COT 5615 Math for Intelligent Systems, Fall 2021

Homework 1

1.7 (10 pts) We have $y = 2x - 1$. To see this, we note that $y_i = 2x_i - 1$. When $x_i = 1$, we have $y_1 = 2 \cdot 1 - 1 = +1$; when $x_i = 0$, we have $y_1 = 2 \cdot 0 - 1 = -1$.

Conversely, we have $x = (1/2)(y + \mathbf{1})$.

1.13 (20 pts)

(a) The total population is $\mathbf{1}^T x$.

(b) The total number of people aged 65 or over is given by $a^T x$, where $a = (\mathbf{0}_{65}, \mathbf{1}_{35})$. (The subscripts give the dimensions of the zero and ones vectors.)

(c) The sum of the ages across the population is $(0, 1, 2, \dots, 99)^T x$. And so the average age is given by

$$\frac{(0, 1, 2, \dots, 99)^T x}{\mathbf{1}^T x}.$$

1.17 (15 pts) We are asked to write the T-vector

$$c = (1, 0, \dots, 0, -(1+r)^{T-1})$$

as a linear combination of the $T - 1$ vectors

$$l_t = (0, \dots, 0, 1, -(1+r), 0, \dots, 0), \quad t = 1, \dots, T-1.$$

In the definition of l_t there are $t - 1$ leading and $T - t - 1$ trailing zeros, *i.e.*, the element 1 is in position t. There is only one way to do this:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -(1+r)^{T-1} \end{bmatrix} = \begin{bmatrix} 1 \\ -(1+r) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + (1+r) \begin{bmatrix} 0 \\ 1 \\ -(1+r) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$+ (1+r)^2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -(1+r) \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \cdots + (1+r)^{T-2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ -(1+r) \end{bmatrix}$$

In other words,

$$c = l_1 + (1+r)l_2 + (1+r)^2 l_3 + \cdots + (1+r)^{T-2} l_{T-1}$$

The coefficients in the linear combination are $1, 1+r, (1+r)^2, \cdots, (1+r)^{T-1}$.

The idea is that you extend the length of an initial loan by taking out a new loan each period to cover the amount that you owe. So after taking out a loan for \$1 in period 1, you take out a loan for $\$(1+r)$ in period 2, and end up owing $\$(1+r)^2$ in period 3. Then you take out a loan for $\$(1+r)^2$ in period 3, and end up owing $\$(1+r)^3$ in period 4, and so on.

A1.2 (20 pts)

(a) $a = (0, 0, 0, 0, 1, 0, 0, 0, 0, 0)$

(b) $a = (0.3, 0.4, 0.3)$

(c) $a = (0, 0, 0, 1, 0, 0, -1, 1, 0, 0, 0, 1, 0, -1, 0, 1, 0, 0, 0, 1, -1, 0)$

(d) $a = (0, 0, 0, 1/5, 1/5, 1/5, 1/5, 1/5, 0, 0, 0)$

A1.6 (10 pts) Computing the inner product of two n-vectors takes $2n-1$ flops, n scalar multiplications and $n-1$ scalar additions.

So the complexity of the inner product of two 10^7-vectors is about 10 times the complexity of the inner product of two 10^6-vectors, which takes around 0.01 second.

2.4 (10 pts) The correct answer is: *ϕ cannot be linear.* To see this, we note that the third point $(1, -1, -1)$ is the negative of the second point $(-1, 1, 1)$. If ϕ were linear, the two values of ϕ would need to be negatives of each other. But they are 1 and 1, not negatives of each other.

2.8 (15 pts)

(a) The integral is

$$\int_\alpha^\beta p(x)dx = c_1(\beta - \alpha) + \frac{c_2}{2}(\beta^2 - \alpha^2) + \frac{c_3}{3}(\beta^3 - \alpha^3) + \cdots + \frac{c_n}{n}(\beta^n - \alpha^n)$$

Therefore

$$a = (\beta - \alpha, \frac{\beta^2 - \alpha^2}{2}, \frac{\beta^3 - \alpha^3}{3}, \ldots, \frac{\beta^n - \alpha^n}{n}).$$

(b) The derivative at $\hat{x}$ is

$$p'(\alpha) = c_2 + 2c_3\alpha + 3c_4\alpha^2 + \cdots + c_n(n-1)\alpha^{n-2}.$$

Therefore

$$b = (0, 1, 2\alpha, 3\alpha^2, \ldots, (n-1)\alpha^{n-2}).$$