# SEMI-INVARIANTS OF GENTLE ALGEBRAS BY DEFORMATION METHOD AND SPHERICITY.

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ABSTRACT. We investigate the algebras of semi-invariants for gentle algebras applying so-called Deformation method and using our observation that varieties of complexes and circular complexes are spherical. This yields an explicit description of generators and relations for the algebras of semi-invariants.

## 1. INTRODUCTION

Let $Q$ be a quiver, that is an oriented graph with the sets $Q_0$ of vertices and $Q_1$ of arrows together with two maps, $t, h : Q_1 \to Q_0$ such that $t\varphi$ and $h\varphi$ are the tail and the head of an arrow $\varphi \in Q_1$. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0 and let $\mathbf{k}Q$ denote the *path algebra* of the quiver with all oriented paths in $Q$ as the basis and with the multiplication defined via concatenation of paths. This algebra is finite dimensional if and only if $Q$ does not have an oriented cycles. Denote by $\mathbf{k}Q_1 \subseteq \mathbf{k}Q$ the two sided ideal generated by all arrows. By *bound quiver algebra* we mean a quotient $\mathbf{k}Q/I$, where $I$ is any ideal contained in $\mathbf{k}Q_1^2$. For any dimension vector $\alpha : Q_0 \to \mathbf{Z}_+$ we define the representation space $R(Q, I, \alpha)$ of algebra $\mathbf{k}Q/I$ as the set of all representations of $Q$ of dimension $\alpha$ such that the induced representation of $\mathbf{k}Q$ vanishes on $I$.

**Definition 1.1.** [AS] The bound quiver algebra $\mathbf{k}Q/I$ is called *gentle algebra* if the following conditions hold:

(1) for each vertex $i \in Q_0$ there are at most two arrows with head $i$, and at most two arrows with the tail $i$.

(2) $I$ is generated by paths of length 2

(3) for any $\varphi \in Q_1$, if there are two arrows $\psi_1 \neq \psi_2 \in Q_1$ with $h\psi_1 = h\psi_2 = t\varphi$ (resp. with $t\psi_1 = t\psi_2 = h\varphi$), then precisely one of $\varphi\psi_1$ and $\varphi\psi_2$ (resp. $\psi_1\varphi$ and $\psi_2\varphi$) belongs to $I$.

A gentle algebra is called *triangular* if $Q$ has no oriented cycles.

Gentle algebras are tame and their representation theory is well described. Following [CW] we use the definition of gentle algebras in terms of *arrow coloring*:

**Definition 1.2.** A coloring of a quiver $Q$ is a map $c : Q_1 \to S$, where $S$ is a finite set of colors such that for any $s \in c(Q_1)$ the preimage $c^{-1}(s)$ constitutes an oriented path. The ideal $I_c \subseteq \mathbf{k}Q$ is generated by all unicolored paths of length two.

One can notice that oriented paths in the above definition fall in three cases:

($a$) A simple path without self-intersection.

($b$) A circular path with tail equal to head but without interior self-intersection

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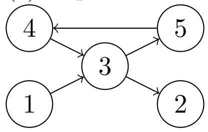
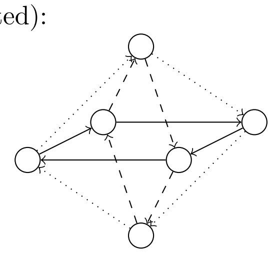
$(c)$ A path with interior self-intersection like $(1,3,5,4,3,2)$ below:




The case $(c)$ has the singularity that some vertices are incident to more than 2 arrows. For example, the above vertex 3 is incident to four arrows of the path and all four 2-hop sub-paths $(1,3,5),(4,3,2)$,$(1,3,2),(4,3,5)$ belong to $I_c$ by Definition 1.2. In order to apply our results from [Sh02], we need to limit the paths from Definition 1.2. One way is to consider triangular gentle algebras, that is, require $Q$ to have no oriented cycles, hence, only case $(a)$ is allowed. However, we want to consider both cases $(a)$ and $(b)$ and we claim an obvious statement as follows:

**Proposition 1.3. 1.** *For a quiver $Q$, consider a coloring $c : Q_1 \to S$ such that:*
*$(i)$ For every $s \in S, \varphi, \psi \in c^{-1}(s)$ holds $t\varphi \neq t\psi$ and $h\varphi \neq h\psi$*
*$(ii)$ Every vertex is incident to arrows of at most two colors.*
*Then the bound quiver algebra $\mathbf{k}Q/I_c$ is a gentle algebra.*
**2.** *Conversely, every triangular gentle algebra is equal to $\mathbf{k}Q/I_c$ for some coloring $c : Q_1 \to S$ with properties $(i)$ and $(ii)$.*

In what follows we develop methods that describe the semi-invariants of bound quiver algebras $\mathbf{k}Q/I_c$ matching the conditions $(i),(ii)$ from Proposition 1.3.1, for example the following octahedron (three colors are depicted with the type of arrow, normal, dashed, and dotted):



For a bound quiver algebra $\mathbf{k}Q/I$ and a dimension vector $\alpha$, the representation space $R(Q,I,\alpha)$ is a closed affine subvariety (not necessarily irreducible) of the vector space $R(Q,\alpha)$ of all representations of $Q$. The latter vector space is acted upon by the group $GL(\alpha) = \prod_{i\in Q_0} GL_{\alpha(i)}$ and the subvariety $R(Q,I,\alpha)$ is clearly $GL(\alpha)$-stable so that the $GL(\alpha)$-orbits are the isomorphism classes of reperesenations. To classify those orbits, the semi-invariant functions are useful.

If the dimension vector $\alpha$ is *sincere*, which means, $\alpha(i) > 0$ for each $i \in Q_0$, then the character group $\mathcal{X}(GL(\alpha))$ is a free Abelian group generated by determinants $\det^i, i \in Q_0$ of the components of $GL(\alpha)$. For any character $\chi \in \mathcal{X}(GL(\alpha))$, we consider the module of semi-invariant functions of weight $\chi$:

$$\mathbf{k}[R(Q,I,\alpha)]_\chi^{(GL(\alpha))} = \{f \in \mathbf{k}[R(Q,I,\alpha)] | gf = \chi(g)f, \forall g \in GL(\alpha)\} \qquad (1)$$

The group $SL(\alpha) = \{g \in GL(\alpha) | \det^i(g) = 1, \forall i \in Q_0\}$ is the commutator subgroup of $GL(\alpha)$ and a decomposition holds, as follows:

$$\mathbf{k}[R(Q,I,\alpha)]^{SL(\alpha)} = \bigoplus_{\chi\in\mathcal{X}(GL(\alpha))} \mathbf{k}[R(Q,I,\alpha)]_\chi^{(GL(\alpha))}. \qquad (2)$$

The algebra $\mathbf{k}[R(Q,I,\alpha)]^{SL(\alpha)}$ is called the algebra of semi-invariants for the representation of gentle algebras and it is our main interest in this paper. Three papers, [KW],[CW], and [CC] are our benchmarks in this topic. Out of them, [KW] is focused on one special case, when $Q$ is the double of an equioriented quiver of type $A_n$ with two equal arrows instead of each of $A_n$. Note that the variety $R(Q,I,\alpha)$ is acted upon by the direct product $GL(\alpha)\times GL(\alpha)$. Let $\Lambda$ be the semi-group of highest weights of simple $GL(\alpha)\times GL(\alpha)$-modules in $\mathbf{k}[R(Q,I,\alpha)]$ containing $SL(\alpha)$-invariants. In that paper, [KW], a specific model is derived to prove that $\mathbf{k}[R(Q,I,\alpha)]^{SL(\alpha)}$ is isomorphic to the semi-group algebra $\mathbf{k}[\Lambda]$. Later a similar statement was proved in [CW] in the whole generality. The proof extensively uses the coordinates in $\mathbf{k}[R(Q,I,\alpha)]$ related to Young diagrams.

Our approach is inspired by the fact that the rank gentle varieties are direct products of varieties of complexes or circular complexes and on observation from [Sh02] that the latter varieties are spherical in the sense of [VK]. Moreover, the algebra of invariants for the maximal unipotent subgroup $U\subseteq GL(\alpha)$ is polynomial and explicitly described in [Sh02]. These observations put the problem to the setup of Deformation method in style of [Pa], which describes the associated graded algebra of the algebra of invariants $\mathbf{k}[Z]^H$ for a spherical subgroup $H\subseteq G$ of a reductive group $G$ acting on $Z$ as the algebra of invariants $\mathbf{k}[Z]^{I(\Gamma(G/H))}$, where $I(\Gamma(G/H))$ is the *horospherical contraction* of $H$ in $G$. This method applies if the algebra of invariants $\mathbf{k}[Z]^U$ has an explicit description. In our case, thanks to [Sh02] the action $G:Z$ is spherical and we may use the explicit description of the algebraically independent generators of $\mathbf{k}[Z]^U$. Proposition 2.7 yields an isomorphism of $\mathbf{k}[Z]^{I(\Gamma(G/H))}$ and $\mathbf{k}[\Lambda]$ for the case, when the action of $G$ on $Z$ is spherical so that the isomorphism from [CW] is mostly explained by the Deformation method, but in a weak form. We try to get a stronger version for the particular case of $G/H$ being a spherical homogeneous space of a certian narrow class (and $Z$ spherical w.r.t. $G$). We provide Theorem **2.10**, which yields an explicit $Z(G)$-equivariant vector space isomorphism of $\mathbf{k}[Z]^{I(\Gamma(G/H))}$ over $\mathbf{k}[Z]^H$ such that the image of generators of the former algebra generate the latter one. Applying this general statement to the rank gentle varieties, we get Theorem **4.9** claiming that $\mathbf{k}[R(Q,I,\alpha)]^{SL(\alpha)}$ is a *deformation* of the above semi-group algebra $\mathbf{k}[\Lambda]$ and the generators of the former algebra are explicit from the latter one. Moreover, our approach yields a new description of the algebra of semi-invariants in terms of the $U$-invariants of varieties of complexes and circular complexes. Using that data, we develop a graph-theoretical approach to the semi-invariants, inspired by *up an down graphs* from [CC], where those graphs were introduced with different target: to describe generic representations and modules of rank gentle varieties. In our case, we introduce a bidirected graph in the sense of [EJ] such that this graph structure is straightforward from the rank gentle variety and by Theorem **5.4** the generating semi-invariants are the *complete walks* along this graph. This makes computation of $\mathbf{k}[R(Q,I,\alpha)]^{SL(\alpha)}$ explicit and we show several examples of this techniques. In particular, we give an explicit description of semi-invariants for irreducible components of the variety consisting of quadruples $\{A_1,B_1,A_2,B_2|A_1,A_2\in \mathrm{Hom}(L,M),B_1,B_2\in\mathrm{Hom}(M,L)\}$ of linear maps subject to the constraints $A_1B_1=0=B_1A_1, A_2B_2=0=B_2A_2$.

## 2. Deformation method and spherical actions.

We apply to gentle algebras the approach introduced in [Pa]. Let $G$ be a connected reductive group with a maximal torus and a Borel subgroup $T \subseteq B \subseteq G$. Denote by $\mathcal{X}_T$ the character group of $T$ and by $\mathcal{X}_+ \subseteq \mathcal{X}_T$ the sub-semi-group of dominant weights with respect to $B$. For $\lambda \in \mathcal{X}_+$ denote by $V_\lambda$ the corresponding simple module. Let $H \subseteq G$ be an algebraic subgroup. Set

$$\Gamma(G/H) = \{\lambda \in \mathcal{X}_+ | V_\lambda^H \neq \{0\}.$$

A sub-semi-group $\Gamma \subseteq \mathcal{X}_+$ is called *saturated* if the subgroup in $\mathcal{X}_T$ generated by $\Gamma$ interects $\mathcal{X}_+$ exactly by $\Gamma$. Let $U = B^U$ be the maximal unipotent subgroup of $B$. For any $\lambda \in \mathcal{X}_+$ denote by $\langle v_\lambda \rangle \subseteq V_\lambda$ the unique $U$-invariant line of highest vectors. For any sub-semi-group $\Gamma \subseteq \mathcal{X}_+$ set

$$I(\Gamma) = \{g \in G | gv_\lambda = v_\lambda,\ for\ any\ \lambda \in \Gamma\}. \tag{3}$$

Obviously, we have $U \subseteq I(\Gamma)$. The overgroups of $U$ are called *horospherical.* Thus, $I(\Gamma(G/H))$ is called the *horospherical contraction* of $H$.

