Drake Implementation of Rigid Body Dynamics Algorithms with Constraints

Russ Tedrake

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More complete write-up may follow. For now, just some quick notes.

1 Bilateral Position Constraints

Consider constraint equation

$$\phi(q) = 0.$$

These arise, for instance, when the system has a closed kinematic chain. The (floating base) equations of motion can be written

$$H(q)\ddot{q} + C(q,\dot{q}) = Bu + J(q)^T\lambda,$$

where $J(q) = \frac{\partial \phi}{\partial q}$ and λ is the constraint force.

To solve for λ, observe that when the constraint is imposed, $\phi(q) = 0$ and therefor $\dot{\phi} = 0$ and $\ddot{\phi} = 0$. Writing this out, we have

$$\dot{\phi} = J(q)\dot{q} = 0,$$
$$\ddot{\phi} = J(q)\ddot{q} + \dot{J}(q)\dot{q} = 0.$$

Inserting the dynamics and solving for λ yields

$$\lambda = -(JH^{-1}J^T)^{+}(JH^{-1}(Bu - C) + \dot{J}\dot{q}).$$

The $^+$ notation refers to a Moore-Penrose pseudo-inverse. In most cases, there are less constraints than degrees of freedom, in which case the inverse has a unique solution (and the traditional inverse could have been used). But the pseudo-inverse also works in cases where the system is over-constrained.

For numerical stability, I would like to add a restoring force to this constraint in the event that the constraint is not satisfied to numerical precision. To accomplish this, I'll ask for

$$\ddot{\phi} = -\frac{2}{\epsilon}\dot{\phi}(q) - \frac{1}{\epsilon^2}\phi(q).$$

Carrying this through yields

$$\lambda = -(JH^{-1}J^T)^{+}(JH^{-1}(Bu - C) + (\dot{J} + \frac{2}{\epsilon}J)\dot{q} + \frac{1}{\epsilon^2}\phi).$$

2 Bilateral Velocity Constraints

Consider the constraint equation

$$\psi(q, \dot{q}) = 0,$$

where $\frac{\partial \psi}{\partial \dot{q}} = \neq 0$. These are less common, but arise when, for instance, a joint is driven through a prescribed motion. Here, the manipulator equations are given by

$$H(q)\ddot{q} + C = Bu + \frac{\partial \psi}{\partial \dot{q}}^T \lambda.$$

To solve for λ, we take

$$\dot{\psi} = \frac{\partial \psi}{\partial q}\dot{q} + \frac{\partial \psi}{\partial \dot{q}}\ddot{q} = 0,$$

which yields

$$\lambda = -\left(\frac{\partial \psi}{\partial \dot{q}} H^{-1} \frac{\partial \psi}{\partial \dot{q}}\right)^+ \left[\frac{\partial \psi}{\partial \dot{q}} H^{-1}(Bu - C) + \frac{\partial \psi}{\partial q}\dot{q}\right].$$

Again, for numerical stability, we as instead for $\dot{\psi} = -\frac{1}{\epsilon}\psi$, which yields

$$\lambda = -\left(\frac{\partial \psi}{\partial \dot{q}} H^{-1} \frac{\partial \psi}{\partial \dot{q}}\right)^+ \left[\frac{\partial \psi}{\partial \dot{q}} H^{-1}(Bu - C) + \frac{\partial \psi}{\partial q}\dot{q} + \frac{1}{\epsilon}\psi\right].$$

3 Unilateral Position Constraints

Consider the constraint equation

$$\phi(q) \geq 0.$$

One common example of this, for instance, is a joint limit. The dynamics of unilateral constraints contain to pieces: the continuous dynamics when the constraint is inactive ($\phi(q) > 0$) or active ($\phi(q) = 0$), but also an impulsive event when the constraint becomes active ($\phi(q(t)) = 0, \phi(q(t-\epsilon)) > 0$). We model this as a hybrid transition. There is no corresponding event when the constraint transitions to inactive.

3.1 Continuous Dynamics

The continuous equations are governed by

$$H\ddot{q} + C = Bu + J^T\lambda,$$

where $J = \frac{\partial \phi}{\partial q}$. Let us consider the solution for different cases.

