Linear Algebra: Homework #4

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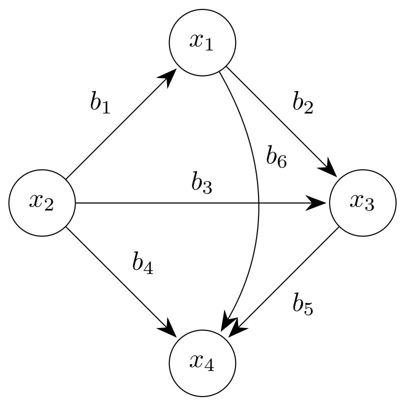
0130

Since in this homework we repeatedly use the fact that if the columns of A is linear independent then we have the matrix A^TA is invertible, I want to prove this at the beginning.

So the core is that I want to show that x must be $\mathbf{0}$ if the columns of A is independent and $A^TAx = 0$, the method in lecture 16 is that we can multiply the transpose of x in the both sides of the equation, then we get $x^TA^TAx = (Ax)^T(Ax) = 0$, then we know $Ax = 0$. But we can also think of that since the columns of A is independent, we have $\dim \mathbf{N}(A^T) = 0$, which means Ax must be the zero vector. Then using the assumption of the independency of columns, we know x must be the zero vector.

Problem 1

Find the **incidentce matrix** and its rank and one vector in each subspace for this complete graph—all six edges included.



Solution

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2

If $\mathbf{A^TAx = 0}$ then $\mathbf{Ax = 0}$. Reason: $A\mathbf{x}$ is in nullspace of A^T and also in the ____ of A and those spaces are ____. Conclusion: $A\mathbf{x} = \mathbf{0}$ and thereforeA^TA has the same nullspace as A. This key fact will be repeated when we need it.

Solution

1. Column space.
2. Orthogonal.

Problem 3

Compute the projection matrices $\mathbf{a}\mathbf{a}^T/\mathbf{a}^T\mathbf{a}$ onto lines through $\mathbf{a}_1 = (-1, 2, 2)$ and $\mathbf{a}_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is.

Solution

$$P_1 = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}, P_2 = \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}, P_1P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The product P_1P_2 is the zero matrix because the lines through $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal.

Problem 4

Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Solution

1. The shape of matrix P is 4 by 4.

2. The columns of A is a basis of the column space of A. Let $\mathbf{p}$ represents the projection of $\mathbf{b}$, then error can be represented as $\mathbf{e} = \mathbf{b} - \mathbf{p}$, and we know $C(A)$ and $\mathbf{e}$ are orthogonal so $\mathbf{e} \in \mathbf{N}(A^T)$ therefore $A^T\mathbf{e} = 0$. Since $\mathbf{p}$ is in the column space of A, $\mathbf{p}$ can be represented as a linear combination of the columns of A, therefore $\mathbf{p}$ can be written as $\mathbf{p} = A\hat{\mathbf{x}}$, and we have:

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 5

What linear combination of (1, 2, -1) and (1, 0, 1) is closest to $\mathbf{b} = (2, 1, 1)$?

Solution

Similar to Problem 4, let $\mathbf{e}, \mathbf{p}$ represent error and the projection of $\mathbf{b}$ seperately, and use $\hat{\mathbf{x}}$ represents the coeffcients of the cloest linear combination, we have

$$\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} 1/3 \\ 3/2 \end{bmatrix}$$

so the cloest linear combination of $\mathbf{b}$ is $\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Problem 6

To find the projection matrix onto the plane $x - y - 2z = 0$, choose two vectors in that plane and make them the columns of A. The plane will be the column space of A! Then compute $P = A(A^T A)^{-1} A^T$.

OR

To find the projection matrix P onto the same plane $x - y - 2z = 0$, write down a vector $\mathbf{e}$ that is perpendicular to the plane. Compute the projection $Q = \mathbf{e}\mathbf{e}^{\mathbf{T}}/\mathbf{e}^{\mathbf{T}}\mathbf{e}$ and then $P = I - Q$.

Solution

1. First method. Choose $\mathbf{v}_1 = \begin{bmatrix}1 & 1 & 0\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}0 & 2 & -1\end{bmatrix}$ as a basis of the plane. Let $A = \begin{bmatrix}1 & 0 \\ 1 & 2 \\ 0 & -1\end{bmatrix}$, then compute projection matrix

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix}5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3\end{bmatrix}$$

