手推神经网络

焦瑞强

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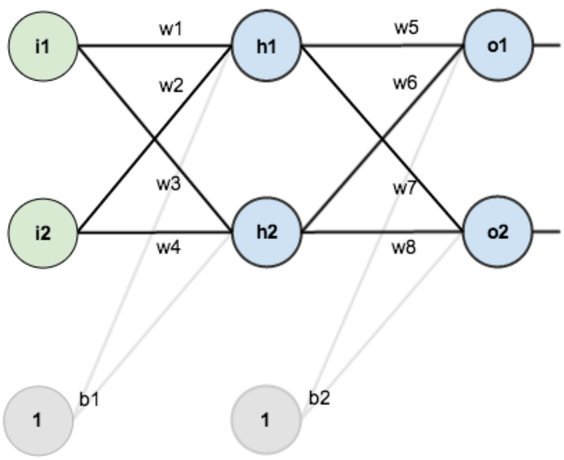
摘　要

本文主要参考 Matt Mazur 的例子。

关键词：神经网络，MSE，BP

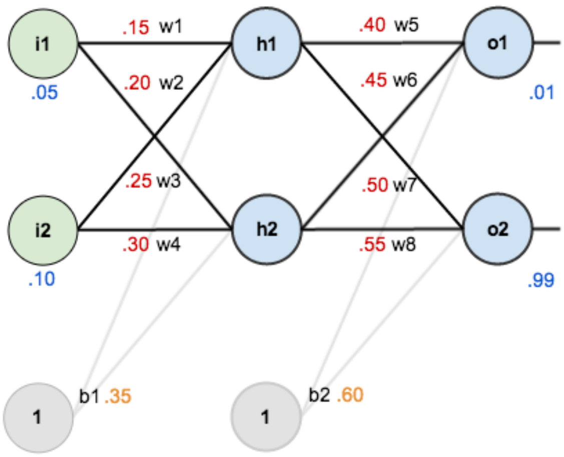
1　前向传播



1.1　初始化权重和偏差

令 $w_1 = 0.15, w_2 = 0.2, w_3 = 0.25, w_4 = 0.3, w_5 = 0.4, w_6 = 0.45, w_7 = 0.5, w_8 = 0.55, b_1 = 0.35, b_2 = 0.6$

如下图



1.2 隐藏层计算

- 计算 h_1 层的输入，计算如下

$$\begin{aligned} \text{net}_{h_1} &= w_1 * i_1 + w_2 * i_2 + b_1 * 1 \\ &= 0.15 * 0.05 + 0.2 * 0.1 + 0.35 * 1 \\ &= 0.3775 \end{aligned} \tag{1}$$

- 使用逻辑函数计算 h_1 层的输出，计算如下

$$\begin{aligned} \text{out}_{h_1} &= \frac{1}{1 + exp^{-\text{net}_{h_1}}} \\ &= \frac{1}{1 + exp^{-0.3775}} \\ &= 0.5932699921071872 \end{aligned} \tag{2}$$

同理，可得

$$\begin{aligned} \text{out}_{h_2} &= \frac{1}{1 + exp^{-\text{net}_{h_2}}} \\ &= \frac{1}{1 + exp^{-w_3 * i_1 + w_4 * i_2 + b_1 * 1}} \\ &= \frac{1}{1 + exp^{-1*(0.25*0.05+0.3*0.1+0.35*1)}} \\ &= \frac{1}{1 + exp^{-0.39249999999999996}} \\ &= 0.596884378259767 \end{aligned} \tag{3}$$

1.3 输出层计算

输出层的计算重复隐藏层的计算过程，如下

$$
\begin{aligned}
\text{out}_{o_1} &= \frac{1}{1+exp^{-\text{net}_{o_1}}} \\
&= \frac{1}{1+exp^{-(w_5*\text{out}_{h_1}+w_6*\text{out}_{h_2}+b_2*1)}} \\
&= \frac{1}{1+exp^{-1*(0.4*0.5932699921071872+0.45*0.596884378259767+0.6*1)}} \\
&= \frac{1}{1+exp^{-0.39249999999999996}} \\
&= 0.7513650695523157
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\text{out}_{o_2} &= \frac{1}{1+exp^{-\text{net}_{o_2}}} \\
&= \frac{1}{1+exp^{-(w_7*\text{out}_{h_1}+w_8*\text{out}_{h_2}+b_2*1)}} \\
&= \frac{1}{1+exp^{-1*(0.5*0.5932699921071872+0.55*0.596884378259767+0.6*1)}} \\
&= \frac{1}{1+exp^{-1.2249214040964653}} \\
&= 0.7729284653214625
\end{aligned} \tag{5}
$$

