COT 5615 Math for Intelligent Systems, Fall 2021

Midterm 2

1. (a) Formulate the following problem as a set of linear equations. Find a point $\boldsymbol{x} \in \mathbb{R}^n$ at equal distance to $n+1$ given points $\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_{n+1} \in \mathbb{R}^n$:

$$\|\boldsymbol{x} - \boldsymbol{y}_1\| = \|\boldsymbol{x} - \boldsymbol{y}_2\| = \cdots = \|\boldsymbol{x} - \boldsymbol{y}_{n+1}\|.$$

   Write the equations in matrix form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

   (b) Show that the solution in part (a) is unique if the $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_{n+1} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

   is invertible

2. Formulate the following problems as least squares problems. For each problem, give a matrix $\boldsymbol{A}$ and a vector $\boldsymbol{b}$ such that the problem can be expressed as

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2$$

   (a) Minimize $x_1^2 + 2x_2^2 + 3x_3^2 + (x_1 - x_2 + x_3 + 1)^2 + (-x_1 - 4x_2 + 2)^2$

   (b) Minimize $(-6x_2 + 4)^2 + (-4x_1 + 3x_2 - 1)^2 + (x_1 + 8x_2 - 3)^2$

   (c) Minimize $2(-6x_1 + 4)^2 + 3(-4x_1 + 3x_2 - 1)^2 + 4(x_1 + 8x_2 - 3)^2$

   (d) Minimize $\boldsymbol{x}^\top\boldsymbol{x} + \|\boldsymbol{B}\boldsymbol{x} - \boldsymbol{d}\|^2$ where the $p \times n$ matrix $\boldsymbol{B}$ and p-vector $\boldsymbol{d}$ are given.

   (e) Minimize $\|\boldsymbol{B}\boldsymbol{x} - \boldsymbol{d}\|^2 + 2\|\boldsymbol{F}\boldsymbol{x} - \boldsymbol{g}\|^2$. The $p \times n$ matrix $\boldsymbol{B}$, $l \times n$ matrix $\boldsymbol{F}$, the p-vector $\boldsymbol{d}$ and the l-vector $\boldsymbol{g}$ are given.

   (f) Minimize $\boldsymbol{x}^\top\boldsymbol{D}\boldsymbol{x} + \|\boldsymbol{B}\boldsymbol{x} - \boldsymbol{d}\|^2$. $\boldsymbol{D}$ is a $n \times n$ diagonal matrix with positive diagonal elements, $\boldsymbol{B}$ is $p \times n$, and $\boldsymbol{d}$ is a p-vector. $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{d}$ are given.

3. A co-worker develops a classifier of the form $\hat{y} = \text{sign}(\boldsymbol{x}^\top\boldsymbol{w} + v)$, with $v < 0$, where the n-vector $\boldsymbol{x}$ is the feature vector, and the n-vector $\boldsymbol{w}$ and scalar v are the classifier parameters. The classifier is evaluated on a given test daata set. The false positive rate is the fraction of the test data points with $y = -1$ for which $\hat{y} = +1$. (We will assume there is at least one data point with $y = -1$.)

   Are each of the following statements true of false? True means it always holds, with no other assumptions on the data set or model; false meaans that it need not hold.

   (a) Replacing v with zero with reduce, or not increase, the false positive rate.

   (b) Replacing $\boldsymbol{w}$ with zero with reduce, or not increase, the false positive rate.

(c) Halving v (i.e., replacing v with $v/2$) with reduce, or not increase, the false positive rate.

(d) Halving $\boldsymbol{w}$ (i.e., replacing $\boldsymbol{w}$ with $\boldsymbol{w}/2$) with reduce, or not increase, the false positive rate.

4. Formulate the following problem as a least squares problem. Find a polynomial

$$p(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3$$

that satisfies the following conditions.

- The values $p(t_i)$ at 4 given points t_1, t_2, t_3, t_4 in the interval $[0, 1]$ is approximately equal to given values y_i:
$$p(t_i) \approx y_i, \quad i = 1, 2, 3, 4.$$
The points t_i are given and distinct ($t_i \neq t_j$ for $i \neq j$). The values y_i are also given.
- The derivatives of p at $t = 0$ and $t = 1$ are small:
$$p'(0) \approx 0, \quad p'(1) \approx 0.$$
- The average value of p over the interval $[0, 1]$ is approximately equal to the value at $t = 1/2$:
$$\int_0^1 p(t)dt \approx p(1/2).$$

To determine coefficients x_i that satisfy these conditions, we minimize

$$E(\boldsymbol{x}) = \frac{1}{4}\sum_{i=1}^{4}(p(t_i) - y_i)^2 + p'(0)^2 + p'(1)^2 + \left(\int_0^1 p(t)dt - p(1/2)\right)^2.$$

Given matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$ such that $E(\boldsymbol{x}) = \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2$. Clearly state the dimensions of $\boldsymbol{A}$ and $\boldsymbol{b}$, and what their elements are.

5. *Inverse of a block matrix.* Consider the $(n + 1) \times (n + 1)$ matrix

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{a} \\ \boldsymbol{a}^\top & 0 \end{bmatrix},$$

where $\boldsymbol{a}$ is a n-vector.

(a) When is $\boldsymbol{A}$ invertible? Give your answer in terms of $\boldsymbol{a}$. Justify your answer.

(b) Assuming the condition you found in part (a) holds, give an expression for the inverse matrix $\boldsymbol{A}^{-1}$.