

trees

are lists enough?

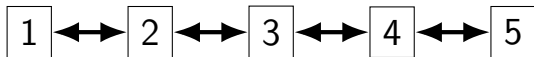
for correctness — sure

want to efficiently access items

better than linear time to find something

want to **represent relationships** more naturally

inter-item relationships in lists



List: *nodes* related to predecessor/successor

trees

trees: allow representing more relationships

(but not arbitrary relationships — see graphs later in semester)

restriction: single path from *root* to every node

implies single path from every node to every other node (possibly through root)

natural trees: phylogenetic tree

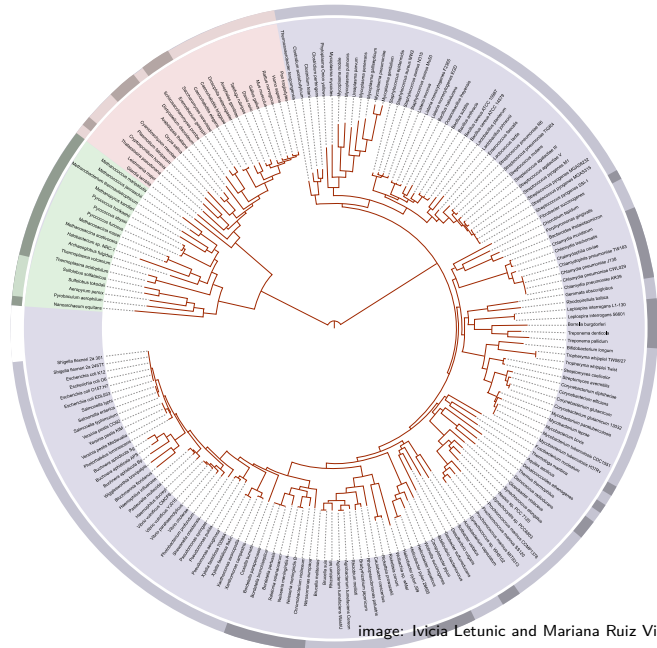
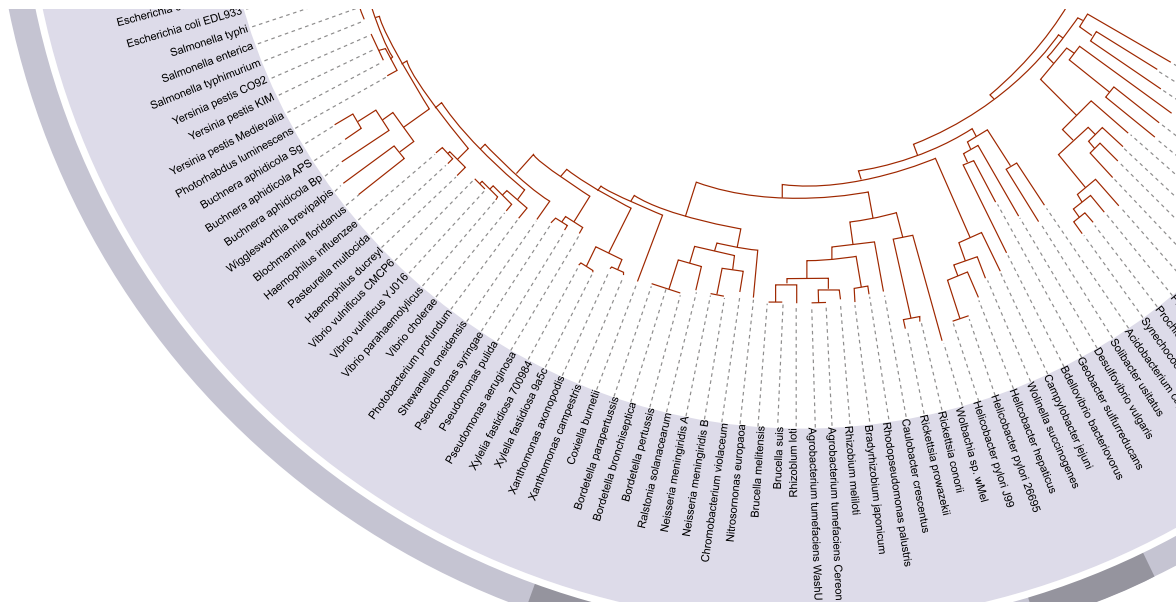


image: Ivicia Letunic and Mariana Ruiz Villarreal, via the tool iTOL (Iterative Tree of Life), via Wikipedia

natural trees: phylogenetic tree (zoom)



natural trees: Indo-European languages

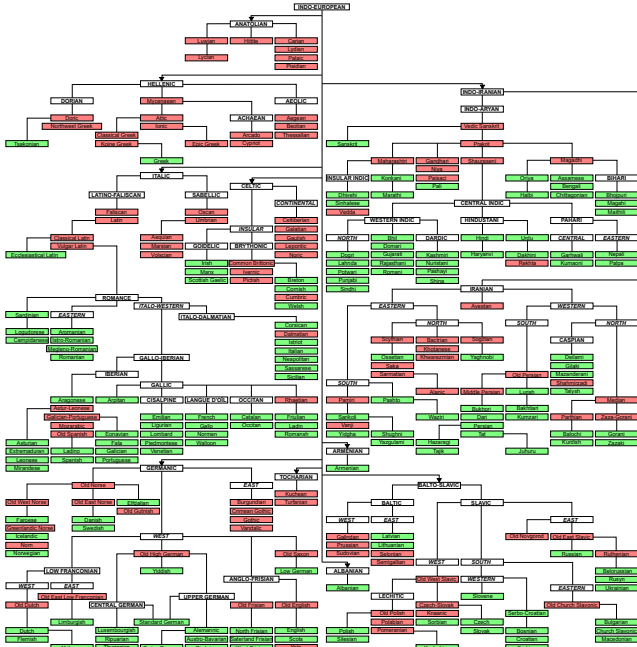
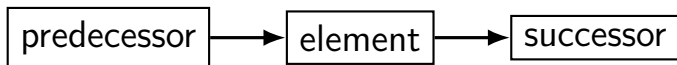


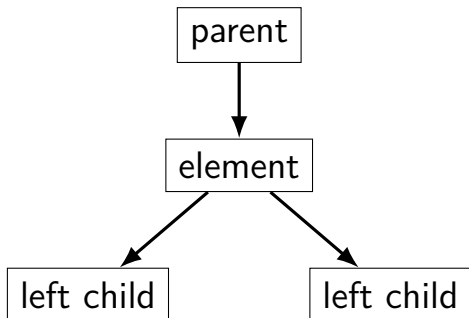
image: via Wikipedia/Mandrak

list to tree

list — up to 2 related nodes

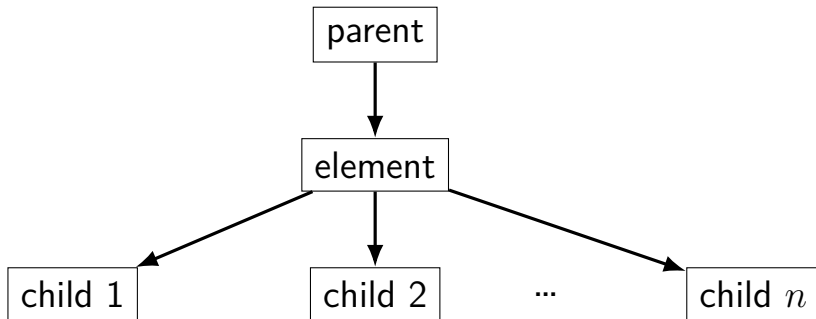


binary tree — up to 3 related nodes (list is special-case)

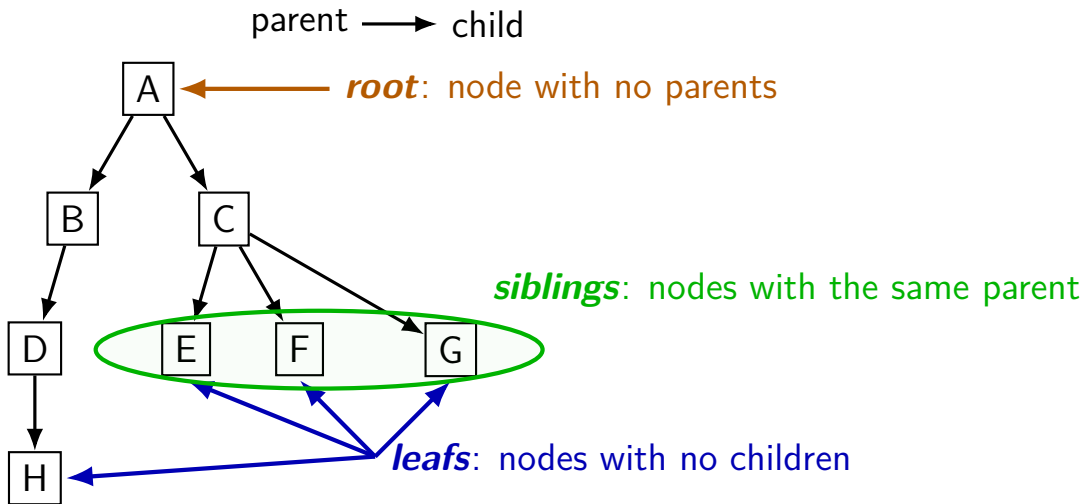


more general trees

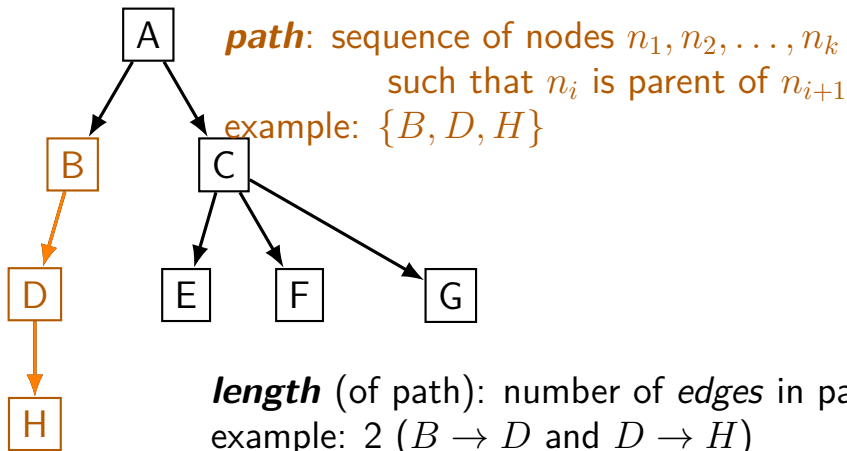
tree — any number of relationships (binary tree is special case)
at most one parent



tree terms (1)



