base-10 numbers

$$
\begin{aligned}
12345 &= 1 \cdot 10^4 + 2 \cdot 10^3 + 3 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0 \\
987.65 &= 9 \cdot 10^2 + 8 \cdot 10^1 + 7 \cdot 10^0 + 6 \cdot 10^{-1} + 5 \cdot 10^{-2}
\end{aligned}
$$

base-2 numbers

$$\begin{aligned}
20_{\textsf{TEN}} \quad (\textsf{or } 20_{10}) &= 11101_{\textsf{TWO}} \quad (\textsf{or } 11101_2) \\
&= 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\
4_{\textsf{TEN}} &= 100_{\textsf{TWO}} \\
&= 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 \\
1.25_{\textsf{TEN}} &= 1.01_{\textsf{TWO}} \\
&= 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2}
\end{aligned}$$

base-16 numbers

`0 1 2 3 4 5 6 7 8 9 A B C D E F`

$$
\begin{aligned}
15_{\mathsf{TEN}} &= & \mathtt{F}_{\mathsf{SIXTEEN}} &= & 15 \cdot 16^0 \\
100_{\mathsf{TEN}} &= & \mathtt{64}_{\mathsf{SIXTEEN}} &= & 6 \cdot 16^1 + 4 \cdot 16^0 \\
0.5_{\mathsf{TEN}} &= & \mathtt{0.8}_{\mathsf{SIXTEEN}} &= & 8 \cdot 16^{-1}
\end{aligned}
$$

integers in C++

15_{TEN}	`15`
17_{EIGHT}	`017`
$\mathrm{F}_{\text{SIXTEEN}}$	`0xF`

99_{TEN}	`99`
143_{EIGHT}	`0143`
63_{SIXTEEN}	`0x63`

16_{TEN}	`16`
20_{EIGHT}	`020`
10_{SIXTEEN}	`0x10`

terminology

base-X number — X is the radix

I will call components of base X number 'digits'

  but not a great term — digit sometimes implies base-10

  sometimes "radit"

  base-2 digit $=$ bit

  base-16 digit $=$ nibble (sometimes)

base-10 $=$ decimal

base-2 $=$ binary

base-8 $=$ octal

base-16 $=$ hexadecimal

convert to decimal

$$42_{\mathsf{FIVE}} =$$

$$121_{\mathsf{THREE}} =$$

convert to decimal

$$
\begin{aligned}
42_{\mathsf{FIVE}} &= 4 \cdot 5^1 + 2 \cdot 5^0 \\
&=
\end{aligned}
$$

$$
121_{\mathsf{THREE}} =
$$

convert to decimal

$$
\begin{aligned}
42_{\mathsf{FIVE}} &= 4 \cdot 5^1 + 2 \cdot 5^0 \\
&= 20_{\mathsf{TEN}} + 2 = 22_{\mathsf{TEN}}
\end{aligned}
$$

$$
121_{\mathsf{THREE}} =
$$

convert to decimal

$$
\begin{aligned}
42_{\textsf{FIVE}} &= 4 \cdot 5^1 + 2 \cdot 5^0 \\
&= 20_{\textsf{TEN}} + 2 = 22_{\textsf{TEN}}
\end{aligned}
$$

$$
\begin{aligned}
121_{\textsf{THREE}} &= 1 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0 \\
&=
\end{aligned}
$$

convert to decimal

$$
\begin{aligned}
42_{\mathsf{FIVE}} &= 4 \cdot 5^1 + 2 \cdot 5^0 \\
&= 20_{\mathsf{TEN}} + 2 = 22_{\mathsf{TEN}}
\end{aligned}
$$

$$
\begin{aligned}
121_{\mathsf{THREE}} &= 1 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0 \\
&= 9 + 6 + 1 = 16_{\mathsf{TEN}}
\end{aligned}
$$

convert to something (1)

$$42_{\mathsf{TEN}} \text{ as radix } 5 \quad =$$

convert to something (1)

$$
\begin{aligned}
42_{\text{TEN}} \text{ as radix 5} &= \underline{\hspace{2em}}\textcolor{red}{2} \\
\\
42 \div 5 &= 8 + \ldots \\
42 \bmod 5 &= 2 \\
42 &= 8 \cdot 5 + \textcolor{red}{2}
\end{aligned}
$$

convert to something (1)

$$
\begin{aligned}
42_{\mathsf{TEN}} \text{ as radix 5} &= \underline{}{\color{red}3}2 \\
\\
42 \div 5 &= 8 + \ldots \\
42 \bmod 5 &= 2 \\
42 &= 8 \cdot 5 + 2 \\
8 &= 1 \cdot 5 + {\color{red}3}
\end{aligned}
$$

convert to something (1)

$$\begin{aligned} 42_{\mathsf{TEN}} \text{ as radix 5} &= 132_{\mathsf{FIVE}} \\ \\ 42 \div 5 &= 8 + \ldots \\ 42 \bmod 5 &= 2 \\ 42 &= 8 \cdot 5 + 2 \\ 8 &= 1 \cdot 5 + 3 \\ 1 & \end{aligned}$$

convert to something (2)

121_{TEN} as radix 11 $=$

convert to something (2)

$$
\begin{aligned}
121_{\mathsf{TEN}} \text{ as radix } 11 &= \underline{\hspace{2em}}{\color{red}0}_{ELEVEN} \\
121 \div 11 &= 11 \\
121 \bmod 11 &= 0 \\
121 &= 11 \cdot 11 + {\color{red}0}
\end{aligned}
$$

convert to something (2)

$$\begin{aligned}
121_{\text{TEN}} \text{ as radix } 11 &= _00_{ELEVEN} \\
\\
121 \div 11 &= 11 \\
121 \bmod 11 &= 0 \\
121 &= 11 \cdot 11 + 0 \\
11 &= 1 \cdot 11 + 0
\end{aligned}$$

convert to something (2)

$$\begin{aligned} 121_{\textsf{TEN}} \text{ as radix } 11 &= 100_{ELEVEN} \\ \\ 121 \div 11 &= 11 \\ 121 \bmod 11 &= 0 \\ 121 &= 11 \cdot 11 + 0 \\ 11 &= 1 \cdot 11 + 0 \\ 1 & \end{aligned}$$

special case: base-16 to base-2

each “nibble” (hexadecimal digit) $=$ 4 binary bits

1	2	3	4_{SIXTEEN}

special case: base-16 to base-2

each “nibble” (hexadecimal digit) $=$ 4 binary bits

1	2	3	4_{SIXTEEN}
0001	0010	0011	0100_{TWO}

special case: base-16 to base-2

each "nibble" (hexadecimal digit) = 4 binary bits

1	2	3	4_{SIXTEEN}
0001	0010	0011	0100_{TWO}

special case: base-16 to base-2

each "nibble" (hexadecimal digit) = 4 binary bits

1	2	3	4_{SIXTEEN}
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special case: base-16 to base-2

each "nibble" (hexadecimal digit) = 4 binary bits

1	2	3	4_{SIXTEEN}
0001	0010	0011	0100_{TWO}

1101	1110	0011	0000_{TWO}

special case: base-16 to base-2

each "nibble" (hexadecimal digit) $=$ 4 binary bits

1	2	3	4_{SIXTEEN}
0001	0010	0011	0100_{TWO}

1101	1110	0011	0000_{TWO}
C	D	3	0_{SIXTEEN}

a note on bytes

one byte = one "octet" =
two nibbles (hexadecimal digits) =
eight bits

this class — byte is always eight bits

(some very old machines sometimes called different sizes "bytes")

a note on bytes

one byte = one "octet" =
two nibbles (hexadecimal digits) =
eight bits

this class — byte is always eight bits
(some very old machines sometimes called different sizes "bytes")

a note on bytes

one byte = one "octet" =
two nibbles (hexadecimal digits) =
eight bits

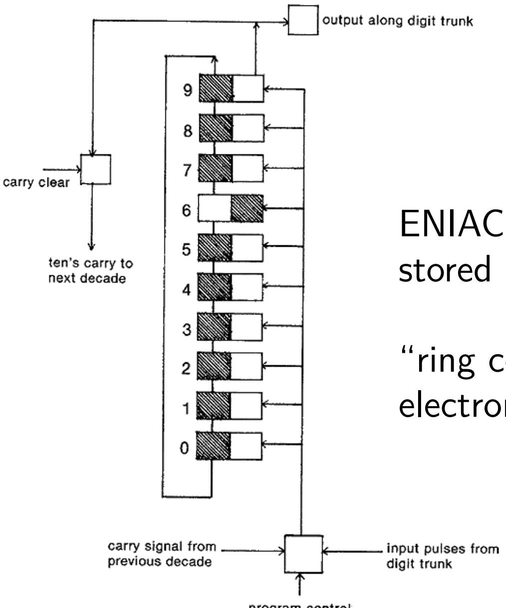
this class — byte is always eight bits
(some very old machines sometimes called different sizes "bytes")

integer representation

modern machine represent integers as series of bits (base-2)

why not base-10?

