Linear Algebra: Homework #1

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0130

Problem 1

How could you decide if the vectors $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (0, 1, 1)$ and $\mathbf{w} = (a, b, c)$ are linearly independent or dependent?

Solution

If $a + b - c \neq 0$, then these vectors are linearly independent, otherwise they are linearly dependent.

Problem 2

How many corners $(\pm 1, \pm 1, \pm 1, \pm 1)$ does a cube of side 2 have in 4 dimensions? What is its volume? How many 3D faces? How many edges? Find one edge.

Solution

1. The cube has 16 corners.
2. The volume of the cube is 16.
3. There are 8 3D faces.
4. There are 32 edges.
5. $\{(-1, -1, -1, -1), (-1, -1, -1, 1)\}$ is one edge.

Problem 3

The **triangule inequality** says: (length of $\mathbf{v} + \mathbf{w}$) $\leq$ (length of $\mathbf{v}$) + (length of $\mathbf{w}$).

1. Show that $||\mathbf{v} + \mathbf{w}||^2 = ||\mathbf{v}||^2 + 2\mathbf{v} \cdot \mathbf{w} + ||\mathbf{w}||^2$.
2. Increase that $\mathbf{v} \cdot \mathbf{w}$ to $||\mathbf{v}||\ ||\mathbf{w}||$ to show that $||\textbf{side 3}||$ cannot exceed $||\textbf{side 1}|| + ||\textbf{side 2}||$:

Solution

1. $||\mathbf{v} + \mathbf{w}||^{\mathbf{2}} = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v}^{\mathbf{2}} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w}^{\mathbf{2}} = ||\mathbf{v}||^{\mathbf{2}} + 2\mathbf{v} \cdot \mathbf{w} + ||\mathbf{w}||^{\mathbf{2}}$
2. $||\textbf{side 3}|| = ||\textbf{side 1} - \textbf{side 2}|| \leq ||\textbf{side 1}|| + || - \textbf{side 2}|| = ||\textbf{side 1}|| + ||\textbf{side 2}||$

Problem 4

Which numbers q would leave A with two independent columns ?

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 9 \\ 5 & 0 & q \end{bmatrix}, A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & q \end{bmatrix}, A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & q \end{bmatrix}$$

Solution

1. $q = 10$.
2. $q = 9$.
3. $q \neq 0$.

Problem 5

If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d). This is surprisingly important; two columns are failing on one line. You could use numbers first to see how a, b, c, d are related. The question will lead to:

$$\text{If } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has dependent rows, then it also has dependent columns.}$$

Solution

Suppose (a, c) is k times (b, d), then we have $\frac{a}{b} = \frac{kc}{kd} = \frac{c}{d}$.

Problem 6

Why is it impossible for a matrix A with 7 columns and 4 rows to have 5 independent columns? This is not a trivial or useless question.

Solution

This question is equivalent to asking the rank of a matrix is less than or equal to the lesser of them. Recall the process of elimination, if we have 5 independent columns, then we should have 5 pivot, which conflicts to the fact that we only have 4 rows.

Problem 7

Going from left to right, put each column of A into the matrix C if that column is not a combination of earlier columns:

$$A = \begin{bmatrix} 2 & -2 & 1 & 6 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 3 & -3 & 0 & 6 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & & \\ 1 & & \\ 3 & & \end{bmatrix}$$

Find R in Problem 6 so that $A = CR$. If your C has r columns, then R has r rows. The 5 columns of R tell how to produce the 5 columns of A from the columns in C.

Solution

$$R = \begin{bmatrix} -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 8

Complete these 2 by 2 matrices to meet the requirements printed underneath:

$$\underset{\text{rank one}}{\begin{bmatrix} 3 & 6 \\ 5 & \end{bmatrix}} \quad \underset{\text{orthogonal columns}}{\begin{bmatrix} 6 & 7 \\ 6 & \end{bmatrix}} \quad \underset{\text{rank 2}}{\begin{bmatrix} 3 & \\ 3 & 6 \end{bmatrix}} \quad \underset{A^2 = I}{\begin{bmatrix} 3 & 4 \\ & -3 \end{bmatrix}}$$

Problem 8 continued on next page...

Solution

$$\begin{bmatrix} 3 & 6 \\ 5 & 10 \end{bmatrix}, \begin{bmatrix} 6 & 7 \\ 7 & -6 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$$

Problem 9

True or false, with a reason:

1. If 3 by 3 matrices A and B have rank 1, then AB will always have rank 1.
2. If 3 by 3 matrices A and B have rank 3, then AB will always have rank 3.
3. Suppose $AB = BA$ for every 2 by 2 matrix B. Then $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = cI$ for some number c. Only those matrices $A = cI$ commute with every B.

Solution

(a) No.consider the following matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Yes, rank (AB) = rank $(A) = 3$ since rank $(B) = 3$.

(c) Omitted.

Problem 10

How many small multiplications for $(AB)C$ and $A(BC)$ if those matrices have sizes ABC $= (4\times3)(3\times2)(2\times1)$? The two counts are different.

Solution

32;24