2 计算误差

接下来我们使用均方误差函数计算每一个输出神经元的误差并得到总的误差

$$E_{total} = \sum \frac{1}{2}(target - output)^2 \tag{6}$$

如上图，o_1 的原始输出为 0.01，而神经网络的输出为 0.7513650695523157，则其误差为

$$
\begin{aligned}
E_{o_1} &= \frac{1}{2}\big(target_{o_1} - output_{o_1}\big)^2 \\
&= \frac{1}{2}(0.01 - 0.7513650695523157)^2 \\
&= 0.274811083176155
\end{aligned} \tag{7}
$$

o_2 的原始输出为 0.99，神经网络的输出为 0.7729284653214625，则其误差为

$$
\begin{aligned}
E_{o_2} &= \frac{1}{2}\big(target_{o_2} - output_{o_2}\big)^2 \\
&= \frac{1}{2}(0.99 - 0.7729284653214625)^2 \\
&= 0.023560025583847746
\end{aligned} \tag{8}
$$

综上所述，总的误差为

$$E_{total} = E_{o_1} + E_{o_2} = 0.274811083176155 + 0.023560025583847746 = 0.2983711087600027 \tag{9}$$

$$
\begin{aligned}
E_{total} &= E_{o_1} + E_{o_2} \\
&= \frac{1}{2}\left(target_{o_1} - out_{o_1}\right)^2 + \frac{1}{2}\left(target_{o_2} - out_{o_2}\right)^2 \\
&= \frac{1}{2}\left(target_{o_1} - \frac{1}{1+exp^{-(w_5*\mathrm{out}_{h_1}+w_6*\mathrm{out}_{h_2}+b_2*1)}}\right)^2 + \frac{1}{2}\left(target_{o_2} - \frac{1}{1+exp^{-(w_7*\mathrm{out}_{h_1}+w_8*\mathrm{out}_{h_2}+b_2*1)}}\right)^2 \\
&= \frac{1}{2}\left(target_{o_1} - \frac{1}{1+exp^{-(w_5*\frac{1}{1+exp^{-\mathrm{net}_{h_1}}}+w_6*\frac{1}{1+exp^{-\mathrm{net}_{h_2}}}+b_2*1)}}\right)^2 \\
&+ \frac{1}{2}\left(target_{o_2} - \frac{1}{1+exp^{-(w_7*\frac{1}{1+exp^{-\mathrm{net}_{h_1}}}+w_8*\frac{1}{1+exp^{-\mathrm{net}_{h_2}}}+b_2*1)}}\right)^2 \\
&= \frac{1}{2}\left(target_{o_1} - \frac{1}{1+exp^{-(w_5*\frac{1}{1+exp^{-(w_1*i_1+w_2*i_2+b_1*1)}}+w_6*\frac{1}{1+exp^{-(w_3*i_1+w_4*i_2+b_1*1)}}+b_2*1)}}\right)^2 \\
&+ \frac{1}{2}\left(target_{o_2} - \frac{1}{1+exp^{-(w_7*\frac{1}{1+exp^{-(w_1*i_1+w_2*i_2+b_1*1)}}+w_8*\frac{1}{1+exp^{-(w_3*i_1+w_4*i_2+b_1*1)}}+b_2*1)}}\right)^2
\end{aligned}
\tag{10}
$$

3 反向传播

3.1 输出层参数更新

运用链式法则更新输出层相关参数 w_5, w_6, w_7, w_8

$$
\begin{aligned}
\frac{\mathrm{d}E_{total}}{\mathrm{d}w_5} &= \frac{\mathrm{d}E_{o_1}}{\mathrm{d}out_{o_1}}\frac{\mathrm{d}out_{o_1}}{\mathrm{d}net_{o_1}}\frac{\mathrm{d}net_{o_1}}{\mathrm{d}w_5} \\
&= (2*\frac{1}{2}*\left(target_{o_1} - out_{o_1}\right)*-1)*\frac{1}{1+exp^{-\mathrm{net}_{o_1}}}\left(1-\frac{1}{1+exp^{-\mathrm{net}_{o_1}}}\right)*\mathrm{out}_{h_1} \\
&= (out_{o_1} - target_{o_1})\frac{1}{1+exp^{-\mathrm{net}_{o_1}}}\left(1-\frac{1}{1+exp^{-\mathrm{net}_{o_1}}}\right)*\mathrm{out}_{h_1} \\
&= (0.7513650695523157 - 0.01)*0.7513650695523157*(1-0.7513650695523157)*0.593269992107187 \\
&= 0.08216704056423077
\end{aligned}
\tag{11}
$$

