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MIS

Mid-Term 2

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(a) $||x-y_1|| = ||x-y_2|| = \dots = ||x-y_{n+1}||$

⇒ finding a point x ~~to~~ equidistant to $n+1$ given points $y_1, y_2, \dots y_{n+1} \in R^n$ it can be expressed as $Ax = b$ as follows:

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$$\cancel{b = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n+1} \end{bmatrix} \quad (n+1)\times 1}$$

Since y is a n-vector

$$b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{bmatrix} \Rightarrow \underline{n(n+1) \times 1}$$

$$A = \begin{bmatrix} I \\ I \\ I \\ \vdots \\ I \end{bmatrix} \rightarrow \underline{(n+1) \text{ times}}$$

where each I is a Identity Matrix of size $\underline{n}$

$\Rightarrow$ dimension$(A) = \underline{n(n+1) \times n}$

$\boxed{x \text{ will be a } \underline{n \times 1} \text{ } \underline{\text{vector}}.}$

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ⓐ $$\underset{x}{\text{minimize}}\ \|Ax-b\|^2$$

$$x_1^2 + 2x_2^2 + 3x_3^2 + (x_1 - x_2 + x_3 + 1)^2$$
$$+ (-x_1 - 4x_2 + 2)^2$$

Representing it as a multi-objective least square problem, we get:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & -1 & 1 \\ -1 & -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -2 \end{bmatrix}$$

$$\underline{A} \qquad x = \underline{b}$$

ⓑ Similarly,

$$\begin{bmatrix} 0 & -6 & 0 \\ -4 & 3 & 0 \\ 1 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ +1 \\ +3 \end{bmatrix}$$

$$\underline{A} \qquad x = \underline{b}$$

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(c) Similarly,

$$\left(-6\sqrt{2}x_1 + 4\sqrt{2}\right)^2 + \left(-4\sqrt{3}x_1 + 3\sqrt{3}x_2 - \sqrt{3}\right)^2$$

$$+ \left(2x_1 + \overset{16}{\cancel{8}}x_2 - 6\right)^2$$

$$\begin{bmatrix} -6\sqrt{2} & 0 & 0 \\ -4\sqrt{3} & 3\sqrt{3} & 0 \\ 2 & 16 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4\sqrt{2} \\ \sqrt{3} \\ 6 \end{bmatrix}$$

$$\underline{A} \cdot x = \underline{b}$$

(d) Here we got,

$$\|x\|^2 + \|Bx - d\|^2$$

$$\begin{bmatrix} I_n \\ B \end{bmatrix} \begin{bmatrix} x \end{bmatrix}_{n\times 1} = \begin{bmatrix} 0_n \\ d \end{bmatrix}$$

$$\underline{A} \quad x \qquad \underline{b}$$

$$\underline{(P+n)\times n} \qquad \underline{(P+n)\times 1}$$

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c

(c) $\|Bx-d\|^2+\|\sqrt{2}Fx-\sqrt{2}g\|^2$

$$\begin{bmatrix} B \\ \sqrt{2}F \end{bmatrix} \begin{bmatrix} x \end{bmatrix}_n = \begin{bmatrix} d \\ \sqrt{2}g \end{bmatrix}$$

$$\underbrace{\underline{\underline{A}}}_{(p+l)\times n} \; x = \underbrace{\underline{\underline{b}}}_{(p+l)\times 1}$$

(d)

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3 $\hat{y} = \text{Sign}(x^T w + v)$

$v < 0$

(a) $v = 0$, Here v is already -ve,
thus it will increase the false positive rate.

$\boxed{\text{Ans} = \text{False}}$

$\hat{y} = \text{Sign}(x^T w)$

(b) $w = 0$, $\hat{y} = \text{Sign}(v)$

Here as v is -ve, everything will be negative.

Thus, there will be no false positives.

Thus, $\boxed{\text{Ans} = \text{True}}$

(c) $v = v/2$, $\hat{y} = \text{Sign}(x^T w + v/2)$

Here as v is -ve and we decrease it, the overall false positive rate will increase

Thus, $\boxed{\text{Ans} = \text{False}}$

d $w = w/2 \Rightarrow \hat{y} = \text{sign}\left(x^T \frac{w}{2} + v\right)$

Here, the change in w will either make the overall false positive rate constant or, will decrease it.

Thus, $\boxed{\text{Ans} = \text{True}}$

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4 $F(x) = \frac{1}{4}\sum_{i=1}^{4}\left(P(t_i) - y_i\right)^2 + P'(0)^2 + P'(1)^2$

$$+ \left(\int_0^1 P(t)\,dt - P(1/2)\right)^2$$

$\Rightarrow$ Multi-Objective least-square solution.

