Hadamard Transform :→

⇒ The 1-D, forward Hadamard Kernel is given by the relation

$$g(x,u) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1} b_i(x)\, b_i(u)}$$

i.e. $$g(x,u) = \frac{1}{N}(-1)^{\sum\limits_{i=0}^{n-1} b_i(x)\, b_i(u)}$$

⇒ The 1-D Hadamard transform is given by

$$H(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)(-1)^{\sum_{i=0}^{n-1} b_i(x)\, b_i(u)}$$

where $N=2^n$, and u has values in the range $0, 1, 2, \ldots, N-1$

⇒ The Hadamard Kernel forms a matrix having orthogonal rows and columns

⇒ The inverse Hadamard Kernel is given by

$$h(x,u) = (-1)^{\sum_{i=0}^{n-1} b_i(x)\, b_i(u)}$$

⇒ The inverse Hadamard Transform

$$f(x) = \sum_{u=0}^{N-1} H(u)\,(-1)^{\sum_{i=0}^{n-1} b_i(x)\, b_i(u)}$$

for $x = 0, 1, 2, \ldots, N-1$

The 2-D kernels are given by

$$g(x,y,u,v) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

and

$$h(x,y,u,v) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

⇒ The 2-D Hadamard transform pair

$$H(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u) + b_i(y)b_i(v)]}$$

and

$$f(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} H(u,v)(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u) + b_i(y)b_i(v)]}$$

⇒ The Hadamard matrix of lowest order $(N=2)$ is

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

⇒ Letting H_N represent the matrix of order N, the recursive relationship is

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix}$$

where H_{2N} is the Hadamard matrix of order $2N$ and $N=2^n$ is assumed

The transformation matrix is given by

$$A = \frac{1}{\sqrt{N}} H_N$$

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & +1 \end{bmatrix}$$

and

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$$

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & +1 & 1 & -1 & -1 & +1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & +1 & -1 & +1 \\ 1 & 1 & -1 & -1 & -1 & -1 & +1 & +1 \\ 1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \end{bmatrix}$$

THE HAAR TRANSFORM

The haar functions $h_K(x)$ are defined on a continuous interval, $x \in [0, 1]$, and for $K = 0, \ldots N-1$, where $N = 2^n$. The integer K can be uniquely decomposed as $\boxed{K = 2^p + q - 1}$

where $0 \leq p \leq n-1$; $q = 0, 1$ for $p = 0$ and $1 \leq q \leq 2^p$ for $p \neq 0$. For Eg:→ when $N=4$, we have

K	0	1	2	3
p	0	0	1	1
q	0	1	1	2

$N = 2^n$
$N = 4$
$2^n = 4$, 2^2, $n = 2$

Representing K by (p,q), the Haar functions are defined as

$$h_0(x) \equiv h_{0,0}(x) = \frac{1}{\sqrt{N}}, \quad \underbrace{x \in [0, 1]}_{0}$$

$$h_K(x) = h_{p,q}(x) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2}, & \frac{q-1}{2^p} \leq x \leq \frac{q - \frac{1}{2}}{2^p} \\ -2^{p/2}, & \frac{q - \frac{1}{2}}{2^p} \leq x < \frac{q}{2^p} \\ 0 & \text{otherwise for} \end{cases}$$

Haar Transform when $N=2$

$\therefore k = 0, 1$

k	0	1
p	0	0
q	0	1

$\Rightarrow$ The first row ~~is computed~~ of 2×2 Haar matrix is computed by using $h_0(x)$ with $x = 0/2, 1/2$

$\Rightarrow$ $h_0(x)$ is equal to $1/\sqrt{2}$, independent of x

$\Rightarrow$ So the first row of the matrix has two identical elements $1/\sqrt{2}$

$\Rightarrow$ The second row is obtained by computing $h_1(x)$ for $x = 0/2, 1/2$

$\Rightarrow$ When $k=1$, $p=0$ & $q=1$

So,

$$h_1(x) = h_{01}(x) = \frac{1}{\sqrt{2}} \begin{cases} 2^0 = 1 & 0 \le x < 1/2 \\ -2^0 = -1 & 1/2 \le x < 1 \\ 0 & \text{otherwise for } x \in [0,1] \end{cases}$$

$\therefore h_1(0/2) = h_1(0) = 1/\sqrt{2}$

$h_1(1/2) = -1/\sqrt{2}$

$$\therefore A_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Following similar procedure yield the matrix for N=4

$$A_4 = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}}\begin{array}{c} \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{array}\right] \end{array} \begin{array}{l} \varphi_0(t) \\ \psi_0(t) \\ \psi_{1,0}(t) \\ \psi_{1,1}(t) \\ \psi_{2,0}(t) \\ \psi_{2,1}(t) \\ \psi_{2,2}(t) \\ \psi_{2,3}(t) \end{array}$$

The Cosine Transform :→

The N×N cosine transform matrix $C = \{c(k,n)\}$ also called the discrete cosine transform (DCT) is defined as

$$c(k,n) = \begin{cases} \dfrac{1}{\sqrt{N}} & k=0,\ 0 \le n \le N-1 \\ \sqrt{\dfrac{2}{N}} \cos \dfrac{\pi(2n+1)k}{2N} & 1 \le k \le N-1,\ 0 \le n \le N-1 \end{cases}$$

The one-dimensional DCT of a sequence $\{u(n), 0 \leq n \leq N-1\}$ is defined as

$$v(k) = \alpha(k) \sum_{n=0}^{N-1} u(n) \cos\left[\frac{\pi(2n+1)k}{2N}\right], \quad 0 \leq k \leq N-1$$

where

$$\alpha(0) = \sqrt{\frac{1}{N}} \qquad \alpha(k) = \sqrt{\frac{2}{N}} \text{ for } 1 \leq k \leq N-1$$

The inverse transformation is given by

$$u(n) = \sum_{k=0}^{N-1} \alpha(k)\, v(k) \cos\left[\frac{\pi(2n+1)k}{2N}\right], \quad 0 \leq n \leq N-1$$

Properties :→

① The cosine transform is real and orthogonal, i.e., $C = C^* \Rightarrow C^{-1} = C^T$

② The cosine transform has excellent energy compaction for highly correlated data

Properties Of Hadamard Transform:

(1) It is real, symetric and orthogonal
i.e

$$H = H^{*} \Rightarrow H^{-1} = H^{T}$$

(2) It is a fast transform and 1-D Hadamard Transform can be implemented in $O(N\log_2 N)$ addition and subtraction.

(3) It has only binary values i.e 1 or -1 in its kernel matrix. No multiplications are required in the transform.

(4) It is used for digital image processing and digital signal processing.

(5) It has good energy compaction for highly co-related image.

Properties Of Haar Transform:

(1) It is symmetric, seperable unitary transform that uses haar function for its basis.

(2) It is orthogonal and real.
ie

$$T^{-1} = T^{T} \Rightarrow H_r = H_r^{*}$$

(3) It is a fast transform and can be implemented in $O(N)$ operation. Where, N is a number of samples.

(4) It exist for $N = 2^n$, where 'n' is an integer.

(5) It has poor energy compaction property.