Linear Algebra: Homework #3

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0130

Problem 1

1. Which rules are broken if we keep only the positive numbers $x > 0$ in $\mathbf{R}^1$? Every c must be allowed. This half-line is not a subspace.

2. The positive numbers with $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$ redefined to equal the usual xy and x^c do satisfy the eight rules. Test rule 7 when $c = 3, x = 2, y = 1$. (Then $\mathbf{x} + \mathbf{y} = 2$) and $c\mathbf{x} = 8$. Which number acts as the "zero vector" in this space?

Solution

1. Rule 4.

2. 1 acts as the "zero vector" in this space.

Problem 2

M is the space of 2 by 2 matrices.

1. Describe a subspace of M that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

2. If a subspace of M does contain A and B, must it contain the identity matrix?

3. Describe a subspace of M that contains no nonzero diagonal matrices.

Solution

1. $\left\{ Q \mid Q = c \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, c \in \mathbb{R} \right\}$

2. Yes, because identity matrix is a linear combination of A and B.

3. $\left\{ Q \mid Q = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}, a, b \in \mathbb{R} \right\}$

Problem 3

The columns of AB are combinations of the columns of A. This means: The column space of AB is contained in (possibly equal to) the column space A. Give an example where the columns spaces of A and AB are not equal.

Solution

We can consider the case where the matrix B is not full of rank and give the following example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 4

(nullspace of A) Create a 2 by 4 matrix R whose special solutions to $R\mathbf{x} = 0$ are $\mathbf{s}_1$ and $\mathbf{s}_2$:

$$\mathbf{s}_1 = \begin{bmatrix} -3 \\ \mathbf{1} \\ 0 \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathbf{s}_2 = \begin{bmatrix} -2 \\ \mathbf{0} \\ -6 \\ \mathbf{1} \end{bmatrix}$$

pivots columns 1 and 3 free variables x_2 and x_4, x_2 and x_4 are 1,0 and 0,1 in the "special solutions".

Describe all 2 by 4 matrices with this nullspace $\mathbf{N}(A)$ spanned by $\mathbf{s}_1$ and $\mathbf{s}_2$.

Solution

$$\mathbf{N}(A) = \{c_1\mathbf{s}_1 + c_2\mathbf{s}_2 \mid c_1, c_2 \in \mathbb{R}\} = \left\{ c_1 \begin{bmatrix} -3 \\ \mathbf{1} \\ 0 \\ \mathbf{1} \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ \mathbf{0} \\ -6 \\ \mathbf{1} \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

Problem 5

Suppose an m by n matrix has r pivots. The number of special solutions (basis for the nullspace) is ____ by the Counting Theorem. The nullspace contains only $\mathbf{x} = \mathbf{0}$ when $r =$ ____. The column space is all of $\mathbf{R}^m$ when the rank is $r =$ ____.

Solution

1. $n - r$
2. n
3. m

Problem 6

Construct a matrix whose column space contains (1,1,5) and (0,3,1) and whose nullspace contains (1,1,2).

Solution

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

Problem 7

If $AB = 0$ then the column space of B is contained in the ____ of A. Why?

Solution

Problem 7 continued on next page...

The column space of B is contained in the nullspace of A. Because if we rewrite matrix B as the following form:

$$B = [B_1, B_2, \dots, B_q]$$

where the column number of $B_i (i \in \{1, \dots, q\})$ is 1,then AB can be represented as $[AB_1, AB_2, \dots, AB_q]$, and we know $B_1, \dots, B_q$ are in the nullspace of A since $AB = 0$. Then we know the space spanned by $B_1, \dots, B_q$ (which is the column space of B) is contained in the nullspace of A.

Problem 8

How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Solution

$\mathbf{N}(C) = \mathbf{N}(A) \cap \mathbf{N}(B)$.

Problem 9

$A = C \begin{bmatrix} I & F \end{bmatrix} = C \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$. C has 2 independent columns.