$J_1 = \frac{1}{4}\sum_{i=1}^{4}\left(P(t_i) - y_i\right)^2,$

$J_2 = P'(0)^2$ $\quad J_4 = \left(\int_0^1 P(t)\,dt - P(1/2)\right)^2$

$J_3 = P'(1)^2$

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$\rightarrow J_1 \Rightarrow$

$$\underbrace{\begin{bmatrix} \frac{1}{2} & \frac{t_1}{2} & \frac{t_1^2}{2} & \frac{t_1^3}{2} \\ \frac{1}{2} & \frac{t_2}{2} & \frac{t_2^2}{2} & \frac{t_2^3}{2} \\ \frac{1}{2} & \frac{t_3}{2} & \frac{t_3^2}{2} & \frac{t_3^3}{2} \\ \frac{1}{2} & \frac{t_4}{2} & \frac{t_4^2}{2} & \frac{t_4^3}{2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} y_1/2 \\ y_2/2 \\ y_3/2 \\ y_4/2 \end{bmatrix}}_{b}$$

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$\rightarrow J_2 \Rightarrow \quad p'(0)$

$x_2 + 2x_3(0) + 3x_4(0) = 0$

$x_2 = 0$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b} \quad \begin{bmatrix} 0 \end{bmatrix}$$

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$\rightarrow J_3$

$$p'(1) = 0$$

$$x_2 + 2x_3 + 3x_4 = 0$$

$$\underline{\begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underline{\begin{bmatrix} 0 \end{bmatrix}}_{b} \rightarrow B$$

$\rightarrow J_4$

$$\left[x_1 t + \frac{x_2 t^2}{2} + \frac{x_3 t^3}{3} + \frac{x_4 t^4}{4} \right]_0^1$$

$$\Rightarrow - \left[x_1 + \frac{x_2}{2} + \frac{x_3}{4} + \frac{x_4}{8} \right] = [0]$$

$$\Rightarrow \cancel{x_1} + \cancel{\frac{x_2}{2}} + \frac{x_3}{3} + \frac{x_4}{4}$$

$$- \cancel{x_1} - \cancel{\frac{x_2}{2}} - \frac{x_3}{4} - \frac{x_4}{8} = [0]$$

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$$\Rightarrow x_3 \frac{1}{12} + x_4 \frac{1}{8} = 0$$

$$\Rightarrow \underline{\underset{A}{\begin{bmatrix} 0 & 0 & \frac{1}{12} & \frac{1}{8} \end{bmatrix}}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underline{\underset{b}{\begin{bmatrix} 0 \end{bmatrix}}} \rightarrow 5$$

Combine 1, 2, 3, 4

$$\begin{bmatrix} \frac{1}{2} & \frac{t_1}{2} & \frac{t_1^2}{2} & \frac{t_1^3}{2} \\ 1/2 & t_2/2 & t_2^2/2 & t_2^3/2 \\ 1/2 & t_3/2 & t_3^2/2 & t_3^3/2 \\ 1/2 & t_4/2 & t_4^2/2 & t_4^3/2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1/12 & 1/8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1/2 \\ y_2/2 \\ y_3/2 \\ y_4/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{(7 \times 4)}{\underline{A}} \; x = \underset{(7 \times 1)}{\underline{b}}$$

$\boxed{5}$ A is invertible if and only if $a \neq 0$

Using square matrix's property,

$Ax = 0$ implies $x = 0$

Assume $x = (y, z)$

$$Ax = \begin{bmatrix} 1 & a \\ a^T & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y + az \\ a^T x \end{bmatrix}$$

$y =$ n-vector
$z =$ scalar

If $a = 0$, $y = 0, z = 1 \Rightarrow$ A is not invertible

If $a \neq 0$, implies $z = 0$. If $z = 0$, $y = -za = 0$. Thus A is invertible

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b

$A^{-1} \Rightarrow$

$$\begin{bmatrix} I & a \\ a^T & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$x_1 + a x_2 = b_1 \Rightarrow x_1 = b_1 - x_2 a$

$a^T x_1 = b_2$

$b_2 = a^T(b_1 - x_2 a) \Rightarrow a^T b_1 - x_2 \|a\|^2$

$\Rightarrow$ solve for x_2,

$$x_2 = (a^T b_1 - b_2) / \|a\|^2, \quad x_1 = b_1 - \left(\frac{a^T b_1 - b_2}{\|a\|^2}\right) a$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\|a\|^2} \begin{bmatrix} \|a\|^2 I - a a^T & a \\ a^T & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\boxed{\Rightarrow A = \frac{1}{\|a\|^2} \begin{bmatrix} \|a\|^2 I - a a^T & a \\ a^T & -1 \end{bmatrix}}$$

$\boxed{2}$ f) $x^T D x + \|Bx - d\|^2$

$\|\sqrt{D}\, x\|^2 + \|Bx - d\|^2$

$$\underbrace{\begin{bmatrix} \sqrt{D} \\ B \end{bmatrix}}_{\substack{A \\ (n+p) \times n}} \begin{bmatrix} x \end{bmatrix} = \underbrace{\begin{bmatrix} 0_n \\ d \end{bmatrix}}_{(n+p) \times 1}$$

$x = b$

$\sqrt{D}$ = root each value of D [i.e. root of each diag. value]