MA6453-PROBABILITY AND QUEUEING THEORY

UNIT I – RANDOM VARIABLES

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Random Variables

Random variable

A real variable (X) whose value is determined by the outcome of a random experiment is called a random variable.

(e.g) A random experiment consists of two tosses of a coin. Consider the random experiment which is the number of heads (0, 1 or2)

Outcome:	HH	HT	TH	TT
Value of X:	2	1	1	0

One Dimensional Random Variable and Two dimensional Random Variable

- A real valued function defined on S and talking values in $R(-\infty, \infty)$ is called a one-dimensional random variable.
- If the values are ordered pairs of real numbers, the function is said to be two-dimensional random variable.
- Note: In unit-I random variables and their probability distributions we restricted ourselves to one dimensional sample spaces.

Discrete Random Variable

A random variable x which takes a countable number of real values is called a discrete random variable.

(e.g) 1. number of telephone calls per unit time

2. marks obtained in a test

3. number of printing mistakes in each page of a book

Probability Mass Function

If X is a discrete random variable taking atmost a countably infinite number of values x_1, x_2, .., we associate a number $P_i = P(X = x_i) = P(x_i)$, called the probability mass function of X. The function $P(x_i)$ satisfies the following conditions:

(i) $P(x_i) \geq 0 \ \forall \ i = 1, 2, ..., \infty$

(ii) $$\sum_{i=1}^{\infty} P(x_i) = 1$$

Continuous Random Variable

A random variable X is said to be continuous if it can take all possible values between certain limits.

(e.g.)1. The length of a time during which a vacuum tube installed is a continuous random variable .

2. number of scratches on a surface, proportion of defective parts among 1000 tested,

3. number of transmitted in error.

Probability Density Function

Consider a small interval (x, x+dx) of length dx. The function f(x)dx represents the probability that X falls in the interval (x, x+dx)

i.e., $P(x \leq X \leq x+dx) = f(x)\,dx$.

The probability function of a continuous random variable X is called as probability density function and it satisfies the following conditions.

(i) $f(x) \geq 0 \;\forall\; x$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Distribution Function

The distribution function of a random variable X is denoted as F(X) and is defined as F(x) = P(X $\leq$ x). The function is also called as the cumulative probability function.

$$F(x) = P(X \leq x) = \sum_{x=-\infty}^{x} P(x) \quad \text{when X is discrete}$$

$$= \int_{-\infty}^{x} F(x)dx \quad \text{when X is continuous}$$

Properties on Cumulative Distribution

1. If $a \leq b$, $F(a) \leq F(b)$, where a and b are real quantities.

2. If F is the distribution function of a one-dimensional random variable X, then $0 \leq F(x) \leq 1$.

3. If F is the distribution function of a one dimensional random variable X, then $F(-\infty) = 0$ and $F(\infty) = 1$.

Problems

1. If a random variable X takes the values 1, 2, 3, 4 such that $2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)$. Find the probability distribution of X.

Solution:

Assume $P(X=3) = \alpha$ By the given equation

$$P(X=1) = \frac{\alpha}{2} \quad P(X=2) = \frac{\alpha}{3} \quad P(X=4) = \frac{\alpha}{5}$$

For a probability distribution (and mass function)

$\Sigma P(x) = 1$

$P(1)+P(2)+P(3)+P(4) = 1$

$$\frac{\alpha}{2}+\frac{\alpha}{3}+\alpha+\frac{\alpha}{5}=1 \qquad \Rightarrow \qquad \frac{61}{30}\alpha=1 \qquad \Rightarrow \alpha=\frac{30}{61}$$

$$P(X=1)=\frac{15}{61}\ ; P(X=2)=\frac{10}{61};\ P(X=3)=\frac{30}{61};\ P(X=4)=\frac{6}{61}$$

The probability distribution is given by

X	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

2. Let X be a continuous random variable having the probability density function

$$f(x) = \begin{cases} \dfrac{2}{x^3}, & x \geq 1 \\ 0, & otherwise \end{cases}$$

Find the distribution function of x.

Solution:

$$F(x) = \int_1^x f(x)\, dx = \int_1^x \frac{2}{x^3} dx = \left[-\frac{1}{x^2} \right]_1^x = 1 - \frac{1}{x^2}$$

3. A random variable X has the probability density function *f(x)* given by

$$f(x) = \begin{cases} cxe^{-x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

Find the value of c and CDF of X.

Solution:

$$\int_0^{\infty} f(x)\, dx = 1$$

$$\int_0^{\infty} cxe^{-x}\, dx = 1$$

$$c\left[-xe^{-x} - e^{-x}\right]_0^{\infty} = 1$$

$$c(1) = 1$$

$$c = 1$$

$$F(x) = \int_0^{x} f(x)\, dx$$

$$= \int_0^{x} cxe^{-x}\, dx$$

$$= \int_0^{x} xe^{-x}\, dx$$

$$= \left[-xe^{-x} - e^{-x}\right]_0^{x}$$

$$= 1 - xe^{-x} - e^{-x}$$

4. A continuous random variable X has the probability density function *f(x)* given by

$$f(x) = ce^{-|x|}, -\infty < x < \infty$$

Find the value of c and CDF of X.

Solution:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} c\, e^{-|x|}\, dx = 1$$

$$2\int_{0}^{\infty} c\, e^{-|x|}\, dx = 1$$

$$2\int_{0}^{\infty} c\, e^{-x}\, dx = 1$$

$$2c\left[-e^{-x}\right]_{0}^{\infty} = 1$$

$$2c(1) = 1$$

$$c = \frac{1}{2}$$

$Case(i)\ x < 0$

$$F(x) = \int_{-\infty}^{x} f(x)\ dx$$

$$= \int_{-\infty}^{x} c\, e^{-|x|}\ dx$$

$$= c \int_{-\infty}^{x} e^{x}\ dx$$

$$= c\left[\ e^{x}\ \right]_{-\infty}^{x}$$

$$= \frac{1}{2} e^{x}$$

$Case(ii)\ x > 0$

$$F(x) = \int_{-\infty}^{x} f(x)\ dx$$

$$= \int_{-\infty}^{x} c\, e^{-|x|}\ dx$$

$$= c \int_{-\infty}^{0} e^{x}\ dx + c\int_{0}^{x} e^{-x} dx$$

$$= c\left[\ e^{x}\ \right]_{-\infty}^{0} + c\left[-e^{-x}\ \right]_{0}^{x}$$

$$= c - ce^{-x} + c$$

$$= c\left(2 - e^{-x}\right)$$

$$= \frac{1}{2}\left(2 - e^{-x}\right)$$

$$F(x) = \begin{cases} \frac{1}{2} e^{x}, & x > 0 \\ \frac{1}{2}\left(2 - e^{-x}\right), & x < 0 \end{cases}$$

5. A random variable X has the following probability distribution.

X:	0	1	2	3	4	5	6	7
f(x):	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) the value of k (ii) p(1.5 < X < 4.5 | X >2) and (iii) the smallest value of λ such that p(X≤λ) > 1/2.

Solution

(i) $$\sum P(x) = 1$$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0 \quad \Rightarrow k = -1, \frac{1}{10}$$

$$k = \frac{1}{10} = 0.1$$

(ii) $A = 1.5 < X < 4.5 = \{2,3,4\}$

$B = X > 2 = \{3,4,5,6,7\}$

$A \cap B = \{3,4\}$

$$p(1.5 < X < 4.5 | X > 2) = p(A | B) = \frac{p(A \cap B)}{p(B)} = \frac{p(3,4)}{p(3,4,5,6,7)}$$

$$= \frac{2k + 3k}{2k + 3k + k^2 + 2k^2 + 7k^2 + k} = \frac{5k}{10k^2 + 6k} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

(iii)

X	p(X)	F(X)
0	0	0
2	2k = 0.2	0.3
3	2k = 0.2	0.5
4	3k = 0.3	0.8
5	k^2=0.01	0.81
6	$2k^2$ = 0.02	0.83
7	$7k^2$+k = 0.17	1.00

From the table for X = 4,5,6,7 p(X) > and the smallest value is 4

Therefore λ = 4.

Expectation of a Random Variable

The expectation of a random variable X is denoted as E(X). It returns a representative value for a probability distribution.

For a discrete probability distribution

$$E(X) = \sum x\, p(x).$$

For a continuous random variable X which assumes values in (a, b)

$$E(X) = \int_a^b x f(x) dx$$

Properties on Expectation

1. Expectation of a constant is a constant.

2. $E[aX] = aE(X)$, where a is a constant.

3. $E(aX + b) = aE(X) + b$, where a and b are constants.

4. $|E(X)| \leq E|X|$, for any random variable X.

5. If $X \leq Y$, $E(X) \leq E(Y)$.

Variance of a Random Variable

The variance of a Random variable X, which is represented as V(X) is defined as the expectation of squares of the derivations from the expected value.

$$V(X) = E(X^2) - (E(X))^2$$

Properties On Variance

1. Variance of a constant is 0
2. $Var(aX + b) = a^2 Var(X)$, where a is a constant.

Moments and Other Statistical Constants

Raw Moments

Raw moments about origin

$$\mu'_r = \int_a^b x^r f(x)dx$$

Raw moments about any arbitrary value A

$$\mu'_r = \int_a^b (x - A)^r f(x)dx$$

Central moments

$$\mu_r = E[X - E(X)]^r = \int_a^b (X - E(X))^r f(x)dx$$

Relationship between Raw Moments and Central Moments

$$\mu_1 = 0 \ \ (\text{always})$$

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

Moment Generating Function (M.G.F)

It is a function which automatically generates the raw moments. For a random variable X, the moment generating function is denoted as $M_X(t)$ and is derived as $M_X(t) = E(e^{tX})$.

Reason for the name M.G.F

$$M_X(t) = E(e^{tx})$$

$$= E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^2}{3!} + \cdots\right]$$

$$= E(1) + E(tx) + E\left(\frac{t^2X^2}{2!}\right) + \cdots$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots$$

Here μ_1' = coefficient of t in $M_X(t)$

μ_2' = coefficient of $\frac{t^2}{2!}$ in $M_X(t)$

In general μ_r' = coefficient of $\frac{t^2}{2!}$ in $M_X(t)$.

Problems

1. The p.m.f of a RV X, is given by Find MGF, mean and variance.

Solution

$$M_X(t) = E\left(e^{tX}\right) = \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{1}{2^x}$$

$$= \sum_{x=0}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \left(\frac{e^t}{2}\right)^4 + \ldots.$$

$$= \frac{e^t}{2}\left(1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \left(\frac{e^t}{2}\right)^4 + ..\right)$$

$$= \frac{e^t}{2} \frac{1}{1 - \frac{e^t}{2}} = \frac{e^t}{2 - e^t}$$

Differentiating twice with respect to t

$$M'_X(t) = \frac{\left(2-e^t\right)\left(e^t\right)-e^t\left(-e^t\right)}{\left(2-e^t\right)^2} = \frac{2e^t}{\left(2-e^t\right)^2}$$

$$M''_X(t) = \frac{\left(2-e^t\right)^2\left(2e^t\right)-2e^t\,2\left(2-e^t\right)\left(-e^t\right)}{\left(2-e^t\right)^4} = \frac{4e^t+2e^{2t}}{\left(2-e^t\right)^3}$$

put t = 0 above $E(X) = M'_X(0) = 2$

$$E\left(X^2\right) = M''_X(0) = 6$$

$$Variance = E\left(X^2\right) - E(X)^2 = 6-4 = 2$$

2. Find MGF of the RV X, whose pdf is given by and hence find the first four central moments.

Solution

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_{0}^{\infty} e^{-(\lambda - t)x}\, dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \frac{\lambda}{(\lambda - t)}$$

Expanding in powers of t

$$M_X(t) = \frac{\lambda}{(\lambda - t)} = \frac{1}{1-\left(\frac{t}{\lambda}\right)} = 1 + \left(\frac{t}{\lambda}\right) + \left(\frac{t}{\lambda}\right)^2 + \left(\frac{t}{\lambda}\right)^3 + \dots$$

Taking the coefficient we get the raw moments about origin

$$E(X) = (coefficient\ of\ t)1! = \frac{1}{\lambda}$$

$$E\left(X^2\right) = \left(coefficient\ of\ t^2\right)2! = \frac{2}{\lambda^2}$$

$$E\left(X^3\right) = \left(coefficient\ of\ t^3\right)3! = \frac{6}{\lambda^3}$$

$$E\left(X^4\right) = \left(coefficient\ of\ t^4\right)4! = \frac{24}{\lambda^4}$$

and the central moments are

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - 2C_1\mu_1'\mu_1' + \mu_1'^2$$

$$= \frac{2}{\lambda^2} - 2\frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\mu_3 = \mu_3' - 3C_1\mu_2'\mu_1' + 3C_2\mu_1'\mu_1'^2 - \mu_1'^3$$

$$= \frac{6}{\lambda^3} - 3\frac{2}{\lambda^2}\frac{1}{\lambda} + 3\frac{1}{\lambda}\frac{1}{\lambda^2} - \frac{1}{\lambda^3} = \frac{2}{\lambda^3}$$

$$\mu_4 = \mu_4' - 4C_1\mu_3'\mu_1' + 4C_2\mu_2'\mu_1'^2 - 4C_3\mu_1'^4 + \mu_1'^4$$

$$= \frac{24}{\lambda^4} - 4\frac{6}{\lambda^3}\frac{1}{\lambda} + 6\frac{2}{\lambda^2}\frac{1}{\lambda^2} - 4\frac{1}{\lambda^4} + \frac{1}{\lambda^4} = \frac{9}{\lambda^4}$$

3. If the MGF of a (discrete) RV X is $\frac{1}{5-4e^t}$ find the distribution of X and p (X = 5 or 6).

Solution

$$M_X(t) = \frac{1}{5-4e^t} = \frac{1}{5\left(1-\frac{4e^t}{5}\right)}$$

$$= \frac{1}{5}\left[1+\left(\frac{4e^t}{5}\right)+\left(\frac{4e^t}{5}\right)^2+\left(\frac{4e^t}{5}\right)^3+\ldots\right]$$

By definition

$$M_X(t) = E\left(e^{tX}\right) = \sum e^{tx}p(x)$$

$$=1 + e^{t0}p(0) + e^{t}p(1) + e^{t2}p(2) + \ldots$$

Standard Distributions

Binomial Distribution

Assumptions

1. The random experiment corresponds to only two possibly outcomes.
2. The number of trials is finite.
3. The trials are independent.
4. The probability of success is a constant from trial to trial.

Notations

n – number of trials

p – probability of success

q – probability of failure

X – A random variable which represents the number of successes

Binomial Distribution

A discrete random variable X is said to follow Binomial distribution if it's probability mass function is

$$P(x) = {}_nC_x p^x q^{n-x}, \quad x = 0,1,2,\ldots,n$$

Moment Generating Function

The M.G.F of a Binomial variate is

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x=0}^{n} e^{tx} P(x)$$

$$M_X(t) = \sum_{x=0}^{n} e^{tx} \, {}_nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}_nC_x (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n$$

Raw Moments

$$\mu_1' = \frac{d}{dt} M_X(t)\Big|_{t=0}$$

$$= \frac{d}{dt}(pe^t + q)^n\Big|_{t=0}$$

$$= n(pe^t + q)^{n-1} pe^t\Big|_{t=0}$$

$$= n(p+q)^{n-1} p = np \qquad [\because p+q=1]$$

Hence mean of Binomial distribution = np

$$\mu_2' = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0}$$

$$= \frac{d}{dt}[npe^t(pe^t + q)^{n-1}]_{t=0}$$

$$= np[e^t(n-1)(pe^t + q)^{n-2} pe^t + (pe^t + q)^{n-1} e^t]_{t=0}$$

$$= np[(n-1)(p+q)^{n-2} p + (p+q)^{n-1}]$$

$$= np[(n-1)p + 1] \qquad [\because p + q = 1]$$

$$= n(n-1)p^2 + np$$

Variance of Binomial Distribution

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

$$= npq$$

Additive Property of Binomial Distribution

Let X_1 follow binomial distribution with parameters n_1 and p_1. Let X_2 follow Binomial distribution with parameters n_2 and p_2. Further let X_1 and X_2 be independent.

$$M_{X_1}(t) = (q_1 + p_1 e^t)^{n_1}$$

$$M_{X_2}(t) = (q_2 + p_2 e^t)^{n_2}$$

Consider $M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) \quad [\because X_1 \text{ and } X_2 \text{ are independent}]$

$$= (q_1 + p_1 e^t)^{n_1} (q_2 + p_2 e^t)^{n_2}$$

Which is not of the form $(q+pe^t)^n$. Hence $X_1 + X_2$ is not a binomial variate.

But if $p_1 = p_2 = p$, then $q_1 = q_2 = q$. Equation (1) becomes

$$M_{X_1+X_2}(t) = (q + pe^t)^{n_1}(q + pe^t)^{n_2}$$

$$= (q + pe^t)^{n_1+n_2}$$

Which is of the form $(q + pe^t)^n$.

Hence $X_1 + X_2$ follows Binomial distribution when $p_1 = p_2 = p$.

i.e., Binomial distribution has additive property when $p_1 = p_2 = p$.

Problems Based On Binomial Distribution

1. It has been claimed that in 60 % of all solar heat installation the utility bill is reduced by atleast one-third. Accordingly what are the probabilities that the utility bill will be reduced by atleast one-third in atleast four of five installation.

Solution Given n = 5, p = 60% = 0.6 and

q = 1-p = 0.4

$$p(x \geq 4) = p[x = 4] + p[x = 5]$$

$$= 5c_4 (0.6)^4 (0.4)^{5-4} + 5c_5 (0.6)^5 (0.4)^{5-5}$$

$$= 0.337$$

2. The mean and variance of a binomial variate are 6 and 2 respectively. Find $P(X \geq 2)$.

Solution

Given that $E(X) = 6$ i.e., $np = 6$ (1)

and $V(X) = 2 \Rightarrow npq = 2$ (2)

Dividing Equation (2) by Equation (1) gives,

$$\frac{npq}{np} = \frac{2}{6} \Rightarrow q = \frac{1}{3}$$

$$p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

Use $p = \frac{2}{3}$ in Equation (1)

$$n\left(\frac{2}{3}\right) = 6 \Rightarrow n = 9$$

$$\therefore \quad P(x) = {}_nC_x p^x q^{n-x} = {}_9C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{9-x}$$

$$\begin{aligned} P(X \geq 1) &= 1 - P(X < 1) \\ &= 1 - P(X = 0) \\ &= 1 - {}_9C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{9-0} \\ &= 1 \end{aligned}$$

3. An unbiased die is rolled 10 times. Getting an outcome greater than 4 in a die is termed as a success. What is the chance of getting at least 8 successes?

Solution

X : number of successes

p : P (number of success) = P (outcome is greater than 4 in a die) $= \frac{{}_2C_1}{{}_6C_1} = \frac{1}{3}$

$$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

n = number of trials = 10

$$P(x) = {}_{n}C_{x} p^{x} q^{n-x} = {}_{10}C_{x} \left(\frac{1}{3}\right)^{x} \left(\frac{2}{3}\right)^{10-x}, \quad x = 0,1,2,\ldots 10$$

P (atleast 8 successes) = P (X ≥ 8)

$$= \sum_{x=8}^{10} P(x) = \sum_{x=8}^{10} {}_{10}C_{x} \left(\frac{1}{3}\right)^{x} \left(\frac{2}{3}\right)^{10-x}$$

$$= {}_{10}C_{8} \left(\frac{1}{3}\right)^{8} \left(\frac{2}{3}\right)^{2} + {}_{10}C_{9} \left(\frac{1}{3}\right)^{9} \left(\frac{2}{3}\right)^{1} + {}_{10}C_{10} \left(\frac{1}{3}\right)^{10} \left(\frac{2}{3}\right)^{0}$$

$$= 3.37 \times 10^{-3}$$

Poisson Distribution

The application of Binomial distribution will be invalid when $n \to \infty$ and $p \to 0$. Hence in the presence of the above two conditions we need a theoretical distribution which overcomes the drawback of Binomial distribution. The Binomial distribution tends to poisson distribution when

i. The number of trials is indefinitely large i.e., $n \to \infty$
ii. The probability of success is very small i.e., $p \to 0$
iii. np is a constant i.e., $np = \lambda$.

Poisson Distribution – pmf

A discrete random variable X is said to follow Poisson distribution with parameter λ if its probability mass function is $P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\ldots,\infty$

Moment Generating Function

$$
\begin{aligned}
M_X(t) = E[e^{tX}] &= \sum_{x=0}^{\infty} e^{tx} P(x) \\
&= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda\lambda^{t})^{x}}{x!} \\
&= e^{-\lambda} e^{-\lambda e^{t}} \qquad \left[\because \sum_{x=0}^{\infty} \frac{a^{x}}{x!} = e^{a} \right] \\
&= e^{\lambda(e^{t}-1)}
\end{aligned}
$$

Raw Moments of Poisson Distribution

$$
\begin{aligned}
\mu_1' &= \frac{d}{dt} M_X(t)\bigg|_{t=0} \\
&= \frac{d}{dt} e^{-\lambda} e^{\lambda e^t}\bigg|_{t=0} \\
&= e^{-\lambda} e^{\lambda e^t} \lambda e^t\Big|_{t=0} \\
&= e^{-\lambda} e^{\lambda} \lambda \\
&= \lambda
\end{aligned}
$$

i.e., mean of Poisson distribution = λ.

Second Order Raw Moment

$$\mu_2' = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0}$$

$$= \left.\frac{d}{dt} \cdot \frac{dM_X(t)}{dt}\right|_{t=0}$$

$$= \frac{d}{dt}[e^{-\lambda} e^{\lambda e^t} \lambda e^t]_{t=0}$$

$$= \lambda e^{-\lambda}[e^{\lambda e^t} e^t + e^t e^{\lambda e^t} \lambda e^t]_{t=0}$$

$$= \lambda e^{-\lambda}[e^{\lambda} + e^{\lambda}\lambda]$$

$$= \lambda e^{-\lambda} e^{\lambda} + \lambda^2 e^{-\lambda} e^{\lambda}$$

$$= \lambda^2 + \lambda$$

Variance of Poisson distribution

Variance = Second order central moment

$$= \mu_2 = \mu_2' - \mu_1'^2$$

$$= \lambda^2 + \lambda - \lambda^2 - \lambda$$

Thus in Poisson distribution mean = Variance = λ.

Additive Property of Poisson Distribution

Let X_1 and X_2 be two independent Poisson variates with parameter λ_1 and λ_2 respectively. Then

$$M_{X_1}(t) = e^{\lambda_1(e^t - 1)} \quad \text{and} \quad M_{X_2}(t) = e^{\lambda_2(e^t - 1)}$$

Consider the variate $X_1 + X_2$

Now

$$M_{X_1+X_2}(t) = M_{X_1}(t) \ldots M_{X_2}(t) \quad [\because X_1 \text{ and } X_2 \text{ are independent}]$$

$$= e^{\lambda_1(e^t - 1)} e^{\lambda_2(e^t - 1)}$$

$$= e^{(e^t - 1)(\lambda_1 + \lambda_2)}$$

Which is of the form $e^{\lambda(e^t - 1)}$. Hence $X_1 + X_2$ f ollows Poisson distribution with parameter $\lambda_1 + \lambda_2$. i.e., the Poisson distribution has additive property.

Prove that poisson distribution is the limiting case of Binomial distribution.

(or)

Poisson distribution is a limiting case of Binomial distribution under the following conditions

(i) n, the no. of trials is indefinitely large, i.e, n $\to \infty$

(ii) p, the constant probability of success in each trial is very small, i.e p $\to$ 0

(iii) $np = \lambda\, is \inf inite \;\; or \;\; p = \frac{\lambda}{n} and \;\; q = 1 - \frac{\lambda}{n}, \lambda is \; positive \; real$

Solution If X is binomial r.v with parameter n & p, then

$$p(X=x)=nc_x\,p^x q^{n-x}\,,\; x=0,1,2,\ldots.n$$

$$=\frac{n!}{(n-x)!\,x!}\,p^x(1-p)^{n-x}$$

$$=\frac{n(n-1)(n-2)\ldots.(n-(x-1))(n-x)!}{(n-x)!\,x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$=\frac{n(n-1)(n-2)\ldots.(n-(x-1))}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$=\frac{\lambda^x}{n^x\,x!}\,n.n\left(1-\frac{1}{n}\right)n\left(1-\frac{2}{n}\right)\ldots..n\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$=\frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots..\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

Taking limit as n $\to \infty$ on both sides

$$\lim_{n\to\infty} p(X=x) = \lim_{n\to\infty} \frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right) \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}(1.1....1).\left(e^{-\lambda}\right).1 = \frac{e^{-\lambda}\lambda^x}{x!} \quad , \; x = 0,1,2,.....$$

$$\therefore \; p(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad , \; x = 0,1,2,.....$$ and it is poisson distbn.

Hence the proof.

Prove that the sum of two independent poisson variates is a poisson variate, while the difference is not a poisson variate.

Solution Let X_1 and X_2 be independent r.v.s that follow poisson distbn. with Parameters λ_1 and λ_2 respectively.

