Linear Algebra: Homework #2

Gilbert Strang

0130

Problem 1

What multiple ℓ of equation 1 should be subtracted from equation 2 to remove c?

$$ax + by = f$$
$$cx + dy = g.$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot? What is y? The second pivot is missing when $ad = bc$: singular.

Solution

1. $\ell = \frac{c}{a}$.
2. formula: $(d - \frac{c}{b})y = g - \frac{c}{a}f$
3. $y = \frac{ag-cf}{ad-bc}$

Problem 2

For which three numbers k does elimination break down? Which is fixed by a row exchange? Is the number of solutions 0 or 1 or ∞?

$$kx + 3y = 6$$
$$3x + ky = -6$$

Solution

1. $k = 0, 3, -3$
2. if $k = 0$, a row exchange can be used to fix the elimination.
3. (a) $k = 0$ has 1 solution.
   (b) $k = 3$ has 0 solution.
   (c) $k = -3$ has ∞ solutions.

Problem 3

Which number d forces a row exchange, and what is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot) ?

$$2x + 5y + z = 0$$
$$4x + dy + z = 2$$
$$y - z = 3$$

Solution

Problem 3 continued on next page...

1. $d = 10$ forces a row exchange.
2. $$\begin{bmatrix} 2 & 5 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$
3. $d = 11$ makes the system singular.

Problem 4

Write down the 3 by 3 matrices that produce these elimination steps:

1. E_{21} subtracts 5 times row 1 from row 2.
2. E_{32} subtracts -7 times row 2 from row 3.
3. P exchanges rows 1 and 2, then rows 2 and 3.

Solution

1. $$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
2. $$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$$
3. $$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Problem 5

Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

and $E_{32}E_{31}E_{21}A = EA = U$.

Multiply those E's to get one elimination matrix E. What is $E^{-1} = L$?

Include $\mathbf{b} = (1, 0, 0)$ as a fourth column to produce $[A\ \mathbf{b}]$. Carry out the elimination steps on this augmented matrix to solve $A\mathbf{x} = \mathbf{b}$.

Solution

Problem 5 continued on next page...

1.
$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

2.
$$E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}, L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

3. Apply the elimination steps to $[A\ \mathbf{b}]$, then we get $[U\ \ L\mathbf{b}]$:
$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$
then we know $\mathbf{x} = [-1, 2, 0]^T$.

Problem 6

Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}?

Solution

Yes, B is invertible. And since $B = P_{12}A$, $P_{12}^{-1} = P_{12}$, we have $B^{-1} = A^{-1}P_{12}$, where P_{12} is the permutation matrix. And the formula implies that B^{-1} can be found by exchange the first two columns of A^{-1}.

Problem 7

1. What 3 by 3 matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.
2. What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

Solution

1.
$$E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

2.
$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Problem 8

Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots of A^{-1}). Then find three numbers c so that C is not invertible.

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}, C = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

Solution

1. $$A^{-1} = \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

   The pivots of A^{-1} are $a, a-b, a-b$, so A is invertible.

2. $c = 0, 2, 7$ makes C not invertible.

Problem 9

What three elimination matrices E_{21}, E_{31}, E_{32} put A into its upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by $E_{32}^{-1}, E_{31}^{-1}, E_{21}^{-1}$ to factor A into L times U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

and $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$.

Solution

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

then we can represent A as $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 10

1. How many entries of S can be chosen independently, if $S = S^T$ is 5 by 5?
2. How do L and D (still 5 by 5) give the same number of entries in LDL^T?
3. How many entries of A can be chosen if A is skew-symmetric? ($A^T = -A$).
4. Why does $A^T A$ have no negetive numbers on its diagonal?

Problem 10 continued on next page...

Solution

1. 15.
2. Ommited (Not sure what's the meaning of the question).
3. Suppose A is n by n, then the answer is n.
4. Because if we consider the i-i entry of $A^T A$, we have:

$$(A^T A)_{ii} = \sum_{j=1}^{n} {A^T}_{ij} A_{ji} = \sum_{j=1}^{n} {A_{ji}}^2 \geq 0$$