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VYOM PATHAK

MIS FINAL

96703101

1 Cauchy-Schwarz inequality:

$$\Rightarrow |a^T b| \leq ||a|| \, ||b||$$

$$\Rightarrow \left(\sum_{K=1}^{n} a_K b_K\right)^2 \leq \left(\sum_{K=1}^{n} a_K^2\right)\left(\sum_{K=1}^{n} b_K^2\right)$$

we have $x_K > 0 \quad \forall \; 1 \leq K \leq n$

Suppose $a_K = \dfrac{1}{\sqrt{x_K}}$, $b_K = \sqrt{x_K}$

$\Rightarrow$ Use this in cauchy-Schwarz inequality as follows:

$$\left(\sum_{K=1}^{n} \frac{1}{\sqrt{x_K}} \sqrt{x_K}\right)^2 \leq \left(\sum_{K=1}^{n} \left(\frac{1}{\sqrt{x_K}}\right)^2\right)\left(\sum_{K=1}^{n} (\sqrt{x_K})^2\right)$$

$$\left(\sum_{K=1}^{n} 1\right)^2 \leq \left(\sum_{K=1}^{n} \frac{1}{x_K}\right)\left(\sum_{K=1}^{n} x_K\right)$$

$$[n]^2 \leq \left(\sum_{K=1}^{n} \frac{1}{x_K}\right)\left(\sum_{K=1}^{n} x_K\right) \quad \left[\because \sum_{K=1}^{n} 1 = n\right]$$

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$$n^2 \leq \left(\sum_{k=1}^{n} \frac{1}{x_k}\right)\left(\sum_{k=1}^{n} x_k\right)$$

$$1 \leq \left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k}\right)\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)$$

$$1 \leq \left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k}\right)\cdot\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)$$

$$\boxed{\left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k}\right)^{-1} \leq \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)}$$

Hence Proved

2

We can write vector c as a product of matrix $T_c(a)$ & b as follows:

$$c = a \circledast b = \begin{bmatrix} a_1 & a_n & a_{n-1} & \cdots & a_3 & a_2 \\ a_2 & a_1 & a_n & \cdots & a_4 & a_3 \\ a_3 & a_2 & a_1 & \cdots & a_5 & a_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

for $k = n$, $i + j = n+1$ or $2n+1$
but $i + j \leq 2n$
($\because j = 1, 2, \ldots, n$ & $i = 1, 2, \ldots, n$)

$\Rightarrow$ $\underline{i + j = n+1}$

So for a given value a, as $c = a \circledast b$ can be represented as matrix $T_c(a)$ & b as $\boxed{T_c(a)\, b = c}$ forall $\underline{b}$.

$\Rightarrow$ Thus, $c\,[a \circledast b]$ is a linear function of $\underline{b}$.

B+

3

$A = QR$

$\Rightarrow$ Q forms orthonormal basis of columns of A which is a non-singular matrix.

Hence, $\underline{A = Q}$ & $\underline{R = I}$

$\Rightarrow$ $\boxed{\text{b \& c are correct}}$

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4 It can be shown in ~~julia~~ matrix form as follows:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1/1.05 & 1/1.1025 \\ 1 & 1/1.1 & 1/1.21 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{b}$$

$\Rightarrow$ we can get x as $x = A \backslash b$ in julia.

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$\Rightarrow C_1 = -41.99$

$C_2 = 66.099$

$C_3 = -23.099$

[Code in the .ipynb file]

5

(a) Validation

(b) K-means

(c) Validation

(d) Regularization

(e) Least Squares

6

$P(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3$

$q(t) = x_5 + x_6 t + x_7 t^2 + x_8 t^3$

$\Rightarrow$ The eight conditions can be represented in $Ax = b$ format as follows:

$$\underbrace{\begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 & 0 & 0 & 0 & 0 \\ 1 & t_2 & t_2^2 & t_2^3 & 0 & 0 & 0 & 0 \\ 1 & t_3 & t_3^2 & t_3^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t_5 & t_5^2 & t_5^3 \\ 0 & 0 & 0 & 0 & 1 & t_6 & t_6^2 & t_6^3 \\ 0 & 0 & 0 & 0 & 1 & t_7 & t_7^2 & t_7^3 \\ 1 & t_4 & t_4^2 & t_4^3 & -1 & -t_4 & -t_4^2 & -t_4^3 \\ 0 & 1 & 2t_4 & 3t_4^2 & 0 & -1 & -2t_4 & -3t_4^2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}}_{x}$$

$$= \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_5 \\ y_6 \\ y_7 \\ y_4 = 0 \\ y_4 = 0 \end{bmatrix}}_{b}$$

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See .ipynb to get the values of $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ & x_8.
Also, plotted in 1 graph for both polynomials.