Let $X = X_1 + X_2$

$$p(X=n)=p(X_1+X_2=n)$$

$$=\sum_{r=0}^{n} p[X_1=r].p[X_2=n-r] \quad \text{since } X_1 \,\&\, X_2 \text{ are independent}$$

$$=\sum_{r=0}^{n} \frac{e^{-\lambda_1}.\lambda_1{}^{r}}{r!}.\frac{e^{-\lambda_2}.\lambda_2{}^{n-r}}{(n-r)!}$$

$$=e^{-\lambda_1}\,e^{-\lambda_2}\sum_{r=0}^{n}\frac{.\lambda_1{}^{r}}{r!}.\frac{1}{n!}\frac{n!.}{(n-r)!}\lambda_2{}^{n-r}$$

$$=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{r=0}^{n}.\frac{n!.}{r!\,(n-r)!}\lambda_1{}^{r}\lambda_2{}^{n-r}$$

$$=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{r=0}^{n}.n c_r \lambda_1{}^{r}\lambda_2{}^{n-r} \quad =\frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^{n}$$

This is poisson with parameter $(\lambda_1 + \lambda_2)$

(ii) Difference is not poisson

Let X = $X_1 - X_2$

$$E(X) = E[X_1 - X_2]$$
$$= E(X_1) - E(X_2)$$
$$= \lambda_1 - \lambda_2$$
$$E(X^2) = E\left[(X_1 - X_2)^2\right]$$
$$= E\left[X_1^{\ 2} + X_1^{\ 2} - 2X_1X_2\right]$$
$$= E\left[X_1^{\ 2}\right] + E\left[X_2^{\ 2}\right] - 2E[X_1]E[X_2]$$
$$= (\lambda_1^{\ 2} + \lambda_1) + (\lambda_2^{\ 2} + \lambda_2) - 2(\lambda_1\lambda_2)$$
$$= (\lambda_1 - \lambda_2)^2 + (\lambda_1 + \lambda_2)$$
$$\neq (\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_2)$$

It is not poisson.

Prove that poisson distribution is the limiting case of Binomial distribution.

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(iii) $np = \lambda \text{ is infinite or } p = \frac{\lambda}{n} \text{ and } q = 1 - \frac{\lambda}{n}, \lambda \text{ is positive real}$

Solution If X is binomial r.v with parameter n & p, then

$$p(X=x)=nc_x\, p^x q^{n-x}\ ,\ x=0,1,2,\ldots.n$$

$$=\frac{n!}{(n-x)!\,x!}\,p^x(1-p)^{n-x}$$

$$=\frac{n(n-1)(n-2)\ldots.(n-(x-1))(n-x)!}{(n-x)!\,x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$=\frac{n(n-1)(n-2)\ldots.(n-(x-1))}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$=\frac{\lambda^x}{n^x\,x!}\,n.n\left(1-\frac{1}{n}\right)n\left(1-\frac{2}{n}\right)\ldots..n\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$=\frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots..\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

Taking limit as n $\rightarrow \infty$ on both sides

$$\lim_{n\to\infty} p(X=x) = \lim_{n\to\infty} \frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right)\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}(1.1....1).\left(e^{-\lambda}\right).1 = \frac{e^{-\lambda}\lambda^x}{x!} \qquad , \; x = 0,1,2,.....$$

$$\therefore\; p(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad , \; x = 0,1,2,.....$$ and it is poisson distbn.

Hence the proof.

Problems Based on Poisson Distribution

1. The no. of monthly breakdowns of a computer is a r.v. having poisson distbn with mean 1.8. Find the probability that this computer will function for a month with only one breakdown.

Solution

$$p(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad , \quad given\ \lambda = 1.8$$

$$p(x = 1) = \frac{e^{-1.8}(1.8)^{1}}{1!} = 0.2975$$

2. It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the no. of packets containing atleast, exactly, atmost 2 defectives in a consignment of 1000 packets using poisson.

Solution

Give $n = 20$, $p = 0.05$, $N = 1000$

Mean $\lambda = np = 1$

Let X denote the no. of defectives.

$$p[X = x] = \frac{e^{-\lambda}.\lambda^{x}}{x!} = \frac{e^{-1}.1^{x}}{x!} = \frac{e^{-1}.}{x!} \quad x = 0,1,2,\ldots$$

$$p[x \geq 2] = 1 - p[x < 2]$$

$$= 1 - [p(x = 0) + p(x = 1)]$$

$$= 1 - \left[\frac{e^{-1}}{0!} + \frac{e^{-1}}{1!}\right] = 1 - 2e^{-1} = 0.2642$$

Therefore, out of 1000 packets , the no. of packets containing atleast 2 defectives

$$= N.p[x \geq 2] = 1000 * 0.2642 \cong 264 \ packets$$

(ii) $p[x = 2] = \frac{e^{-1}}{2!} = 0.18395$

Out of 1000 packets,=N*p[x=2]=184 packets

(iii) $$p[x \leq 2] = p[x = 0] + p[x = 1] + p[x = 2]$$

$$= \frac{e^{-1}}{0!} + \frac{e^{-1}}{1!} + \frac{e^{-1}}{2!} = 0.91975$$

For 1000 packets = 1000*0.91975=920 packets approximately.

3. The atoms of radio active element are randomly disintegrating. If every gram of this element , on average, emits 3.9 alpha particles per second, what is the probability during the next second the no. of alpha particles emitted from 1 gram is

(i) atmost 6 (ii) atleast 2 (iii) atleast 3 and atmost 6?

Solution

Given $\lambda = 3.9$

Let X denote the no. of alpha particles emitted

$$(i)\ p(x \le 6) = p(x=0) + p(x=1) + p(x=2) + \ldots\ldots + p(x=6)$$

$$= \frac{e^{-3.9}(3.9)^0}{0!} + \frac{e^{-3.9}(3.9)^1}{1!} + \frac{e^{-3.9}(3.9)^2}{2!} + \ldots\ldots + \frac{e^{-3.9}(3.9)^6}{6!}$$

$$= 0.898$$

$$(ii) \quad p(x \geq 2) = 1 - p(x < 2)$$

$$= 1 - \left[p(x = 0) + p(x = 1) \right]$$

$$= 1 - \left[\frac{e^{-3.9}(3.9)^0}{0!} + \frac{e^{-3.9}(3.9)^1}{1!} \right]$$

$$= 0.901$$

$$(iii) \quad p(3 \leq x \leq 6) = p(x = 3) + p(x = 4) + p(x = 5) + p(x = 6)$$

$$= \frac{e^{-3.9}(3.9)^3}{3!} + \frac{e^{-3.9}(3.9)^4}{4!} + \frac{e^{-3.9}(3.9)^5}{5!} + \frac{e^{-3.9}(3.9)^6}{6!}$$

$$= 0.645$$

Geometric Distribution

A discrete random variable X which represents the number of failures preceding the first, success is said to follow Geometric distribution if its probability mass function is

$$P(x) = q^x p, \quad x = 0, 1, \ldots, \infty$$

Moment Generating Function

$$M_x(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} q^x p$$

$$= p \sum_{x=0}^{\infty} (qe^t)^x$$

$$= p[1 + qe^t + (qe^t)^2 + \cdots]$$

$$= p[1 - qe^t]^{-1}$$

Raw Moments of Geometric Distribution

$$\mu_1' = \frac{d}{dt} M_X(t)\bigg|_{t=0}$$

$$= \frac{d}{dt} p(1-qe^t)^{-1}\bigg|_{t=0}$$

$$= p(-1)(1-qe^t)^{-2}(0-qe^t)\bigg|_{t=0}$$

$$= pqe^t(1-qe^t)^{-2}\bigg|_{t=0}$$

$$= pq(1-q)^{-2} = \frac{pq}{p^2} = \frac{q}{p}$$

$$\mu_2' = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0}$$

$$= \left.\frac{d}{dt}\, pqe^t (1-qe^t)^{-2}\right|_{t=0}$$

$$\Rightarrow \mu_2' = pq[e^t(-2)(1-qe^t)^{-3}(0-qe^t) + (1-qe^t)^{-2}e^t]_{t=0}$$

$$= pq[2q(1-q)^{-3} + (1-q)^{-2}]$$

$$= pq\left[\frac{2q}{p^3} + \frac{1}{p^2}\right]$$

$$= \frac{2q^2}{p^2} + \frac{q}{p}$$

Variance of Geometric Distribution

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2}$$

$$= \frac{q^2}{p^2} + \frac{q}{p} = \frac{q}{p}\left(\frac{q}{p} + 1\right)$$

$$= \frac{q}{p}\left(\frac{q+p}{p}\right)$$

$$= \frac{q}{p^2} \quad [\because q + p = 1]$$

Expectation of Geometric Distribution (without using MGF)

$$E(X) = \sum_{x=0}^{\infty} xP(x) = \sum_{x=0}^{\infty} xq^{x}p$$

$$= p[0 + q + 2q^{2} + \ldots]$$

$$= pq[1 + 2q + 3q^{2} + \ldots]$$

$$= pq[1 - q]^{-2}$$

$$= \frac{pq}{p^{2}} = \frac{q}{p}$$

Variance of Geometric Distribution

$$V(X) = E(X^2) - [E(X)]^2$$

$$E[X^2] = \sum_{x=0}^{\infty} x^2 P(x)$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x]P(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)P(x) + \sum_{x=0}^{\infty} xP(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)q^x p + \tfrac{q}{p}$$

$$= p[0 + 0 + 2q^2 + 6q^3 + 8q^4 + \ldots] = \tfrac{q}{p}$$

$$= 2pq^2[1 + 3q + 4q^2 + \ldots] + \tfrac{q}{p}$$

$$= 2pq^2[1 - q]^{-3} + \tfrac{q}{p}$$

$$= \frac{2pq^2}{p^3} + \frac{q}{p} = 2\frac{q^2}{p^3} + \frac{q}{p}$$

$$
\begin{aligned}
V(X) &= E(X^2) - [E(X)]^2 \\
&= 2\frac{q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} \\
&= \frac{q^2}{p^2} + \frac{q}{p} = \frac{q}{p}\left(1 + \frac{q}{p}\right) \\
&= \frac{q}{p}\left(1 + \frac{q}{p}\right) = \frac{q}{p^2} \qquad [\because p + q = 1]
\end{aligned}
$$

Establish the memoryless property of geometric distbn.

Solution If X is a discrete r.v. following a geometric distbn.

$$\therefore\ p(X = x) = pq^{x-1} \qquad , x = 1,2,.....$$

$$p(x > k) = \sum_{x=k+1}^{\infty} pq^{x-1}$$

$$= p\left[q^{k} + q^{k+1} + q^{k+2} + ...\right]$$

$$= p\,q^{k}\left[1 + q + q^{2} + ...\right] = p\,q^{k}(1-q)^{-1}$$

$$= p\,q^{k}\,p^{-1} = q^{k}$$

Now

$$p\left[x > m+n / x > m\right] = \frac{p\left[x > m+n \ and \ \ x > m\right]}{p\left[\,x > m\right]}$$

$$= \frac{p\left[x > m+n\ \right]}{p\left[\,x > m\right]} = \frac{q^{m+n}}{q^{m}} = q^{n} = p\left[x > n\right]$$

$$\therefore\ p\left[x > m+n / x > m\right] = p\left[x > n\right]$$

Problem Based on Geometric Distribution

1. Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.7.

(i) What is the probability that the target would be hit in 10 th attempt?

(ii) What is the probability that it takes him less than 4 shots?

(iii) What is the probability that it takes him an even no. of shots?

(iv) What is the average no. of shots needed to hit the target?

Solution Let X denote the no. of shots needed to hit the target and X follows geometric distribution with pmf $p[X = x] = p\,q^{x-1}\ ,\ x = 1,2,....$

(i) $$p[x=10]=(0.7)(0.3)^{10-1} = 0.0000138$$

(ii) $$p[x<4]= p(x=1)+ p(x=2)+ p(x=3)$$
$$= (0.7)(0.3)^{1-1} + (0.7)(0.3)^{2-1} + (0.7)(0.3)^{3-1}$$
$$=0.973$$

(iii) $$p[x\, is\, an\, even\, number]= p(x=2)+ p(x=4)+ \ldots\ldots\ldots$$
$$= (0.7)(0.3)^{2-1} + (0.7)(0.3)^{4-1} + \ldots\ldots$$
$$= (0.7)(0.3)\left[1+(0.3)^2 + (0.3)^4 \ldots\ldots\ldots\right]$$
$$= 0.21\left[1+\left((0.3)^2\right)+\left((0.3)^2\right)^2 + \ldots\ldots\ldots\right]$$
$$= 0.21\left[1-(0.3)^2\right]^{-1} = (0.21)(0.91)^{-1}$$
$$=\frac{0.21}{0.91} = 0.231$$

(iv) $$\text{Average no. of shots} = \mathrm{E(X)} = \frac{1}{\mathrm{p}} = \frac{1}{0.7} = 1.4286$$

Uniform Distribution

Continuous Uniform Distribution or Rectangular Distribution

A continuous random variable X defined in the interval (a,b) is said to follow uniform distribution if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

Note: and b are the parameters of the distribution.

Moment Generating Function of a Uniform Distribution

$$M_X(t) = E(e^{tX})$$

$$= \int_a^b e^{tx} f(x) dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

MGF

$$= \frac{1}{t(b-a)} (e^{bt} - e^{at})$$

Raw Moments of Rectangular Distribution

$$\mu_r' = \int_a^b x^r f(x)dx = \int_a^b x^r \frac{1}{b-a} dx$$

$$= \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b$$

$$\Rightarrow \mu_r' = \frac{1}{b-a}\left[\frac{b^{r+1} - a^{r+1}}{r+1}\right] \qquad (1)$$

When r = 1, Equation (1) becomes

$$\mu_1' = \frac{1}{b-a}\left[\frac{b^2 - a^2}{2}\right]$$

Mean $$= \frac{(b-a)(b+a)}{(b-a)2} = \frac{b+a}{2}$$

When r = 2, Equation (1) becomes

$$\mu_2' = \frac{1}{b-a}\frac{b^3-a^3}{3}$$
$$= \frac{1}{b-a}\frac{(b-a)(b^2+ab+a^2)}{3}$$
$$= \frac{b^2+ab+a^2}{3}$$

Variance of Uniform Distribution

$$V(X) = E(X^2) - [E(X)]^2$$
$$= \mu_2' - \mu_1'^2$$
$$= \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$
$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$
$$= \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$$

Problems Based on Uniform Distribution

1.Show that for the uniform distribution

$f(x)=\frac{1}{2a}, -a<x<a$, the mgf about origin is $\frac{\sinh at}{at}$.

Solution: Given $f(x)=\frac{1}{2a}, -a<x<a$

$$MGF \quad M_x(t)=E\left[e^{tx}\right]$$

$$=\int_{-\infty}^{\infty} e^{tx} f(x)dx=\int_{-a}^{a} e^{tx}\frac{1}{2a}dx$$

$$=\frac{1}{2a}\int_{-a}^{a} e^{tx}dx=\frac{1}{2a}\left[\frac{e^{tx}}{t}\right]_{-a}^{a}$$

$$=\frac{1}{2at}\left[e^{at}-e^{-at}\right]=\frac{1}{2at}2\sinh at=\frac{\sinh at}{at}$$

$$M_x(t)=\frac{\sinh at}{at}$$

2. The number of personel computer (pc) sold daily at a computer world is uniformly distributed with a minimum of 2000 pc and a maximum of 5000 pc. Find

(1) The probability that daily sales will fall between 2500 and 3000 pc

(2)What is the probability that the computer world will sell atleast 4000 pc’s?

(3) What is the probability that the computer world will sell exactly 2500 pc’s?

Solution Let $X \sim U(a , b)$, then the pdf is given by

$$f(x) = \frac{1}{b-a} \quad , \; a < x < b$$

$$= \frac{1}{5000-2000} \; , \quad 2000 < x < 5000$$

$$= \frac{1}{3000} \quad , \quad 2000 < x < 5000$$

(1)

$$p[2500 < x < 3000] = \int_{2500}^{3000} f(x)\,dx$$

$$= \int_{2500}^{3000} \frac{1}{3000}\,dx = \frac{1}{3000}[x]_{2500}^{3000}$$

$$= \frac{1}{3000}[3000-2500] = 0.166$$

(2)

$$p[x \geq 4000] = \int_{4000}^{5000} f(x)\,dx$$

$$= \int_{4000}^{5000} \frac{1}{3000}\,dx = \frac{1}{3000}[x]_{4000}^{5000}$$

$$= \frac{1}{3000}[5000-4000] = 0.333$$

(3) $p[x = 2500] = 0$ (i.e) it is particular point, the value is zero.

3. Starting at 5.00 am every half an hour there is a flight from San Fransisco airport to Losangles. Suppose that none of three planes is completely sold out and that they always have room for passengers. A person who wants to fly to Losangles arrive at a random time between 8.45 am and 9.45 am. Find the probability that she waits

(a) Atmost 10 min (b) atleast 15 min

Solution Let X be the uniform r.v. over the interval (0, 60). Then the pdf is given by

$$f(x) = \frac{1}{b-a} \quad , a < x < b$$

$$= \frac{1}{60} \quad , \quad 0 < x < 60$$

(a) The passengers will have to wait less than 10 min. if she arrives at the airport

$$= p(5 < x < 15) + p(35 < x < 45)$$

$$= \int_{5}^{15} \frac{1}{60}\,dx + \int_{35}^{45} \frac{1}{60}\,dx$$

$$= \frac{1}{60}\left[x\right]_{5}^{15} + \frac{1}{60}\left[x\right]_{35}^{45}$$

$$= \frac{1}{3}$$

(b) The probability that she has to wait atleast 15 min.

$$= p(15 < x < 30) + p(45 < x < 60)$$

$$= \int_{15}^{30} \frac{1}{60}\,dx + \int_{45}^{60} \frac{1}{60}\,dx$$

$$= \frac{1}{60}\left[x\right]_{15}^{30} + \frac{1}{60}\left[x\right]_{45}^{60}$$

$$= \frac{1}{2}$$

Exponential Distribution

A continuous random variable X is said to follow exponential distribution if its pdf is given by

$$f(x) = \lambda e^{-\lambda x}, x \geq 0$$

Mean and variance of an Exponential distribution:

$$Mean = E[x] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} x\lambda e^{-\lambda x}\, dx = \lambda \int_{0}^{\infty} x e^{-\lambda x}\, dx$$

$$= \lambda \left[\frac{-xe^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty}$$

$$= \lambda \left[(0-0) - \left(0 - \frac{1}{\lambda^2} \right) \right] = \lambda \left(\frac{1}{\lambda^2} \right) = \frac{1}{\lambda}$$

$$Mean = \frac{1}{\lambda}$$

$$E\left[x^2\right] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^{\infty} x^2 \lambda\, e^{-\lambda x}\, dx = \lambda \int_0^{\infty} x^2\, e^{-\lambda x}\, dx$$

$$= \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} - \frac{2xe^{-\lambda x}}{\lambda^2} - \frac{2e^{-\lambda x}}{\lambda^3} \right]_0^{\infty}$$

$$= \lambda \left[(0-0-0) - \left(0 - 0 - \frac{2}{\lambda^3} \right) \right] = \lambda \left(\frac{2}{\lambda^3} \right) = \frac{2}{\lambda^2}$$

$$Variance = E\left(x^2\right) - \left[E(x)\right]^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Moment Generating Function of an Exponential Distribution

$$M_X(t) = E[e^{tX}]$$

$$= \int_0^\infty e^{tx} f(x)dx$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-x(\lambda - t)} dx$$

$$= \lambda \left[\frac{e^{-x(\lambda - t)}}{-(\lambda - t)} \right]_0^\infty$$

$$= \frac{-\lambda}{\lambda - t}[0 - 1] = \frac{\lambda}{\lambda - t}$$

Establish the memory less property of an exponential distribution.

Solution If X is exponentially distributed, then

$$p[x > s + t / x > s] = p[x > t] \;\; for\ any\ s, t > 0$$

The pdf of exponential distn is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \; x \geq 0 \\ 0 & , \; otherwise \end{cases}$$

$$p(x > k) = \int_k^\infty f(x)\, dx$$

$$= \int_k^\infty \lambda e^{-\lambda x}\, dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_k^\infty$$

$$= -\left[0 - e^{-\lambda k}\right] = e^{-\lambda x} \; ---------(1)$$

$$p[x > s + t / x > s] = \frac{p[x > s + t \;\; and \;\; x > s]}{p[x > s]}$$

$$= \frac{p[x > s + t \;]}{p[x > s]} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = p[x > t]$$

$$\therefore \quad p[x > s + t / x > s] = p[x > t] \;\; for\ any\ s, t > 0$$

Problems based on Exponential Distribution

The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$.

(a)What is the probability that the repair time exceeds 2 hrs ?

(b)What is the conditional probability that a repair takes atleast 11 hrs given that Its direction exceeds 8 hrs ?

Solution If X represents the time to repair the machine, the density function Of X is given by

$$f(x) = \lambda e^{-\lambda x} \qquad , x \geq 0$$

$$= \frac{1}{2} e^{-x/2} \qquad x \geq 0$$

$$p(x>2)=\int_2^{\infty} f(x)\,dx=\int_2^{\infty}\lambda e^{-\lambda x}\,dx$$

$$=\int_2^{\infty}\frac{1}{2}e^{\frac{-x}{2}}\,dx=\frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_2^{\infty}$$

$$=-\left[0-e^{-1}\right]=0.3679$$

$$p\left[x\geq 11/x>8\right]=p\left[x>3\right]$$

$$=\int_3^{\infty} f(x)\,dx=\int_3^{\infty}\lambda e^{-\lambda x}\,dx$$

$$=\int_3^{\infty}\frac{1}{2}e^{\frac{-x}{2}}\,dx=\frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_3^{\infty}$$

$$=-\left[0-e^{-\frac{3}{2}}\right]=e^{-\frac{3}{2}}=0.2231$$

Gamma Distribution

A continuous random variable X is said to follow Gamma distribution with parameter λ if its probability density function is

$$f(x) = \begin{cases} \dfrac{e^{-x}x^{\lambda-1}}{\Gamma\lambda} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Moment generating function of gamma distribution

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} f(x)dx$$

$$= \int_0^\infty e^{tx} \frac{e^{-x}x^{\lambda-1}}{\sqrt{\lambda}} dx$$

$$= \frac{1}{\sqrt{\lambda}} \int_0^\infty e^{-x(1-t)} x^{\lambda-1} dx$$

$$= \frac{1}{\sqrt{\lambda}} \frac{\sqrt{\lambda}}{(1-t)^{\lambda}}$$

$$= (1-t)^{-\lambda}$$

Raw moments of gamma distribution

$$\begin{aligned}\mu_1' &= \frac{d}{dt} M_X(t)\Big|_{t=0} \\ &= \frac{d}{dt}(1-t)^{-\lambda}\Big|_{t=0} \\ &= -\lambda(1-t)^{-\lambda-1}(-1)\Big|_{t=0} \\ &= \lambda(1-t)^{-\lambda-1}\Big|_{t=0} \\ &= \lambda\end{aligned}$$

$$\begin{aligned}\mu_2' &= \frac{d^2 M_X(t)}{dt^2}\Big|_{t=0} \\ &= \frac{d}{dt}\lambda(1-t)^{-\lambda-1}\Big|_{t=0} \\ &= \lambda(-\lambda-1)(1-t)^{-\lambda-2}(-1)\Big|_{t=0} \\ &= \lambda(\lambda+1) = \lambda^2 + \lambda\end{aligned}$$

Variance of gamma distribution

$$\begin{aligned}\mu_2 &= \mu_2' - \mu_1'^2 \\ &= \lambda^2 + \lambda - \lambda^2 = \lambda\end{aligned}$$

Hence in gamma distribution mean = variance = λ

Variance of gamma distribution

$$V(X) = E(X^2) - [E(X)]^2$$
$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

Additive property of Gamma Distribution

Consider two independent gamma variates X_1 and X_2 with parameters λ_1 and λ_2 respectively.

$$M_{X_1}(t) = (1-t)^{-\lambda_1}$$
$$M_{X_2}(t) = (1-t)^{-\lambda_2}$$
$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) \quad [\because X_1 \text{ and } X_2 \text{ are independent}]$$
$$= (1-t)^{-\lambda_1}(1-t)^{-\lambda_2}$$
$$= (1-t)^{-(\lambda_1+\lambda_2)}$$

Which is of the form $(1-t)-\lambda$. Hence $X_1 + X_2$ is also a Gamma variate with parameter $\lambda_1+\lambda_2$. Hence Gamma distribution has additive property.

Problems Based on Gamma Distribution

1. In a certain city the daily consumption of electric power in millions of kilowatt hrs can be treated as central gamma distn with $\lambda = \frac{1}{2}, k = 3$. If the power plant has a daily capacity of 12 million kilowatt hours. What is the probability that the power supply will be inadequate on any given day.

Solution Let X be the daily consumption of electric power Then the density function of X is given by

$$f(x)=\frac{\lambda^k x^{k-1}e^{-\lambda x}}{\Gamma_k}$$

$$=\frac{\left(\frac{1}{2}\right)^3 x^{3-1}e^{-\frac{x}{2}}}{\Gamma_3}=\frac{\left(\frac{1}{8}\right)x^2e^{-\frac{x}{2}}}{2!}$$

$$=\frac{x^2e^{-\frac{x}{2}}}{16}$$

p[the power supply is inadequate]=p[x>12]

$$=\int_{12}^{\infty} f(x)\,dx=\int_{12}^{\infty}\frac{x^2e^{-\frac{x}{2}}}{16}\,dx$$

$$=\frac{1}{16}\int_{3}^{\infty} x^2e^{-\frac{x}{2}}\,dx$$

$$=\frac{1}{16}\left[-2x^2e^{-\frac{x}{2}}-8xe^{-\frac{x}{2}}-16e^{-\frac{x}{2}}\right]_{12}^{\infty}$$

$$=0.0625$$

2. The daily consumption of milk in a city in excess of 20,000 liters is approximately distributed as an Gamma distn with parameter $\lambda = \frac{1}{10000}, k = 2$. The city has a daily stock of 30,000 liters. What is the probability that the stock is insufficient on a particular day.

