COT 5615 Math for Intelligent Systems Fall 2021 Homework #5

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Problem 10.11

Trace of matrix-matrix product

Solution

a Trace represents diagonal entries, thus trace of $A^T B$ can be shows as follows:

$$\sum_{i=1}^{m}(A^T B)_{ii} = \sum_{i=1}^{m}\sum_{j=1}^{n}(A^T)_{ij}(B)_{ji} = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij}$$

The complexity of the algorithm is $2mn$ flops, as we only need mn multiplications and $mn-1$ additions.

b The matrix $B^T A$ is the transpose of $A^T B$, so it has the same diagonal entries and same trace. Hence, $tr(B^T A) = tr(A^T B)$

c From (a), we can derive $A^T A$ as follows:

$$A^T A = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2 = \|A\|^2$$

d Here the following derivations shows the proof:

$$tr(BA^T) = \sum_{i=1}^{m}(BA^T)_{ii} = \sum_{i=1}^{m}\sum_{j=1}^{n}(B)_{ij}(A^T)_{ji} = \sum_{i=1}^{m}\sum_{j=1}^{n} B_{ij}A_{ij}$$

Problem 10.13

Laplacian matrix of a graph

Solution

a The Dirichlet energy can be derived as follows:

$$v^T Lv = v^T AA^T v = (A^T v)^T(A^T v) = \left\|A^T v\right\|^2 = D(v)$$

b Each entry of L_{ij} where $i = j$ i.e. L_{ii} is the degree of node i and when $i \neq j$ i.e. L_{ij} is the negative of the number of edges between nodes i and j.

Problem 10.31

Diameter of a graph

Solution

a From the figure, it is clear that number of paths from j to i of length n is shown as $(A^k)_{ij}$. Thus, total number of paths of length no more than k in-terms of A can be shown as follows: $P = I + A + A^2 + A^3 + \ldots + A^k$

b Using equation from (a), we can find the diameter by calculating P for different values of k until all the desired entries are positive and thus get the final result.

Problem 11.3

Matrix cancellation

Solution

a $A = (1,0)^T, X = (0,1),\ and\ Y = (0,0)$

b Let C be a left inverse of A, then by muliply $AX = AY$ on the left side by C we get th following: $CAX = CAY \implies X = Y\ [\because CA = I]$

c As A is non-invertible, its columns are linearly dependent i.e. $Ax = 0$ where x is non-zero. Also, $Ay = 0$ where $y = 0$. Thus, $Ax = Ay$ where $x \neq y$. Hence proved.

Problem 11.11

Interpolation of rational functions

Solution

The five interpolation equation can be written. as follows:

$$\begin{aligned} c_1 + c_2 + c_3 &= 2(1 + d_1 + d_2) \\ c_1 + 2c_2 + 4c_3 &= 5(1 + 2d_1 + 4d_2) \\ c_1 + 3c_2 + 9c_3 &= 9(1 + 3d_1 + 9d_2) \\ c_1 + 4c_2 + 16c_3 &= -1(1 + 4d_1 + 16d_2) \\ c_1 + 5c_2 + 25c_3 &= -4(1 + 5d_1 + 25d_2) \end{aligned}$$

These equations can be represented in matrix form as follows:

$(1,1,1,1,1;1,2,3,4,5;1,4,9,16,25;-2,-10,-27,4,20;-2,-20,-81,16,100)(c_1,c_2,c_3,d_1,d_2) = (2,5,9,-1,-4)$

The solution of the above equation is:

$$c_1 = 0.62962,\ c_2 = 0.60493,\ c_3 = -0.19753,\ d_1 = -0.56790,\ d_2 = 0.08641$$