Find the 2 special solutions to $A\mathbf{x} = \mathbf{0}$ of the form $(x_1, x_2, 1, 0)$ and $(x_1, x_2, 0, 1)$.

Note: If $A\mathbf{x} = \mathbf{b}$ has a solution $x = x_p$ then all its solutions have the form $x = x_p + x_n$. Here x_p is the only solution in the row space of A and x_n is in the nullspace of A (so $Ax_n = 0$).

Solution

Let $R = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$, then $A\mathbf{x} = 0$ has the same solutions of $R\mathbf{x} = 0$. We can find two special solutions by specifying the free variables to (1, 0) and (0, 1):

$$\mathbf{v}_1 = \left(-2, -3, 1, 0\right), \mathbf{v}_2 = \left(-2, -1, 0, 1\right)$$

Problem 10

Write the complete solution as $\mathbf{x}_p$ plus any multiple of $\mathbf{s}$ in the nullspace:

$$\begin{aligned} x + 3y &= 7 \\ 2x + 6y &= 14 \end{aligned} \qquad \begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned}$$

Solution

1. $\mathbf{x}_p = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix}, \ c \in \mathbb{R}$

2. $\mathbf{x}_p = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \ c \in \mathbb{R}$

Problem 11

Under what conditions on b_1, b_2, b_3 are these systems solvable? Include $\mathbf{b}$ as a fourth column in elimination. Find all solutions when that solvability condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3 \end{aligned} \qquad \begin{aligned} 2x + 2z &= b_1 \\ 4x + 4y &= b_2 \\ 8x + 8y &= b_3 \end{aligned}$$

Solution

1. $b_3 - b_2 - 2b_1 = 0.$

2. $b_3 - 2b_2 = 0.$

Problem 12

Construct a 2 by 3 system $A\mathbf{x} = \mathbf{b}$ with particular solution $\mathbf{x}_p = \big(2, 4, 0\big)$ and homogeneous solution $\mathbf{x}_n =$ any multiple of (1,1,1).

Solution

The augmented matrix of this system is given below:

$$\begin{bmatrix} 1 & 2 & -3 & 10 \\ 4 & 3 & -7 & 20 \end{bmatrix}$$

Problem 13

Give examples of matrices A for which the number of solutions to $A\mathbf{x} = \mathbf{b}$ is

1. 0 or 1, depending on $\mathbf{b}$

2. ∞, regardless of $\mathbf{b}$

3. 0 or ∞, depending on $\mathbf{b}$

4. 1, regardless of $\mathbf{b}$

Solution

The number of solutions is determined by the rank of A, suppose A is a n by m matrix.

1. rank(A) = n < m: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Problem 13 continued on next page...

2. rank(A) = m < n: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. rank (A) < m; rank (A) < n: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. rank (A) = n = m: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 14

Find a basis (independent vectors that span the subspace) for each of these subspaces of $\mathbf{R}^4$:

1. All vectors whose components are equal.
2. All vectors whose components add to zero.
3. All vectors that are perpendicular to (1, 1, 0, 0) and (1, 0, 1, 1).
4. The column space and the nullspace of I (4 by 4).

Find a basis (and the dimension) for each of these subspaces of 3 by 3 matrices:

1. All symmetric matrices ($A^T = A$).
2. All skew-symmetric matrices ($A^T = -A$).

Solution

1. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

2. To get the dimension of this subspace, one way is to think of this subspace as a nullspace of $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, and then we know the dimension of this subspace is 3, assign 1 to free variables one by one, we have:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, the question is, find a basis of the nullspace of A? Using elimination, a basis is given below:

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

4. (a) A basis of the column space of I:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Problem 14 continued on next page...

(b) The nullspace of I doesn't have any bases.

5. Let S represents the subspace of all symmetric matrices. A basis of this subspace is given below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

   And we know $\dim(S) = 6$.

6. Let S represents the subspace of all skew-symmetric matrices, a basis of this subspace is given below:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

   And we know $\dim(S) = 3$.