COT 5615 Math for Intelligent Systems, Fall 2021

Midterm 1

A2.1 (14 pts)

$$f(x) = (\boldsymbol{e}_m - \frac{1}{n}\mathbf{1}_n)^T x.$$

3.25 (22 pts) We went over this question during the lecture on Sept. 7.

(a) (10 pts) The mean return of the portfolio is the average of the vector p:

$$\begin{aligned}
\mathbf{avg}(p) &= \mathbf{avg}(\theta r + (1-\theta)\mu^{\text{rf}}\mathbf{1}) \\
&= \theta\mathbf{avg}(r) + (1-\theta)\mu^{\text{rf}}\mathbf{avg}(\mathbf{1}) \\
&= \theta\mu + (1-\theta)\mu^{\text{rf}}.
\end{aligned}$$

On the last line we use $\mathbf{avg}(r) = \mu$, and $\mathbf{avg}(\mathbf{1}) = 1$.
The risk is the standard deviation of the vector p:

$$\begin{aligned}
\mathbf{std}(p) &= \|p - \mathbf{avg}(p)\mathbf{1}\|/\sqrt{T} \\
&= \|\theta r + (1-\theta)\mu^{\text{rf}}\mathbf{1} - (\theta\mu + (1-\theta)\mu^{\text{rf}})\mathbf{1}\|/\sqrt{T} \\
&= \|\theta(r - \mu\mathbf{1})\|/\sqrt{T} \\
&= |\theta|\|(r - \mu\mathbf{1})\|/\sqrt{T} \\
&= |\theta|\mathbf{std}(r) \\
&= |\theta|\sigma.
\end{aligned}$$

On line 2 we use the expression for $\mathbf{avg}(p)$ that we derived in part (a). The last step is the definition of $\sigma = \mathbf{std}(r)$.

(b) (6 pts) To achieve the target risk σ^{tar}, we need $|\theta| = \sigma^{\text{tar}}/\sigma$, so there are two choices:

$$\theta = \sigma^{\text{tar}}/\sigma, \quad \theta = -\sigma^{\text{tar}}/\sigma.$$

To choose the sign of θ we consider the portfolio return

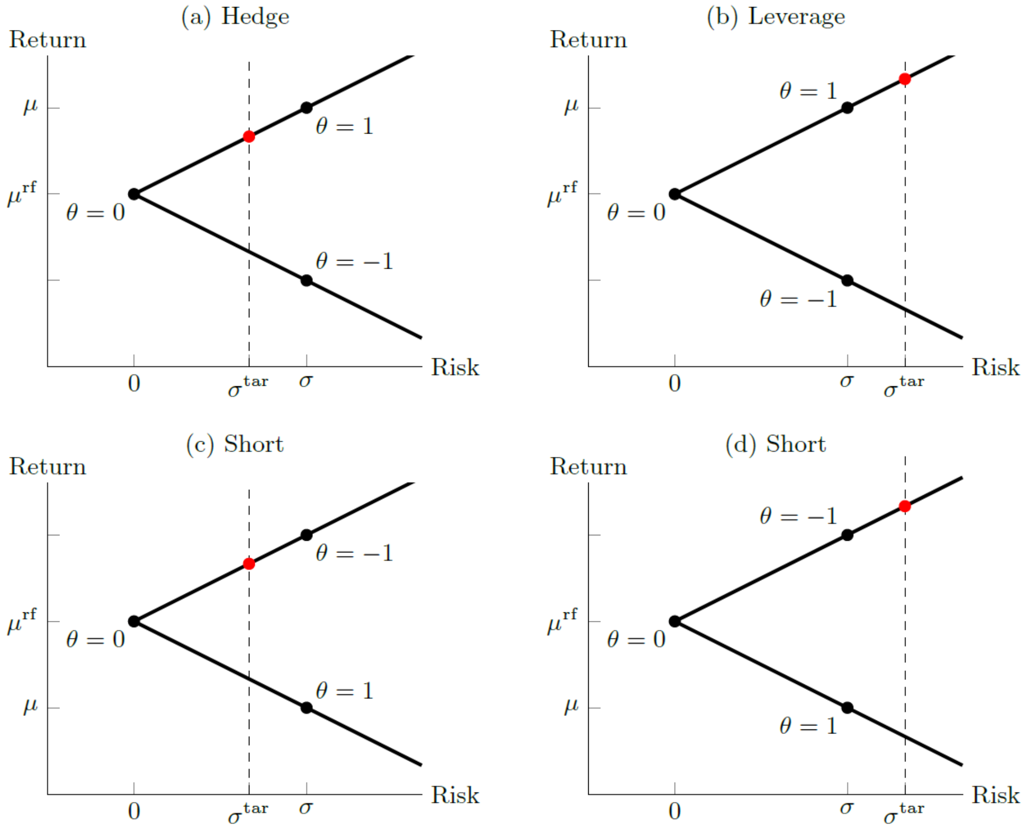
$$\theta\mu + (1-\theta)\mu^{\text{rf}} = \mu^{\text{rf}} + \theta(\mu - \mu^{\text{rf}}).$$

