Chapter 6 Algebra and Analysis

6.1: A little Algebra

Problem 6.1.1:- Show that if n is a positive integer then n^3-n is divisible by 3.

Sol^n: Given n^3-n

$=n\ (n^2-1)$

$=n\ (n+1)\ (n-1)$

These three consecutive number of positive integer, out of these three one number must divisible by 3.Hence n^3-n is divisible by 3.

Problem 6.1.2:- Show that if n is a positive integer then n^5-n always divisible by 5.

Sol^n: Given n^5-n

$=n\ (n^4-1)$

$=n\ (n^2-1)\ (n^2+1)$

$=n\ (n-1)(n+1)(n^2+1)$

If n is integer ending with 0,1,4,5,6,9 then one of n (n+1)(n-1) is divisible by 5.If n is integer ending with 2,3,7,8 then n^2+1 is divisible by 5.Hence for every n,n^5-n is divisible by 5.

Problem 6.1.3:-Show that if n is a positive integer then n^7-n is divisible by 7.

Sol^n: Given n^7-n

$=n\ (n^6-1)$

$=n\ (n^3-1)(n^3+1)$

$=n\ (n-1)(n^2+n+1)(n-1)(n^2-n+1)$

For n=0, 0/7=0, which is divisible by 7.

For n=1, 0/7=0, which is divisible by 7.

For n=2, $n^2+n+1/7=2^2+2+1/7=1$, which is divisible by 7.

For n=3, $n^2-n+1/7=3^2-3+1/7=1$, which is divisible by 7.

For n=4, $n^2+n+1/7=4^2+4+1/7=3$, which is divisible by 7.

For n=5, $n^2-n+1/7=5^2-5+1/7=3$, which is divisible by 7.

For n=6, n+1/7=6+1/7=1 which is divisible by 7.

For n=8, n-1/7=8-1/7=1 which is divisible by 7.

For n=9, $n^2+n+1/7=9^2+9+1/7=13$ which is divisible by 7.

Hence for all n, n^7-n is divisible by 7.

Problem 6.1.4:- Verify combinational identify.

${}^kc_m+{}^kc_{m+1}={}^{k+1}c_{m+1}$

Sol^n: L.H.S. $={}^kc_m+{}^kc_{m+1}$ $({}^nc_r=n!/(n-r)!r!)$

$=k!/(k-m)m! + \quad k!/(k-m-1)!(m+1)!$

$= k!(m+1)/(k-m)(m+1)! + k!\,(k-m)/(k-m)!(m+1)!$

$=k!(m+1+k-m)/(k-m)!(m+1)!$

$=k!(k+1)/(k-m)!(m+1)!$

$=(k+1)!/(k-m)!(m+1)!$

$={}^{k+1}C_{m+1}$

Therefore L.H.S. =R.H.S.

Hence Proved

Problem 6.1.6:- Which is greater $\alpha=(1+0.00001)^{1000000}$ or 2?

Sol^n: Here given,

$\alpha=(1+1/1000000)^{1000000}$

We put k=1000000, then $\alpha=(1+1/k)^k$ which is **e** whose value is 2.718. Hence, α is greater than 2.

Problem 6.1.7:- Which is greater? $(1000)^{1000}$ or $(1001)^{999}$.

Sol^n:- We have (1001)999

$=(1000+1)^{999}$

[Formula$(x+y)^n={}^nc_o x^n y^o+{}^nc_1\ x^{n-1}y^1+{}^nc_2\, x^{n-2}y^2+....+{}^nc_n x^0 y^n$]

$={}^{999}c_o 1000^{999}+{}^{999}c_1\ 1000^{999-1}1^1+{}^{999}c_2\, 1000^{999-2}1^2+.... +{}^{999}c_{999}1000^0 1^{999}$

$=1.\ 1000^{999}+999.1000^{998}+1......+1 < \underline{1.1000^{1000}+1000^{999}+ ....+1000^{900}}$

(1000 terms)

$=1000^{999}*1000$

$=1000^{1000}$

Therefore, $[1000+1]^{999}<1000^{1000}$

$[1001]^{999}<1000^{1000}$

Problem 6.1.8:- Assume that k is a positive integer.Calculate

1/1.2 + 1/2.3 +.............+1/(k-1)k +1\k(k+1)