**Example 2.1.** Let $H$ be a connected reductive group and consider $H$ as a diagonal subgroup in $G = H \times H$. By definition, $\Gamma(G/H)$ consists of pairs $\lambda = (\mu_1, \mu_2)$ of $H$-dominant weights such that $V_{\mu_1} \otimes V_{\mu_2}$ contains an $H$-invariant point. Schur Lemma then implies that $V_{\mu_2}$ is isomorphic to $V_{\mu_1}^*$. In other words, $\Gamma(G/H) = \{(\mu, \mu^*) | \mu \in \mathcal{X}_+(H)\}$. Clearly, such a subgroup is saturated. Now, what is $I(\Gamma(G/H))$? As a subgroup it depends on the choice of the maximal unipotent group in $G$. Let $U$ be a maximal unipotent subgroup in $H$ and $B = UT$ be a Borel subgroup. There exists an opposite Borel subgroup, denoted by $B^{op} \subseteq G$ such that $B \cap B^{op} = T$. For example, if $H = GL(V)$, and $B$ consists of upper triangular matrices, then $B^{op}$ consists of lower triangular matrices. Denote by $U^{op}$ the unipotent radical of $B^{op}$. For any simple $H$-module $V_\mu$ the highest vector with respect to $U^{op}$ is the lowest vector with respect to $U$ with the weight $-\mu^*$. Therefore, by definition (3), the group $I(\Gamma(G/H))$ has the following form:

$$I(\Gamma(G/H)) = (U \times U^{op})T^{diag}, T^{diag} = \{(t,t) \in T \times T\}. \tag{4}$$

Recall the following definitions and properties:

**Definition 2.2.** An action $G : Z$ is called spherical if a Borel subgroup $B \subseteq G$ acts on $Z$ with a dense orbit. A subgroup $H \subseteq G$ is called spherical if the homogeneous G-variety $G/H$ is spherical.

The following Proposition from [VK] characterizes sphericall actions and subgroups in terms of $G$-module properties:

**Proposition 2.3. 1.** *An affine $G$-variety $Z$ is spherical if and only if $G$-module $\mathbf{k}[Z]$ is multiplicity free.*

**2.** *If $H$ is a spherical subgroup of $G$ and $V$ is an irreducible $G$-module of finite dimension, then* $\dim V^H \leq 1$.

Consider an affine $G$-variety $Z$. The main idea of [Pa] is that for any spherical subgroup $H \subseteq G$, the algebras of invariants $\mathbf{k}[Z]^H$ and $\mathbf{k}[Z]^{I(\Gamma(G/H))}$ are related. To state this result, recall that the character group $\mathcal{X}_T$ carries an important partial order such that $\lambda \prec \mu$ if $\mu - \lambda$ is a non-negative integer linear combination of simple roots. In particular, the highest weight $\lambda$ is maximal in this order among all torus weights in the corresponding irreducible module. Applying this order to the highest weights, we get a partial order on irreducible factors of $\mathbf{k}[Z]$. Moreover, if $V_1 \cong V_\lambda$ and $V_2 \cong V_\mu$ are irreducible factors of $\mathbf{k}[Z]$, then $V_1V_2$ is a sum of components smaller than or equal to $\lambda+\mu$ in the above partial order. This implies that the subspaces $\mathbf{k}[Z]_{\preceq\lambda}$ yield a natural $G$-invariant filtration.

**Theorem 2.4.** [Pa, Theorem 0.2] *Let $H \subseteq G$ be a spherical subgroup of a connected reductive group $G$ acting on an affine variety $Z$. Assume that $\Gamma(G/H)$ is saturated. The associated graded algebra $\mathfrak{gr}\mathbf{k}[Z]^H$ with respect to the filtration $\mathbf{k}[Z] = \cup_{\lambda\in\mathcal{X}_+}\mathbf{k}[Z]_{\preceq\lambda}$ is isomorphic to $\mathbf{k}[Z]^{I(\Gamma(G/H))}$.*

**Example 2.5.** We continue Example 2.1 and consider a reducible action of $H$ on a direct product $Z = X \times Y$ of two affine $H$-varieties. The action $H : Z$ can be thought of as a restriction of that for $G = H \times H$. Then applying Theorem 2.4 to this context, we get Theorem 4.2 from [Pa]:

$$\mathfrak{gr}\mathbf{k}[X \times Y]^H \cong (\mathbf{k}[X]^U \otimes \mathbf{k}[Y]^{U^{op}})^{T^{diag}} \tag{5}$$

The results like Theorem 2.4 or (5) are useful as is shown in [Pa], because the singularities of an algebra $A$ are not "worse" than those of associated graded $\mathfrak{gr}A$:

**Theorem 2.6.** *In the conditions of 2.4 the following statements hold:*

**1.** [Pa, 3.5] *If $\mathbf{k}[Z]^{I(\Gamma(G/H))}$ is a complete intersection, then so is $\mathbf{k}[Z]^H$.*

**2.** [Pa, 3.7] *The number of elements in a minimal system of homogeneous generators for $\mathbf{k}[Z]^{I(\Gamma(G/H))}$ is greater than or equal to that for $\mathbf{k}[Z]^H$. In particular, if $\mathbf{k}[Z]^{I(\Gamma(G/H))}$ is a polynomial ring, then so is $\mathbf{k}[Z]^H$.*

The situation would be of course much simpler if we had an isomorphism $\mathbf{k}[Z]^H \cong \mathbf{k}[Z]^{I(\Gamma(G/H))}$. The case, when the latter algebra is polynomial guarantees an isomorphism, because by 2.6.2 the former one is also a polynomial algebra of the same dimension. However, this isomorphism is defined in external terms. Is it possible to build an isomorphism in internal terms?

In general this is impossible because, as shown in the example [Pa, 4.3] the algebras can be non-isomorphic. In the rest of this section we restrict consideration to the most favourable case and try to clarify a bit possible relations between both algebras.

From now on we assume the variety $Z$ to be spherical itself. By 2.3.1 we can decompose the $G$-module $\mathbf{k}[Z] = \oplus_{\lambda\in\Delta}V_\lambda$ for some sub-semi-group $\Delta \subseteq \mathcal{X}_+$. Restricting this decomposition to the algebra of $U$-invariants, we get $\mathbf{k}[Z]^U = \oplus_{\lambda\in\Delta}V_\lambda^U$. Thus $\mathbf{k}[Z]^U$ is a direct sum of one-dimensional subspaces corresponding to the elements of $\Delta$. Moreover, it is known that:

**Proposition 2.7.** *$\mathbf{k}[Z]^U$ is isomorphic to the semi-group algebra $\mathbf{k}[\Delta]$.*

Though the above fact is well known by specialists, we present here a simple and explicit isomorphism of the algebras due to D.Timashev:

*Proof.* Let $B \subseteq G$ be the Borel subgroup such that $B^U = U$. Then for any $\lambda \in \Delta$, $V_\lambda^U$ consists of $B$-semi-invariant functions of weight $\lambda$. Fix a point $z_0 \in Z$ such that the orbit $Bz_0$ is dense in $Z$. If some $B$-semi-invariant function $g$ vanishes at $z_0$, then it totally vanishes on $Z$. Therefore for any $\lambda \in \Delta$ there exists a unique $f_\lambda \in V_\lambda^U$ such that $f_\lambda(z_0) = 1$. Thus the set $\{f_\lambda|\lambda \in \Delta\}$ is a $\mathbf{k}$-basis of $\mathbf{k}[Z]^U$ closed under multiplication. Sending elements of this basis to the tautological basis of $\mathbf{k}[\Delta]$ gives rise to an algebra isomorphism of $\mathbf{k}[Z]^U$ onto $\mathbf{k}[\Delta]$. □

**Corollary 2.8.** $\mathbf{k}[Z]^{I(\Gamma(G/H))}$ *is isomorphic to* $\mathbf{k}[\Delta\cap\Gamma(G/H)]$.

*Proof.* Since $I(\Gamma(G/H))$ contains $U$, the algebra $\mathbf{k}[Z]^{I(\Gamma(G/H))}$ is spanned as a vector space by highest vectors $v_\lambda$ with $\lambda\in\Delta\cap\Gamma(G/H)$. Then, this subalgebra in $\mathbf{k}[Z]^U$ is isomorphich to the subalgebra $\mathbf{k}[\Delta\cap\Gamma(G/H)]\subseteq\mathbf{k}[\Delta]$. $\square$

We need to slightly generalize Example 2.1 in order to control semi-invariants. In what follows we denote by $Z(K)$ the center of an algebraic group $K$.

**Proposition 2.9.** *Let $K$ be a connected reductive group and $H\subseteq K$ be a connected subroup containing the commutator subgroup: $(K,K)\subseteq H\subseteq K$. For a Borel subgroup $B=TU\subseteq K$ consider the maximal subgroup $U\times U^{op}$ and the maximal torus $T\times T$ of $K\times K$. Set $T_H=T\cap H$. Then we have:*

**1.** *The diagonal embedding of $H$ to $K\times K$ is a spherical subgroup in $K\times K$*

**2.** *The semi-group $\Gamma((K\times K)/H)$ consists of all weights $(\mu_1,\mu_2)$ such that the restrictions of $\mu_1$ and $\mu_2^*$ to $T_H$ are equal. This semi-group is saturated.*

**3.** *Holds $I(\Gamma((K\times K)/H))=(U\times U^{op})T_H$.*

*Proof.* **1** Since the diagonal embedding of $K$ to $K\times K$ is spherical, there is a Borel subgroup $B_0$ of $K\times K$ such that $B_0K$ is dense in $K\times K$. Because $B_0$ contains $Z(K)\times Z(K)$, $B_0H$ is equal to $B_0K$. The proof of **2** and **3** follows Example 2.1. $\square$

For a reductive group $H$ and a finite dimensional $H$-module $V$ we denote by $\natural : V\to V^H$ the Reynolds operator, which is the unique projection along the sum of non-trivial irreducible $H$-sub-modules. The existence and uniqueness of $\natural$ follow from the fact that every finite dimensional representation of $H$ is completely reducible. It is well known that every $H$-stable submodule is stable with respect to Reynolds operator.

**Theorem 2.10.** *For a connected reductive subgroup $H\subseteq G$ assume that $G/H$ is isomorphic to the direct product: $G/H\cong\prod_{l=1,\cdots,L}(K_l\times K_l)/H_l$, where for $l=1,\cdots,L$ holds $(K_l,K_l)\subseteq H_l\subseteq K_l$. Then we have:*

**1.** *$\natural:\mathbf{k}[Z]^{I(\Gamma(G/H))}\to\mathbf{k}[Z]^H$ is $Z(G)$-equivariant isomorphism of vector spaces.*

**2.** *For generators $\lambda_1,\cdots,\lambda_m$ of $\Delta\cap\Gamma(G/H)$, $v^\natural_{\lambda_1},\cdots,v^\natural_{\lambda_m}$ generate $\mathbf{k}[Z]^H$.*

*Proof.* By Proposition 2.9, $H$ is a spherical subroup in $G$. Hence, by Proposition 2.3.2, for any simple $G$-submodule $V_\lambda\subseteq\mathbf{k}[Z]$, either $\dim V_\lambda^H=0$ or $\dim V_\lambda^H=1$, and the latter is equivalent to $\lambda\in\Delta\cap\Gamma(G/H)$. We claim that for any $\lambda\in\Delta\cap\Gamma(G/H)$ and highest vector $v_\lambda\in V_\lambda$ holds:

$$v^\natural_\lambda=\alpha_\lambda v_\lambda+w_\lambda,\alpha_\lambda\neq0,w_\lambda\in\bigoplus_{\mu\in\mathcal{X},\mu\prec\lambda}V^\mu_\lambda, \tag{6}$$

where by $V^\mu_\lambda$ we denote the $T$-weight subspace of weight $\mu$ in $V_\lambda$, eventually zero. Indeed, the assumption about $G/H$ implies that $\lambda$ is the sum of weights: $\lambda=\lambda_1+\cdots+\lambda_L$ such that $\lambda_l$ is a dominant weight of $K_l\times K_l$ and $V_{\lambda_l}^{H_l}\neq\{0\}$. Moreover,$V_\lambda$ is a tensor product, $V_\lambda\cong V_{\lambda_1}\otimes\cdots V_{\lambda_L}$, and both $V_\lambda^U$ and $V_\lambda^H$ are the tensor products of one dimensional corresponding spaces $V_{\lambda_l}^U$ and $V_{\lambda_l}^{H_l}, l=1,\cdots,L$. Therefore it is sufficient to prove the claim in the case when $\lambda=\lambda_l$ for some $l$. In other words, we may assume $G/H=(K\times K)/H$ as in Proposition 2.9.

