Linear Algebra: Homework #5

Gilbert Strang

0130

Before start this homework, let's review the process of Gram-Schmidt method. So the reason why we want to get orthonormal basis is that, in the chapter of the projection of a vector onto a subspace, we know that if we want to project a vector to the column space of a matrix A, the corresponding projection matrix is

$$P = A(A^TA)^{-1}A^T$$

if we have an orthonormal basis for the column space, the above formula becomes:

$$P = QQ^T$$

which is much easier to compute. And remember, here the columns of A must be linearly independent, otherwise we can't do the Gram-Schmidt process.

Let's think of Gram-Schmidt process in the 2 dimension case first. We pick the first column vector $\mathbf{a}_1/||\mathbf{a}_1||$ as the first vector $\mathbf{e}_1$, and try to find the second vector that is orthogonal to the first one. What we do is to project the second column vector $\mathbf{a}_2$ to the first one, decompose $\mathbf{a}_2$ into two parts $\mathbf{v}_2 + \mathbf{p}_2$, where $\mathbf{v}_2 \perp \mathbf{e}_1$ and $\mathbf{p}_2 \in$ span$(\mathbf{e}_1)$, we then have

$$\begin{aligned} &\mathbf{e}_1^T(\mathbf{a}_2 - \mathbf{p}_2) = 0 \\ &\mathbf{p}_2 = c_2\mathbf{e}_1, c_2 \in \mathbb{R} \end{aligned}$$

after some calculation, we get $c_2 = \mathbf{e}_1^T\mathbf{a}_2/\mathbf{e}_1^T\mathbf{e}_1$, so $\mathbf{v}_2$ can be computed as:

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{e}_1^T\mathbf{a}_2}{\mathbf{e}_1^T\mathbf{e_1}}\mathbf{e}_1 = \mathbf{a}_2 - \mathbf{e}_1^T\mathbf{a}_2\mathbf{e}_1$$

then we can get the second vector $\mathbf{e}_2 = \mathbf{v}_2/||\mathbf{v}_2||$.

After discussing the 2 dimension case, we can talk about the general case which is similar to the 2-dim case.

So suppose we have a sequence of vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$, these vectors are linearly independent and they are orthonormal, now I have a vector $\mathbf{a}_n$ which is also linearly independent to these vectors, how can I find a vector $\mathbf{e}_n$ that is orthonormal to $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ and we also have span$(\mathbf{e}_1, \ldots, \mathbf{e}_n) =$ span$(\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}, \mathbf{a}_n)$?

A natural idea is to throw away the part of $\mathbf{a}_n$ that is in the span of $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$, as we know, $\mathbf{a}_n$ can we written as $\mathbf{v}_n + \mathbf{p_n}$, define the matrix A_{n-1} equals to $\begin{bmatrix}\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}\end{bmatrix}$, then we can represent $\mathbf{v}_n$ as $\mathbf{v}_n = \mathbf{a}_n - A_{n-1}\mathbf{x}$, since $\mathbf{v}_n \perp C(A_{n-1})$, we know $\mathbf{v}_n \in \mathbf{N}(A_{n-1}^T)$, then we have

$$\begin{aligned} A_{n-1}^T\mathbf{v}_n =& A_{n-1}^T(\mathbf{a}_n - \mathbf{p}_n) \\ =& A_{n-1}^T(\mathbf{a}_n - A_{n-1}\mathbf{x}) \\ =& 0 \end{aligned}$$

Then we get $\mathbf{x} = (A_{n-1}^TA_{n-1})^{-1}A_{n-1}^T\mathbf{a}_n$, and we get $\mathbf{v}_n = \mathbf{a}_n - A_{n-1}(A_{n-1}^TA_{n-1})^{-1}A_{n-1}^T\mathbf{a}_n$.