因此，

$$
\begin{aligned}
w_5^+ &= w_5 - \eta * \frac{\mathrm{d}E_{total}}{\mathrm{d}w_5} \\
&= 0.4 - 0.5 * 0.08216704056423077 \\
&= 0.35891647971788465
\end{aligned}
\tag{12}
$$

同理可得，

$$
\begin{aligned}
w_6^+ &= w_6 - \eta * \frac{\mathrm{d}E_{total}}{\mathrm{d}w_6} \\
&= 0.45 - 0.5 * (out_{o_1} - target_{o_1})\frac{1}{1+exp^{-\mathrm{net}_{o_1}}}\left(1 - \frac{1}{1+exp^{-\mathrm{net}_{o_1}}}\right) * \mathrm{out}_{h_2} \\
&= 0.45 - 0.5 * (0.7513650695523157 - 0.01) * 0.7513650695523157 * (1 - 0.7513650695523157) * 0.59688437 \\
&= 0.4086661860762334
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
w_7^+ &= w_7 - \eta * \frac{\mathrm{d}E_{total}}{\mathrm{d}w_7} \\
&= 0.5 - 0.5 * (out_{o_2} - target_{o_2})\frac{1}{1+exp^{-\mathrm{net}_{o_2}}}\left(1 - \frac{1}{1+exp^{-\mathrm{net}_{o_2}}}\right) * \mathrm{out}_{h_1} \\
&= 0.5 - 0.5 * (0.7729284653214625 - 0.99) * 0.7729284653214625 * (1 - 0.7729284653214625) * 0.593269992 \\
&= 0.5113012702387375
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
w_8^+ &= w_8 - \eta * \frac{\mathrm{d}E_{total}}{\mathrm{d}w_8} \\
&= 0.55 - 0.55 * (out_{o_2} - target_{o_2})\frac{1}{1+exp^{-\mathrm{net}_{o_2}}}\left(1 - \frac{1}{1+exp^{-\mathrm{net}_{o_2}}}\right) * \mathrm{out}_{h_2} \\
&= 0.55 - 0.5 * (0.7729284653214625 - 0.99) * 0.7729284653214625 * (1 - 0.7729284653214625) * 0.59688437 \\
&= 0.5613701211079891
\end{aligned} \tag{15}
$$

3.2 隐藏层参数更新

运用链式法则更新隐藏层相关参数 w_1，w_2，w_3，w_4

$$
\begin{aligned}
\frac{\mathrm{d}E_{total}}{\mathrm{d}w_1} &= \frac{\mathrm{d}E_{total}}{\mathrm{d}out_{h_1}}\frac{\mathrm{d}out_{h_1}}{\mathrm{d}net_{h_1}}\frac{\mathrm{d}net_{h_1}}{\mathrm{d}w_1} \\
&= \frac{\mathrm{d}E_{total}}{\mathrm{d}out_{h_1}} out_{h_1}(1 - out_{h_1}) i_1
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\frac{\mathrm{d}E_{total}}{\mathrm{d}out_{h_1}} &= \frac{\mathrm{d}E_{o_1}}{\mathrm{d}out_{h_1}} + \frac{\mathrm{d}E_{o_2}}{\mathrm{d}out_{h_1}} \\
&= \frac{\mathrm{d}E_{o_1}}{\mathrm{d}net_{o_1}}\frac{\mathrm{d}net_{o_1}}{\mathrm{d}out_{h_1}} + \frac{\mathrm{d}E_{o_2}}{\mathrm{d}net_{o_2}}\frac{\mathrm{d}net_{o_2}}{\mathrm{d}out_{h_1}} \\
&= \frac{\mathrm{d}E_{o_1}}{\mathrm{d}out_{o_1}}\frac{\mathrm{d}out_{o_1}}{\mathrm{d}net_{o_1}}\frac{\mathrm{d}net_{o_1}}{\mathrm{d}out_{h_1}} + \frac{\mathrm{d}E_{o_2}}{\mathrm{d}out_{o_2}}\frac{\mathrm{d}out_{o_2}}{\mathrm{d}net_{o_2}}\frac{\mathrm{d}net_{o_2}}{\mathrm{d}out_{h_1}} \\
&= \frac{\mathrm{d}E_{o_1}}{\mathrm{d}out_{o_1}}\frac{\mathrm{d}out_{o_1}}{\mathrm{d}net_{o_1}} w_5 + \frac{\mathrm{d}E_{o_2}}{\mathrm{d}out_{o_2}}\frac{\mathrm{d}out_{o_2}}{\mathrm{d}net_{o_2}} w_7 \\
&= (0.7513650695523157 - 0.01) * 0.7513650695523157 * (1 - 0.7513650695523157) * 0.4 + (0.772928465 \\
&= 0.05539942465142278 - 0.019049118258278118 \\
&= 0.03635030639314466
\end{aligned} \tag{17}
$$

