Mathematical Biostatistics Boot Camp: Lecture 2, Probability

Brian Caffo

Department of Biostatistics
Johns Hopkins Bloomberg School of Public Health
Johns Hopkins University

June 26, 2012

Table of contents

Probability measures

A **probability measure**, P, is a real valued function from the collection of possible events so that the following hold

1. For an event $E \subset \Omega$, $0 \leq P(E) \leq 1$
2. $P(\Omega) = 1$
3. If E_1 and E_2 are mutually exclusive events $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

Additivity

Part 3 of the definition implies **finite additivity**

$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$$

where the $\{A_i\}$ are mutually exclusive.
This is usually extended to **countable additivity**

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Note

- P is defined on $\mathcal{F}$ a collection of subsets of Ω
- Example $\Omega = \{1, 2, 3\}$ then

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

- When Ω is a continuous set, the definition gets much trickier. In this case we assume that $\mathcal{F}$ is sufficiently rich so that any set that we're interested in will be in it.

Consequences

You should be able to prove all of the following:

- $P(\emptyset) = 0$
- $P(E) = 1 - P(E^c)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- if $A \subset B$ then $P(A) \leq P(B)$
- $P\left(A \cup B\right) = 1 - P(A^c \cap B^c)$
- $P(A \cap B^c) = P(A) - P(A \cap B)$
- $P(\cup_{i=1}^n E_i) \leq \sum_{i=1}^n P(E_i)$
- $P(\cup_{i=1}^n E_i) \geq \max_i P(E_i)$

Example

Proof that $P(E) = 1 - P(E^c)$

$$\begin{aligned} 1 &= P(\Omega) \\ &= P(E \cup E^c) \\ &= P(E) + P(E^c) \end{aligned}$$

$\blacksquare$

Example

Proof that $P(\cup_{i=1}^n E_i) \leq \sum_{i=1}^n P(E_i)$

$$
\begin{aligned}
P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\
&\leq P(E_1) + P(E_2)
\end{aligned}
$$

Assume the statement is true for $n - 1$ and consider n

$$
\begin{aligned}
P(\cup_{i=1}^n E_i) &\leq P(E_n) + P(\cup_{i=1}^{n-1} E_i) \\
&\leq P(E_n) + \sum_{i=1}^{n-1} P(E_i) \\
&= \sum_{i=1}^{n} P(E_i)
\end{aligned}
$$

$\blacksquare$

Example

The National Sleep Foundation (`www.sleepfoundation.org`) reports that around 3% of the American population has sleep apnea. They also report that around 10% of the North American and European population has restless leg syndrome. Similarly, they report that 58% of adults in the US experience insomnia. Does this imply that 71% of people will have at least one sleep problems of these sorts?

Example continued

Answer: No, the events are not mutually exclusive. To elaborate let:

$$
\begin{aligned}
A_1 &= \{\text{Person has sleep apnea}\} \\
A_2 &= \{\text{Person has RLS}\} \\
A_3 &= \{\text{Person has insomnia}\}
\end{aligned}
$$

Then (work out the details for yourself)

$$
\begin{aligned}
P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\
&- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\
&+ P(A_1 \cap A_2 \cap A_3) \\
&= .71 + \text{Other stuff}
\end{aligned}
$$

where the "Other stuff" has to be less than 0

Random variables

- A **random variable** is a numerical outcome of an experiment.
- The random variables that we study will come in two varieties, **discrete** or **continuous**.
- Discrete random variable are random variables that take on only a countable number of possibilities.
  - $P(X = k)$
- Continuous random variable can take any value on the real line or some subset of the real line.
  - $P(X \in A)$

Examples of variables that can be thought of as random variables

- The $(0-1)$ outcome of the flip of a coin
- The outcome from the roll of a die
- The BMI of a subject four years after a baseline measurement
- The hypertension status of a subject randomly drawn from a population

PMF

A probability mass function evaluated at a value corresponds to the probability that a random variable takes that value. To be a valid pmf a function, p, must satisfy

1. $p(x) \geq 0$ for all x
2. $\sum_x p(x) = 1$

The sum is taken over all of the possible values for x.

