

# Linear Algebra: Homework #7

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0130

The lectures from 18 to 28 contains a lot of information, so before we start the homework, let's give a quick review of those stuffs.

**First of all**, let's talk about difference equations. The question is, given  $u_0$ , and the recursive formula  $u_{k+1} = Au_k$ , how can I compute  $u_k$ ?

Suppose that  $A$  is diagonalizable, this means that the eigenvectors  $x_1, x_2, \dots, x_n$  of  $A$  is a basis of the whole space, so we can project  $u_0$  onto the space spanned by these eigenvectors:

$$u_0 = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

using the fact that  $Ax_i = \lambda_i x_i$ , we have

$$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \dots + c_n \lambda_n^k x_n$$

a very interesting example is the Fibonacci sequence  $F_{n+1} = F_n + F_{n-1}$ , a trick to solve this is to add a equation  $F_n = F_n$  and define  $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$  and  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ , then we have  $u_{k+1} = Au_k$ , and we can compute  $u_k$  using the method above.

**Secondly**, let's talk about the differential equations. What we are considering is the equation  $u' = Au$ , projecting  $u$  onto the space spanned by the eigenvectors of  $A$ , we then have  $u = Sv$ , the differential equation becomes  $Sv' = ASv$ , multiplying  $S^{-1}$  on both sides, we have  $v' = S^{-1}ASv = \Lambda v$ , and the solution of  $v' = \Lambda v$  is  $v = e^{\Lambda t}v(0)$ , where the exponential of a matrix is defined as  $e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$ , and the form of  $e^{\Lambda t}$  is very simple:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

and we have  $u = Se^{\Lambda t}v(0) = Se^{\Lambda t}S^{-1}u(0) = e^{At}u(0)$ .

Now consider the two order differential equation, for example,  $y'' + by' + ky = 0$

using the same trick in difference equations, we can add an equation  $y' = y'$ , then we have

$$\begin{cases} y'' + by' + ky = 0 \\ y' = y' \end{cases} \xRightarrow{u = \begin{bmatrix} y' \\ y \end{bmatrix}} u' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} u$$

then we can solve this using the method above.

**Thirdly**, let's talk about Markov matrices, the special properties of Markov matrices are:

1. Every entry is non-negative.
2. Every column sums to 1.

Markov matrices have a very special property: it has an eigenvalue 1, this can be proved by the fact that the sum of every column is 1, so if we consider the matrix  $A - I$ , this matrix has the property that every

column sums to 0, then we have  $(A - I) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \Rightarrow A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ . This property can be used to calculate the eigenvalues of Markov matrices.

**The last thing** of the application of eigenvalues and eigenvectors we need to talk about is Fourier series, instead of considering the traditional vector space, we consider the space of functions, and the basis of this space is  $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots$ , what we are trying to do is to project a function onto this space, so we are trying to rewrite function  $f(x)$  as:

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots + a_n \cos nx + b_n \sin nx + \dots$$

and the great property of this basis is that they are orthogonal (the inner product of two functions  $f(x)$  and  $g(x)$  is  $\int_0^{2\pi} f(x)g(x)dx$ ), and then if we want to determine the value of parameters, for example,  $a_2$ , we can do inner product with  $\cos 2x$  on both sides, then we have:

$$\int_0^{2\pi} f(x) \cos 2x dx = a_2 \int_0^{2\pi} \cos^2 2x dx \Rightarrow a_2 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos 2x dx$$

After talking about the applications of eigenvalues and eigenvectors, we are going to talk about the eigenvalues and eigenvectors of real symmetric matrices, they have special properties:

1. All eigenvalues are real.
2. Eigenvectors can be chosen to be orthogonal.

further more, we can require the eigenvectors to be orthonormal, then we have  $QQ^T = I$  where  $Q$  is orthonormal eigenvector matrix, then we can decompose  $A$  into  $A = Q\Lambda Q^T$  and this is called **spectrum theorem**.

