

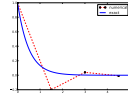
## On Schemes for Exponential Decay

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## Goal

The primary goal of this demo talk is to demonstrate how to write talks with **DocOnce** and get them rendered in numerous HTML formats.

### Layout

This version utilizes beamer slides with the theme `red_plain`.

## Problem setting and methods



We aim to solve the (almost) simplest possible differential equation problem

$$\begin{aligned} u'(t) &= -au(t) & (1) \\ u(0) &= I & (2) \end{aligned}$$

Here,

- ▶  $t \in (0, T]$
- ▶  $a$ ,  $I$ , and  $T$  are prescribed parameters
- ▶  $u(t)$  is the unknown function
- ▶ The ODE (1) has the initial condition (2)



## The ODE problem is solved by a finite difference scheme

- ▶ Mesh in time:  $0 = t_0 < t_1 < \dots < t_N = T$
- ▶ Assume constant  $\Delta t = t_n - t_{n-1}$
- ▶  $u^n$ : numerical approx to the exact solution at  $t_n$

The  $\theta$  rule,

$$u^{n+1} = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t} u^n, \quad n = 0, 1, \dots, N-1$$

contains the **Forward Euler** ( $\theta = 0$ ), the **Backward Euler** ( $\theta = 1$ ), and the **Crank-Nicolson** ( $\theta = 0.5$ ) schemes.

## The Forward Euler scheme explained

<http://youtube.com/PtJrPEIHJw>

## Implementation

### Implementation in a Python function:

```
def solver(I, a, T, dt, theta):
    """Solve u' = -au, u(0)=I, for t in (0,T]; step: dt."""
    dt = float(dt) # avoid integer division
    N = int(round(T/dt)) # no of time intervals
    T = N*dt # adjust T to fit time step dt
    u = zeros(N+1) # array of u[n] values
    t = linspace(0, T, N+1) # time mesh

    u[0] = I # assign initial condition
    for n in range(0, N): # n=0,1,...,N-1
        u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]
    return u, t
```

## How to use the solver function

### A complete main program

```
# Set problem parameters
I = 1.2
a = 0.2
T = 8
dt = 0.25
theta = 0.5
|\\pause|

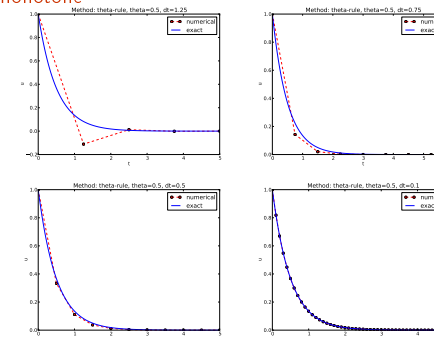
from solver import solver, exact_solution
u, t = solver(I, a, T, dt, theta)
|\\pause|

import matplotlib.pyplot as plt
plt.plot(t, u, t, exact_solution)
plt.legend(['numerical', 'exact'])
plt.show()
```

## Results



The Crank-Nicolson method shows oscillatory behavior for not sufficiently small time steps, while the solution should be monotone



## The artifacts can be explained by some theory

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t}.$$

Key results:

- ▶ Stability:  $|A| < 1$
- ▶ No oscillations:  $A > 0$
- ▶  $\Delta t < 1/a$  for Forward Euler ( $\theta = 0$ )
- ▶  $\Delta t < 2/a$  for Crank-Nicolson ( $\theta = 1/2$ )

### Concluding remarks:

Only the Backward Euler scheme is guaranteed to always give qualitatively correct results.