Week 6

Coalotional Game theory

- politics, political parties
- companies
- marriage, building a house, etc

basic unit - a team.

so the focus is on a group of agents rather than on individuals

Transferable utility - we can assign some payoff to the whole coalotion

Coalotional game with transferable utility

is a pair (N, v) where

- N - finite set of players, indexed by i
- $v : 2^N \to \mathbb{R}$ associates with each coalition $S \subseteq N$

utility function

for every coalition S

a real-valued payoff $v(S)$ that coalition's member can distribute among themselves

$v(\emptyset) = 0$

Questions?

- Which coalition will form?
- How coalition should divide its payoff among its members?
  (fair vs stable)

Superadditive Coalition Game -

A game $G = (N, v)$ if for all

$S, T \subset N$ } ~~for~~ for all pairs (S, T)

if $S \cap T = \emptyset$ } each has different set of agents

then $v(S \cup T) \geq v(S) + v(T)$

(if we form a coalition of S and T, then its payoff is at least as good as sum of their payoffs separately)

$\Downarrow$

It will lead to a formation of a Grand Coalition

E.g. $N = 3$, $v(1) = v(2) = v(3) = 1$ ← super additive game

$v(1,2) = 3$
$v(1,3) = 4$
$v(2,3) = 5$

$\underbrace{}_{\text{greater than}}$
$v(1) + v(2)$
$v(1) + v(3)$
$v(2) + v(3)$

$\underbrace{v(1,2,3)}_{\text{greater than any of these}} = 7$

+1

The Shapley Value

Question: what is a "fair" way for a coalition to divide its payoff?

(how we define "fairness"?)

So we can define some set of axioms to define it.

Lloyd Shapley's ideas: members receive payoff proportional to their contribution
(marginal)

- Symmetry

  if 2 agents i and j, when contributed, give the same amount of payoff,

  they are interchangeable

  for all S which contains neither i nor j,

  $$v(S \cup \{i\}) = v(S \cup \{j\})$$

  if we add i or j, it will give same payoff

  So they should receive the same amount of payments

- Dummy player

i is a dummy player if

for all S : $v(S \cup \{i\}) = v(S)$

i.e. i doesn't give anything

dummy player should receive nothing

- Additivity

if we can separate a game into 2 parts, we should be able to decompose payments: with $v = v_1 + v_2$,

So, with all these axioms

For game $G(N, v)$

the Shapley Value divides payoffs among the players according to:

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} \overbrace{|S|!\,(|N|-|S|-1)!}^{\text{weighting}} \cdot \underbrace{\left[v(S \cup \{i\}) - v(S) \right]}_{\text{how much } i \text{ add to a coalition without him}}$$

for each i

a coalit. without i

weighting it by how many different ways we could come up with this calculation

so we average over all these things

Theorem

Given a coalitional game (N, v), there is a unique payoff division

$x(v) = \phi(N, v)$ that divides the payoff

and satisfies the Symmetry, Dummy player and Additivity axioms:

the Shapley Value.

Shapley Value

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|! \, (|N| - |S| - 1)! \cdot \left[\underbrace{v(S \cup \{i\}) - v(S)}_{} \right]$$

- this captures the "marginal contribution of agent i
- averaging over all different sequences according to which the grand coalition could be built up

$\downarrow$ (1, 12, 123, ...)

- for any such sequence, look at agent's marginal contribution:

$$[v(S \cup \{i\}) - v(S)]$$

- Weight this quantity by the $|S|!$ ways the set S could be formed

  prior i's addition

  by the $(|N|-|S|-1)!$ ways the remaining players could be added

- Sum over all possible sets of S

- and average by dividing by $|N|!$, the number of possible orderings of all the agents

Eg.

$$v(\{1\}) = 1$$
$$v(\{2\}) = 2$$

$$\phi_1 = 1.5 \qquad \phi_2 = 2.5$$

Eg2. Suppose $N=2$,

$$v(1)=0, \; v(2)=2$$
$$v(1,2)=2$$

$$\Downarrow$$

$$\underbrace{\phi_1(N,v)=0}_{} \qquad \phi_2(N,v)=2$$

contributes nothing $\Rightarrow$ gets 0.

The Core

The Shapley value – how to ~~def~~ divide in a fair way

but it ignores the question of stability.

I.e. would agents be willing to form the grand coalition?

Or they would like to form smaller?

(sometimes smaller is more attractive)

Example. (Voting game)

A parlament is made up of 4 parties, A, B, C, D which have 45, 25, 15, 15 representatives

number of seats

They are to vote

51 should vote for to pass, otherwise all get nothing

majority

Shapley value is (50, 16.67, 16.67, 16.67)

But A and B can form a coalition and get more (75 + 25)!

so they have incentive to defect

- So under what playment divisions would the agents want to form the grand coalition?

Core

a payoff vector x is in the core of a coalitional game (N, v) iff

$$\underset{\text{for every coalition they could form}}{\forall S \subseteq N,} \quad \underbrace{\sum_{i \in S} x_i}_{\text{value they would get if form a G.C.}} \geq \underbrace{v(S)}_{\text{if they deviate and don't form Grand Coalition}}$$

- The sum of payoffs to the agents in any subcoalition S is at least as much as they could learn on their own

- Analogous to NE

Core is sometimes empty
(not always exists)
$\uparrow$
stable way of splitting

And Core is not always unique

Simple Game
A game $G=(N,v)$ is simple if
$$\forall S \subset N,\ v(S) \in \{0,1\}$$
for all coalitions — they gain either 0 or 1

(Vote game is simple!
they either get money or not)

Veto player
a player i is a veto player
if $v(N \setminus \{i\})=0$

(participation of i is necessary if
a coalition wants to produce any
value)

Theorem.

In a simple game the core is empty iff there is no veto player.

If there are veto players, the core consists of all payoff vectors in which the nonveto players get 0.

Example (Airport Game)

Several cities airports

Options: Big regional airport + share costs
vs own airports

N - set of cities
$v(S)$ - sum of costs of building runways for each city in S — cost of the largest runway required by any city in S

Convex game

A game $G = (N, v)$ is convex if for all $(S, T) \subset N$, $v(S \cup T) \geq \underline{v(S)} + \underline{v(T)} - \underline{v(S \cap T)}$

all coalitions in N

can achieved by itself

common

Stronger condition than superadditivity

Airport game is a convex game

<u>Th</u> Every convex game has a nonempty core

<u>Th</u> In every convex game, the Shapley value is in the core

(stable and fair division - possible to do both)

<u>Eg</u>. Suppose $N=3$

$$v(1)=v(2)=v(3)=0$$

$$v(1,2)=v(2,3)=v(3,1)=2/3$$

$$v(1,2,3)=1.$$

What is the core?

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is the core

it satisfies

for every $i \neq j$: $x_i + x_j \geq \frac{2}{3}$