Sol^n: Here ,

$1/k(k+1) = 1/k – 1/(k+1)$

Let, $S_k =$ [1-1\2]+[1/2-1/3]+[1/3-1/4]+....+[1/(k-1)-1/k]+[1/k-1/(k+1)]

$= 1 - \cancel{1/2} + \cancel{1/2} - \cancel{1/3} + \cancel{1/3} - \cancel{1/4} + \cancel{1/4} +..........+ \cancel{1/k-1} - \cancel{1/k} + \cancel{1/k} - 1/(k-1)$

$= 1 - 1/k+1$

$= k/k+1$

Problem 6.1.10:- Calculate the sum (1.2)+(2.3)+(3.4)+.......+n(n+1)

Sol^n: Let $\quad S_n =1.2+2.3+3.4+........+n(n+1)$

$=2(1+3) + 3(2+4) + 4(3+5) + n\,[(n-1)+(n+1)]$

$=2[(1.2+2.3+3.4+.....+n\,(n+1)] - 1.2 – n(n+1)$

We subtracted 1.2 and n(n+1) because it uses only one time.

$2.4 + 3.6 + 4.8 + \ldots.+ n.2n = S_n – 1.2 – n(n+1)$

or, $2^2 . 2 + 3^2 . 2 + 4^2 . 2 +n^2 .2 = S_n – 2 – n(n+1)$

or, $2\{2^2 + 3^2 + 4^2 \ \ +n^2\} = S_n – 2 – n(n+1)$

or, $2 \{1^2 + 2^2 + 3^2 + 4^2 \ \ +n^2\} = S_n – \cancel{2} – n(n+1) + \cancel{2}$

or, $2 n (n+1)(2n+1)/6 = S_n – n(n+1)$

or, $S_n = n (n+1)(2n+1)/3 + n(n+1)$

or, $S_n = [n (n+1)(2n+1) + 3n(n+1)]/3$

or, $S_n = [n (n+1) (2n+1+3)] / 3$

or, $S_n = [2n(n+1)(n+2)]/3$

Challenging Problem 6.1.11:- Calculate the sum 1.2.3 + 2.3.4 + ...+ n (n+1) (n+2)

Sol^n: Given, 1.2.3 + 2.3.4 + ...+ n (n+1) (n+2)*Type equation here*.

So, it's n^{th} term is $t_n = n (n+1) (n+2)$

$$S_n = \sum_{k=1}^{n} k(k+1)(k+2)$$

$$= \sum_{k=1}^{n} (k^2+k)(k+2)$$

$$= \sum_{k=1}^{n} (k^3+2k^2+k^2+2k)$$

$$= \sum_{k=1}^{n} (k^3+3k^2+2k)$$

$$= \sum_{k=1}^{n} (k^3) + 3 \sum_{k=1}^{n} (k^2) + 2\sum_{k=1}^{n} (k)$$

$$= (n(n+1¿¿2)^2¿ + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \frac{n^2+(n+1)^2}{4} + \frac{(n+1)(2n+1)n}{2} + n (n+1)$$

$$= n(n+1) \left[\frac{n(n+1)}{4} + \frac{2n+1}{2} +1 \right]$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

Therefore, $S_n = \frac{n(n+1)(n+2)(n+3)}{4}$

Problem 6.2.1:- If a and b are positive real numbers then show that ab $\le \frac{a^2+b^2}{2}$

Sol^n: Given, ab ≤ $\frac{a^2+b^2}{2}$

or, 2ab ≤ $a^2 + b^2$

or, $a^2 - 2ab + b^2 \geq 0$

or, $0 \leq (a-b)^2$

The square of any value always positive, so the expression $0 \leq (a-b)^2$ is always true. Hence the inequality ab ≤ $\frac{a^2+b^2}{2}$ hold.