Example

Let X be the result of a coin flip where $X = 0$ represents tails and $X = 1$ represents heads.

$$p(x) = (1/2)^x (1/2)^{1-x} \quad \text{for} \quad x = 0, 1$$

Suppose that we do not know whether or not the coin is fair; Let θ be the probability of a head

$$p(x) = \theta^x (1-\theta)^{1-x} \quad \text{for} \quad x = 0, 1$$

PDF

A probability density function (pdf), is a function associated with a continuous random variable

> *Areas under pdfs correspond to probabilities for that random variable*

To be a valid pdf, a function f must satisfy

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Example

Assume that the time in years from diagnosis until death of persons with a specific kind of cancer follows a density like

$$f(x) = \begin{cases} \frac{e^{-x/5}}{5} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

More compactly written: $f(x) = \frac{1}{5}e^{-x/5}$ for $x > 0$.
Is this a valid density?

1. e raised to any power is always positive
2. $$\int_0^\infty f(x)dx = \int_0^\infty e^{-x/5}/5dx = -e^{-x/5}\Big|_0^\infty = 1$$

Example continued

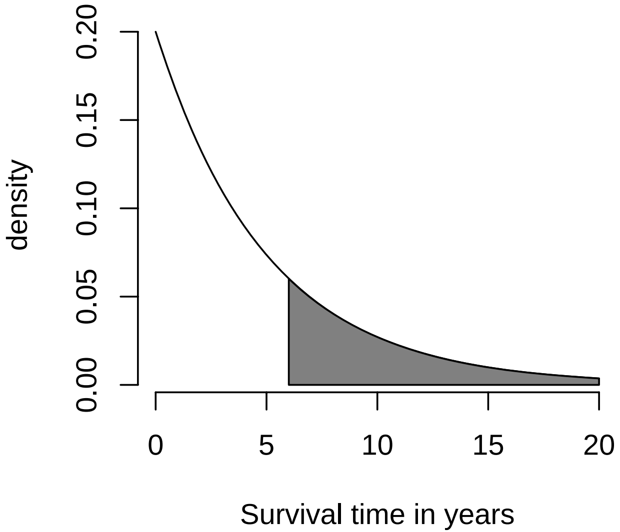
What's the probability that a randomly selected person from this distribution survives more than 6 years?

$$P(X \geq 6) = \int_6^\infty \frac{e^{-t/5}}{5} dt = -e^{-t/5}\Big|_6^\infty = e^{-6/5} \approx .301.$$

Approximation in R

```
pexp(6, 1/5, lower.tail = FALSE)
```

Example continued



CDF and survival function

- The **cumulative distribution function** (CDF) of a random variable X is defined as the function

$$F(x) = P(X \leq x)$$

- This definition applies regardless of whether X is discrete or continuous.
- The **survival function** of a random variable X is defined as

$$S(x) = P(X > x)$$

- Notice that $S(x) = 1 - F(x)$
- For continuous random variables, the PDF is the derivative of the CDF

Example

What are the survival function and CDF from the exponential density considered before?

$$S(x) = \int_x^\infty \frac{e^{-t/5}}{5} dt = -e^{-t/5}\Big|_x^\infty = e^{-x/5}$$

hence we know that

$$F(x) = 1 - S(x) = 1 - e^{-x/5}$$

Notice that we can recover the PDF by

$$f(x) = F'(x) = \frac{d}{dx}(1 - e^{-x/5}) = e^{-x/5}/5$$

Quantiles

- The α^{th}**quantile** of a distribution with distribution function F is the point x_α so that

$$F(x_\alpha) = \alpha$$

- A **percentile** is simply a quantile with α expressed as a percent
- The **median** is the 50^{th} percentile

Example

- What is the 25^{th} percentile of the exponential survival distribution considered before?
- We want to solve (for x)

$$
\begin{aligned}
.25 &= F(x) \\
&= 1 - e^{-x/5}
\end{aligned}
$$

resulting in the solution $x = -\log(.75) \times 5 \approx 1.44$

- Therefore, 25% of the subjects from this population live less than 1.44 years
- R can approximate exponential quantiles for you `qexp(.25, 1/5)`