

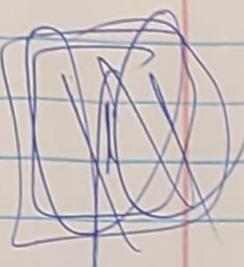
1

MIS

Mid-Term 2

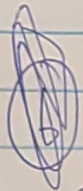
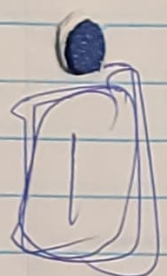
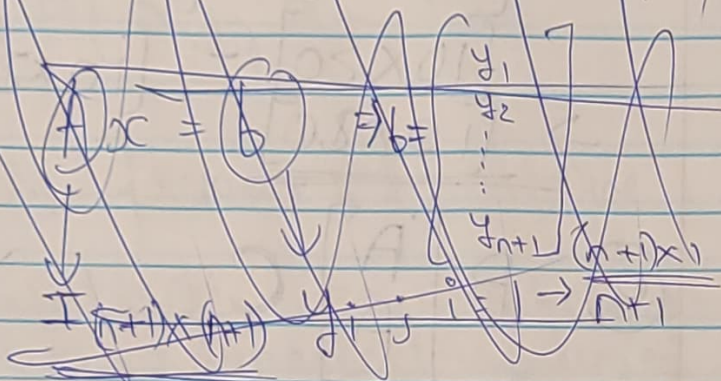
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(a)

$$\|x - y_1\| = \|x - y_2\| = \dots = \|x - y_{n+1}\|$$



(a)  $\|x - y_1\| = \|x - y_2\| = \dots = \|x - y_{n+1}\|$

$\Rightarrow$  finding a point  $x$  to equidistant to  $n+1$  given points  $y_1, y_2, \dots, y_{n+1} \in \mathbb{R}^n$  it can be expressed as  $Ax = b$  as follows :

②

~~$$b = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n+1} \end{bmatrix}$$

$$(n+1) \times 1$$~~

Since  $y$  is a  $n$ -vector

$$b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{bmatrix}$$

$$\Rightarrow \underline{n(n+1) \times 1}$$

$$A = \begin{bmatrix} I \\ I \\ I \\ \vdots \\ I \end{bmatrix}$$

$\Rightarrow \underline{(n+1) \text{ times}}$

where each  $I$  is a Identity Matrix of size  $n$

$$\Rightarrow \underline{\text{dimension}(A) = n(n+1) \times n}$$

$x$  will be a  $n \times 1$  vector.



3

2

a) minimize  $\|Ax - b\|^2$   
 $x$

$$x_1^2 + 2x_2^2 + 3x_3^2 + (x_1 - x_2 + x_3 + 1)^2 + (-x_1 - 4x_2 + 2)^2$$

Representing it as a multi-objective least square problem, we get:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & -1 & 1 \\ -1 & -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -2 \end{bmatrix}$$

$$\underline{A} \quad x = \underline{b}$$

b) Similarly,

$$\begin{bmatrix} 0 & -6 & 0 \\ -4 & 3 & 0 \\ 1 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ +1 \\ +3 \end{bmatrix}$$

$$\underline{A} \quad x = \underline{b}$$

④

© Similarly,

$$(-6\sqrt{2}x_1 + 4\sqrt{2})^2 + (-4\sqrt{3}x_1 + 3\sqrt{3}x_2 - \sqrt{3})^2 + (2x_1 + 16x_2 - 6)^2$$

$$\begin{bmatrix} -6\sqrt{2} & 0 & 0 \\ -4\sqrt{3} & 3\sqrt{3} & 0 \\ 2 & 16 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4\sqrt{2} \\ \sqrt{3} \\ 6 \end{bmatrix}$$

$$\underline{A} \quad x = \underline{b}$$

⑤ Here we get,

$$\|x\|^2 + \|Bx - d\|^2$$

$$\begin{bmatrix} I_n \\ B \end{bmatrix} \begin{bmatrix} x \end{bmatrix}_{n \times 1} = \begin{bmatrix} 0_n \\ d \end{bmatrix}$$

$$\underline{A} \quad x$$

$$\underline{b}$$

$$\underline{(P+n) \times n}$$

$$\underline{(P+n) \times 1}$$

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$$(c) \|Bx - d\|^2 + \|\sqrt{2}Fx - \sqrt{2}g\|^2$$

$$\begin{bmatrix} B \\ \sqrt{2}F \end{bmatrix} \begin{bmatrix} x \end{bmatrix}_n = \begin{bmatrix} d \\ \sqrt{2}g \end{bmatrix}$$

$$\underline{A} \quad x = \underline{b}$$

$$\underline{(P+1) \times n}$$

$$\underline{(P+1) \times 1}$$



6.

3

$$\hat{y} = \text{Sign}(x^T w + v)$$

$$\underline{v < 0}$$

- (a)  $v = 0$ , Here  $v$  is already  $-ve$ , thus it will increase the false positive rate.

Ans = False

$$\hat{y} = \text{Sign}(x^T w)$$

(b)  $w = 0$ ,  $\hat{y} = \text{Sign}(v)$

Here as  $v$  is  $-ve$ , everything will be negative.

Thus, there will be no false positives.

Thus, Ans = True

(c)  $v = 1/2$ ,  $\hat{y} = \text{Sign}(x^T w + 1/2)$

Here as  $v$  is  $-ve$  and we decrease it, the overall false positive rate will increase

Thus, Ans = False

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d

$$w = \frac{w}{2} \Rightarrow \hat{y} = \text{Sign}\left(x^T \frac{w}{2} + v\right)$$

Here, the change in  $w$  will either make the overall false positive rate constant or, will decrease it.

