COT 5615 Math for Intelligent Systems, Fall 2021

Homework 5

10.11 (20 pts)

(a) The diagonal entries of $A^T B$ are

$$\left(A^T B\right)_{jj} = \sum_{i=1}^{m} \left(A^T\right)_{ji} B_{ij} = \sum_{i=1}^{m} A_{ij} B_{ij}$$

The complexity is $2mn$ flops. We need mn multiplications and $mn-1$ additions. Note that this is lower than the $2mn^2$ complexity of the entire matrix-matrix product $A^T B$.

(b) The matrix $B^T A$ is the transpose of $A^T B$, so it has the same diagonal entries and trace.

(c) From part (a), $A^T A = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2 = \|A\|^2$.

(d) We have

$$\mathbf{tr}\left(BA^T\right) = \sum_{i=1}^{m} \left(BA^T\right)_{ii} = \sum_{i=1}^{m} \sum_{j=1}^{n} B_{ij} \left(A^T\right)_{ji} = \sum_{i=1}^{m} \sum_{j=1}^{n} B_{ij} A_{ij}$$

the same expression as in part (a).

10.13 (15 pts)

(a) We have

$$v^T L v = v^T A A^T v = \left(A^T v\right)^T \left(A^T v\right) = \left\|A^T v\right\|^2 = \mathcal{D}(v)$$

(b) L_{ii} is the degree of node i. L_{ij} is minus the total number of edges that connect nodes i and j, in either direction. (Note that the direction of the edge connecting i and j does not affect L_{ij}.)

To see this, we first consider the expression for L_{ii}

$$L_{ii} = \left(AA^T\right)_{ii} = \sum_{k=1}^{m} A_{ik}^2$$

Each term A_{ik}^2 in the sum is zero or one. It is one if edge k arrives at node i $(A_{ik}=1)$ or leaves node $i\,(A_{ik}=-1)$, and it is zero if edge k is not incident to node i. Therefore L_{ii} is the degree of node i (the total number of edges connected to node i).

Suppose $i \neq j$. Then

$$L_{ij} = \left(AA^T\right)_{ij} = \sum_{k=1}^{m} A_{ik} A_{jk}$$

Each term is a product of two entries in column k of the incidence matrix. Now, each column has exactly two nonzero entries, $A_{ik} = 1$ if i is the node at which edge j ends and $A_{ik} = -1$ if i is the node at which it leaves. The kth term $A_{ik}A_{jk}$ in the sum is therefore -1 or 0. It is equal to -1 only if edge k is between nodes i and j. By summing over k we obtain the negative of the number of edges between nodes i and j.

10.31 (15 pts)

(a) The number of paths from j to i of length ℓ is given by $(A^\ell)_{ij}$. So the total number of paths of length $\leq \ell$ is P_{ij} , where

$$P = I + A + A^2 + \cdots + A^k.$$

(b) To find the diameter, we evaluate P for $k = 1, 2, \cdots$ and stop the first time P has all entries positive.

11.3 (15 pts) Here is an

(a) Take $A = [1, 0] \in \mathbb{R}^{1\times 2}$, and $X = (0, 1), Y = (0, 0) \in \mathbb{R}^{2\times 1}$.

(b) Let B be a left inverse of A. Multiply $AX = AY$ on the left by B to get $B(AX) = B(AY)$. Re-associate as $(BA)X = (BA)Y$. From $BA = I$ we conclude that $X = Y$.

(c) Now suppose that A is not left-invertible. This means that its columns are linearly dependent. This means that there is a nonzero vector x for which $Ax = 0$. Then we have $Ax = Ay$, with $y = 0$, but $x \neq y$. This is a counterexample, where we simply consider x and y as $n \times 1$ matrices.

11.11 (18 pts) We write the five interpolation conditions as

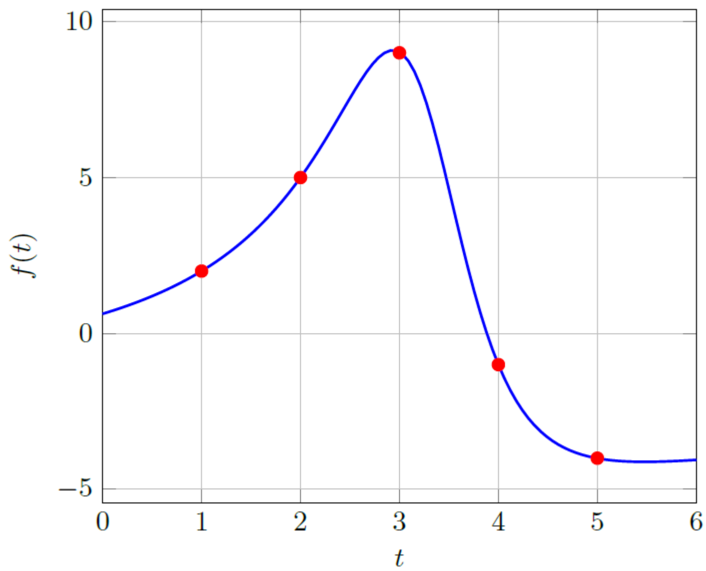
$$\begin{aligned}
c_1 + c_2 + c_3 &= 2(1 + d_1 + d_2) \\
c_1 + 2c_2 + 4c_3 &= 5(1 + 2d_1 + 4d_2) \\
c_1 + 3c_2 + 9c_3 &= 9(1 + 3d_1 + 9d_2) \\
c_1 + 4c_2 + 16c_3 &= -(1 + 4d_1 + 16d_2) \\
c_1 + 5c_2 + 25c_3 &= -4(1 + 5d_1 + 25d_2)
\end{aligned}$$

This is a set of linear equations in five variables

$$\begin{bmatrix} 1 & 1 & 1 & -2 & -2 \\ 1 & 2 & 4 & -10 & -20 \\ 1 & 3 & 9 & -27 & -81 \\ 1 & 4 & 16 & 4 & 16 \\ 1 & 5 & 25 & 20 & 100 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \\ -1 \\ -4 \end{bmatrix}.$$

The solution is

$$c_1 = 0.6296, \quad c_2 = 0.6049, \quad c_3 = -0.1975, \quad d_1 = -0.5679, \quad d_2 = 0.0864.$$



11.21 (17 pts) We solve the equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^3 & t_2^3 & t_3^3 & t_4^3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix},$$

where

$$b_k = \int_{-1}^{1} t^{k-1} dt = \begin{cases} 2/k & k \text{ is odd} \\ 0 & k \text{ is even.} \end{cases}$$

The solution is

$$w_1 = 0.9167, \quad w_2 = 0.0833, \quad w_3 = 0.0833, \quad w_4 = 0.9167$$

For $f(t) = e^x$, we know $\int_{-1}^{1} e^x dx = e^1 - e^{-1}$. We find

$$\alpha = 2.3504, \quad \hat{\alpha} = 2.3434.$$

This is a reasonably good approximation.