What we are talking above is the case that the matrix is real, but what if the matrix is complex? Can we decompose  $A$  into a similar form? The answer is: if  $A = A^H$  where  $A^H = \bar{A}^T$ , then there exists a unitary matrix  $U$  such that  $A = U\Lambda U^H$ , and the eigenvalues are still real. (Unitary matrix means  $U^H U = I$ )

Since we already talked about the complex matrices, we can think of the Fast Fourier Transform (**FFT**), in previous discussion, in order to compute the coefficients, for example,  $a_2$ , we need to compute the integral:  $a_2 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos 2x dx$  which is very time-consuming, in practice, we use Distributed Fourier Transform (**DFT**) to fit the result of these coefficients, let  $F_k = \pi(a_k + ib_k) \approx \sum_{n=0}^{N-1} f(x_n) e^{-ik2\pi n/N}$ , where  $x_k = \frac{2\pi k}{N}$ ,  $N$  is the number of sampled points, and  $f(x_k)$  is sampled value, we can see the time complexity is  $O(N^2)$  in order to compute  $n$  coefficients. Consider using FFT, define the Fourier matrix as below:

$$M_N = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{-i2\pi/N} & \dots & e^{-i2\pi(N-1)/N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-i2\pi(N-1)/N} & \dots & e^{-i2\pi(N-1)(N-1)/N} \end{bmatrix}$$

where  $M_N(i, j) = e^{-i2\pi(i-1)(j-1)/N}$ , then we have:

$$M_N \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \end{bmatrix} = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{N-1} \end{bmatrix}$$

to reduce the times of multiplication, we can decompose  $M_N$  into  $M_N = \begin{bmatrix} I & D_{N/2} \\ I & -D_{N/2} \end{bmatrix} \begin{bmatrix} M_{N/2} & 0 \\ 0 & M_{N/2} \end{bmatrix} \begin{bmatrix} O_{N/2} \\ E_{N/2} \end{bmatrix}$ ,

where  $D_N = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{-i2\pi/N} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-i2\pi(N-1)/N} \end{bmatrix}$ ,  $E_N$  is the matrix whose elements with even corner labels

are all 1's while  $O_N$  is the matrix whose elements with odd corner labels are all 1's. Multiplying  $\begin{bmatrix} O_{N/2} \\ E_{N/2} \end{bmatrix}$  costs little time since it just reshuffles the elements, multiplying the leftmost matrix costs  $O(N)$  time since there the special structure of the matrix  $D_N$ . Then we can further decompose the intermediate matrices into smaller matrices and eventually we can reduce the time complexity to  $O(N \log N)$ .

Come back to real matrices, currently, we know that all eigenvalues are real, but are they positive or negative? A theorem tells us that # positive pivots = # positive eigenvalues, this is useful, for example, in differential equations, the positive eigenvalues means that the system is unstable.

We are already to talk about positive definite matrices, the positive definite matrices are all symmetric matrices, and have one of the following properties (actually they are equivalent):

1. All eigenvalues are positive.
2. All pivots are positive.
3. All sub-determinants (upper-left) are positive.
4.  $x^T A x > 0$ .

and we call  $x^T A x$  the quadratic form of  $A$ . Further more, we know that if the Hessian of function  $f$  is positive-definite at  $x$ , then  $f$  attains an isolated local minimum at  $x$ .

## Problem 1

1. Find a symmetric matrix  $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$  that has a negative eigenvalue.
2. How do you know it must have a negative pivot?
3. How do you know it can't have two negative eigenvalues?

## Solution

1.  $b = 2$ .
2. Because we know signs of pivots are the same as signs of eigenvalues.
3. Because the first pivot is positive.

## Problem 2

Every 2 by 2 symmetric matrix is  $\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T = \lambda_1 P_1 + \lambda_2 P_2$ .

Explain  $P_1 + P_2 = \mathbf{x}_1 \mathbf{x}_1^T + \mathbf{x}_2 \mathbf{x}_2^T = I$  from columns times rows of  $Q$ . Why is  $P_1 P_2 = 0$ ?

