

# Ellipse

**Definition:** An ellipse is defined as set of points such that sum of the distances from the two fixed points is the same for all points. If the distance to two fixed points from any point P(x,y) on ellipse are  $d_1, d_2$  then general equation of an ellipse is  $d_1 + d_2 = \text{constant}$

Or expressing distance  $d_1, d_2$  in terms of focal coordinates  $F_1 = (x_1, y_1)$  and  $F_2 = (x_2, y_2)$

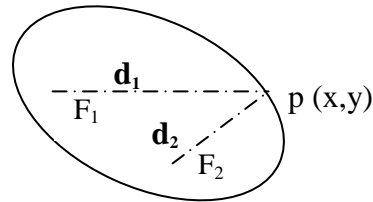
We have,

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2} = \text{constant}$$

Mid point ellipse method is applied throughout first quadrant in two parts( according to the slope of ellipse)

The equation of an ellipse is given by

$$x^2 / r_x^2 + y^2 / r_y^2 = 1$$



or  $F_{\text{ellipse}}(x,y) = r_y^2 x^2 + r_x^2 y^2 - r_x^2 r_y^2$

now  $F_{\text{ellipse}}(x,y) < 0$  if(x,y) is inside the ellipse boundary  
 $= 0$  if(x,y) is on the ellipse boundary  
 $> 0$  if(x,y) is outside the ellipse boundary

This ellipse function  $F_{\text{ellipse}}(x,y)$  serves as the decision parameter

We select next pixel along the ellipse path according to the sign of ellipse function evaluated at the midpoint between two candidate pixels.

Start at  $(0, r_y)$  take unit steps in 'x' direction until we reach boundary region 1 and 2 then switch to unit steps in 'y' direction for remainder of curve in the first quadrant.

At each step test the value of the slope of the curve. The ellipse slope is given by

$$2 r_y^2 x + 2 r_x^2 y \, dy/dx = 0$$

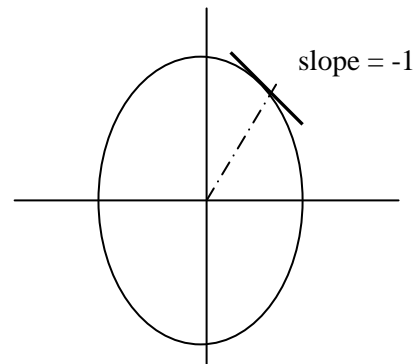
$$dy/dx = - 2 r_y^2 x / 2 r_x^2 y$$

At the boundary between region 1 and 2  $dy/dx = -1$

$$\text{so, } 2 r_y^2 x = 2 r_x^2 y$$

and we move out of the region 1 when

$$2 r_y^2 x \geq 2 r_x^2 y$$



Assuming position  $(x_k, y_k)$  has been selected at previous step we determine next position  $(x_{k+1}, y_{k+1})$  either  $(x_{k+1}, y_k)$  or  $(x_{k+1}, y_k - 1)$  along elliptic path by evaluating the decision parameter(elliptic function)

$$P1_k = F_{\text{ellipse}}(x_k + 1, y_k - 1/2) \\ = r_y^2 (x_k + 1)^2 + r_x^2 (y_k - 1/2)^2 - r_x^2 r_y^2 \dots\dots\dots (i)$$

At the next sampling position  $(x_{k+1} + 1 = x_k + 2)$ , the decision parameter for region 1 is evaluated as

$$P1_{k+1} = F_{\text{ellipse}}(x_{k+1} + 1, y_{k+1} - 1/2) \\ = r_y^2 [(x_k + 1) + 1]^2 + r_x^2 (y_{k+1} - 1/2)^2 - r_x^2 r_y^2 \dots\dots\dots (ii)$$

Now subtracting eq (i) and (ii),

$$P1_{k+1} = P1_k + 2r_y^2 (x_k + 1) + r_y^2 + r_x^2 [(y_{k+1} - 1/2)^2 - (y_k - 1/2)^2] \dots\dots\dots (iii)$$

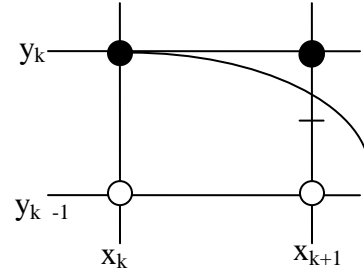
where  $y_{k+1}$  is either  $y_k$  or  $y_k - 1$  depending on the sign of  $P1_k$ .