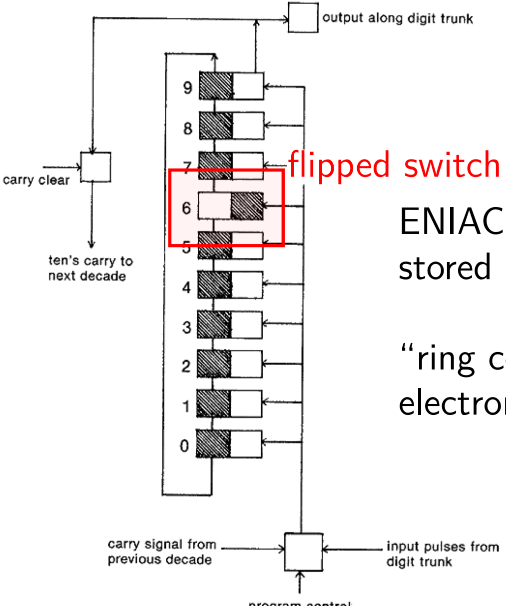
ENIAC: base-10 representation



ENIAC: 1946 computer
stored base-10 digits

"ring counter" of ten
electronic switches per digit

ENIAC: base-10 representation

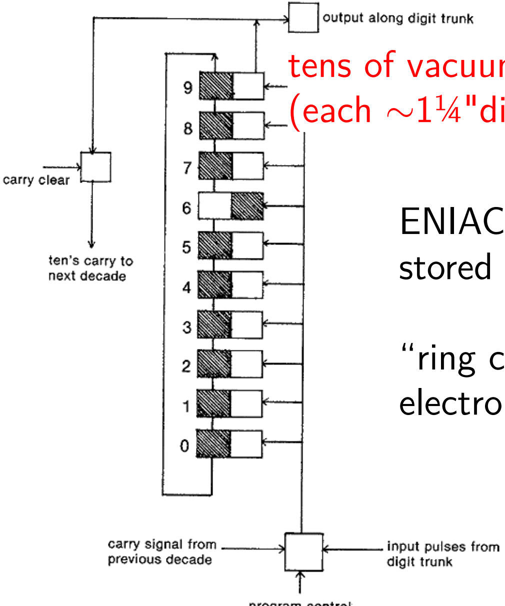


flipped switch indicates digit stored

ENIAC: 1946 computer
stored base-10 digits

"ring counter" of ten
electronic switches per digit

ENIAC: base-10 representation



tens of vacuum tubes total
(each ∼1¼"diameter by 2¾"height)

ENIAC: 1946 computer
stored base-10 digits

“ring counter” of ten
electronic switches per digit

base-2 representation

base 2 — each switch represents one “digit”

- much more efficient use of switches

used in some pre-ENIAC electronic computers

- Atanasoff-Berry computer (1937, Ohio State)
- Z3 (1941, German Laboratory for Aviation)

quote: “A Survey of Digital Computer Memory Systems”, Proceedings of the Institute of Radio Engineers, October 1953

base-2 representation

base 2 — each switch represents one "digit"

- much more efficient use of switches

used in some pre-ENIAC electronic computers

- Atanasoff-Berry computer (1937, Ohio State)
- Z3 (1941, German Laboratory for Aviation)

why not used in ENIAC?

- Eckert (ENIAC designer), 1953: "Although [binary-based digit counters] were known at the time of the construction of the ENIAC, it was not used because it required stable resistors, which were then much more expensive than they are now."
- also, important to input/output decimal digits directly

quote: "A Survey of Digital Computer Memory Systems", Proceedings of the Institute of Radio Engineers, October 1953

base-2 bit addition

+	0	1
0	00	01
1	01	10

base-2 bit addition

+	0	1
0	00	01
1	01	10

exactly one set to 1 — result is 1; otherwise 0

base-2 bit addition

+	0	1
0	00	01
1	01	10

exactly one set to 1 — result is 1; otherwise 0

both set to 1 — carry is 1; otherwise 0

base-2 capacity

$$
\begin{aligned}
n\text{-bit number:} \qquad & b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0 \\
= \; & \sum_{i=0}^{n-1} b_i \cdot 2^i \\
\leq \; & \sum_{i=0}^{n-1} 1 \cdot 2^i = 2^{n-1}
\end{aligned}
$$

base-2 capacity

n-bit number: $\quad b_{n-1}b_{n-2}b_{n-3}\ldots b_2 b_1 b_0$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^i$$

$$\leq \sum_{i=0}^{n-1} 1 \cdot 2^i = 2^{n-1}$$

base-2 capacity

$$
\begin{aligned}
n\text{-bit number:} \qquad & b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0 \\
= \;\; & \sum_{i=0}^{n-1} b_i \cdot 2^i \\
\leq \;\; & \sum_{i=0}^{n-1} 1 \cdot 2^i = 2^{n-1}
\end{aligned}
$$

missing pieces:

- negative numbers?
- non-whole numbers?
- what is n?

base-2 capacity

$$
\begin{aligned}
n\text{-bit number:} \quad & b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0 \\
= \quad & \sum_{i=0}^{n-1} b_i \cdot 2^i \\
\leq \quad & \sum_{i=0}^{n-1} 1 \cdot 2^i = 2^{n-1}
\end{aligned}
$$

missing pieces:

- negative numbers?
- non-whole numbers?
- what is n?

integer size in C++

varies between machines

- compiler uses what makes most sense on each machine?

type	size in bits	
	minimum	on lab machines
`unsigned char`	8	8
`unsigned short`	16	16
`unsigned int`	16	32
`unsigned long`	32	64

integer size in C++

varies between machines

compiler uses what makes most sense on each machine?

type	size in bits: minimum	size in bits: on lab machines
`unsigned char`	8	8
`unsigned short`	16	16
`unsigned int`	16	32
`unsigned long`	32	64

"unsigned" — can't be negative (no sign)

integer size in C++

varies between machines

- compiler uses what makes most sense on each machine?

type	size in bits: minimum	size in bits: on lab machines
`unsigned char`	8	8
`unsigned short`	16	16
`unsigned int`	16	32
`unsigned long`	32	64

minimum size required by standard for all C++ compilers
all allowed to be bigger

querying sizes in C++

```
#include <climits>  // C: <limits.h>
...
ULONG_MAX or UINT_MAX or USHRT_MAX or UCHAR_MAX
// e.g. USHRT_MAX == 65535 on lab machines
```

```
#include <limits>
...
std::numeric_limits<unsigned long>::max()
    // == ULONG_MAX
...
```

```
sizeof(unsigned long)  // number of *bytes*
    // == 8 on lab machines
...
```

numbering bits

option 1: n-bit number: $b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^i$$

option 2: n-bit number: $b_0b_1b_2\ldots b_{n-3}b_{n-2}b_{n-1}$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^{n-i-1}$$

numbering bits

option 1: n-bit number: $b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^i$$

option 2: n-bit number: $b_0b_1b_2\ldots b_{n-3}b_{n-2}b_{n-1}$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^{n-i-1}$$

two viable ways to number bits

numbering bits

$$\begin{aligned}
\text{option 1: } n\text{-bit number:} \quad & b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0 \\
= \quad & \sum_{i=0}^{n-1} b_i \cdot 2^i \\
\text{option 2: } n\text{-bit number:} \quad & b_0b_1b_2\ldots b_{n-3}b_{n-2}b_{n-1} \\
= \quad & \sum_{i=0}^{n-1} b_i \cdot 2^{n-i-1}
\end{aligned}$$

two viable ways to number bits

does it matter which I use?