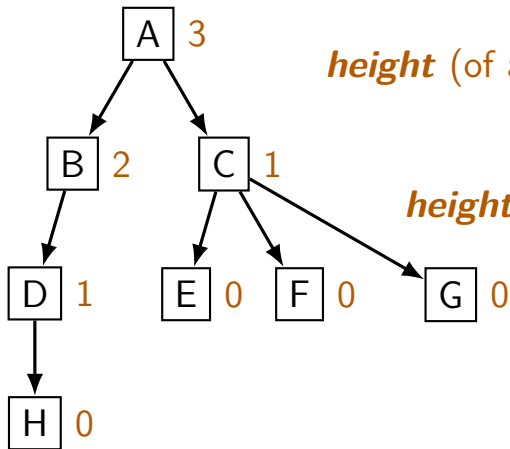
paths and path lengths



internal path length: sum of depth of nodes
example: $6 = 1 + 2 + 3$

tree/node height

parent \longrightarrow child

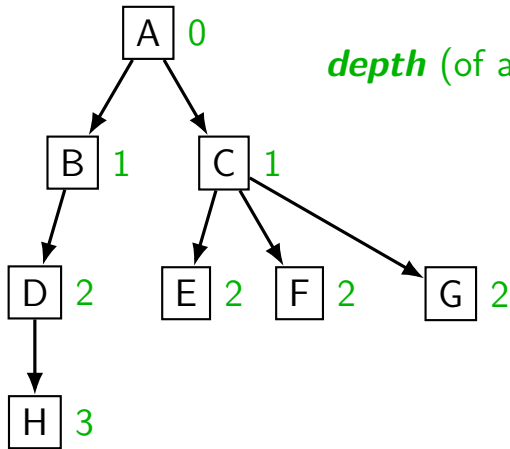


height (of a node): length of longest path to leaf

height (of a tree): height of tree's root
(this example: 3)

tree/node depth

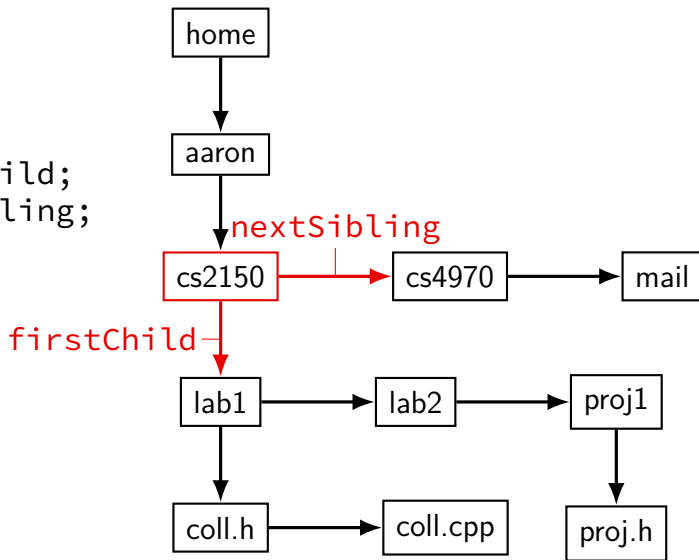
parent \longrightarrow child



depth (of a node): length of path to root

first child/next sibling

```
class TreeNode {  
    private:  
        string element;  
        TreeNode *firstChild;  
        TreeNode *nextSibling;  
    public:  
        ...  
};
```

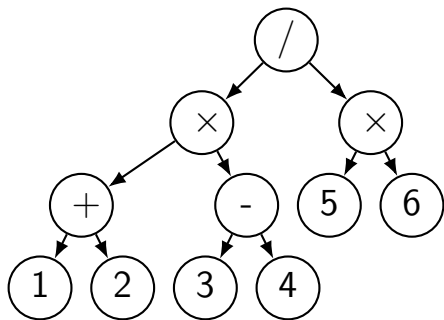


another tree representations

```
class TreeNode {  
    private:  
        string element;  
        vector<TreeNode *> children;  
    public:  
        ...  
};
```

// and more --- see when we talk about graphs

tree traversal



pre-order: / * + 1 2 - 3 4 * 5 6

in-order: (((1+2) * (3-4)) / (5*6)) (parenthesis optional?)

post-order: 1 2 + 3 4 - * 5 6 * /

pre/post-order traversal printing

(this is pseudocode)

```
TreeNode::printPreOrder() {  
    this->print();  
    for each child c of this:  
        c->printPreOrder()  
}
```

```
TreeNode::printPostOrder() {  
    for each child c of this:  
        c->printPostOrder()  
    this->print();  
}
```

in-order traversal printing

(this is pseudocode)

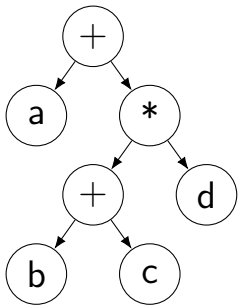
```
BinaryTreeNode::printInOrder() {  
    if (this->left)  
        this->left->printInOrder();  
    cout << this->element << "_";  
    if (this->right)  
        this->right->printInOrder();  
}
```

post-order traversal counting

(this is pseudocode)

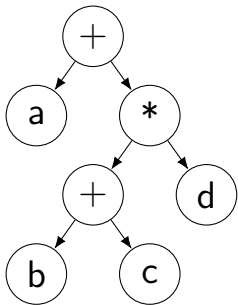
```
int numNodes(TreeNode *tnode) {  
    if ( tnode == NULL )  
        return 0;  
    else {  
        sum=0;  
        for each child c of tnode  
            sum += numNodes(c);  
        return 1 + sum;  
    }  
}
```

expression tree and traversals



$(a + ((b + c) * d))$

expression tree and traversals



infix: $(a + ((b + c) * d))$

postfix: a b c + d * +

prefix: + a * + b c d

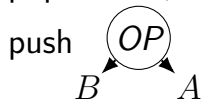
postfix expression to tree

use a stack of trees

number $n \rightarrow \text{push}(\textcircled{n})$

operator $OP \rightarrow$

pop into A, B ; then

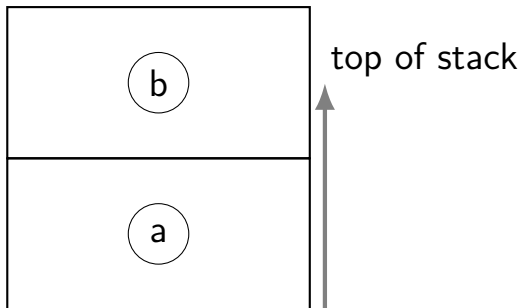


example

a b + c d e + * *

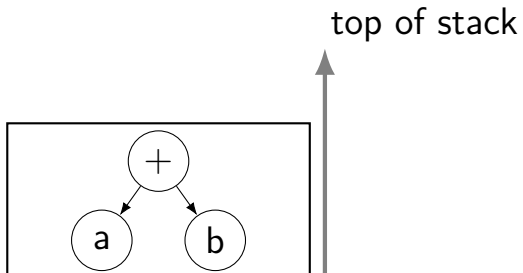
example

a b + c d e + * *



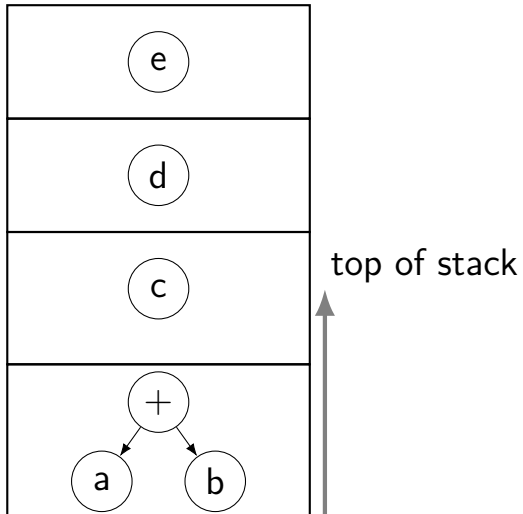
example

a b + c d e + * *



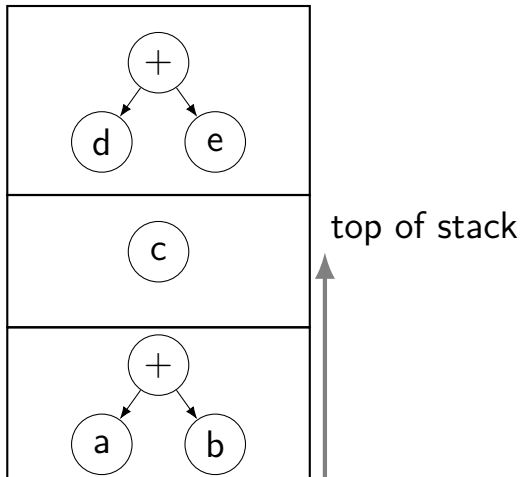
example

a b + c d e + * *



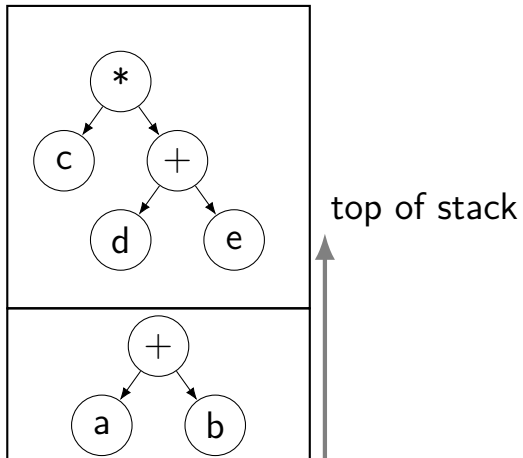
example

a b + c d e + * *



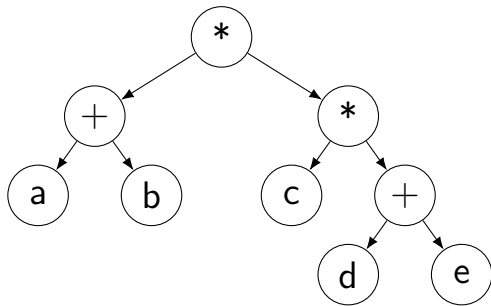
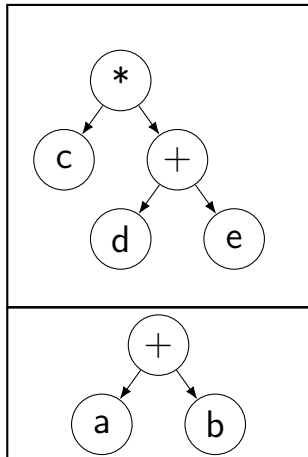
example

a b + c d e + * *



example

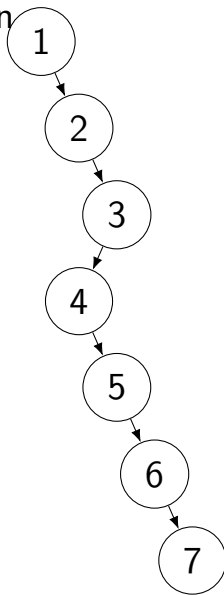
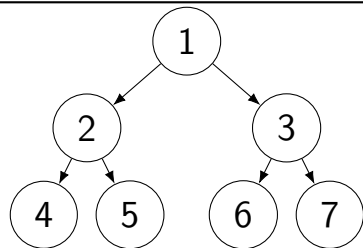
a b + c d e + * *



binary trees

all nodes have *at most* 2 children

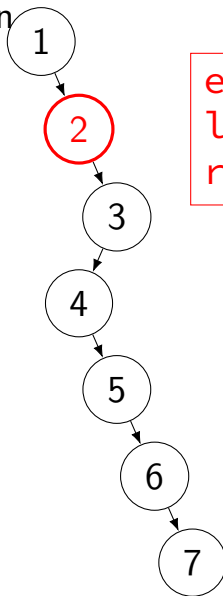
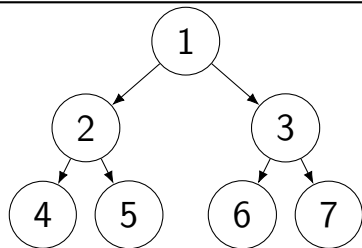
```
class BinaryNode {  
    ...  
    int element;  
    BinaryNode *left;  
    BinaryNode *right;  
};
```



binary trees

all nodes have *at most* 2 children

```
class BinaryNode {  
    ...  
    int element;  
    BinaryNode *left;  
    BinaryNode *right;  
};
```