Problem 6.2.3:- Prove that, $2 < 1/\log_2^{\pi} + 1/\log_5^{\pi}$

Formula=$\log_b^a = \ln(a)/\ln(b)$

Sol^n: Here given $2 < 1/\log_2^{\pi} + 1/\log_5^{\pi}$

or, $2 < 1/\ln(\pi)/\ln(2) + 1/\ln(\pi)/\ln(5)$

or, $2 < \ln 2/\ln\pi + \ln 5/\ln\pi$

or, $2 < \ln 2 + \ln 5/\ln\pi$

or, $2 \ln\pi < \ln(2.5)$

or, $\ln\pi^2 < \ln 10$

or, $\pi^2 < 10$

Here , π=3.14 whose squaring value is always less than 10.

Hence the given inequality always hold.

Problem 6.2.5:- Show that, |cosx + sinx|≤√2 with equality only if sin2x=1.

(Formula: $|\alpha|=\sqrt{\alpha^2}$

Sol^n: |sinx + cosx|=$\sqrt{(sinx+cosx)^2}$

$=\sqrt{\sin^2 x+2\,sinx\,.\,cosx+\cos^2 x}$

$= \sqrt{1+\sin 2x}$

Clearly greater value of sin2x is 1.

Therefore, |cosx + sinx|≤$\sqrt{2}$

Challenging Problem 6.2.6:- Show that |cosx-sinx|≤$\sqrt{2}$

Sol^n: |cosx-sinx|= $\sqrt{(cosx-sinx)^2}$

$= \sqrt{\sin^2 x+\cos^2 x-2\,sinx\,.\,cosx}$

$=\sqrt{1-\sin 2x}$

Clearly least value of sin2x is -1.

Therefore, |cosx-sinx|≤$\sqrt{2}$

Problem 6.2.7:- Which is greater sin(cosx) or cos(sinx)?

Sol^n: Let cos($\frac{\pi}{2}+cosx$)

=cos$\frac{\pi}{2}$.cos(cosx) -sin$\frac{\pi}{2}$.sin(cosx)

= -sin(cosx)

Therefore, cosx(sinx)- sin(cosx)

$= \cos x(\sin x) + \cos x(\cos x + \frac{\pi}{2})$

Again,

cosx+cosy= 2cos(x+y/2) – cos(x-y/2)

Then from equ^n (i) and (ii),

Therefore ,

$\cos(\sin x) - \sin(\cos x) = \cos(\sin x) + \cos(\cos + \frac{\pi}{2})$

$$= 2\cos\frac{\sin x + \cos x + \frac{\pi}{2}}{2} . \cos\frac{\sin x - \cos x - \frac{\pi}{2}}{2}$$

Now,

$$0 < \left|\frac{\sin x + \cos x + \frac{\pi}{2}}{2}\right| \le \left|\frac{\sin x + \cos x}{2}\right| + \left|\frac{\frac{\pi}{2}}{2}\right|$$

$$\le \frac{\sqrt{2}}{2} + \frac{\pi}{4}$$

$$< 1.5 < \frac{\pi}{2}$$

Again,

$$0 < \left|\frac{\sin x - \cos x - \frac{\pi}{2}}{2}\right| \le \left|\frac{\sin x - \cos x}{2}\right| + \left|\frac{\frac{\pi}{2}}{2}\right|$$

$$\le \frac{\sqrt{2}}{2} + \frac{\pi}{4}$$

$$< 1.5 < \frac{\pi}{2}$$

Hence from equ^n (iii)

$\cos(\sin x) - \sin(\cos x) < 2\cos(\frac{\pi}{2}).\cos(\frac{\pi}{2})$

$\le 2.1.1$

Therefore,

Cos(sinx) – sin (cosx) < (+ve value)

Hence, difference of two value is +ve, so cos(sinx) is greater than sin(cosx)

6.3 Trigonometry and Related Ideas

Problem 6.3.1:-

Suppose that α is an angle and $\tan(\frac{\alpha}{2})$ is rational.Verify that sinα and cosα are both rational.

Sol^n: We know that,

$$1+\tan^2\frac{\alpha}{2} = \sec^2\frac{\alpha}{2}$$

$$= 1/\cos^2\frac{\alpha}{2}$$

Since $\tan(\frac{\alpha}{2})$ is rational.Here left hand side is rational so by equality right hand side is also rational. Hence $\cos^2 \alpha/2$ is also rational.