Remember that the matrix A_{n-1} is composed of orthonormal vectors, so $A_{n-1}^TA_{n-1} = I$, then we have

$$\mathbf{v}_n = \mathbf{a}_n - A_{n-1}A_{n-1}^T\mathbf{a}_n = \mathbf{a}_n - \mathbf{e}_1^T\mathbf{a}_n\mathbf{e}_1 - \mathbf{e}_2^T\mathbf{a}_n\mathbf{e}_2 - \cdots - \mathbf{e}_{n-1}^T\mathbf{a}_n\mathbf{e}_{n-1}$$

if we want to do the normalization at the end, we can rewrite the above formula as below:

$$\mathbf{v}_n = \mathbf{a}_n - B_{n-1}(B_{n-1}^TB_{n-1})^{-1}B_{n-1}^T\mathbf{a}_n = \mathbf{a}_n - \frac{\mathbf{u}_1^T\mathbf{a}_n}{\mathbf{u}_1^T\mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2^T\mathbf{a}_n}{\mathbf{u}_2^T\mathbf{u}_2}\mathbf{u}_2 - \cdots - \frac{\mathbf{u}_{n-1}^T\mathbf{a}_n}{\mathbf{u}_{n-1}^T\mathbf{u}_{n-1}}\mathbf{u}_{n-1}$$

Here $\mathbf{u}_i/||\mathbf{u}_i|| = \mathbf{e}_i$, and matrix B_{n-1} is the matrix composed of $\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$.

Problem 1

If $\mathbf{q}_1$ and $\mathbf{q}_2$ are orthonormal vectors in $\mathbf{R}^5$, what combination ____$\mathbf{q}_1$ + ____$\mathbf{q}_2$ is closest to a given vector $\mathbf{b}$?

Solution

$\underline{\mathbf{q}_1^T\mathbf{b}}\ \mathbf{q}_1 + \underline{\mathbf{q}_2^T\mathbf{b}}\ \mathbf{q}_2$.

Problem 2

What multiple of $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should be subtracted from $\mathbf{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to make the result $\mathbf{B}$ orthogonal to $\mathbf{a}$? Sketch a figure to show $\mathbf{a}, \mathbf{b}$ and $\mathbf{B}$.

Complete the Gram-Schmidt process for this problem by computing $\mathbf{q}_1 = \mathbf{a}/||\mathbf{a}||$ and $\mathbf{q}_2 = \mathbf{B}/||\mathbf{B}||$ and factoring into QR:

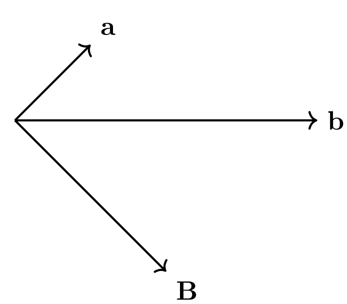
$$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} ||\mathbf{a}|| & ? \\ ||0|| & ||\mathbf{B}|| \end{bmatrix}$$

Solution

As shown at the beginning of this homework, we can get $\mathbf{B}$ as below:

$$\mathbf{B} = \mathbf{b} - \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\mathbf{a} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

The multiple of $\mathbf{a}$ that should be subtracted from $\mathbf{b}$ is 2.
The picture is shown below:



After computation we get $\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, then we have QR decomposition of the matrix A:

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}$$

Problem 3

Find $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ (orthonormal) as combinations of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (independent columns). Then write A as QR:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

Solution

$\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ are shown below:

$$\begin{aligned}
\mathbf{q}_1 =& \mathbf{a} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \\
\mathbf{q}_2 =& (\mathbf{b} - \mathbf{q}_1^T \mathbf{b} \mathbf{q}_1)/||\mathbf{b} - \mathbf{q}_1^T \mathbf{b} \mathbf{q}_1|| = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \\
\mathbf{q}_3 =& (\mathbf{c} - \mathbf{q}_1^T \mathbf{c} \mathbf{q}_1 - \mathbf{q}_2^T \mathbf{c} \mathbf{q}_2)/||\mathbf{c} - \mathbf{q}_1^T \mathbf{c} \mathbf{q}_1 - \mathbf{q}_2^T \mathbf{c} \mathbf{q}_2|| = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T
\end{aligned}$$

and then we can get the QR decomposition of the matrix A:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Problem 4

Choose c so that Q is an orthogonal matrix:

$$Q = c \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Project $\mathbf{b} = (1, 1, 1, 1)$ onto the first column. Then project $\mathbf{b}$ onto the plane of the first two columns.