numbering bytes

$$\begin{aligned}
\text{option 1: 4-byte number:} \quad & B_3 B_2 B_1 B_0 \\
= \quad & \sum_{i=0}^{3} B_i \cdot 256^i \\
\text{option 2: } n\text{-bit number:} \quad & B_0 B_1 B_2 B_3 \\
= \quad & \sum_{i=0}^{n3} b_i \cdot 256^{3-i}
\end{aligned}$$

numbering bytes

option 1: 4-byte number: $B_3 B_2 B_1 B_0$

$$= \sum_{i=0}^{3} B_i \cdot 256^i$$

option 2: n-bit number: $B_0 B_1 B_2 B_3$

$$= \sum_{i=0}^{n3} b_i \cdot 256^{3-i}$$

two viable ways to number bytes

numbering bytes

$$
\begin{aligned}
\text{option 1: 4-byte number:} \quad & B_3B_2B_1B_0 \\
= \quad & \sum_{i=0}^{3} B_i \cdot 256^i \\
\text{option 2: } n\text{-bit number:} \quad & B_0B_1B_2B_3 \\
= \quad & \sum_{i=0}^{n3} b_i \cdot 256^{3-i}
\end{aligned}
$$

two viable ways to number bytes

does it matter which I use?

memory

memory
(as 64-bit values)

addr.	value (64-bit)
0	123999
8	323232
16	434093
…	…

memory

if big endian (as 8-bit values)

addr.	value (8-bit)
0	0
1	0
2	0
3	0
4	0
5	1
6	228
7	95
8	0
9	0
10	0
11	…
…	…

memory (as 64-bit values)

addr.	value (64-bit)
0	123999
8	323232
16	434093
…	…

if little endian (as 8-bit values)

addr.	value (8-bit)
0	95
1	228
2	1
3	0
4	0
5	0
6	0
7	0
8	160
9	238
10	4
11	…
…	…

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

---

little endian (e.g. lab machine):