联立上面两式可得

$$
\begin{aligned}
\frac{\mathrm{d}E_{total}}{\mathrm{d}w_1} &= 0.03635030639314466 * 0.5932699921071872 * (1 - 0.5932699921071872) * 0.05 \\
&= 0.0004385677344743465
\end{aligned} \tag{18}
$$

基于上面的推导更新 w_1

$$
\begin{aligned}
w_1^+ &= w_1 - \eta * \frac{\mathrm{d}E_{total}}{\mathrm{d}w_1} \\
&= 0.15 - 0.5 * 0.0004385677344743465 \\
&= 0.1497807161327628
\end{aligned} \tag{19}
$$

同方法可得

$$
\begin{aligned}
w_2^+ &= w_2 - \eta * \frac{\mathrm{d}E_{total}}{\mathrm{d}w_2} \\
&= 0.2 - 0.5 * \frac{\mathrm{d}E_{total}}{\mathrm{d}out_{h_1}} out_{h_1}(1 - out_{h_1}) i_2 \\
&= 0.2 - 0.5 * 0.03635030639314466 * 0.5932699921071872 * (1 - 0.5932699921071872) * 0.1 \\
&= 0.19956143226552567
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\frac{\mathrm{d}E_{total}}{\mathrm{d}out_{h_2}} &= \frac{\mathrm{d}E_{o_1}}{\mathrm{d}out_{h_2}} + \frac{\mathrm{d}E_{o_2}}{\mathrm{d}out_{h_2}} \\
&= \frac{\mathrm{d}E_{o_1}}{\mathrm{d}net_{o_1}}\frac{\mathrm{d}net_{o_1}}{\mathrm{d}out_{h_2}} + \frac{\mathrm{d}E_{o_2}}{\mathrm{d}net_{o_2}}\frac{\mathrm{d}net_{o_2}}{\mathrm{d}out_{h_2}} \\
&= \frac{\mathrm{d}E_{o_1}}{\mathrm{d}out_{o_1}}\frac{\mathrm{d}out_{o_1}}{\mathrm{d}net_{o_1}}\frac{\mathrm{d}net_{o_1}}{\mathrm{d}out_{h_2}} + \frac{\mathrm{d}E_{o_2}}{\mathrm{d}out_{o_2}}\frac{\mathrm{d}out_{o_2}}{\mathrm{d}net_{o_2}}\frac{\mathrm{d}net_{o_2}}{\mathrm{d}out_{h_2}} \\
&= \frac{\mathrm{d}E_{o_1}}{\mathrm{d}out_{o_1}}\frac{\mathrm{d}out_{o_1}}{\mathrm{d}net_{o_1}}w_6 + \frac{\mathrm{d}E_{o_2}}{\mathrm{d}out_{o_2}}\frac{\mathrm{d}out_{o_2}}{\mathrm{d}net_{o_2}}w_8 \\
&= (0.7513650695523157 - 0.01) * 0.7513650695523157 * (1 - 0.7513650695523157) * 0.45 + (0.77292846 \\
&= 0.0413703226487447
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
w_3^+ &= w_3 - \eta * \frac{\mathrm{d}E_{total}}{\mathrm{d}w_3} \\
&= 0.25 - 0.5* \\
&= 0.25 - 0.5 * \frac{\mathrm{d}E_{total}}{\mathrm{d}out_{h_2}} out_{h_2}(1 - out_{h_2})i_1 \\
&= 0.25 - 0.5 * \frac{\mathrm{d}E_{total}}{\mathrm{d}out_{h_2}} * 0.596884378259767 * (1 - 0.596884378259767) * 0.05 \\
&= 0.25 - 0.5 * 0.0413703226487447 * 0.596884378259767 * (1 - 0.596884378259767) * 0.05 \\
&= 0.24975114363236958
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
w_4^+ &= w_4 - \eta * \frac{\mathrm{d}E_{total}}{\mathrm{d}w_4} \\
&= 0.3 - 0.5 * \frac{\mathrm{d}E_{total}}{\mathrm{d}out_{h_2}} out_{h_2}(1 - out_{h_2})i_2 \\
&= 0.3 - 0.5 * 0.0413703226487447 * 0.596884378259767 * (1 - 0.596884378259767) * 0.1 \\
&= 0.29950228726473915
\end{aligned}
\tag{23}
$$