Hence, because $T_H$ is a maximal torus of $H$ (here and below we use the notation of 2.9), 2.9.2 means $V_\lambda$ is $H$-isomorphic to $W\otimes W^*$, where $W$ is a simple $H$-module. The highest vector $v_\lambda$ with respect to the maximal unipotent subgroup

$U \times U^{op}$ is the tensor product $v_\lambda = w \otimes w^*$ of $H$-highest vector in $W$ and $H$-lowest vector in $W^*$. Let's include $w$ as the first vector to a $T$-weight basis of $W$ and consider $W \otimes W^*$ as the matrices in this basis. Then $w \otimes w^*$ is the diagonal matrix $diag(1, 0, \cdots, 0)$. We have a decomposition $diag(1, 0, \cdots, 0) = (\dim W)^{-1}(Id + diag(\dim W - 1, -1, \cdots, -1))$, where the first summand is $GL(W)$-invariant and the second one has trace 0, so belongs to $GL(W)$-stable supplementary subspace. Since $\dim V_\lambda^H = 1$, this decomposition is also a sum of a $H$-invariant vector and a vector in the sum of $H$-irrreducible non-trivial summands of $V_\lambda$. Hence, $v_\lambda^\natural = (\dim W)^{-1} Id = (\dim W)^{-1}(v_\lambda + diag(0, 1, \cdots, 1)) = (\dim V_\lambda)^{-0.5} v_\lambda + w_\lambda,\ w_\lambda \in \bigoplus_{\mu \in \mathcal{X}, \mu \prec \lambda} V_\lambda^\mu$. Thus (6) holds.

So we have a clear situation, as follows. The $G$-submodule $\oplus_{\lambda \in \Gamma(G/H)} \mathbf{k}[Z]_\lambda$ contains both $\mathbf{k}[Z]^H$ and $\mathbf{k}[Z]^{I(\Gamma(G/H))}$ and both subalgebras are the direct sums of 1-dimensional subspaces contained in each component $\mathbf{k}[Z]_\lambda, \lambda \in \Delta \cap \Gamma(G/H)$. Reynolds operator maps the $H$-stable submodule $\mathbf{k}[Z]_\lambda$ to itself. Hence, formula (6) shows that $\natural : \mathbf{k}[Z]_\lambda^{I(\Gamma(G/H))} \to \mathbf{k}[Z]_\lambda^H$ is an isomorphism of 1-demensional vector spaces. Thus we have an isomorphism of graded vector spaces, $\natural : \mathbf{k}[Z]^{I(\Gamma(G/H))} \xrightarrow{\sim} \mathbf{k}[Z]^H$. Because the action of $Z(G)$ commutes with that of $H$, the projection $\natural$ to $\mathbf{k}[Z]^H$ is $Z(G)$-equivariant.

In order to prove **2** it is sufficient to check the inclusion $v_\lambda^\natural \in \mathbf{k}[v_{\lambda_1}^\natural, \cdots, v_{\lambda_m}^\natural]$ for any $\lambda \in \Delta \cap \Gamma(G/H)$. We apply induction on the weight. The basis of induction are the minimal weights $\lambda_1, \cdots, \lambda_m$, for which the statment is trivial. For any $\lambda = \sum_{i=1,\cdots,m} p_i \lambda_i$ the condition (6) implies that

$$v_\lambda^\natural - \alpha_\lambda \prod_{i=1,\cdots,m} \alpha_{\lambda_i}^{-p_i} (v_{\lambda_i}^\natural)^{p_i} \in \bigoplus_{\mu \in \mathcal{X}, \mu \prec \lambda} \mathbf{k}[Z]^\mu \tag{7}$$

and we are done by induction. □

*Remark* 2.1. The Theorem basically states the same as Theorem 2.4, but in this particular case the effect is very clear: the Reynolds operator gives a $Z(G)$-equivariant isomorphism of vector spaces and maps generators to generators. Practically, this is a method to describe $Z(G)H$-semi-invariant generators of $\mathbf{k}[Z]^H$, provided we know the $U$-invariants in $\mathbf{k}[Z]$. Unfortunately, this map has no chances to be an isomorphism of algebras. Indeed, for that we would need the equality $\alpha_{\lambda+\mu} = \alpha_\lambda \alpha_\mu$. But the latter is not the case, because we showed (at least in some cases) $\alpha_\lambda = (\dim V_\lambda)^{-0.5}$, and $\dim V_{\lambda+\mu}$ is typically smaller than $\dim V_\lambda \dim V_\mu$.

## 3. Varieties of complexes and circular complexes

Consider an equioriented circular quiver $\widetilde{A_n}$ with $n \geq 2$ vertices $\{1, 2, \cdots, n\}$. The quiver $\widetilde{A_n}$ has $n$ arrows $\varphi_i : i \to i+1, i = 1, \cdots, n$, where, abusing notation, we confuse the index $n+1$ with 1. Consider the relation ideal $I \subseteq \mathbf{k}\widetilde{A_n}$ generated by $\varphi_{i+1}\varphi_i = 0, i = 1, \cdots, n$. For any dimension vector $\alpha \in \mathbf{Z}_+^n$ the variety $R(\widetilde{A_n^0}, \alpha)$ of $\alpha$-dimensional representations of the bound quiver algebra $\widetilde{A_n^0} = \mathbf{k}\widetilde{A_n}/I$ consists of $n$ linear operators $A_i : \mathbf{k}^{\alpha(i)} \to \mathbf{k}^{\alpha(i+1)}$ (by our notation, $A_n$ maps $\mathbf{k}^{\alpha(n)}$ to $\mathbf{k}^{\alpha(1)}$!) subject to the relations $A_{i+1}A_i = 0, i = 1, \cdots, n$. This affine variety is in general reducible and one has a nice description of the irreducible components, as follows. Clearly, the relation $A_{i+1}A_i = 0$ implies the inequality $\mathrm{rank} A_i + \mathrm{rank} A_{i+1} \leq \alpha(i+1)$.

For any rank vector $r \in \mathbf{Z}_+^n$ subject to the inequalities:

$$r(i) + r(i+1) \leq \alpha(i+1) \tag{8}$$

the subvariety $R(\widetilde{A_n^0}, \alpha, r)$ consisting of representations with rank upper bounds:

$$R(\widetilde{A_n^0}, \alpha, r) \subseteq R(\widetilde{A_n^0}, \alpha) : \mathrm{rank} A_i \leq r(i), i = 1, \cdots, n \tag{9}$$

is non-empty. This variety can be alternatively described as follows. Let $V_i$ be a representation of $\widetilde{A_n^0}$ of dimension vector having value 1 for vertices $i$ and $i+1$ and zero for other vertices, and $V_i(\varphi_i) = 1$. Then $V_i$ is an indecomposable representation of $\widetilde{A_n^0}$. Let $S_i$ be the irreducible representation of dimension vector having value 1 for the vertex $i$ and zero for other vertices. Consider the representation

$$V(\alpha, r) = \sum_{i \in Q_0} r_i V_i + (\alpha_i - r_i - r_{i-1}) S_i.$$

Then the variety $R(\widetilde{A_n^0}, \alpha, r)$ is the closure of the $GL(\alpha)$-orbit of $V(\alpha, r)$. In particular, this variety is irreducible. The maximal subsets among those $R(\widetilde{A_n^0}, \alpha, r)$ are the same as the subsets with maximal rank vector $r$ and are clearly the irreducible components. We call the varieties $R(\widetilde{A_n^0}, \alpha, r)$ the varieties of circular complexes.

The varieties of complexes $R(A_n^0, \alpha, r)$ are derived from the equioriented quiver $A_n$ the same way as the circular complexes from $\widetilde{A_n}$. On the other hand, $R(A_n^0, \alpha, r)$ is the same as $R(\widetilde{A_{n+1}^0}, \tilde{\alpha}, \tilde{r})$, where we add to $A_n$ the vertex $n+1$ and 2 arrows, $n \to n+1$ and $n+1 \to 1$. We extend $\alpha$ to $\tilde{\alpha}$ setting $\tilde{\alpha}(i) = \alpha(i), i = 1, \cdots, n, \tilde{\alpha}(n+1) = 0$ and similarly extend $r$ to $\tilde{r}$. From this settings the equality $R(A_n^0, \alpha, r) = R(\widetilde{A_{n+1}^0}, \tilde{\alpha}, \tilde{r})$ is straightforward. Hence, the varieties of complexes are a particular case of those for circular complexes.

Many research papers consider the algebraic and geometric properties of varieties of complexes, $R(A_n^0, \alpha, r)$. In particular, in [DS] it was proven that $R(A_n^0, \alpha, r)$ are normal, Cohen-Macaulay with rational singularities. Besides, the $GL(\alpha)$-module $\mathbf{k}[R(A_n^0, \alpha, r)]$ is described. On the other hand, all those results can be deduced from the explicit description of the algebra $\mathbf{k}[R(\widetilde{A_n^0}, \alpha, r)]^{U(\alpha)}$ of invariants of a maximal unipotent subgroup $U(\alpha) \subseteq GL(\alpha)$. This was remarked in [B] and then clarified and generalized in [Sh02]. For convenience we present this result here.

Fix a basis in every vector space $\mathbf{k}^{\alpha(i)}$. Let $B(\alpha) \subseteq GL(\alpha)$ be the upper triangular matrices in the chosen basises. Let $U(\alpha) \subseteq B(\alpha)$ be matrices with units on the diagonal. Let $\rho_j^i, j = 1, \cdots, \alpha(i)$ be the fundamental dominant weights of the factor $GL(\mathbf{k}^{\alpha(i)}) \subseteq GL(\alpha)$, namely $\rho_j^i = \varepsilon_1 + \cdots + \varepsilon_j$, where $\varepsilon_k$ is the $k$-th diagonal element of an operator. Note also that $\rho_{\alpha(i)}^i$ is nothing but the determinant $\det^i$ of the matrix. For every $i = 1, \cdots, n$ and every $j = 1, \cdots, r(i)$ consider the left lower $j \times j$ minor $M_j(A_i)$ of the matrix of $A_i$ in the chosen basises as a regular function on $R(\widetilde{A_n^0}, \alpha, r)$. The following Theorem collects the necessary results from [Sh02]:

**Theorem 3.1.** **1.** *$M_j(A_i)$ is a $B(\alpha)$-eigenvector with the weight*

$$\rho_j^i + \rho_{\alpha(i+1)-j}^{i+1} - \det{}^{i+1} \tag{10}$$

**2.** *$\mathbf{k}[R(\widetilde{A_n^0}, \alpha, r)]^{U(\alpha)}$ is generated by $M_j(A_i), j = 1, \cdots, r(i), i = 1, \cdots, n$.*

**3.** *The weights $\rho_j^i + \rho_{\alpha(i+1)-j}^{i+1} - \det^{i+1}, j = 1, \cdots, r(i), i = 1, \cdots, n$ are linear independent. In particular, $\mathbf{k}[R(\widetilde{A_n^0}, \alpha, r)]^{U(\alpha)}$ is a polynomial algebra.*

Theorem 3.1 has several important consequences. First, 3.1.3 and general relationship of singularities of an affine $G$-variety $X$ and the spectrum of $\mathbf{k}[X]^U$ for a maximal unipotent subgroup $U \subseteq G$ (see, e.g.,[K, Satz 2,p.194.]) imply:

**Corollary 3.2.** *$R(\widetilde{A^0_n}, \alpha, r)$ is normal, Cohen-Macaulay with rational singularities.*

Second, the linear independence of the weights implies:

**Corollary 3.3.** *$R(\widetilde{A^0_n}, \alpha, r)$ is a spherical $GL(\alpha)$-variety.*

Let $p_{\varphi_i}, i = 1, \cdots, n$ be the projection taking $(A_1, \cdots, A_n)$ to $A_i$. The image $p_{\varphi_i}(R(\widetilde{A^0_n}, \alpha, r))$ is clearly the set $X(\varphi_i)$ of all $\alpha(i+1) \times \alpha(i)$ matrices having rank smaller than or equal to $r(i)$. The fact that $\mathbf{k}[R(\widetilde{A^0_n}, \alpha, r)]^{U(\alpha)}$ is generated by the functions depending on just one matrix $A_i$ implies:

**Corollary 3.4.** $\mathbf{k}[R(\widetilde{A^0_n}, \alpha, r)]^{U(\alpha)} = \otimes_{i=1}^n p^*_{\varphi_i} \mathbf{k}[X(\varphi_i)]^{U(\alpha)}$.