Solution Let X be the daily consumption, so, the r.v. Y=X-20000. Then

$$f_Y(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{\Gamma_k}$$

$$= \frac{\left(\frac{1}{10000}\right)^2 y^{2-1} e^{-\frac{y}{10000}}}{\Gamma_2} = \frac{y e^{-\frac{y}{10000}}}{(10000)^2 \, 1!}$$

$$= \frac{y e^{-\frac{y}{10000}}}{(10000)^2}$$

p[insufficient stock]=p[X>30000]

=p[Y>10000]

$$p[Y > 10000] = \int_{10000}^{\infty} f(y)\,dy = \int_{10000}^{\infty} \frac{ye^{-\frac{y}{10000}}}{(10000)^2}\,dy$$

$$= \frac{1}{(10000)^2} \int_{3}^{\infty} ye^{-\frac{y}{10000}}\,dy$$

$$= 2e^{-1} \qquad , \left[By\ substitution\ method\ ,\ put\ t = \frac{y}{10000} \right]$$

$$= 0.7357$$

Normal Distribution

The English mathematician De-Moivre, obtained a continuous distribution as a limiting case of binomial distribution in the year 1733. This distribution was named normal distribution. The first person who made reference to this distribution was Gauss who used it to study the distribution of errors in Astronomy.

Probability Density Function

A continuous random variable X is said to follow normal distribution with parameters μ (mean) and σ^2 (variance), it its density function is given by the probability law:

$$f(x) = \frac{1}{\sigma\sqrt{2\lambda}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, -\infty < \mu < \infty \text{ and } \sigma > 0$$

Remarks

A random variable X which follows Normal distribution with mean μ and variance σ^2 is represented as $X \sim N(\mu, \sigma^2)$.

If X is a normal variate, $z = \frac{x - \mu}{\sigma}$ is called as a standard normal variate. If $X \sim N(\mu, \sigma^2)$, then $z \sim N(0, 1)$.

The pdf of a standard normal variate Z is given by,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-Z^2/2}, \quad -\infty < Z < \infty$$

Moment Generating Function of Normal Distribution

If X follows normal distribution with mean μ and variance σ^2, its moment generating function is derived as follows:

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\text{Let} \quad \frac{x-\mu}{\sigma} = z \Rightarrow dx = \sigma dz$$

$$x = -\infty \Rightarrow z = -\infty$$

$$x = \infty \Rightarrow z = \infty$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t + t\sigma z} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z - \frac{1}{2}z^2} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz$$

$$\Rightarrow M_X(t) = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z + t^2\sigma^2 - t^2\sigma^2)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z + t^2\sigma^2)} e^{\frac{1}{2}t^2\sigma^2} dz$$

$$= \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - t\sigma\sigma^2} dz$$

$$z - t\sigma = A$$

$$\Rightarrow dz = dA$$

$$z = -\infty \Rightarrow A = -\infty$$

$$z = \infty \Rightarrow A = \infty$$

$$M_X(t) = \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}A^2} dA$$

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}A^2}}{\sqrt{2\pi}} dA = 1 \text{ because } \frac{e^{-\frac{1}{2}A^2}}{\sqrt{2\pi}} \text{ is the pdf of a standard normal variate.}$$

$$M_X(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$$

Generating the raw moments using MGF

$$M_X(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$$

$$= e^{\mu t} e^{\frac{1}{2}t^2\sigma^2}$$

$$\mu_1' = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left. e^{\mu t} e^{\frac{t^2\sigma^2}{2}} \frac{\sigma^2}{2}(2t) + e^{\frac{t^2\sigma^2}{2}} e^{\mu t}\mu \right|_{t=0} = \mu$$

$$\mu_2' = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0}$$

$$= \left[\sigma^2 \left\{ e^{\mu t} e^{\frac{t^2\sigma^2}{2}}(1) + e^{\mu t} e^{\frac{t^2\sigma^2}{2}} \frac{\sigma^2}{2}(2t) + e^{\frac{t^2\sigma^2}{2}} t e^{\mu t}\mu + \right\} \right.$$

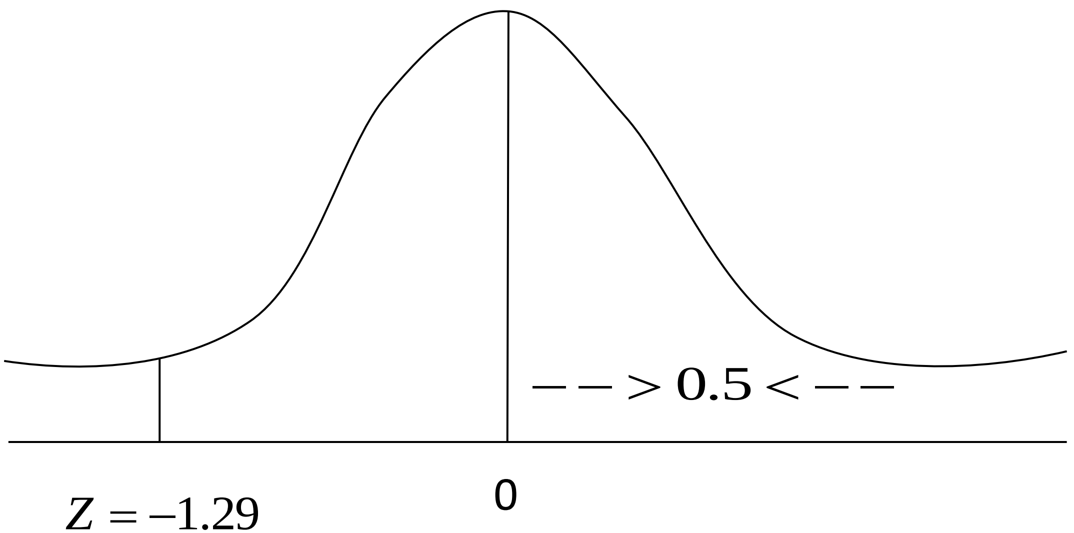
$$\left. \mu \left\{ e^{\frac{t^2\sigma^2}{2}} e^{\mu t}\mu + e^{\mu t} e^{\frac{t^2\sigma^2}{2}} \frac{\sigma^2}{2}(2t) \right\} \right]_{t=0}$$

$$= \sigma^2 + \mu^2$$

Problems based on Normal Distribution

1. Find P (z > -1.29)

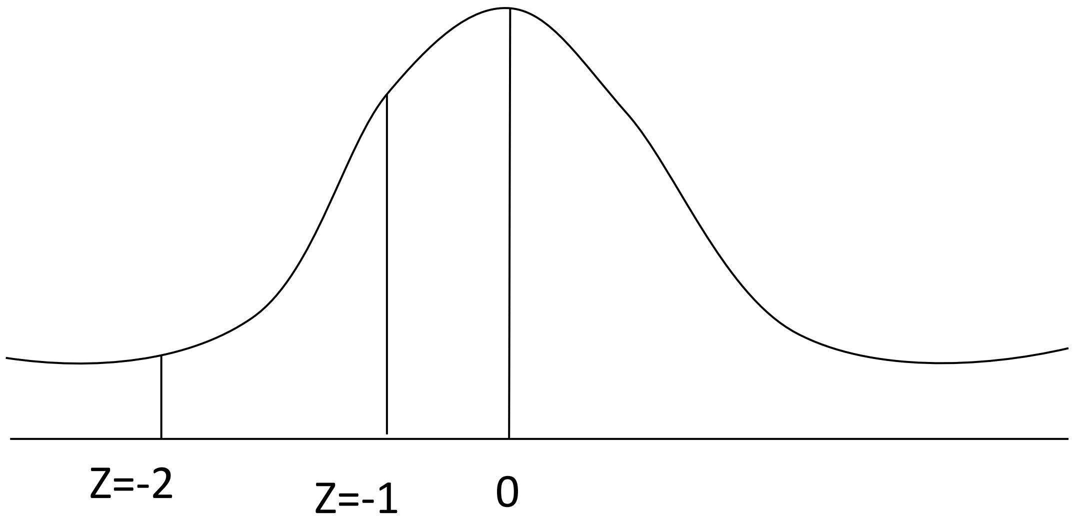
Solution



$$P(z > -1.29) = P(-1.29 < z < 0) + 0.5$$
$$= P(0 < z < 1.29) + 0.5$$
$$= 0.4015 + 0.5$$
$$= 0.9015$$

2. Find P $(-1 \leq z \leq -2)$

Solution



$$P(-1 \leq z \leq -2) = P(0 \leq z \leq -2) - P(0 \leq z \leq -1)$$

$$= 0.4772 - 0.3413$$

$$= 0.1359$$

3. 20% of the observations in the normal distribution are below 60. 80% of the observations are below 120. Compute the mean and standard deviation of the distribution.

Solution Let X be the normal variate

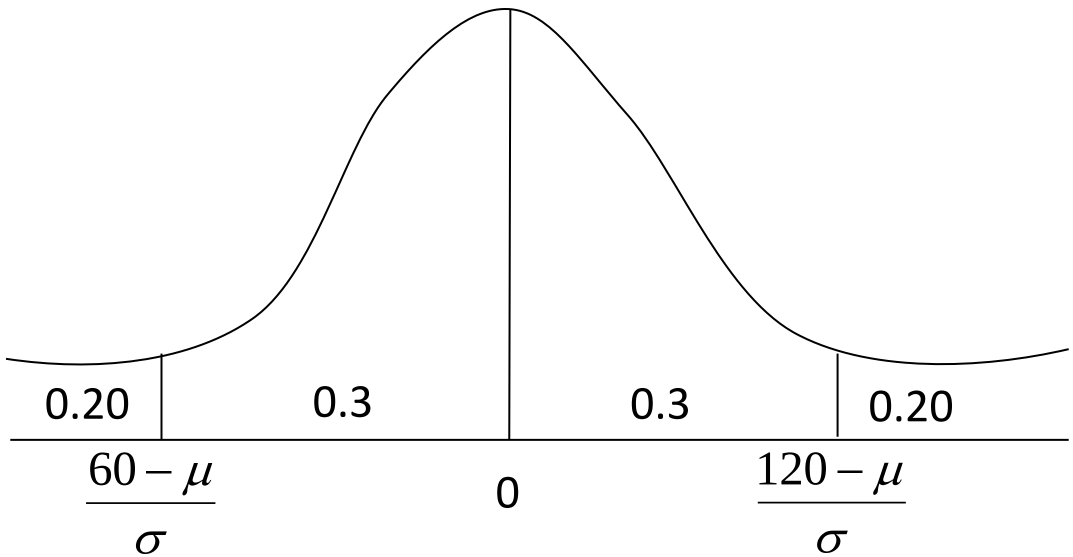
Given that P(X ≤ 60) = 0.20

$$\Rightarrow \quad P\left(z \leq \frac{60-\mu}{\sigma}\right) = 0.20 \qquad (1)$$

Also given that P(X ≤ 120) = 0.80

$$\Rightarrow \quad P\left(z \leq \frac{120-\mu}{\sigma}\right) = 0.80 \qquad (2)$$

Representing Equations (1) and (2) in the diagram



From the diagram $P\left(0 < z < \frac{120-\mu}{\sigma}\right) = 0.3$

$$\Rightarrow \quad \frac{120-\mu}{\sigma} = 0.85 \quad \text{[From the tables]}$$

$$\Rightarrow \quad 120-\mu = 0.85\sigma \tag{3}$$

Again from the diagram, $P\left(\frac{60-\mu}{\sigma} < z < 0\right) = 0.3$

$$\Rightarrow \quad P\left(0 < z < \frac{\mu-60}{\sigma}\right) = 0.3 \quad \text{[By symmetry]}$$

$$\Rightarrow \quad \frac{\mu-60}{\sigma} = 0.85 \quad \text{[From tables]}$$

$$\Rightarrow \quad \mu - 60 = 0.85\sigma \qquad (4)$$

Adding Equations (3) and (4)

$$\Rightarrow \quad 60 = 1.7\sigma \Rightarrow \sigma = 35.29$$

Putting $\sigma = 35.29$ in $120 - \mu = 0.85\sigma$

$$\Rightarrow \quad \mu = 90 \quad \text{and}$$

Hence $\mu = 90$ and $\sigma = 35.29$

Summarizing the Unit

Random Variables: 1. Discrete R.V

2. Continous R.V

Standard Distribution

Under Discrete - 1. Binomial Distn

2. Poisson Distn

3. Geometric Distn

Under Continuous- 1. Uniform Distn

2. Exponential Distn

3. Gamma Distn

4. Normal Distn

3) FORMULAE

Sl. No.	Discrete random variable	Continuous random variable
1.	$\sum_{i=-\infty}^{\infty} p(x_i) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$
2.	$F(x) = P[X \le x]$	$F(x) = P[X \le x] = \int_{-\infty}^{x} f(x)dx$
3.	$\text{Mean} = E[X] = \sum_i x_i p(x_i)$	$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x f(x)dx$
4.	$E[X^2] = \sum_i x_i^2 p(x_i)$	$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx$
5.	$Var(X) = E(X^2) - [E(X)]^2$	$Var(X) = E(X^2) - [E(X)]^2$
6.	$\text{Moment} = E[X^r] = \sum_i x_i^r p_i$	$\text{Moment} = E[X^r] = \int_{-\infty}^{\infty} x^r f(x)dx$
7.	M.G.F. $M_X(t) = E[e^{tX}] = \sum_x e^{tx} p(x)$	M.G.F $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx$

4) $E(aX + b) = aE(X) + b$

5) $Var(aX + b) = a^2 Var(X)$

6) $Var(aX \pm bY) = a^2\ Var(X) + b^2\ Var(Y)$

7) Standard Deviation = $\sqrt{Var(X)}$

8) $f(x) = F'(x)$

9) $p(X > a) = 1 - p(X \leq a)$

10) $p(A/B) = \dfrac{p(A \cap B)}{p(B)},\ p(B) \neq 0$

11) If A and B are independent, then

$p(A \cap B) = p(A).p(B).$

(1).P.D.F, M.G.F, Mean and Variance of all the distributions:

Sl. No.	Distribution	P.D.F. (P(X = x))	M.G.F	Mean	Variance
1.	Binomial	$nC_x p^x q^{n-x}$	$(q+pe^t)^n$	np	npq
2.	Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}$	$e^{\lambda(e^t-1)}$	λ	λ

Sl. No.	Distribution	P.D.F. (P(X = x))	M.G.F	Mean	Variance
3.	Geometric	$q^{x-1}p$ (or) $q^x p$	$\frac{pe^t}{1-qe^t}$	$\frac{1}{p}$	$\frac{q}{p^2}$
4	uniform	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{bt} - e^{at}}{(b-a)t}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
5	Exponential	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
6	Gamma	$f(x) =$ $f(x) = \frac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)}, 0 < x < \infty, \lambda > 0$	$\frac{1}{(1-t)^{\lambda}}$	λ	λ
.					

Thank you

TWO DIMENSIONAL RANDOM VARIABLES

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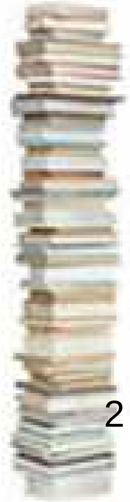
MA 6453 PROBABILITY AND QUEUEING THEORY

(Regulation 2013)

UNIT SYLLABUS

UNIT II TWO - DIMENSIONAL RANDOM VARIABLES

Joint distributions – Marginal and conditional distributions – Covariance – Correlation and Linear regression – Transformation of random variables



JOINT DISTRIBUTIONS
&
MARGINAL AND CONDITIONAL DISTRIBUTIONS

TWO – DIMENSIONAL RANDOM VARIABLES

Let S be the sample space associated with a random experiment E. Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcomes $s \in S$. Then (X, Y) is called a two - dimensional random variable.

Examples :

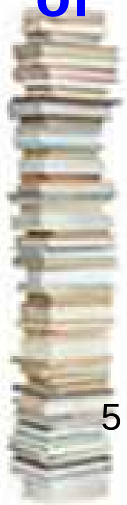
1. **Signal transmission : X is high quality signals and Y low quality signals.**

TWO – DIMENSIONAL DISCRETE RANDOM VARIABLES

If the possible values of (X, Y) are finite or countably infinite, then (X, Y) is called a two - dimensional discrete random variables.

Example:

(Sample space for dice) Consider the experiment of tossing two fair dice. The sample space for this experiment has 36 equally likely points. Let X = sum of the two dice and Y = |deference of two dice|. The random variable (X, Y) defined above is called a discrete random vector because it has only a countable (in this case, finite) number of possible values.



JOINT PROBABILITY MASS FUNCTION (PMF)

When both X and Y are discrete random variables, we define their joint PMF or simply the Probability Function of (X, Y) as follows :

$$p_{XY}(x_i, y_j) = P[X = x_i, Y = y_j]$$

provided the following conditions are satisfied.

$$(i) \quad 0 \le p_{XY}(x_i, y_j)) \le 1 \; \forall \; i, \; j$$

$$(ii) \quad \sum_i \sum_j p_{XY}(x_i, y_j) = 1$$

MARGINAL PROBABILITY DISTRIBUTION
(For discrete random variables)

If (X,Y) is a two-dimensional discrete RV, then the probability distribution of X, also called marginal probability mass function of X, denoted by $p_X(x_i)$, is defined as

$$p_X(x_i) = \sum_j p_{XY}(x_i, y_j) = P[X = x_i]$$

Similarly, the marginal probability mass function of Y, denoted by $p_Y(y_j)$, is defined as

$$p_Y(y_j) = \sum_j p_{XY}(x_i, y_j) = P[Y = y_j]$$

CONDITIONAL PROBABILITY DISTRIBUTION
(For discrete random variables)

Let (X, Y) be a discrete two - dimensional discrete random variables with the joint PMF $p_{XY}(x, y)$. Then the conditional PMF of Y, given $X = x$, *is given by*

$$p_{Y/X}(y/x) = \frac{P[X = x, Y = y]}{P[X = x]} = \frac{p_{XY}(x, y)}{p_X(x)} \text{ provided } p_X(x) > 0.$$

Similarly,

the conditional PMF of X, given $Y = y$, *is given by*

$$P_{X/Y}(x/y) = \frac{P[X = x, Y = y]}{P[Y = y]} = \frac{p_{XY}(x, y)}{p_Y(y)} \text{ provided } p_Y(y) > 0.$$

INDEPENDENT RANDOM VARIABLES
(For discrete random variables)

Let (X, Y) be a two - dimensional discrete random variable.

Then X and Y are independent random variables iff

$$P_{X,Y}(x, y) = P_X(x).P_Y(y)$$

where $P_{X,Y}(x, y)$ **is the joint p.m.f of (X, Y) and** $P_X(x)$ **and** $P_Y(y)$ **are the marginal p.m.f of X and Y.**

CONDITIONAL MEANS
(For discrete random variables)

If X and Y are discrete random variables with the joint PMF $p_{XY}(x, y)$, the conditional expected value of Y, given that $X = x$, is defined by $E[Y/X = x] = \sum_{y} y\, p_{Y/X}(y/x)$

Similarly,

the conditional expected value X, given $Y = y$,

***is* given by** $E[X/Y = y] = \sum_{x} x\, p_{X/Y}(x/y)$

Example :

The joint probability mass function of (X, Y) is given by $p(x, y) = k(2x + 3y), x = 0,1,2;\ y = 1,2,3$. Find all the marginal and conditional probability distributions.

Solution :

The joint probability mass function of (X, Y) is given below

Y / X	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

$Since\ p(x,y)$ **is a probability mass function, we have**

$$\sum p(x,y) = 1$$

$$3k + 6k + 9k + 5k + 8k + 11k + 7k + 10k + 13k = 1$$

$$72k = 1 \Rightarrow k = \frac{1}{72}.$$

Marginal probability distribution of X

$$P[X=0] = 3k + 6k + 9k = 18k = \frac{18}{72} = \frac{1}{4}$$

$$P[X=1] = 5k + 8k + 11k = 24k = \frac{24}{72} = \frac{1}{3}$$

$$P[X=0] = 7k + 10k + 13k = 30k = \frac{30}{72} = \frac{5}{12}$$

Marginal probability distribution of Y

$$P[Y=1] = 3k+5k+7k = 15k = \frac{15}{72} = \frac{5}{24}$$

$$P[Y=2] = 6k+8k+10k = 24k = \frac{24}{72} = \frac{1}{3}$$

$$P[Y=3] = 9k+11k+13k = 33k = \frac{33}{72} = \frac{11}{24}$$

Conditional distribution of X *given* $Y = 1$

$$P[X=0/Y=1] = \frac{P[X=0\,,\,Y=1]}{P[Y=1]} = \frac{3k}{15k} = \frac{3}{15} = \frac{1}{5}$$

$$P[X=1/Y=1] = \frac{P[X=1\,,\,Y=1]}{P[Y=1]} = \frac{5k}{15k} = \frac{5}{15} = \frac{1}{3}$$

$$P[X=2/Y=1] = \frac{P[X=2\,,\,Y=1]}{P[Y=1]} = \frac{7k}{15k} = \frac{7}{15}$$

Conditional distribution of X given $Y = 2$

$$P[X=0/Y=2]=\frac{P[X=0\ ,\ Y=2]}{P[Y=2]}=\frac{6k}{24k}=\frac{6}{24}=\frac{1}{4}$$

$$P[X=1/Y=2]=\frac{P[X=1\ ,\ Y=2]}{P[Y=2]}=\frac{8k}{24k}=\frac{8}{24}=\frac{1}{3}$$

$$P[X=2/Y=2]=\frac{P[X=2\ ,\ Y=2]}{P[Y=2]}=\frac{10k}{24k}=\frac{5}{12}$$

Conditional distribution of X *given* $Y = 3$

$$P\left[X=0/Y=3\right]=\frac{P\left[X=0\ ,\ Y=3\right]}{P\left[Y=3\right]}=\frac{9k}{33k}=\frac{9}{33}=\frac{3}{11}$$

$$P\left[X=1/Y=3\right]=\frac{P\left[X=1\ ,\ Y=3\right]}{P\left[Y=3\right]}=\frac{11k}{33k}=\frac{11}{33}=\frac{1}{3}$$

$$P\left[X=2/Y=3\right]=\frac{P\left[X=2\ ,\ Y=3\right]}{P\left[Y=3\right]}=\frac{13k}{33k}=\frac{13}{33}$$

Conditional distribution of Y given $X = 0$

$$P[Y=1/X=0] = \frac{P[X=0\ ,\ Y=1]}{P[X=0]} = \frac{3k}{18k} = \frac{3}{18} = \frac{1}{6}$$

$$P[Y=2/X=0] = \frac{P[X=0\ ,\ Y=2]}{P[X=0]} = \frac{6k}{18k} = \frac{6}{18} = \frac{1}{3}$$

$$P[Y=3/X=0] = \frac{P[X=0\ ,\ Y=3]}{P[X=0]} = \frac{9k}{18k} = \frac{9}{18} = \frac{1}{2}$$

Conditional distribution of Y given $X = 1$

$$P[Y=1/X=1]=\frac{P[X=1\,,Y=1]}{P[X=1]}=\frac{5k}{24k}=\frac{5}{24}$$

$$P[Y=2/X=1]=\frac{P[X=1\,,Y=2]}{P[X=1]}=\frac{8k}{24k}=\frac{8}{24}=\frac{1}{3}$$

$$P[Y=3/X=1]=\frac{P[X=1\,,Y=3]}{P[X=1]}=\frac{11k}{24k}=\frac{11}{24}$$

Conditional distribution of Y *given* $X = 2$

$$P[Y=1/X=2] = \frac{P[X=2\,,Y=1]}{P[X=2]} = \frac{7k}{30k} = \frac{7}{30}$$

$$P[Y=2/X=2] = \frac{P[X=2\,,Y=2]}{P[X=2]} = \frac{10k}{30k} = \frac{10}{30} = \frac{1}{3}$$

$$P[Y=3/X=2] = \frac{P[X=2\,,Y=3]}{P[X=2]} = \frac{13k}{30k} = \frac{13}{30}$$

Example :

The Joint p.m.f of X and Y is

p(x,y)		Y		
		0	1	2
X	0	0.1	0.04	0.02
	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

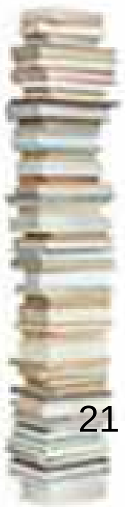
Determine if X and Y are independent.

Solution :

The marginal p.m.f of X and Y are given by

p(x,y)		Y			p(x) = P(X=x)
		0	1	2	
X	0	0.1	0.04	0.02	0.16
	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.5
p(y) = P(Y=y)		0.24	0.38	0.38	1

If $p_{XY}(x,y) = p_X(x).p_Y(y)$ for all x and y,
we can say that X and Y are independent.



We have $p(X=0)\times p(Y=0)=0.16\times 0.24=0.0384$

$$\neq p(x=0,y=0)=0.1$$

Similarly we can verify that

$p(X=1)\times p(Y=1)\neq p(x=1,y=1)$

$p(X=2)\times p(Y=2)\neq p(x=2,y=2)$ and so on.

Hence the RVs X and are not independent.

TWO – DIMENSIONAL CONTINUOUS RANDOM VARIABLES

If (X, Y) can assume all values in a specified region R in the xy - plane, (X, Y) is called a two - dimensional continuous random variables.

Example:

Suppose *X* denotes the duration of an eruption (in second) of Volcano, and *Y* denotes the time (in minutes) until the next eruption. We might want to know if there is a relationship between *X* and *Y*. Or, we might want to know the probability that *X* falls between two particular values *a* and *b*, and *Y* falls between two particular values *c* and *d*. That is, we might want to know $P(a < X < b, c < Y < d)$. In this case (X, Y) is called two dimensional continuous random variables.

JOINT PROBABILITY DENSITY FUNCTION (PDF)

If (X,Y) is a two - dimensional continuous random variable such that

$$P\left\{x-\frac{dx}{2}\le X\le x+\frac{dx}{2},\ y-\frac{dy}{2}\le Y\le y+\frac{dy}{2}\right\}=f(x,y)\,dxdy,$$

then $f(x,y)$ is called the joint p.d.f of (X, Y), provided $f(x,y)$ satisfies the following conditions:

(i) $f(x,y)\ge 0$ *for all* $(x,y)\in R$, whre R is the range space.