To maximize this, for given $|\theta|$, we choose θ positive if $\mu > \mu^{\text{rf}}$ and θ negative if $\mu < \mu^{\text{rf}}$. This means we short the asset when its return is less than the risk-free return.

(c) (6 pts) We can distinguish the cases shown in the figure below. The solid lines show

$$(\text{risk, return}) = (\mathbf{std}(p), \mathbf{avg}(p)) = (|\theta|\sigma, \mu^{\text{rf}} + \theta(\mu - \mu^{\text{rf}}))$$

for all values of θ. The red dot shows the portfolio with the highest return for the given target value of risk.



In case (a), the asset return is more than the risk-free return ($\mu > \mu^{\mathrm{rf}}$) and its risk is higher than the target risk ($\sigma > \sigma^{\mathrm{tar}}$). In this case we hedge ($0 < \theta < 1$). In case (b), the asset return is more than the risk-free return ($\mu > \mu^{\mathrm{rf}}$) and its risk is less than the target risk ($\sigma < \sigma^{\mathrm{tar}}/\sigma$). In this case we leverage ($\theta > 1$). In cases (c) and (d), the asset return is less than the risk-free return ($\mu < \mu^{\mathrm{rf}}$). In this case we short ($\theta < 0$).

To summarize, we short the asset when its return is less than the risk-free return. We hedge when the asset return is more than the risk-free return and the asset risk is higher than the target risk. We leverage when the asset return is more than the risk-free return and the asset risk is less than the target risk.

A8.1 (15 pts)

$$p(0) = p(1) \Rightarrow c_1 = c_1 + c_2 + c_3 + c_4 + c_5.$$

Besides,

$$p'(x) = c_2 + 2c_3x + 3c_4x^2 + 4c_5x^3,$$

so

$$p'(0) = p'(1) \Rightarrow c_2 = c_2 + 2c_3 + 3c_4 + 4c_5.$$

The set of linear equations

$$\begin{aligned} c_2 + c_3 + c_4 + c_5 &= 0 \\ 2c_3 + 3c_4 + 4c_5 &= 0 \end{aligned}$$

can be written as $Ac = b$ with

$$A_{2\times 5} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}, \quad b_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

8.8 (15 pts) The interpolation conditions

$$\frac{c_1 + c_2 t_i + c_3 t_i^2}{1 + d_1 t_i + d_2 t_i^2} = y_i, \quad i = 1, \cdots, K,$$

can be written as linear equations

$$c_1 + c_2 t_i + c_3 t_i^2 - y_i d_i t_i - y_i d_2 t_i^2 = y_i, \quad i = 1, \cdots, K.$$

If we define $\theta = (c_1, c_2, c_3, d_1, d_2)$ then this can be written as $A\theta = b$ with

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & -y_1 t_1 & -y_1 t_1^2 \\ 1 & t_2 & t_2^2 & -y_2 t_2 & -y_2 t_2^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_K & t_K^2 & -y_K t_K & -y_K t_K^2 \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_K \end{bmatrix}.$$

9.4 (14 pts)

$$z_{t+1} = \begin{bmatrix} A_1 & A_2 \\ I & 0 \end{bmatrix} z_t.$$

A10.5 (20 pts)

(a) Makes sense. Dimension is 10×10.

(b) Makes sense. Dimension is 20×10.

(c) Doesn't make sense.

(d) Makes sense. Dimension is 10×20.

(e) Makes sense. Dimension is 20×10.