## 4. Gentle algebras

From now on $Q$ will stand for a colored quiver with a fixed coloring $c : Q_1 \to S$ that fulfills the condition of Proposition 1.3. By Proposition 1.3, $\mathbf{k}Q/I_c$ is a gentle algebra and all triangular gentle algebras can be described this way. It is clear from the Definition 1.2 and the previous section that representation spaces $R(Q, I_c, \alpha)$ are constructed from varieties of complexes and circular complexes. We make it explicit in the following definitions and statements.

**Definition 4.1.** For any $s \in S$ define the subquiver $Q^s \subseteq Q$ by the condition $Q^s_1 = c^{-1}(s)$ and set $I^s = I_c \cap \mathbf{k}Q^s$. The bound quiver $(Q^s, I^s)$ is isomorphic either to $A^0_{n(s)}$, if $Q^s$ is a simple oriented path or to $\widetilde{A^0_{n(s)}}$, if $Q^s$ is a circular path, where $n(s) = |Q^s_0|$. To make formulae uniform, we use the term $\hat{A}_{n(s)}$ for both cases. For any dimension vector $\alpha$ set $\alpha^s$ to be the restriction of $\alpha$ to $Q^s_0$.

**Proposition 4.2.** *For any $\alpha \in \mathbf{Z}_+^{Q_0}$ holds: $R(Q, I_c, \alpha) = \prod_{s \in S} R(\hat{A}_{n(s)}, \alpha^s)$.*

**Definition 4.3.** A rank vector $r : Q_1 \to \mathbf{Z}_+$ is called admissible for a dimension vector $\alpha : Q_0 \to \mathbf{Z}_+$ if for any $\varphi, \psi \in Q_1$ with $t\varphi = h\psi$ and $c(\varphi) = c(\psi)$ holds $r(\varphi) + r(\psi) \leq \alpha(h\psi)$. We introduce the *rank gentle varieties*:

$$R(Q, c, \alpha, r) = \{V \in R(Q, I_c, \alpha) | \mathrm{rank} V(\varphi) \leq r(\varphi), \varphi \in Q_1\}. \qquad (11)$$

For any admissible rank vector $r$ set $r^s$ to be the restriction of $r$ to $Q^s_1$.

**Proposition 4.4. 1.** *$R(Q, c, \alpha, r) = \prod_{s \in S} R(\hat{A}_{n(s)}, \alpha^s, r^s)$.*

**2.** *The rank gentle varieties $R(Q, c, \alpha, r)$ for maximal admissible rank vectors are the irreducible components of $R(Q, I_c, \alpha)$.*

**3.** *For any admissible rank vector $r$, $R(Q, c, \alpha, r)$ is normal, Cohen-Macaulay with rational singularities.*

*Proof.* **1** follows from definitions, **3** follows from **1** and Corollary 3.2. **2** follows from Proposition 4.2 and the remark that $r$ is maximal if and only if $r^s$ is $\alpha^s$-maximal admissible rank vector. □

Our goal is to describe the algebra of semi-invariants $\mathbf{k}[R(Q, c, \alpha, r)]^{SL(\alpha)}$. By Proposition 4.4 and Corollary 3.3 the variety $R(Q, c, \alpha, r)$ is a direct product of spherical varieties of (circular) complexes, and we know the $U$-invariants of varieties of (circular) complexes from Theorem 3.1. Therefore we are in position to apply the Deformation method. Let's start with the particular example, which was considered in a separate paper[KW].

Let $Q$ be the double of an equioriented $A_n$ quiver. For every $i = 1, \cdots, n-1$ we denote by $\varphi_i, \psi_i$ the arrows from $i$ to $i+1$. Let the coloring $c$ be as follows: $c(\varphi_i) = 1, c(\psi_i) = 2, i = 1, \cdots, n-1$. Then $A_n^2 = \mathbf{k}Q/I_c$ is a triangular gentle algebra. In this particular case the rank vector $r$ is just a pair $(r_1, r_2)$ with $r_1, r_2$ being the rank vectors of the color 1 and color 2 arrows, respectively.So in this case we can write the rank gentle varieties $R(Q, c, \alpha, r) = R(A_n^2, \alpha, r_1, r_2)$ as $R(A_n^0, \alpha, r_1) \times R(A_n^0, \alpha, r_2)$.

Let's introduce a notation for the weights of $U$-invariants for a variety $R(A_n^0, \alpha, r)$. By $\Omega(n, \alpha, r)$ we denote the weights of $M_j(A_i)$ from the formula (10) over all $i, j$:

$$\Omega(n, \alpha, r) = \{\rho_j^i + \rho_{\alpha(i+1)-j}^{i+1} - \det{}^{i+1} | j = 1, \cdots, r(i), i = 1, \cdots, n-1\} \tag{12}$$

For a torus $T$ and a weight set $\Omega$ denote by $\mathbf{k}^\Omega$ the $T$-module with the weights from the set $\Omega$. Let $T(\alpha)$ be the group of diagonal matrices. Then Theorem 3.1 states:

$$(T(\alpha), \mathbf{k}[R(A_n^0, \alpha, r)]^{U(\alpha)}) \cong (T(\alpha), \mathbf{k}[\mathbf{k}^{\Omega(n,\alpha,r)}]). \tag{13}$$

We now apply Theorem 2.10 to this context. First of all, in our previous notation, $G = GL(\alpha) \times GL(\alpha)$ and $H = SL(\alpha)$ with diagonal embedding to $G$. So by Proposition 2.9 the group $I(\Gamma(G/H))$ is $(U(\alpha) \times U^{op}(\alpha))T_{SL(\alpha)}$, where $T_{SL(\alpha)} \subseteq T(\alpha)$ is the subgroup of unimodular diagonal matrices. The restriction of the weights from (12) to $T_{SL(\alpha)}$ are clear: $\det^i$ are trivial and the $\rho_j^i$ are linear independent. In every simple $GL(\alpha)$-module with $U(\alpha)$-highest weight $\lambda$ the $U^{op}(\alpha)$-highest vector is the $U(\alpha)$-lowest vector with the weight $-\lambda^*$. So we have an $T(\alpha)$-equivariant isomorphism:

$$\mathbf{k}[R(A_n^0, \alpha, r_1)]^{U(\alpha)} \otimes \mathbf{k}[R(A_n^0, \alpha, r_2)]^{U^{op}(\alpha)}] \cong \mathbf{k}[\mathbf{k}^{\Omega(n,\alpha,r_1) \cup -\Omega^*(n,\alpha,r_2)}]. \tag{14}$$

Applying Theorem 2.10 to this situation, we get:

**Theorem 4.5.** *Let $A$ denote the algebra $\mathbf{k}[\mathbf{k}^{\Omega(n,\alpha,r_1) \cup -\Omega^*(n,\alpha,r_2)}]^{T_{SL(\alpha)}}$ and let $f_i \otimes g_i, i = 1, \cdots, m$ are the generators of $A$. Then the Reynolds operator $\natural$ is a $T(\alpha)$-equivariant linear isomorphism of $A$ over $\mathbf{k}[R(A_n^2, \alpha, r_1, r_2)]^{SL(\alpha)}$. Moreover, $(f_i \otimes g_i)^\natural, i = 1, \cdots, m$ generate $R(A_n^2, \alpha, r_1, r_2)]^{SL(\alpha)}$.* □

*Remark* 4.1. **1.** The equivariant isomorphism in the Theorem yields the $T(\alpha)$-weight of the generator $(f_i \otimes g_i)^\natural$ as the sum of $\det^k$ summands in the weights of $f_i$ and $g_i$.

**2.** The above result is partially covered by [KW, Lemma 3], where it is stated, under some extra conditions on $\alpha, r_1, r_2$ (the conditions sound as $R(A_n^2, \alpha, r_1, r_2)$ is a band component, in particular, $r_1 = r_2$ and $\alpha$ is uniquely determined by $r_1 = r_2$), that $\mathbf{k}[R(A_n^2, \alpha, r_1, r_2)]^{SL(\alpha)}$ is isomorphic to the semi-group algebra $\mathbf{k}[\Lambda]$ of the highest weights of irreducible components of $\mathbf{k}[R(A_n^2, \alpha, r_1, r_2)]$ containing $SL(\alpha)$-invariants. By Corollary 2.8, the latter algebra is isomorphic to $A$. This result is then generalized in [CW] for arbitrary triangular gentle algebras (see a discussion below).

Passing from the particular case of $A_n^2$ to the general case, we need to handle the situation, where the rank gentle variety is a direct product of more than two varieties of complexes. However, the condition that every vertex is incident to arrows of at most two colors suggests that actually every factor of $GL(\alpha)$ is involved in at most two varieties of complexes. So let $Z$ be a rank gentle variety, $Z = R(Q, c, \alpha, r) = \prod_{s\in S} R(\hat{A}_{n(s)}, \alpha^s, r^s)$. As in the case of the varieties $Z = X \times Y$, we define the group $G$ to act independently on each direct factor, that is, $G = \prod_{s\in S} GL(\alpha^s)$. In what follows we call a vertex $i \in Q_0$ *single* if it is incident to arrows of just one color, and *coupled*, otherwise. We denote by $Q_0^1$ (resp. $Q_0^2$) the set of single (resp. coupled vertices). We can write the group $G$ as follows:

$$G = \prod_{s\in S} GL(\alpha^s) = \prod_{i\in Q_0^1} GL_{\alpha(i)} \times \prod_{i\in Q_0^2} (GL_{\alpha(i)} \times GL_{\alpha(i)}). \tag{15}$$

The group $H$ is $SL(\alpha)$ naturally embedded in $G$ such that for a single vertex $i$, $H$ contains the whole of $SL(\alpha(i))$ and for coupled vertex $H$ contains the image of diagonal embedding of $SL(\alpha(i))$ to $GL(\alpha(i)) \times GL(\alpha(i))$. A coupled vertex $i$ owns two $GL(\alpha(i))$-factors of $G$ and we need to distinguish these factors as "first" and "second". To this end we follow a convention suggested in [CC] and going back to the original paper [BR]. Let $\mathcal{P}$ denote the set of pairs $(i, s) \in Q_0 \times S$ such that $i$ is incident to an arrow of color $s$. Note that $\mathcal{P}$ is in 1-to-1 correspondance with $GL$ factors of the group $G$ in (15). We then make a non-unique choice of an object of the following type:

**Definition 4.6.** A map $\varepsilon : \mathcal{P} \to \{+, -\}$ is called sign function if $\varepsilon(i, s_1) = -\varepsilon(i, s_2)$ provided $s_1 \neq s_2$.

For any $n > 1$ we denote by $T_n \subseteq GL_n$ the subgroup of diagonal operators in some basis, by $U_n, U_n^{op} \subseteq GL_n$ the subgroups of upper and lower triangular operators with units on the diagonal, in the same basis, respectively. A sign function $\varepsilon : \mathcal{P} \to \{+, -\}$ used in [BR],[CC] to define indecomposable representations of gentle string algebras has in this paper a different meaning: we introduce a maximal unipotent subgroup $U^\varepsilon \subseteq G$ embeded to $G$ as the presentation (15) suggests:

$$U^\varepsilon = \prod_{(i,s)\in\mathcal{P}:\varepsilon(i,s)=+} U_{\alpha(i)} \times \prod_{(i,s)\in\mathcal{P}:\varepsilon(i,s)=-} U_{\alpha(i)}^{op}. \tag{16}$$

If we restrict attention to just one color, $s$, that is $GL(\alpha^s)$ acting on $R(\hat{A}_{n(s)}, \alpha^s, r^s)$, we are naturally interested in the $U^\varepsilon$-invariants on $R(\hat{A}_{n(s)}, \alpha^s, r^s)$. $U^\varepsilon$ acts via its intersection $U^\varepsilon \cap GL(\alpha^s)$, which is a maximal unipotent subgroup in $GL(\alpha^s)$ different from $U(\alpha^s)$ used in Theorem 3.1: for those vertices $i$ such that $\varepsilon(i, s) = -$ the factor $U_{\alpha(i)}$ in $U(\alpha^s)$ is replaced by $U_{\alpha(i)}^{op}$ for $U^\varepsilon$.

We generalize Theorem 3.1 to get $U^\varepsilon$ invariants. Corollary 3.4 yields the decomposition $\mathbf{k}[R(\widetilde{A_n^0}, \alpha, r)]^{U(\alpha)} = \otimes_{i=1}^n p_\varphi^* \mathbf{k}[X(\varphi_i)]^{U(\alpha)}$, where $X(\varphi)$ is the projection of a variety $R(\widetilde{A_n^0}, \alpha, r)$ to the factor of one arrow $\varphi$, the determinantal variety of $\alpha(h\varphi) \times \alpha(t\varphi)$ matrices with rank smaller than or equal to $r(\varphi)$. If we replace $U(\alpha)$ by $U^\varepsilon \cap GL(\alpha_s)$, the decomposition from Corollary 3.4 keeps because these two subgroup are $GL(\alpha_s)$ conjugate. We then just need to find generators of $\mathbf{k}[X(\varphi_i)]^{U^\varepsilon}$ for every arrow $\varphi_i \in Q_1$.

**Proposition 4.7.** *Let $X = X(a_1, a_2, r)$ be the variety of $a_2 \times a_1$ matrices with rank smaller than or equal to $r$. Let $\varepsilon_1, \varepsilon_2 \in \{+, -\}$ be a sign function. Chose a maximal unipotent subgroup $U_i \subseteq GL_{a_i}$ as follows: $U_i = U_{a_i}$ if $\varepsilon_i = +$ and $U_i = U_{a_i}^{op}$, otherwise. For every $j = 1, \cdots, r$ consider a certain $j \times j$ submatrix $X_j$ of the matrix in $X$, which is left, if $\varepsilon_1 = +$ and is right, otherwise; further, $X_j$ is lower, if $\varepsilon_2 = +$ and is upper, otherwise. Set $M_j = \det(X_j)$. Then we have:*

**1.** *$M_j, j = 1, \cdots, r$ are algebraically independent generators of $\mathbf{k}[X]^{U_1\times U_2}$.*

**2.** *The weight of $M_j$ with respect to the maximal torus $T_{a_1}$ is equal to $\rho^1_j$ if $\varepsilon_1 = +$ and $\det^1 - \rho^1_{a_1-j}$, otherwise.*

**3.** *The weight of $M_j$ with respect to the maximal torus $T_{a_2}$ is equal to $\rho^2_{a_2-j} - \det^2$ if $\varepsilon_2 = +$ and $-\rho^2_j$, otherwise.*

The proof of Proposition 4.7 is straightforward. Now we apply the Deformation Theory to the given pair $(G, H)$:

**Proposition 4.8.** **1.** *The homogeneous space $G/H$ is isomorphic to*

$$(\prod_{s\in S} GL(\alpha^s))/SL(\alpha) \cong \prod_{i\in Q^1_0} \mathbf{k}^* \times \prod_{i\in Q^2_0} (GL_{\alpha(i)} \times GL_{\alpha(i)})/SL_{\alpha(i)} \tag{17}$$

**2.** *$H$ is spherical in $G$*

**3.** *Holds: $\Gamma(G/H) = \prod_{i\in Q^1_0}\langle \det_i\rangle \times \prod_{j\in Q^2_0} \Gamma((GL_{\alpha(j)} \times GL_{\alpha(j)})/SL_{\alpha(j)})$.*

**4.** *$\Gamma(G/H)$ is saturated.*

**5.** *$I(\Gamma(G/H)) = \prod_{i\in Q^1_0} SL_{\alpha(i)} \times \prod_{j\in Q^2_0} (U_{\alpha(j)} \times U^{op}_{\alpha(j)})T_{SL(\alpha(j))}$.*

*Proof.* **1** follows from the definition of single and coupled vertices. **2** follows from **1**, because $G/H$ is a direct product of spherical homogeneous varieties. **3** also follows from **1** and **4** is obvious. To prove **5** we consider $U^\varepsilon$ as a maximal unipotent subgroup in $G$ and $I(\Gamma(G/H))$ as introduced in (3). By formula (3), we have:

$$I(\Gamma(GL_n/SL_n)) = I(\langle\det\rangle) = SL_n. \tag{18}$$

This explains the factors of $I(\Gamma(G/H))$ in **5** corresponding to the single vertices and those corresponding to the coupled ones follow from Proposition 2.9. □

Before formulating our main result about $SL(\alpha)$-invariants of rank gentle varieties, we would like to distinguish one degenerated case, which is better to consider separately. There can be an arrow $\varphi \in Q_1$ such that $r(\varphi) = \alpha(t\varphi) = \alpha(h\varphi)$. Then clearly the colored path containing $\varphi$ contains no other arrows. The determinant of the matrix corresponding to $\varphi$ is clearly an $SL(\alpha)$-invariant function to be included into the generators. If moreover at least one of end points of $\varphi$, say $t\varphi$ is a single vertex, then no other $SL(\alpha)$-invariant function (apart of the powers of that determinant) depends on the coordinates of $V(\varphi)$, because $SL_{\alpha(t\varphi)}$ acts on no other arrows in $Q_1$ (c.f. the proof of Theorem 4.9 below). So we may and will restrict our consideration to the cases with no arrows like that.

**Theorem 4.9.** *Consider a rank gentle variety $Z = R(Q, c, \alpha, r)$ and assume that for every arrow $\varphi \in Q_1$ such that $r(\varphi) = \alpha(t\varphi) = \alpha(h\varphi)$ none of $t\varphi, h\varphi$ is a single vertex. Fix a sign function $\varepsilon$. Consider a subalgebra $A \subseteq \mathbf{k}[Z]$ generated by the polynomials $M_i, i = 1, \cdots, r(\varphi)$ defined by Proposition 4.7 for all $\varphi \in Q_1$ such that $t\varphi, h\varphi \in Q^2_0$. Then the restriction of the Reynolds operator $\natural$ to $A^{T_{SL(\alpha)}}$ yields a $Z(GL(\alpha))$-equivariant linear isomorphism of $A^{T_{SL(\alpha)}}$ over $\mathbf{k}[Z]^{SL(\alpha)}$. Moreover, the Reynolds images of the generators of $A^{T_{SL(\alpha)}}$ generate $\mathbf{k}[Z]^{SL(\alpha)}$.*

*Proof.* We apply Theorem 2.10 and only need to show $\mathbf{k}[Z]^{I(\Gamma(G/H))} = A^{T_{SL(\alpha)}}$. This follows from Propositions 4.7 and 4.8.5 with only one addition to clarify: we can ignore all arrows such that at least one end point is single. Indeed, by Proposition 4.8.5 for this single vertex $i$ the group $I(\Gamma(G/H))$ contains a factor $SL_{\alpha(i)}$ and this factor acts on at most two arrows adjacent to $i$ and belonging to one colored path. Consider the action of $SL_{\alpha(i)}$ on $Z$. Assume for example that $i$ is adjacent to two arrows: $\psi : j \to i$ and $\varphi : i \to k$. Let's repesent a point in $Z$ as a triple $(V(\psi), V(\varphi), z')$ such that $SL_{\alpha(i)}$ acts trivially on $z'$. First Fundamental Theorem says for this situation that every $SL_{\alpha(i)}$ -invariant is a polynomial in the coordinates of $z'$, in the components of the composition $V(\varphi)V(\psi)$, and in determinants of $V(\varphi), V(\psi)$. However the relations of the variety of complexes imply $V(\varphi)V(\psi) = 0$ and our assumption yields that none of $V(\varphi), V(\psi)$ is a linear isomorphism. So we see that none $SL_{\alpha(i)}$-invariant function depends on the coordinates of $V(\varphi), V(\psi)$. The case when $i$ is adjacent to just one arrow is similar and simpler. □

*Remark* 4.2. Theorem 4.9 shows that the arrows adjacent to single vertices have very little impact on $SL(\alpha)$ invariants and one could think about simplifying the picture by just removing all these arrows from $Q_1$. However, one can't do it because those arrows are related with others ones by the relations of variety of complexes.

**Corollary 4.10. 1.** *An isomorphism* $\mathbf{k}[\Delta \cap \Gamma(G/H)] \cong A^{T_{SL(\alpha)}}$ *holds.*
**2.** *If* $\mathbf{k}Q/I_c$ *is a triangular gentle algebra, then holds* $A^{T_{SL(\alpha)}} \cong \mathbf{k}[Z]^{SL(\alpha)}$.

*Proof.* By Corollary 2.8, $\mathbf{k}[Z]^{I(\Gamma(G/H))}$ is isomorphic to $\mathbf{k}[\Delta \cap \Gamma(G/H)]$, hence an isomorphism $A^{T_{SL(\alpha)}} \cong \mathbf{k}[\Delta \cap \Gamma(G/H)]$ holds. The semi-group $\Delta \cap \Gamma(G/H)$ is nothing but the semi-group of highest weights of irreducible $G$-modules in the multiplicity free $G$-module $\mathbf{k}[Z]$ containing $SL(\alpha)$-invariant. This semi-group is therefore the same as $\Lambda_{SI}(Q, c, \beta, r)$ from [CW]. Then [CW, Theorem 6.10] states that the corresponding semi-group algebra is isomorphic to $\mathbf{k}[Z]^{SL(\alpha)}$. □

Despite our Theorem 4.9 is a bit weaker than [CW], we believe it is useful for three reasons. First, it covers the case of non-triangular gentle algebra, that is, $\mathbf{k}Q/I_c$ such that for some color $s \in S$, $c^{-1}(s)$ is a cyclic path. Second, we like the simplicity of our approach comparing with the proof in [CW] based on coordinate computations. The isomorphism of [CW] is given explicitly and formulae are very involved. By the way, in accordance with Remark 2.1, the polynomials corresponding to the semi-group basis are not contained in one $G$-irreducible factor. So the isomorphism proof has to be as complicated as the isomorphism, but the fact that one algebra is a deformation of another is clearly explained by our arguments, simple to follow. Finally, our presentation of the algebra of invariants as $A^{T_{SL(\alpha)}}$ allows for proposing a nice computational model, and this is the goal of the next section.

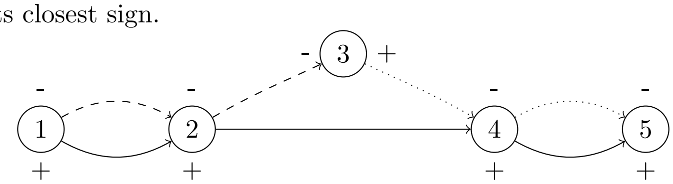
## 5. Graph model of semi-invariants for rank gentle varieties.

Following [EJ] we define a *bidirected graph* as a graph with in/out orientation assigned to each end of each edge. Usual oriented graphs with arcs like $\leftarrow$ and $\rightarrow$ are a particular case. Besides that, introverted$-$ and extraverted $\leftrightarrow$ edges are allowed. We introduce a biderected graph as follows:

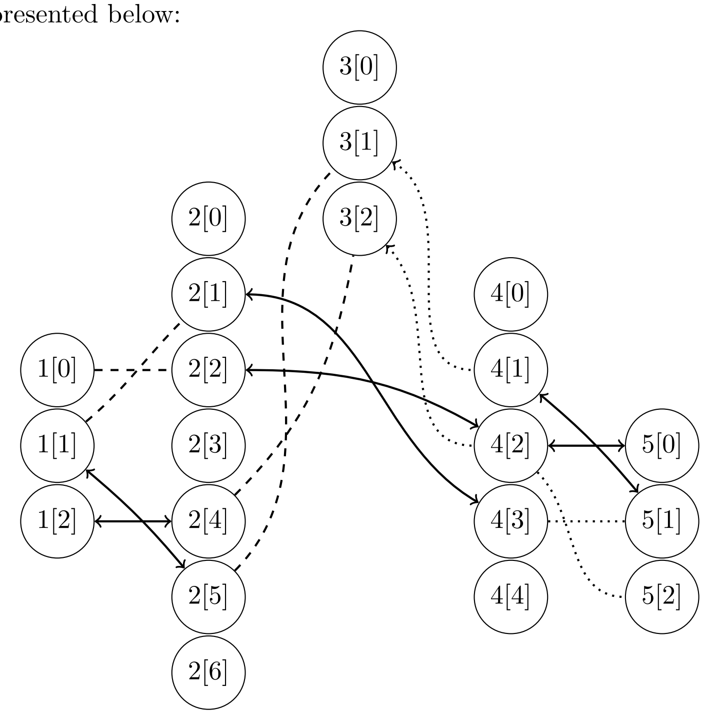
**Definition 5.1.** For a rank gentle variety $R(Q, c, \alpha, r)$ and a fixed sign function $\varepsilon$ consider the graph $UI(Q, c, \alpha, r, \varepsilon)$ with nodes $i[k], i \in Q_0, k = 0, \cdots, \alpha(i)$ and $r(\varphi)$ arcs $\varphi^j, j = 1, \cdots, r(\varphi)$ for every arrow $\varphi$ such that $t\varphi, h\varphi \in Q_0^2$. For such an arrow $\varphi$, set $\varepsilon_t = \varepsilon(t\varphi, c(\varphi)), \varepsilon_h = \varepsilon(h\varphi, c(\varphi))$. The arc $\varphi^j$ connects the node $t\varphi[j]$ (resp. $t\varphi[\alpha(t\varphi) - j]$) if $\varepsilon_t = +$ (resp. if $\varepsilon_t = -$) and $h\varphi[\alpha(h\varphi) - j]$ (resp. $h\varphi[j]$) if $\varepsilon_h = +$ (resp. if $\varepsilon_h = -$). Both endpoints of $\varphi^j$ get orientation defined by $\varepsilon_t$ and

$\varepsilon_h$, respectively: the endpoint is out if and only if $\varepsilon = +$. Besides that, every arc gets degree $\deg(\varphi^j) = j$ and weight $w(\varphi^j) = \delta_t \det^{t\varphi} - \delta_h \det^{h\varphi} \in \mathcal{X}(GL(\alpha)$, where both $\delta_t, \delta_h$ take values 0 or 1 and $\delta_t = 1$ iff $\varepsilon_t = -$ or $j = \alpha(t\varphi)$ and $\delta_h = 1$ iff $\varepsilon_h = +$ or $j = \alpha(h\varphi)$. We distinguish the nodes $i[0], i \in Q_0$ as *zero* nodes and the nodes $i[\alpha(i)], i \in Q_0$ as *det* nodes.