(ii) $\iint\limits_{R} f(x,y)\,dxdy=1$

(iii) If D is a subspace of the range space R, then

$$P\{(X,Y)\in D\}=\iint\limits_{D} f(x,y)\,dxdy.$$

MARGINAL PROBABILITY DISTRIBUTION

(For continuous random variables)

Let (X, Y) be the two-dimensional continuous RV.

Then the marginal density function of X is denoted by $f_X(x)$ *and is defined as,*

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dy$$

Similarly,

the marginal density function of Y is denoted by $f_Y(y)$ *and is defined as,*

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dx$$

CONDITIONAL PROBABILITY DISTRIBUTION
(For continuous random variables)

Consider two continuous random variables X and Y with the joint PDF $f_{XY}(x, y)$.

The conditional PDF of Y, given $X = x$, is given by

$$f_{Y/X}(y/x) = \frac{f_{XY}(x, y)}{f_X(x)} \text{ provided } f_X(x) > 0$$

Similarly,

the conditional PDF of X, $Y = y$, is given by

$$f_{X/Y}(x/y) = \frac{f_{XY}(x, y)}{f_Y(y)} \text{ provided } f_Y(y) > 0$$

INDEPENDENT RANDOM VARIABLES
(For continuous random variables)

Two random variables X and Y with joint p.d.f $f(x, y)$ and marginal p.d.fs $f(x)$ and $f(y)$ respectively are said to be independent iff

$$f_{X/Y}\left(x/y\right) = f_X(x)\,f_Y(y).$$

Example :

Given $f_{XY}(x,y)=\begin{cases} cx(x-y) & ;0<x<2,-x<y<x \\ 0 & ;\text{otherwise} \end{cases}.$

(*i*) Evaluate *c* (*ii*) $f_X(x)$ (*iii*) $f_{Y/X}(y/x)$ *and* (*iv*) $f_Y(y)$.

Solution :

Given $f(x,y)$ *is the joint p.d.f, we have*

$$\iint f(x,y)\,dx\,dy=1$$

$$c\int_0^2\int_{-x}^{x}\left(x^2-xy\right)dy\,dx=1$$

$$c\int_0^2\left[x^2\left(x-(-x)\right)-\frac{x}{2}\left(x^2-x^2\right)\right]dx=1$$

$$\frac{c}{2}[16-0]=1 \quad \Rightarrow 8c=1 \quad \Rightarrow c=\frac{1}{8}$$

$$\therefore \quad f(x,y)=\frac{1}{8}\left(x^2-xy\right);0<x<2,-x<y<x$$

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\frac{1}{8}\int_{-x}^{x}\left(x^2-xy\right)dy$$

$$=\frac{1}{8}\left[x^2\,(y)_{-x}^{x}-x\left(\frac{y^2}{2}\right)_{-x}^{x}\right]$$

$$= \frac{1}{8}\left[x^2 \left(x - (-x) \right) - \frac{x}{2}\left(x^2 - x^2 \right) \right]$$

$$= \frac{1}{8}\left[x^2 (2x) - 0 \right]$$

$$= \frac{x^3}{4}, 0 < x < 2$$

$$f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{8}\left(x^2 - xy \right)}{\frac{x^3}{4}}$$

$$= \frac{4}{8}\,\frac{x(x-y)}{x^3}$$

$$= \frac{x-y}{2x^2}, -x < y < x$$

Example :

Given the joint pdf of (X, Y) as $f(x,y)=\begin{cases} 8xy & ; 0<x<y<1 \\ 0 & , \text{otherwise} \end{cases}$**. Find the marginal and conditional probability density functions of X and Y. Are X and Y are independent?**

Solution :

Marginal density of X is

$$f_X(x)=\int f(x,y)\,dy=\int_x^1 8xy\,dy$$

$$=8x\int_x^1 y\,dy=8x\left(\frac{y^2}{2}\right)_x^1$$

$$=4x\left(1-x^2\right),\ 0<x<1$$

Marginal density of Y is

$$f_Y(y) = \int f(x,y)\,dx = \int_0^y 8\,x\,y\,dx$$

$$= 8\,y\int_0^y x\,dx = 8\,y\left(\frac{x^2}{2}\right)_0^y$$

$$= 4\,y\left(y^2 - 0\right) = 4\,y^3\ ,\ 0 < y < 1$$

$$\text{and}\quad f_X(x).\,f_Y(y) = 4x\left(1 - x^2\right).4\,y^3 \neq 8\,xy = f_{XY}(x,y)$$

Therefore X and Y are not independent.

Conditional density of X given Y is

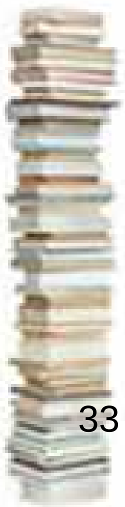
$$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{8\,x\,y}{4\,y^3} = \frac{2\,x}{y^2}\,,\ 0 < x < y$$

Conditional density of Y given X is

$$f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{8xy}{4x(1-x^2)}$$

$$= \frac{2y}{1-x^2}, \; x<y<1$$



JOINT CUMMULATIVE DISTRIBUTION FUNCTION (CDF)

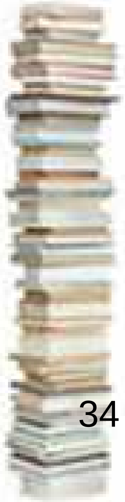
If (X,Y) is a two - dimensional RV(discrete or continuous), then by
$F(x, y) = P[X \leq x, Y \leq y]$ is called the cdf of (X,Y).

In the discrete case,

$$F(x,y) = \sum_{\substack{j \\ y_j \leq y}} \sum_{\substack{i \\ x_i \leq x}} p(x_i, y_j)$$

In the continuous case,

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\,dx\,dy$$



PROPERTIES OF JOINT CDFs

1. $0 \le F_{XY}(x, y) \le 1 \;\; for \; -\infty < x < \infty, \; -\infty < y < \infty.$

2. $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} F_{XY}(x, y) = F_{XY}(\infty, \infty) = 1$

3. $\lim\limits_{x \to -\infty} F_{XY}(x, y) = F_{XY}(-\infty, y) = 0$

4. $\lim\limits_{y \to -\infty} F_{XY}(x, y) = F_{XY}(x, -\infty) = 0$

MARGINAL CDFs

The marginal distribution of X with respect to the joint CDF

$F_{XY}(x, y)$ is $F_X(x) = P[X \le x] = P[X \le x, Y < \infty]$

$$i.e.,\ F_X(x) = \begin{cases} \sum_y P[X \le x, Y = y], & \text{for discrete random variables} \\ \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \right\} dx, & \text{for continuous random variables} \end{cases}$$

and the marginal distribution of Y with respect to the joint CDF

$F_{XY}(x, y)$ is $F_Y(y) = P[Y \le y] = P[X < \infty, Y \le y]$

$$i.e.,\ F_Y(y) = \begin{cases} \sum_x P[X = x, Y \le y], & \text{for discrete random variables} \\ \int_{-\infty}^{y} \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \right\} dy, & \text{for continuous random variables} \end{cases}$$

COVARIANCE

AND

CORRELATION COEFFICIENT

COVARIANCE

COVARIANCE(Definition)

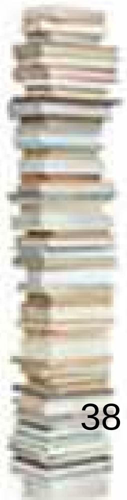
The covariance of X and Y, which is denoted by Cov(X, Y) or σ_{XY}, is defined by $\mathrm{Cov}(X,Y) = E\{[X - E(X)][Y - E(Y)]\}$

$$= E[XY] - E[X]E[Y]$$

Note :

(*i*) *If* $Cov(X, Y) = 0$, we define the two random variables to be uncorrelated.

(ii) $\mathrm{Cov}(aX + b, cY + d) = ac\ \mathrm{Cov}(X, Y)$



Result 1 :

Let X and Y be any two random variables and a,b be constants.

Prove that $Cov\left(aX, bY\right) = ab\,\mathrm{cov}\left(X, Y\right)$

Result 2 :

Let X and Y be any two random variables and a,b be constants.

Prove that $\mathrm{Cov(X+a, Y+b) = Cov(X, Y)}$

Result 3 :

Show that $Cov^2(X, Y) \le Var(X).Var(Y)$

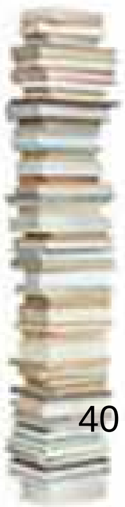
CORRELATION COEFFICIENT

CORRELATION COEFFICIENT (Formula)

We define the correlation coefficient of X and Y, denoted by

$\rho(X,Y)$ *or* ρ_{XY}, **as follows :**

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}, -1 \le \rho_{XY} \le 1$$



Result 1 :

1. Prove that the limits of correlation coefficient is between -1 and 1.

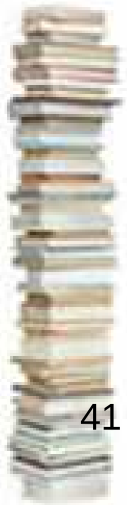
i.e., $-1 \leq \rho_{XY} \leq 1 \ or \ |\rho_{XY}| \leq 1 \ or \ C_{XY} \leq \rho_X \rho_Y$.

Result 2 :

Prove that two independent variables are uncorrelated.

Result 3 :

Prove that correlation coefficient is independent of change of origin and scale.



Example :

If X_1 has mean 4 and variance 9 while X_2 has mean -2 and variance 5 and the two variables independent, find $Var\left(2X_1+X_2-5\right)$.

Solution :

Given $E[X_1]=4 \quad ,Var[X_1]=9$

$E[X_2]=-2 \quad ,Var[X_2]=5$

$$Var\left(2X_1+X_2-5\right)=4Var\,X_1+Var\,X_2$$
$$=4(9)+5=36+5=41.$$

Example :

The joint p.m.f of X and Y is given below :

p(x,y)		X	
		-1	1
Y	0	1/8	3/8
	1	2/8	2/8

Find the correlation coefficient of (X,Y).

Solution :

The marginal p.m.f of X and Y are given by

p(x,y)		X		p(Y= y)
		-1	1	
Y	0	1/8	3/8	4/8
	1	2/8	2/8	4/8
P(X= x)		3/8	5/8	1

$$E[X] = \sum_i x_i \, p(x_i) = (-1)\left(\frac{3}{8}\right) + (1)\left(\frac{5}{8}\right) = \frac{2}{8}$$

$$E[X^2] = \sum_i x_i^2 \, p(x_i) = (-1)^2\left(\frac{3}{8}\right) + (1)^2\left(\frac{5}{8}\right) = 1$$

$$E[Y] = \sum_i y_i \, p(y_i) = (0)\left(\frac{4}{8}\right) + (1)\left(\frac{4}{8}\right) = \frac{1}{2}$$

$$E[Y^2] = \sum_i y_i^2 \, p(y_i) = (0)^2\left(\frac{4}{8}\right) + (1)^2\left(\frac{4}{8}\right) = \frac{1}{2}$$

$$\mathrm{E[XY]} = \sum_{\mathrm{i}} \sum_{\mathrm{j}} \mathrm{x_i} y_j \, p(x_i, y_i)$$

$$= (0)(\text{-}1)\left(\frac{1}{8}\right) + (0)(1)\left(\frac{3}{8}\right) + (1)(\text{-}1)\left(\frac{2}{8}\right) + (1)(1)\left(\frac{2}{8}\right) = 0$$

$$\sigma_{\mathrm{X}}^2 = \mathrm{E[X^2]} - [E(X)]^2 = 1 - \frac{4}{64} = \frac{15}{16}$$

$$\therefore \sigma_X = \frac{\sqrt{15}}{4}$$

$$\sigma_Y^2 = E[Y^2] - [E(Y)]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\therefore \sigma_Y = \frac{1}{2}$$

$$\text{Hence } \rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$$

$$= -0.2582.$$

Example :

Suppose that the two dimensional RVs (X, Y) has the joint p.d.f

$$f(x,y) = \begin{cases} x + y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & otherwise \end{cases}.$$

Obtain the correlation coefficient between X and Y.

Solution :

The marginal density function of X is given by

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_0^1 (x+y)dy = \left(xy + \frac{y^2}{2} \right)_0^1 = x + \frac{1}{2}.$$

Similarly, the marginal density function of Y is given by

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = y + \frac{1}{2},\ 0 < y < 1.$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2} \right) dx$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{4} \right)_0^1 = \frac{7}{12}.$$

$$E[Y] = \int_{-\infty}^{\infty} y f(y)\,dy$$

$$= \int_0^1 \left(y^2 + \frac{y}{2} \right) dy$$

$$= \left(\frac{y^3}{3} + \frac{y^2}{4} \right)_0^1 = \frac{7}{12}.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^1 \left(x^3 + \frac{x^2}{2} \right) dx = \left(\frac{x^4}{4} + \frac{x^3}{6} \right)_0^1$$

$$= \frac{5}{12}.$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_0^1 \left(y^3 + \frac{y^2}{2} \right) dy = \left(\frac{y^4}{4} + \frac{y^3}{6} \right)_0^1$$

$$= \frac{5}{12}.$$

$$E[XY]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dxdy$$

$$=\int_{0}^{1}\int_{0}^{1} xy(x+y)\,dxdy$$

$$=\int_{0}^{1} y\left\{\int_{0}^{1}(x^2+xy)\,dx\right\}dy$$

$$=\int_{0}^{1}\left(\frac{y}{3}+\frac{y^2}{2}\right)dy$$

$$=\frac{1}{3}.$$

$$Var(X) = E[X^2] - [E(X)]^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$$

$$\therefore \sigma_X = \frac{\sqrt{11}}{12}.$$

$$Var(Y) = E[Y^2] - [E(Y)]^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$$

$$\therefore \sigma_Y = \frac{\sqrt{11}}{12}.$$

Hence $$\rho(\mathrm{X,Y}) = \frac{Coc(X,Y)}{\sigma_X \sigma_Y} = \frac{\mathrm{E[XY] - E[X]E[Y]}}{\sigma_X \sigma_Y} = -\frac{1}{11}.$$

LINEAR REGRESSION

LINES OF REGRESSION

The line of regression of y on x is given by

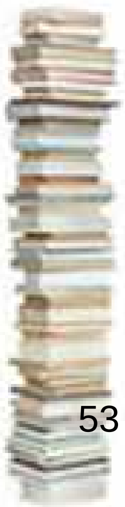
$$\mathbf{y - \bar{y} = \rho \frac{\sigma_y}{\sigma_x}(x - \bar{x})}$$

The line of regression of x on y is given by

$$\mathbf{x - \bar{x} = \rho \frac{\sigma_x}{\sigma_y}(y - \bar{y})}$$

Note

Both the lines of regression passes through $(\bar{x}, \bar{y})$.



REGRESSION COEFFICIENTS

Regression coefficient of y on x is $\rho \dfrac{\sigma_y}{\sigma_x} = b_{yx}$

Regression coefficient of x on y is $\rho \dfrac{\sigma_x}{\sigma_y} = b_{xy}$

PROPERTIES OF REGRESSION

1. Correlation coefficient is the geometric mean between the regression coefficients. i.e $\rho = \pm\sqrt{b_{xy}\, b_{yx}}$.

2. If one of the regression coefficients is greater than unity, then other must be less than unity.

3. Regression coefficients are independent of the change of origin but not of scale.

Result :

The acute angle between the two lines of regression is

$$\tan\theta = \frac{\left(1 - r^2\right)\sigma_x\,\sigma_y}{|r|\,\left(\sigma_x^2 + \sigma_y^2\right)}.$$

Example :

If the equations of the to lines of regression of y on x and x on y are respectively, $7x - 16y + 9 = 0$; $5y - 4x - 3 = 0$, calculate the coefficient of correlation, $\bar{x}$ and $\bar{y}$.

Solution :

Since both the regression lines passes through $(\bar{x}, \bar{y})$, we get

$$7\bar{x} - 16\bar{y} = -9 \;\; \& \;\; 4\bar{x} - 5\bar{y} = -3$$

Solving, we get $\bar{x} = -\dfrac{3}{29}$ **and** $\bar{y} = \dfrac{15}{29}$.

$\therefore$ The mean value of x and y are $-\dfrac{3}{29}$ and $\dfrac{15}{29}$.

Now, the regression equation of y on x is,

$$7x - 16y + 9 = 0$$

i.e $\quad y = \dfrac{7}{16}x + \dfrac{9}{16} \Rightarrow b_{yx} = \dfrac{7}{16}.$

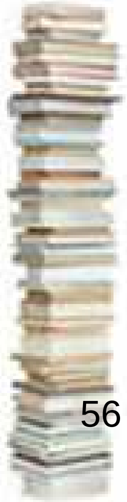
Similarly, the regression equation of x on y is,

$$5y - 4x - 3 = 0$$

i.e $\quad x = \dfrac{5}{4}y - \dfrac{3}{4} \Rightarrow b_{xy} = \dfrac{5}{4}.$

Hence the correlation coefficient between X and Y is given by

$$\rho = \pm\sqrt{b_{yx} \times b_{xy}} = \pm 0.7338.$$



Example

From the following data, find

(i) the two regression equations,

(ii) the coefficient of correlation between the marks in Economics and Statistics and

(iii) the most likely marks in statistics when marks in Economics are 30.

Marks in Economics	25	28	35	32	31	36	29	38	34	32
Marks in Statistics	43	46	49	41	36	32	31	30	33	39

Solution :

Let x = marks in Economics & y = marks in Statistics

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
25	43	-7	5	49	25	-35
28	46	-4	8	16	64	-32
35	49	3	11	9	121	33
32	41	0	3	0	9	0
31	36	-1	-2	1	4	2
36	32	4	-6	16	36	-24
29	31	-3	-7	9	49	21
38	30	6	-8	36	64	-48
34	33	2	-5	4	25	-10
32	39	0	1	0	1	0
320	380	0	0	140	398	-93

$$\text{where } \bar{x} = \frac{\sum x}{n} = 32; \ \bar{y} = \frac{\sum y}{n} = 38$$

$$\text{and } b_{YX} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2} = -0.6643$$

$$b_{XY} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (y-\bar{y})^2} = -0.2337$$

Equation of the line of regression of x on y is

$$x - \bar{x} = b_{XY}(y - \bar{y})$$

$$x = -0.2337\, y + 40.8806$$

Equation of the line of regression of y on x is

$$y - \bar{y} = b_{YX}(x - \bar{x})$$

$$y = -0.6643\, x + 59.2576$$

Hence the correlation coefficient between X and Y is given by

$$\rho = \pm\sqrt{b_{yx} \times b_{xy}} = \pm 0.394$$

$$\textit{When } x = 30,$$

$$y = (-0.6643 \times 30) + 59.2576 = -19.929 + 59.2576 = 39.3286.$$

REGRESSION CURVE OR FUNCTION

Regression curve y on x is given by

$$y = E[Y/X = x] = \int_{-\infty}^{\infty} y\, f(y/x)\, dy$$

Regression curve x on y is given by

$$x = E[X/Y = y] = \int_{-\infty}^{\infty} x\, f(x/y)\, dx$$

Note

A linear regression line has an equation of the form $Y = a + bX$, where X is the explanatory variable and Y is the dependent variable. The slope of the line is 'b' and 'a' is the intercept. (the value of Y when $x = 0$)

Example :

Let (X, Y) have the joint p.d.f given by $f(x,y) = \begin{cases} 1, & \text{if } |y| = x,\ 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$.

Show that the regression of y on x is linear but regression of x on y is not linear.

Solution :

The marginal p.d.f of X is

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad = \int_{-x}^{x} dy = 2x$$

Similarly, The marginal p.d.f of Y is

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \quad = \int_{0}^{1} dx = 1$$

Also, the conditional p.d.f of Y is given X = x is given by

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{1}{2x}$$

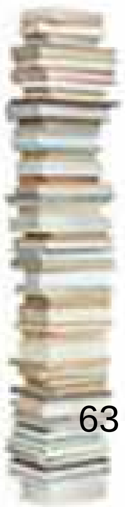
and the conditional p.d.f of X is given Y = y is given by

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{1}{1} = 1.$$

The regression curve y on x is

$$y = E[Y/X = x] = \int_{-\infty}^{\infty} y\, f\left(y/x\right) dy$$

$$= \int_{-x}^{x} y \frac{1}{2x} dy = \frac{1}{2x}\left[\frac{y^2}{2}\right]_{-x}^{x} = \frac{1}{2x}\left[\frac{x^2}{2} - \frac{x^2}{2}\right] = 0$$



i.e $y = 0$, which is a straight line.

$\therefore$ The regression curve y on x is linear.

Regression curve x on y is given by

$$x = E[X/Y = y] = \int_{-\infty}^{\infty} x\, f(x/y)\, dx$$

$$= \int_{0}^{1} x\, dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}.$$

Hence the regression curve s on y is not linear.

TRANSFORMATION OF RANDOVARIABLES

TRANSFORMATION OF TWO-DIMENSIONAL RANDOM VARIABLE

If (X, Y) is a two-dimensional random variable with joint pdf $f_{XY}(x,y)$ and if $U = g(x,y)$ and $V = h(x,y)$ are two other random variables then the joint pdf of (U, V) is given by

$$f_{UV}(u,v) = f_{XY}(x,y)\,.|J|$$

where $|J| = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ is called the jacobian of the transformation.

Result

Assume that $U = g(X, Y)$, and we are required to find the p.d.f of U. We can use the above transformation method by defining an auxiliary function $W = X$ or Y so we can obtain the joint PDF $f_{UW}(u, w)$ of U and W.

Then we obtain the required marginal PDF $f_U(u)$ as follows :

$$f_U(u) = \int_{-\infty}^{\infty} f_{UW}(u, w)\, dw$$

Example :

If X and Y are independent RVs with pdf's $e^{-x}, x \geq 0$, $e^{-y}, y \geq 0$, respectively, find the density functions of $U = \dfrac{X}{X+Y}$ and $V = X + Y$. Are U and V independent?

Solution :

Since X and Y are independent, $f_{X/Y}\left(x/y\right) = f_X(x) f_Y(y) = e^{-(x+y)}$.

Solving the equations $u = \dfrac{x}{x+y}$ and $v = x + y$,

we get $x = uv$ and $y = v(1-u)$.

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & (1-u) \end{vmatrix} = v.$$

The joint pdf of (U,V) is given by

$$f_{UV}\left(u,v\right) = f_{XY}(x,y).|J|$$

$$= e^{-(x+y)} v = e^{-v} v.$$

The range space of (U, V) is obtained as follows :

Since x and y ≥ 0, uv ≥ 0 and $1 - u \geq 0$.

$\therefore$ either $u \geq 0$ and $v \geq 0$ and $1 - u \geq 0$ i.e $0 \leq u \leq 1$ and $v \geq 0$

or $u \leq 0$, $v \leq 0$ and $1 - u \leq 0$, i.e $u \leq 0$ *and* $u \geq 1$, which is absurd.

Therefore range space of (U, V) is given by $0 \leq u \leq 1$ and $v \geq 0$.

$\therefore \quad f_{UV}(u, v) = e^{-v} v; \; 0 \leq u \leq 1$ and $v \geq 0$.

PDF of U is given by $f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u, v)\,dv$

$$= \int_0^{\infty} e^{-v} v\,dv \quad = \left[(v)\left(\frac{e^{-v}}{-1}\right) - (1)\left(\frac{e^{-v}}{1}\right) \right]_0^{\infty}$$

$$= 1, \; 0 \leq u \leq 1$$

$$\text{PDF of V is given by } f_V(v) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,du$$

$$= \int_0^1 e^{-v} v\,du$$

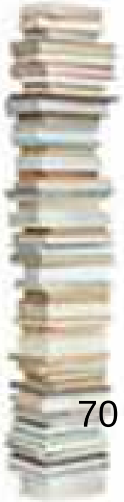
$$= e^{-v} v(1-0) = e^{-v} v \qquad v \geq 0.$$

$$\text{Now } f_U(u) \times f_V(v) = (1) \times e^{-v} v$$

$$= e^{-v} v$$

$$= f_{UV}(u,v)$$

Therefore, U and V are independent RVs.



Example:

If X and Y are independent RVs each normally distributed with mean zero and variance σ^2, find the density functions of

$$R = \sqrt{X^2 + Y^2} \text{ and } \theta = \tan^{-1}\left(\frac{Y}{X}\right).$$

Solution:

Since X and Y are independent N(0,σ),

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} \quad -\infty < x,y < \infty \text{ and } r = \sqrt{x^2 + y^2} \text{ and}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$ **are the transformations from cartesians to polars.**

Therefore, the inverse transformations are given by $x = r\cos\theta$ and $y = r\sin\theta$

$$|J| = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r.$$

The joint pdf of (R,θ) is given by

$$f_{R\theta}(r,\theta) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \qquad r \ge 0,\ \ 0 \le \theta \le 2\pi.$$

Since $-\infty < x, y < \infty$ and $r \ge 0,\ \ 0 \le \theta \le 2\pi$ both represent the entire xy - plane.

The density function of R is given by

$$f_R(r) = \int_0^{2\pi} f_{R\theta}(r,\theta)\,d\theta = \int_0^{2\pi} \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\,d\theta$$

$$= \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \left(\theta\right)_0^{2\pi}$$

$$= \frac{r}{\sigma^2} \times e^{-r^2/2\sigma^2} \qquad r \ge 0$$

The density function of θ is given by

$$f_\theta(\theta) = \int_0^\infty \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \, dr$$

Putting $t = \dfrac{r^2}{2\sigma^2}$.

$$\therefore f_\theta(\theta) = \frac{1}{2\pi} \int_0^\infty e^{-t} \, dt$$

$$= \frac{1}{2\pi} \left(\frac{e^{-t}}{-1} \right)_0^\infty = \frac{1}{2\pi}, \; 0 \le \theta \le 2\pi$$

THANK YOU

UNIT III

RANDOM PROCESSES

by

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Introduction:

The concept of a random process is an extension of the concept of a random variable to include time. Thus, instead of thinking of a random variable X that maps an event s belongs S to some number X(s), We think of how the random variable maps the event to different numbers at different times.

