Linear Algebra: Homework #6

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0130

Problem 1

The example at the start of the chapter has powers of this matrix A:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors. Show from A how a row exchange can produce different eigenvalues.

Solution

1. The eigenvalues of A are: $1/2, 1$.
2. The eigenvalues of A^2 are: $1/4, 1$.
3. The eigenvalues of A^∞ of $0, 1$.

If we exchange two rows of A, then the eigenvalues of the resulting matrix will be: $-1/2, 1$.

Problem 2

Find three eigenvectors for this matrix P (projection matrices have $\lambda = 1$ and 0):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

Solution

The eigenvectors corresponding to the eigenvalue $\lambda = 1$ are: $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The eigenvector corresponding to the eigenvalue $\lambda = 0$ is: $\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}^T$.

A eigenvector of P with no zero components is: $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$.

Problem 3

A 3 by 3 matrix B is known to have eigenvalues 0,1,2. This information is enough to find three of these (give the answer where possible):

1. the rank of B
2. the determinant of $B^T B$
3. the eigenvalues of $B^T B$
4. the eigenvalues of $(B^2 + I)^{-1}$

Solution

1. The rank of B is 2.

Problem 3 continued on next page...

2. The determinant of $B^T B$ is 0.
3. The eigenvalues of $B^T B$ are unknown.
4. The eigenvalues of $(B^2 + I)^{-1}$ are 1, 1/2, 1/5.

Problem 4

The matrix is singular with rank one. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$$

Solution

Three eigenvalues are 0, 0, 6. The corresponding eigenvectors are: $\begin{bmatrix} -1 & 2 & 0 \end{bmatrix}^T$, $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$.

Problem 5

Find the rank and the four eigenvalues of A and C:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution

1. The rank of A is one, the eigenvalues are 0, 0, 0, 4.
2. The rank of C is 2, the eigenvalues are 0, 0, 2, 2.

Problem 6

Suppose A has eigenvalues 0,3,5 with independent eigenvectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

1. Give a basis for the nullspace and a basis for the column space.
2. Find a particular solution to $A\mathbf{x} = \mathbf{v} + \mathbf{w}$. Find all solutions.
3. $A\mathbf{x} = \mathbf{u}$ has no solution. If it did then ____ would be in the column space.

Solution

1. A basis for the nullspace is $\mathbf{u}$,a basis for the column space is $\{\mathbf{v}, \mathbf{w}\}$.
2. A particular solution to $A\mathbf{x} = \mathbf{v} + \mathbf{w}$ is $\mathbf{x}^* = 1/3\mathbf{v} + 1/5\mathbf{w}$ and all solutions have the form $\mathbf{x} = c\mathbf{u} + 1/3\mathbf{v} + 1/5\mathbf{w},\ c \in \mathbb{R}$.
3. $a\mathbf{v} + b\mathbf{w} \in\ C(A), a, b \in \mathbb{R}$.

Problem 7

1. Factor these two matrices into $A = X\Lambda X^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

2. If $A = X\Lambda X^{-1}$ then $A^3 = (\ )(\ )(\ )$ and $A^{-1} = (\ )(\ )(\ )$.

Solution

1. The factorization of first $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, the factorization of second $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -3/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$.
2. $A^3 = X\Lambda^3 X^{-1}$, $A^{-1} = X\Lambda^{-1}X^{-1}$.

Problem 8

True or false: If the column of X (eigenvectors of A) are linearly independent, then

1. A is invertible.
2. A is diagonalizable.
3. X is invertible.
4. X is diagonalizable.

Solution

1. False. Consider **Problem 6**.
2. True.
3. True.
4. True.

Problem 9

$A^k = X\Lambda^k X^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ____ . Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

(Recommended) Find Λ and X to diagonalize A_1 in the above problem. What is the limit of Λ^k as $k \to \infty$? In the columns of this limiting matrix you see the ____.

Solution

Problem 9 continued on next page...

1. $|\lambda| < 1$.

2. The eigenvalues of A_1 are 1 and -0.3, the eigenvalues of A_2 are 0.9 and 0.3, since the absolute value of the eigenvalues of A_2 are less than 1, we know $A_2^k \to 0, k \to \infty$.

3. A_1 can be diagonalized as $A_1 = \begin{bmatrix} 9 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.3 \end{bmatrix} \begin{bmatrix} 1/13 & 1/13 \\ -4/13 & 9/13 \end{bmatrix}$, $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

   the columns with eigenvalues less than 1 become the zero columns.

Problem 10

Show that trace $XY = \text{trace}YX$ by adding the diagonal entries of XY and YX:

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} q & r \\ s & t \end{bmatrix}$$

Now choose Y to be ΛX^{-1}. Then $X\Lambda X^{-1}$ has the same trace as $\Lambda X^{-1}X = \Lambda$. This proves that the trace of A equals the trace of Λ = the sum of the eigenvalues. **AB − BA = I is impossible** since the left side has trace ____.

Solution

1. $(XY)_{11} + (XY)_{22} = (aq + bs) + (cr + dt) = (qa + rc) + (sb + td) = (YX)_{11} + (YX)_{22}$

2. 0.

Problem 11

1. If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ then the determinant of $A - \lambda I$ is $(\lambda - a)(\lambda - d)$. Check the "Cayley-Hamilton Theorem" that $(A - aI)(A - dI) =$ zero matrix.

2. Test the Cayley-Hamilton Theorem on Fibonacci's $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.

Solution

(a) $(A - aI)(A - dI) = \begin{bmatrix} 0 & b \\ 0 & d - a \end{bmatrix} \begin{bmatrix} a - d & b \\ 0 & 0 \end{bmatrix} = \mathbf{0}$.

(b) The characteristic polynomial of Fibonacci matrix is $\lambda^2 - \lambda - 1 = 0$, we have

$$A^2 - A - I = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The result checks the Cayley-Hamilton Theorem is true.