

Big-Oh

asymptotic growth rate or *order*

compare two functions, but...

ignore constant factors, small inputs

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g grows faster — eventually much bigger than f

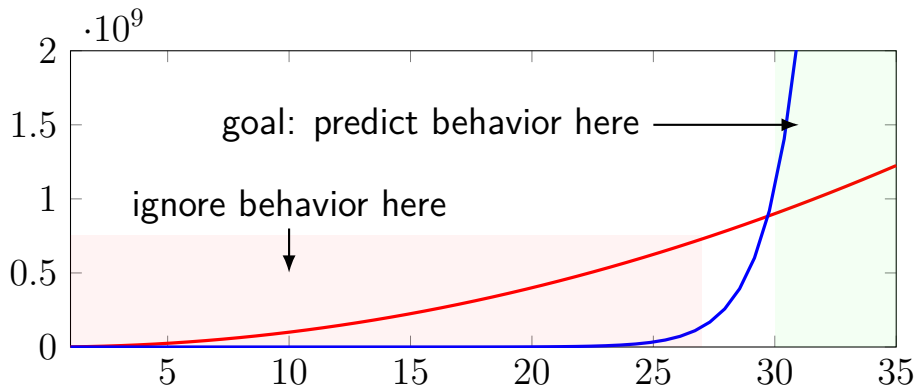
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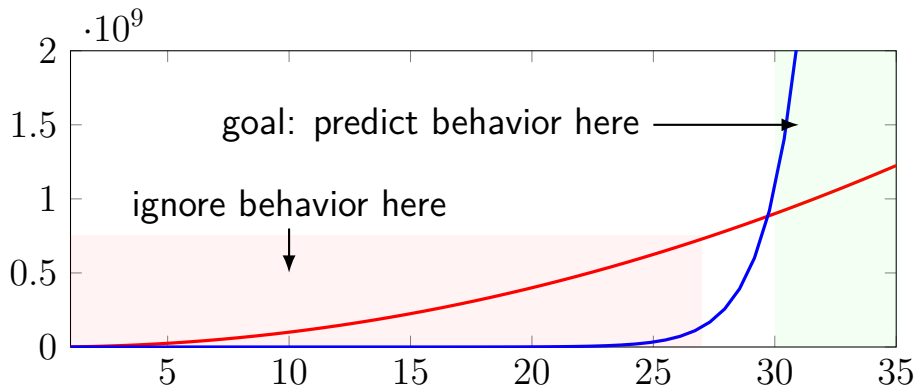
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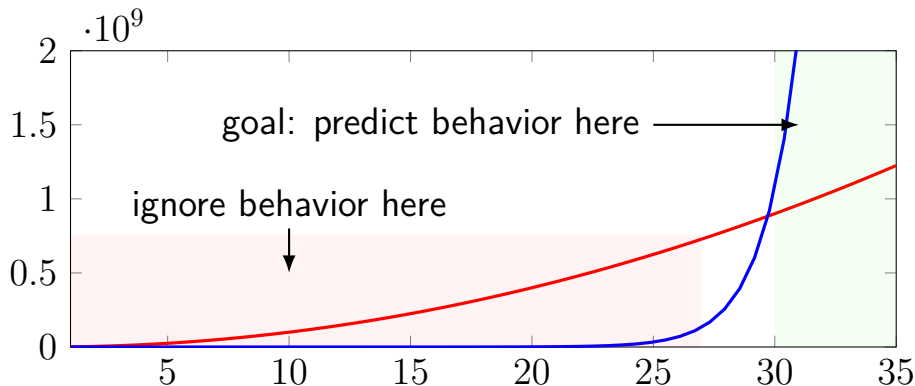
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preview: what functions?

example: comparing sorting algorithms

runtime = $f(\text{size of input})$

e.g. seconds to sort = $f(\text{number of elements in list})$

e.g. # operations to sort = $f(\text{number of elements in list})$

space = $f(\text{size of input})$

e.g. number of bytes of memory = $f(\text{number of elements in list})$

theory, not empirical

yes, you can make *guesses* about big-oh behavior from measurements

but, no, graphs \neq big-oh comparison
what happens further to the right?
might not have tested big enough

want to write down **formula**

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what happens further to the right?
might not have tested big enough

want to write down **formula**

example: summing a list of n items:

exactly n addition operations

assume each one takes k unit of time

runtime = $f(n) = kn$

recall: comparing list data structures

List benchmark (from intro slides) w/ 100000 elements

Data structure	Total	Insert	Search	Delete
Vector	87.818	0.004	63.202	24.612 s
ArrayList	87.192	0.010	62.470	24.712 s
LinkedList	263.776	0.006	196.550	67.439 s
HashSet	0.029	0.022	0.003	0.004 s
TreeSet	0.134	0.110	0.017	0.007 s
Vector, sorted	2.642	0.009	0.024	2.609 s

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some runtimes get really big as size gets large...

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others seem to remain manageable

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problem: **growth rate** of runtimes with list size

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for Vector (unsorted), ArrayList, LinkedList...
operations grows like n where n is list size

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for HashSet...

operations per search/remove is constant (sort of)

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for TreeSet, sorted Vector...

operations per search grows like $\log(n)$ where n is list size

why asymptotic analysis?

“can my program work when data gets big?”

website gets thousands of new users?

text editor opening 1MB book? 1 GB log file?

music player sees 1 000 song collection? 50 000?

text search on 100 petabyte copy of the text of the web?

why asymptotic analysis?