### Solution

This is because if  $A$  is symmetric, we know that the eigenvectors can be chosen to be orthogonal, and then we can normalize these eigenvectors, then we have  $Q^T Q = I$  which is equivalent to  $P_1 + P_2 = \mathbf{x}_1 \mathbf{x}_1^T + \mathbf{x}_2 \mathbf{x}_2^T = I$ .  $P_1 P_2$  is equal to  $\mathbf{x}_1 \mathbf{x}_1^T \mathbf{x}_2 \mathbf{x}_2^T$ , since  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are orthonormal, we know  $\mathbf{x}_1^T \mathbf{x}_2 = 0$ , therefore  $P_1 P_2 = 0$ .

## Problem 3

Show that this  $A$  (**symmetric but complex**) has only one line of eigenvectors:

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \text{ is not even diagonalizable: eigenvalues } \lambda = 0, 0.$$

$A^T = A$  is not such a special property for complex matrices. The good property is  $A^H = A$ . Then all  $\lambda$ 's are real and the eigenvectors are orthogonal.

### Solution

The characteristic polynomial of  $A$  is  $\det(A - \lambda I) = \lambda^2$ , so the eigenvalues of  $A$  is  $\lambda_1 = \lambda_2 = 0$ , using the elimination method, we can find that the only eigenvector of  $A$  is:  $\begin{bmatrix} i & 1 \end{bmatrix}^T$ .

## Problem 4

Write a 2 by 2 complex matrix with  $S^H = S$  (a “Hermitian matrix”). Find  $\lambda_1$  and  $\lambda_2$  for your complex matrix. Check that  $\mathbf{x}_1^H \mathbf{x}_2 = 0$  (this is complex orthogonality).

### Solution

Consider  $S = \begin{bmatrix} 1 & -2i \\ 2i & 1 \end{bmatrix}$ , the characteristic polynomial of  $S$  is  $\det(S - \lambda I) = \lambda^2 - 2\lambda - 3$ , and we find the eigenvalues are  $\lambda_1 = 3, \lambda_2 = -1$ ,  $\mathbf{x}_1 = \begin{bmatrix} -i & 1 \end{bmatrix}^T, \mathbf{x}_2 = \begin{bmatrix} i & 1 \end{bmatrix}^T$ , then we have  $\mathbf{x}_1^H \mathbf{x}_2 = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = 0$ .

## Problem 5

**True** (with reason) **or false** (with example).

1. A matrix with  $n$  real eigenvalues and  $n$  real eigenvectors is symmetric.
2. A matrix with  $n$  real eigenvalues and  $n$  orthonormal eigenvectors is symmetric.
3. The inverse of an invertible symmetric matrix is symmetric.
4. The eigenvector matrix  $Q$  of a symmetric matrix is symmetric.
5. The main diagonal of a positive definite matrix is all positive.

**Solution**

1. False. Consider the matrix  $\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$ .
2. True. Using the spectrum theorem, we have  $A = Q\Lambda Q^T$ , transpose both sides, we have  $A^T = Q\Lambda Q^T$  which is equal to  $A$ .
3. True.  $(A(A^{-1})^T)^T = A^{-1}A^T = A^{-1}A = I$ , so  $A^{-1} = (A^{-1})^T$  since the inverse of a matrix is unique.
4. False. Consider the matrix  $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ , the eigenvector matrix can be chosen as  $\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$ .
5. False. Consider the matrix  $\begin{bmatrix} 1 & -10 \\ 1 & -2 \end{bmatrix}$ .

**Problem 6**

Which 3 by 3 symmetric matrices  $S$  and  $T$  produce these quadratics?

$$\mathbf{x}^T S \mathbf{x} = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3). \text{ Why is } S \text{ positive definite?}$$

$$\mathbf{x}^T T \mathbf{x} = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3). \text{ Why is } T \text{ semi-definite?}$$

**Solution**

$$S = \begin{bmatrix} 2 & -1/2 & 0 \\ -1/2 & 2 & -1/2 \\ 0 & -1/2 & 2 \end{bmatrix}$$

$S$  is positive definite because  $\mathbf{x}^T S \mathbf{x} = x_1^2 + x_2^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 > 0$  for all  $\mathbf{x} \neq 0$ .

$$T = \begin{bmatrix} 2 & -1/2 & -1/2 \\ -1/2 & 2 & -1/2 \\ -1/2 & -1/2 & 2 \end{bmatrix}$$

$T$  is semi-definite because we only have  $\mathbf{x}^T T \mathbf{x} = (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2 \geq 0$  for all  $\mathbf{x}$ .

