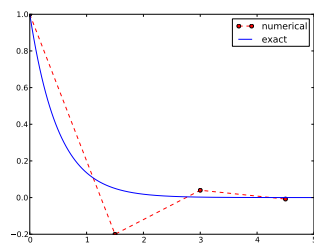# On Schemes for Exponential Decay

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1   Goal

The primary goal of this demo talk is to demonstrate how to write talks with DocOnce and get them rendered in numerous HTML formats.

> **Layout.**
>
> This version utilizes latex document slides with the theme `no theme`.

The talk investigates the accuracy of three finite difference schemes for the ordinary differential equation $u' = -au$ with the aid of numerical experiments. Numerical artifacts are in particular demonstrated.

2   Mathematical problem

$$u'(t) = -au(t), \qquad (1)$$
$$u(0) = I, \qquad (2)$$

- $t \in (0, T]$

- a, I, and T are prescribed parameters

- $u(t)$ is the unknown function

3 Numerical solution method

- Mesh in time: $0 = t_0 < t_1 \cdots < t_N = T$

- Assume constant $\Delta t = t_n - t_{n-1}$

- u^n: numerical approx to the exact solution at t_n

Numerical scheme:

$$u^{n+1} = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t} u^n, \quad n = 0, 1, \dots, N - 1$$

3.1 Forward Euler explained

http://youtube.com/PtJrPEIHNJw

4 Implementation

The numerical method is implemented in a Python function:

```python
def solver(I, a, T, dt, theta):
    """Solve u'=-a*u, u(0)=I, for t in (0,T] with steps of dt."""
    dt = float(dt)          # avoid integer division
    N = int(round(T/dt))    # no of time intervals
```

```
T = N*dt                        # adjust T to fit time step dt
u = zeros(N+1)                  # array of u[n] values
t = linspace(0, T, N+1)         # time mesh

u[0] = I                        # assign initial condition
for n in range(0, N):           # n=0,1,...,N-1
    u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]
return u, t
```
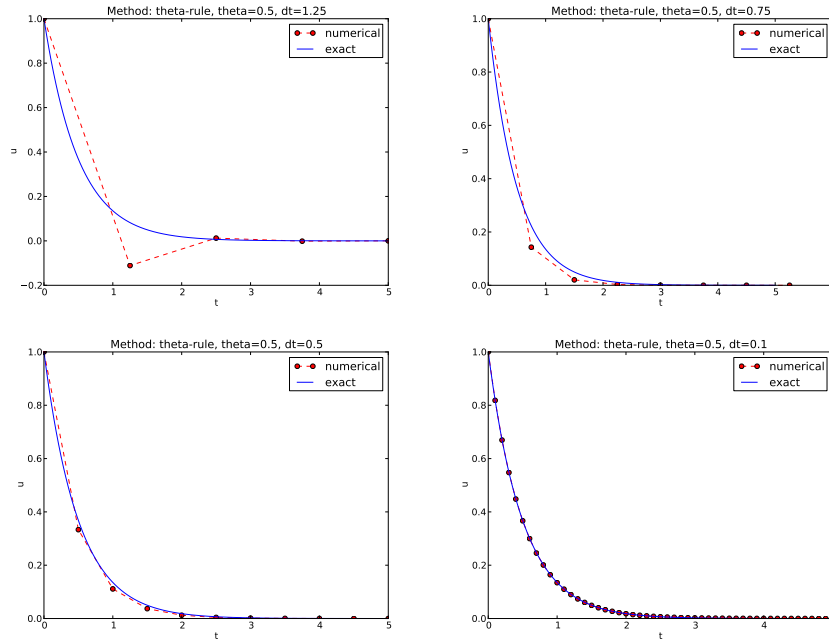
4.1 The Crank-Nicolson method

4.2 The artifacts can be explained by some theory

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1 - (1-\theta)a\Delta t}{1 + \theta a\Delta t} \,.$$

- Stability: $|A| < 1$

- No oscillations: $A > 0$

- Always for Backward Euler ($\theta = 1$)

- $\Delta t < 1/a$ for Forward Euler ($\theta = 0$)

- $\Delta t < 2/a$ for Crank-Nicolson ($\theta = 1/2$)

Concluding remarks:

Only the Backward Euler scheme is guaranteed to always give qualitatively correct results.