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if asymptotic analysis says “no”

can find out **before implementing algorithm**

won't be fixed by, e.g., buying a faster CPU

sets of functions

define sets of functions based on an example f

$\Omega(f)$: grow no slower than f (" $\geq f$ ")

$O(f)$: grow no faster than f (" $\leq f$ ")

$\Theta(f) = \Omega(f) \cap O(f)$: grow as fast as f (" $= f$ ")

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$\Theta(f) = \Omega(f) \cap O(f)$: grow as fast as f (“ $= f$ ”)

examples:

$$n^3 \in \Omega(n^2)$$

$$100n \in O(n^2)$$

$$10n^2 + n \in \Theta(n^2) \text{ — ignore constant factor, etc.}$$

$$\text{and } 10n^2 + n \in O(n^2) \text{ and } 10n^2 + n \in \Omega(n^2)$$

what are we measuring

$f(n)$ = worst case running time

n = input size — as a positive integer

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will compare f to another function $g(n)$

example: $f(n) \in O(g(n))$ (or $f \in O(g)$)

informally: “ f is big-oh of g ”

example $f(n) \notin \Omega(g(n))$ or $(g \notin \Omega(g))$

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intuition: detect if program will *ever* take “forever”

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example: iterating through an array until we find a value

best case: look at one value, it's the one we want

worst case: look at every value, none of them are what we want

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example: iterating through an array until we find a value

best case: look at one value, it's the one we want

worst case: look at every value, none of them are what we want

$f(n)$ is run time of *slowest* input of size n

formal definitions

$f(n) \in O(g(n))$:

there exists $c > 0$ and $n_0 > 0$ such that
for all $n > n_0$, $f(n) \leq c \cdot g(n)$

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$f(n) \in \Theta(g(n))$:

$f(n) \in O(g(n))$ **and** $f(n) \in \Omega(g(n))$

formal definition example (1)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
 $f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $n \in O(n^2)$:

formal definition example (1)

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there exists $c > 0$ and $n_0 > 0$ such that
 $f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $n \in O(n^2)$:

choose $c = 1$, $n_0 = 2$

for $n > 2 = n_0$: $n \leq c \cdot n^2 = n^2$

Yes!

formal definition example (2)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
 $f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $10n \in O(n)$?

formal definition example (2)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
 $f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $10n \in O(n)$?

choose $c = 11$, $n_0 = 2$

for $n > 2 = n_0$: $f(n) = 10n \leq c \cdot g(n) = 11n$

Yes!

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there exists $c > 0$ and $n_0 > 0$ such that
 $f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $10n \in O(n)$?

choose $c = 11$, $n_0 = 2$

for $n > 2 = n_0$: $f(n) = 10n \leq c \cdot g(n) = 11n$

Yes!



don't need to choose smallest possible c

negating formal definitions

$f \in O(g)$: there exists $c, n_0 > 0$ so for all $n > n_0$: $f(n) \leq cg(n)$

$f \notin O(g)$:

there does not exist $c, n_0 > 0$ so for all $n > n_0$: $f(n) \leq cg(n)$

for all c, n_0 , there exists $n > n_0$: $f(n) > cg(n)$

formal definition example (3)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
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Is $n^2 \in O(n)$?

formal definition example (3)

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 $f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $n^2 \in O(n)$?

no — consider any $c, n_0 > 0$

consider $n_{bad} = (c + 100)(n_0 + 100) > n_0$

$$n_{bad}^2 = (c + 100)^2(n_0 + 100)^2 > c(c + 100)(n_0 + 100) = cn_{bad}$$

so can't find c, n_0 that satisfy definition

(i.e. $f(n) = n_{bad}^2 \not\leq c \cdot g(n_{bad}) = cn_{bad}$)

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so can't find c, n_0 that satisfy definition

(i.e. $f(n) = n_{bad}^2 \not\leq c \cdot g(n_{bad}) = cn_{bad}$)

alternative

$$n_{bad} = \max\{c + 100, n_0 + 1\} > n_0$$

formal definition example (4)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
 $f(n) \leq c \cdot g(n)$ for all $n > n_0$

consider: $f(n) = 100 \cdot n^2 + n$, $g(n) = n^2$:

choose $c = 200$, $n_0 = 2$

observe for $n > 2$: $100n^2 + n \leq 101n^2$

for $n > 2 = n_0$: $f(n) = 100n^2 + n \leq 101n^2 \leq c \cdot g(n) = 200n^2$

big-oh proofs generally

if proving yes case:

- look at inequality

- choose* a large enough c and n_0 that it's definitely true

- don't bother finding smallest c, n_0 that work

if proving no case:

- game: given c, n_0 find counter example

- general idea: *choose* $n > n_0$ using a formula based on c

- show that this n never satisfies the inequality

- don't bother showing it's true for all $n' > n$

- don't bother finding smallest n that works

aside: forall/exists

$\forall n > 0$: for all $n > 0$

$\exists n < 0$: there exists an $n < 0$

definition consequences

If $f \in O(h)$ and $g \notin O(h)$, which are true?

1. $\forall m > 0, f(m) < g(m)$
for all m , f is less than g
2. $\exists m > 0, f(m) < g(m)$
there exists an m , so f is less than g
3. $\exists m_0 > 0, \forall m > m_0, f(m) < g(m)$
there exists an m_0 , so for all m larger, f is less than g
4. 1 and 2
5. 2 and 3
6. 1 and 2 and 3

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$$f \in O(h), g \notin O(h) \not\Rightarrow \forall m. f(m) < g(m)$$

counterexample — $f(n) = 5n$; $g(n) = n^3$; $h(n) = n^2$

$f \in O(h)$: $5n \leq cn^2$ for all $n > n_0$ with $c = 6$, $n_0 = 2$

$g \notin O(h)$: $n^3 \leq cn^2$? use $n \approx cn_0$ as counterexample

$$m = 2: f(m) = 10 \not\leq g(m) = 8$$

$$f \in O(h), g \notin O(h) \not\Rightarrow \forall m. f(m) < g(m)$$

counterexample — $f(n) = 5n$; $g(n) = n^3$; $h(n) = n^2$

$f \in O(h)$: $5n \leq cn^2$ for all $n > n_0$ with $c = 6$, $n_0 = 2$

$g \notin O(h)$: $n^3 \leq cn^2$? use $n \approx cn_0$ as counterexample

$$m = 2: f(m) = 10 \not\leq g(m) = 8$$

intuition: big-oh ignores behavior for small n

$$n^3 \notin O(n^2)$$

big-Oh definition requires:

$$n^3 \leq cn^2 \text{ for all } n > n_0$$

choose any $c > 1$ and $n_0 > 1$, then

$n = cn_0$ is a counterexample

$$n^3 = c^3 n_0^3 = cn_0 (cn_0)^2 > cn^2$$

contradicting the definition

(and for $c < 1$, use $n = n_0 + 1$, etc.)

$$f \in O(h), g \notin O(h) \implies \exists m. f(m) < g(m)$$

intuition: should be true for 'big enough' m

assume definition of big-Oh:

$$f \in O(h): \forall n > n_0 : f(n) \leq ch(n) \text{ (for a } n_0, c > 0)$$

$$g \notin O(h): \exists n > n_0 : g(n) > ch(n) \text{ (for any } n_0, c > 0)$$

assume f 's n_0, c

use the n that must exist for g (from definition)

$$f \in O(h), g \notin O(h) \implies ? \exists m_0 \forall m > m_0. f(m) < g(m)$$

intuitively, seems so g must grow faster than f — for big m :

$$f(m) < c_1 \cdot h(m)$$

$$g(m) < c_2 \cdot h(m)$$

but some corner case counterexamples:

$$f(n) = n$$

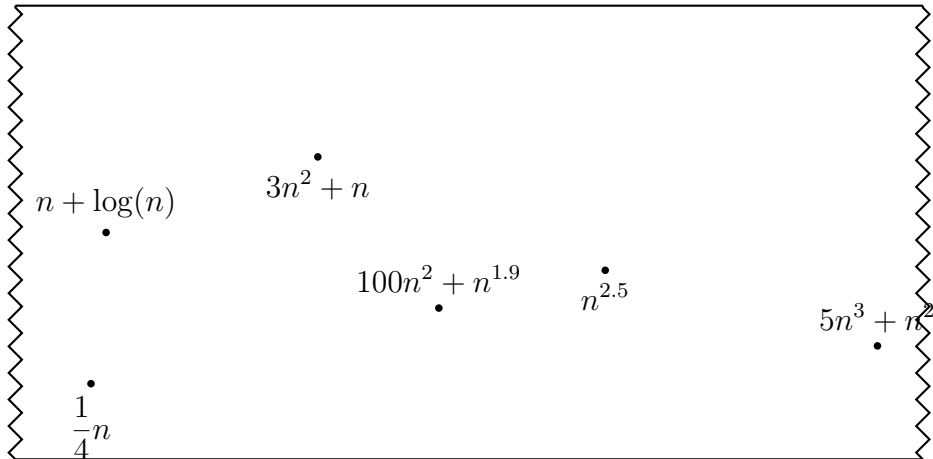
$$g(n) = \begin{cases} 1 & n \text{ odd} \\ n^2 & n \text{ even} \end{cases}$$

$$h(n) = n$$

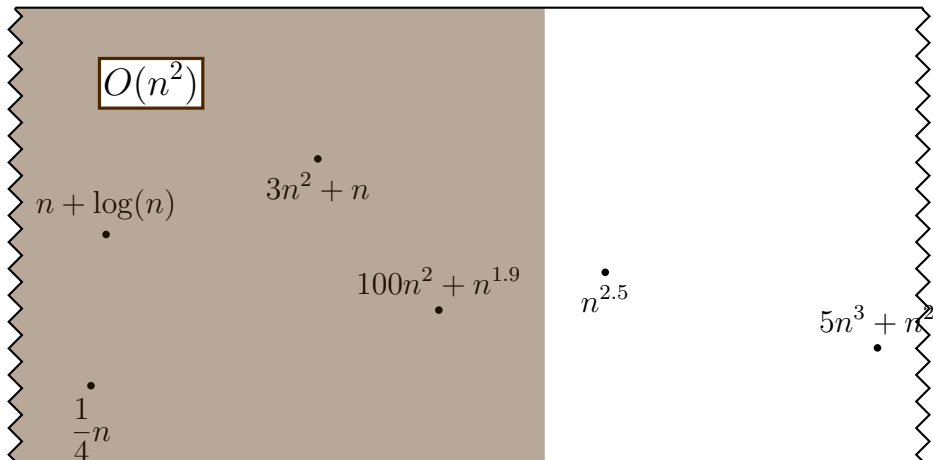
true with additional restriction:

$$f, g \text{ monotonic } (g(n) \leq g(n+1), \text{ etc.})$$

function hierarchy

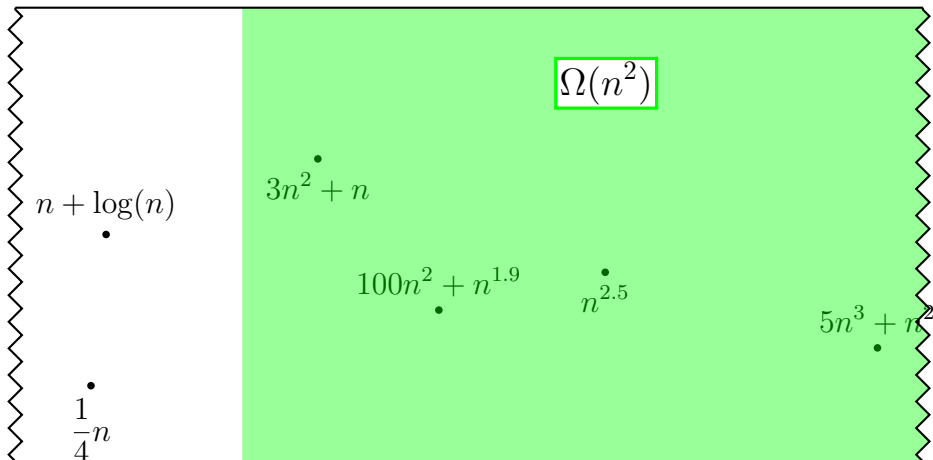


function hierarchy



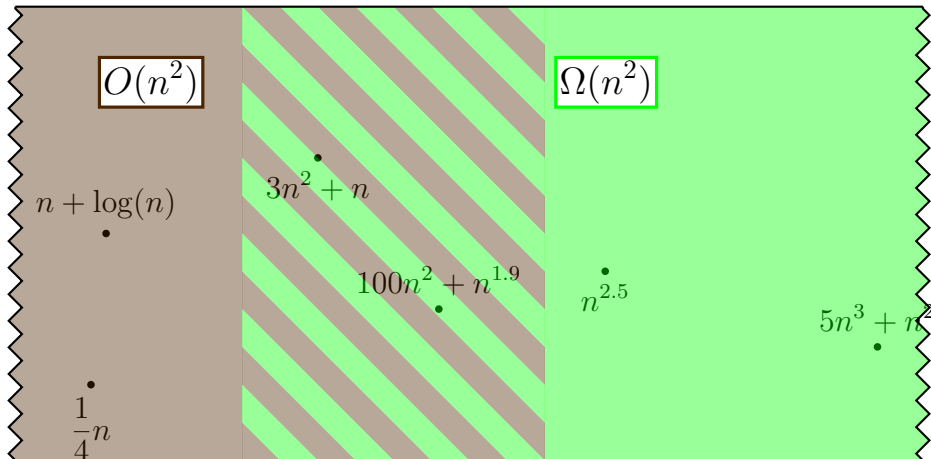
O — upper bound (" \leq ")

function hierarchy



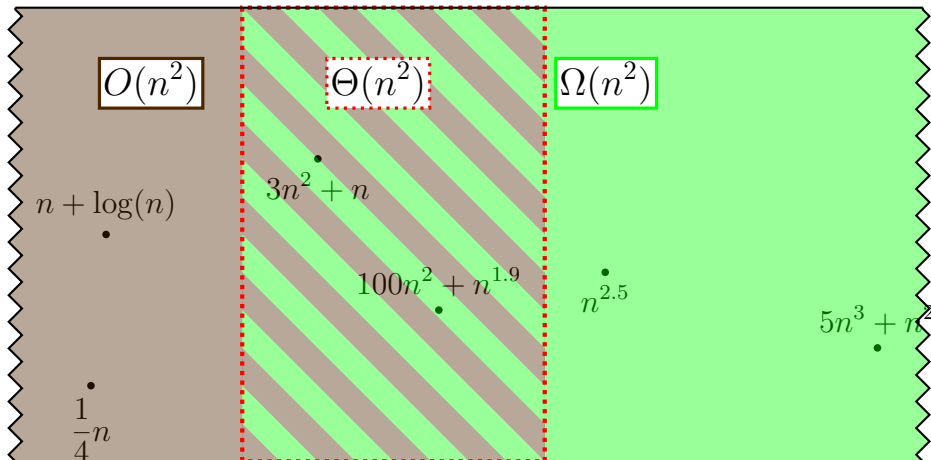
Ω — lower bound (“ \geq ”)

function hierarchy



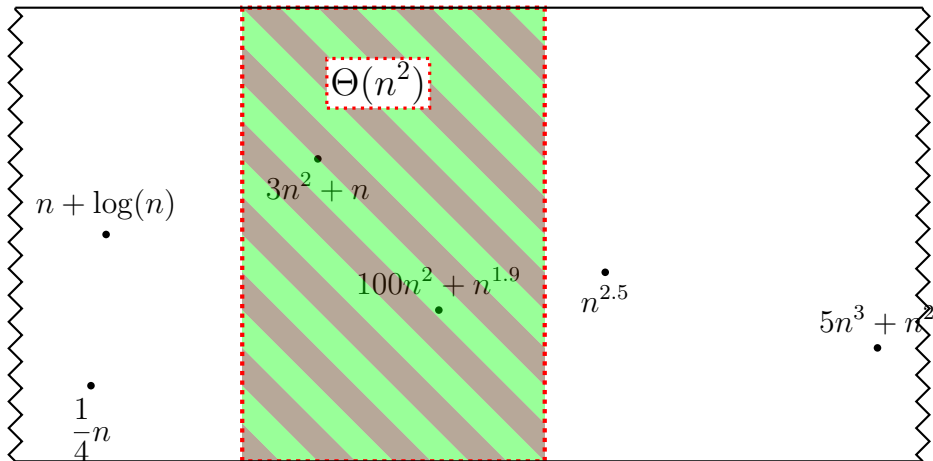
O and Ω overlap

function hierarchy



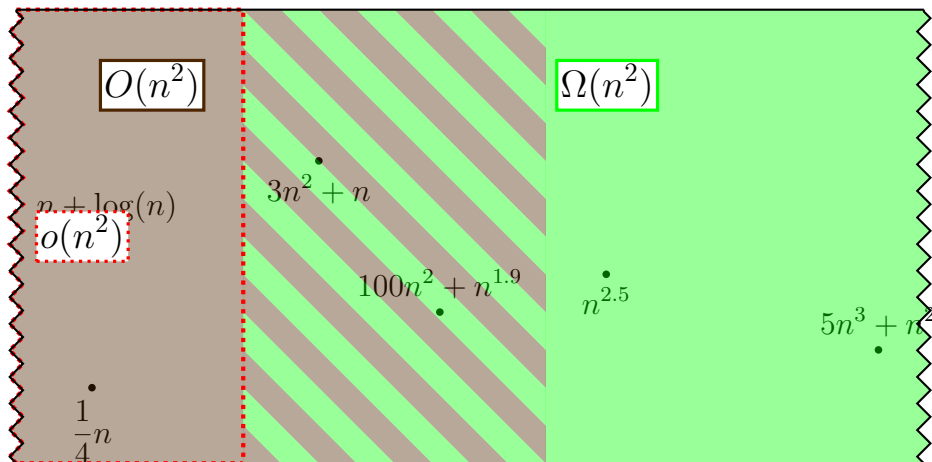
Θ — tight bound (“=”) — O and Ω

function hierarchy



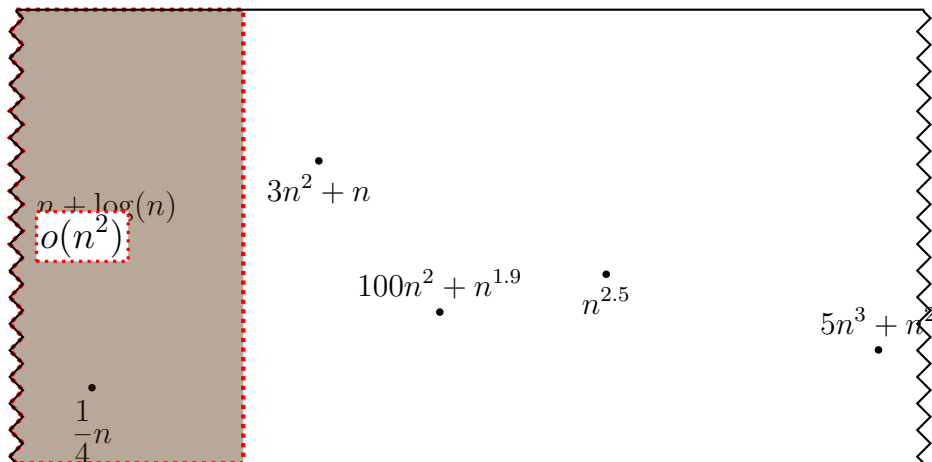
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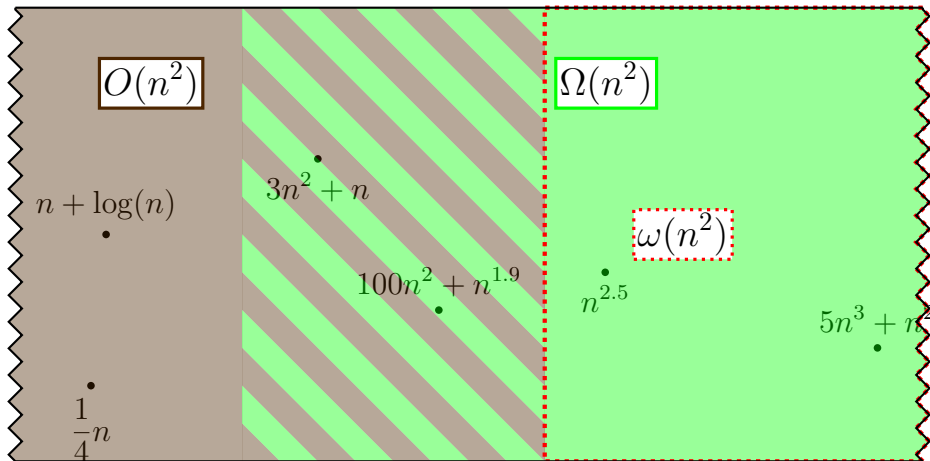
$g \in o(f)$ (“little-oh”)— strict upper bound
 $f(n) < c \cdot g(n)$ (all c); (versus $O(f)$: $f(n) \leq c \cdot g(n)$)

function hierarchy



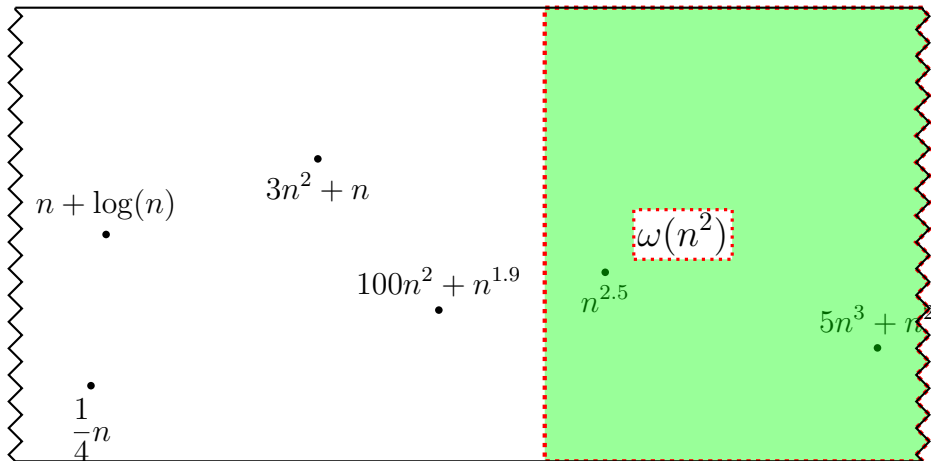
$g \in o(f)$ (“little-oh”)— strict upper bound
 $f(n) < c \cdot g(n)$ (all c); (versus $O(f)$: $f(n) \leq c \cdot g(n)$)

function hierarchy



$g \in \omega(f)$ — strict lower bound
 $f(n) > c \cdot g(n)$ (all c); (versus $\Omega(f)$: $f(n) \geq c \cdot g(n)$)

function hierarchy



$g \in \omega(f)$ — strict lower bound

$f(n) > c \cdot g(n)$ (all c); (versus $\Omega(f)$: $f(n) \geq c \cdot g(n)$)

big-Oh variants

- $O(f)$ asymptotically less than or equal to f
- $o(f)$ asymptotically less than f
- $\Omega(f)$ asymptotically greater than or equal to f
- $\omega(f)$ asymptotically greater than f
- $\Theta(f)$ asymptotically equal to f

limit-based definition

$$\limsup_{n \rightarrow \infty} \frac{f(n)}{g(n)} = X$$

if only if...

$$X < \infty: f \in O(g)$$

$$X > 0: f \in \Omega(g)$$

$$0 < X < \infty: f \in \Theta(g)$$

$$X = 0: f \in o(g)$$

$$X = \infty \text{ (and } \liminf): f \in \omega(g)$$

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lim sup?

$\limsup \frac{f(n)}{g(n)}$ — “limit superior”
equal to normal \lim if it is defined

only care about upper bound

e.g. n^2 in $f(n) = \begin{cases} 1 & n \text{ odd} \\ n^2 & n \text{ even} \end{cases}$

usually glossed over (including in Bloomfield's/Floryan's slides from prior semesters)

some big-Oh properties (1)

for O and Ω and Θ :

$$O(f + g) = O(\max(f, g))$$

$$f \in O(g) \text{ and } g \in O(h) \implies f \in O(h)$$

also holds for o (little-oh), ω

$$f \in O(f)$$

some big-Oh properties (2)

$$f \in O(g) \leftrightarrow g \in \Omega(f)$$

$$f \in \Theta(g) \leftrightarrow g \in \Theta(f)$$

does *not* hold for O , Ω , etc.

Θ is an **equivalence relation**

reflexive, transitive, etc.

a note on $=$

informally, sometimes people write $5n^2 = O(n^2)$

not very precise — O is a *set of functions*

selected asymptotic relationships

for $k > 0$, $c > 1$, $\epsilon > 0$:

$n^k \in o(c^n)$ (polynomial always smaller than exponential)

$n^k \in o(n^k \log n)$ (adding log makes something bigger)

$\log_k(n) \in \Theta(\log_l(n))$ (all log bases are the same)

$n^k + cn^{k-1} \in \Theta(n^k)$ (only polynomial degree matters)

some names

$\Theta(1)$ — constant (some fixed maximum)
read k th element of array

$\Theta(\log n)$ — logarithmic
binary search a sorted array

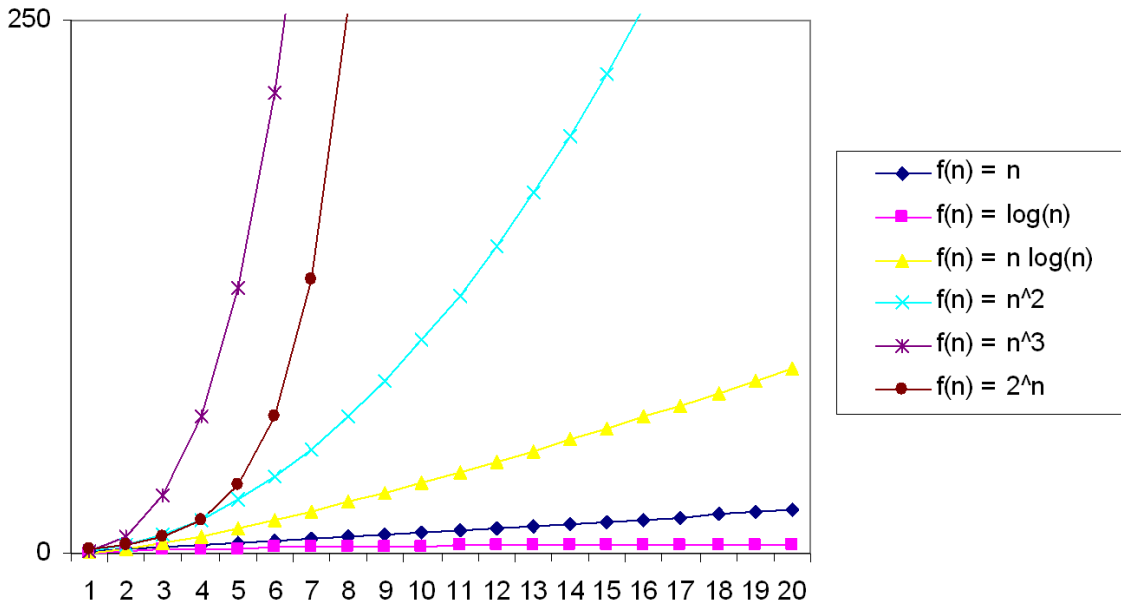
$\Theta(n)$ — linear
searching an unsorted array

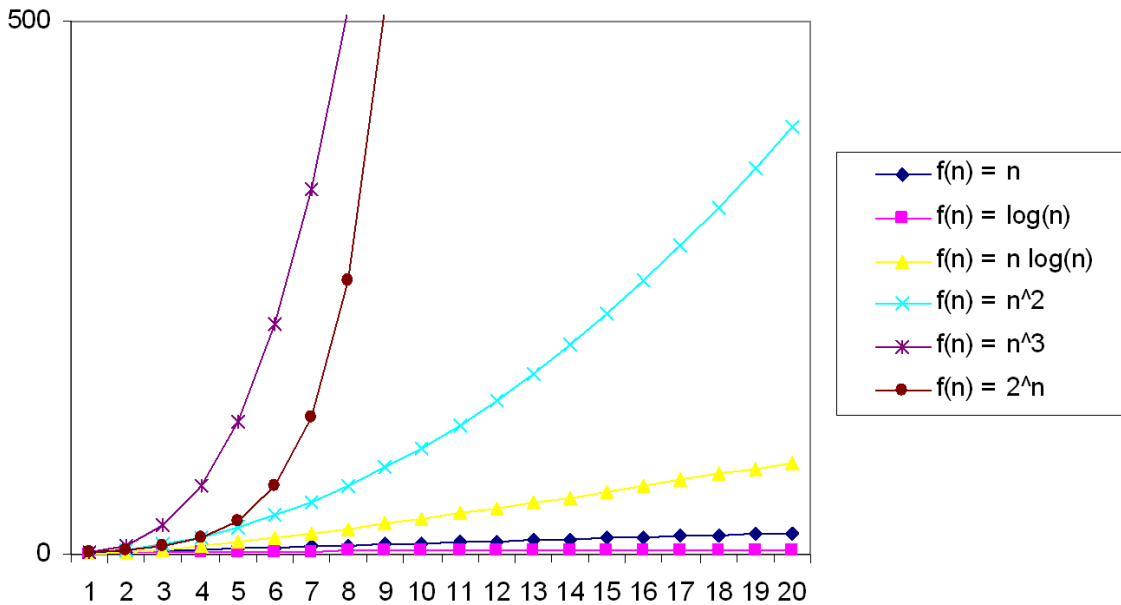
$\Theta(n \log n)$ — log-linear
sorting an array by comparing elements