**Problem 7**

**Important!** Suppose  $S$  is positive definite with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ .

1. What are the eigenvalues of the matrix  $\lambda_1 I - S$ ? Is it positive semi-definite?
2. How does it follow that  $\lambda_1 \mathbf{x}^T \mathbf{x} \geq \mathbf{x}^T S \mathbf{x}$  for every  $\mathbf{x}$ ?
3. Draw this conclusion: **The maximum value of  $\mathbf{x}^T S \mathbf{x} / \mathbf{x}^T \mathbf{x}$  is \_\_\_\_.**

**Solution**

1. The eigenvalues of  $\lambda_1 I - S$  are  $0, \lambda_1 - \lambda_2, \dots, \lambda_1 - \lambda_n$ , and it is positive semi-definite.
2. Since  $\lambda_1 I - S$  is semi-definite, we have  $\mathbf{x}^T (\lambda_1 I - S) \mathbf{x} \geq 0 \Rightarrow \lambda_1 \mathbf{x}^T \mathbf{x} \geq \mathbf{x}^T S \mathbf{x}$ .
3.  $\lambda_1$ .

## Problem 8

Find the eigenvalues and eigenvectors of the Hermitian matrix  $S = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$ .

### Solution

The eigenvalues of  $S$  are  $\lambda_1 = 3, \lambda_2 = 0$ , the eigenvectors are  $[1+i \quad -1]^T, [1+i \quad 2]^T$ .

## Problem 9

If  $Q^H Q = I$  (unitary matrix = complex orthogonal) and  $Q\mathbf{x} = \lambda\mathbf{x}$ , show that  $|\lambda| = 1$ .

### Solution

Conjugate and transpose both sides of  $Q\mathbf{x} = \lambda\mathbf{x}$ , we have  $\mathbf{x}^H Q^H = \lambda^H \mathbf{x}^H$ , combining two equations, we have  $\mathbf{x}^H Q^H Q \mathbf{x} = \lambda^H \lambda \mathbf{x}^H \mathbf{x} \Rightarrow 1 = \lambda^H \lambda \Rightarrow |\lambda| = 1$ .

## Problem 10

Find two  $\lambda$ 's and  $\mathbf{x}$ 's so that  $\mathbf{u} = e^{\lambda t} \mathbf{x}$  solves  $\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \mathbf{u}$ . What combination  $\mathbf{u} = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2$  starts from  $\mathbf{u}(0) = (5, -2)$ ?

### Solution

1.  $\lambda_1 = 4, \mathbf{x}_1 = [1 \quad 0]^T, \lambda_2 = 1, \mathbf{x}_2 = [-1 \quad 1]^T$ .
2.  $\mathbf{u} = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

## Problem 11

1. If every column of  $A$  adds to zero, why is  $\lambda = 0$  an eigenvalue?
2. With negative diagonal and positive off-diagonal adding to zero,  $\mathbf{u}' = A\mathbf{u}$  will be a "continuous" Markov equation. Find the eigenvalues and eigenvectors, and the steady state as  $t \rightarrow \infty$

Solve  $\frac{d\mathbf{u}}{dt} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \mathbf{u}$  with  $\mathbf{u}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ . What is  $\mathbf{u}(\infty)$ ?

### Solution

1. Yes. Because this property shows that  $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$ .
2. The eigenvalues are  $0, -5$  and the corresponding eigenvectors are  $\begin{bmatrix} 3 & 2 \end{bmatrix}^T, \begin{bmatrix} -1 & 1 \end{bmatrix}^T$ . Substituting  $\mathbf{u}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ , we know that the general solution of  $\mathbf{u}$  is  $\mathbf{u}(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ , let  $t \rightarrow \infty$ , we get the steady state  $\mathbf{u}(\infty) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ .