1 ZMP LQR Riccati Equation

Using $z(t)$ as the 2D position of the ZMP, we formulate:

$$
\begin{aligned}
\underset{u(t)}{\text{minimize}} \quad & \int_0^\infty \left[\|z(t) - z_d(t)\|_2^2 + \|u(t)\|_R^2\right] dt, \\
\text{subject to} \quad & R = R' > 0, \\
& z_d(t) = z_d(t_f), \quad \forall t \geq t_f \\
& \dot{x}(t) = Ax(t) + Bu(t), \quad z(t) = Cx(t) + Du(t) \\
& A = \begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{2\times2} \\ I_{2\times2} \end{bmatrix} \\
& C = \begin{bmatrix} I_{2\times2} & 0_{2\times2} \end{bmatrix}, \quad D = -\frac{h}{g} I_{2\times2}
\end{aligned}
$$

This can be rewritten as a cost on state, *in coordinates relative to the final conditions*, $\bar{x} = x - \begin{bmatrix} z_d^T(t_f) & 0 & 0 \end{bmatrix}^T$, $\bar{z}_d(t) = z_d(t) - z_d(t_f)$:

$$
\begin{aligned}
\underset{u(t)}{\text{minimize}} \quad & \int_0^\infty \bar{x}^T Q_1 \bar{x} + \bar{x}^T q_2(t) + q_3(t) + u^T R_1 y + u^T r_2(t) + 2\bar{x}^T N u \\
\text{subject to} \quad & Q_1 = \text{diag}(1 \quad 1 \quad 0 \quad 0), \quad q_2(t) = \begin{bmatrix} -2\bar{z}_d(t) \\ 0_{2\times1} \end{bmatrix}, \quad q_3(t) = \|\bar{z}_d(t)\|_2^2 \\
& R_1 = R + \left(\frac{h}{g}\right)^2 I_{2\times2}, \quad r_2(t) = 2\bar{z}_d(t)\frac{h}{g}, \quad N = -\frac{h}{g}\begin{bmatrix} I_{2\times2} \\ 0_{2\times2} \end{bmatrix} \\
& \dot{x}(t) = Ax(t) + Bu(t) \\
& A = \begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{2\times2} \\ I_{2\times2} \end{bmatrix}
\end{aligned}
$$

Note that this implies that $\bar{x}(\infty) = 0$ in order for the cost to be finite.

The resulting cost-to-go is given by

$$
J = \bar{x}^T S_1(t) \bar{x} + \bar{x}^T s_2(t) + s_3(t),
$$

with the corresponding Riccati differential equation given by

$$
\begin{aligned}
\dot{S}_1 &= -\left(Q_1 - (N + S_1 B) R_1^{-1} (B^T S_1 + N^T) + S_1 A + A^T S_1\right) \\
\dot{s}_2 &= -\left(q_2(t) - 2(N + S_1 B) R^{-1} r_s(t) + A^T s_2\right), \quad r_s(t) = \frac{1}{2}(r_2(t) + B^T s_2(t)) \\
\dot{s}_3 &= -\left(q_3(t) - r_s(t)^T R^{-1} r_s(t)\right)
\end{aligned}
$$

Note that S_1 has no time-dependent terms, and therefore $S_1(t)$ is a constant, given by the steady-state solution of the algebraic Riccati equation (e.g. from time-invariant LQR). Given this, the affine terms in the Riccati differential equation are given by the linear differential equations:

$$\dot{s}_2(t) = A_2 s_2(t) + B_2 \bar{z}_d(t), \quad s_2(t_f) = 0$$

with

$$A_2 = (N + S_1 B)R^{-1}B^T - A^T, \quad B_2 = \begin{bmatrix} 2I_{2\times 2} \\ 0_{2\times 2} \end{bmatrix} + 2\frac{h}{g}(N + S_1 B)R^{-1}$$

Assuming $\bar{z}_d(t)$ is described by a *continuous* piecewise polynomial of degree k with $n+1$ breaks at t_j (with $t_0 = 0$ and $t_n = t_f$):

$$\bar{z}_d(t) = \sum_{i=0}^{k} c_{j,i}(t - t_j)^i, \quad \text{for } j = 0, ..., n-1, \text{ and } \forall t \in [t_j, t_{j+1}),$$

this system has a closed-form solution given by:

$$s_2(t) = e^{A_2(t-t_j)}\alpha_j + \sum_{i=0}^{k} \beta_{j,i}(t - t_j)^i, \quad \forall t \in [t_j, t_{j+1}),$$

with α_j and $\beta_{j.i}$ vector parameters to be solved for. Taking

$$\begin{aligned} \dot{s}_2(t) =& A_2 e^{A_2(t-t_j)}\alpha_j + \sum_{i=0}^{k} A_2\beta_{j,i}(t - t_j)^i + \sum_{i=0}^{k} B_2 c_{j,i}(t - t_j)^i \\ =& A_2 e^{A_2(t-t_j)}\alpha_j + \sum_{i=1}^{k} i\beta_{j,i}(t - t_j)^{i-1} \end{aligned}$$

forces that

$$A_2\beta_{j,i} + B_2 c_{j,i} = (i+1)\beta_{j,i+1}, \quad \text{for } i = 0, ..., k-1$$
$$A_2\beta_{j,k} + B_2 c_{j,k} = 0.$$

Note: need to prove that A_2 is full rank (it appears to be in practice). Solve backwards ($i = k, k-1, ..., 0$) for $\beta_{j,i}$. Finally, the continuity and the terminal boundary condition $s(t_f) = 0$ gives

$$e^{A(t_{j+1}-t_j)}\alpha_j + \sum_{i=0}^{k} \beta_{j,i}(t_{j+1} - t_j)^{i+1} = s(t_{j+1}).$$