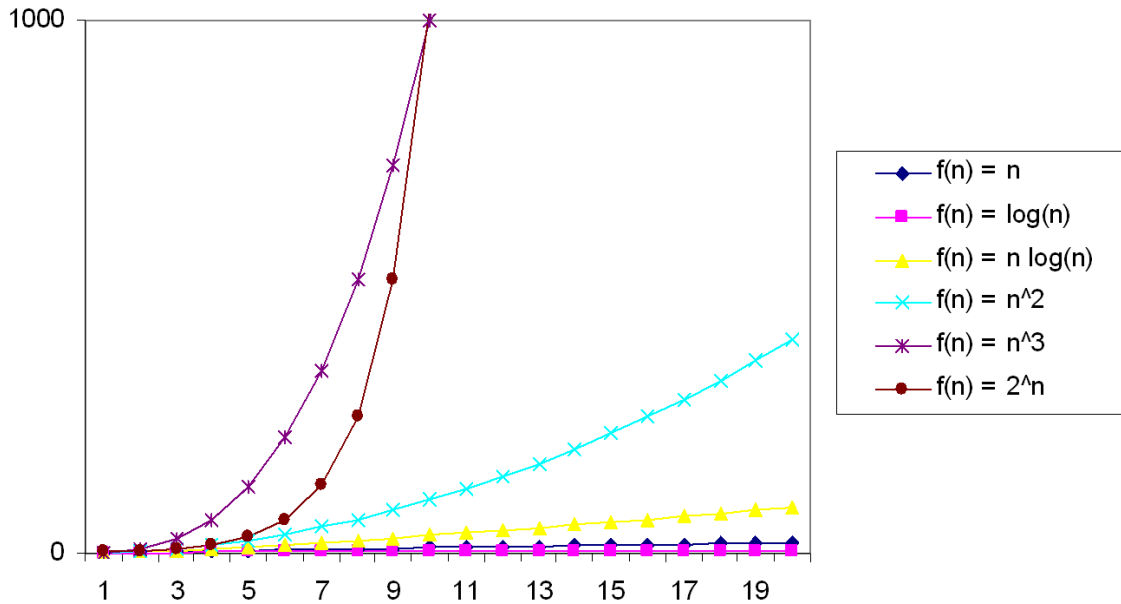
$\Theta(n^2)$ — quadratic

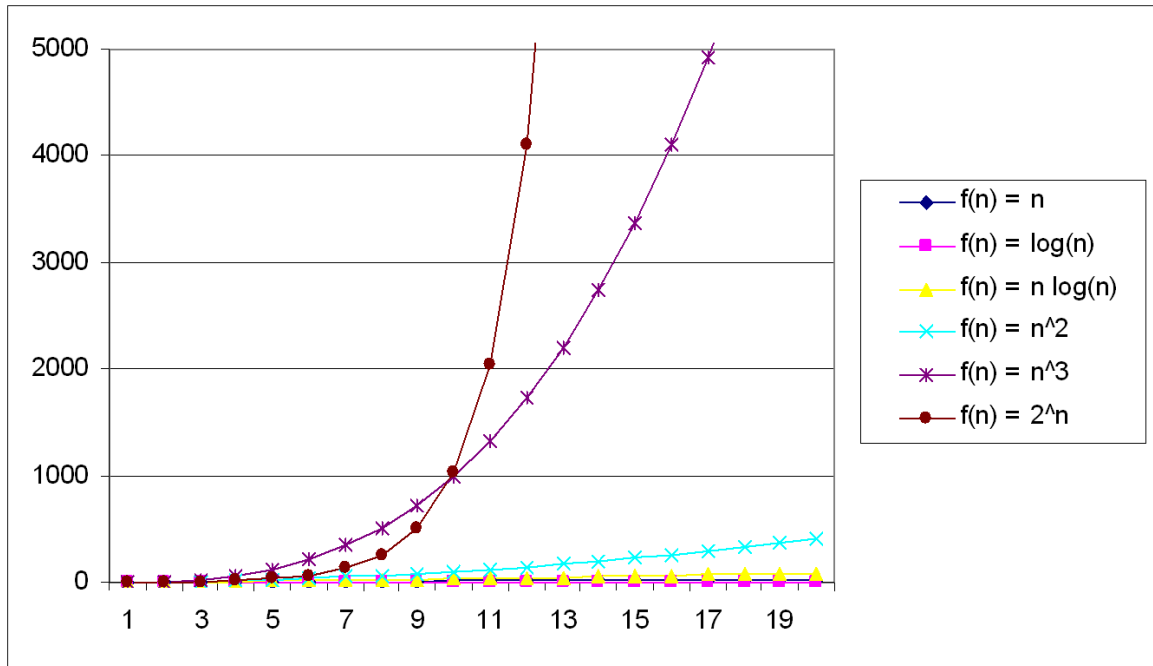
$\Theta(n^3)$ — cubic

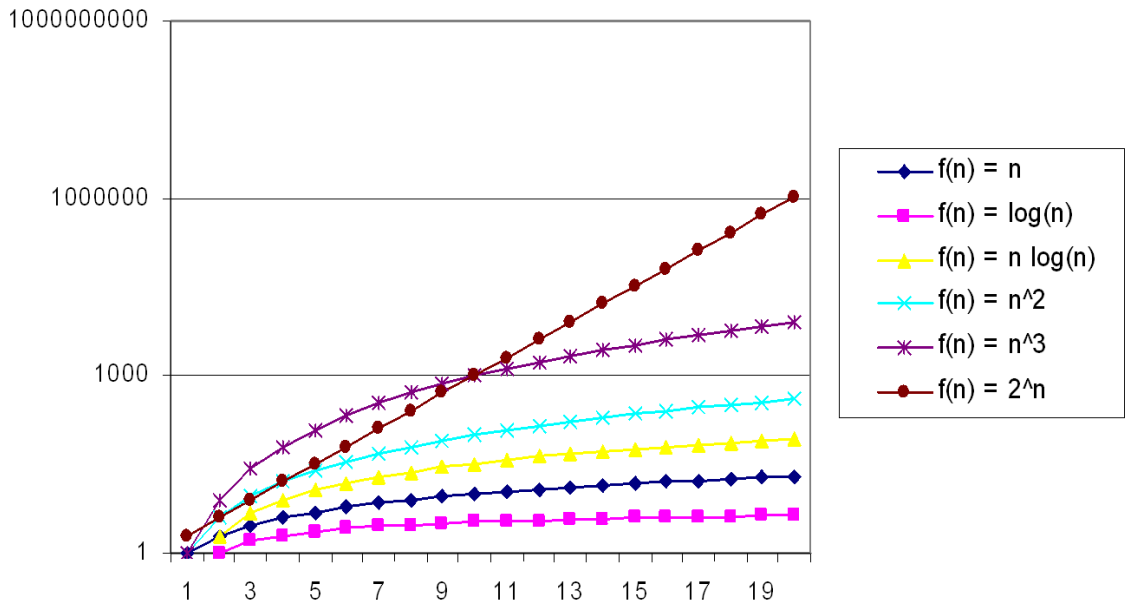
$\Theta(2^n), \Theta(c^n)$ — exponential