Random process:(Stochastic process)

A random process is a collection of RVs $\{X(s,t)\}$ that are functions of a real variable, namely time t where $s \,\varepsilon\, S$ (sample space) and $t \,\varepsilon\, T$ (parameter set or index set).

Remarks:

- The set of possible values of any individual member of random process is called state space and any individual member is called sample function.
- If 's' and 't' are fixed then $\{X(s,t)\}$ is a number
- If 's' and 't' are variable then $\{X(s,t)\}$ is the collection of RV which are time function
- If 's' is fixed then $\{X(s,t)\}$ is a single time function
- If 't' is fixed then $\{X(s,t)\}$ is a RV
- Since the dependence of random process $\{X(s,t)\}$ on 's' is obvious, we may drop the variable 's' and write the short form X(t)

Classification of Random Process:

Depending on the continuous or discrete nature of the state space S and parameter set T, a random process can be classified into four types:

a) If both T and S are discrete, the random process is called a discrete random sequence.

For example, if X_n represents the outcome of the n^{th} toss of a fair die, then $\{X_n, n \geq 1\}$ is a discrete random sequence, since $T = \{1,2,3,\ldots\}$ and $S = \{1,2,3,4,5,6\}$.

b) If T is discrete and S is continuous, the random process is called a continuous random sequence.

For example, if X_n represents the temperature at the end of the n^{th} hour of a day, then $\{X_n, 1 \leq n \leq 24\}$ is a continuous random sequence, since temperature can take any value in an interval and hence continuous.

c) If T is continuous and S is discrete, the random process is called a discrete random process.

For example, if $X(t)$ represents the number of telephone calls received in the interval *(0,t)* then $\{X(t)\}$ is a discrete random process, since $S = \{0,1,2,3,....\}$

d) If T and S are continuous, the random process is called a continuous random process.

For example, if $X(t)$ represents the maximum temperature at a place in the interval *(0,t)* , $\{X(t)\}$ is a continuous random process.

Mean of the Random processes:

Mean of the process $\{X(t)\}$ is the expected value of a typical member $X(t)$ of the process.

$$\mu(t) = E[X(t)]$$

Auto Correlation of the Random processes:

Autocorrelation of the process $\{X(t)\}$ denoted by $R_{XX}(t_1,t_2)$ is the expected value of the product of any two members $X(t_1)$ and $X(t_2)$ of the process

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$$

Auto Covariance of the Random processes:

Auto covariance of the process $\{X(t)\}$ denoted by $C_{XX}(t_1,t_2)$ is defined as

$$C_{XX}(t_1,t_2) = E\left[\{X(t_1) - \mu(t_1)\}\{X(t_2) - \mu(t_2)\}\right]$$

$$= R_{XX}(t_1,t_2) - \mu(t_1)\mu(t_2)$$

Cross Correlation of the Random processes:

Cross correlation of two processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$R_{XY}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right]$$

Cross Covariance of the Random processes:

Cross covariance of two processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2)$$

Stationary processes:

If certain probability distribution or averages do not depend on t, then the random process $\{X(t)\}$ is called a stationary process.

Example:

A Bernoulli process is a stationary stochastic process as the joint probability distributions are independent of time.

Remarks:

1. A random process $\{X(t)\}$ is said to be first order stationary if its first order density function is invariant under translation of time parameter

2. A random process $\{X(t)\}$ is said to be second order stationary if its second order density function is invariant under translation of time parameter

Strict Sence Stationary (OR)
Strongly Stationary processes: (SSS)

A random process is called a strict sense stationary process or strongly stationary process if all its finite dimensional distributions are invariant under translation of time parameter

Bernoulli's process is an example for strict sense stationary random process

Wide Sence Stationary (OR) Weak Stationary processes (OR) Covarience Stationary Process: (WSS)

A random process $\{X(t)\}$ with finite first and second order moments is called a weakly stationary process or covariance stationary process or wide-sense stationary process if its mean is a constant and the auto correlation depends only on the time difference. i.e,

if i) $E[X(t)] = \mu$ (constant)

and ii) $E[X(t)X(t-z)] = R(z)$

Jointly WSS processes:

Two processes $\{X(t)\}$ and $\{Y(t)\}$ are called jointly WSS if each is WSS and their cross correlation depends only on time difference

1. The probability distribution of the process $\{X(t)\}$ is given by

$$P[X(t)=n]=\begin{cases}\dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n=1,2,3,\ldots\\[2ex] \dfrac{at}{1+at} & n=0\end{cases}$$

Show that it is not stationary.

Ans.

The probability distribution of $X(t)$ is

X(t) =n	**0**	**1**	**2**	**3**	**...**
P_n	$\dfrac{at}{1+at}$	$\dfrac{1}{(1+at)^2}$	$\dfrac{at}{(1+at)^3}$	$\dfrac{(at)^2}{(1+at)^4}$	....

$$E[X(t)] = \sum_{n=0}^{\infty} np_n$$

$$= 0p_0 + 1p_1 + 2p_2 + 3p_3 + \ldots.$$

$$= \frac{1}{(1+at)^2} + 2\frac{at}{(1+at)^3} + 3\frac{(at)^2}{(1+at)^4} + \ldots.$$

$$= \frac{1}{(1+at)^2}\left[1 + 2\frac{at}{1+at} + 3\frac{(at)^2}{(1+at)^2} + \ldots.\right]$$

$$= \frac{1}{(1+at)^2}\left[1 + 2\left(\frac{at}{1+at}\right) + 3\left(\frac{at}{1+at}\right)^2 + \ldots.\right]$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}(1+at)^2 \quad =1$$

$$\therefore E\left[X(t)\right] \quad = 1 \quad (\text{constant})$$

$$E\left[X^2(t)\right] = \sum_{n=0}^{\infty} n^2 p_n$$

$$= \sum_{n=1}^{\infty} n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=1}^{\infty} (n^2 + n - n) \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=1}^{\infty} \left(n(n+1) - n\right) \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=1}^{\infty} n(n+1) \frac{(at)^{n-1}}{(1+at)^{n+1}} - \sum_{n=1}^{\infty} n \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= 1.2. \frac{(at)^0}{(1+at)^2} + 2.3. \frac{(at)^1}{(1+at)^3} + 3.4. \frac{(at)^2}{(1+at)^4} + \ldots\ldots\ldots - 1$$

$$= \frac{2}{(1+at)^2} + 2.3.\frac{(at)^1}{(1+at)^3} + 6.2.\frac{(at)^2}{(1+at)^4} + \ldots\ldots\ldots - 1$$

$$= \frac{2}{(1+at)^2}\left[1 + 3.\frac{(at)^1}{(1+at)} + 6.\frac{(at)^2}{(1+at)^2} + \ldots\ldots\right] - 1$$

$$= \frac{2}{(1+at)^2}\left[1 + 3.\left(\frac{at}{1+at}\right) + 6\left(\frac{at}{1+at}\right)^2. + \ldots\ldots\right] - 1$$

$$= \frac{2}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}(1+at)^3 - 1$$

$$= 2 + 2at - 1$$

$$= 1 + 2at$$

$$\therefore E[X^2(t)] = 1 + 2at$$

$$Var X(t) = E[X^2(t)] - (E[X(t)])^2$$

$$= 1 + 2at - 1^2$$

$$= 2at$$

If $\{X(t)\}$ is a stationary process, $\therefore E[X(t)]$ and $Var X(t)$ are constants.

Since $Var X(t)$ is a function of t, the given process is not stationary.

2. Verify if the sine wave process $\{X(t)\}$, where $\{X(t) = Y \cos \omega t\}$ and Y is uniformly distributed in $(0,1)$ is a strict sense stationary process.

Ans.

Given Y is uniformly distributed in $(0,1)$, the density

function of Y is $$f(y) = \frac{1}{1-0}, 0 < y < 1$$

$$i.e, f(y) = 1, \quad 0 < y < 1$$

$$E[X(t)] = E[Y \cos \omega t]$$

$$= \cos \omega t E[Y]$$

$$= \cos \omega t \int_0^1 y f(y) dy \qquad = \cos \omega t \int_0^1 y.1.dy$$

$$= \cos \omega t \left[\frac{y^2}{2} \right]_0^1$$

$$= \cos \omega t \left[\frac{1}{2} - 0 \right]$$

$$= \frac{1}{2} \cos \omega t, \quad \text{a function of t.}$$

If $\{X(t)\}$ is to be a SSS process, its mean must be a constant.

Therefore, $\{X(t)\}$ is not a strict sense stationary process.

3. Verify whether the random process $X(t) = A\cos(\omega t + \theta)$

· is wide sense stationary when A and ω are constants and

θ is uniformly distributed on the interval $\left(0, \frac{\pi}{2}\right)$.

Ans.

Given θ is uniformly distributed on the interval $\left(0, \frac{\pi}{2}\right)$,

the density function is given by $f(\theta) = \dfrac{1}{\frac{\pi}{2} - 0} = \dfrac{2}{\pi}, 0 < \theta < \dfrac{\pi}{2}$

$$E[X(t)] = E[A\cos(\omega t + \theta)]$$

$$= AE[\cos(\omega t + \theta)]$$

$$= A\int_0^{\frac{\pi}{2}} \cos(\omega t + \theta) f(\theta) d\theta$$

$$= A\int_0^{\frac{\pi}{2}} \cos(\omega t + \theta) \frac{2}{\pi} d\theta$$

$$= A\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(\omega t + \theta) d\theta$$

$$= A\frac{2}{\pi}\left[\sin(\omega t + \theta)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2A}{\pi}\left[\sin\left(\omega t + \frac{\pi}{2}\right) - \sin(\omega t + 0)\right]$$

$$= \frac{2A}{\pi}\left[\sin\omega t \cos\frac{\pi}{2} + \cos\omega t \sin\frac{\pi}{2} - \sin\omega t\right]$$

$$= \frac{2A}{\pi}\left[\sin\omega t(0) + \cos\omega t - \sin\omega t\right]$$

$$= \frac{2A}{\pi}\left[\cos\omega t - \sin\omega t\right], \quad \text{a function of } t$$

$E\left[X(t)\right]$ is a function of t. Therefore, the random process $X(t)$ is not a wide sense stationary process.

4. Show that the random process $X(t) = A\cos(\omega t + \theta)$ is wide sense stationary where A and ω are constants and θ is uniformly distributed on the interval $(0,2\pi)$

Ans.

Given θ is uniformly distributed on the interval $(0,2\pi)$

The density function is

$$f_\theta(\theta) = \frac{1}{2\pi - 0} = \frac{1}{2\pi} 0 < \theta < 2\pi$$

$$E[X(t)] = E[A\cos(\omega t + \theta)]$$

$$= A\int_0^{2\pi} \cos(\omega t + \theta)\frac{1}{2\pi}d\theta$$

$$= \frac{A}{2\pi} \int_0^{2\pi} \cos(\omega t + \theta) d\theta$$

$$= \frac{A}{2\pi} \left[\sin(\omega t + \theta)\right]_0^{2\pi} = \frac{A}{2\pi} \left[\sin(\omega t + 2\pi) - \sin(\omega t + 0)\right]$$

$$= \frac{A}{2\pi} \left[\sin \omega t \cos 2\pi + \cos \omega t \sin 2\pi - \sin \omega t\right]$$

$$= \frac{A}{2\pi} \left[\sin \omega t + 0 - \sin \omega t\right]$$

$$= \frac{A}{2\pi} [0]$$

$$= 0$$

$$E\left[X(t)\right] = 0 = \text{constant}$$

$$\text{Autocorrelation} = R_{XX}(t, t+\tau)$$

$$= E[X(t)X(t+\tau)]$$

$$= E[A\cos(\omega t + \theta)\, A\cos(\omega(t+\tau) + \theta)]$$

$$= A^2 \frac{1}{2} E[\cos(2\omega t + \omega\tau + 2\theta) + \cos(-\omega\tau)]$$

$$= A^2 \frac{1}{2}\cos\omega\tau + \frac{A^2}{2}\int_0^{2\pi} \cos(2\omega t + \omega\tau + 2\theta) f_\theta(\theta)\, d\theta$$

$$= A^2 \frac{1}{2}\cos\omega\tau + \frac{A^2}{2}\int_0^{2\pi} \cos(2\omega t + \omega\tau + 2\theta)\frac{1}{2\pi}\, d\theta$$

$$= A^2 \frac{1}{2}\cos\omega\tau + \frac{A^2}{4\pi}\int_0^{2\pi} \cos(2\omega t + \omega\tau + 2\theta)\, d\theta$$

$$= A^2 \frac{1}{2} \cos \omega\tau + \frac{A^2}{4\pi}(0) \quad \text{since} \quad \int_0^{2\pi} \cos(\theta + K) d\theta = 0$$

$$= A^2 \frac{1}{2} \cos \omega\tau$$

[a function of time difference (τ)]

Autocorrelation is a function of time difference (τ)

Therefore the process $\{X(t)\}$ is a wide sense stationary process.

5. Show that the process $X(t) = A \cos \lambda t + B \sin \lambda t$ where A and B are random variables is wide sense stationary if

$$(i) E[A] = E[B] = 0; (ii) E[A^2] = E[B^2]; (iii) E[AB] = 0$$

Ans.

Given $E[A] = E[B] = 0; E[AB] = 0$

$$E[A^2] = E[B^2] = k(say)$$

$$E[X(t)] = E[A\cos\lambda t + B\sin\lambda t]$$

$$= \cos\lambda t E[A] + \sin\lambda t E[B]$$

$$= \cos\lambda t(0) + \sin\lambda t(0)$$

$$= 0 \quad = \text{a constant}$$

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= E[(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)]$$

$$= E\left[\left(\begin{aligned} A^2 \cos\lambda t_1 \cos\lambda t_2 + AB\cos\lambda t_1 \sin\lambda t_2 \\ + AB\sin\lambda t_1 \cos\lambda t_2 + B^2 \sin\lambda t_1 \sin\lambda t_2 \end{aligned}\right)\right]$$

$$= \cos\lambda t_1 \cos\lambda t_2 E\left[A^2\right] + \cos\lambda t_1 \sin\lambda t_2 E[AB] + \sin\lambda t_1 \cos\lambda t_2 E[AB] + \sin\lambda t_1 \sin\lambda t_2 E\left[B^2\right]$$

$$= \cos\lambda t_1 \cos\lambda t_2 (k) + \cos\lambda t_1 \sin\lambda t_2 (0) + \sin\lambda t_1 \cos\lambda t_2 (0) + \sin\lambda t_1 \sin\lambda t_2 (k)$$

$$= k\left(\cos\lambda t_1 \cos\lambda t_2 + \sin\lambda t_1 \sin\lambda t_2\right)$$

$$= k\cos(\lambda t_1 - \lambda t_2) = k\cos\lambda(t_1 - t_2)$$

$$= \text{a function of } (t_1 - t_2)$$

Therefore the process $\{X(t)\}$ is a WSS process.

6. Two random processes $X(t)$ and $Y(t)$ are defined by

$$X(t) = A\cos\omega_0 t + B\sin\omega_0 t \quad \text{and} \quad Y(t) = B\cos\omega_0 t - A\sin\omega_0 t$$

Show that $X(t)$ and $Y(t)$ are jointly wide sense stationary, if A and B are uncorrelated RV's with zero means and the same variance and ω_0 is a constant.

Ans.

Given: $E[A] = E[B] = 0$ $\qquad \sigma_A^2 = \sigma_B^2$

$$E[A^2] - (E[A])^2 = E[B^2] - (E[B])^2$$

$$E[A^2] - 0 = E[B^2] - 0$$

$$E[A^2] = E[B^2] = k(say)$$

Since A and B are uncorrelated RVs, we have

$$E[AB] = E[A]E[B] = 0$$

By the previous problem, $\{X(t)\}$ and $\{Y(t)\}$ are individually WSS processes.

Now $R_{XX}(t_1, t_2) = E[X(t_1)Y(t_2)]$

$$= E[(A\cos\omega_0 t_1 + B\sin\omega_0 t_1)(B\cos\omega_0 t_2 - A\sin\omega_0 t_2)]$$

$$= E\begin{bmatrix} AB\cos\omega_0 t_1 \cos\omega_0 t_2 - A^2\cos\omega_0 t_1 \sin\omega_0 t_2 \\ + B^2\sin\omega_0 t_1 \cos\omega_0 t_2 - AB\sin\omega_0 t_1 \sin\omega_0 t_2 \end{bmatrix}$$

$$= \cos\omega_0 t_1 \cos\omega t_2 E[AB] - \cos\omega_0 t_1 \sin\omega_0 t_2 E[A^2]$$
$$+ \sin\omega_0 t_1 \cos\omega_0 t_2 E[B^2] - \sin\omega_0 t_1 \sin\omega_0 t_2 E[AB]$$

$$= \cos \omega_0 t_1 \cos \omega t_2 (0) - \cos \omega_0 t_1 \sin \omega_0 t_2 (k)$$
$$+ \sin \omega_0 t_1 \cos \omega_0 t_2 (k) - \sin \omega_0 t_1 \sin \omega_0 t_2 (0)$$

$$= k\left(- \cos \omega_0 t_1 \sin \omega_0 t_2 + \sin \omega_0 t_1 \cos \omega_0 t_2 \right)$$

$$= k \sin(\omega_0 t_1 - \omega_0 t_2)$$

$$= k \sin \omega_0 (t_1 - t_2) = \text{a function of } (t_1 - t_2)$$

Hence $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS processes

Markov process:

A random process $\{X(t)\}$ is said to be Markov process if for any $t_0 < t_1 < \ldots\ldots < t_n$ then

$$\forall n, P\left[X(t_n) \leq x_n \big/ X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \ldots\ldots X(t_0) = x_0 \right]$$
$$= P\left[X(t_n) \leq x_n \big/ X(t_{n-1}) = x_{n-1} \right]$$

Markov process is one in which the future behavior of the process depends only on the present value and not on the past value.

Example:

The probability of raining today depends on previous day weather and not on past day weather condition.

Remark:

Though the above definition holds for continuous Markov process, we are interested in the discrete state Markov processes known as Markov chain where the system can occupy only a finite or countable number of states.

Markov Chain:

Let $\{X(t)\}$ be a Markov process with states $X(t_r) = X_r$ $t_0 < t_1 < \ldots\ldots < t_n$

If $\forall n, P\left[{X_n = a_n}/{X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots\ldots X_0 = a_0}\right] = P\left[{X_n = a_n}/{X_{n-1} = a_{n-1}}\right]$

then the process $\{X_n\}$, $n=0,1,2,\ldots.$ is called a Markov chain, where $a_0, a_1, a_2 \ldots\ldots a_n$ are states of the Markov chain.

Transition Probabilities of Markov chain:

The conditional probability $P\left[X_n = a_j \big/ X_m = a_i\right] = P_{ij}(m,n)$

represents the probability that system goes in to state a_j at time $t = t_n$ given that it was in state a_i at time $t = t_m$ The numbers $P_{ij}(m,n)$ represents the transition probabilities of the Markov chain.

Remarks:

1. $P[X_m = a_i] = P_i(m)$ represents the probability that the system occupies the state a_i at time $t = t_m$

2. The probabilities $P[X_m = a_i] = P_i(m)$ and the transition probabilities $P_{ij}(m,n)$ completely determine the system, for if m < n < r then the joint probability

$$P[X_r = a_i, X_n = a_j, X_m = a_k] = P[X_r = a_i / X_n = a_j]$$
$$P[X_n = a_j / X_m = a_k]$$
$$P[X_m = a_k]$$

$$= P_{ji}(n,r) P_{kj}(m,n) P_k(m)$$

3. The conditional probability $P\left[X_n = a_j / X_0 = a_i\right]$ is called n-step transition probability and it is denoted by $P_{ij}^{(n)}$

One step Transition Probabilities:

The conditional probability $P\left[X_{n+1} = a_j \big/ X_n = a_i\right]$ is called the one step transition probability from state a_i at time t_n to state a_j at time t_{n+1} in one step. It is denoted by $P_{ij}(n, n+1)$

Homogeneous Markov chain:

A Markov chain is said to be homogeneous in time if one step transition probabilities are independent of the step

i.e $P_{ij}(n, n+1) = P_{ij}(m, m+1) \quad \forall\, m, n \quad and \quad i, j$

and it is denoted by P_{ij}

Transition Probability Matrix:(TPM)

$P = [P_{ij}]$ is a square matrix called the transition probability matrix (TPM) where P_{ij} 's are called one step transition probabilities.

Probability vector and Stochastic Matrix:

A vector (row matrix) $[P_1, P_2, \ldots\ldots P_n]$ is called a probability vector if

$$i)\ \ P_i \geq 0 \ \ \forall i \qquad ii)\ \ \sum_{i=1}^{n} P_i = 1$$

A square matrix $P = [P_{ij}]$ is called a stochastic matrix if each row of P is a probability vector. The TPM of a finite homogeneous Markov chain is a stochastic matrix.

Remark:

If A and B are stochastic matrices, then AB is a stochastic matrix. In particular A^n is a stochastic matrix for all n=2,3,….

State Probability distribution:

If the probability that the process is in state a_i is $p_i^{(n)}$ i=1,2,3….k at step 'n', then the row vector

$$P^{(n)} = (p_1^{(n)} p_2^{(n)} p_3^{(n)} \ldots\ldots p_k^{(n)})$$

is called state probability distribution of the process at time 't'.

The initial state probability distribution i.e. at t=0 is denoted by

$$P^{(0)} = (p_1^{(0)} p_2^{(0)} p_3^{(0)} \ldots\ldots p_k^{(0)})$$

Regular Matrix and Regular Markov chain:

A stochastic matrix is P said to be regular if for some positive integer 'm' all entries of the matrix P^m are positive.

A Markov chain is said to be regular if its one step transition probability matrix is regular.

Remarks:

1. A Markov chain is completely specified by the one-step transition probability matrix P and initial state probability distribution $P^{(0)}$ i.e. n^{th} state probability distribution.

$$P^{(1)} = P^{(0)} P$$

$$P^{(2)} = P^{(1)} P$$

$$P^{(n)} = P^{(n-1)} P$$

2. If a homogeneous Markov chain is regular, then every sequence of state probability distribution approach a fixed probability distribution

$$\pi = (\pi_1, \pi_2, \ldots. \pi_k) \quad \text{and} \quad \sum \pi_k = 1$$

where π is called **stationary distribution** (or) **steady state distribution** (or) **long run probability** of the Markov chain

i.e. $\pi = \lim_{n \to \infty} \{P^{(n)}\}$ where $P^{(n)} = (p_1^{(n)} p_2^{(n)} p_3^{(n)} \ldots\ldots p_k^{(n)})$

3. If P is TPM of regular Markov chain then $\pi P = \pi$

By using this property we can find steady state distribution (or) invariant probabilities π .

1. The transition probability matrix of a Markov chain $\{X_n\}$, three states 1,2 and 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $p^{(0)} = (0.7, 0.2, 0.1)$ Find (i) $P[X_2 = 3]$

(ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

Ans. By Chapman Kolmogorov equation

$$P^{(n)} = [P]^n$$

$$P^{(2)} = [P]^2$$

$$= \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

(i) $$P[X_2 = 3] = \sum_{i=1}^{3} P[X_2 = 3 / X_0 = i] P[X_0 = i]$$

$$= P[X_2 = 3 / X_0 = 1] P[X_0 = 1]$$
$$+ P[X_2 = 3 / X_0 = 2] P[X_0 = 1]$$
$$+ P[X_2 = 3 / X_0 = 1] P[X_0 = 3]$$

$$= p_{13}^{2} P[X_0 = 1] + p_{23}^{2} P[X_0 = 2] + p_{33}^{2} P[X_0 = 3]$$
$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1$$
$$= 0.182 + 0.068 + 0.029$$
$$= 0.279$$

(ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

$$= P[X_3 = 2/X_2 = 3, X_1 = 3, X_0 = 2]$$
$$P[X_2 = 3, X_1 = 3, X_0 = 2]$$

Since by definition of conditional probability

$$P(A \cap B) = P\left(A/_B\right)P(B)$$

$$= P[X_3 = 2/X_2 = 3]P[X_2 = 3/X_1 = 3, X_0 = 2].$$
$$P[X_1 = 3, X_0 = 2]$$

(by Markov property)

$$= P[X_3 = 2/X_2 = 3]P[X_2 = 3/X_1 = 3]$$
$$P[X_1 = 3/X_0 = 2]P[X_0 = 2]$$

$$= p_{32}^{(1)} \cdot p_{33}^{(1)} p_{23}^{(1)} \cdot P[X_0 = 2]$$

$$= 0.4 \times 0.3 \times 0.2 \times 0.2$$

$$= 0.0048$$

$$\therefore P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2] = 0.0048$$

2. A man either drives a car or catches a train to got to office each day. He never goes 2 days in a row by train, but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work iff a 6 appeared.

Find (i) the probability that he takes a train on the third day and

(ii) the probability that he drives to work in the long run

Ans.

The travel pattern is a Markov chain, with state space = (train,car)

The TPM of the chain is $P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(i) The initial state probability distribution is $p^{(1)} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}$

since

P(traveling by car)=P(getting 6 in the toss of the die)= $\frac{1}{6}$

P(traveling by train)= $\frac{5}{6}$

$$p^{(2)} = p^{(1)}P = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix}$$

$$p^{(3)} = p^{(2)} P = \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{11}{24} & \frac{13}{24} \end{pmatrix}$$

Therefore, P(the man travels by train on the third day)= $\frac{11}{24}$

(ii) Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution or stationary state distribution of the Markov chain.