**Example 5.2.** Consider the running example from [CW], a quiver $Q$ with 5 vertices and 7 arrows colored in 3 colors, normal, dotted, and dashed (see below). The sign function $\varepsilon$ is also drawn with assumption that every arrow endpoint gets signed by its closest sign.




Consider the dimension vector $\alpha = (2, 6, 2, 4, 2)$ and the rank function such that the rank of every arrow is equal to 2. Then the biderected graph $UI(Q, c, \alpha, r, \varepsilon)$ is presented below:




*Remark* 5.1. Despite its cumbersome nature, the sense of the graph $UI(Q, c, \alpha, r, \varepsilon)$ can be easily explained: the graph represents the $T(\alpha)$-characters of the minors $M_i$, which generate the algebra $A$ in Theorem 4.9. Namely, the nodes $i[k], i \in Q_0, k = 1, \cdots, \alpha(i) - 1$ stand for the fundamental weights $\rho^i_k$ of $GL_{\alpha(i)}$ with respect to the chosen Borel subgroup; the zero nodes $i[0]$ stand for zero weight and the det nodes $i[\alpha(i)]$ stand for $\rho^i_{\alpha(i)} = \det^i$. Then the arcs control the weights of the functions $M_i$, as follows. The weight of $M_j$ in Proposition 4.7 is equal to the sum of the fundamental weights corresponding to the endpoints of the arc $\varphi^j$ with the corresponding sings and a det summand described in 4.7. So we encode the non-$GL(\alpha)$-semi-invariant part of the weight of $M_j$ graphically, as that arc. As for the part belonging to $\mathcal{X}(GL(\alpha)$, the arc $\beta = \varphi^j$ is labelled with the weight $w(\beta)$. Note that the summands $\rho^i_{\alpha(i)}$ for the case, when the arc is incident to the det node $i[\alpha(i)]$ is also counted in $w(\beta)$. The degree $\deg(\varphi^j)$ is equal to the minor size, as should be. Below we denote by $M(\varphi)$ the minor corresponding to the arc.

**Definition 5.3.** A *walk* on the bidirected graph is a sequence $(\beta_1, \cdots, \beta_l)$ of arcs such that for every $i = 1, \cdots, l-1$ the arcs $\beta_i$ and $\beta_{i+1}$ have a common endpoint $b_i$ and either $b_i$ is in-oriented at $\beta_i$ and out-oriented at $\beta_{i+1}$ or vice-versa $b_i$ is out-oriented at $\beta_i$ and in-oriented at $\beta_{i+1}$. The *source* of the walk is the endpoint $b_0$ of $\beta_1$ different from $b_1$. Analogously, the *target* is the endpoint $b_l$ of $\beta_l$ different from $b_{l-1}$. The walk is called *cyclic* if $b_0 = b_l$ and $b_0$ is either in-oriented at $\beta_1$ and out-oriented at $\beta_l$, or vice versa, is out-oriented at $\beta_1$ and in-oriented at $\beta_l$. The walk is called *complete* if either it is cyclic or both the source and the target are zero or det nodes. A complete walk is called *minimal* if it does not allow a proper subsequence $(\beta_p, \cdots, \beta_q)$, which is also a complete walk.

**Theorem 5.4.** *For any complete walk $(\beta_1, \cdots, \beta_l)$ the product $M(\beta_1)\cdots M(\beta_l)$ is $GL(\alpha)$-semi-invariant. Furthermore, the algebra generated by these products over all minimal complete walks is isomorphic to $A^{T_{SL(\alpha)}}$.*

*Proof.* By definition of the walk, the consecutive arcs, $\beta_i$ and $\beta_{i+1}$ share some node $b_i = j[k]$ and the signs corresponding to that node on both arcs are different. Hence, the fundamenal weight $\rho^j_k$ occurs with different signs in the weights of the minors corresponding to both arcs, so $\rho^j_k$ does not occur in the weight of the product. Therefore, if the walk is complete, then all non-$SL(\alpha)$-invariant summands annihilate each other (see Remark 5.1) and the product $M(\beta_1)\cdots M(\beta_l)$ is $GL(\alpha)$-semi-invariant of weight equal to the sum $w(\beta_1) + \cdots + w(\beta_l)$.

It is well-known that vector space torus invariants are generated by the invariant monomials in the coordinates and the minimal set of generators consists of all monomials non-divisible by other invariant monomials. Assume that a monomial $Q \in A^{T_{SL(\alpha)}}$ is $SL(\alpha)$-invariant. We claim that $Q$ has a divisor equal to the product of minors corresponding to the arcs of some minimal complete walk. Applying induction, we deduce the statement from that claim.

Now we prove the claim. In the rest of the proof we view $Q$ as a set of arcs with multiplicities corresponding to the minor factors of $Q$. Clearly, any endpoint $a = j[k]$ of any arc in $Q$ has the property as follows:
*Either $a$ is zero or* det *node, or the number of arcs (with multiplicities) in $Q$, incident to $a$, and in-oriented at $a$ is equal to the number of those out-oriented.*
Otherwise the weight of $Q$ would contain a non semi-invariant summand $q\rho^j_k, q \neq 0$.

Then, we build a complete walk out of the arcs in $Q$. Assume first that no zero or det-nodes are endpoints of the arcs in $Q$. We define counters $In(a), Out(a)$ for each node $a$ of $UI(Q, c, \alpha, r, \varepsilon)$ that count the in- and out-multiplicity of arcs incident to $a$ and occuring in $Q$. The above property means that the equality $In(a) = Out(a)$ holds for every $a$.

Starting from any arc $\beta = \beta_1$ with nodes $b_0$ and $b_1$, we consider it as the initial state of a walk with source $s = b_0$ and target $t = b_1$. We also subtract 1 from $In(b_0)$ (resp. $Out(b_0)$) if $\beta$ is in- (resp. out-)oriented at $b_0$, and do the same for $In(b_1)$ or $Out(b_1)$. Finally, we exclude one copy of $\beta$ out of $M$. At this moment and throughout the further process we have a walk from $s$ to $t$ with $n$ hops such that $In(a) = Out(a)$ for all nodes except for $s, t$ and if $\beta_n$ is in-oriented at $t$, then $Out(t) = In(t) + 1$ and similarly for the case, when $\beta_n$ is out-oriented at $t$. Then the latter observation implies that there is at least one arc incident to $t$ that can be appended to the walk. We also update $In, Out$ and $Q$ as above. Therefore the process can't finish at a node $a$ with $In(a) = Out(a)$, so the process finishes when $s = t$ and besides that $\beta_1, \beta_n$ have different in-/out-property at $t$, and in this case the walk will be complete. If we stop the first time like that, then this complete walk will be minimal.

For the case, when a zero or det-node $a$ is an endpoint of $\beta \in Q$ we apply the same process as above starting from $\beta$ and taking $a$ as $s$. This process only can stop if we get $t = s$ or if $t$ is a zero or det-node. In both cases we are done. $\square$

*Remark* 5.2. The above proof is similar to that for the well known existance theorem of an Euler cycle in any graph such that every node has even degree.

The following observation clarifies the topology of $UI(Q, c, \alpha, r, \varepsilon)$ at zero and det nodes:

**Proposition 5.5.** **1.** *For an arc $\varphi^j$ and a vertex $i \in Q_0$ holds:*
*$\varphi^j$ is adjacent either to $i[0]$ or to $i[\alpha(i)]$ $\Leftrightarrow$ $j = \alpha(i)$ and either $i = t\varphi$ or $i = h\varphi$. If the above conditions hold and if $\psi \in Q_1$ is such that $c(\psi) = c(\varphi)$ and either $i = t\psi$ or $i = h\psi$, then $r(\psi) = 0$.*

**2.** *Any zero or det node $b$ is adjacent to at most 2 arcs and if they are 2, then one is in- another is out-oriented at $b$.*

**3.** *For every minimal complete walk $(\beta_1, \cdots, \beta_l)$ with nodes $(b_0, \cdots, b_l)$ holds:*
*(i) either $b_0 \neq b_l$ and $b_1, \cdots, b_{l-1}$ are neither zero nor det*
*(ii) or the walk is cyclic with at most one node being zero or det.*

*Proof.* The first statement of **1** is straighforward from Definition 5.1. If $\psi \in Q_1$ has the same color as $\varphi$ and is adjacent to $i$, then $j = \alpha(i)$ implies $r(\varphi) = \alpha(i)$ and by Definition 4.3 we have $r(\psi) = 0$. **1** and the property that every vertex is adjacent to arows of at most two colors implies **2**. To prove **3** observe that if $b_0 \neq b_l$, then no interior node can be zero or det, otherwise we can split the walk. Otherwise, if $b_0 = b_1$ and this is a zero or det vertex, then by **2** the walk is cyclic. If a cyclic walk has more than 1 zero or det node, it can be splitted. $\square$

For a complete walk $(\beta_1, \cdots, \beta_l)$ the Theorem yields the form $w(\beta_1) + \cdots + w(\beta_l)$ of the $GL(\alpha)$-weight of the corresponding semi-invariant. It is possible to write down this weight in more suggestive form, as follows. Let $b_0, \cdots, b_l$ be the nodes such that $b_i$ is the common node of $\beta_i$ and $\beta_{i+1}, i = 1, \cdots, l-1$ and $b_0$ (resp. $b_l$) is the node of $\beta_1$ (resp. $\beta_l$) different from $b_1$ (resp. $b_{l-1}$). If the walk is cyclic and $b_0 = b_l$, set $\beta_{l+1} = \beta_1$ and $\beta_0 = \beta_l$. For non-cyclic walk this notation is void. Let $\varphi_i \in Q_1$ be the underlying arrow of $\beta_i = \varphi_i^{q_i}, i = 0, \cdots, l+1$. Similarly, let $a_k \in Q_0$ be the underlying vertex of the node $b_k = a_k[r_k], k = 0, \cdots, l$.

**Proposition 5.6.** *The $GL(\alpha)$-weight of $M(\beta_1) \cdots M(\beta_l)$ is equal to*

$$\sum_{k=0,\cdots,l: t\varphi_k = a_k = t\varphi_{k+1}} \det{}^{a_k} - \sum_{k=0,\cdots,l: h\varphi_k = a_k = h\varphi_{k+1}} \det{}^{a_k}. \tag{19}$$

*Proof.* Recall the weight part of Definition 5.1:the arc $\beta = \varphi^j$ has the weight $w(\varphi^j) = \delta_t \det^{t\varphi} - \delta_h \det^{h\varphi}$, where both $\delta_t, \delta_h$ take values 0 or 1 and holds:

$$\delta_t = 1 \Leftrightarrow (\varepsilon_t = - \ \vee \ j = \alpha(t\varphi)), \delta_h = 1 \Leftrightarrow (\varepsilon_h = + \ \vee \ j = \alpha(h\varphi)). \tag{20}$$

We inspect all cases for $\varphi_k, a_k, \varphi_{k+1}$ and the signs of $a_k$ at $\varphi_k$ and $\varphi_{k+1}$ (different by definition of the walk!) and check the $\det^{a_k}$ coordinate of the weight.

