(a_i \mid \theta_i)}_{\text{what other players will be doing}} \Big) \cdot \underbrace{u_i(a, \theta_i, \theta_{-i})}_{\text{utilities evaluated with respect to those types}}$$

ex ante expected utility

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i)\, EU_i(s \mid \theta_i)$$

Bayesian Equilibrium

that's a mixed strategy profile s that satisfies

$$s_i \in \arg\max_{s_i'} \underbrace{EU_i(s_i', s_{-i} \mid \theta_i)}_{\text{maximizing the expected utility}}$$

each individual should choose the best responce

for each i and $\theta_i \in \Theta_i$

(based on interim maximization)

equivalent to an ex ante formulation

if $p(\theta_i) > 0$ for all $\theta_i \in \Theta_i$, then

$$s_i \in \arg\max_{s_i'} EU_i(s_i', s_{-i}) =$$

$$= \arg\max_{s_i'} \sum_{\theta_i} p(\theta_i) EU_i(s_i', s_{-i} \mid \theta_i)$$

for each i

So, it

- explicitly models behavior in uncertain environment
- players choose strategies to maximize their payoffs in response to others,

  accounting for:

  - strategic uncertainty about how others will play
  - payoff uncertainty about the value to their actions

Another Example

so Bayesian (Nash) Equilibrium:

- a plan of actions for each player as a function of types that maximizes each type's expected utility

  - expecting over the actions of other players
  - expecting over the types of other players

A Sheriff's Dilemma

- a sheriff faces an armed suspect and they each must (simultaneously) decide whether to shoot or not

- a suspect is a criminal with probability p and not a criminal with prob. $1-p$

- the sheriff would rather shoot if suspect shoots, and not shoot otherwise

- the criminal would rather shoot even if the sheriff doesn't – he doesn't want to be caught

- the innocent would rather not shoot even if sheriff does

Sheriff

Innocent

	Shoot	Not	
~~Shoot~~	~~-3, -1~~	~~-1, -2~~	$1-p$
Not	-2, -1	0, 0	

good → strictly dominated strategy of ~~not~~ shooting

guilty

	Shoot	Not	
Shoot	0, 0	2, -2	p
~~Not~~	~~-2, -1~~	~~-1, 1~~	

bad → not shooting is dominated strategy

shoot strictly dominates

What's sheriff's ~~[illegible]~~ best reply?

$$-1(1-p) = -2p$$

$p > \frac{1}{3}$ → you're likely to shoot.

So if p is greater than $\frac{1}{3}$, the Sheriff shoots

$p > \frac{1}{3}$: shoot

$p < \frac{1}{3}$: not shoot

$p = \frac{1}{3}$ any

Summary:

- NE explicitly models behavior in an uncertain environment

- Players choose strategies to maximize their payoffs in ~~repe~~ response to others considering
  - strategic uncertainty about how others will play
  - payoff uncertainty about the value to their actions

Examples

(1) in the following two-player Bayesian game, the payoffs to player 2 depends on whether 2 is a friendly player (with probability p) or a foe (with probability $1-p$)

p

Friend	L	R
L	3,1	0,0
R	2,1	1,0

$1-p$

Foe	L	R
L	3,0	0,1
R	2,0	1,1

player 2 knows if he is a foe or friend, but player 1 doesn't know that

if player 2 uses Left when a friend
Right when a foe

What is player 1's expected utility?

it's $3p$ when 1 chooses Left.

if 1 chooses Left, with probability p
player 2 is a friend and chooses Left
and then 1 earns 3

and with probability $1-p$ 2 is a foe and
chooses R and 1 earns 0,

Thus, the expected payoff is $3p + 0(1-p) = 3p$

(2) Consider the conflict game

Strong	Fight	Not
Fight	1, -2	2, -1
Not	-1, 2	0, 0

with probability p

Weak	Fight	Not
Fight	-2, 1	2, -1
Not	-1, 2	0, 0

with probability $1-p$

Let p^* be the threshold such that player 1
fights when strong and doesn't fight when weak

then

if $p > p^*$ player 2 prefers "Not"

if $p < p^*$ player 2 prefers "Fight"

What is p^* in this modified game?

To find it, we need to write down the payoff of 2 when choosing "Fight" and "Not Fight", and equalize it to get p^*

Conditional on 1 fighting when strong and not fighting when weak,

the payoff of 2

when choosing Not: $-1p + 0(1-p)$

when choosing Fight: $-2p + 2(1-p)$

$$\Downarrow$$

$$-1p = -2p + 2(1-p), \quad p^* = \frac{2}{3}$$