Thus, Ans = True

4

$$F(x) = \frac{1}{4} \sum_{i=1}^4 (P(x_i) - y_i)^2 + P'(0)^2 + P'(1)^2 + \left( \int_0^1 P(t) dt - P(1/2) \right)^2$$

$\Rightarrow$  Multi-Objective least-square solution.

$$J_1 = \frac{1}{4} \sum_{i=1}^4 (P(x_i) - y_i)^2$$

$$J_2 = P'(0)^2 \quad J_4 = \left( \int_0^1 P(t) dt - P(1/2) \right)^2$$

$$J_3 = P'(1)^2$$



⑧

$$\rightarrow J_1 \Rightarrow \begin{bmatrix} \frac{1}{2} & \frac{t_1}{2} & \frac{t_1^2}{2} & \frac{t_1^3}{2} \\ \frac{1}{2} & \frac{t_2}{2} & \frac{t_2^2}{2} & \frac{t_2^3}{2} \\ \frac{1}{2} & \frac{t_3}{2} & \frac{t_3^2}{2} & \frac{t_3^3}{2} \\ \frac{1}{2} & \frac{t_4}{2} & \frac{t_4^2}{2} & \frac{t_4^3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1/2 \\ y_2/2 \\ y_3/2 \\ y_4/2 \end{bmatrix}$$

①  $\underline{A}$   $x = \underline{b}$

$$\rightarrow J_2 \Rightarrow P'(0)$$

$$x_2 + 2x_3 + 3x_4(0) = 0$$

$$x_2 = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{A}$   $x = \underline{b}$

$\hookrightarrow$  ②



(9)

 $\rightarrow J_3$ 

$$P'(1) \approx 0$$

$$x_2 + 2x_3 + 3x_4 = 0$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \rightarrow \textcircled{3}$$

A~~x~~b $\rightarrow J_4$ 

$$\begin{aligned} & \left[ x_1 t + \frac{x_2 t^2}{2} + \frac{x_3 t^3}{3} + \frac{x_4 t^4}{4} \right]_0^1 \\ & \Rightarrow \left[ x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \frac{x_4}{4} \right] = \begin{bmatrix} 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} & \Rightarrow \cancel{x_1} + \cancel{\frac{x_2}{2}} + \cancel{\frac{x_3}{3}} + \cancel{\frac{x_4}{4}} \\ & \quad - \cancel{x_1} - \cancel{\frac{x_2}{2}} - \cancel{\frac{x_3}{3}} - \cancel{\frac{x_4}{4}} = \begin{bmatrix} 0 \end{bmatrix} \end{aligned}$$

(10)

$$\Rightarrow x_3 \frac{1}{12} + x_4 \frac{1}{8} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & \frac{1}{12} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \rightarrow \textcircled{4}$$

A

x

b

Combine  $\textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}$

$$\left[ \begin{array}{cccc|c} \frac{1}{2} & \frac{d_1}{2} & \frac{d_2}{2} & \frac{d_3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{d_2}{2} & \frac{d_3}{2} & \frac{d_4}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{d_3}{2} & \frac{d_4}{2} & \frac{d_5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{d_4}{2} & \frac{d_5}{2} & \frac{d_6}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & \frac{1}{12} & \frac{1}{8} & 0 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} \\ \frac{4}{2} \\ \frac{4}{2} \\ \frac{4}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A

x

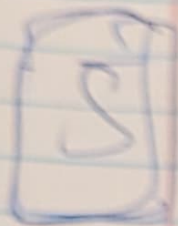
b

(7 x 4)

(7 x 1)



(11)



$A$  is invertible if and only if  $a \neq 0$

Using square matrix's property,

$$Ax = 0 \text{ implies } x = 0$$

$$\text{Assume } x = (y, z)$$

$$Ax = \begin{bmatrix} 1 & a \\ a^T & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y + az \\ a^T x \end{bmatrix}$$

$$y = n\text{-vector}$$

$$z = \text{scalar}$$

If  $a = 0$ ,  $y = 0, z = 1 \Rightarrow A$  is not invertible

If  $a \neq 0$ , implies  $z = 0$ . If  $z = 0$ ,  
 $y = -za = 0$ . Thus  $A$  is invertible

17

6

$$A^{-1} \Rightarrow$$

$$\begin{bmatrix} 1 & a \\ a^T & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x_1 + ax_2 = b_1 \Rightarrow x_1 = b_1 - x_2 a$$

$$a^T x_1 = b_2$$

$$b_2 = a^T (b_1 - x_2 a) \Rightarrow a^T b_1 - x_2 \|a\|^2$$

$\Rightarrow$  solve for  $x_2$ ,

$$x_2 = (a^T b_1 - b_2) / \|a\|^2, x_1 = b_1 -$$

$$\left( \frac{a^T b_1 - b_2}{\|a\|^2} \right) a$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\|a\|^2} \begin{bmatrix} \|a\|^2 I - aa^T & a \\ a^T & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow A = \frac{1}{\|a\|^2} \begin{bmatrix} \|a\|^2 I - aa^T & a \\ a^T & -1 \end{bmatrix}$$



(13)

$$\boxed{2} \quad x^T D x + \|Bx - d\|^2$$

$$\| \sqrt{D} x \|^2 + \|Bx - d\|^2$$

$$\begin{bmatrix} \sqrt{D} \\ B \end{bmatrix} [x] = \begin{bmatrix} 0_n \\ d \end{bmatrix}$$

$$\begin{matrix} (n+p) \times n & = & (n+p) \times 1 \end{matrix}$$

$\sqrt{D} \Rightarrow$  root each value of D [i.e. root of each diag. value]