heaps and Huffman codes

priority queues: motivation

dynamically changing list of events with dates

- want to find next event quickly

list of running programs, some more important (e.g. what user will notice being slow)

- choose most important to run first
- want to find most important quickly

list of connections, some interactive (video call), some not (download)

- want quick way to choose which one to service

data structure: priority queue

priority queue ADT

`insert(priority, item)`

`findMin()` — return item with lowest (first) priority

`deleteMin()` — remove item with lowest (first) priority

priority queue implementations

structure	insert	findMin	deleteMin
unsorted vector	$\Theta(1)$ (amortized)	$\Theta(n)$	$\Theta(n)$
unsorted linked list	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
sorted vector	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$
sorted linked list	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$
balanced tree	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
binary heap	$\Theta(\log n)$	$\Theta(1)$	$\Theta(\log n)$
Fibannoci heap	amortized $\Theta(1)$	$\Theta(1)$	amortized $\Theta(\log n)$
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not necessary to have a priority queue, but useful...

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- remove — remove value with given index/pointer

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aside: min v max

can also have ADT with findMax/etc. *instead of* findMin/etc.

same complexities, etc. (use different comparisons)

terms for heaps: “min-heap” (findMin version) or “max-heap” (findMax version)

binary heaps

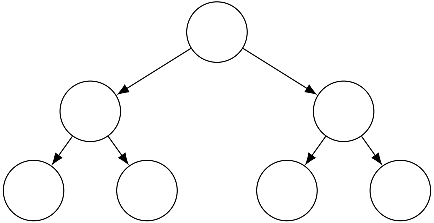
binary heap is a binary tree

binary tree is not a binary search tree

structure: almost a perfect tree

ordering: parent $<$ child (everywhere in tree)

perfect binary trees



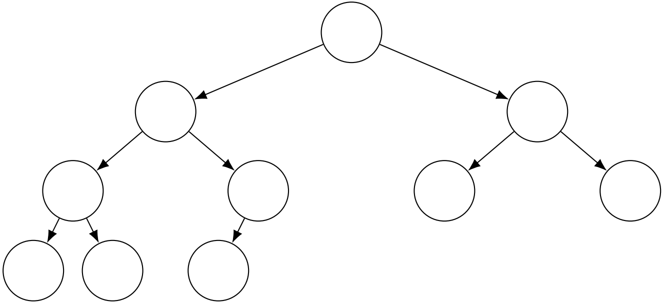
a binary tree is perfect or complete if

all leaves have same depth

all nodes have zero children (leaf) or two children

exactly the trees that achieve $2^h - 1$ nodes

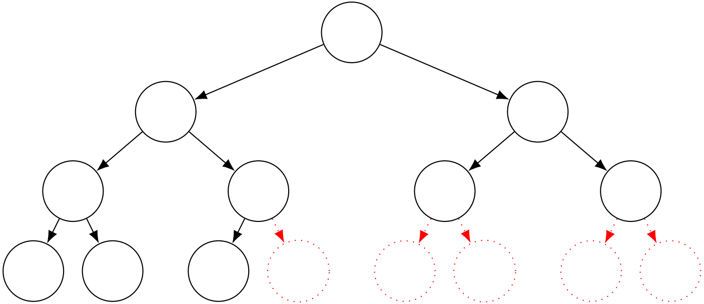
almost perfect/complete binary trees



heaps are almost complete trees

only missing bottom-rightmost slots

almost perfect/complete binary trees



heaps are almost complete trees

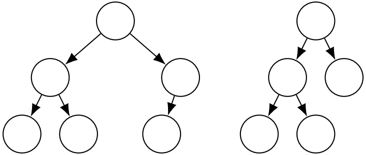
only missing bottom-rightmost slots

almost complete formally

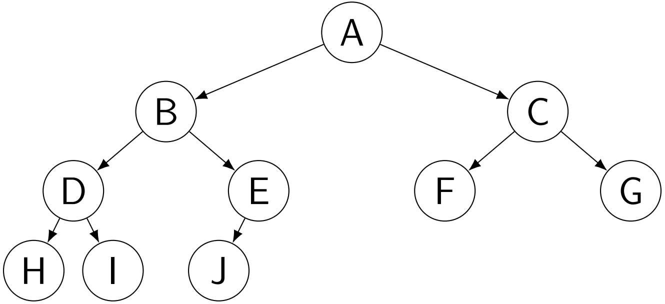
single node tree is almost complete

otherwise: almost complete if either

- left child is complete with height h and right child almost complete with height h; *OR*
- left child is almost complete with height h and right child is complete with height $h-1$



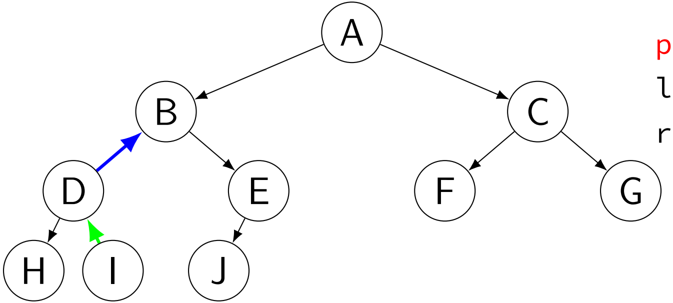
trees as arrays



node		A	B	C	D	E	F	G	H	I	J						
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

```
string theTree[17] = {"", "A", "B", ....}
```

trees as arrays




```
parentIndex = index / 2
leftChild = index * 2
rightChild = index * 2 + 1
```

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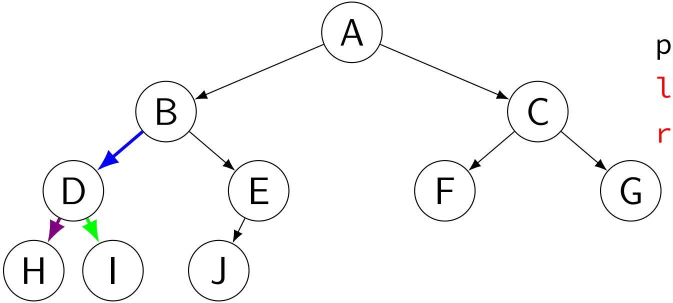


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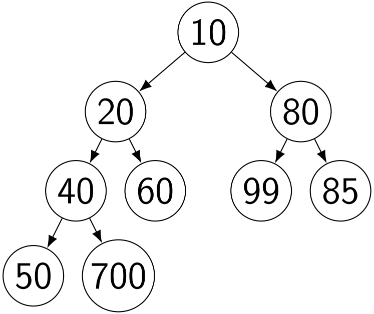
```
string theTree[17] = {"", "A", "B", ....}
```

why arrays

single array — less storage/memory allocation

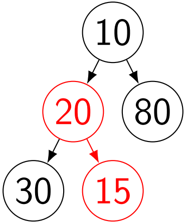
represent tree as single vector

the heap property



heap property: parent $\leq$ any of its children

a non-heap

heap property: parent $\leq$ any of its children 

heap code

linked off slides page of repo

```
class binary_heap {
    ...
private:
    // heap[1] is root
        // leftChildIndex  = index * 2
        // rightChildIndex = index * 2 + 1
        // parentIndex     = index / 2
    vector<int> heap;
    int heap_size;
}
```

heap insert

add new node as leaf node

while new node $<$ parent node: swap with parent

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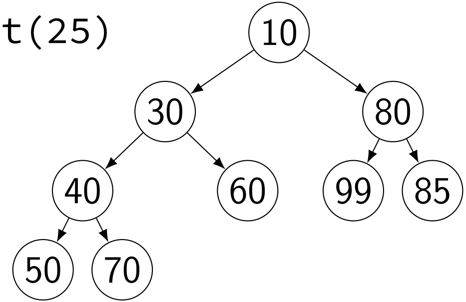
while new node < parent node: swap with parent

`insert(25)`



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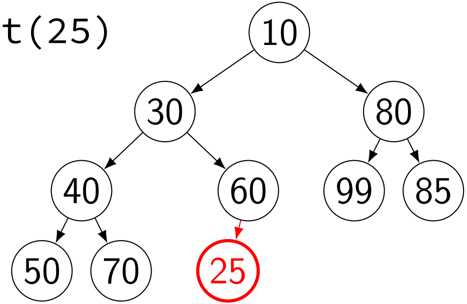

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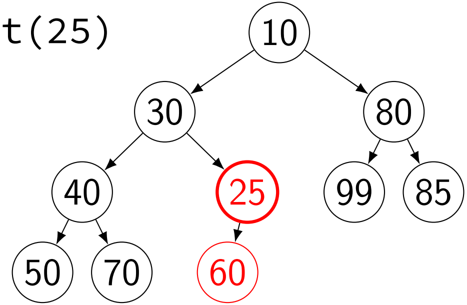

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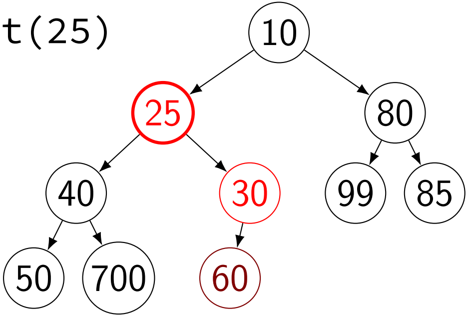

insert(int)

```
void binary_heap::insert(int x) {
    ++heap_size;
    heap.push_back(x);
    percolateUp(x);
}
```

percolateUp(int)

```
void binary_heap::percolateUp(int index) {
    int newValue = heap[index];
    // while not at root and
    //       less than parent...
    while (index > 1 && newValue < heap[index / 2]) {
        // move parent down
        heap[index] = heap[index / 2];
        // advance up the tree
        index /= 2;
    }
    heap[index] = newValue;
}
```

insert runtime

worst case: $\log_2 N$ nodes changed

insert average case?

average case is better assuming random keys:

- intuition: leafs have bottom half of values (on average)
- ...so usually don't need to move up
- ...and if we do, parents of leafs have 25th to 50th percentile of values
- ...so need to move up two steps even less
- about 2 steps moved up on average

heap deleteMin

replace root with last leaf node

while node greater than children: swap with *smallest child*

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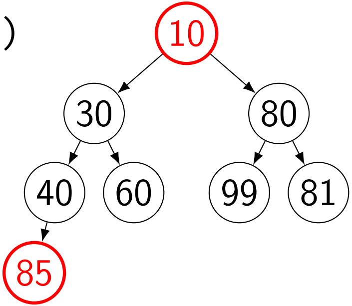
`deleteMin()`



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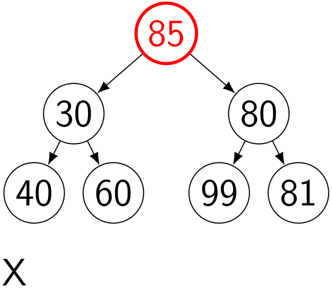

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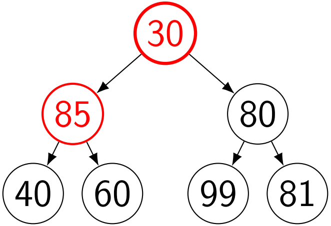

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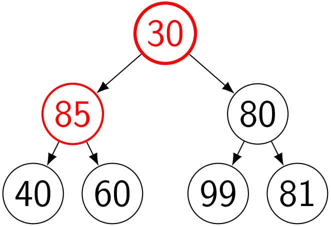
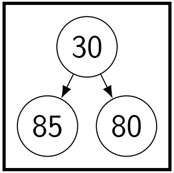
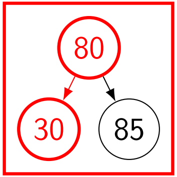




is a heap



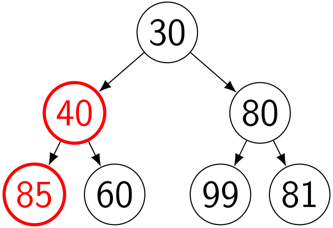
not a heap

heap deleteMin

replace root with last leaf node

while node greater than children: swap with *smallest child*

`deleteMin()`



deleteMin code

```
int binary_heap::deleteMin() {
    if (heap_size == 0)
        throw ...;
    int result = heap[1];
    heap[1] = heap[heap_size--];
    heap.pop_back();
    percolateDown(1);
    return result;
}
```

precolateDown code

```
int binary_heap::percolateDown(int index) {
    int value = heap[index];
    // while left child exists
    while (index * 2 <= heap_size) {
        int left = index * 2, right = index * 2 + 1;

        // set child to smallest child that exists
        int child = left;
        if (right <= heap_size && heap[right] < heap[left])
            child = right;

        // if less than smallest, done
        if (value < heap[child]) break;
        // otherwise:
        heap[index] = heap[child]; // move child up
        index = child;             // and traverse down
    }
    heap[index] = value;
}
```

deleteMin runtime

worst case $\Theta(\log N)$ — move nodes from root to leaf

other heap operations?

`decreaseKey`/`increaseKey`

- change value, then percolateUp/Down
- slow ($\Theta(N)$) if you have to find the value
- fast ($\Theta(\log N)$) if you already know where value is
- (one method: keep track of its index)
- faster (amortized $\Theta(1)/\Theta(1)$) in Fibanocci/strict Fibanocci heaps

`remove`

- `decreaseKey`, then `deleteMin`

core heap operations

insert — $\Theta(\log N)$ worst case, better on "average"

deleteMin — $\Theta(\log N)$

findMin — $\Theta(1)$

heap sort

```
void heapSort(vector<T>& values) {
    binary_heap<T> heap;
    for (T x : values)
        heap.insert(x);
    values.clear();
    while (!heap.empty()) {
        values.push_back(heap.deleteMin());
}
```

$\Theta(N \log N)$ sort

can be done in place with more careful implementation

(use `values` as the max-heap's array,
place sorted elements starting at end)

mostly not as fast in practice as comparable *unstable* sorts

compression

compression

50KB webpage as 5KB download (a lot faster!)

100MB of machine code as 50MB download?

movie of 24 1MB pictures/second into 10MB/minute file?

...

lossy compression

for audio, pictures, video, *lossy compression* is common

intuition: you won't notice if we make the pixel 0.25% darker

...and it had "noise" from camera sensor, etc. anyways

idea: model human perception

write down most important parts of audio/image/etc.

important = noticed by humans

lossless compression

lossless compression — reproduce original file

rely on patterns

example: text file has many more ‘e’s than ’!’s

…so choose shorter encoding for ‘e’ than ‘!’

example: computer-drawn images have lots of white space

…so have a way to represent “a big white rectangle” (instead of specifying each pixel)

typical compression results

ratio = original size:final size

note: usually a compression ratio/speed tradeoff (not shown)

lossless:

- for English text or source code: about 4:1
- for CD-quality audio: about 2:1
- for photographs: about 2:1
- for computer-drawn diagrams: about 5:1 to 20:1

lossy: (making a guess at what is "close enough" in quality)

- for CD-quality audio: about 4:1
- for standard definition TV video+audio: about 1:40

a prefix code

letter	code
a	`0`
b	`100`
c	`101`
d	`11`

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prefix code

no code is prefix of another (no ambiguity)

shorter codes for more frequent values (hopefully)

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prefix code

no code is prefix of another (no ambiguity)

shorter codes for more frequent values (hopefully)

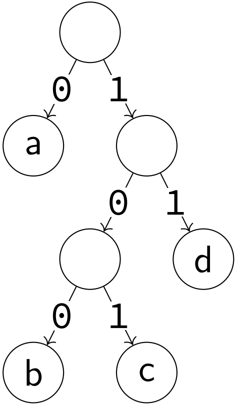
b	a	a	a	c	d	a

→

`100`	`0`	`0`	`0`	`101`	`100`	`0`

prefix codes as trees

letter	code
a	0
b	100
c	101
d	11



prefix code cost

letter	code	frequency
a	`0`	$5/12$
b	`100`	$1/6$
c	`101`	$1/12$
d	`11`	$1/3$

$$\text{cost} = \sum_i p_i r_i = \frac{5}{12} \cdot 1 + \frac{1}{6} \cdot 3 + \frac{1}{12} \cdot 3 + \frac{1}{3} \cdot 2 = \frac{11}{6} \text{ (bits per symbol)}$$

p_i: probability symbol i occurs

r_i: length of code for i

prefix code cost

letter	code	frequency
a	`0`	5/12
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p_i: probability symbol i occurs

r_i: length of code for i

versus a=`00`,b=`01`,c=`10`,d=`11`: cost $= 2$ (bits per symbol)

how to find minimum cost prefix code (given frequencies)?

high-level compression steps

read file, find symbol frequencies

choose best prefix code (called *Huffman code*) based on frequencies

best = assuming each code maps to one symbol

write prefix code to output

read file, convert to preifx code, write to output

high-level compression steps

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write prefix code to output

read file, convert to preifx code, write to output

input file	→	chosen prefix code	input file using prefix code

finding the best prefix code

build prefix code tree from bottom up

intuition 1: least frequent thing at bottom $\rightarrow$ use it first

use case for a priority queue

intuition 2: combine less frequent symbols into more frequent group

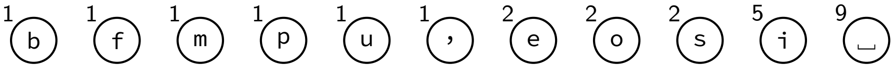
work with partial prefix trees

running example and frequencies

`if it is to be, it is up to me`

symbol	frequency	symbol	frequency
`b`	1	`p`	1
`e`	2	`s`	2
`f`	1	`t`	4
`i`	5	`u`	1
`m`	1	`,` (comma)	1
`o`	2	␣ (space)	9

building the Huffman tree (1)

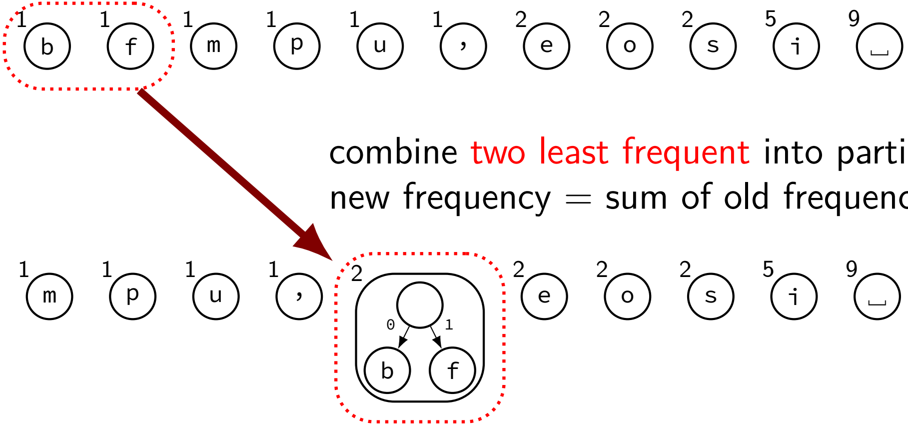


list of partial prefix trees

labelled with total frequency of contained symbols

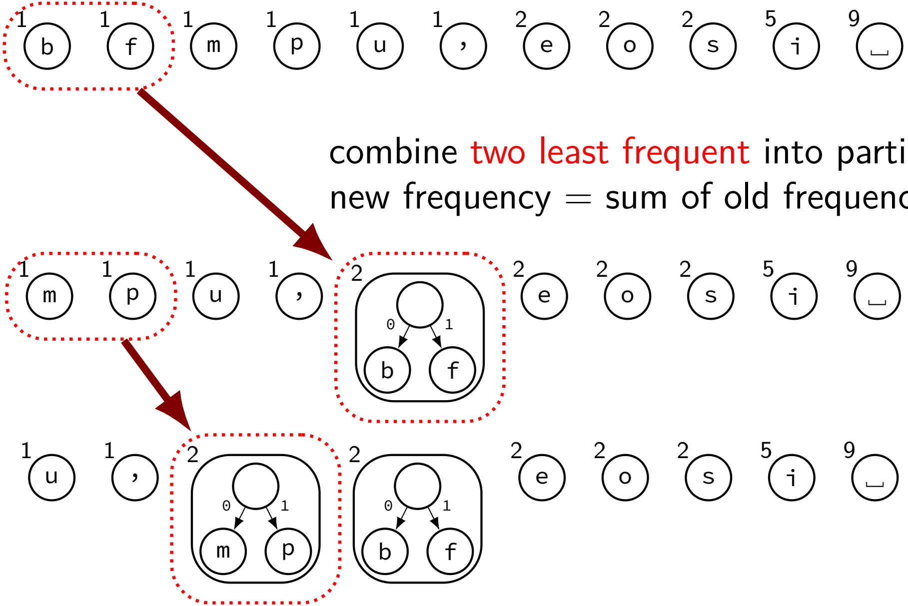
goal: combine these into one prefix tree

building the Huffman tree (1)



combine two least frequent into partial prefix tree
new frequency = sum of old frequencies

building the Huffman tree (1)



combine two least frequent into partial prefix tree
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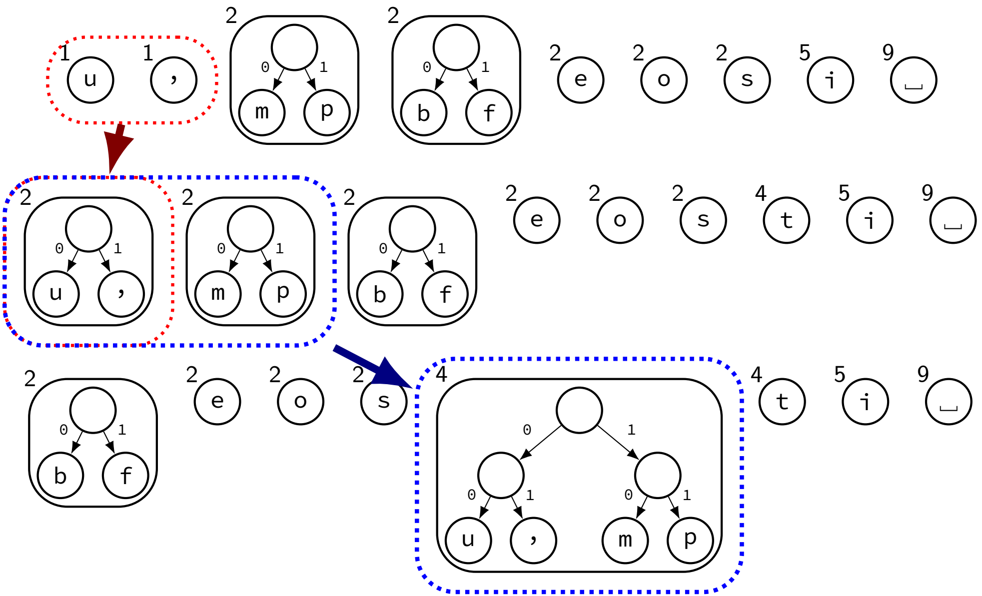
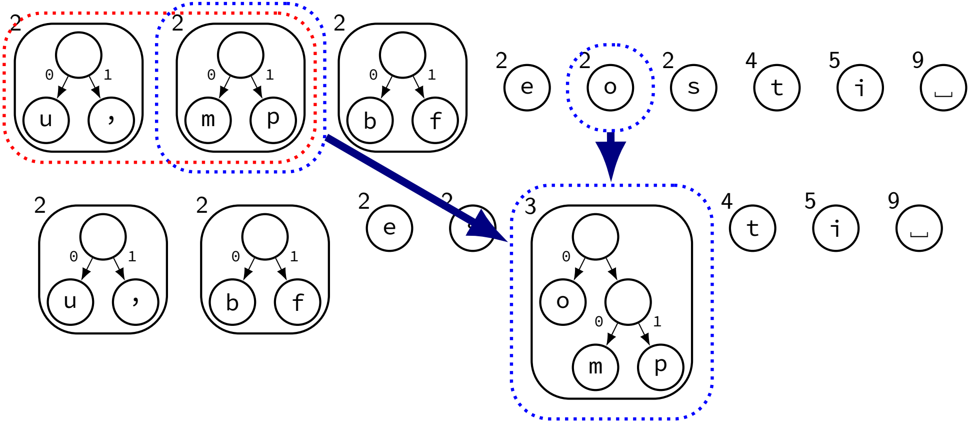
building the Huffman tree (2)

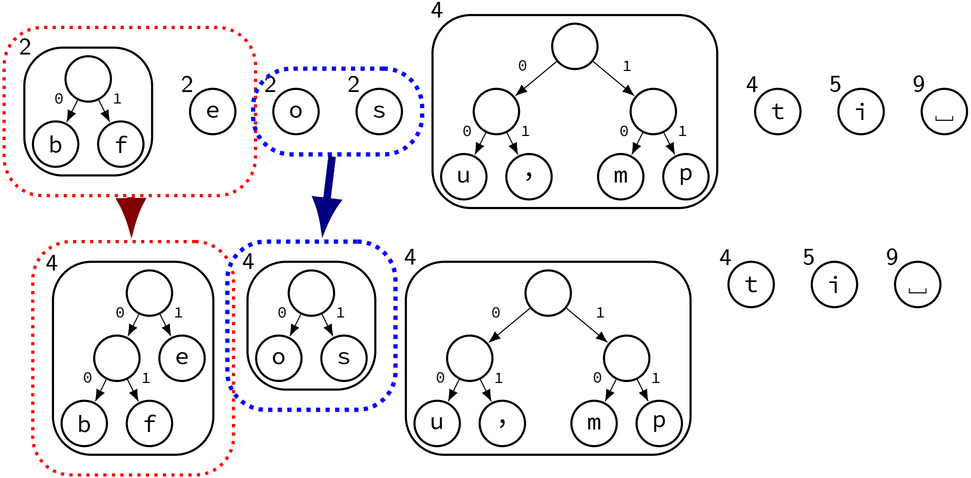


building the Huffman tree: alternatives

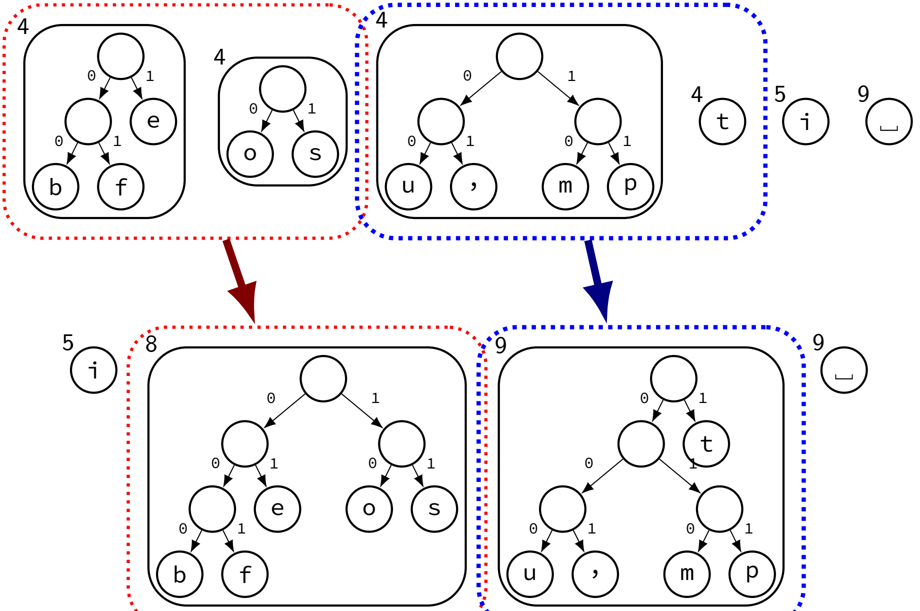


multiple choices of what to combine
proof not shown: produce same quality prefix tree

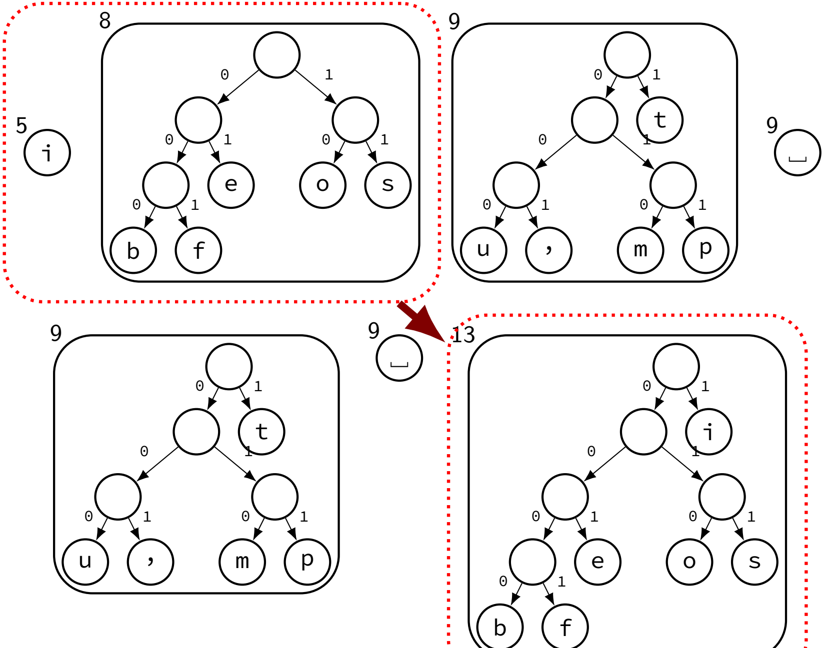
building the Huffman tree (3)



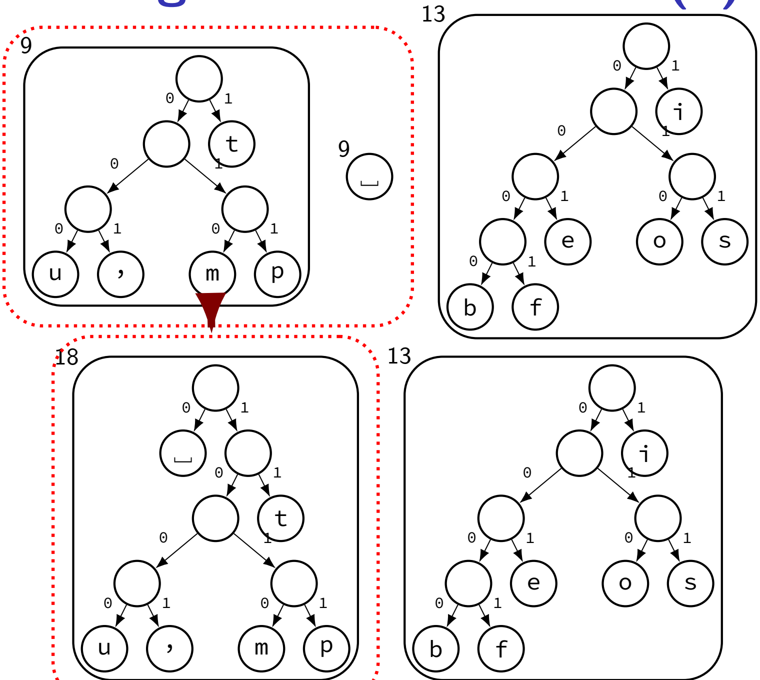
building the Huffman tree (4)



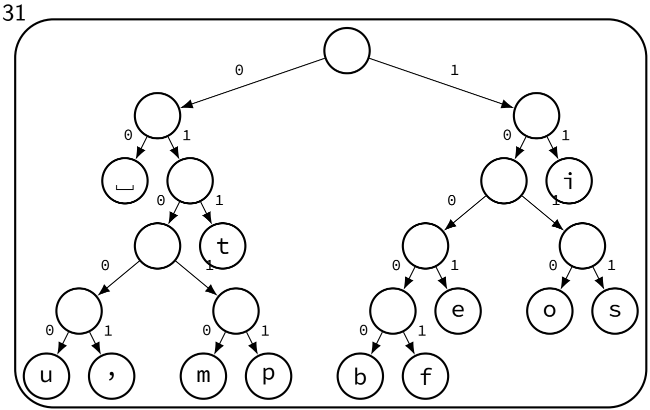
building the Huffman tree (5)



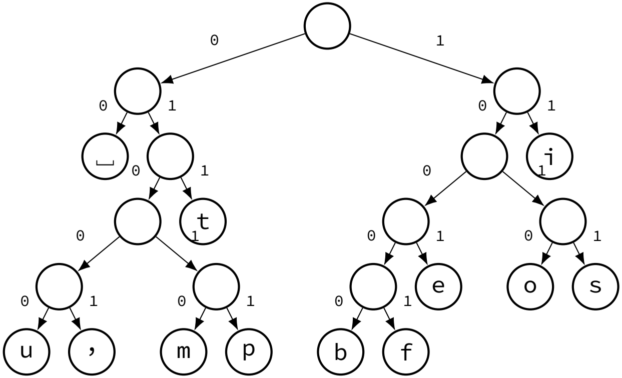
building the Huffman tree (6)



building the Huffman tree (7)



the final Huffman tree



letter	code
␣	00
u	01000
,	01001
m	01010
p	01011
t	011
b	10000
f	10001
e	1001
o	1010
s	1011
i	11

tree-building pseudocode

```
class PrefixTree {
    ...
    PrefixTree(char c, int frequency);
    PrefixTree(PrefixTree rightSide, PrefixTree leftSide);
    PrefixTree(const PrefixTree &other);
    ...
};
...
  PriorityQueue<PrefixTree> queue;
  for (char c, frequency f  in  inputFile) {
      queue.insert(PrefixTree(c, f));
  }
  while (queue.size() > 1) {
      PrefixTree first = queue.deleteMin();
      PrefixTree second = queue.deleteMin();
      queue.insert(PrefixTree(first, second));
  }
  return queue.deleteMin();
...
```

storing the prefix code

file format for the lab:

```
space 00
u 01000
, 01001
m 01010
p 01011
t 011
b 10000
f 10001
e 1001
o 1010
s 1011
i 11
```

real format?

does this save space?

probably if input file is big enough...

but real compression formats use a more compact encoding

not having you do in lab to ease debugging/etc.

what about the data?

in lab: the text `01111110011110`...

obviously wastes a lot of space...

real compression: sequence of bytes, 8 bits per

extra work to extract bit-by-bit, match with prefix code

decoding

load the code into a prefix code tree

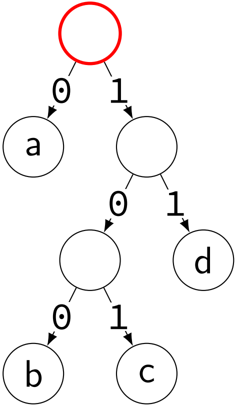
then, read bits, traversing tree until leaf

psuedocode:

```
while (there are more bits) {
    PrefixTreeNode *current = root;
    while (current is not a leaf) {
        if (next bit is 0)
            current = current->left;
        else
            current = current->right;
    }
    output(current->symbol);
}
```

example

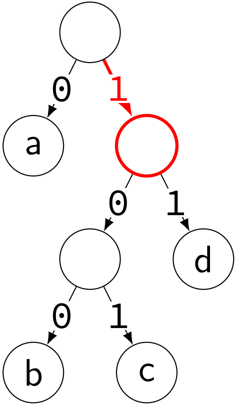
letter	code
a	0
b	100
c	101
d	11



11 100 0 101 0 0 11 = dba:w caad

example

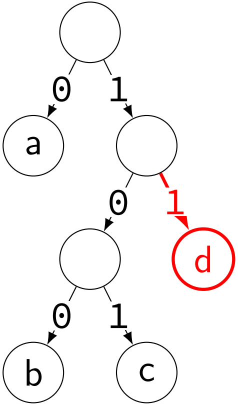
letter	code
a	0
b	100
c	101
d	11



11 100 0 101 0 0 11 = dba:w caad

example

letter	code
a	0
b	100
c	101
d	11



`11 100 0 101 0 0 11` = dba:w caad

example

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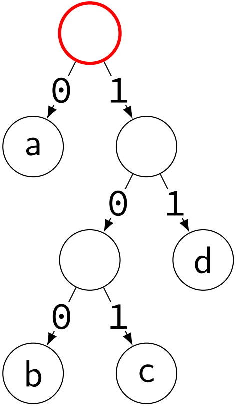
b	100
c	101
d	11



`11 100 0 101 0 0 11` = dba:w caad

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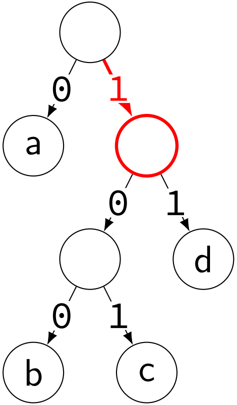

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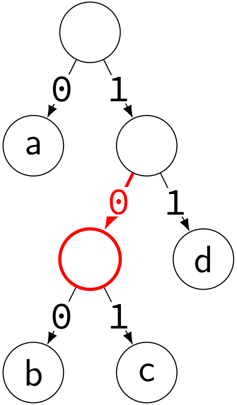
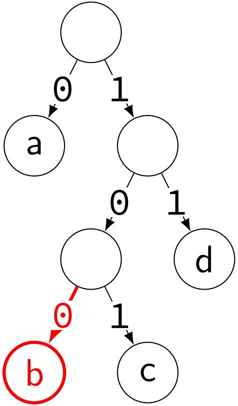


11 100 0 101 0 0 11 = dba:w caad

example

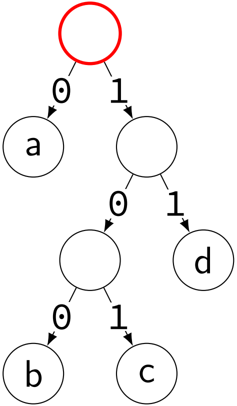
letter	code
a	`0`
b	`100`
c	`101`
d	`11`



`11 100 0 101 0 0 11` = dba:w caad

example

letter	code
a	`0`
b	`100`
c	`101`
d	`11`



`11 100 0 101 0 0 11` = dba:w caad

example

letter	code
a	0
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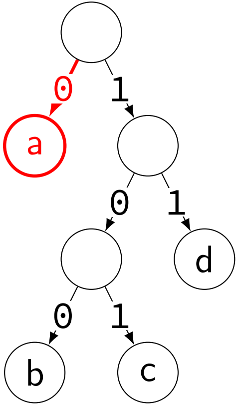
c	101
d	11



11 100 0 101 0 0 11 = dba:w caad

lab preview

pre-lab: compression

in-lab: decompression

post-lab report

pre-lab

write a program to...

calculate letter frequencies of input

use binary heap to build huffman tree

output encoding mapping (format specified in lab)

output encoded message

pre-lab tools

heap code supplied in slides

file I/O code provided (`fileio.cpp`)

- or see `getWordInTable.cpp` from lab 6
- or see `http://www.cplusplus.com/doc/tutorial/files/`
- or see ifstream documentation

a note on ASCII

the American standard character codes

- 7-bit charcters (extra bit left over in bytes)
- ASCII or superset used to represent English text

128 characters (95 printable, 33 non-printable)

Wikipedia article as table/details

ASCII codes

for lab: only worry about "printable" ASCII characters

byte values `0x20` to `0x7e`

special case: `0x20` = 'space'

no other whitespace characters used

(output character in table as itself...)

heap example

linked off slides page as

`binary_heap.h`

`binary_heap.cpp`

you may use for lab

heap declaration: public

```
class binary_heap {
public:
    binary_heap();
    binary_heap(vector<int> vec);
    ~binary_heap();

    void insert(int x);
    int findMin();
    int deleteMin();
    unsigned int size();
    void makeEmpty();
    bool isEmpty();
    void print();
    ...
};
```

heap declaration: private

```
class binary_heap {
    ...
private:
    vector<int> heap;
    unsigned int heap_size;
    void percolateUp(int hole);
    void percolateDown(int hole);
};
```

vector heap

`vector<int> heap` — vector representing binary tree, using rules shown before

- `heap[0]` is unused
- `heap[1]` is root
- `heap[i * 2]` is left child of node i
- `heap[i * 2 + 1]` is right child of node i

`int heap_size` is its size

- (even though `heap.size() - 1` could have been used instead...)

binary_heap::binary_heap(vec)

constructor to initialize from *unsorted* vector

equivalent to repeated insertion...

```
binary_heap::binary_heap(vector<int> vec) :
        heap_size(vec.size()) {
    heap = vec;
    heap.push_back(heap[0]);
    heap[0] = 0;
    for ( int i = heap_size/2; i > 0; i-- )
        percolateDown(i);
}
```

binary_heap::binary_heap(vec)

constructor to initialize from *unsorted* vector

equivalent to repeated insertion...

recall: in-place heap sort — similar to what's happening here...

```
binary_heap::binary_heap(vector<int> vec) :
        heap_size(vec.size()) {
    heap = vec;
    heap.push_back(heap[0]);
    heap[0] = 0;
    for ( int i = heap_size/2; i > 0; i-- )
        percolateDown(i);
}
```

findMin/size/etc.

```
int binary_heap::findMin() {
    if ( heap_size == 0 )
        throw "findMin() called on empty heap";
    return heap[1];
}

unsigned int binary_heap::size() {
    return heap_size;
}

bool binary_heap::isEmpty() {
    return heap_size == 0;
}

void binary_heap::makeEmpty() {
    heap_size = 0;
```

print

```
void binary_heap::print() {
    cout << "(" << heap[0] << ")␣";
    for ( int i = 1; i <= heap_size; i++ ) {
        cout << heap[i] << "␣";
        // next line from from http://tinyurl.com/mf9tbgm
        bool isPow2 = (((i+1) & ~(i))==(i+1))? i+1 : 0;
        if ( isPow2 )
            cout << endl << "\t";
    }
    cout << endl;
}
```