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

---

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << " ";
    }
    ...
}
```

get pointer to byte with lowest address in `value`

little endian (e.g. lab m

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

unless you do something like this
won't see endianness

little endian (e.g. la

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

use pointer to get ith byte of value
(cast to int to output as number, not character)

---

little endian

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

---

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

little endian: byte 0 is least significant
(affects overall value the least)

---

little endian (e.g.

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

---

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

big endian: byte 0 is most significant (affects overall value the most)

little endian (e.g.

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

but we don't write numbers in a different order based on which end we call "part 0"

little endian

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

little versus big endian

little endian — least significant part has lowest address

i.e. index 0 is the one's place

big endian — most significant part has the lowest address

i.e. index $n-1$ is the one's place

endianness in the real world

today and this course: little endian is dominant

- e.g. x86, *typically* ARM

historically: big endian was dominant

- e.g. *typically* SPARC, POWER, Alpha, MIPS, …
- still commonly used for networking because of this

many architectures have switchable endianness

- e.g. ARM, SPARC, POWER, MIPS
- usually, OS chooses endianness

middle endian

sometimes not just big/little endian

e.g. number bytes most to least significant as
5, 6, 7, 8, 1, 2, 3, 4

e.g. doubles on little-endian ARM

generally some sort of historical accident

- e.g. ARM floating point designed for big endian?

endianness is about addresses

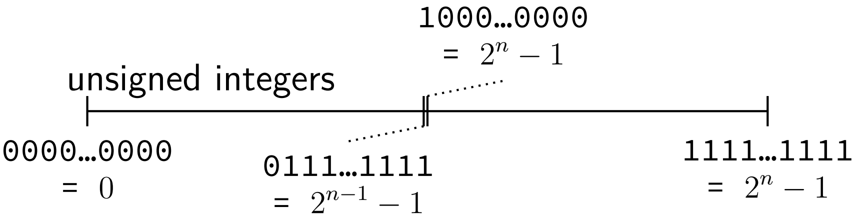
endianness is about numbering,
not (necessairily) placement on the page

addr.	value
0	95
1	228
2	1
3	0
4	0
5	0
6	0
7	0
8	160
9	238
10	4
11	…
…	…

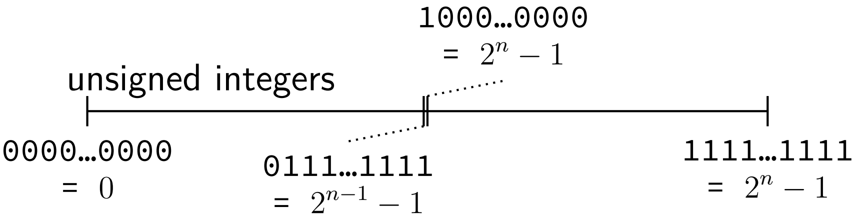
=

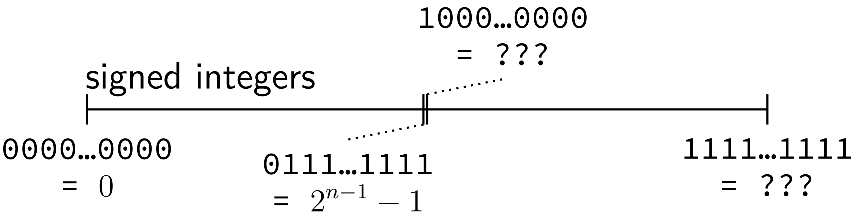
addr.	value
…	…
11	…
10	4
9	238
8	160
7	0
6	0
5	0
4	0
3	0
2	1
1	228
0	95

representing negative numbers

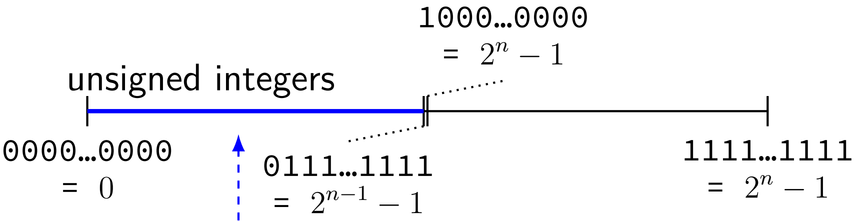


representing negative numbers



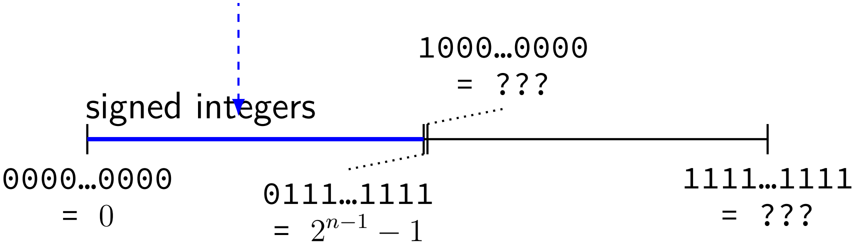


representing negative numbers

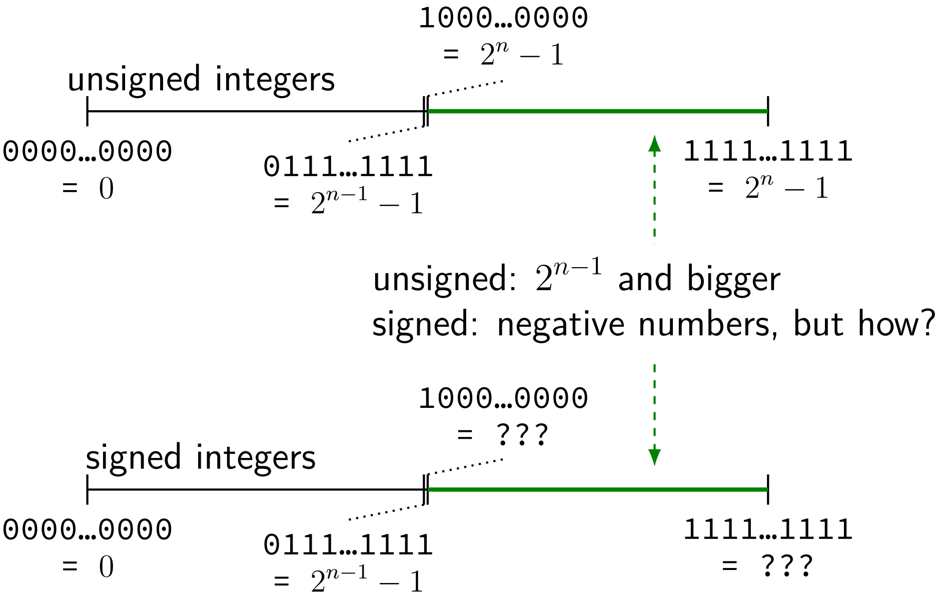


positive numbers up to $2^n - 1$

goal: same bits, signed or not

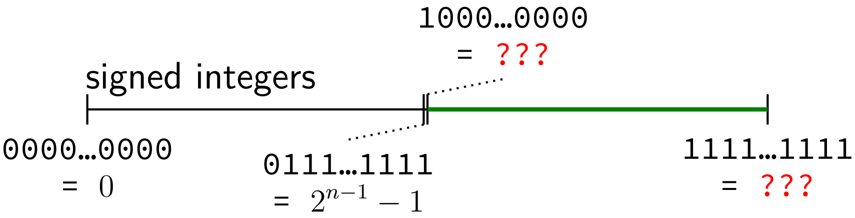


representing negative numbers



representing negative numbers

	sign & magnitude	1's complement	2's complement
`000...000`	0	0	`0`
`011...111`	$2^{n-1}-1$	$2^{n-1}-1$	$2^{n-1}-1$
`100...000`	0	$-2^{n-1}+1$	-2^{n-1}
`111...111`	$-2^{n-1}+1$	0	-1

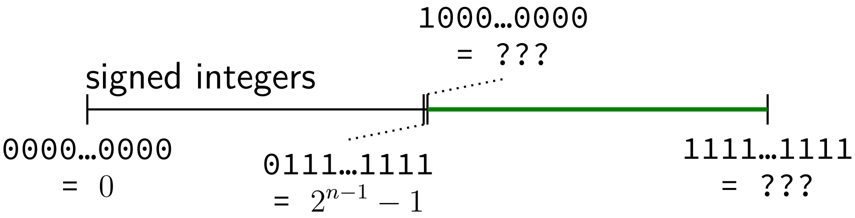


representing negative numbers

	sign & magnitude	1's complement	2's complement
`000...000`	0	0	`0`
`011...111`	$2^{n-1}-1$	$2^{n-1}-1$	$2^{n-1}-1$
`100...000`	0	$-2^{n-1}+1$	-2^{n-1}
`111...111`	$-2^{n-1}+1$	0	-1

two representations of zero?

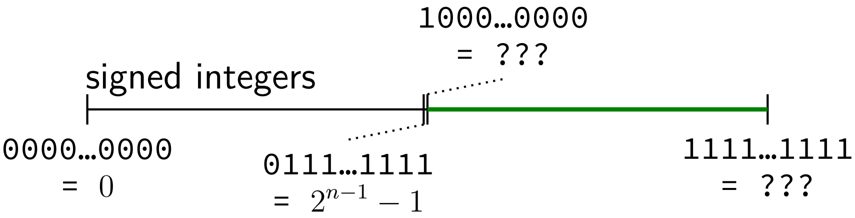
x == y needs to do something special



representing negative numbers

	sign & magnitude	1's complement	2's complement
`000…000`	0	0	`0`
`011…111`	$2^{n-1}-1$	$2^{n-1}-1$	$2^{n-1}-1$
`100…000`	0	$-2^{n-1}+1$	-2^{n-1}
`111…111`	$-2^{n-1}+1$	0	-1

more negative values than positive values?



representing negative numbers

	sign & magnitude	1's complement	2's complement
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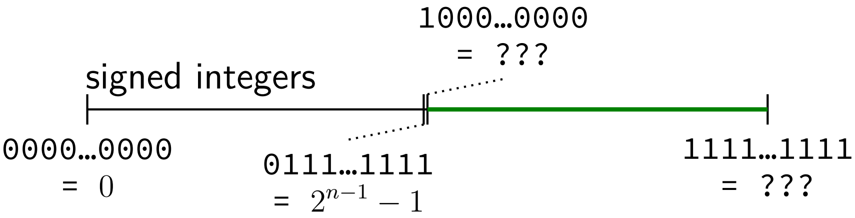
all 1's — least negative?



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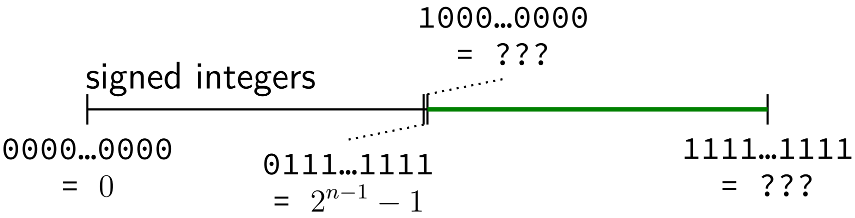
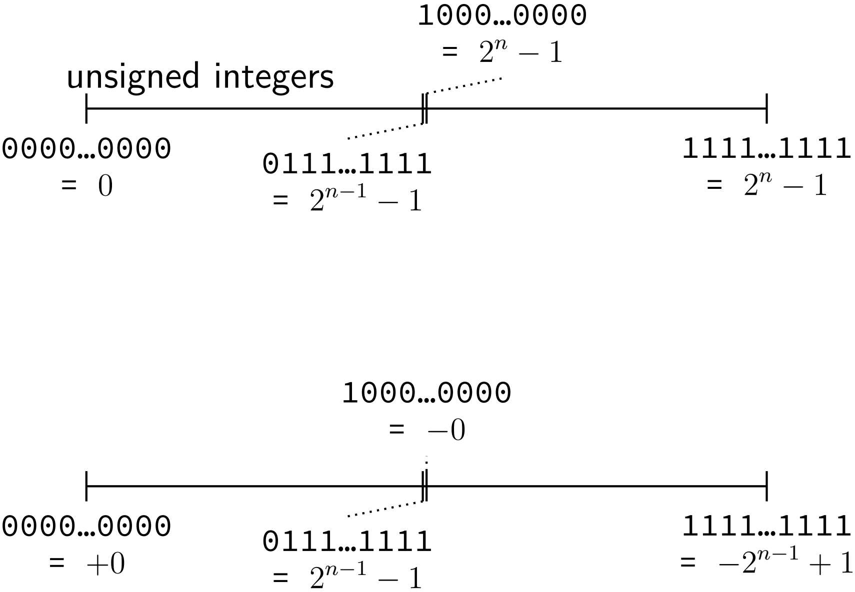
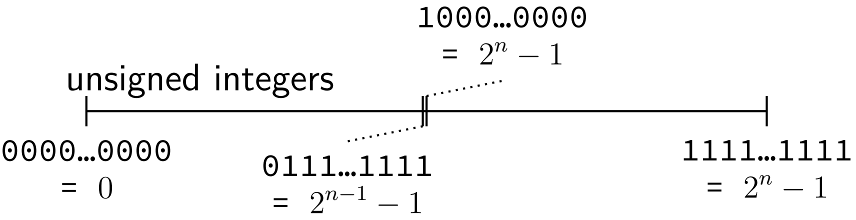
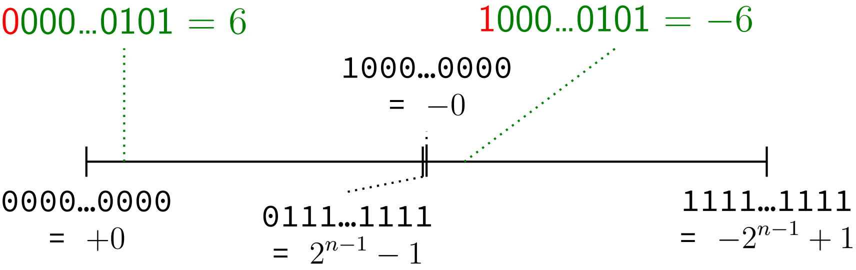


sign and magnitude



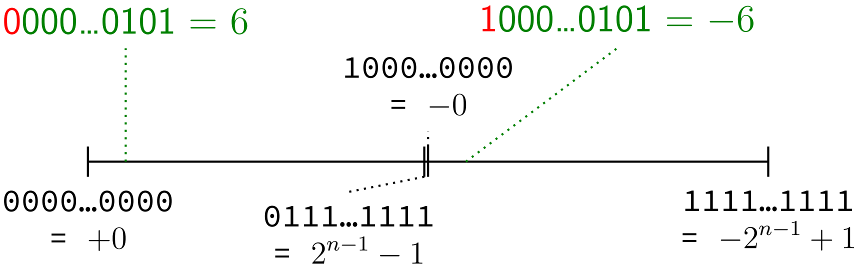
sign and magnitude





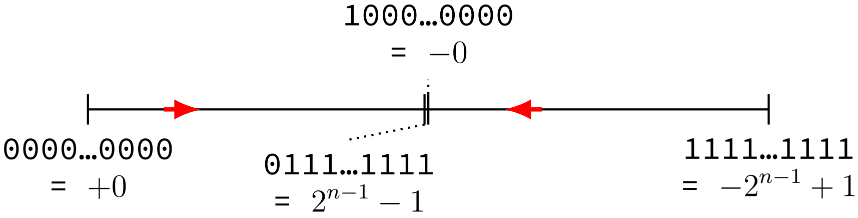
sign and magnitude

first bit is “sign bit” — 0 = positive, 1 = negative
flip sign bit to negate number

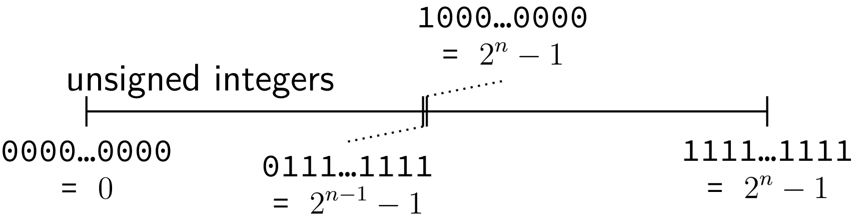


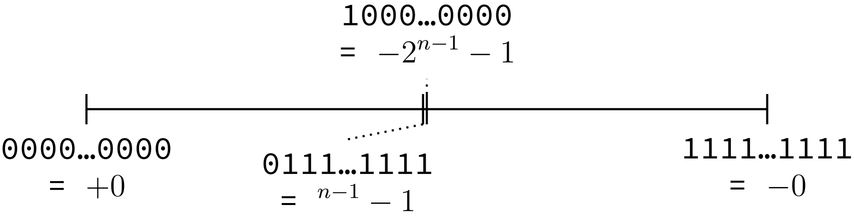
sign and magnitude

adding 1
different direction if negative

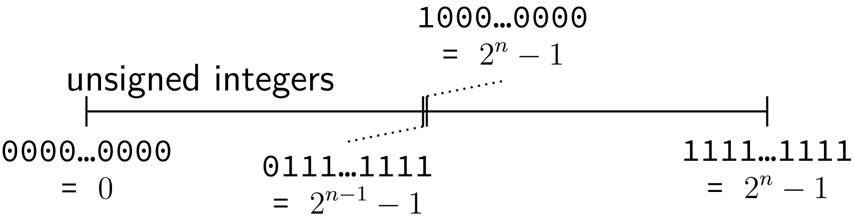


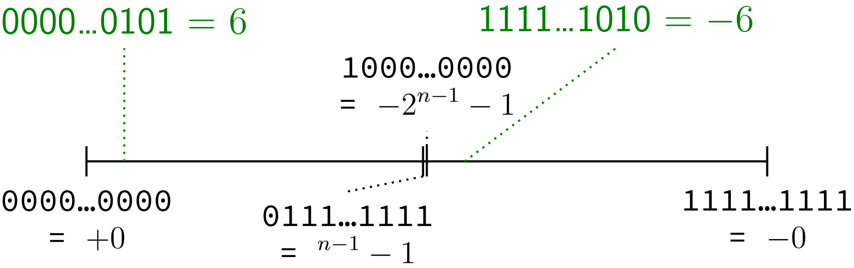
1's complement





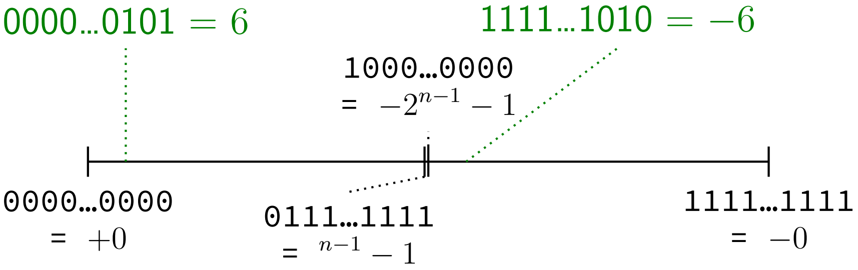
1's complement





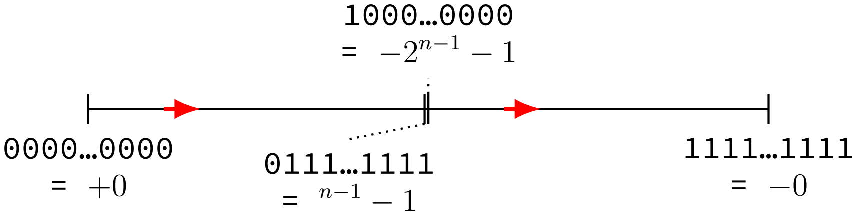
1's complement

flip all bits to negate number

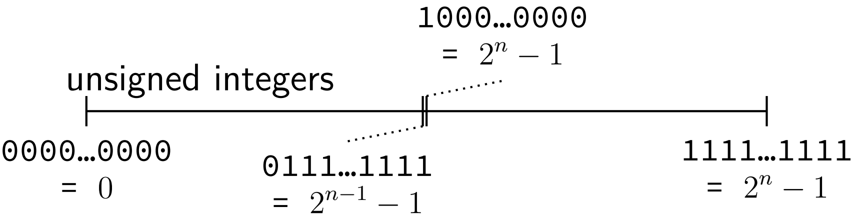


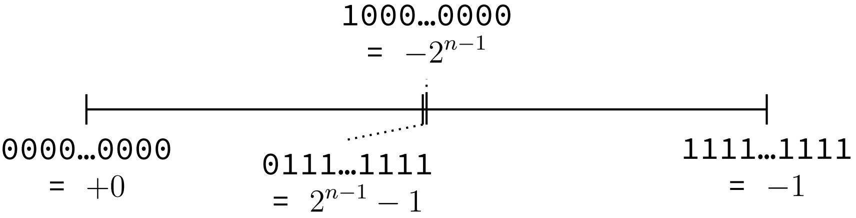
1's complement

adding 1
same direction, no matter original sign

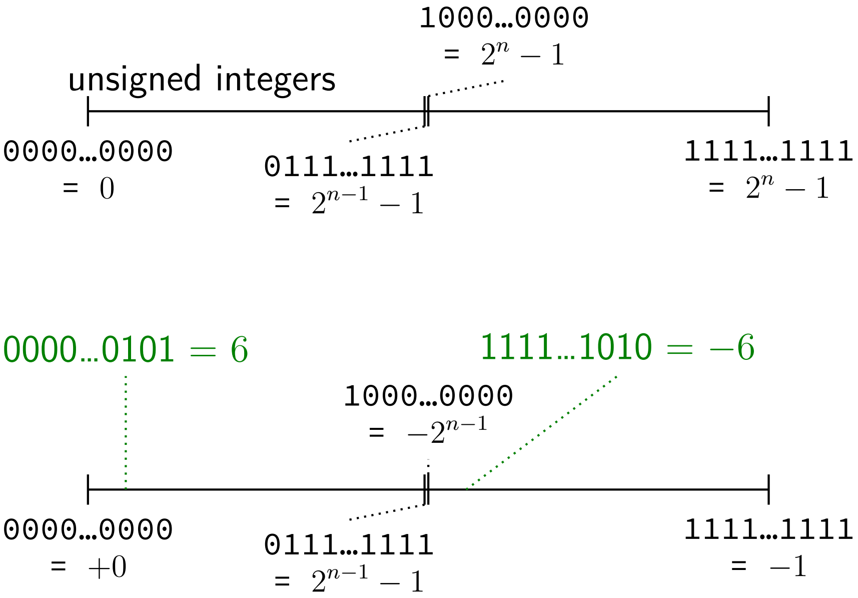


two's complement



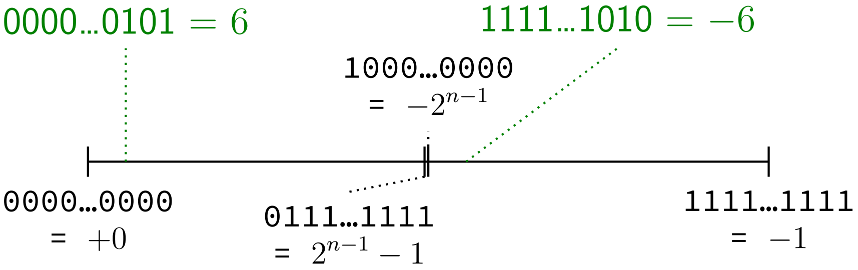


two's complement



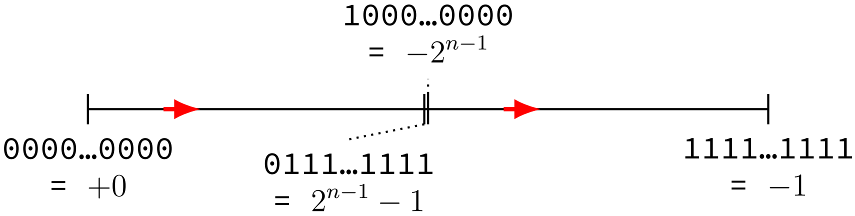
two's complement

flip all bits and add 1 to negate number



two's complement

adding 1
same direction, no matter original sign



2's complement (alt. perspective)

2's complement (5 bit)

$+10 =$	`0`	`1`	`0`	`1`	`0`
	$0 \cdot (-2^4)$ +	$1 \cdot 2^3$ +	$0 \cdot 2^2 +$ +	$1 \cdot 2^1 +$ +	$0 \cdot 2^0$
	0 +	2^3 +	0 +	2^1 +	$0 = 10$
-10 $=$	`1`	`0`	`1`	`1`	`0`
	$1 \cdot (-2^4)$ +	$0 \cdot 2^3$ +	$1 \cdot 2^2$ +	$1 \cdot 2^1$ +	$0 \cdot 2^0$
	-2^4 +	0 +	2^2 +	2^1 +	$0 = -10$

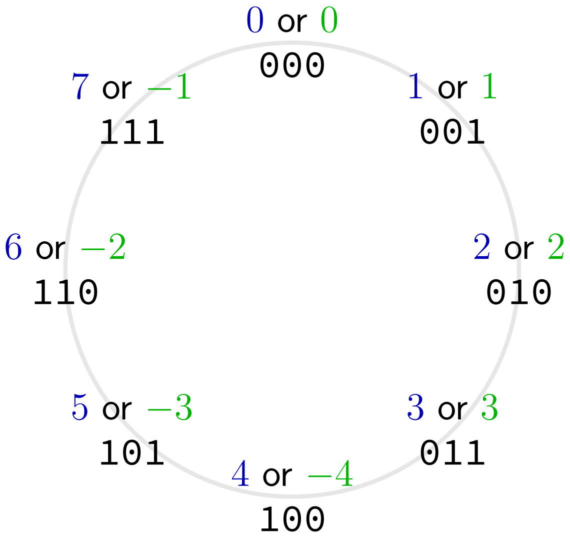
2's complement (alt. perspective)

2's complement (5 bit)

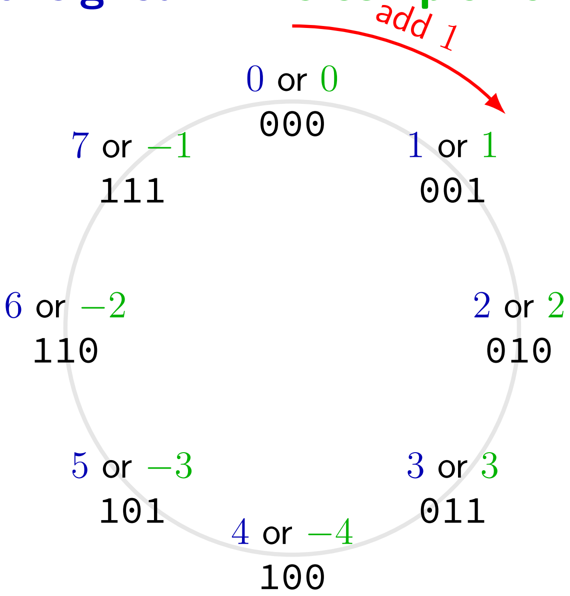
$+10 =$	0	1	0	1	0
	$0 \cdot (-2^4)$ +	$1 \cdot 2^3$ +	$0 \cdot 2^2 +$ +	$1 \cdot 2^1 +$ +	$0 \cdot 2^0$
	0 +	2^3 +	0 +	2^1 +	$0 = 10$
-10 $=$	1	0	1	1	0
	$1 \cdot (-2^4)$ +	$0 \cdot 2^3$ +	$1 \cdot 2^2$ +	$1 \cdot 2^1$ +	$0 \cdot 2^0$
	-2^4 +	0 +	2^2 +	2^1 +	$0 = -10$

"-2^4s place"

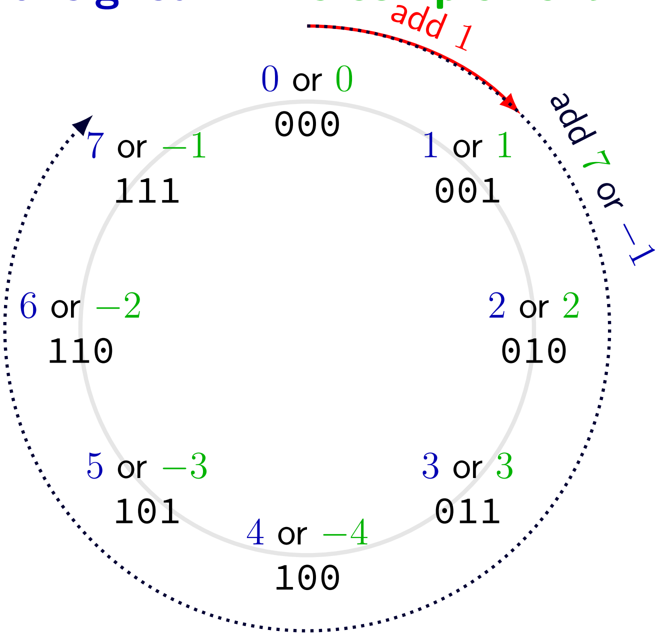
unsigned v. 2's complement



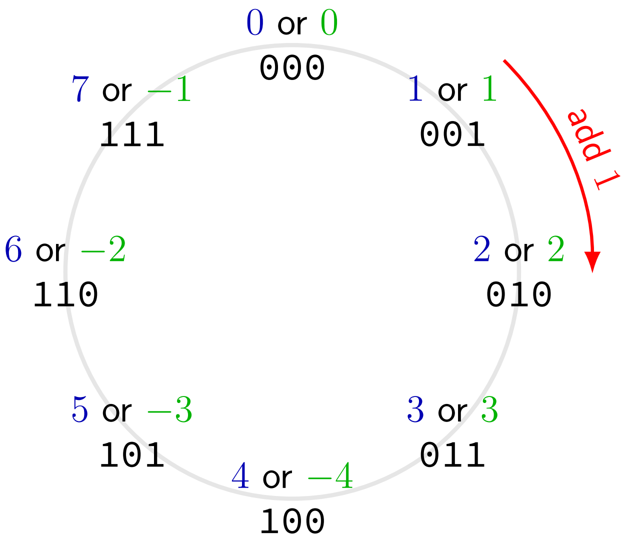
unsigned v. 2's complement



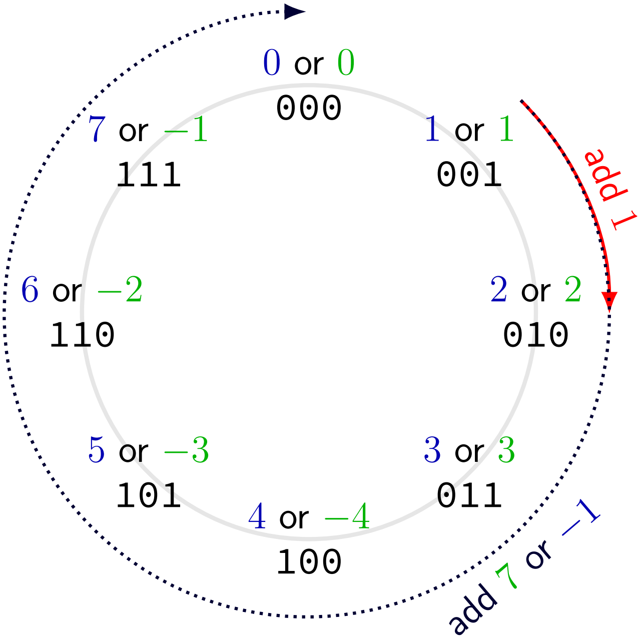
unsigned v. 2's complement



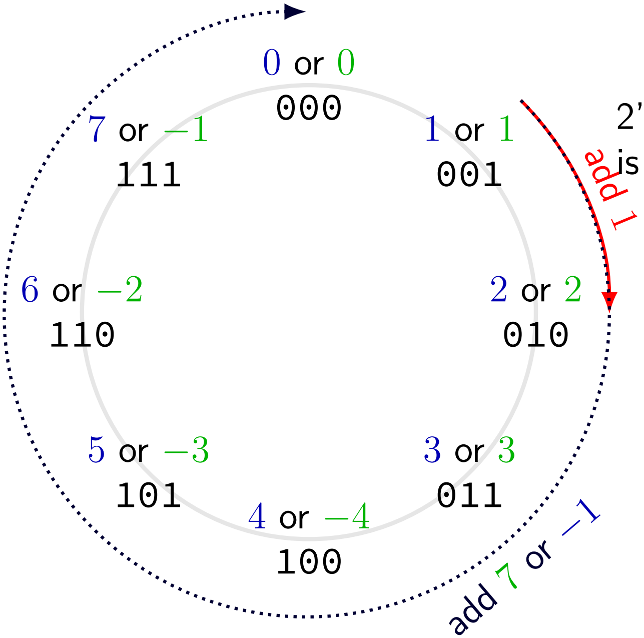
unsigned v. **2's complement**



unsigned v. 2's complement



unsigned v. 2's complement



2's complement addition
is same as unsigned addition

converting to 2's complement (version 1)

take absolute value, convert to bits

if negative, flip all the bits and add one

$-14 \to -00001110 \to 11110001 + 1 \to 11110010$

$-127 \to -01111111 \to 10000000 + 1 \to 10000001$

$-128 \to -10000000 \to 01111111 + 1 \to 10000000$

converting to 2's complement (version 2)

if negative, take absolute value, subtract from 2^n, encode that

$-14 \rightarrow 2^8 - 14 = 242 \rightarrow 11110010$

$-127 \rightarrow 2^8 - 127 = 129 \rightarrow 10000001$

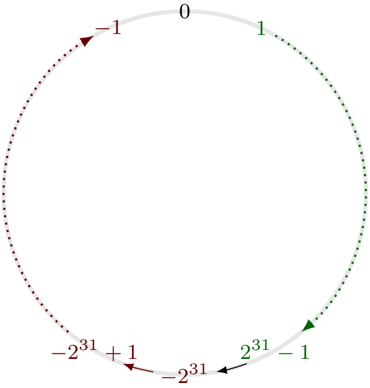
$-128 \rightarrow 2^8 - 127 = 129 \rightarrow 10000000$

two's complement summary

$$
\begin{array}{rccccccc}
 & -2^{31} & +2^{30} & +2^{29} & & +2^{2} & +2^{1} & +2^{0} \\
-1 = & 1 & 1 & 1 & \ldots & 1 & 1 & 1
\end{array}
$$

two's complement summary

$$-1 = \begin{array}{ccccccc} -2^{31} & +2^{30} & +2^{29} & & +2^{2} & +2^{1} & +2^{0} \\ 1 & 1 & 1 & \ldots & 1 & 1 & 1 \end{array}$$



two's complement summary

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\begin{array}{lccccccc}
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-1 = & 1 & 1 & 1 & \ldots & 1 & 1 & 1
\end{array}
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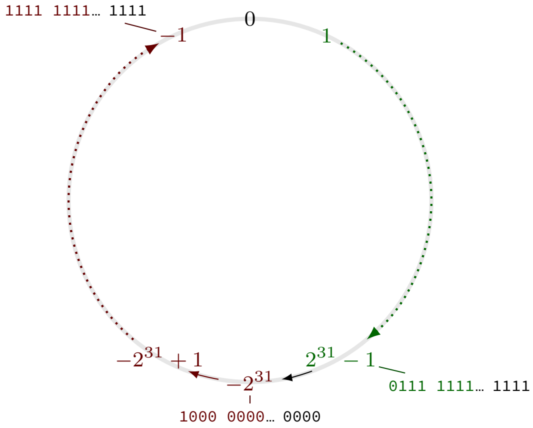

some real numbers

$$\frac{1}{3}$$

$$-\frac{100}{7}$$

$$\pi$$

$$0.1$$

$$\sqrt{2}$$

...

want to represent these: accurately? compactly? efficiently?

fixed point

$$\begin{aligned}
\frac{1}{3} &= 0.101010101\ldots_{\textsf{TWO}} \\
&\approx +0000.1010_{\textsf{TWO}} \text{— represent as } \texttt{00000 1010} \\
\frac{100}{7} &= 1110.001001001\ldots_{\textsf{TWO}} \\
&\approx -1110.0010_{\textsf{TWO}} \text{— represent as } \texttt{01110 0010}
\end{aligned}$$

fixed point

$$
\begin{aligned}
\frac{1}{3} &= 0.101010101\ldots_{\text{TWO}} \\
&\approx +0000.1010_{\text{TWO}} \text{— represent as } \texttt{00000 1010} \\
\frac{100}{7} &= 1110.001001001\ldots_{\text{TWO}} \\
&\approx -1110.0010_{\text{TWO}} \text{— represent as } \texttt{01110 0010}
\end{aligned}
$$

$x \approx y/2^K$ — represent with fixed-sized signed integer y

this case: $y/2^4$ and y is 9 bits.

why fixed-point?

$x \approx y/2^K$ (y fixed-sized singed integer)

math similar to integer math:

- addition/subtraction — same
- multiplication — same except divide by 2^K
- division — same except multiply by 2^K

easy to understand what values are represented well

why not fixed-point?

pretty small range of numbers for space used

hard to choose a 2^K that works for lots of applications

recall (?): scientific notation

$$
\begin{aligned}
+\frac{1}{3} &= +0.33333333\ldots \\
&\approx +3.33 \cdot 10^{-1} \\
-\frac{100}{7} &= -14.285714\ldots \\
&\approx -1.42 \cdot 10^{+1}
\end{aligned}
$$

recall (?): scientific notation

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&\approx -1.42 \cdot 10^{+1}
\end{aligned}
$$

$\pm$mantissa $\cdot$ base$^{\text{exponent}}$

recall (?): scientific notation

$$\begin{aligned} +\frac{1}{3} &= +0.33333333\ldots \\ &\approx +3.33 \cdot 10^{-1} \\ -\frac{100}{7} &= -14.285714\ldots \\ &\approx -1.42 \cdot 10^{+1} \end{aligned}$$

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$\pm\text{mantissa} \cdot \text{base}^{\text{exponent}}$

recall (?): scientific notation

$$
\begin{aligned}
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&\approx -1.42 \cdot 10^{+1}
\end{aligned}
$$

$\pm$mantissa $\cdot$ base$^{\text{exponent}}$

base-2 scientific notation

$$
\begin{aligned}
\frac{1}{3} &= 0.101010101\ldots_{\text{TWO}} \\
&\approx 0.1010101010_{\text{TWO}} = +1.0101010101_{\text{TWO}} \cdot 2^{-}1 \\
\frac{100}{7} &= 1110.001001001\ldots_{\text{TWO}} \\
&\approx -1110.0010010_{\text{TWO}} = -1.1100010010_{\text{TWO}} \cdot 2^{3}
\end{aligned}
$$

IEEE half-precision floating point

$$-1.1100010010_{\text{TWO}} \cdot 2^{3}$$

IEEE half-precision floating point

$-1.1100010010_{\text{TWO}} \cdot 2^{3}$

sign (1 bit)

0 for $+$
1 for $-$

mantissa (10 bits)

don't store leading "1."
(because always present)

exponent (5 bits)

store "$3+15$"
15 is 'bias'

IEEE half-precision floating point

$-1.1100010010_{\text{TWO}} \cdot 2^{3}$

sign (1 bit)

0 for $+$
1 for $-$

mantissa (10 bits)

don't store leading "$1.$"
(because always present)

exponent (5 bits)

store "$3 + 15$"
15 is 'bias'

`1 10010 1100010010`

IEEE half-precision floating point

$$-1.1100010010_{\text{TWO}} \cdot 2^{3}$$

sign (1 bit)

0 for $+$
1 for $-$

mantissa (10 bits)

don't store leading "1."
(because always present)

exponent (5 bits)

store "$3 + 15$"
15 is 'bias'

`1 10010 1100010010`

on typical little endian system:
byte 0: `00010010`
byte 1: `11001011`

IEEE half precision float

1 sign bit (1 for negative)

5 expontent bits

bias of 15 — if bits as unsigned are e, exponent is $E = e - 127$

10 mantissa bits

leading "1." not stored

$$\text{value} = (1 - 2 \cdot \text{sign}) \cdot (1.\text{mantissa}_{\text{TWO}}) \cdot 2^{\text{exponent}-15}$$

other IEEE precisions

	half	single	double	quad
C++*/Java type	—	`float`	`double`	—
sign bits	1	1	1	1
exponent bits	5	8	11	15
exponent bias	15 $(2^5 - 1)$	127 $(2^7 - 1)$	1023 $(2^{10} - 1)$	16383 $(2^{14} - 1)$
mantissa bits	10	23	52	112
total bits	16	32	64	128

(* = typical C++ type; might vary in some implementations)

`float` example: manually (1)

$$25.25 = \frac{101}{4} = \frac{101}{2^2}$$

largest power of two < 101? $128 = 2^6$

$$\begin{aligned} \frac{101}{4} \cdot \frac{2^4}{2^4} &= \frac{101 \cdot 2^4}{2^6} \\ &= \frac{101}{2^6} \times 2^4 \\ &= \frac{1100101_{TWO}}{2^6} \times 2^4 \\ &= 1.000101_{TWO} \times 2^4 \end{aligned}$$

float example: manually (2)

$$25.25 = \frac{101}{4} = 11001.01_{\mathsf{TWO}} =$$

$$+1.1001\,0100\,0000\,0000\,0000\,000_{\mathsf{TWO}} \cdot 2^{4}$$

`float` example: manually (2)

$$25.25 = \frac{101}{4} = 11001.01_{\text{TWO}} =$$

$$+1.1001\,0100\,0000\,0000\,0000\,000_{\text{TWO}} \cdot 2^{4}$$

sign (1 bit)

0 for $+$

mantissa (23 bits)

(leading "$1.$" not stored)

exponent (5 bits)

store "$4 + 127 =$
$1000\,0011_{\text{TWO}}$"
127 is bias for `float`

`float` example: manually (2)

$$25.25 = \frac{101}{4} = 11001.01_{\mathsf{TWO}} =$$

$$+1.1001\,0100\,0000\,0000\,0000\,000_{\mathsf{TWO}} \cdot 2^{4}$$

sign (1 bit)

0 for +

mantissa (23 bits)

(leading "$1.$" not stored)

exponent (5 bits)

store "$4 + 127 =$ $1000\,0011_{\mathsf{TWO}}$"

127 is bias for `float`