2. Second method. We can pick the coeffcients of the plane equation as a vector which is perpendicular to to the plane, this is because for a plane equation, we can think of it as being the dot product of two vectors, one of which is an arbitrary vector in the plane (x, y, z) while the other is a vector perpendicular to the plane. So $\mathbf{e} = (1, -1, -2)$. Since a vector $\mathbf{b}$ can be represented as two part $\mathbf{p}' + \mathbf{e}'$ where $\mathbf{p}' \in C(A)$ and $\mathbf{e}' \in \mathbf{N}(A^T)$, therefore $P\mathbf{b} = \mathbf{b} - Q\mathbf{b}$, where Q is the projection matrix of $\mathbf{b}$ onto $\mathbf{e}$, and we all know the formula of Q is

$$Q = \mathbf{e}\mathbf{e}^{\mathbf{T}}/\mathbf{e}^{\mathbf{T}}\mathbf{e} = \begin{bmatrix}1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3\end{bmatrix}, P = I - Q = \begin{bmatrix}5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3\end{bmatrix}$$

Problem 7

The first three Chebyshev polynomials are given by
$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1$

1. Show that $\{T_0, T_1, T_2\}$ is a basis for $\mathbb{P}_2$, the space of polynomials of degree ≤ 2 (more generally, the space of polynomials of degree $\leq$ n is denoted by $\mathbb{P}_n$).

2. Check that differentiation defines a linear transformation $T_D : \mathbb{P}_2 \to \mathbb{P}_1$ and write down the matrix of each linear transformation in the Chebyshev basis. Similarly, check that integration is a linear transformation $T_S = \mathbb{P}_1 \to \mathbb{P}_2$.

3. Let D and S be the differentiation and integration matrices from part(b). Compute the matrix product DS and SD. Interpret the results using calculus: choose a suitable polynomial in $\mathbb{P}_2$, differentiate it and then integrate it.

4. Write down bases for the null spaces and column spaces of D and S. Provide the corresponding polynomials. Can you interpret your results about D and S in light of what you know about differentiation and integration in calculus?

Problem 7 continued on next page...

Solution

Let the entries of column represents the coeffcients of $1, x, x^2, \ldots$ from top to bottom, and $\mathbf{v}_i$ represents the vector of T_i.

1. The matrix of $\{T_0, T_1, T_2\}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, and its rank equals to 3. Since the a natural basis of $\mathbb{P}_2$ is $\{1, x, x^2\}$ which implies the dimension of this space is 3, we know $\{T_0, T_1, T_2\}$ is a basis of $\mathbb{P}_2$.

2. (a) Differentiation.

$$\begin{aligned} &\frac{d}{dx}\left\{(a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2)\right\} \\ &= 2a_1x + b_1 + 2a_2x + b_2 \\ &= 2(a_1 + a_2)x + (b_1 + b_2) \\ &= \frac{d}{dx}(a_1x^2 + b_1x + c_1) + \frac{d}{dx}(a_2x^2 + b_2x + c_2) \end{aligned}$$

The process above shows that differentiation is a linear transformation. The matrix of each linear transformation in the Chebyshev basis is shown below:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) Integration. The process of checking that integretion is a linear transform is similar to the process above and we omit the details here. The matrix of each linear transformation in the Chebyshev basis is shown below:

$$\begin{bmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} : a, b, c \in \mathbb{R}$$

3. (a) Consider the matrix product DS, the important thing is that, here the differentiation if a linear transformation from $\mathbb{P}_3 \to \mathbb{P}_2$, so the corresponding matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

what we need to know additionally is that $T_3 = 4x^3 - 3x$, then we can compute the product DS as

$$DS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Consider the matrix product SD, similarly, we should notice that here the integration is a linear transformation from $\mathbb{P}_1 \to \mathbb{P}_2$, so the corresponding matrix is

$$\begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix} : a, b \in \mathbb{R}$$

and we can compute the product SD as

$$SD = \begin{bmatrix} 0 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : a, b \in \mathbb{R}$$

This is very interesting, as we know in calculus, in we do the differentiation after the integration, we can preserve all the information. But if we do the differentiation before the integration, we will lose the information of the contant order number. For example, we apply differentiation and then integration to $x - 1$, we have

$$\int \frac{d}{dx}(x-1)dx = \int 1dx = x + c, c \in \mathbb{R}$$

but if we do the integration and then differentiation to $x - 1$, we have

$$\frac{d}{dx}\int (x-1)dx = \frac{d}{dx}(\frac{x^2}{2} - x) = x - 1$$

4. (a) A basis of nullspace of D is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and the corresponding polynomials is the constant 1.

   (b) A basis of column space of D is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, the corresponding polynomials is 1 and x.

   (c) A basis of nullspace of S is $\{\mathbf{0}\}$, there is no corresponding polynomial.

   (d) A basis of column space of S is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, the corresponding polynomials is $1, x, x^2, x^3$.

These results can be interpreted using calculus. (a) is because the constant is the only polynomial that has a derivative of 0, and (b) is because the whole space of $\mathbb{P}_1$ is spanned by $\{1, x\}$, (c) is result of the fact that there does not exist any polynomial whose integral is zero polynomial and finally (d) is reasonable because any polynomial with order ≥ 0 (not zero polynomial) can be obtained by the integration of a polynomial with order lower than it.