By the property of π ,

$$\pi P = \pi \quad \Rightarrow \quad (\pi_1, \pi_2) \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$$

$$\frac{1}{2}\pi_2 = \pi_1 \text{ -------------------(1)}$$

$$\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \text{ -------------(2)}$$

Equations (1) and (2) are one and the same.

Therefore, consider (1) or (2) with $\pi_1 + \pi_2 = 1$ since π is a probability distribution.

$$\frac{1}{2}\pi_2 + \pi_2 = 1 \quad \Rightarrow \quad \frac{3}{2}\pi_2 = 1 \quad \Rightarrow \quad \pi_2 = \frac{2}{3}$$

$$\pi_1 + \frac{2}{3} = 1 \quad \Rightarrow \quad \pi_1 = 1 - \frac{2}{3} = \frac{1}{3}$$

Therefore, P[the man travels by car in the long run]= $\frac{2}{3}$

Classification of states of a Markov chain:

Accessible (or) Reachable state:

State 'j' is said to be reachable from state 'i' if the chain can reach 'j' in finite number of transition. i.e. $p_{ij}^{(n)} > 0$ for some integer n > 0

Irreducible Markov chain:

A Markov chain is said to be irreducible if every state is reachable from every other state.

i.e. $p_{ij}^{(n)} > 0$ for some n and for all 'i' and 'j'. otherwise the chain is said to be reducible.

The transition probability matrix of an irreducible chain is an irreducible matrix.

Return state:

State 'i' of a Markov chain is called a return state if $p_{ii}^{(n)} > 0$ for some $n>1$.

Absorbing state:

A state 'i' of a Markov chain is called an absorbing state if no other state is accessible from it.

Note:

1. For an absorbing state 'i' $p_{ii} = 1,\ p_{ij} = 0\ \ i \neq j$
2. Once the system enters the absorbing state, it gets trapped there.

Period of state:

The period d_i of a return state 'i' is defined as the greatest common divisor of all m such that $p_{ii}^{(m)} > 0$

$$i.e., \quad d_i = GCD\left\{m : p_{ii}^{(m)} > 0\right\}$$

State 'i' is said to be periodic with period d_i if $d_i > 1$ and aperiodic if $d_i = 1$

Obviously, state 'i' is aperiodic if $p_{ii} \neq 0$

First return time probability:

The probability that the chain returns to state 'i', having started from state 'i', for the first time at the n^{th} step is denoted by $f_{ii}^{(n)}$ and called the first return time probability or the recurrence time probability.

i.e., $f_{ii}^{(n)} = P[X_n = i, X_m \neq i, m = 1,2,\ldots n-1 \mid X_0 = i]$

$\{n, f_{ii}^{(n)}\} n = 1,2,3,\ldots$ is the distribution of recurrence times of the state 'i'.

If $F_{ii} = \sum_{n=1}^{\infty} f_{ii} = 1$, the return to state '$i$' is certain.

$\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$ is called the mean recurrence time of the state 'i'

Recurrent state:

A state 'i' is said to be persisent or recurrent if the return to state 'i' is certain i.e., if $F_{ii} = 1$

Transient state:

The state 'i' *is* said to be transient if the return to the state 'i' is uncertain,i.e.,if $F_{ii} < 1$

i.e. there is a +ve probability that the process will never return to state 'i' again after it leaves 'i'.

Non-null and null recurrent state:

The state 'i' *is* said to be nonnull persistent if its mean recurrence time μ_{ii} is finite and null persistent if $\mu_{ii} = \infty$

Ergotic state:

A nonnull recurrent and aperiodic state is called ergodic.

A Markov chain is said to be an ergotic chain if all its states are ergotic.

Remarks:

1. A state 'i' is recurrent iff $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$ and

   transient iff $\sum_{n=0}^{\infty} p_{ii}^{(n)}$ is finite.

2. If a Markov chain is irreducible then all its states are of same type.

   i.e. they are all transient (or) recurrent, all are periodic with same period (or) aperiodic.

3. A finite state Markov chain which is irreducible and aperiodic is ergodic.

4. In a finite Markov chain it is impossible to have all states are transient. Further if it is irreducible then all its states are recurrent and non-null.

Class:

Two state 'i' and 'j' which are accessible to each other are said to communicate if

i) state 'i' communicate with itself for all 'i'

ii) If state 'i' communicates with state 'j' and 'j' communicate with state 'k' then 'i' communicate with state 'k'

The set of all states communicating with each other form a class.

Here the relation communication is an equivalence relation.

Remarks:

- Two states that communicate are in the same class
- Two classes of states are either identical or disjoint.
- The Markov chain is irreducible if there is only one class. i.e. all states communicate with each other.

1. Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states.

Ans.

The transition probability matrix of the process $\{X_n\}$ is given below.

$$P = \begin{array}{cc} & \text{states of } X_n \\ \text{states of } X_{n-1} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

States of X_n depends only on states of X_{n-1} but not on states of X_{n-2}, X_{n-3}, ... or earlier states. Therefore, $\{X_n\}$ is a Markov chain.

$$\text{Now, } P^2 = P.P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}; \quad P^4 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \quad P^5 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}; \quad P^6 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{bmatrix}$$

and so on

$p_{11}^{(3)} > 0, p_{13}^{(2)} > 0, p_{21}^{(2)} > 0, p_{22}^{(2)} > 0, p_{33}^{(2)} > 0$ and all other

$p_{ij}^{(1)} > 0$. Therefore, the chain is irreducible.

We note that $p_{ii}^{(2)}, p_{ii}^{(3)}, p_{ii}^{(5)}, p_{ii}^{(6)}$ etc are >0 for i=2,3 and GCD of 2,3,5,6,....=1

Therefore, the states 2 and 3 are periodic with period 1 i.e., aperiodic

We note that $p_{11}^{(2)}, p_{11}^{(3)}, p_{11}^{(5)}, p_{11}^{(6)}$ etc., are>0 and GCD of 3,5,6,...=1.Therefore, the state 1 is periodic with period 1, i.e, aperiodic.

Since the chain is finite, aperiodic and irreducible all its states are nonnull persistent.

Moreover, all the states are ergodic.

Poisson process:

If $X(t)$ represents the number of occurrences of a certain event in *(0,t)*, then the discrete process $\{X(t)\}$ is called the Poisson process provided the following postulates are satisfied.

(i) $P[1\,occurence\,in\,(t,t+\Delta t)] = \lambda\Delta t + O(\Delta t)$

(ii) $P[0\,occurence\,in\,(t,t+\Delta t)] = 1 - \lambda\Delta t + O(\Delta t)$

(iii) $P[2\,or\,more\;\,occurences\,in\,(t,t+\Delta t)] = O(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior to and after the interval *(0,t)*

(v) The probability that the event occurs a specified number of times in $(t_0, t_0 + t)$ depends only on t, but not on t_0

Probability law of Poisson process:

Let λ be the rate of occurrence or the number of occurrences of the event in unit time and $P_n(t)$ be the probability of n occurrences of the event in the interval (0,t) is Poisson distribution with parameter λt

$$P_n(t) = P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0,1,2,.....\infty$$

Proof:

Let λ be the number of occurrences of the event in unit time

Let $P_n(t) = P[X(t) = n]$ ----------------------------------(1)

$$\therefore P_n(t + \Delta t) = P[X(t + \Delta t) = n]$$

$$= P[(n-1) occurence\, in\, (0,t)\, and\, 1\, occurence\, in\, (t + \Delta t)]$$

$$P[n\, occurences\, in\, (0,t)\, and\, 0\, occurence\, in\, (t + \Delta t)]$$

$$= P_{n-1}(t)\lambda\Delta t + P_n(t)(1 - \lambda\Delta t) \text{ ------------------------(2)}$$

(2)-(1) implies

$$\therefore P_n(t + \Delta t) - P_n(t) = P_{n-1}(t)\lambda\Delta t + P_n(t) - \lambda\Delta t P_n(t) - P_n(t)$$

$$= \lambda\Delta t[P_{n-1}(t) - P_n(t)]$$

$$\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = \lambda[P_{n-1}(t)-P_n(t)]$$

Taking limit as $\Delta t \to 0$,

$$\lim_{\Delta t\to 0}\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = \lambda[P_{n-1}(t)-P_n(t)]$$

$$\frac{d}{dt}P_n(t) = \lambda[P_{n-1}(t)-P_n(t)]$$

$$P_n^{'}(t) = \lambda[P_{n-1}(t)-P_n(t)] \text{ ------------------------(3)}$$

Let the solution of equation (3) be

$$P_n(t) = \frac{\lambda^n t^n}{n!} f(t) \text{ ----------------------------(4)}$$

Differentiating (4) with respect to t, we have,

$$P_n'(t) = \frac{\lambda^n}{n!}\left[nt^{n-1}f(t) + t^n f'(t)\right] \text{------------------}(5)$$

Using (4)and (5) in (3), we have,

$$\frac{\lambda^n}{n!}\left[nt^{n-1}f(t) + t^n f'(t)\right] = \lambda\left[\frac{\lambda^{n-1}t^{n-1}}{(n-1)!}f(t) - \frac{\lambda^n t^n}{n!}f(t)\right]$$

$$\frac{\lambda^n}{n!}nt^{n-1}f(t) + \frac{\lambda^n t^n}{n!}f'(t) = \frac{\lambda\lambda^{n-1}t^{n-1}}{(n-1)!}f(t) - \frac{\lambda\lambda^n t^n}{n!}f(t)$$

$$\frac{\lambda^n}{n(n-1)!}nt^{n-1}f(t) + \frac{\lambda^n t^n}{n!}f'(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}f(t) - \frac{\lambda\lambda^n t^n}{n!}f(t)$$

$$\frac{\lambda^n t^{n-1}}{(n-1)!} f(t) + \frac{\lambda^n t^n}{n!} f'(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} f(t) - \frac{\lambda \lambda^n t^n}{n!} f(t)$$

$$\frac{\lambda^n t^n}{n!} f'(t) = -\lambda \frac{\lambda^n t^n}{n!} f(t)$$

$$\frac{f'(t)}{f(t)} = -\lambda$$

Integrating with respect to t, we have

$$\int \frac{f'(t)}{f(t)} dt = -\lambda \int dt$$

$$\log f(t) = -\lambda t + \log k$$

$$\log f(t) - \log k = -\lambda t$$

$$\log \frac{f(t)}{k} = -\lambda t$$

$$\frac{f(t)}{k} = e^{-\lambda t}$$

$$f(t) = ke^{-\lambda t} \quad \text{-----------}(6)$$

Substituting $n=0$ in (4), we have

$$P_0(t) = \frac{\lambda^0 t^0}{0!} f(t)$$

$$P_0(t) = f(t) \quad \text{-----------}(7)$$

Substituting $t=0$ in (7), we have

$$P_0(0) \quad = f(0)$$

$$f(0) \quad = P[X(0)=0]$$

$$= P[no\ event\ occurs\ in\ (0,0)]$$

$$=1 \text{---------------} (8)$$

Substituting $t=0$ in (6), we have

$$f(0) \quad = ke^{-\lambda(0)}$$

$$f(0) \quad = k \qquad \Rightarrow k=1$$

Therefore, $f(t) = 1.e^{-\lambda t} = e^{-\lambda t}$ --------------(9)

Using (9) in (4), we have

$$P_n(t) = \frac{\lambda^n t^n}{n!} e^{-\lambda t}$$

Thus the probability distribution of $X(t)$ is the Poisson distribution with parameter λt

Remarks:

1. We have assumed that the rate of occurrence of the event λ is constant , but it can be function of t also. When λ is constant, the process is called **homogeneous Poisson process**. Unless specified otherwise, the Poisson process will be assumed homogeneous.

2. The probability law of the poisson process $\{X(t)\}$ is same as that of a poisson distribution with parameter λt. Therefore mean and variance are same.

$$E[X(t)] = Var[X(t)] = \lambda t$$

3. Let $\{X(t)\}$ be a Poisson process with parameter λ then the Second order- probability function of a Poisson process

$$P[X(t_1) = n_1, X(t_2) = n_2]$$

$$= \begin{cases} \dfrac{e^{-\lambda t_2} \lambda^{n_2} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1}}{n_1!(n_2 - n_1)!}, & n_2 \geq n_1 \\ 0 & , \textit{otherwise} \end{cases}$$

Similarly third order- probability function of a Poisson process

$$P[X(t_1)=n_1, X(t_2)=n_2, X(t_3)=n_3]$$

$$= \begin{cases} \dfrac{e^{-\lambda t_3}\lambda^{n_3} t_1^{n_1}(t_2-t_1)^{n_2-n_1}(t_3-t_2)^{n_3-n_2}}{n_1!(n_2-n_1)!(n_3-n_2)!}, & n_3 \geq n_2 \geq n_1 \\ 0, & otherwise \end{cases}$$

Properties of Poisson process:

1. Prove that the sum of two independent Poisson processes is also Poisson processes.

Ans.

Let $X(t) = X_1(t) + X_2(t)$ where $X_1(t)$ *and* $X_2(t)$ be two independent Poisson process with parameter λ_1 *and* λ_2

$$P[X(t) = n] = P[X_1(t) + X_2(t) = n]$$

$$= \sum_{r=0}^{n} P[X_1(t) = r] P[X_2(t) = n - r]$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{r!} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= e^{-\lambda_1 t} e^{-\lambda_2 t} \sum_{r=0}^{n} \frac{\lambda_1^r t^r \lambda_2^{n-r} t^{n-r}}{r!(n-r)!}$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{n} \frac{nC_r}{n!} t^n \lambda_1^r \lambda_2^{n-r}$$

$$= e^{-(\lambda_1+\lambda_2)t} \frac{t^n}{n!} \left[\lambda_2^n + nC_1 \lambda_2^{n-1} \lambda_1 + nC_2 \lambda_2^{n-2} \lambda_1^2 + \ldots\ldots\ldots \lambda_1^n\right]$$

$$= e^{-(\lambda_1+\lambda_2)t} \frac{t^n}{n!} (\lambda_1 + \lambda_2)^n$$

$$= e^{-(\lambda_1+\lambda_2)t} \frac{((\lambda_1+\lambda_2)t)^n}{n!}$$

Therefore, $X_1(t) + X_2(t)$ is a Poisson process with parameter $(\lambda_1 + \lambda_2)t$

2. Prove that the difference of two independent Poisson processes is not a Poisson process.

Ans.

Let $X(t) = X_1(t) - X_2(t)$ where $X_1(t),\ X_2(t)$ be two independent Poisson process with parameter is λ_1 *and* λ_2

$$E[X(t)] = E[X_1(t) - X_2(t)]$$

$$= E[X_1(t)] - E[X_2(t)]$$

$$= \lambda_1 t - \lambda_2 t$$

$$= (\lambda_1 - \lambda_2)t$$

$$E\left[X^2(t)\right]=E\left[\left(X_1(t)-X_2(t)\right)^2\right]$$

$$=E\left[X_1^{\,2}(t)\right]+E\left[X_2^{\,2}(t)\right]-2E\left[X_1\ (t)\right]E\left[X_2\ (t)\right]$$

(by independence)

$$=(\lambda_1^2\,t^2+\lambda_1\,t)+(\lambda_2^2\,t^2+\lambda_2\,t)-2(\lambda_1\,t)\ (\lambda_2\,t)$$

$$=(\lambda_1+\lambda_2)t+(\lambda_1-\lambda_2)^2\,t^2$$

$$\neq(\lambda_1-\lambda_2)t+(\lambda_1-\lambda_2)^2\,t^2$$

Since $E\left[X^2(t)\right]=\lambda\ t+\lambda^2\,t^2$ for a Poisson process $X(t)$ with parameter λ

Therefore $X(t)=X_1(t)-X_2(t)$ is not a poisson process

3. Mean and Variance of Poisson process

Ans.

$$E[X(t)] = \sum_{n=0}^{\infty} nP_n(t) \quad = \sum_{n=0}^{\infty} n \ \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} n \ \frac{(\lambda t)^n}{n(n-1)!}$$

$$= e^{-\lambda t}[\lambda t + \frac{(\lambda t)^2}{1!} + \frac{(\lambda t)^3}{2!} + \ldots\ldots]$$

$$= e^{-\lambda t} \lambda t[1 + \frac{(\lambda t)}{1!} + \frac{(\lambda t)^2}{2!} + \ldots\ldots]$$

$$= e^{-\lambda t} \lambda t \, e^{\lambda t}$$

$$= \lambda t$$

Therefore Mean= $E[X(t)] = \lambda t$ -------(1)

$$E\left[X^2(t)\right] = \sum_{n=0}^{\infty} n^2 P_n(t) = \sum_{n=0}^{\infty} [n(n-1)+n] P_n(t)$$

$$= \sum_{n=0}^{\infty} n(n-1) P_n(t) + \sum_{n=0}^{\infty} n P_n(t)$$

$$= \sum_{n=0}^{\infty} [n(n-1) P_n(t)] + \lambda t \qquad (by\ (1))$$

$$= \sum_{n=0}^{\infty} n(n-1) e^{-\lambda t} \frac{(\lambda t)^n}{n!} + \lambda t$$

$$= e^{-\lambda t} \sum_{n=2}^{\infty} n(n-1) \frac{(\lambda t)^n}{n!} + \lambda t$$

$$= e^{-\lambda t} \sum_{n=2}^{\infty} \frac{(\lambda t)^n}{(n-2)!} + \lambda t$$

$$= e^{-\lambda t}[(\lambda t)^2 + \frac{(\lambda t)^3}{1!} + \frac{(\lambda t)^4}{2!} + ...] + \lambda t$$

$$= e^{-\lambda t}(\lambda t)^2[1 + \frac{(\lambda t)}{1!} + \frac{(\lambda t)^2}{2!} + ...] + \lambda t$$

$$= e^{-\lambda t}(\lambda t)^2 e^{\lambda t} + \lambda t$$

$$= (\lambda t)^2 + \lambda t$$

$$E\left[X^2(t)\right] = (\lambda t)^2 + \lambda t$$

$$Var(X(t)) = E\left[X^2(t)\right] - \{E\left[X(t)\right]\}^2$$

$$= (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t$$

4. Let $X(t)$ be a Poisson process with rate λ. Find Auto correlation, and auto covariance

Ans.

Auto correlation :

$$R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$

$$= E\left[X(t_1)\left(X(t_2) - X(t_1) + X(t_1)\right)\right]$$

$$= E\left[X(t_1)\left(X(t_2) - X(t_1)\right)\right] + E\left[X^2(t_1)\right]$$

$$= E\left[X(t_1)\right]E\left[X(t_2) - X(t_1)\right] + E\left[X^2(t_1)\right]$$

$$= E\left[X(t_1)\right]E\left[X(t_2 - t_1)\right] + E\left[X^2(t_1)\right]$$

$$= \lambda t_1(\lambda(t_2 - t_1) + \lambda t_1 + \lambda^2 t_1{}^2$$

$$= \lambda^2 t_1 t_2 - \lambda^2 t_1{}^2 + \lambda^2 t_1{}^2 + \lambda t_1$$

$$= \lambda^2 t_1 t_2 + \lambda t_1$$

$$\therefore E[X(t_1)X(t_2)] = \lambda^2 t_1 t_2 + \lambda t_1 \qquad for\ t_2 \geq t_1$$

$$= \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2) \quad for\, any\ t_2, t_1$$

Auto covariance:

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - E[X(t_1)X(t_2)]$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda t_1 \lambda t_2 \quad = \lambda t_1$$

$$\therefore \quad C_{XX}(t_1, t_2) = \lambda t_1 \qquad for\ t_2 \geq t_1$$

$$= \lambda \min(t_1, t_2) \quad for\, any\ t_2, t_1$$

5. The inter-arrival time of a Poisson process (i.e) the interval between two successive occurrences of Poisson process with parameter λ has an exponential distribution with mean $1/\lambda$

Ans.

The two consecutive occurrences of the event be $E_i \; and \; E_{i+1}$

Let E_i take place at time instant t_i and T be the interval between the occurrences of $E_i \; and \; E_{i+1}$. Therefore T is a continuous RV

$$P[T > t] = P\{no\ event\ occurs\ in\ an\ \mathrm{int}erval\ of\ length\ t\}$$

$$= P\{X(t) = 0\}$$

$$= e^{-\lambda t}$$

Therefore, the cdf of T is given by

$$F(t) = P[T \le t]$$
$$= 1 - P[T > t]$$
$$= 1 - e^{-\lambda t}$$

Pdf of T is f(t)

$$f(t) = \frac{d}{dt}\{F(t)\} = \lambda e^{-\lambda t} \quad t \ge 0$$

which is the pdf of an exponential RV with mean $1/\lambda$

Therefore the inter-arrival time of a Poisson process with parameter λ has an exponential distribution with mean $1/\lambda$

6. Poisson process is a Markov Process:

Ans.

Consider

$$P[X(t_3)=n_3 / X(t_2)=n_2, X(t_1)=n_1]$$

$$=\frac{P[X(t_3)=n_3, X(t_2)=n_2, X(t_1)=n_1]}{P[X(t_2)=n_2, X(t_1)=n_1]}$$

$$=\frac{\dfrac{e^{-\lambda t_3}\lambda^{n_3} t_1^{n_1}(t_2-t_1)^{n_2-n_1}(t_3-t_2)^{n_3-n_2}}{n_1!(n_2-n_1)!(n_3-n_2)!}}{\dfrac{e^{-\lambda t_2}\lambda^{n_2} t_1^{n_1}(t_2-t_1)^{n_2-n_1}}{n_1!(n_2-n_1)!}}$$

$$= \frac{e^{-\lambda t_3} \lambda^{n_3} (t_3 - t_2)^{n_3 - n_2}}{e^{-\lambda t_2} \lambda^{n_2} (n_3 - n_2)!}$$

$$= \frac{e^{-\lambda(t_3 - t_2)} \lambda^{n_3 - n_2} (t_3 - t_2)^{n_3 - n_2}}{(n_3 - n_2)!}$$

$$= P[X(t_3) = n_3 / X(t_2) = n_2]$$

Therefore Poisson process is Markov process

1. Suppose that customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute. Find the probability that during a time interval of 2 mins

   (a) exactly 4 customers arrive

   (b) more than 4 customers arrive

Ans.

Mean of the Poisson process = λt

Mean arrival rate=mean number of arrivals per minute = λ

Given $\lambda = 3$

$$P[X(t) = k] = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

(a) $P\left[X(2)=4\right] = \dfrac{e^{-3(2)}(3.2)^4}{4!}$

$$= \frac{e^{-6}6^4}{4!} = 0.133$$

(b) $P\left[X(2)>4\right]$

$$= 1 - P\left[X(2) \le 4\right]$$

$$= 1 - \left\{ \begin{array}{l} P\left[X(2)=0\right] + P\left[X(2)=1\right] \\ + P\left[X(2)=2\right] + P\left[X(2)=3\right] + P\left[X(2)=4\right] \end{array} \right\}$$

$$= 1 - \left[\frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!} + \frac{e^{-6}(6)^4}{4!} \right]$$

$$= 0.715$$

2. A machine goes out of order whenever a component fails. The failure of this part follows a Poisson process with a mean rate of 1 per week. Find the probability that 2 weeks have elapsed since last failure. If there are 5 spare parts of this component in an inventory and that the next supply is not due in 10 weeks, find the probability that the machine will not be out of order in the next 10 weeks.

Ans.

Here the unit of time is 1 week.

Mean failure rate=mean number of failures in a week = $\lambda = 1$

P[no. of failures in the 2 weeks since last failure]

$$= P[X(2) = 0]$$

$$= \frac{e^{-2\lambda}(2\lambda)^0}{0!} \quad = e^{-2(1)}$$

$$= e^{-2} = 0.135$$

There are only 5 spare parts and the machine should not go out of order in the next 10 weeks.

$$P[for\ this\ event] = P[X(10) \leq 5]$$

$$= \frac{e^{-10}(10)^0}{0!} + \frac{e^{-10}(10)^1}{1!} + \frac{e^{-10}(10)^2}{2!} \frac{e^{-10}(10)^3}{3!} \frac{e^{-10}(10)^4}{4!} \frac{e^{-10}(10)^5}{5!}$$

$$= e^{-10}\left[1 + 10 + \frac{100}{2} + \frac{1000}{6} + \frac{10000}{24} + \frac{100000}{120}\right]$$

$$= 0.068$$

3. If customer arrive at a counter in accordance with a Poisson process with mean rate of 2 per min, find the probability that the interval between 2 consecutive arrivals is

(i) More than 1 min (ii) between 1 min and 2 min

(iii) 4 min or less

Ans.

By Property of Poisson process, the interval T between two consecutive arrivals follows an exponential distribution

Given $\lambda = 2$

For an exponential distribution

Pdf of T is f(t) $= \lambda e^{-\lambda t} \quad t \geq 0$ => $f(t) = 2e^{-2t}, \ t \geq 0$

$P(T > t) = e^{-\lambda t} = e^{-2t}$

$$(i) P(T > 1) = e^{-2} = 0.135$$

$$(ii) P(1 < T < 2) = \int_1^2 f(t)dt = \int_1^2 2e^{-2t} dt = 2\left(\frac{e^{-2t}}{-2}\right)$$

$$= -e^{-4} + e^{-2} = 0.117$$

$$(iii) P(T \leq 4) = 1 - P(T > 4) = 1 - e^{-2(4)}$$

$$= 1 - e^{-8}$$

$$= 0.999$$

UNIT IV QUEUEING MODELS

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INTRODUCTION

There are many situations in daily life when a queue is formed.

For example

- machines waiting to be repaired
- patients waiting in a Doctor's room
- cars waiting at a traffic signal and
- passengers waiting to buy tickets in counters form queues.