$\overset{a_{k-1}}{\circ} \xrightarrow{\varphi_k,+} \overset{a_k}{\circ} \xrightarrow{-,\varphi_{k+1}} \overset{a_{k+1}}{\circ}$ Rule (20) yields $-\det^{a_k}$ for $\varphi_k$ and $\det^{a_k}$ for $\varphi_{k+1}$, total 0.

$\overset{a_{k-1}}{\circ} \xrightarrow{\varphi_k,-} \overset{a_k}{\circ} \xrightarrow{+,\varphi_{k+1}} \overset{a_{k+1}}{\circ}$ We have two cases: if $b_k$ is a zero or det node, then by 5.5.1 the degree of both $\beta_k$ and $\beta_{k+1}$ is equal to $\alpha(a_k)$ and then rule (20) yields $-\det^{a_k}$ for $\varphi_k$ and $\det^{a_k}$ for $\varphi_{k+1}$; otherwise it yields 0 for both $\varphi_k, \varphi_{k+1}$, so total 0.

Similar to the above two cases we also consider four cases, where $k = 0$ or $k = l + 1$ but the walk is not cyclic. Then either $\varphi_k$ or $\varphi_{k+1}$ is void but still the claim of Proposition 5.6 aplies.These are four cases because $a_k$ can be either head or tail of the adjacent arrow and can be $+$ or $-$. The 4 cases can be drawn as a left or right hand half of the above 2 cases. By definition of complete walk, $b_k$ is either zero or det node. In both cases the degree of the adjacent arc is equal to $\alpha(a_k)$, so the term $\det^{a_k}$ with appropriate sign goes correctly to the above formula.

It remains to consider the cases when $b_k$ is an interior node of the walk adjacent to two arcs with the same in-/out orientation. By 5.5, since $b_k$ is an interior node, it is neither zero nor det node and the degree of two adjacent arcs is less than $\alpha(a_k)$:

$\overset{a_{k-1}}{\circ} \xrightarrow{\varphi_k,+} \overset{a_k}{\circ} \xleftarrow{-,\varphi_{k+1}} \overset{a_{k+1}}{\circ}$ By (20), $\varphi_k$ yields $-\det^{a_k}$ and $\varphi_{k+1}$ yields zero.

$\overset{a_{k-1}}{\circ} \xleftarrow{\varphi_k,+} \overset{a_k}{\circ} \xrightarrow{-,\varphi_{k+1}} \overset{a_{k+1}}{\circ}$ By (20), $\varphi_k$ yields zero and $\varphi_{k+1}$ yields $\det^{a_k}$. $\square$

**Example 5.7.** Consider the rank gentle variety from Example 5.2. We list all the minimal complete walks and provide their statistics. Let's introduce notation for the arrows, as follows. We have 3 colored paths: arrows in the dotted one are denoted by $\rho$, in the normal one by $\varphi$, in the dashed one by $\psi$. The subscript index increases from left to right. So in total we have 7 arrows: $\varphi_1, \varphi_2, \varphi_3$, $\psi_1, \psi_2$, $\rho_1, \rho_2$. We write the walks as sequences of nodes like $(b_0, b_1, \cdots, b_l)$. By Proposition 5.5, if $b_0 = b_l$ then the walk is cyclic. Also we write the corresponding product of arcs:

$F_1 : (1[0], 2[2], 4[2], 5[2]), \psi_1^2\varphi_2^2\rho_2^2$
$F_2 : (1[0], 2[2], 4[2], 3[2]), \psi_1^2\varphi_2^2\rho_1^2$
$F_3 : (3[2], 4[2], 5[0]), \rho_1^2\varphi_3^2$
$F_4 : (5[0], 4[2], 5[2]), \rho_2^2\varphi_3^2$
$F_5 : (1[2], 2[4], 3[2]), \varphi_1^2\psi_2^2$
$F_6 : (1[1], 2[1], 4[3], 5[1], 4[1], 3[1], 2[5], 1[1]), \psi_1^1\varphi_2^1\rho_2^1\varphi_3^1\rho_1^1\psi_2^1\rho_1^1$

Having described the walks, we write down their degrees and weights. The degrees are just the sum of arcs' superscripts and the weights are calculated from the node sequence applying Proposition 5.6:

$F_1 : (6, \det^1 - \det^5), F_2 : (6, \det^1 + \det^3 - \det^4), F_3 : (4, \det^3 - \det^5),$
$F_4 : (4, \det^4 - 2\det^5), F_5 : (4, \det^1 - \det^3), F_6 : (7, \det^1 - \det^5).$

The 6 monomials are subject to the unique relation $F_1F_3 = F_2F_4$ of degree and weight $(10, \det^1 + \det^3 - 2\det^5)$.

## 6. Computational aspects and open questions.

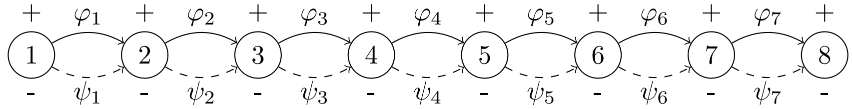
The graph model developed in the previous section is an efficient method to compute the semi-invariants of rank gentle varieties, because it translates the generating semi-invariants to the minimal complete walks on bidirected graphs and also provides degree and weight information. These graphs, $UI(Q, c, \alpha, r, \varepsilon)$ are built from the rank variety data in a straighforward manner. To find the minimal complete walks is a beautiful problem of Computer Science and it can be solved by Johnson's algorithm from [J].

Indeed, the biderected graphs are a generalization of the oriented graphs so that the walks generalize the oriented paths. The difference provided by the bidirection is that the notion of outgoing arc is not well-defined. Going "out of" a node by some arc is possible if the walk went "in" the node with the inverse orientation. But if we build the walk by appending next arc to a previously built walk at one of the endpoints, we certainly know which orientation is needed to append, so there is no difference with building usual oriented paths. Johnson's algorithm [J] is to our knowledge the most efficient in finding all oriented cycles in oriented graphs and it builds the oriented paths by appending new arcs to previous paths, so this procedure applies to bidirected graphs.

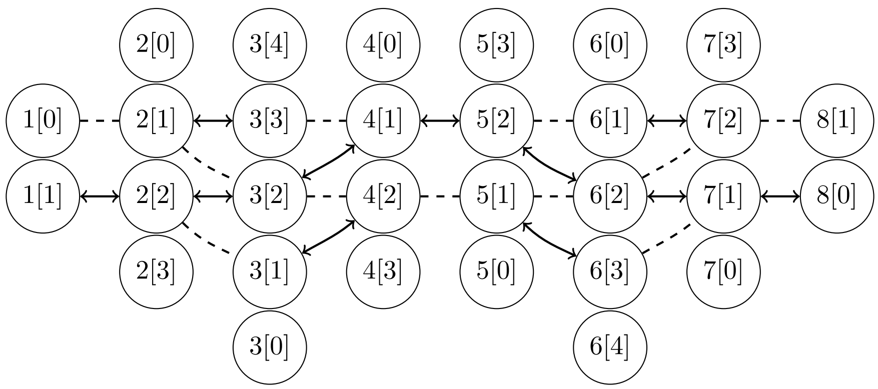
The complete walks that we need are not only cycles but also the walks connecting two different zero or det nodes. Consider the subroutine CIRCUIT in Johnson's algorithm and slightly change it, as follows. In L1 block of CIRCUIT procedure on [J, p.80], we replace the condition $w == s$ (that means, we found a loop) with $(IsZero(w) \vee IsDet(w)) \wedge (s \neq w)$ (that means, we found a walk from the initial node, $s$ to a zero or det node $w$, but not a loop). Applying this way updated CIRCUIT to a zero or det initial node $s$, we get all minimal complete walks starting at $s$. If we apply CIRCUIT to every zero and every det node and after each step we forbid the initial node of that step for visiting in further steps, and take the union of all steps outputs, then we get the list of all minimal complete non-cyclic walks with every walk being printed out exactly one time.

Once all the complete walks are found, Proposition 5.6 allows for getting their weights and degrees. What we need besides that information is the study of the algebraic structure of the algebra generated by the monomials corresponding to the complete walks, which is isomorphic to $A^{T_{SL(\alpha)}}$ by Theorem 5.4, hence to $\mathbf{k}[R(Q, c, \alpha, r)]^{SL(\alpha)}$ by Corollary 4.10, if $\mathbf{k}[Q]/I_c$ is a triangular gentle algebra. The most interesting cases are those, where the spectrum of this algebra is a complete intersection. For example, in [KW] it is proven that the algebra $R(A_n^2, \alpha, r, r)]^{SL(\alpha)}$ decribed in Theorem 4.5 is a complete intersection for $n \leq 7$ and the dimension vector $\alpha$ selected as minimal possible for a given $r$: $\alpha_1 = r_1, \alpha_i = r_{i-1} + r_i, i = 2, n-1, \alpha_n = r_{n-1}$. It is also metioned that for $n > 7$ this is not longer true and there is an example ([KW, Example 8]) but it is presented in terms used in the proof, so $\alpha$ and $r$ are not straighforward from the text. Below we give a clear example and study it with our methods.

**Example 6.1.** Consider the quiver of the algebra $A_8^2$:




There are two colored paths, $(\varphi_1, \cdots, \varphi_7)$ and $(\psi_1, \cdots, \psi_7)$ with $+$ and $-$ signs of all endpoints for the arrows in the first and the second path, respectively. For the rank vector $r = (1, 2, 2, 1, 2, 2, 1)$ and the dimesion vector $\alpha = (1, 3, 4, 3, 3, 4, 3, 1)$ we get $UI(Q, c, \alpha, r, \varepsilon)$ as follows:




For reader's convenience we specify Definition 5.1 in this regular case, as follows: the arc $\varphi_i^j$ is extraverted and connects the nodes $i[j]$ and $(i+1)[\alpha_{i+1}-j]$. The arc $\psi_i^j$ is intraverted (and dashed) and connects the nodes $(i+1)[j]$ and $i[\alpha_i - j]$.

The above graph is more complicated than that from Example 5.2 so in order to find all complete walks like we did in Example 5.7 we need to apply some algorithm (e.g. Johnson's one). If we do this we get the result as follows. The subscript of the semi-invariant presents its degree. First we list the cyclic walks:
$\mathbf{F_6}$: $\psi_2^1\varphi_3^1\psi_3^2\varphi_2^2$, $(2[2],3[1],4[2],3[2],2[2])$, weight $\det^2 - \det^4$.
$\mathbf{G_6}$: $\psi_2^2\varphi_3^2\psi_3^1\varphi_2^1$, $(2[1],3[2],4[1],3[3],2[1])$, weight $\det^2 - \det^4$.
$\mathbf{H_6}$: $\varphi_5^1\psi_6^1\varphi_6^2\psi_5^2$, $(5[1],6[3],7[1],6[2],5[1])$, weight $\det^5 - \det^7$.
$\mathbf{I_6}$: $\varphi_5^2\psi_6^2\varphi_6^1\psi_5^1$, $(5[2],6[2],7[2],6[1],5[2])$, weight $\det^5 - \det^7$.
$\mathbf{F_{18}}$: $\varphi_2^1\psi_3^1\varphi_4^1\psi_5^1\varphi_6^1\psi_6^2\varphi_6^2\psi_6^1\varphi_5^1\psi_4^1\varphi_3^1\psi_2^1\varphi_2^2\psi_2^2$,
$(2[1],3[3],4[1],5[2],6[1],7[2],6[2],7[1],6[3],5[1],4[2],3[1],2[2],3[2],2[1])$, weight $2\det^2 - \det^3 + \det^6 - 2\det^7$.