- If $\phi > 0$ the constraint is inactive, and $\lambda = 0$.
- Otherwise $\phi = 0$, and

  – if $\dot{\phi} > 0$, then the constraint is going inactive, and $\lambda = 0$.
  – otherwise $\dot{\phi} = 0$, and
    * $\ddot{\phi} > 0$, and $\lambda = 0$
    * or $\ddot{\phi} = 0$, and $\lambda > 0$.

For the case when $\ddot{\phi} = 0, \lambda > 0$, we have (as in the bilateral position constraints)

$$\lambda = -(JH^{-1}J^T)^+(JH^{-1}(Bu - C) + \dot{J}\dot{q}).$$

As a result, if $\phi > 0$ or $\dot{\phi}$, then $\lambda = 0$. Otherwise, we have to solve for $\ddot{q}$ and λ simultaneously to determine which constraints are active. We can accomplish this by solving a linear complementarity problem (LCP):

$$\begin{aligned}
\text{find} \quad & \ddot{\phi}, \lambda \\
\text{subject to} \quad & \ddot{\phi} \geq 0, \lambda \geq 0, \\
& \ddot{\phi} = \dot{J}\dot{q} + JH^{-1}(Bu - C + J^T\lambda), \\
& \forall_i \; \ddot{\phi}_i \lambda_i = 0.
\end{aligned}$$

Then $\ddot{q}$ follows from $\ddot{q} = H^{-1}(Bu - C + J^T\lambda)$.

For numerical stability, I must also consider when $\phi < 0$ and/or $\phi = 0, \dot{\phi} < 0$, so I instead ask for

$$\ddot{\phi} \geq -\frac{2}{\epsilon}\dot{\phi} - \frac{1}{\epsilon^2}\phi,$$

given the conditions

- If $\phi > 0$ or $\dot{\phi} > -\frac{1}{\epsilon}\phi$, then $\lambda = 0$.
- Otherwise, take $\alpha = \ddot{\phi} + \frac{2}{\epsilon}\dot{\phi} + \frac{1}{\epsilon^2}\phi$ to write

$$\begin{aligned}
\text{find} \quad & \alpha, \lambda \\
\text{subject to} \quad & \alpha \geq 0, \lambda \geq 0, \\
& \alpha = \dot{J}\dot{q} + JH^{-1}(Bu - C + J^T\lambda) - \frac{2}{\epsilon}\dot{\phi} - \frac{1}{\epsilon^2}\phi, \\
& \forall_i \; \alpha_i \lambda_i = 0.
\end{aligned}$$

3.2 Impulsive Event

The collision event is described by the zero-crossings (from positivie to negative) of the scalar function $\phi(q)$, and that after the impact we impose the constraint that $\phi = 0$. Using

$$H\ddot{q} + C = Bu + J^T\lambda,$$

λ is now an impulsive force that well-defined when integrated over the time of the collision (denoted t_c^- to t_c^+). Integrate both sides of the equation over that (instantaneous) interval:

$$\int_{t_c^-}^{t_c^+} dt \, [H\ddot{q} + C] = \int_{t_c^-}^{t_c^+} dt \left[Bu + J^T\lambda\right]$$

Since q and u are constants over this interval, we are left with

$$H\dot{q}^{+} - H\dot{q}^{-} = J^{T} \int_{t_c^-}^{t_c^+} \lambda dt,$$

where $\dot{q}^{+}$ is short-hand for $\dot{q}(t_c^+)$. Multiplying both sides by JH^{-1}, we have

$$J\dot{q}^{+} - J\dot{q}^{-} = JH^{-1}J^{T} \int_{t_c^-}^{t_c^+} \lambda dt.$$

But the first term on the left is zero because after the collision, $\dot{\phi} = 0$, yielding:

$$\int_{t_c^-}^{t_c^+} \lambda dt = -\left[JH^{-1}J^{T}\right]^{+} J\dot{q}^{-}.$$

Substituting this back in above results in

$$\dot{q}^{+} = \left[I - H^{-1}J^{T}\left[JH^{-1}J^{T}\right]^{+} J\right]\dot{q}^{-}.$$

4 Contact Constraints

4.1 Continuous Dynamics

4.2 Impulsive Event

5 Putting it all together