The figure 1 shows the given rational function with the solution points.

```
using Plots
using LinearAlgebra
Plots.PlotlyBackend()
A = [1 1 1 1 1;1 2 3 4 5;1 4 9 16 25;-2 -10 -27 4 20;-2 -20 -81 16 100]';
C = [2 5 9 -1 -4]';
B = inv(A)C;
display(B);
f(t) = ((B[1])+(B[2]*t)+(B[3]*t*t))/(1+(B[4]*t)+(B[5]*t*t));
plot(f, 0, 6,label = "Rational Function",xlabel="t",ylabel="f(t)")
scatter!([1,2,3,4,5],[2,5,9,-1,-4],label="Interpolation Points")
```

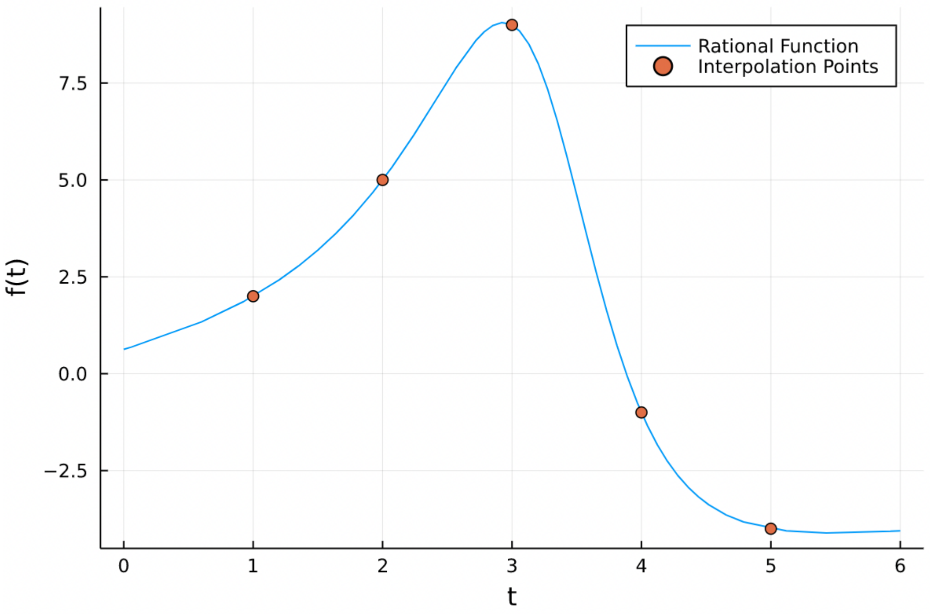

Figure 1: Interpolation of Rational Function

Problem 11.21

Quadrature weights.

Solution

$$(1, t_1, t_1^2, t_1^3; 1, t_2, t_2^2, t_2^3; 1, t_3, t_3^2, t_3^3; 1, t_4, t_4^2, t_4^3)\ (w_1, w_2, w_3, w_4) = (b_1, b_2, b_3, b_4)$$

Here t_i where $i = 1 \ldots 4$ are the values of the vector $t = (-0.6, -0.2, 0.2, 0.6)$, and

$$b_k = \int_{-1}^{1} t^{k-1} dt = \begin{cases} 2/k & k \textit{ is odd} \\ 0k & k \textit{ is even} \end{cases}$$

By solving it using left-inverse and matrix multiplication, we get the following values of w:

$$w_1 = 0.9166\ ,\ w_2 = 0.0833\ ,\ w_3 = 0.0833\ ,\ w_4 = 0.9166$$

Furthermore, for solving $f(t) = e^x$ we get the following values:

$$\alpha = 2.3504\ ,\ \hat{\alpha} = 2.3433$$

Following is the code for the above calculation:

```
A = [1 1 1 1; -0.6 -0.2 0.2 0.6; (-0.6)^2 (-0.2)^2 (0.2)^2 (0.6)^2; (-0.6)^3 (-0.2)^3 (0.2)^3 (0.6)^3];
b = [2 0 (2/3) 0]';
W = A\b;
f(x) = exp(x);
alpha_cap = f.([-0.6 -0.2 0.2 0.6])*W;
alpha = exp(1)-exp(-1);
display(W);
display(alpha_cap);
display(alpha);
```