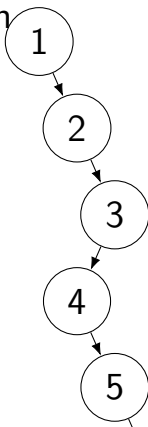
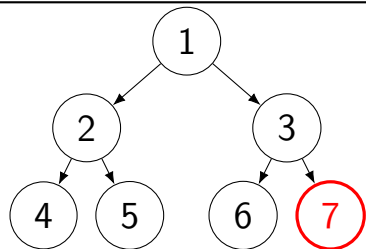


element = 2
left = *NULL*
right = *addr of node 3*

binary trees

all nodes have *at most* 2 children

```
class BinaryNode {  
    ...  
    int element;  
    BinaryNode *left;  
    BinaryNode *right;  
};
```



element = 7
left = NULL
right = NULL

binary search trees

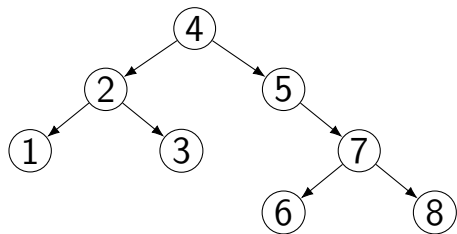
binary tree **and**...

each node has a *key*

for each node:

keys in node's left subtree are less than node's

keys in node's right subtree are greater than node's



binary search trees

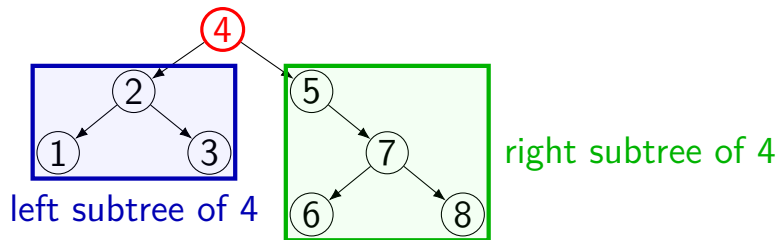
binary tree **and**...

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binary search trees

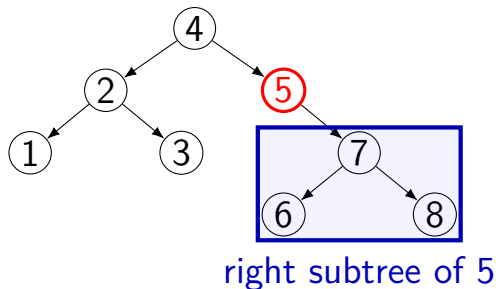
binary tree **and**...

each node has a *key*

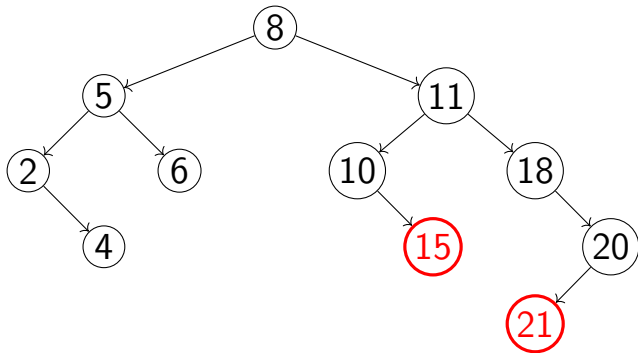
for each node:

keys in node's left subtree are less than node's

keys in node's right subtree are greater than node's



not a binary search tree



binary search tree versus binary tree

binary search trees are a kind of binary tree

...but — often people say “binary tree” to mean “binary search tree”

BST: find

(pseudocode)

```
find(node, key) {  
    if (node == NULL)  
        return NULL;  
    else if (key < node->key)  
        return find(node->left, key)  
    else if (key > node->key)  
        return find(node->right, key)  
    else // if (key == node->key)  
        return node;  
}
```

BST: insert

(pseudocode)

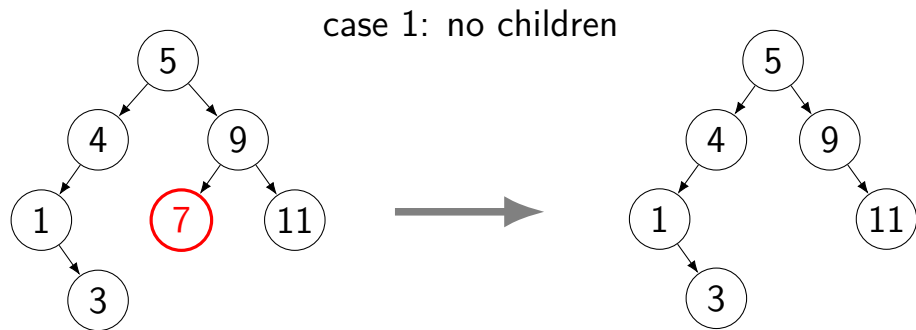
```
insert(Node *&node, key) {  
    if (node == NULL)  
        node = new BinaryNode(key);  
    else if (key < node->key)  
        insert(node->left, key);  
    else if (key < root->key)  
        insert(node->right, key);  
    else // if (key > root->key)  
        ; // duplicate -- no new node needed  
}
```

BST: findMin

(pseudocode)

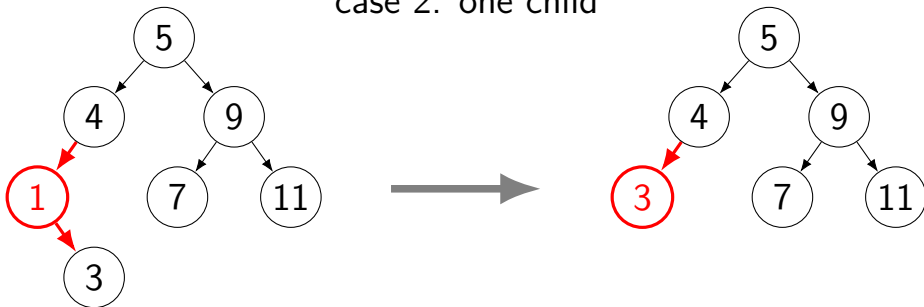
```
findMin(Node *node, key) {  
    if (node->left == NULL)  
        return node;  
    else  
        insert(node->left, key);  
}
```


BST: remove (1)

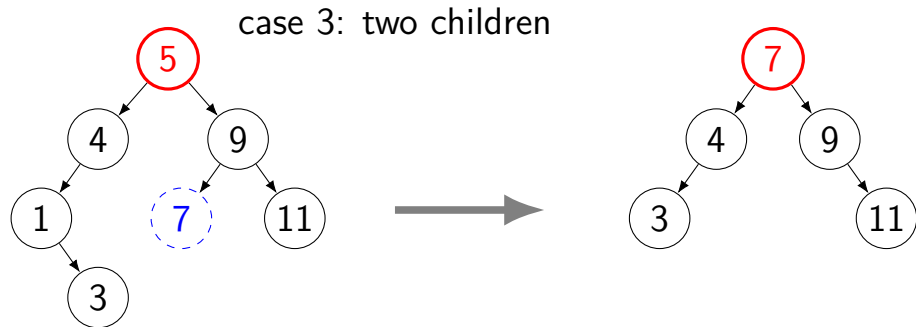


BST: remove (2)

case 2: one child



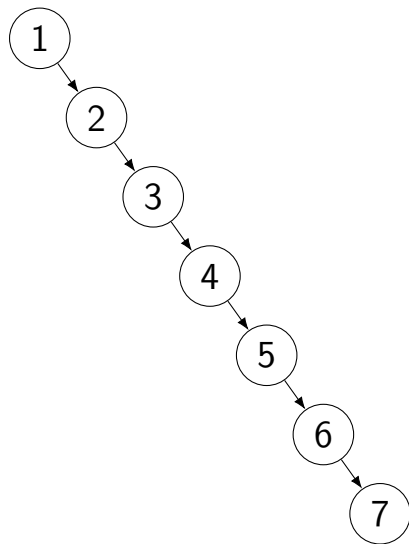
BST: remove (3)



replace with minimum of right subtree
(alternately: maximum of left subtree, ...)

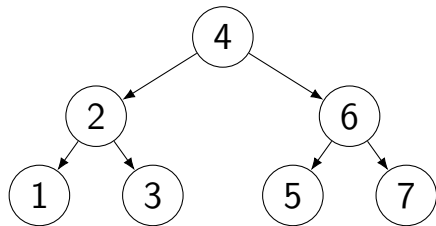
binary tree: worst-case height

n -node BST: worst-case height/depth $n - 1$



binary tree: best-case height

height h : at most $2^{h+1} - 1$ nodes



binary tree: proof best-case height is possible

proof **by induction**: can have $2^{h+1} - 1$ nodes in h -height tree

h = 0: $h = 0$: exactly one node; $2^{h+1} - 1 = 1$ nodes

h = k \rightarrow h = k + 1:

start with *two copies* of a maximum tree of height k

create a new tree as follows:

- create a new root node

- add edges from the root node to the roots of the copies

the height of this new tree is $k + 1$

- path of length k in old tree + either new edge

the number of nodes is

$$2^{(k+1)-1} + 1 = 2^{k+1+1} - 2^{k+1+1-1} = 2^{k+1+1} - 1$$

binary tree: best-case height is best

(informally)

property of trees in root:

- except for the leaves, every node in tree has 2 children

no way to add nodes without increasing height

- add below leaf — longer path to root — longer height

- add above root — every old node has longer path to root

binary tree height formula

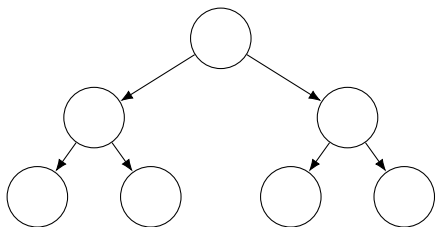
n : number of nodes

h : height

$$\begin{aligned}n + 1 &\leq 2^{h+1} \\ \log_2(n + 1) &\leq \log_2(2^{h+1}) \\ \log(n + 1) &\leq h + 1 \\ h &\geq \log_2(n + 1) - 1\end{aligned}$$

shortest tree of n nodes: $\sim \log_2(n)$ height

perfect binary trees



a binary tree is **perfect** if

- all leaves have same depth

- all nodes have zero children (leaf) or two children

exactly the trees that achieve $2^{h+1} - 1$ nodes

AVL animation tool

[http://webdiis.unizar.es/asignaturas/EDA/
AVLTree/avltree.html](http://webdiis.unizar.es/asignaturas/EDA/AVLTree/avltree.html)

AVL tree idea

AVL trees: one of many **balanced trees** —
search tree *balanced* to keep height $\Theta(\log n)$
avoid “tree is just a long linked list” scenarios

gaurentees $\Theta(\log n)$ for find, insert, remove

AVL = Adelson-Velskii and Landis

AVL gaurentee

the height of the left and right subtrees of *every node* differs by at most one

AVL state

normal binary search tree stuff:

- data; and left, right, parent pointers

additional AVL stuff:

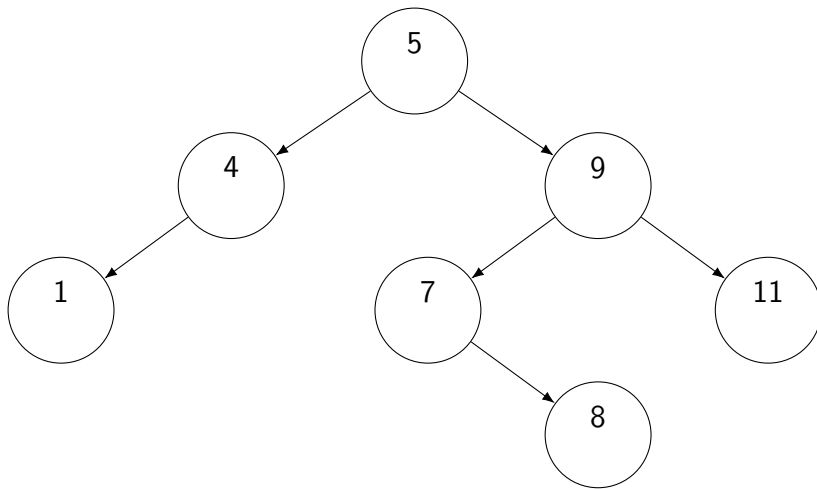
- height of right subtree minus height of left subtree

 - called “balance factor”

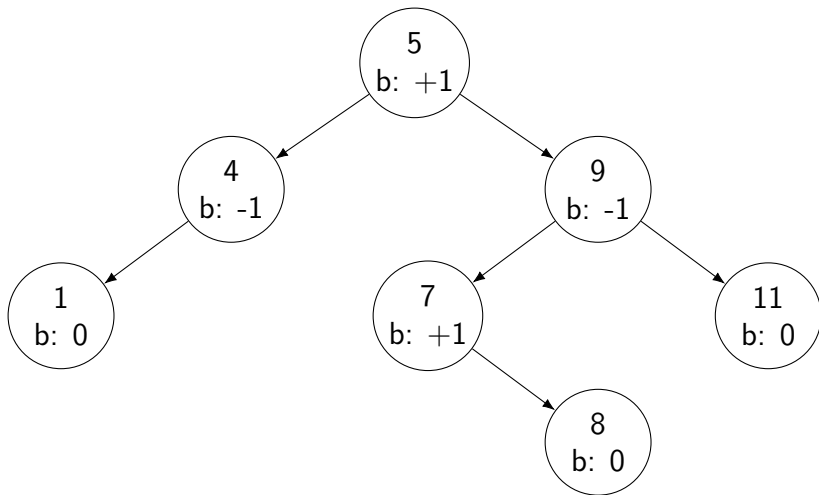
 - 1, 0, +1

- (kept up to date on insert/delete — computing on demand is too slow)

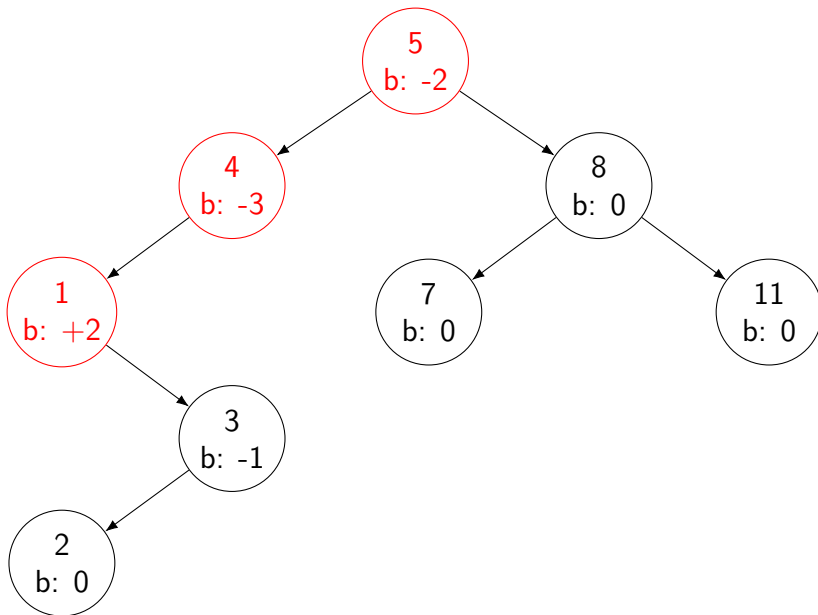
example AVL tree



example AVL tree



example non-AVL tree



AVL tree algorithms

find — exactly the same as binary search tree
just ignore balance factors

insert — two extra steps:
update balance factors
“fix” tree if it became unbalanced

AVL tree algorithms

find — exactly the same as binary search tree
just ignore balance factors

insert — two extra steps:
update balance factors
“fix” tree if it became unbalanced

runtime for both $\Theta(d)$ where d is depth of node found/inserted
max balance factor ± 1 at root
max depth of node is $\Theta(\log_2 n + 1) = \Theta(\log n)$

AVL insertion cases

simple case: tree remains balanced

otherwise:

let x be deepest imbalanced node ($+2/-2$ balance factor)

insert in left subtree of left child of x : single rotation right

insert in right subtree of right child of x : single rotation left

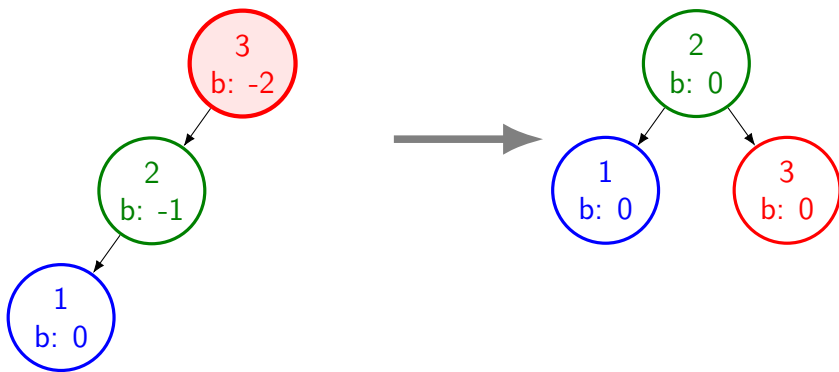
insert in right subtree of left child of x : double left-right rotation

insert in left subtree of right child of x : double right-left rotation

AVL: simple right rotation

just inserted 0

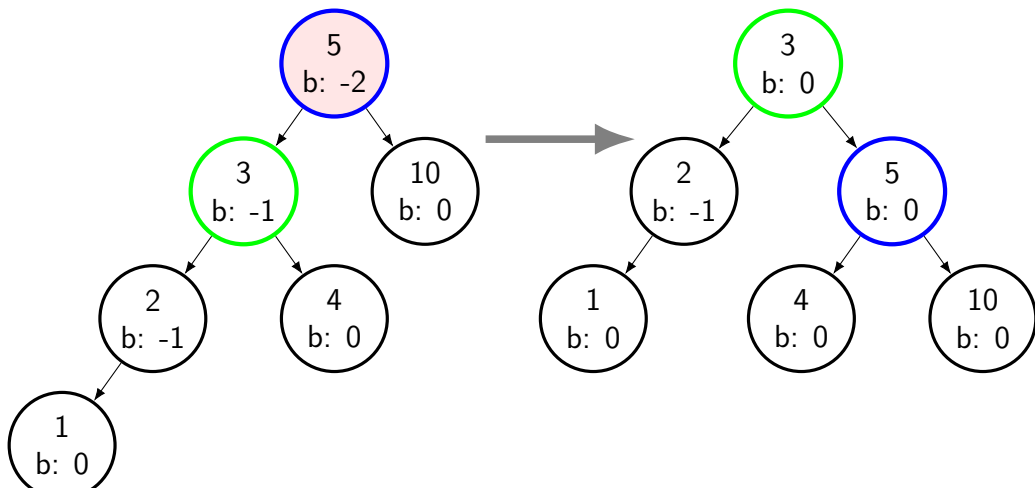
unbalanced root becomes new left child



AVL: less simple right rotation (1)

just inserted 0

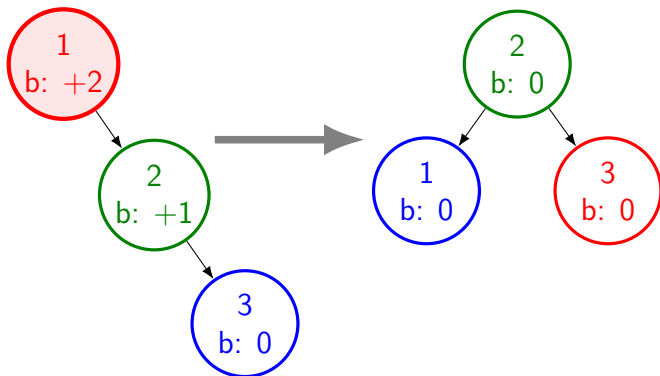
unbalanced root becomes new left child



AVL: simple left rotation

just inserted 1

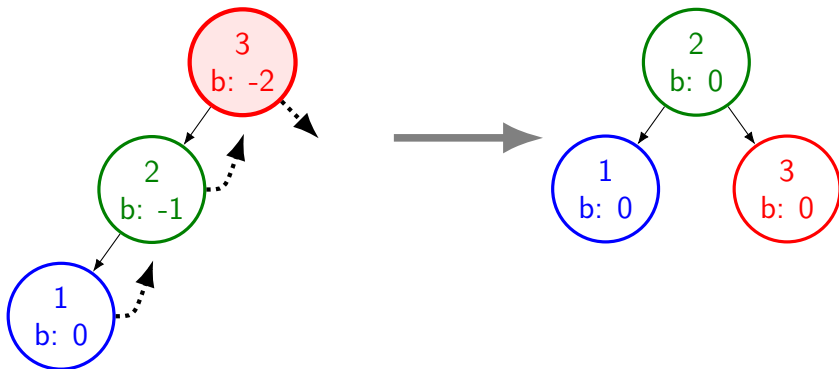
deepest unbalanced node is 3



AVL rotation: up and down

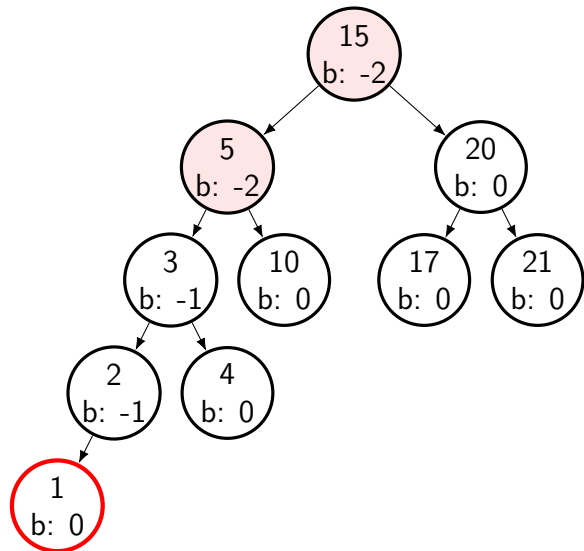
at least one node moves up (this case: 1 and 2)

at least one node moves down (this case: 3)



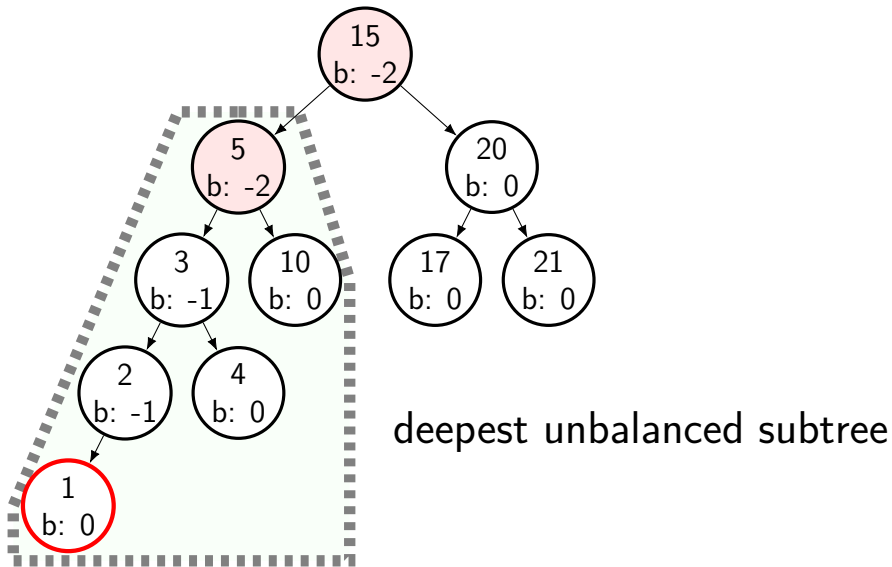
AVL: less simple right rotation (2)

just inserted 1



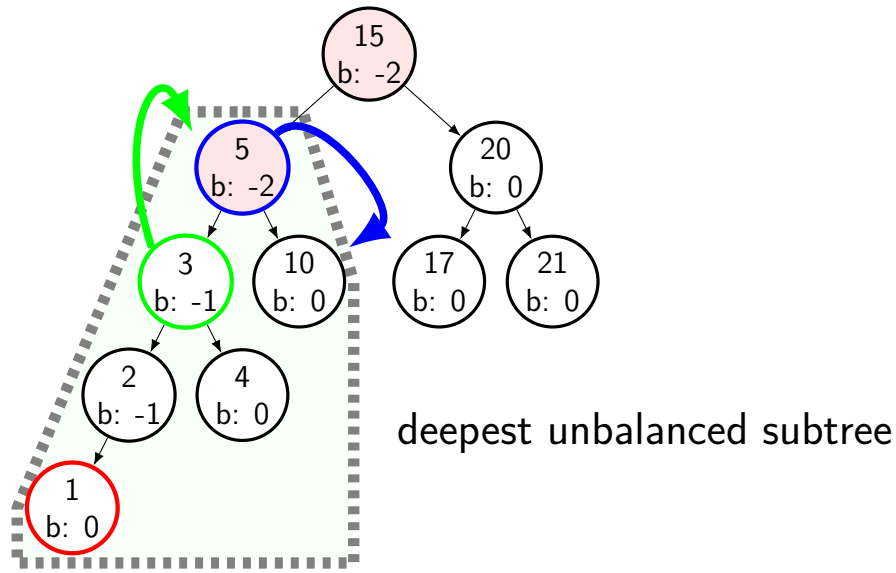
AVL: less simple right rotation (2)

just inserted 1



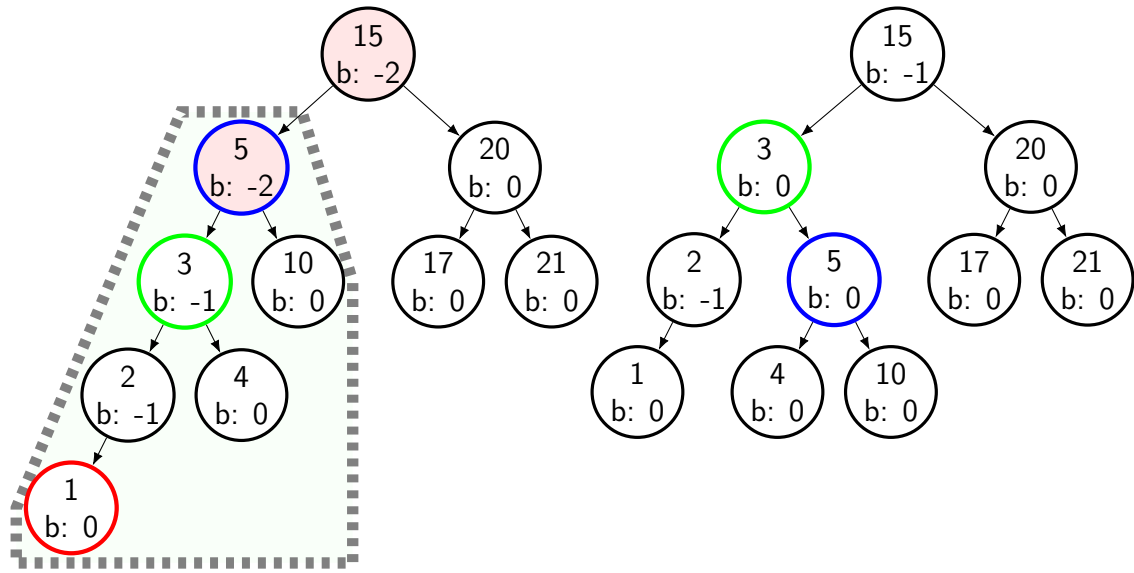
AVL: less simple right rotation (2)

just inserted 1

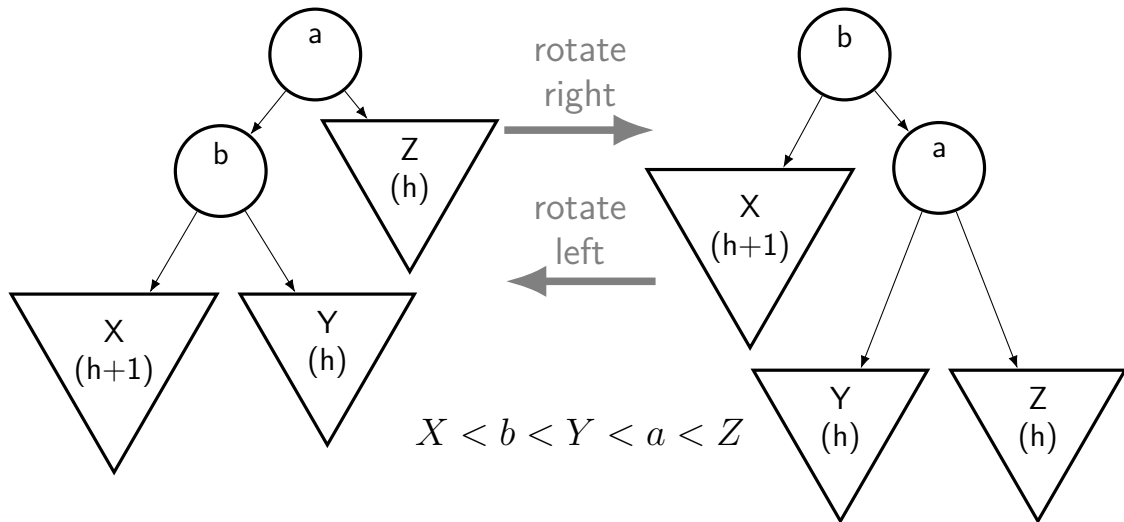


AVL: less simple right rotation (2)

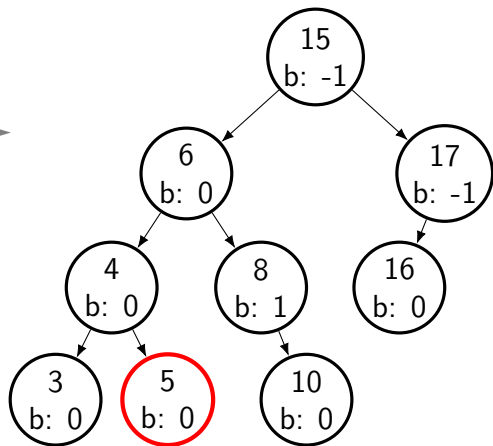
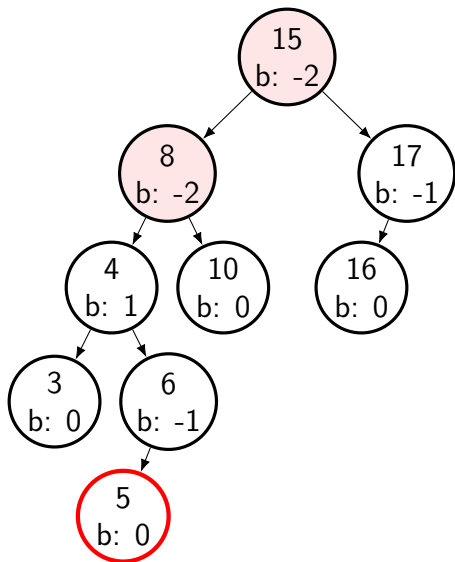
just inserted 1



general single rotation

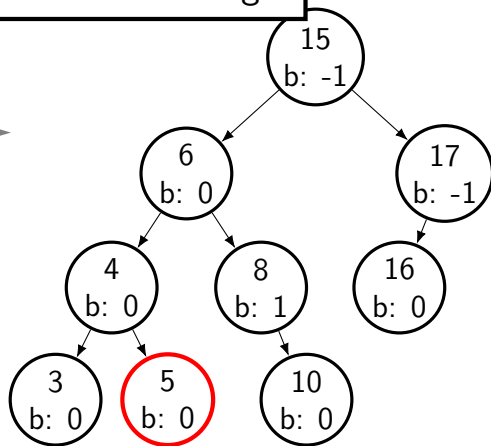
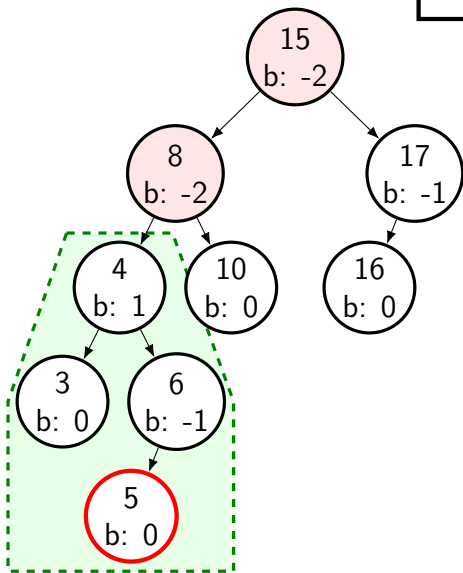


double rotation



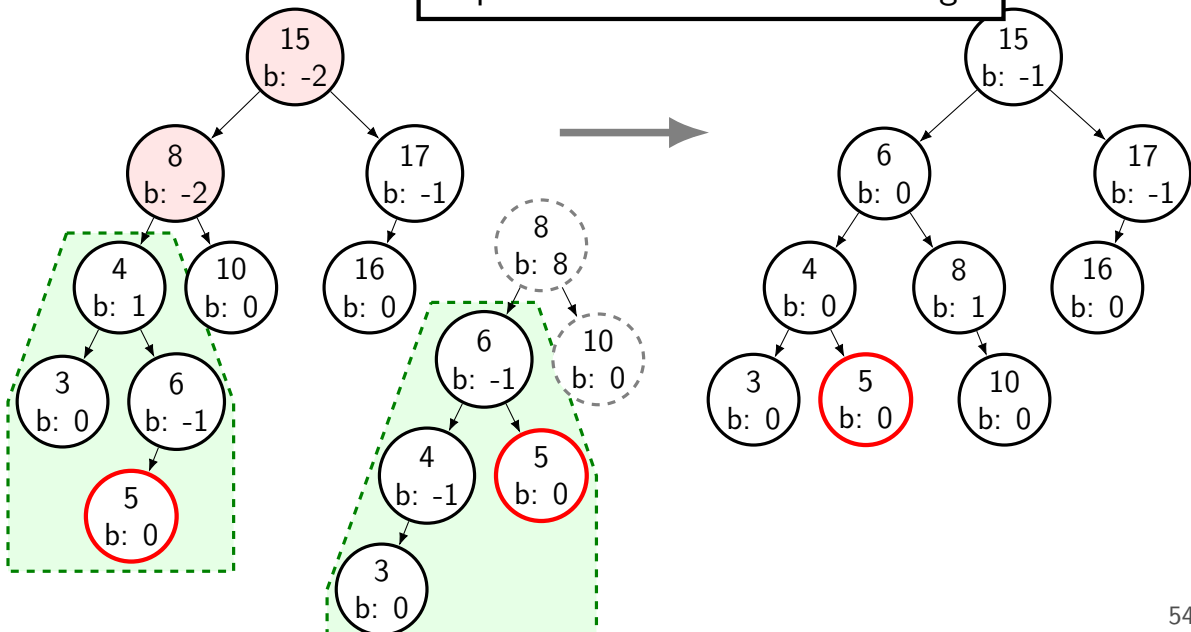
double rotation

step 1: rotate subtree left
step 2: rotate imbalanced tree right



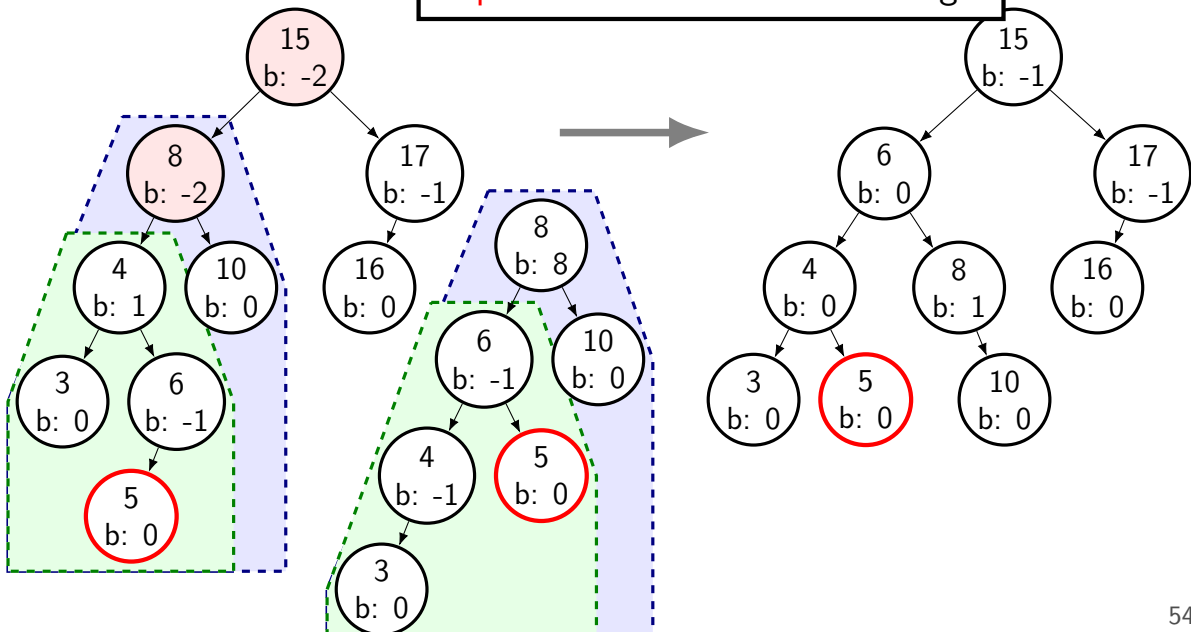
double rotation

step 1: rotate **subtree** left
step 2: rotate **imbalanced tree** right

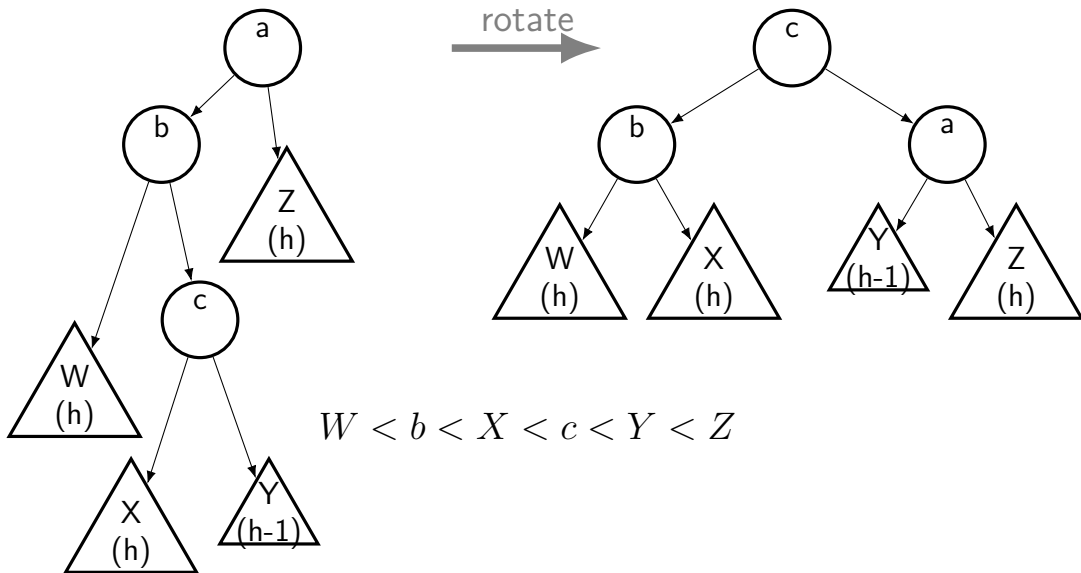


double rotation

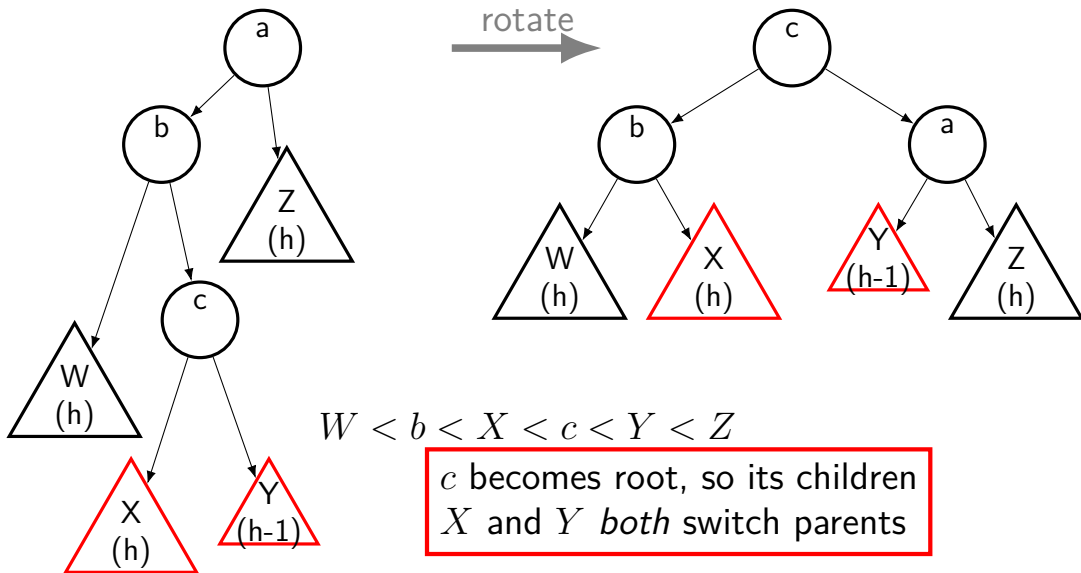
step 1: rotate **subtree** left
step 2: rotate **imbalanced tree** right



general double rotation



general double rotation



double rotation names

sometimes “double left”

first rotation left, or second?

us: “double left-right”

rotate child tree left

rotate parent tree right

“double right-left”

rotate child tree right

rotate parent tree left

AVL insertion cases

simple case: tree remains balanced

otherwise:

let x be deepest imbalanced node ($+2/-2$ balance factor)

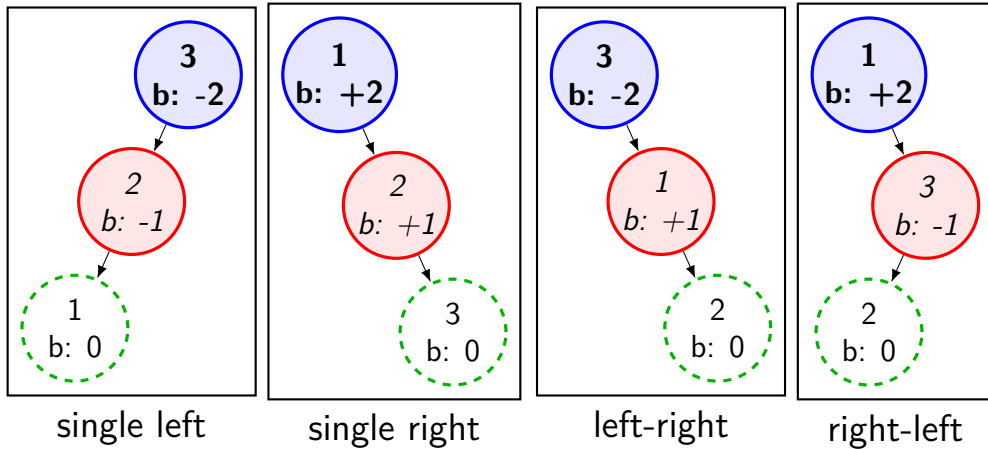
- insert in left subtree of left child of x : single rotation right

- insert in right subtree of right child of x : single rotation left

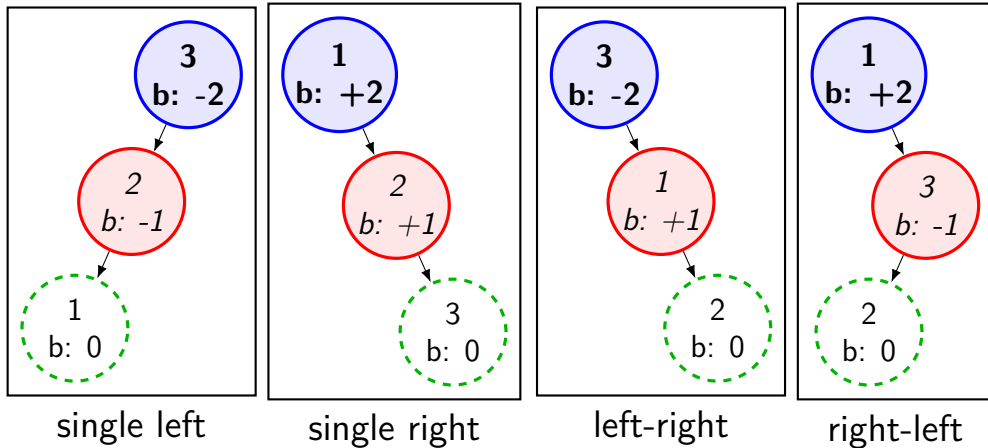
- insert in right subtree of left child of x : double left-right rotation

- insert in left subtree of right child of x : double right-left rotation

AVL insert cases (revisited)

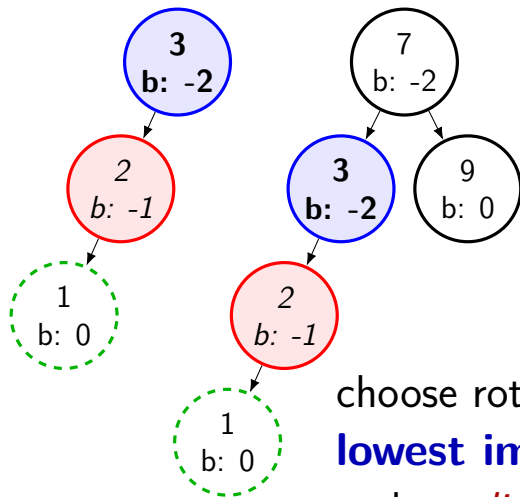


AVL insert cases (revisited)



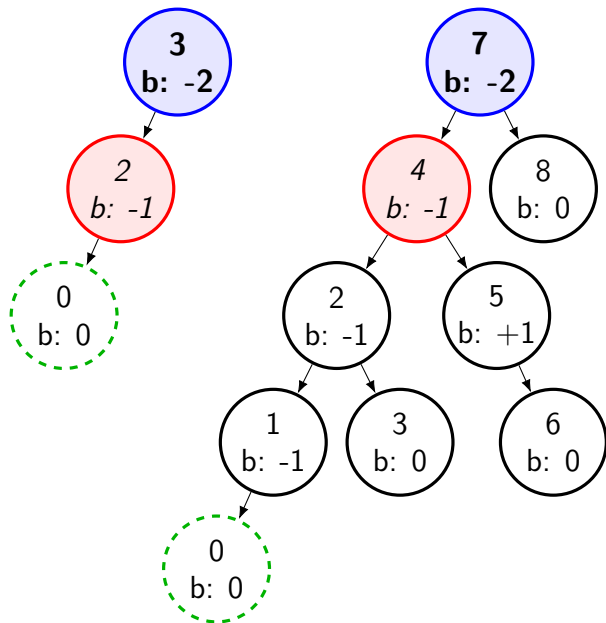
choose rotation based on **lowest imbalanced node**
and on *direction of insertion*
(inserted node is **green+dashed**)

AVL insert case: detail (1)



choose rotation based on
lowest imbalanced node
and on *direction of insertion*
(inserted node is **green+dashed**)

AVL insert case: detail (2)



choose using
lowest imbalanced node
and on *direction of insertion*
(inserted node is
green+dashed)

There are 4 cases in all, choosing which one is made by seeing the direction of the first 2 nodes from the unbalanced node to the newly inserted node and matching them to the top most row.

Root is the initial parent before a rotation and **Pivot** is the child to take the root's place.

<p>Left Left Case</p> <p>Right Rotation</p>	<p>Right Right Case</p> <p>Left Rotation</p>	<p>Left Right Case</p> <p>Left Rotation</p>	<p>Right Left Case</p> <p>Right Rotation</p>
		<p>Right Rotation</p>	<p>Left Rotation</p>

AVL tree: runtime

worst depth of node: $\Theta(\log_2 n + 2) = \Theta(\log n)$

find: $\Theta(\log n)$

worst case: traverse from root to worst depth leaf

insert: $\Theta(\log n)$

worst case: traverse from root to worst depth leaf
then back up (update balance factors)
then perform constant time rotation

remove: $\Theta(\log n)$

left as exercise (similar to insert)

print: $\Theta(n)$

visit each of n nodes

other types of trees

many kinds of *balanced trees*

not all binary trees

different ways of tracking balance factors, etc.

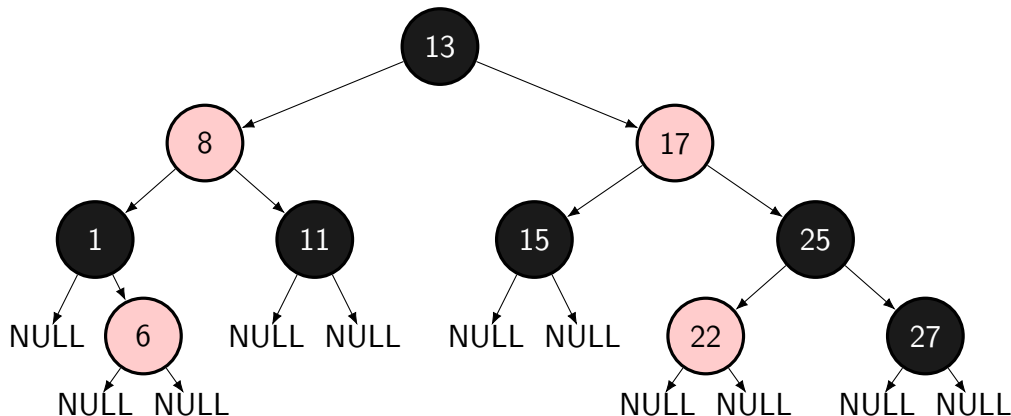
different ways of doing tree rotations or equivalent

red-black trees

each node is **red** or **black**

null leafs considered nodes to aid analysis (still null pointers...)

rules about when nodes can be red/black guarantee maximum depth



red-black tree rules

root is **black**

counting null pointers as nodes, leaves are **black**

a **red** node's children are **black**

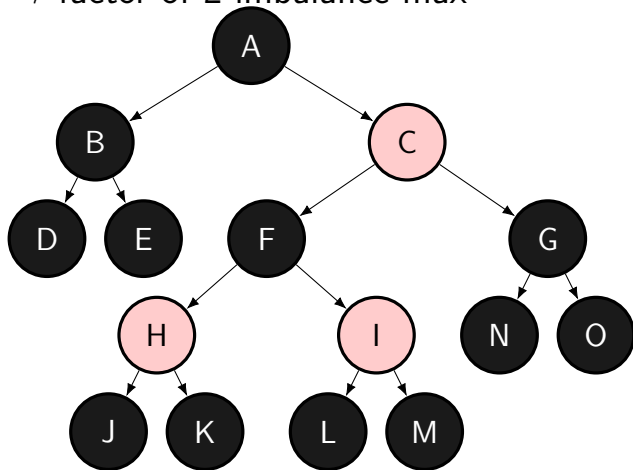
→ a **red** node's parents are **black**

every simple path from node to leaf under it contains same number of black nodes

(property holds regardless of whether null pointers are considered nodes)

worst red-black tree imbalance

same number of black nodes on paths to leaves
→ factor of 2 imbalance max



red-black insert

default: insert as **red**, but...

- (1) if new node is root: color **black**
- (2) if parent is black: keep child **red**
- (3) if parent and uncle is **red**: adjust several colors
- (4) if parent is **red**, uncle is **black**, new node is right child
perform a rotation, then go to case 5
- (5) if parent is **red**, uncle is **black**, new node is left child
perform a rotation

red-black insert

default: insert as **red**, but...

- (1) if new node is root: color **black**
- (2) if parent is **black**: keep child **red**
- (3) if parent and uncle is **red**: adjust several colors
- (4) if parent is **red**, uncle is **black**, new node is right child
perform a rotation, then go to case 5
- (5) if parent is **red** and new node is left child:
perform a rotation, then go to case 4

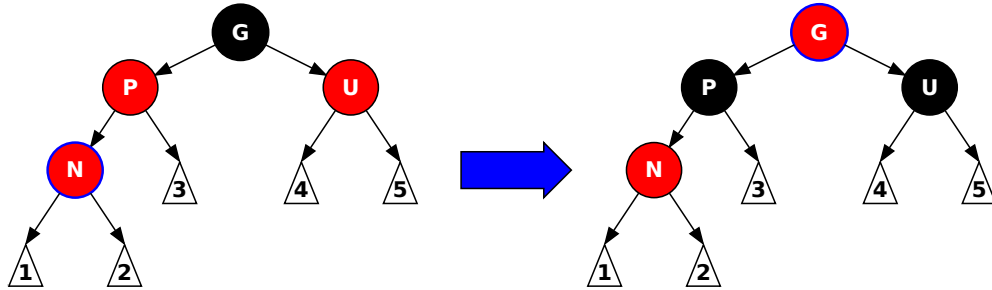
property: “children of **red** node are **black**”
no change in # of **black** nodes on paths

red-black insert

default: insert as **red**, but...

- (1) if new node is root: color **black**
- (2) if parent is black: keep child **red**
- (3) if parent and uncle is **red**: adjust several colors
- (4) if parent is **red**, uncle is **black**, new node is right child
perform a rotation, then go to case 5
- (5) if parent is **red**, uncle is **black**, new node is left child
perform a rotation

case 3: parent, uncle are red

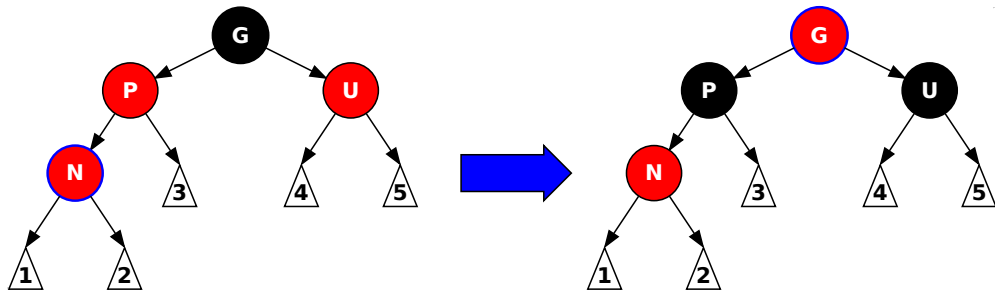


make grandparent **red**, parent and uncle **black**

(property: every path to leaf has same number of black nodes)

just swapped grandparent and parent/uncle in those paths

case 3: parent, uncle are red



make grandparent **red**, parent and uncle **black**

(property: every path to leaf has same number of black nodes)

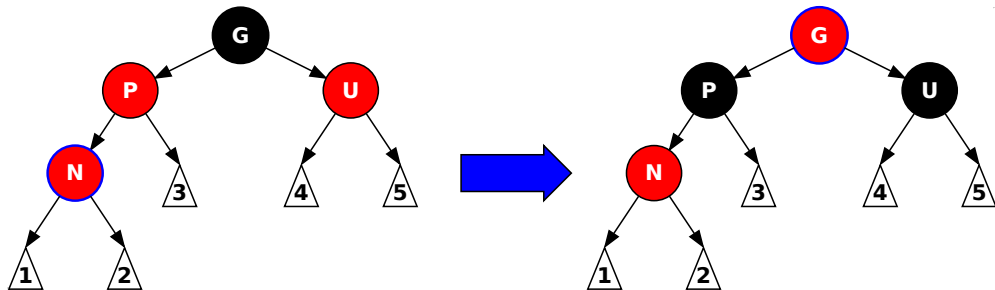
just swapped grandparent and parent/uncle in those paths

but...what if grandparent's parent is red?

(property: children of red node are black)

solution: recurse to the grandparent, as if it was just inserted

case 3: parent, uncle are red



make grandparent **red**, parent and uncle **black**

(property: every path to leaf has same number of black nodes)

just swapped grandparent and parent/uncle in those paths

but...what if grandparent's parent is red?

(property: children of red node are black)

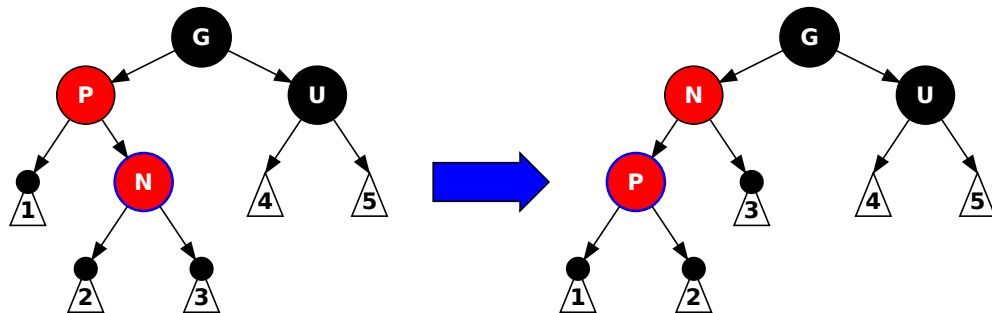
solution: **recurse to the grandparent**, as if it was just inserted

red-black insert

default: insert as **red**, but...