CHARACTERISTICS OF QUEUEING PROCESSES

In most cases, six basic characteristics of queueing processes provide an adequate description of a queueing system:

- **arrival pattern of customers**
- **service pattern of servers**
- **queue discipline**
- **system capacity**
- **number of service channels and**
- **number of service stages.**

SYMBOLIC REPRESENTATION OF A QUEUEING MODEL
(Kendall Notation)

A queueing process is described by a series of symbols and slashes such as **A/B/X/Y/Z**, where

A - indicates the type of distribution of the number of arrivals per unit time,

B - the type of distribution of the service time

X - the number of servers

Y - the capacity of the system

Z - the queue discipline.

BIRTH AND DEATH PROCESS

Definition

If $X(t)$ represents the number of individuals present at time t in a population (or the size of the population at time t) in which two types of events occur – one representing birth which contributes to its increase and the other representing death which contributes to its decrease, then the discrete random process $\{X(t)\}$ is called the birth and death process, provided the two events, viz., birth and death are governed by the following postulates:

If $X(t) = n \ (n > 0)$,

(i) $P[1 \text{ birth in } (t, t+\Delta t)] = \lambda_n(t)\Delta t + O(\Delta t)$

(*ii*) $P[0 \text{ birth in } (t, t+\Delta t)] = 1 - \lambda_n(t)\Delta t + O(\Delta t)$

(iii) $P[2 \text{ or more births in } (t, t+\Delta t)] = O(\Delta t)$

(iv) Births occuring in $(t, t+\Delta t)$ are independent of time since last birth.

(v) $P[1 \text{ death in } (t, t+\Delta t)] = \mu_n(t)\Delta t + O(\Delta t)$

(*vi*) $P[0 \text{ death in } (t, t+\Delta t)] = 1 - \mu_n(t)\Delta t + O(\Delta t)$

(vii) $P[2 \text{ or more deaths in } (t, t+\Delta t)] = O(\Delta t)$

(viii) Deaths occuring in $(t, t+\Delta t)$ are independent of time since last death.

(ix) The birth and death occur independently of each other at any time.

Difference – differential equations related to Birth and Death Process:

$$P_n'(t) = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t)$$

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$$

and

$$\mathrm{P_n} = \frac{\lambda_0.\lambda_1.\lambda_2\ldots\lambda_{n-1}}{\mu_1.\mu_2.\mu_3\ldots\mu_n} P_0 \ \ for \ \ n = 1,2,3\ldots$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0.\lambda_1.\lambda_2\ldots\lambda_{n-1}}{\mu_1.\mu_2.\mu_3\ldots\mu_n}}.$$

Little's formula

(i) $L_s = \lambda W_s$

(ii) $L_q = \lambda W_q$

(iii) $L_s = L_q + \dfrac{\lambda}{\mu}$

(iv) $W_q = W_s - \dfrac{1}{\mu}$

CHARACTERISTICS OF INFINITE CAPACITY, SINGLE SERVER POISSON QUEUE MODEL

MODEL: I : $[(M/M/1):(\infty/FIFO)]$, when $\lambda_n = \lambda$ and $\mu_n = \mu$ $(\lambda < \mu)$

By birth and death process $P(N=n) = P_n = \dfrac{\lambda\,\lambda\,\lambda.......ntimes}{\mu\,\mu\,\mu........ntimes} P_0$

$$= \frac{\lambda^n}{\mu^n} P_0 = \left(\frac{\lambda}{\mu}\right)^n P_0$$

Now sum of all probabilities $= 1$

$$\Rightarrow P_0 + P_1 + P_2 + ... = 1$$

$$\Rightarrow P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \left(\frac{\lambda}{\mu}\right)^3 P_0 + ... = 1$$

$$\Rightarrow P_0\left(1+\left(\frac{\lambda}{\mu}\right)+\left(\frac{\lambda}{\mu}\right)^2+\left(\frac{\lambda}{\mu}\right)^3+\ldots\ldots\ldots\right)=1$$

$$\Rightarrow P_0\left(1-\frac{\lambda}{\mu}\right)^{-1}=1\Rightarrow P_0=1-\frac{\lambda}{\mu}$$

1. **Average number Ls of customers in the system:**

$$Now\ L_s=E(N_s)=\sum_{n=0}^{\infty} n\times P_n$$

$$=\sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$$=\left(1-\frac{\lambda}{\mu}\right)\sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left\{0 + \frac{\lambda}{\mu} + 2.\left(\frac{\lambda}{\mu}\right)^2 + 3\left(\frac{\lambda}{\mu}\right)^3 + \ldots\right\}$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\left(1 - \frac{\lambda}{\mu}\right)^{-2}, \text{ by binomial summation}$$

$$= \left(1 - \frac{\lambda}{\mu}\right)^{-1}\left(\frac{\lambda}{\mu}\right)$$

$$= \frac{\lambda}{\mu - \lambda}.$$

2. Average number Lq of customers in the queue or Average length of the queue:

Using Little's formula, we have

$$L_q = E(N_q) = L_s - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu - \lambda)}.$$

3. Average waiting time of a customer in the system:

Using Little's formula, we have

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu - \lambda}$$

4. Average waiting time of a customer in the queue:

Using Little's formula, we have

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu - \lambda)}$$

5. Average number of customers in the non-empty queue:

$$L_w = \frac{\mu}{\mu - \lambda}$$

6. Probability that the number of customers in the system exceeds 'k'

$$P(N > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

7. Probability density function (PDF) of the waiting time in the system

$$f(w) = (\mu - \lambda)e^{-(\mu-\lambda)w}$$

8. Probability that the waiting time of a customer in the system exceeds 't'

$$P(W_s > t) = e^{-(\mu-\lambda)t}$$

9. Probability density function of the waiting time of a customer in the system exceeds 't'

$$g(w) = \begin{cases} \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)w} & , w > 0 \\ 1-\frac{\lambda}{\mu} & , w = 0 \end{cases}$$

10. Average waiting time of customers in the queue, if he has to wait

$$E(W_q / W_q > 0) = \frac{1}{\mu-\lambda}$$

Example 1:

Arrivals at a telephone booth are considered to be Poisson with an average time of 12 min. between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 4 min.

(a) Find the average number of persons waiting in the system.

(b) What is the probability that a person arriving at the booth will have to wait in the queue?

(c) What is the probability that it will take him more than 10 min. altogether to wait for the phone and complete his call?

Solution:

$$\text{Mean inter - arrival time} = \frac{1}{\lambda} = 12 \text{ min}$$

$$\text{Mean arrival rate} = \lambda = \frac{1}{12} \text{ per minute}$$

$$\text{Mean service time} = \frac{1}{\mu} = 4 \text{ min}$$

$$\therefore \textit{Service rate} = \mu = \frac{1}{4} \textit{ per} \text{ min}$$

$$(a)\ L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{12}}{\frac{1}{4} - \frac{1}{12}} = 0.5 \textit{ customer}$$

$$(b)\ P(W > 0) = 1 - P(W = 0) = 1 - P(\textit{no customer in the system})$$

$$= 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}.$$

$$\textbf{(c) } Wkt,\ P\left(W_s > t\right) = e^{-(\mu-\lambda)t}$$

$$P\left(W > 10\right) = e^{-(\mu-\lambda)\times 10}$$

$$= e^{-5/3} = 0.1889.$$

Example 2:

Customers arrive at a one – man barber shop according to a Poisson process with a mean inner arrival time of 20 min. Customers spend an average of 15 min. in the barber chair. If an hour is used as a unit time, then

(i) What is the probability that a customer need not wait for a hair cut?

(ii) What is the expected number of customers in the barber shop and in the queue?

(iii) How much time can a customer expect to spend in the barber shop?

(iv) Find the average time that the customers spend in the queue.

Solution:

$$\text{Inter arrival time } 20 \text{ min} \Rightarrow \textit{for } 20 \text{ min} = 1 \textit{customer}$$

$$\therefore \textit{ for } 1 \text{min} = \frac{1}{20} \textit{customer} \quad \Rightarrow \lambda = \frac{1}{20} \textit{per} \text{ min}.$$

15 min. service time for 1 customer .

$$\therefore \quad \textit{for } 1 \text{ min} = \frac{1}{15} \textit{ customer} \quad \Rightarrow \mu = \frac{1}{15} \textit{ per } \text{ min}.$$

(i) If no customer in the system, the customer no need to wait

$$\therefore \quad P(N = 0) = 1 - \frac{\lambda}{\mu} = 1 - \frac{3}{4} = \frac{1}{4}.$$

(ii) Barber shop + queue

$$\therefore \quad E(N_s) = L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{20}}{\frac{1}{15} - \frac{1}{20}} = 3.$$

$$(iii)\ \textit{Time to spend in the barber shop (system)}$$

$$\therefore\ W_s = \frac{1}{\mu-\lambda} = \frac{1}{\frac{1}{15}-\frac{1}{20}} = 60 \text{ min}.$$

$$(iv)\ W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\frac{1}{20}}{\frac{1}{15}\left(\frac{1}{15}-\frac{1}{20}\right)} = 45 \text{ min}.$$

CHARACTERISTICS OF INFINITE CAPACITY, MULTIPLE SERVER POISSON QUEUE MODEL

MODEL II $\left[(M/M/s):(\infty/FIFO)\right]$, when $\lambda_n = \lambda$ for all n $(\lambda < s\mu)$

1. Values of P_0 and P_n :

$$P_n = \begin{cases} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n P_0, \text{ if } 0 \le n < s \\[2ex] \dfrac{1}{s!\,s^{n-s}}\left(\dfrac{\lambda}{\mu}\right)^n \times P_0, \text{ if } n \ge s \end{cases}$$

$$P_0 = \frac{1}{\left\{\displaystyle\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\dfrac{1}{s!\left(1-\dfrac{\lambda}{\mu s}\right)}\left(\dfrac{\lambda}{\mu}\right)^s\right\}}.$$

2. Average number of customers in the queue or average queue length

$$L_q = E(N_q) = E(N-s) = \sum_{n=s}^{\infty}(n-s)P_n$$

$$= 0.P_s + 1.P_{s+1} + 2.P_{s+2} + \ldots$$

$$= 0 + \frac{1}{s!s}\left(\frac{\lambda}{\mu}\right)^{s+1} \times P_0 + \frac{2}{s!s^2}\left(\frac{\lambda}{\mu}\right)^{s+2} \times P_0 + \ldots$$

$$= \frac{1}{s!s}\left(\frac{\lambda}{\mu}\right)^{s+1} \times P_0\left\{1 + 2\left(\frac{\lambda}{\mu s}\right) + 3\left(\frac{\lambda}{\mu s}\right)^2 + \ldots\right\}$$

$$= \frac{1}{s!s}\left(\frac{\lambda}{\mu}\right)^{s+1} \times P_0 \times \left(1 - \frac{\lambda}{\mu s}\right)^{-2}$$

$$= \frac{1}{s!s} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^{2}} P_0 .$$

3. Average number of customers in the system

By little's formula,

$$L_s = L_q + \frac{\lambda}{\mu}$$

$$= \frac{1}{s!s} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^{2}} P_0 + \frac{\lambda}{\mu}$$

4. Average time a customer has to spend in the system

By Little's formula,

$$E(W_s) = \frac{L_s}{\lambda}$$

$$= \frac{1}{\mu} + \frac{1}{\mu}\frac{1}{s!s}\cdot\frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1-\frac{\lambda}{\mu s}\right)^2}P_0$$

5. Average time a customer has to spend in the queue

By Little's formula,

$$E(W_q) = \frac{L_q}{\lambda}$$

$$= \frac{1}{\mu}\frac{1}{s!s}\cdot\frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1-\frac{\lambda}{\mu s}\right)^2}P_0$$

6. Probability that an arrival has to wait

$$P(W_s > 0) = P(N \geq s) = \frac{1}{s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1 - \frac{\lambda}{\mu s}\right)} P_0$$

7. Probability that an arrival enters the service without waiting

$$\text{Required probability} = 1 - P(\text{an arrival has to wait})$$

$$= 1 - \frac{1}{s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1 - \frac{\lambda}{\mu s}\right)} P_0$$

8. Mean waiting time in the queue for those who actually wait:

$$E\left(W_q / W_s\right) = \frac{E(W_q)}{P\left(W_s > 0\right)}$$

$$= \frac{1}{\mu s - \lambda}$$

9. Probability that there will be someone waiting:

$$\text{Required probability} = P\left(N \geq s+1\right)$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!} \cdot \frac{\left(\frac{\lambda}{\mu s}\right)}{\left(1 - \frac{\lambda}{\mu s}\right)}$$

10. Average number of customers (in non-empty queues), who have to actually wait:

$$L_w = E(N_q / N_q \geq 1) = \frac{E(N_q)}{P(N \geq s)}$$

$$= \frac{\left(\frac{\lambda}{\mu s}\right)}{1 - \frac{\lambda}{\mu s}}.$$

11. Probability that the waiting time of a customer in the system exceeds 't':

$$P(W > t) = e^{-\mu t}\left\{1 + \frac{\left(\frac{\lambda}{\mu}\right)^s \left[1 - e^{-\mu t\left(s-1-\frac{\lambda}{\mu}\right)}\right] P_0}{s!\left(1 - \frac{\lambda}{\mu s}\right)\left(s - 1 - \frac{\lambda}{\mu}\right)}\right\}$$

Example 1:

There are three typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour,

(a) What fraction of the time all the typists will be busy?

(b) What is the average number of letters waiting to be typed?

(c) What is the average time a letter has to spend for waiting and for being typed?

Solution:

Ther are three typists and No restrictioin about the accommodation of customers.

$$\therefore (M/M/3):(\infty/FIFO) \text{ Model}.$$

Given $\lambda = 15\ /\text{hr.}\ \ \&\ \ \mu = 6\ /\text{hr},\ \ s = 3$

$$\therefore \frac{\lambda}{\mu} = 2.5 \quad \& \quad \frac{\lambda}{s\mu} = \frac{15}{18} = \frac{5}{6}.$$

(a) All the typists will be busy if there are atleast 3 customers (letters) in the system.

$$\therefore P(N \geq 3) = \frac{1}{3!} \cdot \frac{(2.5)^3}{\left(1 - \frac{5}{6}\right)} P_0$$

$$= \frac{1}{6} \frac{(2.5)^3}{\left(1 - \frac{5}{6}\right)} P_0 = (2.5)^3 P_0 ------(1)$$

$$\text{Now } P_0 = \frac{1}{\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right\}}$$

$$= \frac{1}{\left\{\sum_{n=0}^{2} \frac{1}{n!}(2.5)^n\right\} + \left\{\frac{1}{3!\left(1-\frac{5}{6}\right)}(2.5)^3\right\}}$$

$$= \frac{1}{\left\{1+2.5+\frac{1}{2}\times(2.5)^2\right\} + (2.5)^3}$$

$$= \frac{1}{22.25} = 0.0449-----(2)$$

$$\text{Using (2) in (1), we have } P(N \geq 3) = (2.5)^3 \times (0.0449)$$
$$= 0.7016.$$

(*b*) *Waiting to be types (queue)*

$$L_q = \frac{1}{3 \times 6} \times \frac{(2.5)^4}{\left(1 - \frac{5}{6}\right)^2} \times 0.0449 = 3.5078.$$

$$\text{(c) } E(W) = \frac{1}{\lambda} E(N), \text{ by Little's formula}$$

$$W_s = \frac{1}{\lambda}\left[L_q + \frac{\lambda}{\mu}\right]$$

$$= \frac{1}{15}[3.5078 + 2.5] = 0.4005 \text{ hr}$$

Example 2:

A petrol pump station has 4 pumps. The service times follow the exponential distribution with a mean of 6 min and cars arrive for service in a Poisson process at the rate of 30 cars per hour.

(a) What is the probability that an arrival would have to wait in line?

(b) Find the average waiting time, average time spent in the system and the average number of cars in the system.

(c) For what percentage of time would a pump be idle on an average?

Solution:

$$\text{Model}:(M/M/4):(\infty/FIFO)$$

$$\text{Given } \lambda = 4\ /\text{hr},\ \mu = \frac{1}{6}/\text{min.} = 10\ /\text{hr},\ s = 4$$

$$(a)\ P(\text{an arrival has to wait})$$

$$\therefore P(N \geq 4) = \frac{1}{4!} \cdot \frac{(3)^4}{\left(1 - \frac{3}{4}\right)} P_0$$

$$= \frac{1}{24} \frac{(3)^4}{\left(\frac{1}{4}\right)} P_0 \quad = 13.5 \times P_0 \;--(1)$$

$$\text{Now } P_0 = \frac{1}{\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right\}}$$

$$= \frac{1}{\left\{\sum_{n=0}^{3} \frac{1}{n!}(3)^n\right\} + \left\{\frac{1}{4!\left(1-\frac{3}{4}\right)}\right\}}$$

$$= \frac{1}{\left\{1+3+\frac{1}{2}\times 9+\frac{1}{6}\times 27\right\} + \left\{\frac{(3)^4}{24\times\frac{1}{4}}\right\}} = 0.0377 \quad ---(2)$$

$$\text{Using (2) in (1), we have } P(N \geq 4) = 13.5 \times (0.0377)$$

$$= 0.5090.$$

$$(b)\ E(w_q) = \frac{1}{10}\frac{1}{24 \times 4} \cdot \frac{(3)^4}{\left(1 - \frac{3}{4}\right)^2} \times 0.0377 \quad = 0.0509\,\text{h or } 3.05\,\text{min}$$

$$E(W_s) = \frac{1}{\mu} + E(W_q) \ = \frac{1}{10} + 0.0509$$

$$= 0.1 + 0.0509 \ = 0.1509\,\text{h or } 9.054\,\text{min}$$

$$E(N) = L_q + \frac{\lambda}{\mu} \quad = \frac{1}{s!s} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1 - \frac{\lambda}{\mu s}\right)^2} P_0 + \frac{\lambda}{\mu}$$

$$= \frac{1}{24 \times 4} \cdot \frac{(3)^5}{\left(1 - \frac{3}{4}\right)^2} \times 0.0377 + 3 = 4.53 \text{ cars}$$

$$\text{(c) The fraction of time when the pumps are busy} = \text{traffic intensity}$$

$$= \frac{\lambda}{\mu s} = \frac{3}{4}.$$

$$\therefore \text{ The fraction of time when the pumps are idle} = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\text{Therefore, required percentage} = \frac{1}{4} \times 100 = 25\%.$$

THANK YOU

UNIT V

NON-MARKOVIAN QUEUES

by

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SVCE

NON-MARKOVIAN QUEUES

Some of the queueing systems don't have markov property either in arrival or service pattern and known as non-markovian queues.

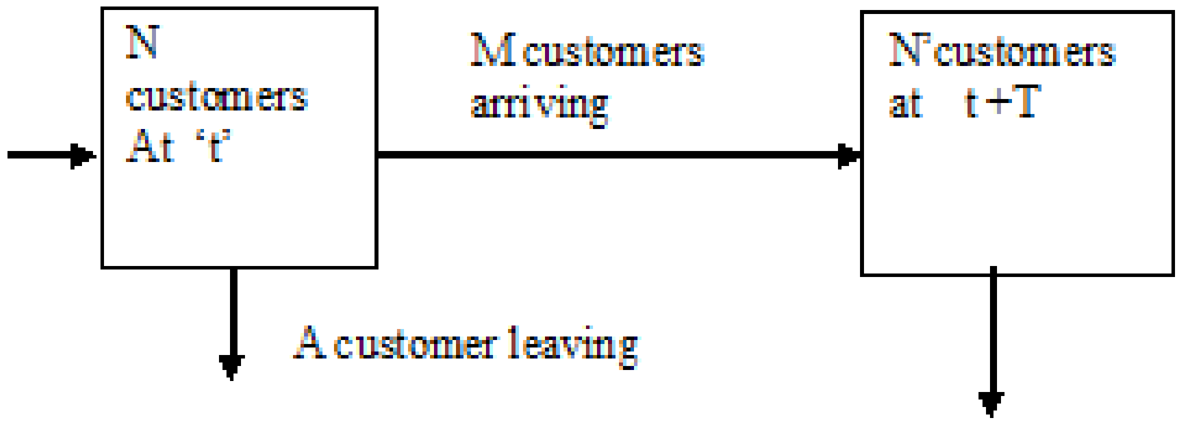
Example:

$M/E_k/1$ – having erlang service time distributions

M/D/1-constant service time distributions.

Pollaczek-Khinchine formula

Let N and N' be the number of customers in the system at times t and t+T, when two consecutive customers have just left the system after getting service.



Thus T is the random service time, which is a continuous random variable. Let $f(t)$, E(T) and var(T) be the p.d.f, mean and variance of the service time T.

Let M customers arriving in the system during service time.

Case (i) If the number of customers in the system is N (=0) during the service time t, then the number of customers in the system during the service time T is M the customers arrived

Case (ii) If the number of customers in the system is N (> 0), then the number of customers in the system during the service time T is N - 1(that is the customer left) + M (the customers arrived)

In notations,

$$N' = \begin{cases} M, & if \;\; N = 0 \\ N - 1 + M, & if \;\; N > 0 \end{cases}$$

where M is a discrete random variable taking the values $0, 1, 2, \ldots$

The same can be written as

$$N' = N - 1 + M + \delta \ldots (1) \quad where\ \delta = \begin{cases} 1, & if\ \ N = 0 \\ 0, & if\ N > 0 \end{cases}$$

Note that by the definition of δ.

$$\delta^2 = \delta$$

$$\text{since } if\ \ \delta = 0 \Rightarrow \delta^2 = 0 \ \ and\ \ \delta = 1 \Rightarrow \delta^2 = 1$$

$$N\delta = \begin{cases} 0 \times 1, & if\ \ N = 0 \\ N \times 0, & if\ N > 0 \end{cases} \quad \Rightarrow N\delta = 0$$

Squaring both sides of 1, we have,

$$N'^2 = N^2 + (M-1)^2 + \delta^2 + 2N(M-1) + 2N\delta + 2(M-1)\delta$$

$$= N^2 + (M-1)^2 + \delta + 2N(M-1) + +2(M-1)\delta$$

$$= N^2 + M^2 - 2M + 1 + 2N(M-1) + +(2M-1)\delta$$

$$2N(1-M) = N^2 - N'^2 + M^2 - 2M + 1 + (2M-1)\delta \ldots\ldots(2)$$

Taking Expectations on the both sides of (1) we get,

$$E(N') = E(N) - 1 + E(M) + E(\delta)$$

In steady state the probability that the number of customers in the system is constant

$$E(N') = E(N) \qquad E(N^2) = E(N'^2) \ldots\ldots.(3)$$

and substituting in previous equation we have

$$E(\delta) = 1 - E(M) \ldots..(4)$$

Taking Expectations on the both sides of (2) we get

$$2E[N(1-M)] = E(N^2) - E(N'^2) + E(M^2) - 2E(M) + 1 + (2E(M)-1)E(\delta).$$

using (3) we have

$$2E[N(1-M)] = E(M^2) - 2E(M) + 1 + (2E(M)-1)E(\delta)..$$

using (4) we have

$$2E[N(1-M)] = E(M^2) - 2E(M) + 1 + (2E(M)-1)(1-E(M))$$

$$= E(M^2) - 2E(M) + 1 + 2E(M) - 2E(M)^2 - 1 + E(M)$$

$$2E[N(1-M)] = E(M^2) - 2E(M)^2 + E(M)$$

Since the number of arrivals (M) to a system is independent of the number of customers already in the system (N) we have

$$2E[N]E[(1-M)] = E(M^2) - 2E(M)^2 + E(M)$$

$$2E(N)[1-E(M)] = E(M^2) - 2E^2(M) + E(M)$$

$$E(N) = \frac{E(M^2) - 2E^2(M) + E(M)}{2[1-E(M)]} \quad \text{......(5)}$$

Since the arrivals follow Poisson process with parameter λ the expected number of arrivals in time T is

$$E(M) = E(\lambda T) = \lambda E(T)$$

$$E(M^2) = E(\lambda^2 T^2 + \lambda T)$$

$$= \lambda^2 E(T^2) + \lambda E(T)$$

$$= \lambda^2 [\mathrm{var}(T) + E^2(T)] + \lambda E(T)$$

$$E^2(M) = \lambda^2 E^2(T)$$

substituting in (5) we have,

$$E(N) = \frac{\lambda^2 [\mathrm{var}(T) + E^2(T)] + \lambda E(T) - 2\lambda^2 E^2(T) + \lambda E(T)}{2[1 - \lambda E(T)]}$$

$$=\frac{\lambda^2\left[\operatorname{var}(T)+E^2(T)\right]+2\lambda E(T)-2\lambda^2E^2(T)}{2\left[1-\lambda E(T)\right]}$$

$$=\frac{\lambda^2\left[\operatorname{var}(T)+E^2(T)\right]+2\lambda E(T)(1-E(T))}{2\left[1-\lambda E(T)\right]}$$

$$=\frac{\lambda^2\left[\operatorname{var}(T)+E^2(T)\right]}{2\left[1-\lambda E(T)\right]}+\frac{2\lambda E(T)(1-E(T))}{2\left[1-\lambda E(T)\right]}$$

$$E(N)=\frac{\lambda^2\left[\operatorname{var}(T)+E^2(T)\right]}{2\left[1-\lambda E(T)\right]}+\lambda E(T) \qquad \text{------------------(A)}$$

Which is called Pollaczek-Khinchine formula (P.K. formula).

