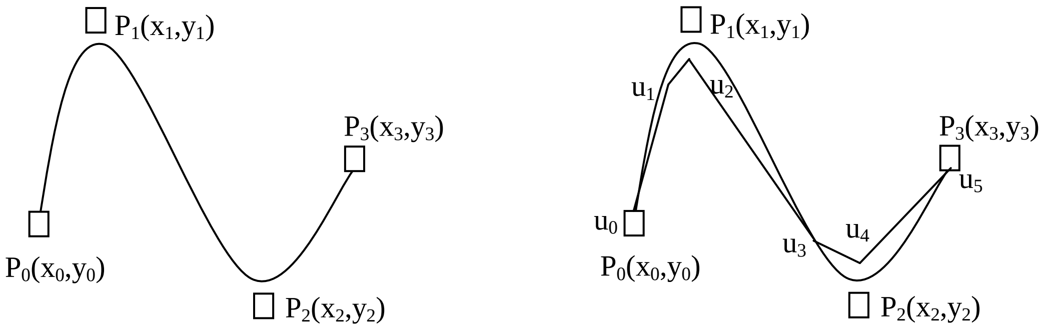
Spline : A spline is a flexible strip that passes thru a designated control points.

Bezier Curve



The above figure shows a smooth curve comprising of a large number of very small line segments. for understanding the concept to draw such a line we deal with a curve as show above which is an approximation of the curve with five line segments only

The approach below is used to draw a curve for any number of control points

Suppose P_0, P_1, P_2, P_3 are four control points

Number of segments in a line segment : nSeg

i = 0 to nSeg

u = i/nSeg [0,1] 0<= u <= 1

u0,u1 ……..u3

$x(u) = \sum_{j=0}^{n} x_j \, BEZ_{j,n}(u)$ n : number of control points

$x(u) = x_0 \, BEZ_{0,3}(u) + x_1 \, BEZ_{1,3}(u) + x_2 \, BEZ_{2,3}(u) + x_3 \, BEZ_{3,3}(u)$

similarly

$y(u) = \sum_{j=0}^{n} y_j \, BEZ_{j,n}(u)$ n : number of control points

$y(u) = y_0 \, BEZ_{0,3}(u) + y_1 \, BEZ_{1,3}(u) + y_2 \, BEZ_{2,3}(u) + y_3 \, BEZ_{3,3}(u)$

The Bezier blending function BEZj,n (u) is defined as,

$$BEZ_{j,n}(u) = \frac{n!}{j!\,(n-j)!} u^j (1-u)^{n-j}$$

$$BEZ_{j,n}(u) = C_{(n,j)} \, u^j (1-u)^{n-j}$$

Where $C_{(n,j)}$ is the Binomial Coefficient

$$C_{(n,j)} = \frac{n!}{j!\,(n-j)!}$$

For each ‘u’ the coordinates x and y are computed and desired curve is produced when the adjacent coordinates (x,y) are connected with a straight line segment

Now

$Q(u) = \; P_0\, BEZ_{0,3}(u) + P_1\, BEZ_{1,3}(u) + P_2\, BEZ_{2,3}(u) + P_3\, BEZ_{3,3}(u)$

Four blending functions must be found based on Bernstein Polynomials

$BEZ_{0,3}(u) = \dfrac{3!}{0!\;3!}\, u^0 (1-u)^3 = (1-u)^3 \qquad BEZ_{1,3}(u) = \dfrac{3!}{1!\,2!}\, u^1 (1-u)^2 = 3u\,(1-u)^2$

$BEZ_{2,3}(u) = \dfrac{3!}{2!\;1!}\, u^2 (1-u) = 3u^2 (1-u) \qquad BEZ_{3,3}(u) = \dfrac{3!}{3!\,0!}\, u^3 (1-u)^0 = u^3$

Normalizing properties apply to blending function s that means thy all add up to one

Substituting these functions in above equation

$Q(u) = \; (1-u)^3 P_0 + 3u\,(1-u)^2 P_1 + 3u^2 (1-u)\, P_2 + u^3 P_3$

When u = 0 then $Q(u) = P_0$ and when u =1 then $Q(u) = P_3$

in Matrix Form

$$Q(u) = \begin{bmatrix}(1-u)^3 & 3u\,(1-u)^2 & 3u^2 (1-u) & u^3\end{bmatrix} \begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\end{bmatrix}$$

or

$$Q(u) = \begin{bmatrix}(1-3u+3u^2-u^3) & (3u-6u^2+3u^3) & (3u^2-3u^3) & u^3\end{bmatrix} \begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\end{bmatrix}$$

or

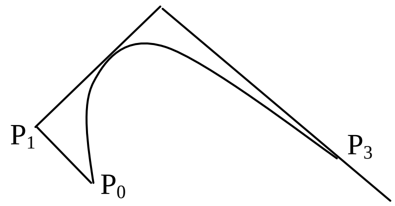
$$Q(u) = \begin{bmatrix}u^3 & u^2 & u^1 & 1\end{bmatrix} \begin{pmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix} \begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\end{bmatrix}$$

Properties of a Bezier Curve

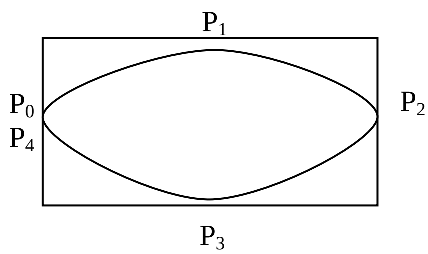
1. Bezier curve lies in the convex hull of the control points which ensure that the curve smoothly follows the control

P_2

Points



2. Four Bezier polynomials are used in the construction of curve to fit four control points
3. It always passes thru the end points
4. Closed curves can be generated by specifying the first and last control points at the same position



5. Specifying multiple control points at a single position gives more weight to that position
6. Complicated curves are formed by piecing several sections of lower degrees together
7. The tangent to the curve at an end point is along the line joining the end point to the adjacent control point