- (1) if new node is root: color **black**
- (2) if parent is black: keep child **red**
- (3) if parent and uncle is **red**: adjust several colors
- (4) if parent is **red**, uncle is **black**, new node is right child
perform a rotation, then go to case 5
- (5) if parent is **red**, uncle is **black**, new node is left child
perform a rotation

case 4: parent red, uncle black, right child



perform left rotation on parent subtree and new node

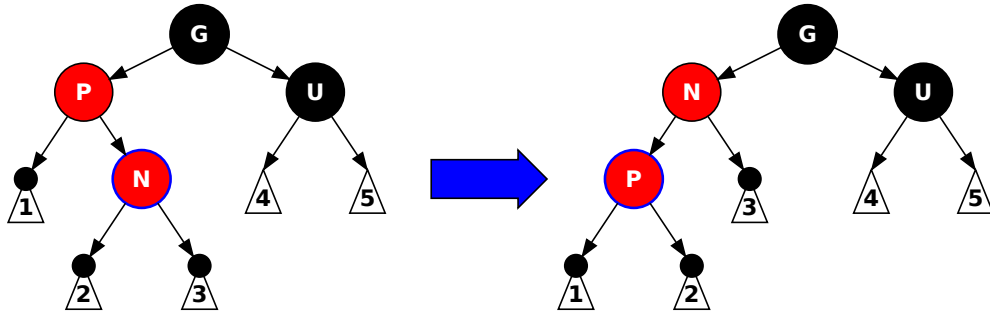
now case 5 (but new node is P , not N)

red-black insert

default: insert as **red**, but...

- (1) if new node is root: color **black**
- (2) if parent is black: keep child **red**
- (3) if parent and uncle is **red**: adjust several colors
- (4) if parent is **red**, uncle is **black**, new node is right child
perform a rotation, then go to case 5
- (5) if parent is **red**, uncle is **black**, new node is left child
perform a rotation

case 5: parent red, uncle black, left child



perform right rotation of grandparent and parent

(property: red parent's children are black)

(property: every path to leaf has same number of black nodes)

RB-tree: removal

start with normal BST remove of x , but...

instead find next highest/lowest node y

can choose node *with at most one child*
("bottom" of a left or right subtree)

swap x and y 's value, then replace y with its child

several cases for color maintenance/rotations

RB tree: removal cases

N: node just replaced with child; S: its sibling; P: its parent

(1): N is new root

(2): S is **red**

(3): P, S, and S's children are **black**

(4): S and S's children are **black**

(5): S is **black**, S's left child is **red**, S's right child is **black**, N is left child of P

(6): S is **black**, S's right child is **red**, N is left child

why red-black trees?

a lot more cases...but

a lot less rotations

...because tree is kept less rigidly balanced

red-black trees end up being faster in practice

splay trees

tree that's fast for **recently used nodes**

self-balancing binary search tree

keeps recent nodes **near the top**

simpler to implement than AVL or RB trees

‘splaying’

every time node is accessed (find, insert, delete)...

“splay” tree around that node

make the node the new tree root

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every time node is accessed (find, insert, delete)...

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make the node the new tree root

$\Theta(h)$ time — where h is tree height

‘splaying’

every time node is accessed (find, insert, delete)...

“splay” tree around that node

make the node the new tree root

$\Theta(h)$ time — where h is tree height

worst-case height: $\Theta(n)$ — linked-list case

amortized complexity

splay tree insert/find/delete is **amortized $O(\log n)$ time**

informally: **average** insert/find/delete: $O(\log n)$

more formally: m operations: $O(m \log n)$ time (where n : max size of tree)

splay tree pro/con

can be *faster* than AVL, RB-trees in practice
take advantage of frequently accessed items

simpler to implement

but worst case find/insert is $\Theta(n)$ time

amortized analysis: vector growth

vector insert algorithm:

- if not big enough, double capacity
- write to end of vector

amortized analysis: vector growth

vector insert algorithm:

if not big enough, double capacity

write to end of vector

doubling size — requires copying! — $\Theta(n)$ time

$\Theta(n)$ worst case per insert

but average...?

counting copies (1)

suppose initial capacity 100 + insert 1600 elements

100 \rightarrow 200: 100 copies

200 \rightarrow 400: 200 copies

400 \rightarrow 800: 400 copies

800 \rightarrow 1600: 800 copies

total: 1500 copies

total operations: 1500 copies + 1600 writes of new elements

about 2 operations per insert

counting copies (2)

more generally: for N inserts

about N copies + N writes

why? K to $2K$ elements: K copies

N inserts: $1 + 2 + 4 + \dots + N/4 + N/2$ copies

(and a bit better if initial capacity isn't 1)

$\Theta(n)$ worst case

but $\Theta(n)$ time for n inserts

→ $O(1)$ amortized time per insert

trees are not great for...

ordered, unsorted lists

list of TODO tasks

being easy/simple to implement

compare, e.g., stack/queue

$\Theta(1)$ time

compare vector

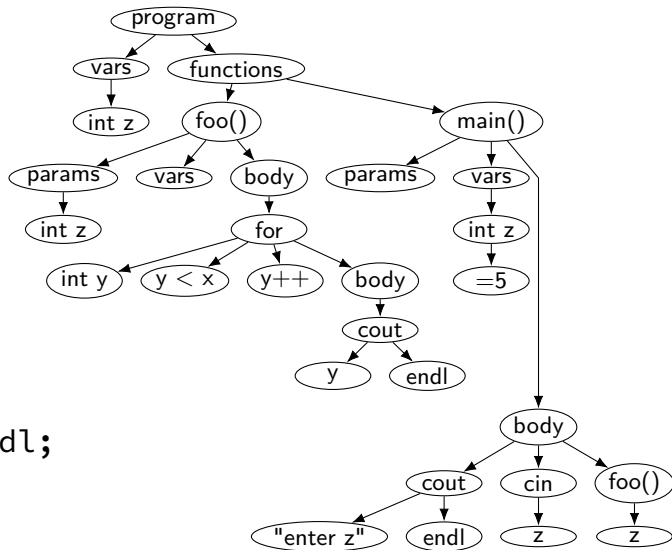
compare hashtables (almost)

programs as trees

```
int z;
```

```
int foo (int x) {  
    for (int y = 0;  
        y < x;  
        y++)  
        cout << y << endl;  
}
```

```
int main() {  
    int z = 5;  
    cout << "enter x" << endl;  
    cin >> z;  
    foo(z);  
}
```

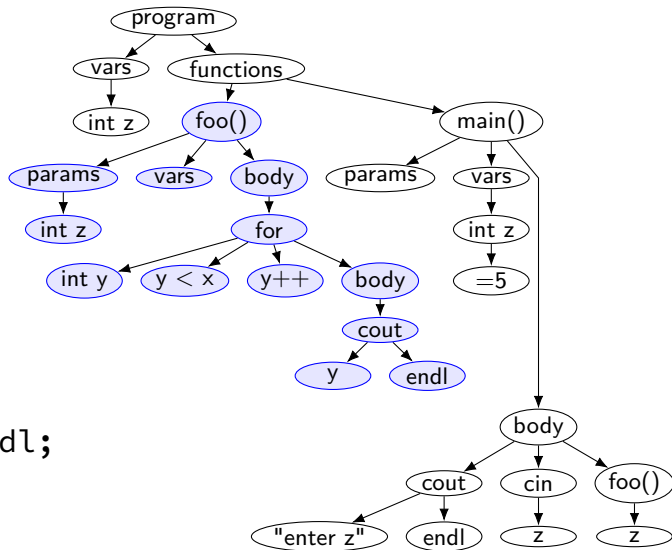


programs as trees

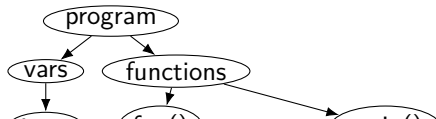
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int z;
```

```
int foo (int x) {  
    for (int y = 0;  
        y < x;  
        y++)  
        cout << y << endl;  
}
```

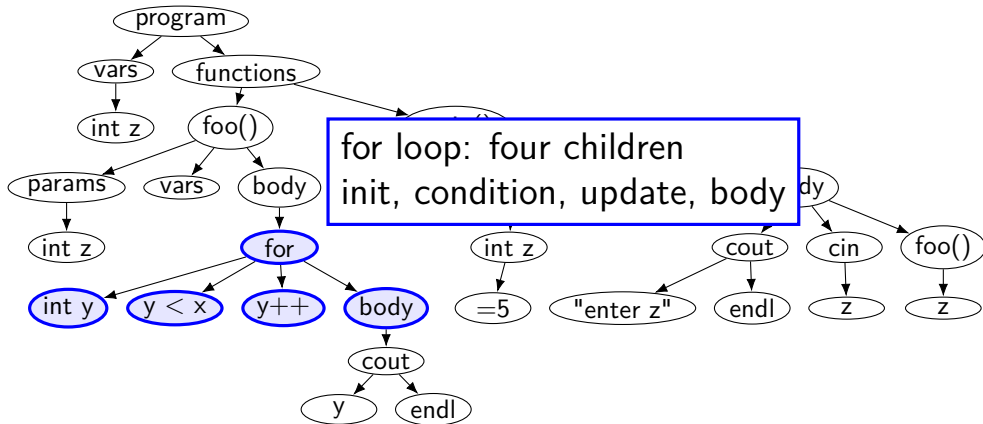
```
int main() {  
    int z = 5;  
    cout << "enter x" << endl;  
    cin >> z;  
    foo(z);  
}
```



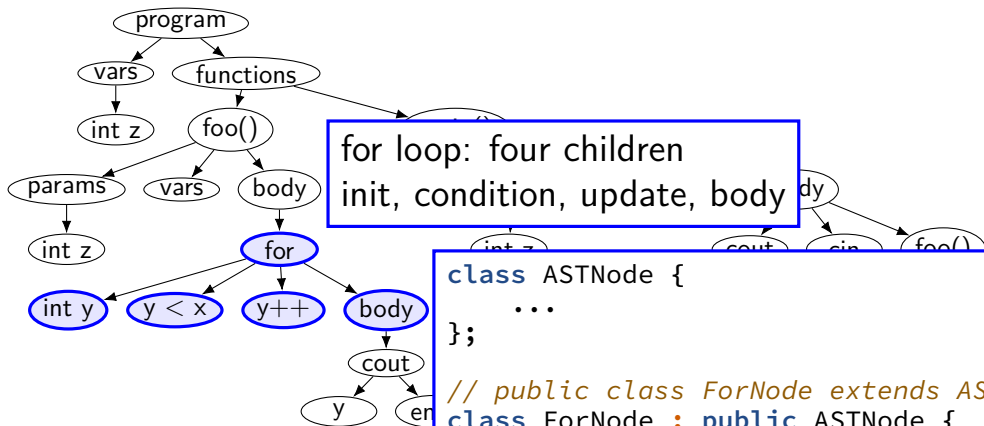
abstract syntax tree



abstract syntax tree



abstract syntax tree



for loop: four children
init, condition, update, body

```
class ASTNode {
    ...
};

// public class ForNode extends ASTNode
class ForNode : public ASTNode {
    ...
private:
    ASTNode *init, *condition,
            *update, *body;
};
```

AST applications

“abstract syntax tree” = “parse tree”

part of how compilers work

do some tree traversal to do...

- code generation — e.g. `ASTNode::outputCode()` method

- optimization

- type checking...

using AST to compare programs

comparing trees is a good way to compare programs...

while ignoring:

- function/method order (e.g. sort function nodes by length)
- variable names (e.g. ignore variable names when comparing)
- comments
- ...

part of many software plagiarism/copy+paste detection tools