Then non-cyclic conecting different pairs of nodes in the set $\{1[0],1[1],8[0],8[1]\}$:
$\mathbf{F_{10}}$: $\psi_1^1\varphi_2^1\psi_3^1\varphi_3^2\psi_3^2\varphi_3^1\psi_2^1\varphi_1^1$, $(1[0],2[1],3[3],4[1],3[2],4[2],3[1],2[2],1[1])$, the weight is $2\det^1 + \det^3 - 2\det^4$.
$\mathbf{F_7}$: $\psi_1^1\varphi_2^1\psi_3^1\varphi_4^1\psi_5^1\varphi_6^1\psi_7^1$, $(1[0],2[1],3[3],4[1],5[2],6[1],7[2],8[1])$, weight $\det^1 - \det^8$.
$\mathbf{F_{16}}$: $\psi_1^1\varphi_2^1\psi_3^1\varphi_4^1\psi_5^1\varphi_6^1\psi_6^2\varphi_6^2\psi_6^1\varphi_5^1\psi_4^1\varphi_3^1\psi_2^1\varphi_1^1$,
$(1[0],2[1],3[3],4[1],5[2],6[1],7[2],6[2],7[1],6[3],5[1],4[2],3[1],2[2],1[1])$, weight $2\det^1 + \det^6 - 2\det^7$.
$\mathbf{G_7}$: $\varphi_1^1\psi_2^1\varphi_3^1\psi_4^1\varphi_5^1\psi_6^1\varphi_7^1$, $(1[1],2[2],3[1],4[2],5[1],6[3],7[1],8[0])$, weight $\det^1 - \det^8$.
$\mathbf{G_{10}}$: $\varphi_7^1\psi_6^1\varphi_5^1\psi_5^2\varphi_5^2\psi_5^1\varphi_6^1\psi_7^1$, $(8[0],7[1],6[3],5[1],6[2],5[2],6[1],7[2],8[1])$, the weight is $2\det^5 - \det^6 - 2\det^8$.
$\mathbf{G_{16}}$: $\varphi_7^1\psi_6^1\varphi_5^1\psi_4^1\varphi_3^1\psi_2^1\varphi_2^2\psi_2^2\varphi_2^1\psi_3^1\varphi_4^1\psi_5^1\varphi_6^1\psi_7^1$,
$(8[0],7[1],6[3],5[1],4[2],3[1],2[2],3[2],2[1],3[3],4[1],5[2],6[1],7[2],8[1])$, weight $2\det^2 - \det^3 - 2\det^8$.
These 11 monomials in variables $\varphi_i^j, \psi_i^j$ generate a subalgebra of dimension 8. One can find some 3 syzygies with degrees and weights, as follows:

$F_{16}G_{10} = F_7G_7H_6I_6$, $(26, 2\det^1 + 2\det^5 - 2\det^7 - 2\det^8)$.

$F_{16}F_6G_6 = F_{10}F_{18}$, $(28, 2\det^1 + 2\det^2 - 2\det^4 + \det^6 - 2\det^7)$.

$F_{16}G_{16} = F_7G_7F_{18}$, $(32, 2\det^1 + 2\det^2 - \det^3 + \det^6 - 2\det^7 - 2\det^8)$.

From the first and the third syzygies we can derive a fourth one:

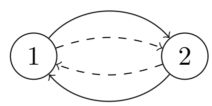
$\frac{F_{16}}{F_7G_7} = \frac{H_6I_6}{G_{10}}$ and $\frac{F_{16}}{F_7G_7} = \frac{F_{18}}{G_{16}} \Rightarrow F_{18}G_{10} = G_{16}H_6I_6$.

Analogously, from the second and the third syzygies we can derive a fifth one:

$F_{10}G_{16} = F_6F_7G_6G_7$.

It is clear that the first three syzygies participate in the homogeneous basis of the ideal of syzygies but the fourth and the fifth ones do not belong to the ideal generated by the first three ones. Then the algebra of semi-invariants is not a complete intersection.

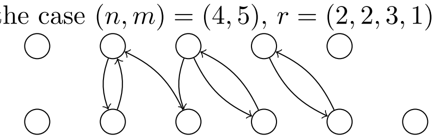
The method introduced in this paper allows for computing semi-invariants for non-triangular gentle algebras $\mathbf{k}Q/I_c$ such that $c^{-1}(s)$ is an oriented cycle for some color $s \in S$. In this case Theorem 4.9 and Corollary 4.10 only imply that $\mathbf{k}[R(Q, I_c, \alpha)]^{SL(\alpha)}$ is a deformation of the semi-group algebra $\mathbf{k}[\Delta \cap \Gamma(G/H)]$ and the question arises, whether the algebras are actually isomorphic? If $\mathbf{k}[\Delta \cap \Gamma(G/H)]$ is a polynomial algebra, then so is $\mathbf{k}[R(Q, I_c, \alpha)]^{SL(\alpha)}$, hence, both algebras are isomorphic. In the particular case of the following quiver we show that the algebras are isomorphic for every rank gentle variety. Let $Q$ be a colored quiver with 2 vertices and 4 arrows, as follows:



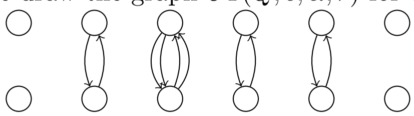

Denote by $\varphi_1$ and $\psi_1$ the normal and dashed arrows from 1 to 2, respectively. Analogously, we introduce $\varphi_2$ and $\psi_2$. The colored paths are $(\varphi_1, \varphi_2)$ and $(\psi_1, \psi_2)$.

**Proposition 6.2.** *The algebra* $\mathbf{k}[\Delta \cap \Gamma(G/H)]$ *is a polynomial algebra unless* $\alpha = (n, n)$ *and* $r(\varphi_1) = k, r(\varphi_2) = n - k, r(\psi_1) = n - k, r(\psi_2) = k, k = 1, \cdots, n-1$. *In the latter case* $\mathbf{k}[R(Q, c, \alpha, r)]^{SL(\alpha)} \cong \mathbf{k}[\Delta \cap \Gamma(G/H)]$ *is a hypersurface.*

*Proof.* We study the algebra $A^{T_{SL(\alpha)}}$, which is isomorphic to $\mathbf{k}[\Delta \cap \Gamma(G/H)]$. Set $(m, n) = \alpha$ and draw the nodes $1[0], \cdots, 1[m]$ and $2[0], \cdots, 2[n]$ of $UI(Q, c, \alpha, r)$ as two aligned point sequence, the first sequence over the second one. Set $\varepsilon(1, \varphi) = + = \varepsilon(2, \psi)$, hence, $\varepsilon(1, \psi) = - = \varepsilon(2, \varphi)$. With such a choice of $\varepsilon$, $UI(Q, c, \alpha, r)$ is a usual oriented graph, so we will describe its arcs as arrows of a quiver. The arrow $\varphi_1^j, j = 1, \cdots, r(\varphi_1)$ goes from $j$-th upper node to the $j$-th lower node if counting from the left. We call this set $r(\varphi_1)$ left aligned down arrows. Analogously, $\varphi_2$ yields $r(\varphi_2)$ right aligned down arrows such that the top right goes from $1[m-1]$ to $2[m-1]$. Moreover, $\psi_1$ and $\psi_2$ yield $r(\psi_1)$ right aligned and $r(\psi_2)$ left aligned up arrows, respectively. We illustrate the above description by the picture of $UI(Q, c, \alpha, r)$ in the case $(n, m) = (4, 5)$, $r = (2, 2, 3, 1)$:



Notice that the nodes $1[0], 2[0], 1[m], 2[n]$ are not incident to any arrow, so the complete walks can only be cyclic. Moreover, the above description implies that each minimal cyclic walk consists of two reverse arrows. An upper node $1[k]$ is incident to no more than 2 down arrows, and no more than 1 unless $r(\varphi_1)+r(\varphi_2) = m$ and $k = r(\varphi_1)$. Analogously, a lower node $2[k]$ is incident to no more than 2 up arrows, and no more than 1 unless $r(\psi_1) + r(\psi_2) = n$ and $k = r(\psi_2)$. Since the algebra $A^{T_{SL(\alpha)}}$ is generated by 2 hops cyclic walks, it is a tensor product of subalgebras corresponding to a pair of nodes connected by at most 2 up and at most 2 down arrows. If for a pair of nodes there is only 1 up or only 1 down arrow, then there are at most 2 cyclic paths on these nodes and the corresponding monomials are algebraically independent. Finally, the only case where there is at least one pair of nodes with 2 up and 2 down arrows is that specific case $m = n, r = (k, n-k, n-k, k)$. We draw the graph $UI(Q, c, \alpha, r)$ for the example $n = 5, r = (2, 3, 3, 2)$:



It is obvious from the above description of $UI(Q, c, \alpha, r)$ that $A^{T_{SL(\alpha)}}$ is as follows:

$$A^{T_{SL(\alpha)}} \cong \mathbf{k}[F_1, \cdots, F_k, G_1, \cdots, G_{n-k}, D_1, D_2]/(F_k G_{n-k} - D_1 D_2), \tag{21}$$

where $F_k, G_{n-k}, D_1, D_2$ are the minors corresponding to four cycles on the nodes $1[k], 2[k]$. Now we write down $n+2$ $GL_n \times GL_n$ semi-invariant polynomials that generate $\mathbf{k}[R(Q, c, \alpha, r)]^{SL_n \times SL_n}$. Set $A_1 = R(\varphi_1), B_1 = R(\varphi_2), A_2 = R(\psi_1), B_2 = R(\psi_2)$. Then these linear maps fulfill the constraints: $A_1 B_1 = 0 = B_1 A_1$, $A_2 B_2 = 0 = B_2 A_2$, $\mathrm{rk}(A_1) \le k$, $\mathrm{rk}(A_2) \le n-k$, $\mathrm{rk}(B_1) \le n-k$, $\mathrm{rk}(B_2) \le k$. The coefficients of characteristic polynomials of the products $A_1 B_2$ and $A_2 B_1$ are $GL_n \times GL_n$ semi-invariant.The above constraints imply $\mathrm{rk}(A_1 B_2) \le k, \mathrm{rk}(A_2 B_1) \le n-k$, hence:

$$\det(A_1 B_2 - \lambda I) = \pm \lambda^{n-k}(\lambda^k + f_1 \lambda^{k-1} + \cdots + f_k), \tag{22}$$

$$\det(A_2 B_1 - \lambda I) = \pm \lambda^{k}(\lambda^{n-k} + g_1 \lambda^{n-k-1} + \cdots + g_{n-k}). \tag{23}$$

Consider the polynomial $\det(\lambda A_1 + \mu A_2)$. In general position the maps $A_1, A_2 : \mathbf{k}^n \to \mathbf{k}^n$ have ranks $k$ and $n-k$, respectively and holds $\mathrm{Im}(A_1) \oplus \mathrm{Im}(A_2) = \mathbf{k}^n$. Hence, the polynomial $\det(\lambda A_1 + \mu A_2)$ is homogeneous of degree $k$ with respect to $\lambda$, and of degree $n-k$ with respect to $\mu$. Hence, the expressions

$$d_1 = \lambda^{-k}\mu^{k-n} \det(\lambda A_1 + \mu A_2), d_2 = \lambda^{k-n}\mu^{-k} \det(\lambda B_1 + \mu B_2) \tag{24}$$

define $GL_n \times GL_n$ semi-invariant polynomials. Reducing polynomials to a generic quadruple $(A_1, B_1, A_2, B_2)$, one can easily check that the relation $f_k g_{n-k} = d_1 d_2$ holds. The degrees and weights of polynomials $f_1, \cdots, f_k, g_1, \cdots, g_{n-k}, d_1, d_2$ match those of $F_1, \cdots, F_k$,$G_1, \cdots, G_{n-k}, D_1, D_2$, hence, Theorem 4.9 implies that an isomorphism $\mathbf{k}[R(Q, c, \alpha, r)]^{SL_n \times SL_n} \cong A^{T_{SL(\alpha)}}$ holds. $\square$

*Remark* 6.1. The above example encourages to conjecture that the isomorphism $\mathbf{k}[R(Q, c, \alpha, r)]^{SL(\alpha)} \cong \mathbf{k}[\Delta \cap \Gamma(G/H)]$ holds in the whole generality of this paper, not just for trangular gentle algebras, as Corollary 4.10 claims.

*Remark* 6.2. The graph model we developed is pretty suitable to compute the generating semi-invariants for the rank gentle varieties $R(Q, c, \alpha, r)$. This model together with Johnson's algorithm (with the above changes) is all one needs to get a tool that provides those generators for any $(Q, c, \alpha, r)$ data. It is an interesting question, which condition ensures this algebra to be a complete intersection? Example 6.1 shows the minimal possible case of non complete intersection for the algebras of $A_n^2$ series: in [KW] it is shown that the complete interstection property holds if $n \le 7$. It would be natural to recover this result in terms of our graph model. Another useful thing would be an algorithm for automatical checking of whether the generating semi-invariants obtained by our method generate a complete intersection or not. Example 6.1 shows that this can be tricky.

*Remark* 6.3. Our definition of the biderected graph $UI(Q, c, \alpha, r, \varepsilon)$ is naturally similar to the so-called *up and down graph* $\Gamma(\alpha, r, \varepsilon)$ introduced in [CC]. The difference is that the up and down graph describes the generic representation in the same rank gentle variety, for which our bidirected graph controls the semi-invariants. We wonder whether this model similarity may have a geometric sense.

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