Note:

1. Since T is RV representing service time, therefore E(T) is the average service time

$$\text{Service rate} \quad \mu = \frac{1}{E(T)} \Rightarrow E(T) = \frac{1}{\mu} \qquad\qquad \rho = \frac{\lambda}{\mu}$$

$$Var(T) = V(T) = \sigma^2 \quad = \text{variance of service time}$$

$$(A) \Rightarrow L_s = E(N) = \frac{\lambda^2\left[\sigma^2 + \left(\frac{1}{\mu}\right)^2\right]}{2\left[1 - \lambda\left(\frac{1}{\mu}\right)\right]} + \lambda\left(\frac{1}{\mu}\right)$$

$$= \rho + \frac{(\lambda^2\sigma^2 + \rho^2)}{2(1-\rho)}$$

Which is average number of customer in the system

Then by little formula,

$$L_q = L_s - \frac{\lambda}{\mu} = L_s - \rho = \frac{(\lambda^2\sigma^2 + \rho^2)}{2(1-\rho)}$$

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu} + \frac{(\lambda^2\sigma^2 + \rho^2)}{2\lambda(1-\rho)} \qquad W_q = \frac{L_q}{\lambda} = \frac{(\lambda^2\sigma^2 + \rho^2)}{2\lambda(1-\rho)}$$

2. Many practical queuing systems are such that service time is constant for all customers. This mean that variance is zero and mean service time $E(T)=1/\mu$

1. A automatic car wash facility operates with only one bay. Cars arrive according to a Poisson fashion with a mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. The parking lot is large enough to accommodate any number of cars. Find L_s, L_q, W_s, W_q if the time for washing and cleaning a car(service time)

(i) is constant and equal to 10 minutes
(ii) follows uniform distribution between 8 and 12 minutes
(iii) follows normal distribution with mean 12 min and S.D 3min.
(iv) follows a discrete distribution with values equal to 4,8,15 minutes and corresponding probabilities 0.2,0.6 and 0.2.

Solution: This is a $(M/G/1)$ *queue* With the arrival rate

$$\lambda = 4 \ \ per \ \ hour = \frac{1}{15} \ per \ \min$$

(i) Service time =constant=10 min

$$\Rightarrow E(T)=\frac{1}{\mu}=10\min$$

and hence there is no variance $\Rightarrow Var(T)=0$

$$L_s = \lambda E(T)+\frac{\lambda^2\left[E^2(t)+Var(T)\right]}{2\left[1-\lambda E(T)\right]}$$

$$=\frac{1}{15}\times 10+\frac{\left(\frac{1}{15}\right)^2\left[10^2+0\right]}{2\left[1-\frac{1}{15}\times 10\right]}=\frac{4}{3}$$

$$=1.33\,car$$

By Little's formula

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{4}{3} - \frac{1/15}{1/10} = \frac{2}{3}\, car$$

$$W_s = \frac{L_s}{\lambda} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{1}{15}\right)} = 20 \text{ min} \qquad W_q = \frac{L_q}{\lambda} = \frac{\left(\frac{2}{3}\right)}{\left(\frac{1}{15}\right)} = 10 \min$$

(ii) When the service time follows uniform distribution between 8 and 12

i.e. a=8 and b=12

$$\Rightarrow E(T) = \frac{(a+b)}{2} = \frac{8+12}{2} = 10 \min \qquad \Rightarrow \mu = \frac{1}{10}\, per \min$$

and $Var(T) = \dfrac{(b-a)^2}{12} = \dfrac{(12-8)^2}{12} = \dfrac{4}{3}$

$$L_s = \lambda E(T) + \frac{\lambda^2\left[E^2(t) + Var(T)\right]}{2\left[1-\lambda E(T)\right]}$$

$$= \frac{1}{15}\times 10 + \frac{\left(\frac{1}{15}\right)^2\left[10^2 + \frac{4}{3}\right]}{2\left[1-\frac{1}{15}\times 10\right]} = 1.3422\,car$$

By Little's formula

$$L_q = L_s - \frac{\lambda}{\mu} = 1.3422 - \frac{1/15}{1/10} = 0.6755car$$

$$W_s = \frac{L_s}{\lambda} = \frac{1.3422}{\left(\frac{1}{15}\right)} = 20.14 \text{ min} \qquad W_q = \frac{L_q}{\lambda} = \frac{0.6755}{\left(\frac{1}{15}\right)} = 10.13 \text{ min}$$

(iii) When the service time follows normal distribution with mean 12 min and SD 3 min

$$\Rightarrow E(T) = 12 \text{ min} \qquad \Rightarrow \mu = \frac{1}{12} \; per \text{ min}$$

$$Var(T) = (SD)^2 = 3^2 = 9$$

$$L_s = \lambda E(T) + \frac{\lambda^2 \left[E^2(t) + Var(T)\right]}{2\left[1 - \lambda E(T)\right]}$$

$$= \frac{1}{15} \times 12 + \frac{\left(\frac{1}{15}\right)^2 \left[12^2 + 9\right]}{2\left[1 - \frac{1}{15} \times 12\right]} = 2.5 car$$

By Little's formula

$$L_q = L_s - \frac{\lambda}{\mu} = 2.5 - \frac{1/15}{1/12} = 1.7 car$$

$$W_s = \frac{L_s}{\lambda} = \frac{2.5}{\left(\frac{1}{15}\right)} = 37.5 \min \qquad W_q = \frac{L_q}{\lambda} = \frac{1.7}{\left(\frac{1}{15}\right)} = 25.5 \min$$

(iv) The service time is a discrete distribution as given below:

T :	4	8	15
p(T) :	0.2	0.6	0.2

So the mean and variance are calculated as

$$E(T) = 4 \times 0.2 + 8 \times 0.6 + 15 \times 0.2 = 8.6 \text{ min}$$

$$E\left(T^2\right) = 4^2 \times 0.2 + 8^2 \times 0.6 + 15^2 \times 0.2 = 86.6 \text{ min}$$

$$Var(T) = E\left(T^2\right) - E(T)^2 = 86.6 - 8.6^2 = 12.64$$

$$E(T) = 8.6 \text{ min} \quad \Rightarrow \mu = \frac{1}{8.6} \, per \text{ min}$$

$$L_s = \lambda E(T) + \frac{\lambda^2\left[E^2(t)+Var(T)\right]}{2\left[1-\lambda E(T)\right]} = \frac{1}{15}\times 8.6 + \frac{\left(\frac{1}{15}\right)^2\left[8.6^2+12.64\right]}{2\left[1-\frac{1}{15}\times 8.6\right]}$$

$$= 1.024\, car$$

By Little's formula

$$L_q = L_s - \frac{\lambda}{\mu} = 1.024 - \frac{\left(\frac{1}{15}\right)}{\left(\frac{1}{8.6}\right)} = 0.4507 \quad cars$$

$$W_s = \frac{L_s}{\lambda} = \frac{1.024}{\left(\frac{1}{15}\right)} = 15.36 \text{ min} \qquad W_q = \frac{L_q}{\lambda} = \frac{0.4507}{\left(\frac{1}{15}\right)} = 6.7605 \text{ min}$$

2. A one-man barber shop takes exactly 25 minutes to complete one hair-cut. If customers arrive at the barber shop in a Poisson fashion at an average rate of one every 40 minutes, how long on the average a customer in the spends in the shop. Also, find the average time a customer must wait for service?

Solution:

In this model the service time is not varying and constant. So this is a $(M/D/1)$ queuing model.

$E(T)=25$ and a constant and there is no variance $Var(T)=0$

the arrival rate $\lambda=\dfrac{1}{40}$ $per\ minute$

$$L_s = \lambda E(T) + \frac{\lambda^2\left[E^2(t) + Var(T)\right]}{2\left[1 - \lambda E(T)\right]}$$

$$= \frac{1}{40} \times 25 + \frac{\left(\frac{1}{40}\right)^2\left[25^2 + 0\right]}{2\left[1 - \frac{1}{40} \times 25\right]} \qquad = \frac{55}{48} = 1.1458 customers$$

By Little's formula

$$W_s = \frac{L_s}{\lambda} = \frac{\left(\frac{55}{48}\right)}{\left(\frac{1}{40}\right)} = 45.8 \text{ minutes}$$

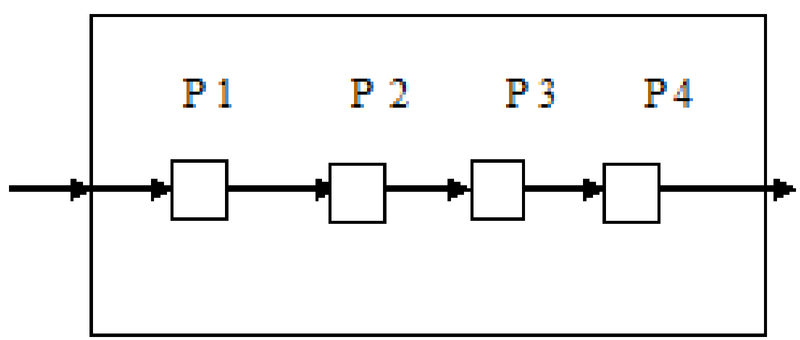
$$W_q = W_s - \frac{1}{\mu} = W_s - \frac{1}{\frac{1}{E(T)}} = 45.8 - 25$$

$$= 20.8 \text{ min}$$

3. A patient who goes to a single doctor clinic for a general check up has to go through 4 phases. The doctor takes on the average 4 minutes for each phase of the check up and the time taken for each phase is exponentially distributed. If the arrivals of the patients at the clinic are approximately Poisson at the average rate of 3 per hour, what is the average time spent by a patient (i) in the examination (ii) waiting in the clinic?

Solution:

The clinic has 4 phases (each having different service nature) in series as follows:



Considering all the phases (each with exponential service time) together as a "one server" we shall take it as a server with Erlang service time (since sum of independent exponential variables is Erlang variable)

So this is a $(M / E_k / 1)$

$$E(T) = \frac{k}{\theta} = \frac{4}{\left(\frac{1}{4}\right)} = 16 \min \qquad Var(T) = \frac{k}{\theta^2} = \frac{4}{\left(\frac{1}{16}\right)} = 64 \min$$

$$\text{the arrival rate } \lambda = \frac{1}{3} \; per \;\; hour = \frac{3}{60} = \frac{1}{20} \, per \; \min ute$$

$$L_s = \lambda E(T) + \frac{\lambda^2 \left[E^2(t) + Var(T)\right]}{2\left[1 - \lambda E(T)\right]}$$

$$= \frac{1}{20} * 16 + \frac{\left(\frac{1}{20}\right)^2 \left[16^2 + 64\right]}{2\left[1 - \frac{1}{20} * 16\right]} \qquad = \frac{14}{5} \, customers$$

By Little's formula

$$W_s \quad = \frac{L_s}{\lambda} = \frac{\left(\frac{14}{5}\right)}{\left(\frac{1}{20}\right)} = 56 \text{ min} \, utes$$

$$W_q = W_s - \frac{1}{\mu} = W_s - \frac{1}{\frac{1}{E(T)}} = 56 - 16 = 40 \text{ min}$$

Queuing Networks

The queuing systems that we have seen so far have a single service facility with one or more servers. But queuing systems that we come across in real life situation are often not isolated but part of organized systems called queuing networks.

(i.e) network of service facilities where customers receive service at some or all of the facilities.

Example:

A factory may have many queues linked together by logical sequence of production process.

A network of queues is a group of (k) nodes where each node represents a service facility with C_i servers at node i (i= 1, 2, 3……k). Customers may enter the system at one node, and after completion of service at one node may move to another node for further service and may leave the system from some other node. They may return to previously visited nodes, skip some nodes and may stay in the system forever.

Hence to find the expected total waiting time, expected number of customers in the system etc., we have to study the entire network.

There are two types of queueing networks

i. Open queueing network
ii. Closed queueing network

Open queueing networks:

In an open queueing network, customers may arrive from outside the system at any node and may depart from the system from any node.

Series Queues (Tandem queues)

These are special type of open network in which there are a series of service facilities where each customer should visit (in the given order) before leaving the system. The nodes form a series system with flow always in a single direction from node to node. Customers enter from only at node 1 and depart only from node k.

Example:

An admission process in a college where the student has to visit a series of official or clerks

Master health check up programme in a hospital where a patient has to undergo a series of tests.

A few kinds of series queues are

i. Series queues with blocking

ii. Two stage tandam queues

Series queues with blocking:

This queue model consisting of two service point S_1, S_2 at each of which there is only one server and no queue is allowed to form at either point.

An entering customer will first go to S_1, after he gets service completed in S_1, he will go to S_2 if it is empty (or) will wait in S_1 until S_2 becomes empty. This means that a potential customer will enter the system only when S_1 is empty, irrespective of whether S_2 is empty or not . Since the model is a sequential model, (i.e) all the customer require service at S_1 and then at S_2.

Steady state probabilities P_{mn} that there is m customer (m=0 or 1) is S_1 and n customer (n=0 or 1) in S_2.

Possible states of the system:

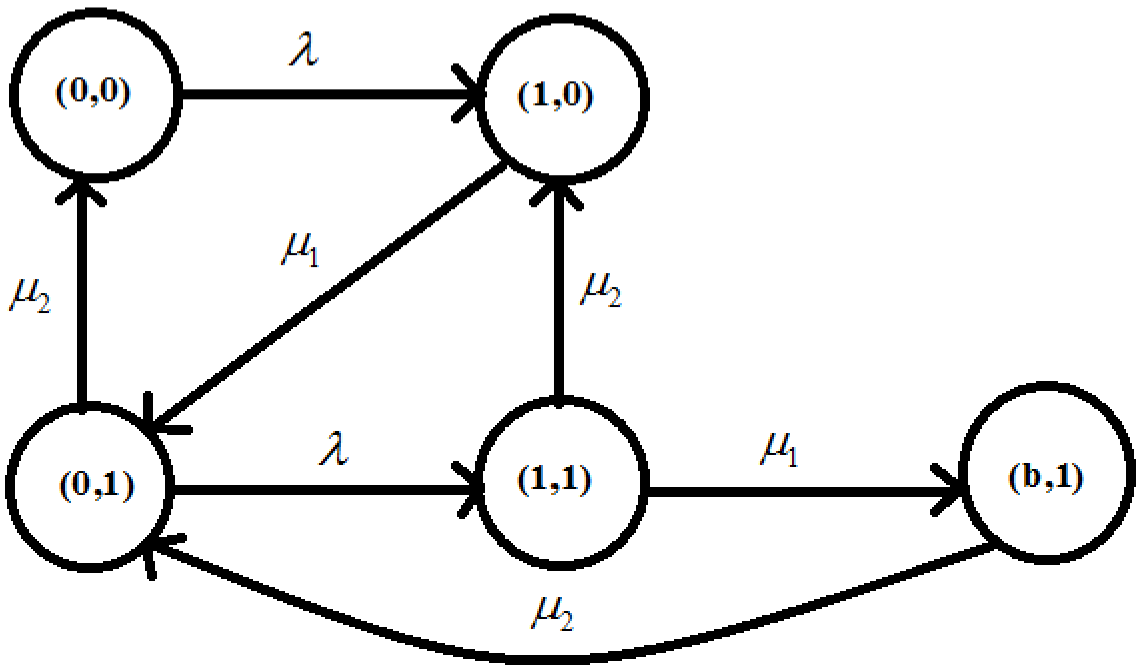
Each station may be either free or busy and station 1 may also be blocked. If 0, 1, and b denote the free, busy, and blocked states respectively and i, j represent the states of station 1 and station 2 respectively, then the possible states of the system are:

i, j	Description
0,0	System is empty
1,0	Customer in station 1 only
0,1	Customer in station 2 only
1,1	Customers in both stations
b,1	Customer have finished getting service at station 1 but is waiting since there is a customer in station 2(i.e. system is blocked)

We assume that potential customers arrive in accordance with a Poisson process with parameter λ and the service time at S_1 and S_2 follows exponential distribution with parameters μ_1 and μ_2 respectively. To get the values of P_{mn}, we shall first write down the steady state balance equations using transition diagram, as given below.

Here small circle refers states and directed lines labeled by the rate at which the process goes from one state to another.

State transition diagram:



Let the balance equation for the 5 states of the system formed by using the above diagram.

States	Balance equation
(0,0)	$\lambda P_{00} = \mu_2 P_{01} \quad \rightarrow (1)$
(1,0)	$\mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11} \quad \rightarrow (2)$
(0,1)	$(\lambda + \mu_2) P_{01} = \mu_1 P_{10} + \mu_2 P_{b1} \quad \rightarrow (3)$
(1,1)	$(\mu_1 + \mu_2) P_{11} = \lambda P_{01} \quad \rightarrow (4)$
(b,1)	$\mu_2 P_{b1} = \mu_1 P_{11} \quad \rightarrow (5)$

$$P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1 \qquad \rightarrow (6)$$

Solving the above six equation, we get the 5 steady state probabilities

Remarks:

1. Probability that an arriving customer enters the system

$$= P_{00} + P_{01}$$

2. Average (or) expected number of customer in the system

$$L_s = 1[P_{10} + P_{01}] + 2[P_{11} + P_{b1}]$$

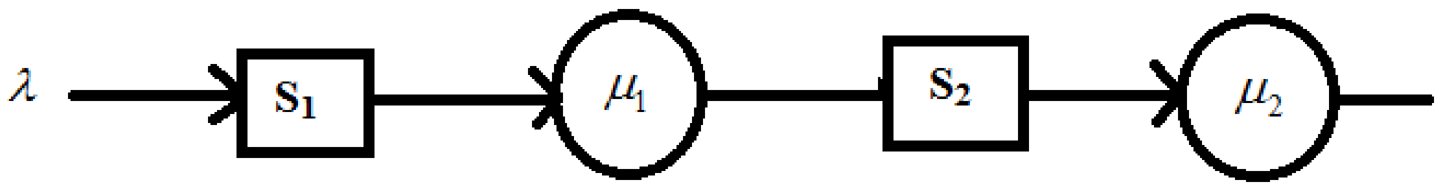
3. Average (or) expected number of customers spends in the system (average waiting time)

$$W_s = \frac{L_s}{\lambda'}$$

Where Effective arrival rate $\lambda' = \lambda(P_{00} + P_{01})$

Two stage tandem queue:

A two stage queueing system in which customer arrive from outside at a Poisson rate λ to S_1. After being served at S_1 they can join the queue in front of S_2. After getting serviced at S_2, they leave the system. So this means that there are infinite waiting at each service point.



Each server serves one customer at a time and the service time at S_1 and S_2 follows exponential distribution with parameter μ_1 and μ_2 respectively.

Remarks:

1. Steady state joint probability P(m,n) of m customers in S_1 and n customers in S_2, where $m \geq 0$ *and* $n \geq 0$.

$$P_{mn} = \left(\frac{\lambda}{\mu_1}\right)^m \left(1-\frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^n \left(1-\frac{\lambda}{\mu_2}\right) \quad m \geq 0 \text{ and } n \geq 0.$$

2. Average number of customer in the system is

$$L_s = \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda}$$

3. Average number of customers spends in the system (Waiting time)

$$W_s = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}$$

Bottleneck of a system:

As the arrival rate λ increases in two stage tandem queue, the node with larger traffic intensity

$$\rho_i = \frac{\lambda_i}{\mu_i}$$

will introduce instability and hence the node with largest traffic intensity is called '**bottleneck**' of the system.

Example:

A two stage tandem network is such that the average service time of node 1 is 1 hour and the average service time for node 2 is 2 hour and the arrival rate is 0.5 per hour. Find the bottleneck of the system.

Given:

$$\lambda = 0.5 \quad \mu_1 = 1 \quad \mu_2 = 2$$

$$\Rightarrow \rho_1 = \frac{\lambda}{\mu_1} = 0.5 \qquad \rho_2 = \frac{\lambda}{\mu_2} = 0.25$$

Since node 1 have larger traffic intensity

Therefore node 1 is bottleneck of the system.

Open Jackson Networks:

In open network customer can join and leave the system anywhere.

An open Jackson network is a network of n nodes or service facility satisfies the following condition:

1. Customer arrive at node i (station) from outside the system follows as Poisson process with mean m_i and join the queue at i and wait for his turn for service
2. At node i the service time are independent and are exponentially distributed with parameter μ_i

3. Once a customer gets the service completed at node i, he next goes to node j with probability P_{ij} (the number of customer waiting at j for service) where i =1,2,3,.....k j= 0,1,2,3..k

4. P_{0i} is the probability that a customer leaves the system from node i after getting serviced.

Let λ_j represent the total arrival rate of customer to j, then λ_j can be defined as

$$\lambda_j = m_j + \sum_{i=1}^{k} \lambda_i P_{ij} \quad j = 1,2,3.....k \quad ....(1)$$

Here, P_{ij} denotes the probability that a departure for node i, joins the queue at j and hence $\lambda_j P_{ij}$ is the rate of arrival to node j from those who are coming out from node i . Equation (1) is called Traffic equation (or) flow balance equation.

Jackson has proved that the steady-state solution of this traffic equation with single server at each node is

$$P(n_1, n_2 \ldots\ldots\ldots n_k) = \prod_{j=1}^{k} \rho_j^{n_j} (1 - \rho_j) \qquad where\, \rho_i = \frac{\lambda_i}{\mu_i}, i = 1,2\ldots k$$

Therefore, Jackson's result shows that network of each node can be viewed as independent (M/M/1) queue with parameter λ_i *and* μ_i

Hence, the network behaves as if its nodes are independent M/M/1 queue model

Remarks:

1 Expected number of customer in the entire system

$$L_s = \sum_{j=1}^{k} \frac{\lambda_j}{\mu_j - \lambda_j}$$

2. Expected time a customer spends in the entire system

$$W_s = \frac{L_s}{\lambda} \qquad where\ \lambda = \sum_{j=1}^{k} m_j$$

3. P(m customers in S_1 and n customers in S_2)

$$P(m,n) = \left(\frac{\lambda_1}{\mu_1}\right)^m \left(1-\frac{\lambda_1}{\mu_1}\right) \left(\frac{\lambda_2}{\mu_2}\right)^n \left(1-\frac{\lambda_2}{\mu_2}\right)$$

1. A repair facility by a large number of machines has two sequential stations with respective rates one per hour and two per hour. The cumulative failure rate of all the machines is 0.5 per hour. Assuming that the system behavior may be approximated by the two-stage tandem queue, determine the average repair time.

Solution:

Given: $\lambda = 0.5 \quad \mu_1 = 1 \quad \mu_2 = 2$

$\rho_1 = 0.5 \quad \rho_2 = 0.25$

Note that each station is a (M/M/1) queue model

The average length of the queue at station 1

$$LS_1 = \frac{\rho_1}{1-\rho_1} = \frac{0.5}{1-0.5} = 1$$

The average length of the queue at station 2

$$LS_2 = \frac{\rho_2}{1-\rho_2} = \frac{0.25}{1-0.25} = \frac{1}{3}$$

By Little's formula

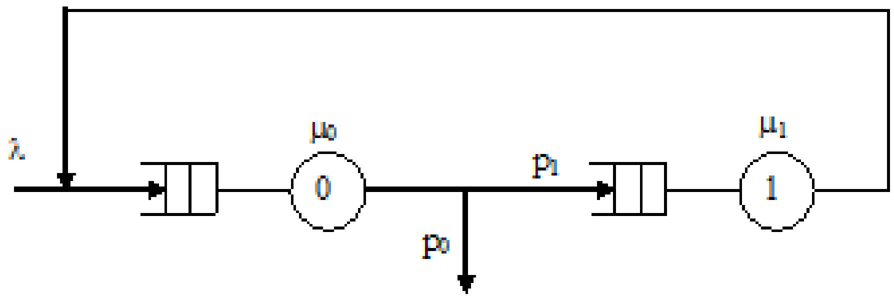
$$W_{s1} = \frac{LS_1}{\lambda} = \frac{1}{0.5} = 2 \qquad W_{s2} = \frac{LS_2}{\lambda} = \frac{1/3}{0.5} = \frac{2}{3}$$

The total repair time of the network is

$$= W_{s1} + W_{s2} = 2 + \frac{2}{3} = \frac{8}{3}\, hours$$

2. Find the average response time for a two-stage tandem open network with feedback.

Solution:



By Jackson's result the two-queues at each node will behave like independent M/M/1 queues and hence the probability that there are k_0 customers at node 0 and k_1 customers at node 1 is given by

$$p(k_0, k_1) = (1-\rho_0)\rho_0^{k_0}(1-\rho_1)\rho_1^{k_1} \quad \text{where} \quad \rho_0 = \frac{\lambda_0}{\mu_0} \qquad \rho_1 = \frac{\lambda_1}{\mu_1}$$

λ_0, λ_1 are the arrival rates at nodes 0 and 1 respectively

In steady state the arrival rate and departure rate at the respective nodes coincide.

The node 0 has arrival from outside with a rate λ and the arrival rate from the feedback (that is departure rate from node 1 which is nothing but λ_1) and so

$$\lambda_0 = \lambda + \lambda_1$$

when a job is completed at node 0, it will reach node 1 therefore the number of jobs at node 1 is p_1 and so the average arrival rate to node 1 is given by

$$\lambda_1 = \lambda_0 p_1$$

thus: $\lambda_0 = \dfrac{\lambda}{1-p_1} = \dfrac{\lambda}{p_0}$ and $\lambda_1 = \dfrac{p_1 \lambda}{p_0}$

which implies that

$$\rho_0 = \frac{\lambda}{\mu_0 p_0} \qquad \rho_1 = \frac{p_1 \lambda}{p_0 \mu_1}$$

B_0 denote the total service time at node 0 for a job then

$$E(B_0) = \frac{\rho_0}{\lambda} = \frac{1}{p_0 \mu_0} \quad \text{Similarly} \quad E(B_1) = \frac{\rho_1}{\lambda} = \frac{p_1}{p_0 \mu_1}$$

The average response time is computed as

$$E(R) = \frac{L_s}{\lambda} = \frac{L_{s_1} + L_{s2}}{\lambda} = \frac{1}{\lambda}\left[\frac{\rho_0}{1-\rho_0} + \frac{\rho_1}{1-\rho_1}\right]$$

$$= \frac{1}{p_0\mu_0 - \lambda} + \frac{1}{\dfrac{p_0\mu_1}{p_1} - \lambda}$$

$$= \frac{E(B_0)}{1 - \lambda E(B_0)} + \frac{E(B_1)}{1 - \lambda E(B_1)}$$