Solution

1. $c = 1/2$.
2. The projection of $\mathbf{b}$ onto the first column $\mathbf{p} = \begin{bmatrix} -1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}^T$.
3. The projection of $\mathbf{b}$ onto the first two columns is $\mathbf{p} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T$.

Problem 5

If you add row 1 $= \begin{bmatrix} a & b & c \end{bmatrix}$ to row 2 $\begin{bmatrix} p & q & r \end{bmatrix}$ to get $\begin{bmatrix} p+a & q+b & r+c \end{bmatrix}$ in row 2, show from formula(1) for $\det A$ that the 3 by 3 determinant does not change. Here is another approach to the rule for adding two rows:

$$\det \begin{bmatrix} \text{row 1} \\ \textbf{row1 + row2} \\ \text{row 3} \end{bmatrix} = \det \begin{bmatrix} \text{row 1} \\ \textbf{row 1} \\ \text{row 3} \end{bmatrix} + \det \begin{bmatrix} \text{row 1} \\ \textbf{row 2} \\ \text{row 3} \end{bmatrix} = \mathbf{0} + \det \begin{bmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{bmatrix}$$

Solution

Problem 6

Do these martices have determinant 0,1,2, or 3?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution

1. $\det A = 1$.
2. $\det B = 2$.
3. $\det C = 0$.
4. $\det D = 0$.

Problem 7

Show that $\det A = 0$, regardless of the five numbers marked by x's:

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$$

What are the cofactors of row 1? What is the rank of A? What are the 6 terms in $\det A$?

Solution

1. The corresponding cofactor matrix is: $\begin{bmatrix} 0 & 0 & 0 \\ -x^2 & x^2 & 0 \\ x^2 & -x^2 & 0 \end{bmatrix}$.
2. If $x \neq 0$, the rank of A is 2, otherwise the rank of A is 0.
3. The 6 terms in $\det A$ are all zero.

Problem 8

Quick proof of Cramer's rule. The determinant is a linear function of column 1. It is zero if two columns are equal. When $\mathbf{b} = A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ goes into the first column of A, we have the matrix B_1 and Cramer's Rule $x_1 = \det B_1 / \det A$:

$$|\mathbf{b} \;\; \mathbf{a}_2 \;\; \mathbf{a}_3| = |x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3| = x_1|\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3| = x_1 \det A.$$

What steps lead to the middle equation?

Solution

Problem 8 continued on next page...

$$
\begin{aligned}
|\mathbf{b} \quad \mathbf{a}_2 \quad \mathbf{a}_3| =& |x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 \quad \mathbf{a}_2 \quad \mathbf{a}_3| \\
=& |x_1\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3| + |x_2\mathbf{a}_2 \quad \mathbf{a}_2 \quad \mathbf{a}_3| + |x_3\mathbf{a}_3 \quad \mathbf{a}_2 \quad \mathbf{a}_3| \\
=& x_1|\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3| + x_2|\mathbf{a}_2 \quad \mathbf{a}_2 \quad \mathbf{a}_3| + x_3|\mathbf{a}_3 \quad \mathbf{a}_2 \quad \mathbf{a}_3|
\end{aligned}
$$

Using the fact that the determinant is zero if two columns are equal, we then have $|\mathbf{b} \quad \mathbf{a}_2 \quad \mathbf{a}_3| = x_1 \det A$.

Problem 9

(prize for the max determinant) If a 3 by 3 matrix has entries 1,2,3,4, ...,9, what is the maximum determinant? I would use a computer to decide. This problem does not seem easy.

Solution

$$
\begin{bmatrix} 1 & 7 & 6 \\ 4 & 0 & 8 \\ 9 & 5 & 2 \end{bmatrix}
$$

The code is shown below:

```
import numpy
import itertools

max = 0

for p in itertools.permutations(range(0,10)):
    matrix = [p[0:3], p[3:6], p[6:9]]
    temp = abs(numpy.linalg.det(matrix))
    if temp > max:
        max = temp
        solution = matrix

print(solution)
```