

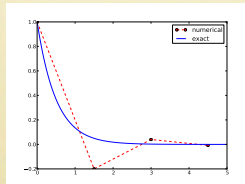
# ON SCHEMES FOR EXPONENTIAL DECAY

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# GOAL

The primary goal of this demo talk is to demonstrate how to write talks with `DocOnce` and get them rendered in numerous HTML formats.

## LAYOUT

This version utilizes beamer slides with the theme `vintage`.

## PROBLEM SETTING AND METHODS

## RESULTS

# PROBLEM SETTING AND METHODS



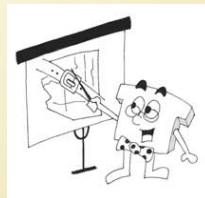
# WE AIM TO SOLVE THE (ALMOST) SIMPLEST POSSIBLE DIFFERENTIAL EQUATION PROBLEM

$$u'(t) = -au(t) \quad (1)$$

$$u(0) = I \quad (2)$$

Here,

- $t \in (0, T]$
- $a$ ,  $I$ , and  $T$  are prescribed parameters
- $u(t)$  is the unknown function
- The ODE (1) has the initial condition (2)



## THE ODE PROBLEM IS SOLVED BY A FINITE DIFFERENCE SCHEME

- Mesh in time:  $0 = t_0 < t_1 \cdots < t_N = T$
- Assume constant  $\Delta t = t_n - t_{n-1}$
- $u^n$ : numerical approx to the exact solution at  $t_n$

The  $\theta$  rule,

$$u^{n+1} = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t} u^n, \quad n = 0, 1, \dots, N - 1$$

contains the Forward Euler ( $\theta = 0$ ), the Backward Euler ( $\theta = 1$ ), and the Crank-Nicolson ( $\theta = 0.5$ ) schemes.

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# THE FORWARD EULER SCHEME EXPLAINED

<http://youtube.com/PtJrPEIHNJw>

# IMPLEMENTATION

## IMPLEMENTATION IN A PYTHON FUNCTION:

```
def solver(I, a, T, dt, theta):  
    """Solve u'=-a*u, u(0)=I, for t in (0,T]; step: dt."""  
    dt = float(dt) # avoid integer division  
    N = int(round(T/dt)) # no of time intervals  
    T = N*dt # adjust T to fit time step dt  
    u = zeros(N+1) # array of u[n] values  
    t = linspace(0, T, N+1) # time mesh  
  
    u[0] = I # assign initial condition  
    for n in range(0, N): # n=0,1,...,N-1  
        u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]  
    return u, t
```

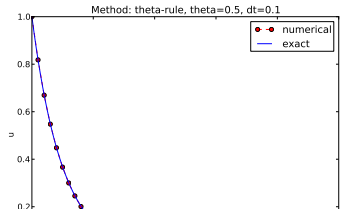
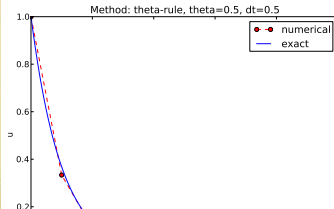
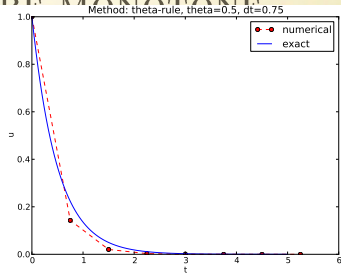
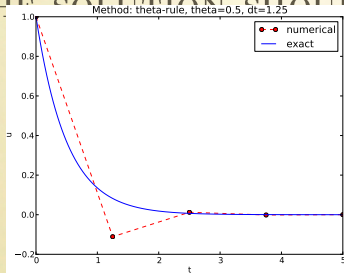
PROBLEM SETTING AND METHODS

RESULTS

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THE CRANK-NICOLSON METHOD SHOWS  
OSCILLATORY BEHAVIOR FOR NOT  
SUFFICIENTLY SMALL TIME STEPS, WHILE  
THE SOLUTION SHOULD BE MONOTONE



# THE ARTIFACTS CAN BE EXPLAINED BY SOME THEORY

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t}.$$

Key results:

- Stability:  $|A| < 1$
- No oscillations:  $A > 0$
- $\Delta t < 1/a$  for Forward Euler ( $\theta = 0$ )
- $\Delta t < 2/a$  for Crank-Nicolson ( $\theta = 1/2$ )

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