

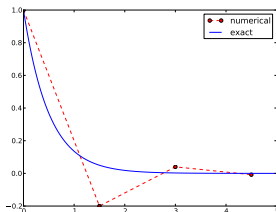
# On Schemes for Exponential Decay

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# Goal

The primary goal of this demo talk is to demonstrate how to write talks with [DocOnce](#) and get them rendered in numerous HTML formats.

## Layout

This version utilizes beamer slides with the theme `cbc`.



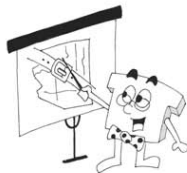
# We aim to solve the (almost) simplest possible differential equation problem

$$u'(t) = -au(t) \quad (1)$$

$$u(0) = I \quad (2)$$

Here,

- ▶  $t \in (0, T]$
- ▶  $a$ ,  $I$ , and  $T$  are prescribed parameters
- ▶  $u(t)$  is the unknown function
- ▶ The ODE (??) has the initial condition (??)



# The ODE problem is solved by a finite difference scheme

- ▶ Mesh in time:  $0 = t_0 < t_1 \cdots < t_N = T$
- ▶ Assume constant  $\Delta t = t_n - t_{n-1}$
- ▶  $u^n$ : numerical approx to the exact solution at  $t_n$

The  $\theta$  rule,

$$u^{n+1} = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t} u^n, \quad n = 0, 1, \dots, N - 1$$

contains the Forward Euler ( $\theta = 0$ ), the Backward Euler ( $\theta = 1$ ), and the Crank-Nicolson ( $\theta = 0.5$ ) schemes.

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# The Forward Euler scheme explained

<http://youtube.com/PtJrPEIHNJw>



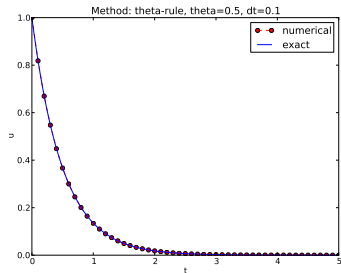
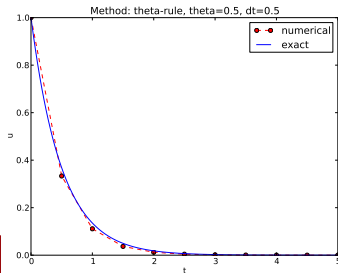
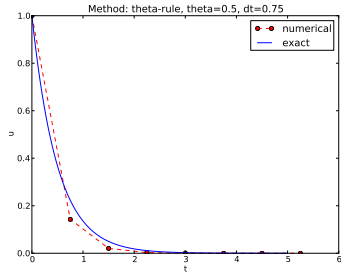
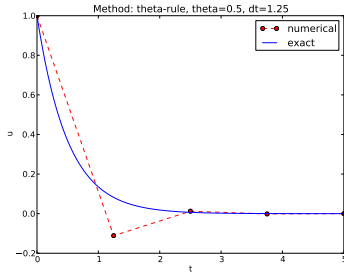
# Implementation

The numerical method is implemented in a Python function:

```
def solver(I, a, T, dt, theta):  
    """Solve  $u' = -a*u$ ,  $u(0) = I$ , for  $t$  in  $(0, T]$  with steps of  $dt$ ."""  
    dt = float(dt)           # avoid integer division  
    N = int(round(T/dt))      # no of time intervals  
    T = N*dt                 # adjust T to fit time step dt  
    u = zeros(N+1)           # array of  $u[n]$  values  
    t = linspace(0, T, N+1)  # time mesh  
  
    u[0] = I                  # assign initial condition  
    for n in range(0, N):     #  $n=0, 1, \dots, N-1$   
        u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]  
    return u, t
```



# The Crank-Nicolson method shows oscillatory behavior for not sufficiently small time steps, while the solution should be monotone



# The artifacts can be explained by some theory

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t}.$$

Key results:

- ▶ Stability:  $|A| < 1$
- ▶ No oscillations:  $A > 0$
- ▶ Always for Backward Euler ( $\theta = 1$ )
- ▶  $\Delta t < 1/a$  for Forward Euler ( $\theta = 0$ )
- ▶  $\Delta t < 2/a$  for Crank-Nicolson ( $\theta = 1/2$ )

Concluding remarks:

Only the Backward Euler scheme is guaranteed to always give qualitatively correct results.

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