

# Linear Algebra: Homework #1

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0130

## Problem 1

How could you decide if the vectors  $\mathbf{u} = (1, 1, 0)$  and  $\mathbf{v} = (0, 1, 1)$  and  $\mathbf{w} = (a, b, c)$  are linearly independent or dependent?

### Solution

If  $a + b - c \neq 0$ , then these vectors are linearly independent, otherwise they are linearly dependent.

## Problem 2

How many corners  $(\pm 1, \pm 1, \pm 1, \pm 1)$  does a cube of side 2 have in 4 dimensions? What is its volume? How many 3D faces? How many edges? Find one edge.

### Solution

1. The cube has 16 corners.
2. The volume of the cube is 16.
3. There are 8 3D faces.
4. There are 32 edges.
5.  $\{(-1, -1, -1, -1), (-1, -1, -1, 1)\}$  is one edge.

## Problem 3

The **triangle inequality** says: (length of  $\mathbf{v} + \mathbf{w}$ )  $\leq$  (length of  $\mathbf{v}$ ) + (length of  $\mathbf{w}$ ).

1. Show that  $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2$ .
2. Increase that  $\mathbf{v} \cdot \mathbf{w}$  to  $\|\mathbf{v}\| \|\mathbf{w}\|$  to show that **side 3** cannot exceed **side 1** + **side 2**:

### Solution

1.  $\|\mathbf{v} + \mathbf{w}\|^2 = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v}^2 + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w}^2 = \|\mathbf{v}\|^2 + 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2$
2.  $\|\mathbf{side\ 3}\| = \|\mathbf{side\ 1} - \mathbf{side\ 2}\| \leq \|\mathbf{side\ 1}\| + \|\mathbf{-side\ 2}\| = \|\mathbf{side\ 1}\| + \|\mathbf{side\ 2}\|$

## Problem 4

Which numbers  $q$  would leave  $A$  with two independent columns?

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 9 \\ 5 & 0 & q \end{bmatrix}, A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & q \end{bmatrix}, A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & q \end{bmatrix}$$

### Solution

1.  $q = 10$ .
2.  $q = 9$ .
3.  $q \neq 0$ .

## Problem 5

If  $(a, b)$  is a multiple of  $(c, d)$  with  $abcd \neq 0$ , show that  $(a, c)$  is a multiple of  $(b, d)$ . This is surprisingly important; two columns are failing on one line. You could use numbers first to see how  $a, b, c, d$  are related. The question will lead to:

If  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  has dependent rows, then it also has dependent columns.

### Solution

Suppose  $(a, c)$  is  $k$  times  $(b, d)$ , then we have  $\frac{a}{b} = \frac{kc}{kd} = \frac{c}{d}$ .

## Problem 6

Why is it impossible for a matrix  $A$  with 7 columns and 4 rows to have 5 independent columns? This is not a trivial or useless question.

### Solution

This question is equivalent to asking the rank of a matrix is less than or equal to the lesser of them. Recall the process of elimination, if we have 5 independent columns, then we should have 5 pivot, which conflicts to the fact that we only have 4 rows.

## Problem 7

Going from left to right, put each column of  $A$  into the matrix  $C$  if that column is not a combination of earlier columns:

$$A = \begin{bmatrix} 2 & -2 & 1 & 6 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 3 & -3 & 0 & 6 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & & & & \\ 1 & & & & \\ 3 & & & & \end{bmatrix}$$

Find  $R$  in Problem 6 so that  $A = CR$ . If your  $C$  has  $r$  columns, then  $R$  has  $r$  rows. The 5 columns of  $R$  tell how to produce the 5 columns of  $A$  from the columns in  $C$ .

### Solution

$$R = \begin{bmatrix} -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

## Problem 8

Complete these 2 by 2 matrices to meet the requirements printed underneath:

$$\begin{array}{cccc} \begin{bmatrix} 3 & 6 \\ 5 & \end{bmatrix} & \begin{bmatrix} 6 & 7 \\ 6 & \end{bmatrix} & \begin{bmatrix} 3 & \\ 3 & 6 \end{bmatrix} & \begin{bmatrix} 3 & 4 \\ & -3 \end{bmatrix} \\ \text{rank one} & \text{orthogonal columns} & \text{rank 2} & A^2 = I \end{array}$$

**Solution**

$$\begin{bmatrix} 3 & 6 \\ 5 & 10 \end{bmatrix}, \begin{bmatrix} 6 & 7 \\ 7 & -6 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$$

**Problem 9**

True or false, with a reason:

1. If 3 by 3 matrices  $A$  and  $B$  have rank 1, then  $AB$  will always have rank 1.
2. If 3 by 3 matrices  $A$  and  $B$  have rank 3, then  $AB$  will always have rank 3.
3. Suppose  $AB = BA$  for every 2 by 2 matrix  $B$ . Then  $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = cI$  for some number  $c$ . Only those matrices  $A = cI$  commute with every  $B$ .

**Solution**

- (a) No, consider the following matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- (b) Yes,  $\text{rank}(AB) = \text{rank}(A) = 3$  since  $\text{rank}(B) = 3$ .  
 (c) Omitted.

**Problem 10**

How many small multiplications for  $(AB)C$  and  $A(BC)$  if those matrices have sizes  $ABC = (4 \times 3)(3 \times 2)(2 \times 1)$ ? The two counts are different.

**Solution**

32;24