big-oh rules of thumb (1)

```
for (int i = 0; i < N; ++i)  
    foo();
```

runtime $\in \Theta(N \times (\text{runtime of foo}))$

```
for (int i = 0; i < N; ++i)  
    for (int j = 0; j < M; ++j)  
        bar();
```

runtime $\in \Theta(N \times (M \times \text{runtime of bar}))$

```
for (int i = 0; i < N; ++i)  
    for (int j = 0; j < i; ++j)  
        foo();
```

runtime $\in \Theta\left(\sum_{i=0}^N i \times \text{runtime of foo}\right) = \Theta(N^2 \cdot \text{runtime of foo})$

big-oh rules of thumb (1)

```
for (int i = 0; i < N; ++i)  
    foo();
```

runtime $\in \Theta(N \times (\text{runtime of foo}))$

time to increment i ?
“constant factor”
ignored by Θ

```
for (int i = 0; i < N; ++i)  
    for (int j = 0; j < M; ++j)  
        bar();
```

runtime $\in \Theta(N \times (M \times \text{runtime of bar}))$

```
for (int i = 0; i < N; ++i)  
    for (int j = 0; j < i; ++j)  
        foo();
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runtime $\in \Theta\left(\sum_{i=0}^N i \times \text{runtime of foo}\right) = \Theta(N^2 \cdot \text{runtime of foo})$

big-oh rules of thumb (1)

```
for (int i = 0; i < N; ++i)  
    foo();
```

runtime $\in \Theta(N \times (\text{runtime of foo}))$

nested loops — work inside out
find time of inner loop (“foo”)
multiply by iterations of outer loop

```
for (int i = 0; i < N; ++i)  
    for (int j = 0; j < M; ++j)  
        bar();
```

runtime $\in \Theta(N \times (M \times \text{runtime of bar}))$

```
for (int i = 0; i < N; ++i)  
    for (int j = 0; j < i; ++j)  
        foo();
```

runtime $\in \Theta\left(\sum_{i=0}^N i \times \text{runtime of foo}\right) = \Theta(N^2 \cdot \text{runtime of foo})$

big-oh rules of thumb (1)

```
for (int i = 0; i < N; ++i)
    foo();
```

runtime $\in \Theta(N \times (\text{runtime of foo}))$

```
for (int i = 0; i < N; ++i)
    for (int j = 0; j < M; ++j)
        bar();
```

runtime $\in \Theta(N \times (M \times \text{runtime of bar}))$

```
for (int i = 0; i < N; ++i)
    for (int j = 0; j < i; ++j)
        foo();
```

runtime $\in \Theta\left(\sum_{i=0}^N i \times \text{runtime of foo}\right) = \Theta(N^2 \cdot \text{runtime of foo})$

at least $N/2$ iterations with
at least $N/2$ calls to foo
 $\implies N/2 \cdot N/2 = N^2/4$
also $\leq N \cdot N = N^2$ calls
 $\implies \# \text{ calls to foo is } \Theta(N^2)$

big-oh rules of thumb (2)

```
foo();  
bar();
```

runtime = runtime of foo + runtime of bar
(but — constant factors don't matter for Θ , O)

```
if (quux()) {  
    foo();  
} else {  
    bar();  
}
```

runtime \approx runtime of quux + \max (runtime of foo, runtime of bar)
(max because we measure the **worst-case**)

$\Theta(1)$: **constant time**

constant time ($\Theta(1)$ time) — runtime does not depend on input

accessing an array element

linked list insert/delete (at known end)

getting a vector's size

...

is that really constant time

is getting vector's size really constant time?

vector stores its size, but, for, e.g. $N = 2^{10000}$, the size itself is huge

our *usual* assumption:

treat “sensible” integer arithmetic as constant time
(anything we'd keep in a `long` or smaller variable in practice?)

can do analysis where we don't assume this, usually not interesting

$\Theta(\log n)$: **logarithmic time**

binary search of sorted array

search space cut in half each iteration — $\lceil \log_2 N \rceil$ iterations

balanced tree search/insert

height of tree (somehow) gaurenteed to be $\Theta(\log N)$

$\Theta(n)$: **linear**

constant # operations/element

printing a list

search in unsorted array

search in linked list

doubling the size of a vector

$\Theta(n \log n)$: **log-linear**

fast comparison-based sorting

merge sort, heap sort, ...

quicksort *if pivot choices are good*

inserting n elements into a balanced tree

$\Theta(n^2)$: quadratic

slow comparison-based sorting

insertion sort, bubble sort, selection sort, ...

quicksort *if pivot choices are bad*

most doubly nested for loops that go up to n

$\Theta(2^{n^c})$, $c \geq 1$: **exponential**

n -bit solution; try every 2^n of the possibilities

crack a combination lock by trying every possibility

finding the best move in an $N \times N$ Go game (with Japanese rules)

checking satisfiability of Boolean expression*

the Traveling Salesman problem*

*known algorithms — maybe can do better?

more?

$\Theta(n^3)$ — find shortest paths between all pairs of n nodes on a fully-connected graph

approx. order $2^{n^{1/3}}$ — best known integer factorization algorithm