```
0 1000 0011 1001 0100 0000 0000 0000 000
```

diversion: 25.25 to binary

$$\begin{aligned}25.25 &= \frac{101}{4} \\ &= \frac{1100101_{\text{TWO}}}{2^2} \\ &= 11001.01_{\text{TWO}}\end{aligned}$$

diversion: 25.25 to binary

$$
\begin{aligned}
25.25 &= 2^4 + 2^3 + (9.25 - 2^3) = 2^4 + 2^3 + 1.25 \\
&\quad (1.25 < 2^2) \\
&\quad (1.25 < 2^1) \\
&= 2^4 + 2^3 + (1.25 - 2^0) = 2^4 + 2^3 + 2^0 + 0.25 \\
&\quad (0.25 < 2^{-1}) \\
&= 2^4 + 2^3 + 2^0 + (0.25 - 2^{-2}) = 2^4 + 2^3 + 2^0 + 2^{-2}
\end{aligned}
$$

float example: from C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
// union: all elements use the *same memory*
union floatOrInt {
    float f;
    unsigned int u;
};
int main() {
    union floatOrInt x;
    x.f = 25.25;
    cout << hex << x.u << endl;
// OUTPUT: 41ca0000
}
```

```
   4    1    c    a    0    0    0    0
0100 0001 1100 1010 0000 0000 0000 0000
```

float example 2: manually

$$0.1_{\mathsf{TEN}} = \frac{1}{16} + 0.0375 = \frac{1}{16} + \frac{1}{32} + 0.00625 =$$
$$\ldots = 0.000110011001100110011\ldots_{\mathsf{TWO}} \approx$$

$$+1.1001\,1001\,1001\,1001\,1001\,10\mathbf{1}_{\mathsf{TWO}} \cdot 2^{-4}$$

float example 2: manually

$$0.1_{\textsf{TEN}} = \frac{1}{16} + 0.0375 = \frac{1}{16} + \frac{1}{32} + 0.00625 =$$
$$\ldots = 0.0001100110011001100110011\ldots_{\textsf{TWO}} \approx$$

$$+1.1001\,1001\,1001\,1001\,1001\,101_{\textsf{TWO}} \cdot 2^{-4}$$

sign (1 bit)

0 for $+$

mantissa (23 bits)

last 1 from rounding

exponent (5 bits)

store "$-4 + 127 =$ $0111\,1011_{\textsf{TWO}}$"

float example 2: manually

$$0.1_{\text{TEN}} = \frac{1}{16} + 0.0375 = \frac{1}{16} + \frac{1}{32} + 0.00625 =$$
$$\ldots = 0.0001100110011001\ldots_{\text{TWO}} \approx$$

$$+1.1001\,1001\,1001\,1001\,1001\,10\mathbf{1}_{\text{TWO}} \cdot 2^{-4}$$

sign (1 bit)

0 for +

mantissa (23 bits)

last **1** from rounding

exponent (5 bits)

store "$-4 + 127 = 0111\,1011_{\text{TWO}}$"

`0 0111 1011 1001 1001 1001 1001 1001 001`

`float` example 2: manually

$$0.1_{\text{TEN}} = \frac{1}{16} + 0.0375 = \frac{1}{16} + \frac{1}{32} + 0.00625 =$$
$$\ldots = 0.00011001100110011\ldots_{\text{TWO}} \approx$$

$$+1.1001\,1001\,1001\,1001\,1001\,101_{\text{TWO}} \cdot 2^{-4}$$

sign (1 bit)

0 for +

mantissa (23 bits)

last **1** from rounding

exponent (5 bits)

store "$-4+127 = 0111\,1011_{\text{TWO}}$"

`0 0111 1011 1001 1001 1001 1001 1001 001`

closest `float` to 0.1 between 0.1 and 0.1000001

float example 2: inaccurate (1)

```
#include <iostream>
using std::cout; using std::endl;

int main(void) {
    int count;
    float base = 0.1f;
    for (count = 0; base * count < 10000000; ++count) {}
    cout << count << endl;
    // OUTPUT: 99999996
    return 0;
}
```

float example 2: inaccurate (2)

```
#include <iostream>
using std::cout; using std::endl;

int main(void) {
    int count = 0;
    for (float f = 0; f < 2000.0; f += 0.1) {
        ++count;
    }
    cout << count << endl;
    // OUTPUT: 20004
    return 0;
}
```

float example 2: inaccurate (3)

```
#include <iostream>
using std::cout; using std::endl;
int main(void) {
    cout.precision(30);
    for (float f = 0; f < 2000.0; f += 0.1) {
        cout << f << endl;
    }
    return 0;
}
```

```
0
0.100000001490116119384765625
0.200000002980232238769531250
…
2.20000004768371582031250
2.2999999523162841796875
…
```

on comparing floats

```
#include <cmath>
using std::fabs;
// or #include <math.h> and use fabs
    // without a using statement
...
    // chose based on expected accuracy
const float EPSILON = 1e−6;
float x, y;
...
if (fabs(x − y) < EPSILON) {
    ...
}
```

floating point accuracy

`float` — about 7 decimal places

`double` — about 15 decimal places

the problem of 0

0 is a very imporant number

can't be represented with implicit "1."

solution: special cases

IEEE float special cases

exponent bits	mantissa bits	meaning
`00000000`	`000...000`	± 0
`00000000`	non-zero	*denormal* number
`11111111`	`000...000`	$\pm\infty$
`11111111`	non-zero	not a number (NaN)

$$(+1/1000000000) \div \text{huge positive number} = +0$$

$$(-1/1000000000) \div \text{huge positive number} = -0$$

$$(+1000000000) \cdot \text{huge positive number} = +\infty$$

$$(-1000000000) \cdot \text{huge positive number} = -\infty$$

$$1 \div 0 = +\infty$$

$$0 \div 0 = \text{NaN}$$

$$\sqrt{-1} = \text{NaN}$$

float min magnitude value

exponent of `0000 0001` (not 0 since that's special)

mantisssa of `000...000`

$1.000000\ldots_{\text{TWO}} \cdot 2^{1-\text{bias}} = 2^{-126}$

float max magnitude value

exponent of `1111 1110` (not all 1s since that's special)

mantisssa of `111…111`

$1.111111\ldots11_{\textsf{TWO}} \cdot 2^{254-\textsf{bias}} = 1.11111\ldots1_{\textsf{TWO}} \cdot 2^{127} = 2^{128} - 2^{104}$

on denormals

denormals — minimum exponent bits, non-zero mantissa

smaller in magntiude than "normal" minimum value

- ignore the "implicit 1." rule

notorious for being superslow on some systems

- some CPUs take 100s of times longer to compute on them

we won't ask you about them

rounding errors (1)

$$2^{100} + 1$$

$2^{100} + 1$ cannot be represented exactly

would need 99 mantissa bits

rounds to 2^{100}

(but 2^{100} and 1 can)

rounding errors (2)

$$(2^{100} + 1) - 2^{100}$$

$$2^{100} - 2^{100}$$

$$0$$

$$(2^{100} - 2^{100}) + 1$$

$$0 + 1$$

$$1$$