$\boxed{8}$

$$l = l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 + l_6^2 + l_7^2$$

$$l_1^2 = (U_1 - U_2)^2 + (V_1 - V_2)^2$$
$$l_2^2 = (U_1 - U_3)^2 + (V_1 - V_3)^2$$
$$l_3^2 = (U_1 + 1)^2 + (V_1)^2$$
$$l_4^2 = (U_2 - 0.5)^2 + (V_2 - 1)^2$$
$$l_5^2 = (U_3)^2 + (V_3 + 1)^2$$
$$l_6^2 = (U_3 - 1)^2 + (V_3 - 0.5)^2$$
$$l_7^2 = (U_2 - 1)^2 + (V_2 - 0.5)^2$$

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$$(U_1-U_2)^2+(U_1-U_3)^2+(U_1+1)^2+(U_2-0.5)^2$$
$$+(U_3)^2+(U_3-1)^2+(U_2-1)^2$$

$$+(V_1-V_2)^2+(V_1-V_3)^2+(V_1)^2$$
$$+(V_2-1)^2+(V_3+1)^2+(V_3-0.5)^2$$
$$+(V_2-0.5)^2=l$$

(d) Multi-Objective least-squares

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0.5 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0.5 \\ 0.5 \end{bmatrix}}_{b}$$

~~8~~ 9

(b)

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0.5 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{b}$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0.5 \\ 0.5 \end{bmatrix}}_{\tilde{b}}$$

(c) Julia solution in .ipynb for (b)

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6 equations

$y_{12} = x_1 + x_2 = y_1$

$y_{13} = x_1 + x_3 = y_2$

$y_{14} = x_1 + x_4 = y_3$

$y_{23} = x_2 + x_3 = y_4$

$y_{24} = x_2 + x_4 = y_5$

$y_{34} = x_3 + x_4 = y_6$

Minimize x ~~for~~ as

$$\|x_i + x_j - y_{ij}\|$$

$$= \|x_1 + x_2 - y_1\|^2 + \|x_1 + x_3 - y_2\|^2$$
$$+ \|x_1 + x_4 - y_3\|^2 + \|x_2 + x_3 - y_4\|^2$$
$$+ \|x_2 + x_4 - y_5\|^2 + \|x_3 + x_4 - y_6\|^2$$

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$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}}_{b}$$

10 (a) Q is a $m \times n$ matrix with orthonormal columns.

& $b \in \mathbb{R}^m$

Q satisfies $Q^T Q = 1$

$\Rightarrow$ To solve $Qx = b$,

we ~~use~~ $Q^T Q x = Q b$

If $Q^T Q = I \Rightarrow \boxed{x = Q^T b}$

but we have

$$\boxed{x = (Q^T Q)^{-1} (Q b)}$$

6) This calculation given $Q_{n \times m}$ & $b_{m \times 1}$ we require $\boxed{mn^2 \text{ flops}}$.

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11) A circulant matrix C is given as:

$$C = F^* \Lambda F$$

$\Rightarrow$ A & B can be written as:

$$A = F^* \, \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \, F$$

$$B = F^* \, \mathrm{Diag}(\rho_1, \rho_2, \ldots, \rho_n) \, F$$

$\Rightarrow$

$$AB = F^* \, \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \, F \cdot F^* \, \mathrm{Diag}(\rho_1, \rho_2, \ldots, \rho_n) \, F$$

[We also have $F^* = F^{-1}$]

$\Rightarrow$

$$AB = F^* \, \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \, \mathrm{Diag}(\rho_1, \rho_2, \ldots, \rho_n) \, F$$

$\boxed{\Rightarrow \text{Eigen values of } AB = \rho_1 \lambda_1, \rho_2 \lambda_2, \ldots, \rho_n \lambda_n}$

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7 $\|y - c_K\| + z = \rho_K$

$\rightarrow \|y - c_1\| + z = \rho_1$

$y^2 - 2c_1 y + c_1^T = \rho_1^2 - 2z\rho_1 + z^2$ $\hookrightarrow$ 1

$\rightarrow \|y - c_2\| + z = \rho_2$

$y^2 - 2c_2 y + c_2^T = \rho_2^2 - 2\rho_2 z + z^2$ $\hookrightarrow$ 2

2 - 1 & doing same for all

$$2\underbrace{\begin{bmatrix} c_1^T - c_2^T & \rho_1 - \rho_2 \\ c_1^T - c_3^T & \rho_1 - \rho_3 \\ c_1^T - c_4^T & \rho_1 - \rho_4 \\ c_1^T - c_5^T & \rho_1 - \rho_5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} \rho_2^2 - \rho_1^2 + c_1^2 - c_2^T \\ \rho_3^2 - \rho_1^2 + c_1^2 - c_3^2 \\ \rho_4^2 - \rho_1^2 + c_1^2 - c_4^2 \\ \rho_5^2 - \rho_1^2 + c_1^2 - c_5^2 \end{bmatrix}}_{b}$$