### Case 1:

if  $P1_k < 0$  then the mid point is inside the ellipse, so pixel on scanline ' $y_k$ ' is closer to the elliptic boundary and  $y_{k+1} = y_k$

so the increment will be  $2r_y^2 x_{k+1} + r_y^2$   
i.e. from equation (iii)

Or  $P1_{k+1} = P1_k + 2r_y^2 x_{k+1} + r_y^2 \dots\dots\dots (a)$



Where  $x_{k+1} = x_k + 1$

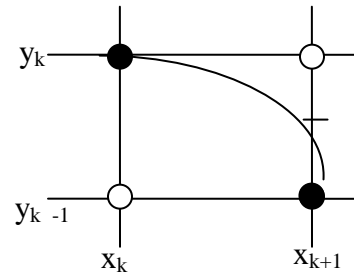
or  $2r_y^2 x_{k+1} = 2r_y^2 x_k + 2r_y^2$

### Case 2:

if  $P1_k \geq 0$  then the mid point is outside or on the boundary of the ellipse, so we select the pixel on scan line ' $y_k - 1$ ' then  $y_{k+1} = y_k - 1$

so the increment will be  $2r_y^2 x_{k+1} - 2r_x^2 y_{k+1} + r_y^2$   
i.e. from equation (iii)

Or  $P1_{k+1} = P1_k + 2r_y^2 x_{k+1} - 2r_x^2 y_{k+1} + r_y^2 \dots\dots\dots (b)$



Where  $2r_x^2 y_{k+1} = 2r_x^2 y_k - 2r_x^2$

or  $2r_y^2 x_{k+1} = 2r_y^2 x_k + 2r_y^2$

**Initial decision parameter for Region 1 =  $P1_0$**

**The starting position is  $(0, r_y)$**

**Next pixel to plot is either  $(1, r_y)$  or  $(1, r_y - 1)$**

**So, midpoint coordinate position is  $(1, r_y - 1/2)$**

$$F_{\text{ellipse}}(1, r_y - 1/2) = r_y^2 + r_x^2 (r_y - 1/2)^2 - r_y^2 r_x^2$$

**Thus,**

$$P1_0 = r_y^2 + 1/4 r_x^2 - r_x^2 r_y$$

**Region 2.**

**Sample at unit steps in 'y' direction, the midpoint is taken between horizontal pixels at each step now. Assuming,  $(x_k, y_k)$  has been plotted, next pixel to plot is  $(x_{k+1}, y_{k+1})$  where**

**$x_{k+1}$  is either  $x_k$  or  $x_{k+1}$**

**and  $y_{k+1}$  is  $y_k - 1$**

**i.e. we choose either  $(x_k, y_k - 1)$  or  $(x_{k+1}, y_k - 1)$**

**So, midpoint coordinate position is  $(x_k + 1/2, y_k - 1)$**

$$F_{\text{ellipse}}(x_k + 1/2, y_k - 1)$$

$$\text{Or, } P2_k = r_y^2 (x_k + 1/2)^2 + r_x^2 (y_k - 1)^2 - r_x^2 r_y^2 \text{ ..... (iv)}$$

**now, at next sampling position, the next pixel to plot will either be**

**$(x_{k+1}, y_{k+1} - 1)$  or  $(x_{k+1} + 1, y_{k+1} - 1)$**

**thus,**

$$F_{\text{ellipse}}(x_{k+1} + 1/2, y_{k+1} - 1)$$

$$\begin{aligned} \text{Or, } P2_{k+1} &= r_y^2 (x_{k+1} + 1/2)^2 + r_x^2 (y_{k+1} - 1)^2 - r_x^2 r_y^2 \\ &= r_y^2 (x_{k+1} + 1/2)^2 + r_x^2 [(y_k - 1) - 1]^2 - r_x^2 r_y^2 \text{ ..... (v)} \end{aligned}$$

