Linear Algebra: Homework #8

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Problem 1

Find the eigenvalues and singular values of this 2 by 2 matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \text{ with } A^T A = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \text{ and } AA^T = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

The eigenvectors $(1, 2)$ and $(1, -2)$ of A are not orthogonal. How do you know the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of $A^T A$ will be orthogonal? Notice that $A^T A$ and AA^T have the same eigenvalues $\lambda_1 = 25$ and $\lambda_2 = 0$.

Solution

1. The eigenvalues of A are 0 and 4.

2. The singular values of A are 5 and 0.

3. Because the matrix $A^T A$ is symmetric, its eigenvectors will be orthogonal.

Problem 2

Find $A^T A$ and AA^T and the singular vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_1, \mathbf{u}_2$ for A:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \text{ has rank } r = 2. \text{ The eigenvalues are } 0, 0, 0$$

Check the equations $A\mathbf{v}_1 = \sigma_1 \mathbf{u}_1$ and $A\mathbf{v}_2 = \sigma_2 \mathbf{u}_2$ and $A = \sigma_1 \mathbf{u_1} \mathbf{v}_1^T + \sigma_2 \mathbf{u_2} \mathbf{v}_2^T$. If you remove row 3 by A (all zeros), show that σ_1 and σ_2 don't change.

Solution

1. $A^T A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{bmatrix}$, $AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the singular values of A are 1 and 8, the singular vectors are $\mathbf{v}_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \mathbf{v}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \mathbf{u}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \mathbf{u}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

2. Omitted.

3. If remove row 3 by A, then $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, we get $A^T A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{bmatrix}$.

Problem 3

If $(A^T A)\mathbf{v} = \sigma^2 \mathbf{v}$, multiply by A. Move the parentheses to get $(AA^T)A\mathbf{v} = \sigma^2(A\mathbf{v})$.
If v is an eigenvector of $\mathbf{A^T A}$, then _____is an eigenvector of $\mathbf{AA^T}$.

Solution

$A\mathbf{v}$ is an eigenvector of AA^T.

Problem 4

If $A = Q$ is an orthogonal matrix, why does every singular value of Q equal 1?

Solution

Since Q is an orthogonal matrix, we have $Q^T Q = I$, then we know:

$$Q^T Q\mathbf{x} = \mathbf{x} = \lambda\mathbf{x} \Rightarrow \lambda = 1$$

Problem 5

1. Why is the trace of $A^T A$ equal to the sum of all a_{ij}^2?
2. For every rank-one matrix, why is $\sigma_1^2 =$ sum of all a_{ij}^2?

Solution

1. $\text{trace}(A^T A) = \sum_i (A^T A)_{ii} = \sum_i \sum_j A_{ij}^T A_{ji} = \sum_{i,j} a_{ij}^2$
2. For rank-one matrix A, the number of singular value is only one, and the square of this singular value σ_1 is the only non-zero eigenvalue of $A^T A$, so $\sigma_1^2 = \text{trace}(A^T A) =$ sum of all a_{ij}^2.

Problem 6

Suppose A_0 holds these 2 measurements of 5 samples:

$$A_0 = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ -1 & 1 & 0 & 1 & -1 \end{bmatrix}$$

Find the average of each row and subtract it to produce the centered matrix A. Compute the sample covariance matrix $S = AA^T/(n-1)$ and find its eigenvalues λ_1 and λ_2. What line through the origin is closest to the 5 samples in columns of A?

Solution

Centered matrix $A = \begin{bmatrix} 2 & 1 & 0 & -1 & -2 \\ -1 & 1 & 0 & 1 & -1 \end{bmatrix}$, sample covariance matrix $S = \begin{bmatrix} 5/2 & 0 \\ 0 & 1 \end{bmatrix}$, the eigenvalues of S are $\lambda_1 = 5/2, \lambda_2 = 1$, the corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then we know the closest line through the origin is $y = 0$.