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LOCALLY SEMI-SIMPLE REPRESENTATIONS OF QUIVERS

D. A. SHMELKIN*

Ostrovitianova, 9-4-187
117437 Moscow, Russia
mitia@mccme.ru

Abstract. We suggest a geometrical approach to the semi-invariants of quivers based on Luna's slice theorem and the Luna–Richardson theorem. The locally semi-simple representations are defined in this spirit but turn out to be connected with stable representations in the sense of GIT, Schofield's perpendicular categories, and Ringel's regular representations. As an application of this method we obtain an independent short proof of a theorem of Skowronski and Weyman about semi-invariants of the tame quivers.

1. Introduction

Let Q be a finite quiver, i.e., an oriented graph. We fix the notation as follows: denote by Q_0 and Q_1 the sets of the vertices and the arrows of Q, respectively. For any arrow $\varphi \in Q_1$ denote by $t\varphi$ and $h\varphi$ its tail and its head, respectively. A representation V of Q over an algebraically closed field $\mathbf{k}$, $\operatorname{char}\mathbf{k} = 0$, consists in defining a vector space $V(i)$ over $\mathbf{k}$ for any $i \in Q_0$ and a $\mathbf{k}$-linear map $V(\varphi) : V(t\varphi) \to V(h\varphi)$ for any $\varphi \in Q_1$. The dimension vector $\dim V$ is the collection of $\dim V(i), i \in Q_0$. For a fixed dimension α we may set $V(i) = \mathbf{k}^{\alpha_i}$. Then the set $R(Q,\alpha)$ of the representations of dimension α is converted into the vector space

$$R(Q,\alpha) = \bigoplus_{\varphi\in Q_1} \operatorname{Hom}(\mathbf{k}^{\alpha_{t\varphi}}, \mathbf{k}^{\alpha_{h\varphi}}). \tag{1}$$

A homomorphism H of a representation U of Q to another representation, V is a collection of linear maps $H(i), U(i) \to V(i), i \in Q_0$, such that for any $\varphi \in Q_1$ holds $V(\varphi)H(t\varphi) = H(h\varphi)U(\varphi)$. The endomorphisms, automorphisms, and isomorphisms are defined naturally. Hence, the isomorphism classes of representations of Q are the orbits of a reductive group $\mathrm{GL}(\alpha) = \prod_{i\in Q_0} \mathrm{GL}(\alpha_i)$ acting naturally on $R(Q,\alpha)$: $(g(V))(\varphi) = g(h\varphi)V(\varphi)(g(t\varphi))^{-1}$. Set $\mathrm{SL}(\alpha) = \prod_{i\in Q_0} \mathrm{SL}(\alpha_i) \subseteq \mathrm{GL}(\alpha)$.

Assume that Q has no oriented cycles. Then for any dimension α the algebra $\mathbf{k}[R(Q,\alpha)]^{\mathrm{GL}(\alpha)}$ of $\mathrm{GL}(\alpha)$-invariant regular functions is trivial and the unique $\mathrm{GL}(\alpha)$-closed orbit is the origin of $R(Q,\alpha)$. It is however interesting to study the $\mathrm{SL}(\alpha)$-invariant functions or the semi-invariants of $\mathrm{GL}(\alpha)$. The aim of this paper is to suggest a geometrical approach to this study in the spirit of Luna's papers [Lu1], [Lu2]. For

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this we need to describe the closed orbits of $\mathrm{SL}(\alpha)$. Consider a more general setting of a connected reductive group G acting on an affine variety X, $x \in X$, $G' \subseteq G$ is the commutant. In Theorem 1 we prove that $G'x$ is closed in X if and only if Gx is closed in an open affine neighborhood $X_f \subseteq X$, where f is semi-invariant. We call such x a *locally semi-simple* point. We prove that there exists a *generic stabilizer of locally semi-simple points* and in Theorem 7 we obtain a special version of the Luna–Richardson theorem (see [Lu2]) that can be called the Luna–Richardson theorem about semi-invariants.

Locally semi-simple representations of quivers turn out to be closely connected with the stable representations in the sense of GIT (see [MF], [Ki]) and the perpendicular categories introduced in [Sch]. Namely, we prove in Theorem 11 that a representation is locally semi-simple if and only if it is a sum of simple objects in a perpendicular category; actually this is just a more strong version of [Ki, Prop. 3.2].

Since the orbit of a locally semi-simple representation V of a quiver Q is closed in an open affine neighborhood of V, we may consider an *étale slice* at V (see [Lu1]) and this is an open affine neighborhood of V in an affine subspace $N \subseteq R(Q, \dim V)$. Luna's étale slice theorem describes the action of the whole group in a stable neighborhood of V in terms of the action of the stabilizer of V on N or, in our case, of $\mathrm{Aut}(V)$ on the space $\mathrm{Ext}(V, V)$ (see, e.g., [Kr]). A crucial observation made in [LBP] is that the linear group $(\mathrm{Aut}(V), \mathrm{Ext}(V, V))$ is isomorphic to $(\mathrm{GL}(\gamma), R(\Sigma_V, \gamma))$, where the quiver Σ_V is defined in terms of the indecomposable summands of V and the dimension vector γ is the collection of their multiplicities. Of course, in [LBP] this was observed for semi-simple representations but the same is obviously true for the locally semi-simple ones.

Recall [Kac] that a decomposition $\alpha = \sum_{i=1}^{t} \beta_i$ with $\alpha, \beta_1, \dots, \beta_t \in \mathbf{Z}_+^{Q_0}$ is called *canonical* if $\beta_1, \dots, \beta_t$ are Schur roots and the set of representations $R_1 + \cdots + R_t$ such that R_i is indecomposable and $\dim R_i = \beta_i$ contains an open dense subset in $R(Q, \alpha)$. On the other hand, Ringel applied in [Ri] the term "canonical decomposition" for a different notion and we need that notion, too. We therefore call the decomposition introduced by Kac *generic* (as, e.g., in [SkW]).

It is well known that the generic decomposition corresponds to the generic stabilizer in the sense that the torus $T \subseteq \mathrm{GL}(\alpha)$ of rank t naturally corresponding to this decomposition is a maximal torus in a generic stabilizer for the action of $\mathrm{GL}(\alpha)$. Analogously, the maximal torus of the generic stabilizer of locally semi-simple points yields another decomposition of α that we call *generic locally semi-simple*. Given a nontrivial locally semi-simple point $V \in R(Q, \alpha)$, we show in Proposition 14 how to describe both decompositions in terms of those for the quiver Σ_V and the dimension vector γ. On the other hand, we reformulate Theorem 7 for the quiver setting in terms of the generic locally semi-simple decomposition and so we get a useful general description Theorem 19 of $\mathbf{k}[R(Q, \alpha)]^{\mathrm{SL}(\alpha)}$.

We apply the above methods to the case when Q is a tame quiver. Ringel introduced in [Ri] the category $\mathcal{R}$ of *regular* representations. For the tame quivers he described explicitly the simple objects of $\mathcal{R}$. An important observation Proposition 20 is that the semi-simple regular representations are locally semi-simple. Moreover, for a generic semi-simple regular representation V the linear group $(\mathrm{GL}(\gamma), R(\Sigma_V, \gamma))$ corresponds to a quiver being a disjoint union of equioriented A_n-type quivers. For the latter quivers we can easily describe the generic and the generic locally semi-simple decomposition.

Together with Proposition 14 this yields a way to determine both decompositions for Q and α (in what concerns the generic decomposition we recover [Ri, Theorem 3.5]).

The algebras of semi-invariants of tame quivers Q have been studied in several papers including [Ri], [HH], [SchW]. In [SkW, Theorem 21] Skowronski and Weyman proved that $\mathbf{k}[R(Q,\alpha)]^{\mathrm{SL}(\alpha)}$ is a *complete intersection* for any α. Using [DW] and Theorem 19, we give an independent short proof of this result (Theorems 23 and 27).

2. Locally semi-simple points

Throughout this section G denotes a connected reductive group, G' stands for its commutant, $T = G/G'$ is a torus, and X is an irreducible affine variety acted upon by G. By Hilbert's theorem on invariants the subalgebra $\mathbf{k}[X]^{G'} \subseteq \mathbf{k}[X]$ of G'-invariant regular functions is finitely generated and the corresponding *quotient* map will be denoted by $\pi_{X,G'} : X \to X/\!\!/G' = \operatorname{Spec}\mathbf{k}[X]^{G'}$. For a T-character $\chi \in \Xi(T)$ denote by $\mathbf{k}[X]^{(G)}_\chi$ the module of eigenvectors of G in $\mathbf{k}[X]$ with weight χ, denote by G_χ the kernel of χ. Clearly, $\mathbf{k}[X]^{G'}$ decomposes as follows: $\mathbf{k}[X]^{G'} = \bigoplus_{\chi\in\Xi(T)} \mathbf{k}[X]^{(G)}_\chi$.

Theorem 1. *The following properties of $x \in X$ are equivalent:*

(i) *for a semi-invariant $f \in \mathbf{k}[X]^{(G)}_\chi$, $f(x) \neq 0$ and Gx is closed in X_f;*
(ii) *for a character $\chi \in \Xi(T)$, the orbit $G_\chi x$ is closed in X;*
(iii) *the orbit $G'x$ is closed in X;*
(iv) *the closure of the orbit $G'x$ in X is contained in Gx.*

Proof. First observe that if $f \in \mathbf{k}[X]^{(G)}_\chi$, $f(x) \neq 0$, and $g \in G$, then $g(G_\chi x) = Gx \cap \{y \in X \mid f(y) = f(gx)\}$. This yields the implication (i) $\Rightarrow$ (ii). Also we note that Gx is a disjoint union of $t(G_\chi x)$ with t running over the one-dimensional coset space G/G_χ. The implication (ii) $\Rightarrow$ (iii) follows from the fact that the subgroup G' is normal in G_χ.

Let us prove the implication (iii) $\Rightarrow$ (i). The torus T acts on the quotient $X/\!\!/G'$; consider a T-equivariant embedding of $X/\!\!/G'$ to a T-module W. Clearly, if $y = \pi_{G'}(x)$ belongs to W^T, then $Ty = y$ implies $Gx = G'x$ and (i) holds with f being a constant function. Otherwise, y is a sum of nonzero T-eigenvectors; let f be the product of the corresponding linear T-eigenfunctions. Then f is a T-semi-invariant function on W with respect to a character $\chi \in \Xi(T)$, and $f(y) \neq 0$. Moreover, the orbit of y with respect to the kernel T_χ of χ is closed, because y is a sum of T_χ-eigenvectors such that the sum of their characters (with respect to T_χ) is zero. Consequently, the orbit $G_\chi x = T_\chi G'x$ is closed in X, because it is equal to the intersection of the closed pull-back $\pi_{G'}^{-1} T_\chi y$ with the closed subset $X_{\dim G'x} = \{z \in X \mid \dim G'z \leqslant \dim G'x\}$. So we got (ii) and, besides, $f(x) \neq 0$ for the above f thought of as a G-semi-invariant function on X. Set $m = \dim G_\chi x$. Then the closure $\overline{Gx}$ of Gx in X is an irreducible variety of dimension $m+1$ and for any $t \in G/G_\chi$ the set $\{y \in \overline{Gx} \mid f(y) = f(tx)\}$ is an equidimensional closed subvariety in X of dimension m with one of the irreducible components being equal to $t(G_\chi x)$. Since G/G_χ acts transitively on the fibers of f, we get that $\overline{Gx}_f$ is an equidimensional variety with one of the irreducible components being equal to Gx. However, the dimension of $\overline{Gx}\setminus Gx$ is less than $\dim Gx$, hence, $\overline{Gx}_f = Gx$, that is, Gx is closed in X_f.

The implication (iii) $\Rightarrow$ (iv) is obvious. If $G'x$ is not closed, then there is $z \in \overline{G'x}$ such that $\dim G'z < \dim G'x$. However, for any $z \in Gx$ we have $\dim G'z = \dim G'x$.

Therefore (iv) implies (iii). $\square$

The points $x \in X$ such that Gx is closed in X are called *semi-simple* in [GV]. If X is a variety of representations of associative algebras, then this is not just a definition, since it is proved that the module corresponding to x is semi-simple if and only if Gx is closed (see, e.g., [Kr]). This motivates

Definition 1. We call $x \in X$ *locally semi-simple* if x fulfills the equivalent conditions of the above theorem.

Remark 1. The property of local semi-simplicity is intermediate between those of *stability* and *semi-stability* introduced by Mumford [MF]. Recall that in our context x is called χ-semi-stable if $f(x) \neq 0$ for a nonconstant semi-invariant $f \in \mathbf{k}[X]^{(G)}_\chi$, and x is called χ-stable if x is χ-semi-stable, the stabilizer of x is equal to the kernel of the action $G : X$, and the orbit Gx is closed in X_f. So locally semi-simple points meeting condition (i) of Theorem 1 are χ-semi-stable and x is χ-stable if and only if x is locally semi-simple with trivial stabilizer.

For a reductive group M acting on an affine variety Y Luna introduced in [Lu1] and [Lu2] the concept of *étale slice* at a semi-simple point $y \in Y$. First of all by Matsushima's criterion [Ma], the stabilizer M_y is reductive. The Luna slice theorem [Lu1] states that there exists an étale slice $S \subseteq Y$ at y such that $S \ni y$ is affine, locally closed, M_y-stable, and the natural map $\varphi_y : M *_{M_y} S \to Y, [m, s] \to ms$ is *excellent* (see the precise definition in [Lu1]), in particular, the image of φ_y is affine and the restriction of φ_y to any fiber of the M-quotient map is an isomorphism. Further, assume that $Y = V$ is a vector space and M acts on V by a linear representation, $v = y$; choose an M_v-stable complementary subspace N to $\mathrm{T}_v Mv$ in V. Then as S we can take $S = v + N_0$ for an open affine subset $N_0 \subseteq N$ containing 0. The representation $\sigma_v : M_v \to \mathrm{GL}(N)$ is called in this case the *slice representation* of v and can be calculated by the formula (Ad stands for the adjoint representation):

$$\sigma_v = (V + \operatorname{Ad} M_v)/(\operatorname{Ad} M)|_{M_v}. \tag{2}$$

Consider a G-equivariant embedding of X to a vector space V with a linear action of G. Clearly, $x \in X$ is locally semi-simple if and only if x is locally semi-simple as a point of V. Furthermore, $X /\!\!/ G'$ is naturally embedded to $V /\!\!/ G'$ such that $\pi_{X,G'}$ is the restriction of $\pi_{V,G'}$ to X, see, e.g., [Kr]. For $\xi \in V /\!\!/ G'$ denote by $\mathcal{O}_\xi$ the unique G'-closed orbit in $\pi^{-1}_{V,G'}(\xi)$. The quotient $V /\!\!/ G'$ carries the Luna stratification [Lu1] by finitely many locally closed subvarieties $(V /\!\!/ G')_{(L)} = \{\xi \in V /\!\!/ G' \mid \mathcal{O}_\xi \cong G'/L\}$, where L is a subgroup in G'. Hence, the strata intersect $X /\!\!/ G'$ by locally closed subvarieties $(X /\!\!/ G')_{(L)}$ and yield a stratification of $X /\!\!/ G'$.

We consider a similar stratification of $X /\!\!/ G'$ with respect to the action of G, as follows. For a subgroup $M \subseteq G$ denote by $(X /\!\!/ G')^G_{(M)}$ the set of all $\xi \in X /\!\!/ G'$ such that G_z is G-conjugate to M for $z \in \mathcal{O}_\xi$. Clearly, if the subgroups M_1 and M_2 are G-conjugate, then $M_1 \cap G'$ and $M_2 \cap G'$ are G'-conjugate, hence each stratum $(X /\!\!/ G')_{(L)}$ is a union of $(X /\!\!/ G')^G_{(M)}$ with $M \cap G'$ being G'-conjugate to L.

Proposition 2. *$(X /\!\!/ G')^G_{(M)}$ is locally closed.*

Proof. For $\xi \in (X/\!\!/G')^G_{(M)}$ we apply the slice theorem for $z \in \mathcal{O}^M_\xi$ and G'. Since z is M-invariant and M normalizes G'_z, the slice S can be chosen to be M-stable. Indeed, one can choose a slice at z with respect to V of form $z+N_0$ such that N is M-stable; by [Lu1] the intersection $(z+N_0)\cap X$ is a slice at z with respect to X. Then the map φ_z is M-equivariant. Denote by $\varphi_z/\!\!/G' : S/\!\!/G'_z \to X/\!\!/G'$ the étale covering of a neighborhood of ξ in $X/\!\!/G'$ given by the slice theorem. Then the M-stratum is covered by S^M, hence is locally closed. □

Proposition 3. *The stratification $X/\!\!/G' = \bigsqcup_M (X/\!\!/G')^G_{(M)}$ is finite.*

Proof. Since the Luna stratification is finite, it is sufficient to show that each stratum $(X/\!\!/G')_{(L)}$ is decomposed into finitely many strata $(X/\!\!/G')^G_{(M)}$. Take $\xi \in (X/\!\!/G')_{(L)}$ and $z \in \mathcal{O}^L_\xi$. Then we have $G_z \subseteq N_G(L)$ and $G_z/L \cong (G/G')_\xi$. So G_z is an extension of L by a diagonalizable subgroup in $N_G(L)$. Choose a maximal torus $A \subseteq N_G(L)$. Then z is $N_G(L)$-conjugate to a point w such that the identity component of G_w is contained in LA. On the other hand, X^L can be divided into finitely many subsets with constant stabilizer with respect to A. So there are finitely many $N_G(L)$-conjugacy classes of stabilizers in G of locally semi-simple points $z \in X^L$ with $G'_z = L$. □

Recall that a subgroup $H_0 \subseteq G'$ is called a *principal isotropy group* if $(X/\!\!/G')_{(H_0)}$ is the unique open and dense Luna stratum. By Propositions 2 and 3 we have

Definition-Proposition 1. A subgroup $H \subseteq G$ is called a *generic stabilizer of a locally semi-simple point* if $(X/\!\!/G')^G_{(H)}$ is the unique open and dense G-stratum of $X/\!\!/G'$. The intersection $H \cap G'$ is a principal isotropy group and the image of H in G/G' is the kernel of the action $G/G' : X/\!\!/G'$.

We now want to describe locally semi-simple points and their stabilizers in terms of the Luna slice theorem with respect to the group G. Indeed, if x is locally semi-simple and Gx is closed in X_f for a semi-invariant f, then there is an étale slice S at x with respect to X_f, and if $X = V$ is a vector space, then an étale slice of type $x + N_0$ exists.

Proposition 4. *If $G : V$ is a linear representation, and $v \in V$ is a locally semi-simple point, then for any $n \in N_0$, $v + n$ is locally semi-simple with respect to G if and only if $n \in N$ is locally semi-simple with respect to G_v.*

Proof. Assume that $G(v+n)$ is closed in V_f for a G-semi-invariant f. Then $\varphi_v^{-1}G(v+n)$ is closed in $\varphi_v^{-1}V_f = G *_{G_v} (v + (N_0)_{f'})$, where $f' \in \mathbf{k}[N]$ is defined as $f'(n') = f(v+n'), n' \in N$, so that f' is G_v-semi-invariant. Moreover, that φ_v is excellent implies that $\varphi_v^{-1}G(v+n)$ is a union of finitely many orbits, hence $G[e, v+n] = G *_{G_v} (v + G_v n)$ is also closed in $G *_{G_v} (v + (N_0)_{f'})$, equivalently, $G_v n$ is closed in $(N_0)_{f'}$. Since N_0 is affine and N is a vector space, $N_0 = N_d$ for some $d \in \mathbf{k}[N]$, and N_0 is G_v-stable implies that d is G_v-semi-invariant. So $G_v n$ is closed in $N_{f'd}$ and we have proved the "only if" part.

Assume that $G_v n$ is closed in $N_{f'}$. Then $G_v n$ is closed in $N_{f'} \cap N_0 = N_{df'}$, so we may assume that $N_{f'}$ is contained in N_0. Then, by the properties of an excellent map, we have that $G(v + n)$ is closed in an open subset V_1 in the image V_0 of φ_v, such that $V_0 \setminus V_1$ is an equidimensional subvariety of codimension 1 in V_0. Since V_0 is affine and V is a vector space, we get $V_1 = V_f$ for some $f \in \mathbf{k}[V]$, and V_0 is G-stable implies f is G-semi-invariant. □

It is well known that the isotropy group G_v for a semi-simple $v \in V$ is principal if and only if the only semi-simple point in $(G_v, N/N^{G_v})$ is 0.

Corollary 5. *Let $N = N^{G_v} \oplus N_+$ be a G_v-stable decomposition. Then v is generic if and only if the only G_v-locally semi-simple point in N_+ is 0.*

Proof. If $N_+ \setminus \{0\}$ contains a G_v-locally semi-simple point, then N contains a G_v-locally semi-simple point n with a proper isotropy subgroup $(G_v)_n \subsetneq G_v$. Multiplying n by a scalar, we may assume $n \in N_0$, hence by the proposition, $v+n$ is G-locally semi-simple with stabilizer $G_{v+n} = (G_v)_n$. Then the closure of $(X/\!\!/G')^G_{(G_{v+n})}$ contains $(X/\!\!/G')^G_{(G_v)}$ so the closure of the latter cannot be equal to $X/\!\!/G'$ and v is not generic. Conversely, if the only G_v-locally semi-simple point in N_+ is 0, then v is a generic locally semi-simple point in the image V_0 of φ_v. Let $V_{\mathrm{lss}} \subseteq V$ be the subset of G-locally semi-simple points. By Theorem 1, V_{lss} is also the union of G'-closed orbits; since $V/\!\!/G'$ is irreducible, the closure $\overline{V_{\mathrm{lss}}}$ is also. Therefore $\overline{V_{\mathrm{lss}}}$ is the closure of its intersection with V_0 and v is generic in V. $\square$

Corollary 6. *If $n \in N_0$ is a generic locally semi-simple point for the action of G_v, then $v+n$ is generic for G acting on V.*

Proof. By the slice theorem $G_{v+n} = (G_v)_n$; by the proposition, $v+n$ is locally semi-simple. Applying formula (2), we get $\sigma_{v+n} = \sigma_n$. Applying Corollary 5, we conclude the proof. $\square$

The notion of the locally semi-simple point can be used in order to describe the semi-invariants of G. Recall that the Luna–Richardson theorem [Lu2] says that if H_1 is a principal isotropy group for the action $G : X$, then the embedding $X^{H_1} \subseteq X$ gives rise to an isomorphism: $\mathbf{k}[X]^G \xrightarrow{\sim} \mathbf{k}[X^{H_1}]^{N_G(H_1)}$. Of course, one could take G' instead of G and apply the Luna–Richardson theorem. But there is another way.

Theorem 7. *Assume that H is a generic stabilizer of a locally semi-simple point for the action $G : X$. Then the embedding $X^H \subseteq X$ and the group isomorphism $\theta_H : N_G(H)/N_{G'}(H) \xrightarrow{\sim} G/G'$ give rise to an isomorphism:*

$$\mathbf{k}[X]^{G'} \xrightarrow{\sim} \mathbf{k}[X^H]^{N_{G'}(H)} \tag{3}$$

of a $\Xi(G/G')$-graded algebra onto a $\Xi(N_G(H)/N_{G'}(H))$-graded algebra. Moreover, a generic $N_{G'}(H)$-orbit in X^H is closed and isomorphic to $N_{G'}(H)/(G' \cap H)$.

Proof. First prove that we have an isomorphism of algebras. Since H normalizes G', HG' is a reductive subgroup in G. Moreover H is a principal isotropy group for HG' acting on X so by the Luna–Richardson theorem $\mathbf{k}[X]^{HG'}$ restricts isomorphically onto $\mathbf{k}[X^H]^{N_{HG'}(H)} = \mathbf{k}[X^H]^{HN_{G'}(H)} = \mathbf{k}[X^H]^{N_{G'}(H)}$. Also we have $\mathbf{k}[X]^{G'} = \mathbf{k}[X]^{HG'}$. Indeed, for any $h \in H, f \in \mathbf{k}[X]^{G'}$ we have $hf \equiv f$ on X^H, hence on $G'X^H$. But X^H intersects all closed HG'-orbits, so any HG'-invariant function is completely defined by its restriction to $HG'X^H = G'X^H$. Thus we got that restricting G'-invariant functions to X^H we have an isomorphism $\mathbf{k}[X]^{G'} \xrightarrow{\sim} \mathbf{k}[X^H]^{N_{G'}(H)}$. Denote by $Z(G)$ the center of G. We have $N_G(H) = N_{G'}(H)Z(G)$. Hence, $N_G(H)/N_{G'}(H) \cong Z(G)/Z(G') \cong G/G'$. Clearly a G-semi-invariant function of weight $\chi \in \Xi(G/G')$ restricts to a $N_G(H)$-semi-invariant function of weight $\theta_H^*(\chi)$. Finally, observe that the generic point $x \in X^H$ has a closed HG'-orbit, because H is a principal isotropy group of HG'. By [Lu2], $N_{HG'}(H)x = N_{G'}(H)x$ is also closed in X^H. $\square$

Remark 2. The description of $\mathbf{k}[X]^{G'}$ given by this theorem is in general different from that given by the Luna–Richardson theorem for G'. For instance, take as G' the group SL_2 acting naturally on $\mathbf{k}^2 \oplus \mathbf{k}^2$, and as G take the extension of G' by $T = \{\mathrm{diag}(t,t,s,s) \mid t,s \in \mathbf{k}^*\}$. The principal isotropy group of G' is trivial so the Luna–Richardson theorem does not give any new information about $\mathbf{k}[X]^{G'}$. On the other hand, $H = \{\mathrm{diag}(u,1,1,u^{-1}) \mid u \in \mathbf{k}^*\}$, $N_{G'}(H) = \mathbf{k}^*, X^H = \mathbf{k}^2$ with $z \in \mathbf{k}^*$ acting as $\mathrm{diag}(z, z^{-1})$ and $(t,s) \in T$ acting as $\mathrm{diag}(t,s)$.

3. Locally semi-simple representations of quivers

Definition 2. A representation V of a quiver Q is called locally semi-simple if V is a locally semi-simple point of $R(Q, \dim V)$ with respect to $\mathrm{GL}(\dim V)$.

We start with observations as follows.

Proposition 8. *Assume that V is a locally semi-simple representation.*

1. *If $V = V_1 + V_2$, then both V_1 and V_2 are locally semi-simple.*
2. *If V is indecomposable, then* $\mathrm{Aut}(V) = \mathbf{k}^*$.
3. *If $V = V_1 + V_2$, both V_1 and V_2 are indecomposable, and $V_1 \ncong V_2$, then we have* $\mathrm{Hom}(V_1, V_2) = 0, \mathrm{Hom}(V_2, V_1) = 0$.

Proof. Assertions 1–3 follow from Theorem 11 below. We give however an independent proof. Set $\alpha = \dim V$.

1. Set $\beta = \dim V_1$. Note that $\mathrm{SL}(\alpha)$ contains a subgroup naturally isomorphic to $\mathrm{SL}(\beta)$ and $\mathrm{SL}(\beta)V_1 + V_2$ is contained in $\overline{\mathrm{SL}(\alpha)V}$. Assuming that V_1 is not locally semi-simple, we get by Theorem 1 that $\overline{\mathrm{SL}(\beta)V_1}$ contains a representation nonisomorphic to V_1, hence $\overline{\mathrm{SL}(\alpha)V}$ contains a representation nonisomorphic to V and V is not locally semi-simple.

2. By Fitting's lemma $\mathrm{End}(V)$ is local. By Matsushima's criterion [Ma], the stabilizer $\mathrm{SL}(\alpha)_V$ is reductive; since $\mathrm{Aut}(V) = \mathrm{GL}(\alpha)_V$ and $\mathrm{GL}(\alpha)_V/\mathrm{SL}(\alpha)_V$ is a subgroup in the center of $\mathrm{GL}(\alpha)$, $\mathrm{Aut}(V)$ is reductive, hence $\mathrm{Aut}(V) = \mathbf{k}^*$.

3. By **2** we know $\mathrm{End}(V_1) = \mathbf{k}$, $\mathrm{End}(V_2) = \mathbf{k}$, and by Matsushima's criterion $\mathrm{End}(V)$ is reductive. The decomposition $\mathrm{End}(V) = \bigoplus_{i,j=1,2} \mathrm{Hom}(V_i, V_j)$ implies that $\mathrm{End}(V)$ is either $\mathbf{k} \oplus \mathbf{k}$ or $\mathrm{End}(\mathbf{k}^2)$. In the latter case, let H_{12} and H_{21} be the generators of $\mathrm{Hom}(V_1, V_2)$ and $\mathrm{Hom}(V_2, V_1)$, respectively. The isomorphism $\mathrm{End}(V) \cong \mathrm{End}(\mathbf{k}^2)$ implies $H_{21}H_{12} \in \mathrm{End}(V_1)$ does not vanish, hence is a scalar operator on V_1. So we have $V_1 \cong V_2$, a contradiction. $\square$

A representation V such that $\mathrm{Aut}(V) = \mathbf{k}^*$ is called *Schurian*. The converse to Proposition 8.2 is not true, i.e., not any Schurian representation is locally semi-simple.

Example 1. Let Q be the quiver with one vertex and two attached loops. Let V be a two-dimensional representation of Q such that the corresponding pair of endomorphisms of $\mathbf{k}^2$ generates the algebra B of the upper triangular matrices, in a basis. Then $\mathrm{End}(V)$ is the centralizer of B in $\mathrm{End}(\mathbf{k}^2)$, so V is Schurian. Since the center of $\mathrm{GL}(\alpha)$ acts trivially, the locally semi-simple representations are in this case just the semi-simple ones. A semi-simple Schurian representation must be simple, but V has a one-dimensional subrepresentation, so V is not semi-simple.

Recall that each quiver Q determines two forms on $\mathbf{Z}^{Q_0}$, the Tits quadratic form $q_Q(\alpha) = \sum_{i\in Q_0} \alpha_i^2 - \sum_{\varphi\in Q_1} \alpha_{t\varphi}\alpha_{h\varphi}$, and the Euler bilinear form:

$$\langle\alpha,\beta\rangle = \sum_{i\in Q_0} \alpha_i\beta_i - \sum_{\varphi\in Q_1} \alpha_{t\varphi}\beta_{h\varphi}. \tag{4}$$

Note also that the Euler form is not symmetric and $\langle\alpha,\alpha\rangle = q_Q(\alpha)$.

Proposition 9. *If V is a Schurian representation and $q_Q(\dim V) = 1$ (in other words, $\alpha = \dim V$ is a real Schur root), then V is locally semi-simple.*

Proof. The hypothesis implies that the $\mathrm{GL}(\alpha)$-orbit of V is dense in $R(Q,\alpha)$, so the generic stabilizer for the action of $\mathrm{SL}(\alpha)$ is trivial. By [Po], generic $\mathrm{SL}(\alpha)$-orbits are closed. Hence, $\mathrm{SL}(\alpha)V$ is closed. □

4. Semi-invariants of quivers and perpendicular categories

The character group of $\mathrm{GL}(\alpha)$ is generated by the determinants of the $\mathrm{GL}(\alpha_a)$-factors, $a \in Q_0$, so is isomorphic to $\mathbf{Z}^{Q_0}$ such that $\chi \in \mathbf{Z}^{Q_0}$ gives rise to the character $\overline{\chi} = \prod_{a\in Q_0,\alpha_a>0} \det_a^{\chi_a}$. We can also think of χ as an integer function on the dimensions of representations such that $\chi(\alpha) = \sum_{a\in Q_0} \chi_a\alpha_a$. We have

$$\mathbf{k}[R(Q,\alpha)]^{\mathrm{SL}(\alpha)} = \bigoplus_{\chi\in\mathbf{Z}^{Q_0}} \mathbf{k}[R(Q,\alpha)]_\chi^{(\mathrm{GL}(\alpha))}, \tag{5}$$

where $\mathbf{k}[R(Q,\alpha)]_\chi^{(\mathrm{GL}(\alpha))} = \{f \in \mathbf{k}[R(Q,\alpha)] \mid gf = \overline{\chi}(g)f\forall g \in \mathrm{GL}(\alpha)\}$. An easy but important observation is that $\mathbf{k}[R(Q,\alpha)]_\chi^{(\mathrm{GL}(\alpha))} \neq \{0\}$ only if $\chi(\alpha) = 0$, because $t \to t^{\chi(\alpha)}$ is the restriction of $\overline{\chi}$ to the subgroup $\mathbf{k}^* \subseteq \mathrm{GL}(\alpha)$ multiplying each $V(a), a \in Q_0$, by the same scalar, hence acting trivially on $R(Q,\alpha)$.

Recall that for dimension vectors $\alpha,\beta \in \mathbf{Z}_+^{Q_0}$ such that $\langle\alpha,\beta\rangle = 0$, Schofield introduced in [Sch] a function c on $R(Q,\alpha)\times R(Q,\beta)$ such that:

(i) $c(V,W) \neq 0$ if and only if $\mathrm{Hom}(V,W) = 0$.
(ii) c is $\mathrm{GL}(\alpha)\times\mathrm{GL}(\beta)$-semi-invariant; if $c(V,\cdot) \not\equiv 0$, then its character is equal to $\langle\alpha,\cdot\rangle$; if $c(\cdot,W) \not\equiv 0$, then its character is equal to $-\langle\cdot,\beta\rangle$.

Derksen and Weyman proved in [DW] that each vector space $\mathbf{k}[R(Q,\alpha)]_\sigma^{(\mathrm{GL}(\alpha))}$ is generated by the functions $c_W = c(\cdot,W)$ such that $-\langle\cdot,\dim W\rangle = \sigma$. Recall the Ringel formula [Ri]:

$$\dim \mathrm{Ext}(V,W) = \dim \mathrm{Hom}(V,W) - \langle\dim V,\dim W\rangle. \tag{6}$$

This formula and the above properties imply that for a given $V \in R(Q,\alpha)$ the semi-invariants c_W such that $c_W(V) \neq 0$ correspond to the representations W such that $\mathrm{Hom}(V,W) = 0, \mathrm{Ext}(V,W) = 0$. Schofield called in [Sch] the set of such representations, $V^\perp$, the *right perpendicular category* of V. The *left* perpendicular category, ${}^\perp V$, is defined similarly. Schofield proved that the perpendicular categories are abelian subcategories. In particular, simple objects in $V^\perp$ are Schurian representations, homomorphisms between nonisomorphic simple objects are trivial, and any representation has a unique Jordan–Hölder decomposition.

On the other hand, we have the notion of χ-stability (see Remark 1) and for representations of quivers King proved in [Ki, Prop. 3.1] that V is χ-stable if and only if $\chi(\dim V) = 0$ and $\chi(\dim V') < 0$ for any subrepresentation $V' \subseteq V$ different from 0 and V.

Proposition 10. *$S \in {}^{\perp} W$ is a simple object if and only if S is $-\langle \cdot, \dim W\rangle$-stable.*

Proof. S is a simple object means that there are no proper subrepresentations $S' \subseteq S$ such that $\mathrm{Hom}(S', W) = 0$ and $\langle \dim S', \dim W\rangle = 0$. So the "if" part follows. Now assume S is simple in ${}^{\perp}W$ and $S' \subseteq S$ is a proper subrepresentation. This means that $c(S, W) \neq 0$ and $c(S', W) = 0$. By [DW, Lemma 1], $\langle \dim S', \dim W\rangle < 0$ would contradict our condition $c(S, W) \neq 0$ and $\langle \dim S', \dim W\rangle = 0$ would imply that $c(S', W) \neq 0$. Thus we have $\langle \dim S', \dim W\rangle > 0$. $\square$

Now we give a criterion for a representation to be locally semi-simple. The above discussion shows that it is sufficient to find out for representations V, W such that $\mathrm{Hom}(V, W) = 0$ and $\mathrm{Ext}(V, W) = 0$ whether the $\mathrm{GL}(\dim V)$-orbit of V is closed in $R(Q, \dim V)_{c_W}$.

Theorem 11. *The orbit of V is closed in $R(Q, \dim V)_{c_W}$ if and only if $V = p_1S_1 + \cdots + p_tS_t$, where $S_1, \ldots, S_t$ are simple objects in ${}^{\perp}W$.*

Remark 3. In view of Proposition 10 one can see that this theorem is similar to [Ki, Prop. 3.2]. However, the property of the orbit to be closed in $R(Q, \dim V)_{c_W}$ is stronger than that to be closed in the open set of semi-stable points as in [Ki].

Proof. Clearly the sum of Jordan–Hölder factors of V in ${}^{\perp}W$ belongs to the closure of the V-orbit and to $R(Q, \dim V)_{c_W}$. So if the orbit is closed, then V is isomorphic to the sum of simple objects. Conversely, assume that V is a sum of simple objects and let U be the closed orbit in the closure of the orbit of V in $R(Q, \dim V)_{c_W}$. Then there is a one-parameter subgroup $g(t) \in \mathrm{GL}(\dim V), t \in \mathbf{k}^*$, such that $\lim_{t\to 0} g(t)V \subseteq U$. Considering the $g(t)$-eigenspace decomposition of $V(i), i \in Q_0$, one easily sees that that the limit exists means that these eigenspaces yield a filtration of V in ${}^{\perp}W$ such that U is the associated graded space of V. Hence, the Jordan–Hölder factors for U are the same as for V, so V belongs to the closure of the orbit of U, in other words, the orbits are equal. $\square$

Remark 4. Clearly, the theorem implies Proposition 8 above.

Example 2. Let $Q = \mathsf{A}_n\colon \circ \to \circ \to \cdots \to \circ \to \circ$, where n stands for the number of vertices. Let $\varepsilon_1, \ldots, \varepsilon_n$ denote the standard basis of $\mathbf{Z}^n$. It is well known that the indecomposable representations of A_n are the representations $S_{ij}, 1 \leqslant i \leqslant j \leqslant n$, such that $\dim S_{ij} = \varepsilon_i + \cdots + \varepsilon_j$ and these representations are Schurian. It can be directly verified that (and follows, e.g., from [Sh, Theorem 10])

$$\mathrm{Hom}(S_{kl}, S_{ij}) \neq 0 \iff i \leqslant k \leqslant j \leqslant l. \tag{7}$$

Hence, the condition $\mathrm{Hom}(S_{ij}, S_{kl}) = 0 = \mathrm{Hom}(S_{kl}, S_{ij})$ is equivalent to

$$j < k, \text{ or } l < i, \text{ or } i < k \leqslant l < j, \text{ or } k < i \leqslant j < l, \tag{8}$$

in other words, the segments $[i, j]$ and $[k, l]$ are either disjoint sets or one of them contains another in the interior.

Proposition 12. *Let* $V = \bigoplus_{p,q} m_{pq} S_{pq}$ *and set* $I = \{(p,q) \mid m_{pq} > 0\}$. *Then* V *is locally semi-simple if and only if any pair* $(i,j),(k,l) \in I$ *satisfies condition* (8).

Proof. The "only if" part follows from Proposition 8. To prove the "if" part we first observe that the condition $\langle \dim S_{kl}, \dim S_{ij}\rangle = 0$ implies $\mathrm{Hom}(S_{kl}, S_{ij}) = 0$, because of (7). Using condition (8), one can show that the dimensions $\alpha_1, \dots, \alpha_t$ of the representations $S_{pq}, (p,q) \in I$, are linear independent and moreover, the sublattice

$$\langle \alpha_1, \dots, \alpha_t\rangle^{\perp} = \{\beta \in \mathbf{Z}^n \mid \langle \alpha_i, \beta\rangle = 0,\ i = 1, \dots, t\}$$

is generated by $n-t$ linear independent roots. By the above observation the corresponding $n-t$ indecomposable representations belong to $V^{\perp}$; let $R_1, \dots, R_k$ be the simple factors of these. Clearly, $\dim R_1, \dots, \dim R_k$ also generate $\langle \alpha_1, \dots, \alpha_t\rangle^{\perp}$; since homomorphisms between simple objects R_i, R_j are trivial for $i \neq j$, this set of dimensions satisfies condition (8), hence $\dim R_1, \dots, \dim R_k$ are linearly independent, so $k = n - t$ and $\alpha_1, \dots, \alpha_t$ is a basis of the sublattice ${}^{\perp}\langle \dim R_1, \dots, \dim R_{n-t}\rangle$. Set $W = R_1 + \dots + R_k$. By construction, $V \in^{\perp} W$. We claim that $S_{pq}, (p,q) \in I$ are simple objects in ${}^{\perp}W$ and this implies the assertion, thanks to Theorem 11. Indeed, assume that a summand S_{ij} is not simple, i.e., a proper subrepresentation $S' \subseteq S_{ij}$ belongs to ${}^{\perp}W$. Any proper subrepresentation of S_{ij} is isomorphic to S_{kj} with $i < k \leqslant j$, so $\dim S_{kj}$ is a linear combination of $\dim S_{pq}, (p,q) \in I$. Using condition (8), one can easily see this is false. $\square$

5. Decompositions and slices

Let V be a locally semi-simple representation of Q, $\dim V = \alpha$; by Proposition 8 we know $V = \bigoplus_{i=1}^{t} m_i S_i$, where S_i are pairwise nonisomorphic Schurian representations with trivial homomorphism spaces between them. Hence, $\mathrm{Aut}(V) \cong \prod_{i=1}^{t} \mathrm{GL}(m_i)$. Note that the group $\mathrm{Aut}(V)$ and its embedding to $\mathrm{GL}(\alpha)$ are completely determined by the decomposition $\alpha = m_1 \dim S_1 + \dots + m_t \dim S_t$.

Definition 3. A decomposition $\alpha = \sum_{i=1}^{t} m_i \beta_i$ with $\alpha, \beta_1, \dots, \beta_t \in \mathbf{Z}_{+}^{Q_0}$, $m_i \in \mathbf{N}$, is called *locally semi-simple* if for each i there exists a representation S_i such that $\dim S_i = \beta_i$, $\dim \mathrm{Hom}(S_i, S_j) = \delta_{ij}$, and $V = \bigoplus_{i=1}^{t} m_i S_i$ is a locally semi-simple representation. If, moreover, V is generic, then we call this decomposition generic.

Note that a locally semi-simple decomposition determines the isomorphism class of the representation if and only if all the components are real Schur roots. Note also that there can be equal summands $\beta_i = \beta_j = \beta$ in such a decomposition; the condition $\mathrm{Hom}(S_i, S_j) = 0$ implies that β is an imaginary root.

Assume that $V = \bigoplus_{i=1}^{t} m_i S_i$ is locally semi-simple. By Ringel's formula (6) we have $\delta_{ij} - \langle \dim S_i, \dim S_j\rangle = \dim \mathrm{Ext}(S_i, S_j) \geqslant 0$. Le Bruyn and Procesi showed in [LBP] that the slice representation for V semi-simple can be expressed in terms of a quiver. Following [LBP], we introduce a quiver Σ_V with vertices $a_1, \dots, a_t$ corresponding to the summands $S_1, \dots, S_t$ and $\delta_{ij} - \langle \dim S_i, \dim S_j\rangle$ arrows from a_i to a_j; set $\gamma = (m_1, \dots, m_t) \in \mathbf{Z}^{(\Sigma_V)_0}$. It is known (see, e.g., [Kr]) that for any representation W the normal space to the isomorphism class of W at W, $R(Q,W)/T_W \mathrm{GL}(\dim W)W$, is isomorphic to $\mathrm{Ext}(W,W)$. Hence, we get a helpful form of the slice representation σ_V

of V (the same as in [LBP] for the semi-simple case):

$$\begin{aligned}(\mathrm{GL}(\alpha)_V, \sigma_V) &= (\mathrm{Aut}(V), \mathrm{Ext}(V, V)) \\ &= (\prod_{i=1}^{t} \mathrm{GL}(m_i), \bigoplus_{i,j=1}^{t} \mathrm{Ext}(S_i, S_j) \otimes \mathrm{Hom}(\mathbf{k}^{m_i}, \mathbf{k}^{m_j})) = (\mathrm{GL}(\gamma), R(\Sigma_V, \gamma)). \quad (9)\end{aligned}$$

Let $D_V : \mathbf{Z}^{(\Sigma_V)_0} \to \mathbf{Z}^{Q_0}$ denote the linear map taking the ith basis vector of $\mathbf{Z}_+^{(\Sigma_V)_0}$ to $\dim S_i, i = 1, \ldots, t$. Note that $D_V(\gamma) = \alpha$. The definition of the Euler form yields:

Proposition 13. *D_V preserves the Euler form: $\langle D_V(\gamma_1), D_V(\gamma_2)\rangle = \langle \gamma_1, \gamma_2\rangle$.*

Proposition 14. *Consider a decomposition $\gamma = \sum_{j=1}^{s} p_j \rho_j$ and the corresponding decomposition $\alpha = \sum_{j=1}^{s} p_j D_V(\rho_j)$ of α.*

1. *If the decomposition of γ is generic, then that of α is.*
2. *If the decomposition of γ is locally semi-simple, then that of α is.*
3. *If the decomposition of γ is generic locally semi-simple, then that of α is.*

Proof. A general remark is that by Luna's slice theorem for any representation $W \in R(\Sigma_V, \gamma)$ there exists a representation $V_1 \in R(Q, \alpha)$ with $\mathrm{Aut}(V_1) = \mathrm{Aut}(W)$, where $\mathrm{Aut}(W) \subseteq \mathrm{GL}(\gamma)$ is embedded to $\mathrm{GL}(\alpha)$ via the embedding $\mathrm{GL}(\gamma) = \mathrm{Aut}(V) \subseteq \mathrm{GL}(\alpha)$. Therefore, if the maximal torus of $\mathrm{Aut}(W)$ corresponds to the given decomposition of γ, then the maximal torus of $\mathrm{Aut}(V_1)$ corresponds to the given decomposition of α. Now **1** follows from the fact that the generic decompositions are determined by generic stabilizers and by Luna's slice theorem if W is generic for $(\mathrm{GL}(\alpha)_V, \sigma_V)$, then V_1 is generic for $(\mathrm{GL}(\alpha), R(Q, \alpha))$. By Proposition 4, W is locally semi-simple implies V_1 is locally semi-simple, so we proved **2**. Applying Corollary 6, we also get **3**. □

Proposition 15.

1. *If $\gamma_1 \in \mathbf{Z}_+^{(\Sigma_V)_0}$ is a (real) Schur root, then $D_V(\gamma_1)$ is.*
2. *If $\gamma_1, \gamma_2 \in \mathbf{Z}_+^{(\Sigma_V)_0}$ are real Schur roots, $W_1 \in R(\Sigma_V, \gamma_1)$, $W_2 \in R(\Sigma_V, \gamma_2)$, and $V_1 \in R(Q, D_V(\gamma_1)), V_2 \in R(Q, D_V(\gamma_2))$ are Schurian representations, then* $\mathrm{Ext}(W_1, W_2) = 0 = \mathrm{Ext}(W_2, W_1)$ *implies* $\mathrm{Ext}(V_1, V_2) = 0 = \mathrm{Ext}(V_2, V_1)$ *and* $\mathrm{Hom}(W_1, W_2) = 0 = \mathrm{Hom}(W_2, W_1)$ *implies* $\mathrm{Hom}(V_1, V_2) = 0 = \mathrm{Hom}(V_2, V_1)$.

Proof. Note that the map D_V depends on the indecomposable summands of V not on V itself. So in **1** we may assume that $\dim V = D_V(\gamma_1)$. Then by Proposition 14.1 $\dim V = \dim V$ is the generic decomposition, that is, $D_V(\gamma_1)$ is a Schur root. By Proposition 13, $q_Q(D_V(\gamma_1)) = q_{\Sigma_V}(\gamma_1)$. So γ_1 is real implies $D_V(\gamma_1)$ is. In **2** we assume $\dim V = \alpha = D_V(\gamma_1) + D_V(\gamma_2)$. Then by [Kac] the condition $\mathrm{Ext}(W_1, W_2) = 0 = \mathrm{Ext}(W_2, W_1)$ implies that $\gamma = \gamma_1 + \gamma_2$ is the generic decomposition. Then, by Proposition 14.1, $\alpha = D_V(\gamma_1) + D_V(\gamma_2)$ is the generic decomposition so again applying [Kac] we get $\mathrm{Ext}(V_1, V_2) = 0 = \mathrm{Ext}(V_2, V_1)$. The condition $\mathrm{Hom}(W_1, W_2) = 0 = \mathrm{Hom}(W_2, W_1)$ yields $\mathrm{Aut}(W_1 + W_2)$ is the corresponding embedding of $(\mathbf{k}^*)^2$ to $\mathrm{GL}(\gamma)$. By Luna's slice theorem there exists a representation $V' \in R(Q, \alpha)$ with $\mathrm{Aut}(V')$ being the image of $\mathrm{Aut}(W_1 + W_2)$ under the embedding $\mathrm{GL}(\gamma) \subseteq \mathrm{GL}(\alpha)$. This means that $V' = V_1' + V_2'$ with V_1', V_2' indecomposable of dimensions $D_V(\gamma_1), D_V(\gamma_2)$ and $\mathrm{Hom}(V_1', V_2') = 0 = \mathrm{Hom}(V_2', V_1')$. Clearly, this is equivalent to what we assert. □

Now we want to describe generic and generic locally semi-simple decompositions for Q being the equioriented A_n-quiver considered in Example 2. Since Q is of finite representation type, there is a dense isomorphism class in $R(Q,\alpha)$ for all α so V is generic is equivalent to V having the dense orbit or $\mathrm{Ext}(V,V)=0$. So we are looking for a sum of S_{ij} with trivial Ext-spaces for summands. Using Ringel's formula (6) and (7), we see that the condition $\mathrm{Ext}(S_{ij},S_{kl})=0=\mathrm{Ext}(S_{kl},S_{ij})$ is equivalent to

$$j<k-1, \text{ or } l<i-1, \text{ or } i\leqslant k\leqslant l\leqslant j, \text{ or } k\leqslant i\leqslant j\leqslant l, \tag{10}$$

so either the distance between the segments $[i,j]$ and $[k,l]$ is at least 2, or one of them contains another. This property yields a simple algorithm for calculating generic decomposition, exactly the same as in [Ri, Lemma 3.3].

Algorithm 16. *For $\alpha\in\mathbf{Z}_+^{Q_0}$ set $m=\min\{\alpha_a\mid a\in Q_0\}$. If $m>0$, then $\alpha=m(1,\dots,1)+\pi,\pi\in\mathbf{Z}_+^{Q_0}$, and the generic decomposition of α is $\alpha=m(1,\dots,1)+$ the terms of the generic decomposition of π. Otherwise, if $\alpha_t=0$, then $\alpha=\pi+\sigma$, $\pi=(\alpha_1,\dots,\alpha_{t-1},0,\dots,0)$, $\sigma=(0,\dots,0,\alpha_{t+1},\dots,\alpha_n)$, and the generic decomposition of α is that of π plus that of σ for the appropriate proper subquivers.*

Now we consider locally semi-simple decompositions. The following observation follows from Proposition 12.

Proposition 17. *If $0<m=\alpha_t<\alpha_i$ for any $i\neq t$, then any locally semi-simple representation V of dimension α decomposes as $V=mS_{tt}+$ other summands.*

Algorithm 18. *For $\alpha\in\mathbf{Z}_+^{Q_0}$ set $m=\min\{\alpha_a\mid a\in Q_0\}$, $t=\min\{a\in Q_0\mid\alpha_a=m\}$, and $s=\max\{a\in Q_0\mid\alpha_a=m\}$. If $m>0$, then $\alpha=\pi+\sigma+\rho+m\dim S_{ts}$, where $\pi=(\alpha_1,\dots,\alpha_{t-1},0,\dots,0)$, $\sigma=(0,\dots,0,\alpha_{s+1},\dots,\alpha_n)$, $\rho\in\langle\varepsilon_{t+1},\dots,\varepsilon_{s-1}\rangle$, and the generic locally semi-simple decomposition of α is $\alpha=m\dim S_{ts}+$ the terms of the decompositions of π,σ,ρ for the appropriate proper subquivers. Otherwise, if $\alpha_t=0$, then $\alpha=\pi+\sigma$, $\pi=(\alpha_1,\dots,\alpha_{t-1},0,\dots,0)$, $\sigma=(0,\dots,0,\alpha_{t+1},\dots,\alpha_n)$, and the generic locally semi-simple decomposition of α is that of π plus that of σ for the appropriate proper subquivers.*

Proof. The second case $m=0$ is obvious. In the first case set $\mu=\alpha-m\dim S_{ts}$ and take a representation $V=mS_{ts}+\mu_1S_{11}+\ldots+\mu_nS_{nn},\dim V=\alpha$. Since $\mu_t=\mu_s=0$, by Proposition 12, V is locally semi-simple. The Ext-spaces for the summands of V are nonzero and one-dimensional only for $\mathrm{Ext}(S_{ii},S_{i+1i+1})$, $\mathrm{Ext}(S_{t-1t-1},S_{ts})$, and $\mathrm{Ext}(S_{ts},S_{s+1s+1})$. So the graph Σ_V is the disjoint union of A_{n-s+t} on the vertices corresponding to $S_{11},\dots,S_{t-1t-1},S_{ts},S_{s+1s+1},S_{nn}$ and A_{s-t-1} (if $s-t\geqslant 2$) on the vertices corresponding to $S_{t+1t+1},\dots,S_{s-1s-1}$. The induced dimension γ is (π,m,σ) on A_{n-s+t} and ρ on A_{s-t-1}. By Proposition 14.3, the generic locally semi-simple decomposition for α is the sum of that for ρ and that for (π,m,σ). By Proposition 17, the generic locally semi-simple decomposition for (π,m,σ) is $m\varepsilon_t+$ the sum of the decompositions for π and σ. Applying the map D_V to the summands of the decomposition, we conclude the proof. $\square$

6. Luna–Richardson theorem for semi-invariants of quivers

Now we want to apply the Luna–Richardson Theorem 7 for semi-invariants to representations of quivers. As we explained in the previous section, generic locally semi-simple representations of dimension α are described in terms of the generic locally semi-simple decomposition of α: $\alpha = \sum_{i=1}^{t} m_i\beta_i$. For a locally semi-simple decomposition we observed above that equal summands $\beta_i = \beta_j$ must be imaginary roots. Besides, in the generic case, the multiplicity m_j of any imaginary root β_j is equal to 1. Indeed, if we have a locally semi-simple representation $V = \bigoplus_{i=1}^{t} m_iS_i$, then by Theorem 11, $S_1, \ldots, S_t$ are simple objects in the category ${}^\perp W$ for some W. Clearly, being perpendicular to W and being a simple object in ${}^\perp W$ are open conditions on $R(Q, \beta_j)$. Since $R(Q, \beta_j)$ contains infinitely many isomorphism classes of indecomposable representations, we could replace m_jS_j with $m_j > 1$ by a sum of m_j generic representations of dimension β_j to get a locally semi-simple representation with a smaller automorphism group.

Theorem 19.

1. *The generic locally semi-simple decomposition has the form*

$$\alpha = \underbrace{\delta_1 + \ldots + \delta_1}_{p_1 \text{ summands}} + \ldots + \underbrace{\delta_r + \ldots + \delta_r}_{p_r \text{ summands}} + m_1\beta_1 + \ldots + m_s\beta_s, \tag{11}$$

where $\delta_1, \ldots, \delta_r$ are pairwise nonequal imaginary Schur roots and $\beta_1, \ldots, \beta_s$ are pairwise nonequal real Schur roots.

2. *Generic stabilizer H of a locally semi-simple point is isomorphic to $\prod_{i=1}^{r}(\mathbf{k}^*)^{p_i} \times \prod_{j=1}^{s} \mathrm{GL}(m_j)$. $N_{\mathrm{GL}(\alpha)}(H)$ is the natural embedding to $\mathrm{GL}(\alpha)$ of the group*

$$\prod_{i=1}^{r}(S_{p_i} \ltimes \mathrm{GL}(\delta_i)^{p_i}) \times \prod_{j=1}^{s} \mathrm{GL}(\beta_j)\,\mathrm{GL}(m_j), \tag{12}$$

where S_{p_i} permutes p_i factors of the direct power $\mathrm{GL}(\delta_i)^{p_i}$, $i = 1, \ldots, r$, $\mathrm{GL}(\beta_j)$ and $\mathrm{GL}(m_j)$ commute, and their intersection is the center of $\mathrm{GL}(m_j)$, $j = 1, \ldots, s$. The linear group $(N_{\mathrm{GL}(\alpha)}(H)/H, R(Q, \alpha)^H)$ is isomorphic to

$$\bigoplus_{i=1}^{r} \underbrace{((\mathrm{GL}(\delta_i), R(Q, \delta_i)) \oplus \cdots \oplus ((\mathrm{GL}(\delta_i), R(Q, \delta_i))}_{S_{p_i} \text{ permutes } p_i \text{ summands}} \oplus \bigoplus_{j=1}^{s}(\mathrm{GL}(\beta_j), R(Q, \beta_j)). \tag{13}$$

3. $\mathbf{k}[R(Q, \alpha)]^{\mathrm{SL}(\alpha)} \cong \mathbf{k}[R(Q, p_1\delta_1) \oplus \cdots \oplus R(Q, p_r\delta_r) \oplus R(Q, \beta_1) \oplus \cdots \oplus R(Q, \beta_s)]^G$, *where $G \subseteq \mathrm{GL}(p_1\delta_1) \times \cdots \times \mathrm{GL}(p_r\delta_r) \times \mathrm{GL}(\beta_1) \times \cdots \times \mathrm{GL}(\beta_s)$ consists of the elements such that for each vertex the product of determinants is 1. Moreover, the generic G-orbit in $R(Q, p_1\delta_1) \oplus \cdots \oplus R(Q, p_r\delta_r) \oplus R(Q, \beta_1) \oplus \cdots \oplus R(Q, \beta_s)$ is closed.*
4. $\mathbf{k}[R(Q, \alpha)]^{\mathrm{SL}(\alpha)} \cong \bigoplus_{\chi \in \Lambda} \mathbf{k}[R(Q, p_1\delta_1)]_\chi^{(\mathrm{GL}(p_1\delta_1))} \otimes \cdots \otimes \mathbf{k}[R(Q, p_r\delta_r)]_\chi^{(\mathrm{GL}(p_r\delta_r))}$, *where $\Lambda = \{\chi \in \mathbf{Z}^{Q_0} \mid \mathbf{k}[R(Q, \beta_j)]_\chi^{(\mathrm{GL}(\beta_j))} \neq 0,\ j = 1, \ldots, s\}$.*

Proof. Assertion 1 is shown above. The form of H in 2 follows from (11). For any vertex $a \in Q_0$ each summand $\rho = \delta_i$ or β_j yields an isotypical component of the H-module $\mathbf{k}^{\alpha_a}$ being the sum of ρ_a irreducible factors of type $(\mathrm{GL}_m, \mathbf{k}^m)$, where $m = 1$ for δ_i and $m = m_j$ for β_j. These isotypical components are stable with respect to the centralizer $Z_{\mathrm{GL}(\alpha)}(H)$ of H and so we get a factor $\mathrm{GL}(\rho)$ of $Z_{\mathrm{GL}(\alpha)}(H)$ such that $\mathrm{GL}(\rho)$ and $\mathrm{GL}(m)$ have the center of $\mathrm{GL}(m)$ as the intersection. Furthermore, ρ yields a direct summand of $R(Q,\alpha)^H$ isomorphic to $R(Q,\rho)$ such that the corresponding factor $\mathrm{GL}(\rho)$ of $Z_{\mathrm{GL}(\alpha)}(H)$ acts on this summand naturally and the other factors act trivially. Elements of $N_{\mathrm{GL}(\alpha)}(H)\backslash HZ_{\mathrm{GL}(\alpha)}(H)$ induce an outer automorphism of the group H and the corresponding permutation of the isotypical components in each space $\mathbf{k}^{\alpha_a}$. Clearly, isotypical components corresponding to nonequal summands cannot be permuted, so $N_{\mathrm{GL}(\alpha)}(H)$ is contained in the extension of $HZ_{\mathrm{GL}(\alpha)}(H)$ by the groups S_{p_i}, $i=1,\dots,r$, and the latter extension does normalize H, so 2 is proved.

By Theorem 7, $\mathbf{k}[R(Q,\alpha)]^{\mathrm{SL}(\alpha)} \cong \mathbf{k}[R(Q,\alpha)^H]^{N_{\mathrm{SL}(\alpha)}(H)}$. Formula (12) presents the group $N_{\mathrm{GL}(\alpha)}(H)$ as a direct product of $r+s$ groups. Consider a subgroup $N \lhd N_{\mathrm{GL}(\alpha)}(H)$ consisting of the elements such that each of $r+s$ components is unimodular in $\mathbf{k}^{\alpha_a}$ for any $a\in Q_0$. Clearly, $N \lhd N_{\mathrm{SL}(\alpha)}(H)$ and $(N, R(Q,\alpha)^H)$ is also a direct sum of $r+s$ independent linear groups. So we have $\mathbf{k}[R(Q,\alpha)^H]^N \cong \big(\bigotimes_{i=1}^r \mathbf{k}[R(Q,\delta_i)\oplus\cdots\oplus R(Q,\delta_i)]^N\big)\otimes\big(\bigotimes_{j=1}^s \mathbf{k}[R(Q,\beta_j)]^N\big)$. Note that $p_i\delta_i=\delta_i+\cdots+\delta_i$ is the generic locally semi-simple decomposition of $p_i\delta_i$ and $(\mathbf{k}^*)^{p_i}=H\cap\mathrm{GL}(p_i\delta_i)$ is the generic stabilizer of a locally semi-simple point in $R(Q,p_i\delta_i)$. Moreover, the restriction of N to $R(Q,\delta_i)\oplus\cdots\oplus R(Q,\delta_i)$ is $N_{\mathrm{SL}(p_i\delta_i)}((\mathbf{k}^*)^{p_i})$. Therefore, by Theorem 7, $\mathbf{k}[R(Q,\delta_i)\oplus\cdots\oplus R(Q,\delta_i)]^N$ and $\mathbf{k}[R(Q,p_i\delta_i)]^{\mathrm{SL}(p_i\delta_i)}$ are isomorphic as $\Xi(\mathrm{GL}(p_i\delta_i))$-graded algebras.

Next, fix $j\in\{1,\dots,s\}$, set $m=m_j,\beta=\beta_j$. Using Theorem 7 as above, we get $\mathbf{k}[R(Q,\beta)]^N \cong \mathbf{k}[R(Q,m\beta)]^{\mathrm{SL}(m\beta)}$. The linear group $(\mathrm{GL}(\beta),R(Q,\beta))$ is prehomogeneous, because β is Schurian, hence $\mathbf{k}[R(Q,\beta)]^{\mathrm{SL}(\beta)}$ is generated by semi-invariants with linear independent weights. Denote by U a Schurian representation of dimension β. Then by [Sch] (or by [DW]), $\mathbf{k}[R(Q,\beta)]^{\mathrm{SL}(\beta)}$ is generated by $c_{S_1},\dots,c_{S_n}$ such that $S_1,\dots,S_n$ are all the simple objects in $U^\perp$. Clearly, $(mU)^\perp=U^\perp$, hence $\mathbf{k}[R(Q,m\beta)]^{\mathrm{SL}(m\beta)}$ is also generated by the same determinantal semi-invariants but as functions on $R(Q,m\beta)$. By [DW, Lemma 1], $c_{S_i}(mU)=c_{S_i}(U)^m$ for any i, hence, $\mathbf{k}[R(Q,\beta)]^N$ is generated by the mth powers of the generators of $\mathbf{k}[R(Q,\beta)]^{\mathrm{SL}(\beta)}$ and we have an isomorphism of $\mathbf{k}[R(Q,\beta)]^{\mathrm{SL}(\beta)}$ onto $\mathbf{k}[R(Q,\beta)]^N$. On the other hand, for each $a\in Q_0$, the determinant $\det_a$ on $\mathrm{GL}(\alpha)$ restricts to the image of $\mathrm{GL}(\beta)$ in $N_{\mathrm{GL}(\alpha)}(H)$ as the mth power of the corresponding determinant on $\mathrm{GL}(\beta)$. So the above isomorphism preserves grading. Thus we proved an isomorphism

$$\mathbf{k}[R(Q,\alpha)^H]^N \cong \bigotimes_{i=1}^r \mathbf{k}[R(Q,p_i\delta_i)]^{\mathrm{SL}(p_i\delta_i)} \otimes \bigotimes_{j=1}^s \mathbf{k}[R(Q,\beta_j)]^{\mathrm{SL}(\beta_j)}, \tag{14}$$

of a $\Xi(N_{\mathrm{GL}(\alpha)}(H)/N)$-graded algebra onto a $\Xi(\prod_{i=1}^r \mathrm{GL}(p_i\delta_i)\times\prod_{j=1}^s \mathrm{GL}(\beta_j))$-graded algebra. The subalgebra $\mathbf{k}[R(Q,\alpha)^H]^{N_{\mathrm{SL}(\alpha)}(H)}$ is the sum of homogeneous components of $\mathbf{k}[R(Q,\alpha)^H]^N$ corresponding to the weights vanishing on $N_{\mathrm{SL}(\alpha)}(H)$ and the same is true for the right-hand side of (14) and the group G. Clearly, the characters of $\prod_{i=1}^r \mathrm{GL}(p_i\delta_i)\times\prod_{j=1}^s \mathrm{GL}(\beta_j)$ vanishing on G correspond to the characters of $N_{\mathrm{GL}(\alpha)}(H)/N$ vanishing on $N_{\mathrm{SL}(\alpha)}(H)$, so $\mathbf{k}[R(Q,\alpha)^H]^{N_{\mathrm{SL}(\alpha)}(H)}$ corresponds under the isomorphism to the algebra of G-invariants. By Theorem 7, generic $N_{\mathrm{SL}(\alpha)}(H)$-orbits are closed in $R(Q,\alpha)^H$; clearly, this implies the same for the G-orbits, and 3 is proved.

Let $\chi=\sigma_1+\cdots+\sigma_r+\chi_1+\cdots+\chi_s$ be a character of $\prod_{i=1}^r \mathrm{GL}(p_i\delta_i)\times\prod_{j=1}^s \mathrm{GL}(\beta_j)$. By definition of G, χ vanishes on G if and only if χ is a linear combination of $\chi_a, a\in Q_0$, where χ_a is the sum of determinants on $\mathrm{GL}(\gamma_a)$ with γ running over all dimensions $\delta_1, \ldots, \delta_r, \beta_1, \ldots, \beta_s$ such that $\gamma_a \neq 0$. This means that $\sigma_1, \ldots, \sigma_r, \chi_1, \ldots, \chi_s$ are equal as elements of $\mathbf{Z}^{Q_0}$. So the algebra of G-invariants is equal to

$$\bigoplus_{\chi\in\Lambda}\left(\bigotimes_{i=1}^{r}\mathbf{k}[R(Q,p_i\delta_i)]_\chi^{(\mathrm{GL}(p_i\delta_i))}\right)\otimes\left(\bigotimes_{j=1}^{r}\mathbf{k}[R(Q,\beta_j)]_\chi^{(\mathrm{GL}(\beta_j))}\right).$$

On the other hand, β_j are real Schur roots, hence, $\dim \mathbf{k}[R(Q,\beta_j)]_\chi^{(\mathrm{GL}(\beta_j))} = 1$ for any j and $\chi \in \Lambda$ and Λ is generated by linear independent generators. So choosing a nonzero semi-invariant in each $\mathbf{k}[R(Q,\beta_j)]_\mu^{(\mathrm{GL}(\beta_j))}$ for each j and each generator μ of Λ, we define an isomorphism of the algebra $\bigoplus_{\chi\in\Lambda}\bigotimes_{i=1}^{r}\mathbf{k}[R(Q,p_i\delta_i)]_\chi^{(\mathrm{GL}(p_i\delta_i))}$ onto $\bigoplus_{\chi\in\Lambda}\left(\bigotimes_{i=1}^{r}\mathbf{k}[R(Q,p_i\delta_i)]_\chi^{(\mathrm{GL}(p_i\delta_i))}\right)\otimes\left(\bigotimes_{j=1}^{r}\mathbf{k}[R(Q,\beta_j)]_\chi^{(\mathrm{GL}(\beta_j))}\right)$. □

7. Regular representations of tame quivers

The tame quivers can be described by several equivalent conditions; in particular, these are the quivers with the underlying graph being an extended Dynkin diagram of type $\widetilde{\mathsf{A}_n}, \widetilde{\mathsf{D}_n}, \widetilde{\mathsf{E}_6}, \widetilde{\mathsf{E}_7}, \widetilde{\mathsf{E}_8}$ (the number of vertices is in all cases the subscript + 1). So let Q be a tame quiver and assume additionally that Q does not have oriented cycles (this is a restriction only for the underlying graph being $\widetilde{\mathsf{A}_n}$).

For quivers without oriented cycles Bernstein, Gelfand, and Ponomarev introduced in [BGP] *Coxeter functors* C^+ and C^- (defined not uniquely) acting on representations of Q. The corresponding linear transformation c is defined by the rule $c(\alpha) = \dim C^+V$ for a representation V of dimension α; note that $\dim C^-(V) = c^{-1}\dim V$. By construction, c is a Coxeter element in the Weyl group corresponding to the underlying graph of Q. Indecomposable representations V such that for some $n \in \mathbf{N}$ holds $C^{+n}V = 0$ are called *preprojective*, the *preinjective* representations being defined symmetrically. Representation having neither preprojective nor preinjective direct summands are called *regular*.

For tame quivers regular indecomposable representations V can be described in terms of a certain *defect* function σ such that V is regular if and only if $\sigma(\dim V) = 0$. This σ is presented explicitly in [Ri] (for special orientations) and one can easily check in all cases that

$$\sigma(\alpha) = \langle \alpha, \delta \rangle, \tag{15}$$

where δ is the nondivisible imaginary root such that $\langle \delta, \delta \rangle = 0$.

In [Ri] Ringel proved that the regular representations form an Abelian subcategory $\mathcal{R}$ closed under direct sums, direct summands, homomorphisms, extensions, etc. Note that by definition and (15) the simple objects in $\mathcal{R}$ are the δ-stable representations. These simple objects are described in [Ri], as follows. In dimension δ there is a one-parameter family of simple regular representations and these are called *homogeneous*. Further, the set $\{e_i \mid i \in I\}$ of the dimensions of simple regular representations different from δ is finite and consists of real Schur roots so that for any $i \in I$ there is a unique simple regular representation E_i of dimension e_i, up to isomorphism. Finally, the set $\{e_i \mid i \in I\}$ is stable with respect to the Coxeter transformation c. Moreover, c has at most three orbits in this set and the sum of dimensions over each c-orbit is equal δ.

Proposition 20. *Semi-simple regular representations are locally semi-simple.*

Proof. Let V be a semi-simple regular representation. By the above description of the simple regular representations, $V = S_1 + \cdots + S_p + \sum_{i\in I} p_i E_i$, where $S_1, \ldots, S_p$ are homogeneous representations. Take S to be a homogeneous simple object nonisomorphic to $S_1, \ldots, S_p$. Since there are no nontrivial homomorphisms between nonisomorphic simple objects, $S_1, \ldots, S_p$ and E_i for all $i \in I$ belong to ${}^{\perp}S$. Since all these are δ-stable, these are also simple in ${}^{\perp}S$, by Proposition 10. So the assertion follows from Theorem 11. □

In (9) we described the slice at a locally semi-simple point V in terms of the quiver Σ_V with dimension γ. For a semi-simple regular representation V, Σ_V has a simple structure. Denote by $E(Q)$ the quiver with $E(Q)_0 = I$ and an arrow from i to j for each pair (i, j) such that $c(e_i) = e_j$. Note that $E(Q)$ is a disjoint union of circular quivers.

Proposition 21. *Consider a representation $V = m_1 S_1 + \cdots + m_q S_q + \sum_{i\in I} r_i E_i$, where $S_1, \ldots, S_q$ are pairwise nonisomorphic homogeneous representations. Then (Σ_V, γ) is a disjoint union of quivers having a unique arrow-loop adjacent to the vertex corresponding to S_t, $t = 1, \ldots, q$, with dimension m_q and $(E(Q), r_i, i \in I)$.*

Proof. The quiver Σ_V is defined in terms of the Euler form or the Ext-spaces for the summands of V. Since $\langle \delta, \delta\rangle = 0$ and $\langle \delta, e_i\rangle = \langle e_i, \delta\rangle = 0$, each of the vertices corresponding to $S_1, \ldots, S_q$ is incident to the unique arrow-loop. It remains to describe $\operatorname{Ext}(E_i, E_j)$. Applying the formula (see, e.g., [Ri, p. 219]):

$$\dim \operatorname{Ext}(U, W) = \dim \operatorname{Hom}(W, C^+U), \tag{16}$$

we get $\dim \operatorname{Ext}(E_i, E_j) = \dim \operatorname{Hom}(E_j, C^+E_i)$. Since $\dim \operatorname{Hom}(E_i, E_j) = \delta_{ij}$, we have that $\dim \operatorname{Ext}(E_i, E_j)$ is either 0 or 1, the latter being equivalent to $c(e_i) = e_j$. □

Denote by $\mathcal{D}_r$ the dimensions of regular representations. If $\alpha \in \mathcal{D}_r$, then α decomposes as $\alpha = p\delta + \sum_{i\in I} p_i e_i$ and there is a unique decomposition of such a type with an additional condition that for every c-orbit there is an element j such that $p_j = 0$. Ringel called this decomposition *canonical*. Consider representations $V = S_1 + \cdots + S_p + \sum_{i\in I} p_i E_i$, where $S_1, \ldots, S_p$ are pairwise nonisomorphic homogeneous representations. These are generic semi-simple regular representations of dimension α, at least in the sense that their automorphism groups are of the minimal possible dimension among the semi-simple regular representations. Indeed, for a representation W having a direct summand E_i for each i from a c-orbit we can replace the sum of E_i over this orbit by a homogeneous summand to get a semi-simple regular representation W', $\dim W' = \alpha$, with $\dim \operatorname{Aut}(W') < \dim \operatorname{Aut}(W)$.

Now we have three ingredients allowing us to calculate the generic and the generic locally semi-simple decompositions for $\alpha \in \mathcal{D}_r$. First, consider a generic semi-simple regular representation V corresponding to the canonical decomposition of α. Proposition 21 yields a description of Σ_V and we see that the group $(\mathrm{GL}(\gamma), R(\Sigma_V, \gamma))$ is isomorphic, up to a p-dimensional invariant subspace to $(\mathrm{GL}(\overline{p}), R(E(Q), \overline{p}))$, where $\overline{p}$ is the tuple of $p_i, i \in I$. Moreover, thanks to the condition $p_j = 0$ for at least one j in each c-orbit, the group $(\mathrm{GL}(\overline{p}), R(E(Q), \overline{p}))$ is isomorphic to a direct sum of groups

$(\mathrm{GL}(\gamma^t), R(A_{n_t}, \gamma^t))$. Second, Proposition 14 reduces both decompositions for Q and α to the same for $E(Q)$ and $\overline{p}$. Third, Algorithms 16 and 18 yield both decompositions for $E(Q)$ and $\overline{p}$.

In what concerns the generic decomposition our algorithm recovers that by Ringel from [Ri, Theorem 3.5]. It should be noted, however, that Ringel used an equivalence of categories instead of the slice theorem.

Example 3. Consider the quiver $Q = \widetilde{E_6}$ (over each vertex we placed the index):

$$\begin{tikzcd}[column sep=small] \widetilde{\mathsf{E}_6}: & \overset{1}{\circ} \arrow[r] & \overset{2}{\circ} \arrow[r] & \overset{7}{\circ} & \overset{4}{\circ} \arrow[l] & \overset{3}{\circ} \arrow[l] \\ & & \overset{5}{\circ} \arrow[r] & \overset{6}{\circ} \arrow[u] & & \end{tikzcd}$$

We have $\delta = (1,2,1,2,1,2,3)$ so that $\sigma(\alpha) = 3\alpha_7 - \alpha_1 - \cdots - \alpha_6$. The sequence of the vertices in the order defined by the indices is *admissible* in the sense of [BGP], i.e., for any arrow φ holds $h\varphi > t\varphi$. So the composition $C^+ = R_1^+ R_2^+ \cdots R_7^+$ of the reflection functors at *sinks* is a Coxeter functor. Hence we have $c = r_1 r_2 \cdots r_7$, where r_i is the reflection at vertex i. There are three c-orbits of dimensions of simple regular representations: $e_3 \to e_2 \to e_1 \to e_3$, $e_6 \to e_5 \to e_4 \to e_6$, $e_8 \to e_7 \to e_8$:

$$e_1 = (1,1,0,1,0,0,1); \quad e_2 = (0,0,1,1,0,1,1); \quad e_3 = (0,1,0,0,1,1,1); \tag{17}$$

$$e_4 = (1,1,0,0,0,1,1); \quad e_5 = (0,0,0,1,1,1,1); \quad e_6 = (0,1,1,1,0,0,1); \tag{18}$$

$$e_7 = (0,1,0,1,0,1,1); \quad e_8 = (1,1,1,1,1,1,2). \tag{19}$$

For example, take $\alpha = (6,10,7,14,5,9,17)$. The canonical decomposition of α is $\alpha = 2\delta + 3e_1 + 2e_2 + 2e_5 + 2e_6 + e_8$. So $(E(Q), \gamma)$ is the direct sum of $(\mathsf{A}_2, (2,3))$, $(\mathsf{A}_2, (2,2))$, and $(\mathsf{A}_1, (1))$. Applying Algorithm 16, we get the generic decomposition for $(E(Q), \gamma)$: $(\mathsf{A}_2, 2(1,1) + (0,1))$, $(\mathsf{A}_2, 2(1,1))$, and $(\mathsf{A}_1, (1))$. So by Proposition 14.1, the generic decomposition of α is $\alpha = 2\delta + 2(e_1 + e_2) + e_1 + 2(e_5 + e_6) + e_8$, where $e_1 + e_2$ and $e_5 + e_6$ are real Schur roots. Next, applying Algorithm 18, we get the generic locally semi-simple decomposition for $(E(Q), \gamma)$: $(\mathsf{A}_2, 2(1,0) + 3(0,1))$, $(\mathsf{A}_2, 2(1,1))$, and $(\mathsf{A}_1, (1))$. So by Proposition 14.3 the generic locally semi-simple decomposition of α is $\alpha = 2\delta + 3e_1 + 2e_2 + 2(e_5 + e_6) + e_8$.

8. Semi-invariants of tame quivers

The algebras of semi-invariants of tame quivers Q have been studied in several papers including [Ri], [HH], [SchW]. In [SkW, Theorem 21] Skowronski and Weyman proved that $\mathbf{k}[R(Q,\alpha)]^{\mathrm{SL}(\alpha)}$ is a *complete intersection* for any α; moreover, in most cases $\mathbf{k}[R(Q,\alpha)]^{\mathrm{SL}(\alpha)}$ is a polynomial algebra and in all other cases is a hypersurface.

Note that after [Kac] it is known that the reflection functors give rise to so-called *castling transforms* of semi-invariants, so given a description of semi-invariants for Q and α, one can describe the semi-invariants for any quiver and dimension obtained by reflection functors. In particular, one may fix a convenient orientation for Q (in the case of $\widetilde{A_n}$, one of the convenient orientations). If $\alpha \notin \mathcal{D}_r$, then by [Ri, Theorem 3.2], $R(Q,\alpha)$ contains a dense orbit, hence $\mathbf{k}[R(Q,\alpha)]^{\mathrm{SL}(\alpha)}$ is a polynomial algebra by the theorem of Sato and Kimura [SK]. Moreover, one can always apply one of the Coxeter functors C^+ or C^- and describe the semi-invariants of Q in dimension α in terms of the castling transforms of those in dimension $\beta = c(\alpha)$ or $c^{-1}(\alpha)$, respectively. It is well known that for $\alpha \notin \mathcal{D}_r$ this process is not cyclic, and in the end we reduce the question to a dimension vector for which the semi-invariants are obvious (see an example of such an approach in [SchW] for the $\widetilde{\mathsf{D}_4}$-quiver). That is why we may and will assume from now on that $\alpha = p\delta + \sum_{i\in I} p_i e_i \in \mathcal{D}_r$.

Ringel described the field $\mathbf{k}(R(Q,\alpha))^{\mathrm{GL}(\alpha)}$ of invariants. Namely, he constructed semi-invariants $f_0, \dots, f_p$ of weight σ and proved in [Ri, Theorem 4.1] that the fractions $f_1/f_0, \dots, f_p/f_0$ generate $\mathbf{k}(R(Q,p\delta))^{\mathrm{GL}(p\delta)}$. Moreover, it is stated in [Ri, p. 237] that $f_0, \dots, f_p$ form a basis of $\mathbf{k}[R(Q,p\delta)]_\sigma^{(\mathrm{GL}(p\delta))}$ and one can actually deduce this from the proof of [Ri, Theorem 4.1].

First consider the homogeneous case $\alpha = p\delta$. The generators of $\mathbf{k}[R(Q,\alpha)]^{\mathrm{SL}(\alpha)}$ can be obtained using the following corollary of the results from [DW].

Proposition 22. *If $W \in V^\perp$ and $m_1S_1 + m_2S_2 + \cdots + m_tS_t$ is the sum of the Jordan–Hölder factors of W in $V^\perp$, then $c_W = c_{S_1}^{m_1} c_{S_2}^{m_1} \cdots c_{S_t}^{m_t}$.*

Proof. We have a filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_d = W$ such that $W_j \in V^\perp$, $W_j/W_{j-1} \cong S_p$, $j = 1, \dots, d$. Applying [DW, Lemma 1], we decompose c_W. $\square$

Denote by n_o the number of c-orbits in $\{e_i \mid i \in I\}$; then $n_o = 2$ if $\Gamma = \widetilde{A_n}$ and $n_o = 3$, otherwise. For each orbit $\mathcal{O}_i, i = 1, \dots, n_o$, denote by P_i the product of c_{E_i} over the orbit; clearly this semi-invariant is of weight σ.

Theorem 23.

1. *The algebra $\mathbf{k}[R(Q,p\delta)]^{\mathrm{SL}(p\delta)}$ is generated by c_{E_i}, $i \in I$, and $f_0, \dots, f_p$.*
2. *A minimal system of generators of $\mathbf{k}[R(Q,p\delta)]^{\mathrm{SL}(p\delta)}$ consists of c_{E_i}, $i \in I$, and $\max(p+1-n_o, 0)$ elements from $f_0, \dots, f_p$. If $p+1 \geqslant n_o$, then these generators are algebraically independent; otherwise, if $p = 1, n_o = 3$, then the generators fulfill a syzygy $c_1P_1 + c_2P_2 + c_3P_3 = 0, c_1, c_2, c_3 \in \mathbf{k}^*$, and the ideal of syzygies is generated by this one.*

Remark 5. This statement is the same as [SkW, Theorem 17].

Proof. Denote by $E_\lambda, \lambda \in \Lambda \subseteq \mathbf{k}$, a one-parameter family of pairwise nonisomorphic simple homogeneous regular representations of Q of dimension δ. A generic representation of dimension $p\delta$ is locally semi-simple and is isomorphic to $E_{\lambda_1} + \cdots + E_{\lambda_p}$. By [DW], $\mathbf{k}[R(Q,\alpha)]^{\mathrm{SL}(\alpha)}$ is generated by the semi-invariants c_W such that $W \in (E_{\lambda_1} + \cdots + E_{\lambda_p})^\perp$ for some collection $\lambda_1, \dots, \lambda_p$; moreover, by Proposition 22 we can assume W to be a simple object of this category, hence, a σ-stable representation, by Proposition 10. The σ-stable representations are the simple regular ones. Note also that for W being homogeneous simple, $c_W \subseteq \mathbf{k}[R(Q,p\delta)]_\sigma^{(\mathrm{GL}(p\delta))} = \langle f_0, \dots, f_p\rangle$. So assertion 1 is proved.

The generic stabilizer of $\mathrm{GL}(p\delta)$ is $(\mathbf{k}^*)^p$, hence the generic stabilizer of $\mathrm{SL}(p\delta)$ is $(\mathbf{k}^*)^{p-1}$. So $\dim \mathbf{k}[R(Q,p\delta)]^{\mathrm{SL}(p\delta)} = \dim R(Q,p\delta) - \dim \mathrm{SL}(p\delta) + (p-1) = q_Q(p\delta) + |Q_0| + (p-1) = n + p$, where $n = |Q_0| - 1$.

Consider the semi-invariants $P_j, j = 1, \dots, n_o$. Set $W_j = \sum_{i\in\mathcal{O}_j} E_i$. By the properties of the Schofield semi-invariants, $P_j(W_k) = 0$ if and only if $\mathrm{Hom}(W_k, E_i) \neq \{0\}$ for some $i \in \mathcal{O}_j$, and the latter is equivalent to $k = j$. Hence, P_i and P_j are nonproportional for $i \neq j$. Moreover, if $n_o = 3$ and $p \geqslant 2$, then P_1, P_2, P_3 are linearly independent because of the values of these on representations $W_1 + W_2, W_1 + W_3, W_2 + W_3$.

Therefore, if $p + 1 \geqslant n_o$, then the semi-invariants $P_1, \dots, P_{n_o}$ are linear independent and one can add $p + 1 - n_o$ elements of $f_0, \dots, f_p$ to get a basis of $\mathbf{k}[R(Q, p\delta)]_\sigma^{(\mathrm{GL}(p\delta))}$. Hence, those $p + 1 - n_o$ elements of $f_0, \dots, f_p$ and c_{E_i}, $i \in I$, constitute a system of generators for $\mathbf{k}[R(Q, p\delta)]^{\mathrm{SL}(p\delta)}$. One can check that $|I| = n + n_o - 1$. So this system of generators consists of $|I| + (p + 1 - n_o) = n + p$ elements, hence, this is a minimal system of algebraically independent generators.

Finally, if $p = 1$, $n_o = 3$, then P_1, P_2, and P_3 are nonproportional elements of two-dimensional vector space $\mathbf{k}[R(Q, p\delta)]_\sigma^{(\mathrm{GL}(p\delta))}$, hence we get a syzygy as in **2**. By **1** $\mathbf{k}[R(Q, p\delta)]^{\mathrm{SL}(p\delta)}$ is generated by $c_{E_i}, i \in I$. Since this algebra is of dimension $n + 1$, $|I| = n + 2$, and because our syzygy is of degree 1 by each of the generators, assertion 2 is proved. $\square$

Now consider the general case: $\alpha = p\delta + \sum_{i \in I} p_i e_i$. The quiver E_Q is the union of circular quivers; for $i \in I$ define by $n(i)$ and $p(i)$ the next and the previous vertex of E_Q so that $c(e_i) = e_{n(i)}, n(p(i)) = i$. A subset $[k, l] = \{k, n(k), \dots, l\} \subseteq I$ will be called an *arc*. By Proposition 15.1, each arc $[k, l]$ with $k \neq l$ yields a real Schur root $e_{k,l} = e_k + \dots + e_l$; pick a Schurian representation $E_{k,l} \in R(Q, e_{k,l})$. By Proposition 14.3 the generic locally semi-simple decomposition of α is $\alpha = p\delta + \sum_{[k,l] \in \Omega} m_{k,l} e_{k,l}$, where Ω is a set of arcs. By Proposition 12 any two arcs in Ω either are disjoint sets or one of them contains another in the interior.

Proposition 24. $\dim \mathbf{k}[R(Q, \alpha)]^{\mathrm{SL}(\alpha)} = \dim \mathbf{k}[R(Q, p\delta)]^{\mathrm{SL}(p\delta)} - |\Omega|$.

Proof. By Theorem 19.3, $\mathbf{k}[R(Q, \alpha)]^{\mathrm{SL}(\alpha)} \cong \mathbf{k}[R(Q, p\delta) \oplus \bigoplus_{[k,l] \in \Omega} R(Q, e_{k,l})]^G$, where $G \subseteq \mathrm{GL}(p\delta) \times \prod_{[k,l] \in \Omega} \mathrm{GL}(e_{k,l})$ consists of the elements with the product of determinants at any vertex being 1; moreover, generic G-orbit is closed. Using the above property of the arcs from Ω, one can see that the dimension vectors $\delta, e_{k,l}, [k, l] \in \Omega$, are linear independent and this property implies that the kernel C of the action of G on $R(Q, p\delta) \oplus \bigoplus_{[k,l] \in \Omega} R(Q, e_{k,l})$ is finite. Since $e_{k,l}$ are real Schur roots, G acts on $\bigoplus_{[k,l] \in \Omega} R(Q, e_{k,l})$ with an open orbit G/K, where K is the kernel of that action. Consequently, the generic K-orbit is closed in $R(Q, p\delta)$ and $\dim \mathbf{k}[R(Q, \alpha)]^{\mathrm{SL}(\alpha)} = \dim \mathbf{k}[R(Q, p\delta)]^K$. Observe that K acts on $R(Q, p\delta)$ as a subgroup $T\mathrm{SL}(p\delta) \subseteq \mathrm{GL}(p\delta)$, where $T \subseteq Z\mathrm{GL}(p\delta), \dim T = |\Omega|$. We have $\dim \mathbf{k}(R(Q, p\delta))^{\mathrm{GL}(p\delta)} = p = \dim \mathbf{k}[R(Q, p\delta)]^{\mathrm{SL}(p\delta)} - \dim ZGL(p\delta) + 1$. Hence, $\dim \mathbf{k}[R(Q, p\delta)]^K = \dim \mathbf{k}[R(Q, p\delta)]^{\mathrm{SL}(p\delta)} - \dim T + \dim T \cap \mathbf{k}^*$, where $\mathbf{k}^*$ is the kernel for the action of $\mathrm{GL}(p\delta)$. But $T \cap \mathbf{k}^*$ is the projection to $\mathrm{GL}(p\delta)$ of C. So $\dim T \cap \mathbf{k}^* = 0$ and this completes the proof. $\square$

By Theorem 19.4, $\mathbf{k}[R(Q, \alpha)]^{\mathrm{SL}(\alpha)}$ is isomorphic to $A = \bigoplus_{\chi \in \Lambda} \mathbf{k}[R(Q, p\delta)]_\chi^{(\mathrm{GL}(p\delta))}$, where $\Lambda \subseteq \mathbf{Z}_+^{Q_0}$ consists of weights such that $\mathbf{k}[R(Q, p\delta)]_\chi^{(\mathrm{GL}(p\delta))} \neq 0$ and for each $[k, l] \in \Omega$, $\mathbf{k}[R(Q, e_{k,l})]_\chi^{(\mathrm{GL}(e_{k,l}))} \neq 0$. By Theorem 23, $\mathbf{k}[R(Q, p\delta)]_\chi^{(\mathrm{GL}(p\delta))} \neq 0$ implies $\chi = -\langle\ , e\rangle$, where $e \in \langle e_i, i \in I\rangle_{\mathbf{Z}_+}$. So in order to determine Λ, we need to find the dimensions $e \in \langle e_i, i \in I\rangle_{\mathbf{Z}_+}$ such that $E_{k,l}^\perp \cap R(Q, e) \neq \varnothing$ for any $[k, l] \in \Omega$.

Proposition 25. $E_{k,l}^\perp \cap R(Q, e) \neq \varnothing$ *if and only if* $e \in \langle e_{k,n(l)}, e_i \mid i \in I, i \neq k, n(l)\rangle_{\mathbf{Z}_+}$.

Proof. Clearly, a necessary condition for $E_{k,l}^{\perp} \cap R(Q,e) \neq \varnothing$ is $\langle e_{k,l}, e\rangle = 0$. By Proposition 13 we have

$$\langle e_{k,l}, \textstyle\sum_{i\in I} q_i e_i\rangle = q_k - q_{n(l)}. \tag{20}$$

Hence, the semi-group $\{e = \sum_{i\in I} q_i e_i, q_i \in \mathbf{Z}_+ \mid \langle e_{k,l}, e\rangle = 0\}$ is generated by dimensions $e_{k,n(l)}, e_i, i \in I\setminus\{k, n(l)\}$. So it is sufficient to check either of the equivalent conditions $\mathrm{Hom}(E_{k,l}, E) = 0$ or $\mathrm{Ext}(E_{k,l}, E) = 0$ for $E = E_{k,n(l)}, E_i, i \neq k, n(l)$. For all E with the exception of $E_{k,n(l)}, E_l$ we have, by (8) and Proposition 15.2, $\mathrm{Hom}(E_{k,l}, E) = 0 = \mathrm{Hom}(E, E_{k,l})$. On the other hand, for $E = E_{k,n(l)}, E_l$, (10) and Proposition 15.2 yield $\mathrm{Ext}(E_{k,l}, E) = 0 = \mathrm{Ext}(E, E_{k,l})$. $\square$

Let $J \subseteq I$ consist of elements being k or $n(l)$ for an arc $[k,l] \in \Omega$. It can happen that J consists of less than $2|\Omega|$ elements because there can be arcs like $[k,l]$ and $[n(l), m]$ in Ω such that their union is again an arc. So we can introduce a new set Δ of arcs such that each arc from Δ is a disjoint union of arcs from Ω, each arc from Ω is contained in an arc from Δ, and for any $[k_1,l_1], [k_2,l_2] \in \Delta$ we have $p(k_1) \neq l_2, n(l_1) \neq k_2$.

Proposition 26. *Λ is generated by $|I|-|\Omega|$ elements $\chi = -\langle\ , e\rangle$, where $e = e_i$, $i \in I\setminus J$, or $e = e_{k,n(l)}, [k,l] \in \Delta$.*

Proof. By formula (20) a necessary condition for a character $\chi = -\langle\ , \sum_{i\in I} q_i e_i\rangle$ to be in Λ is $q_{n(l)} = q_k$ for any arc $[k,l] \in \Omega$. Hence, the semi-group of dimension vectors meeting this condition is generated by $e_i, i \in I\setminus J$, and $e_{k,n(l)}, [k,l] \in \Delta$. On the other hand, by Proposition 25, for each generator e and for each arc $[k,l] \in \Omega$ there is a representation of dimension e perpendicular to $E_{k,l}$. It remains to note that $|J| = |\Delta| + |\Omega|$. $\square$

The subsequent theorem is an analog of [SkW, Theorem 21].

Theorem 27. *Let $\alpha = p\delta + \sum_{i\in I} p_i e_i$, $p > 0$, be the canonical decomposition of α. If $p = 1, n_o = 3$, and for each $\mathcal{O}_j, j = 1, \ldots, 3$, at least two coefficients p_i vanish, then $\mathbf{k}[R(Q,\alpha)]^{\mathrm{SL}(\alpha)}$ is a hypersurface; in all other cases it is a polynomial algebra.*

Proof. Clearly, $A = \bigoplus_{\chi\in\Lambda} \mathbf{k}[R(Q,p\delta)]_{\chi}^{(\mathrm{GL}(p\delta))}$ is generated as an algebra by the subspaces $\mathbf{k}[R(Q,p\delta)]_{\chi}^{(\mathrm{GL}(p\delta))}$, where χ is σ or a generator of Λ from Proposition 26. Moreover, for each $\mathcal{O}_j, j = 1, \ldots, n_o$, δ can be obtained as a nonnegative linear combination of the generators of Λ corresponding to $\mathcal{O}_j$, hence P_j is a corresponding product of generators of A. So, by Proposition 26, A is generated by $c_{E_i}, i \subseteq I\setminus J$, $c_{E_k} \ldots c_{E_{n(l)}}, [k,l] \in \Delta$, and $\max(p+1-n_o, 0)$ elements from $f_0, \ldots, f_p$. Also, by Proposition 26, the number of these generators of A is less than the number of generators of $\mathbf{k}[R(Q,p\delta)]^{\mathrm{SL}(p\delta)}$ by $|\Omega|$, hence, by Proposition 24, if $\mathbf{k}[R(Q,p\delta)]^{\mathrm{SL}(p\delta)}$ is a polynomial algebra, then A is, and if $\mathbf{k}[R(Q,p\delta)]^{\mathrm{SL}(p\delta)}$ is a hypersurface, then A is generated by $\dim A+1$ elements. Moreover, in the latter case, the unique relation between the generators is $c_1P_1 + c_2P_2 + c_3P_3 = 0$. If for some $\mathcal{O}_j$ and some $[k,l] \in \Delta$ holds $P_j = c_{E_k} \cdots c_{E_{n(l)}}$, then the relation says that this generator is redundant, so A is in fact a polynomial algebra. This happens precisely when $n(n(l)) = k$ or, equivalently, $p_{n(l)} = 0$ and $p_i \neq 0$ for all other $i \in \mathcal{O}_j$. This completes the proof. $\square$

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