**Now subtracting eq (iv) and (v),**

$$P2_{k+1} = P2_k - 2r_x^2 (y_k - 1) + r_x^2 + r_y^2 [(x_{k+1} + 1/2)^2 - (x_k + 1/2)^2] \text{ ..... (vi)}$$

**where  $x_{k+1}$  is either  $x_k$  or  $x_k + 1$  depending on the sign of  $P2_k$ .**

**Case 1:**

**if  $P2_k > 0$  then the mid point is**

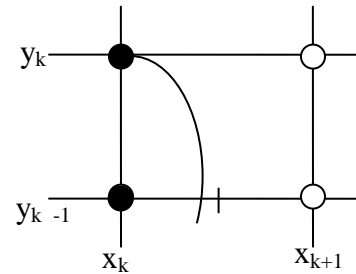
**outside the boundary of the**

**ellipse, so we select the pixel at ' $x_k$ '**

$$\text{Or } P2_{k+1} = P2_k - 2r_x^2 (y_k - 1) + r_x^2$$

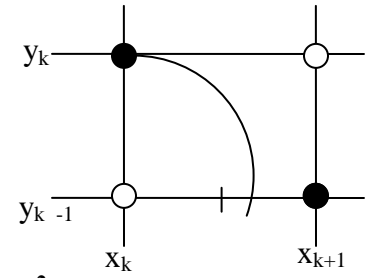
$$= P2_k - 2r_x^2 y_{k+1} + r_x^2 \text{ ..... (c) Where } y_{k+1} = y_k - 1$$

$$\text{or } 2r_x^2 y_{k+1} = 2r_x^2 y_k - 2r_x^2$$



**Case 2:**

if  $P2_k \leq 0$  then the mid point is  
inside or on the boundary of the  
ellipse , so we select pixel at ' $x_k + 1$ '  
i.e. from equation (vi)



$$\begin{aligned}
 \text{Or } P2_{k+1} &= P2_k - 2r_x^2 (y_k - 1) + r_x^2 + r_y^2 [(x_{k+1} + 1/2)^2 - (x_k + 1/2)^2] \\
 &= P2_k - 2r_x^2 (y_k - 1) + r_x^2 + r_y^2 [(x_k + 1) + 1/2]^2 - (x_k + 1/2)^2 \\
 &= P2_k - 2r_x^2 (y_k - 1) + r_x^2 + r_y^2 [(x_k + 3/2)^2 - (x_k + 1/2)^2] \\
 &= P2_k - 2r_x^2 (y_k - 1) + r_x^2 + r_y^2 [x_k^2 + 3x_k + 9/4 - x_k^2 - x_k - 1/4] \\
 &= P2_k - 2r_x^2 (y_k - 1) + r_x^2 + r_y^2 [2x_k + 2] \\
 &= P2_k - 2r_x^2 y_{k+1} + r_x^2 + 2r_y^2 x_{k+1} \dots\dots\dots (d)
 \end{aligned}$$

$$\text{Where } 2r_x^2 y_{k+1} = 2r_x^2 y_k - 2r_x^2$$

$$\text{or } 2r_y^2 x_{k+1} = 2r_y^2 x_k + 2r_y^2$$

For region 2, the initial position  $(x_0, y_0)$  is taken as the last position selected in region 1 and thus the initial decision parameter in region 2 is

$$\begin{aligned}
 P2_0 &= F_{\text{ellipse}}(x_0 + 1/2, y_0 - 1) \\
 &= r_y^2 (x_0 + 1/2)^2 + r_x^2 (y_0 - 1)^2 - r_x^2 r_y^2
 \end{aligned}$$