Again, $\cos\alpha = \cos^2\alpha/2 - \sin^2 \alpha/2$

$= 2 \cos^2\alpha/2 - 1$

Since $\cos^2\alpha/2$ is rational, so right hand side is rational. Again by equality, we can say $\cos\alpha$ is also rational.

Again, $\tan\alpha = \frac{\sin\alpha}{cos\alpha}$

$= (2\sin\alpha/2.\cos\alpha/2)/ (\cos^2\alpha/2 - \sin^2 \alpha/2)$

$= 2 \tan(\frac{\alpha}{2})/ 1- \tan^2\frac{\alpha}{2}$

We know each component of right hand side is rational,so by equality we can say $\tan\alpha$ is rational. But since $\cos\alpha$ is rational.We say that $\sin\alpha$ is also rational.

Problem 6.3.3:- If O is positive, acute angle measure in radians , then show that $\tan\theta > \theta$.

Sol^n: Let, OB= 1 unit (radius of circle)

Here, triangle AOB is similar with triangle COD

From triangle AOB

$\cos \theta = OB/OA = 1/OA$

$OA= 1/ \cos \theta$ = hypotenuse of triangle AOB

Therefore, $\tan\theta=AB/OB = \sin\theta/\cos\theta$ = height of triangle AOB

Also, the height of triangle is greater than the length of the arc of the circle

that is subtended.

$AB > \overset{\frown}{BC} \quad (\theta=l/r= \overset{\frown}{BC})$

$\tan\theta > \theta \quad (\overset{\frown}{BC} = \theta)$

Therefore, $\sin\theta/\cos\theta > \theta$

Problem 6.3.5:- Suppose that α be any angle. Explain why.

$$\cos\frac{\alpha}{2}.\ \cos\frac{\alpha}{4}\ .\ \cos\frac{\alpha}{8} = \frac{sin\alpha}{8\sin\frac{\alpha}{8}}$$

Sol^n: L.H.S.= $\frac{sin\alpha}{8\sin\frac{\alpha}{8}}$

$$= \frac{2\sin\frac{\alpha}{2}.\cos\frac{\alpha}{2}}{8\sin\frac{\alpha}{8}}$$

$$= \frac{4\sin\frac{\alpha}{4}.\cos\frac{\alpha}{4}.\cos\frac{\alpha}{2}}{8\sin\frac{\alpha}{8}}$$

$$= \frac{8\sin\frac{\alpha}{8}.\cos\frac{\alpha}{8}.\cos\frac{\alpha}{4}.\cos\frac{\alpha}{2}}{8\sin\frac{\alpha}{8}}$$

$$= \cos\frac{\alpha}{2}.\cos\frac{\alpha}{4}.\cos\frac{\alpha}{8}$$

= R.H.S. Proved

Problem:- Prove that if $0\leq a,b,c,d\leq 1$, then $(1-a)(1-b)(1-c)(1-d)\geq 1-a-b-c-d$

Sol^n: Proof,

L.H.S.= (1-a)(1-b)(1-c)(1-d)

= (1-b-a+ab)(a-d-c+cd)

=(1-d-c+cd-b+bd+bc-bcd-a+ad+ac-acd+ab-abd-abc+abcd)

= (1-a-b-c+ab+ac+ad+bc+bd+cd-abc-abd-acd-bcd+abcd)

=[1-a-b-c-d+ab(1-c)+bc(1-d)+cd(1-a)+ad(1-b)+ac+bd+abcd]

Therefore , $(1-a)(1-b)(1-c)(1-d)\geq(1-a-b-c-d)$

Problem:- Explain why $11^{10}-1$ is divisible by 100.

Sol^n: Given, $11^{10}-1$

$= (10+1)^{10}-1$

$= [{}^{10}C_0(10)^{10}.1^0+{}^{10}C_1(10)^9.1^1+{}^{10}C_2(10)^8\,1^2+\ldots\ldots\ldots\ldots+{}^{10}C_9\,.10^1\,1^9+{}^{10}C_{10}\,10^0\,.1^{10}\,]$

$=[10^{10}+10^{10}+45.10^8+\ldots\ldots\ldots\ldots+10^2+1-1]$

$=10\,[10^8+10^8+45.10^6+\ldots\ldots\ldots\ldots+1]$

Hence $11^{10}-1$ is divisible by 100.