On Schemes for Exponential Decay

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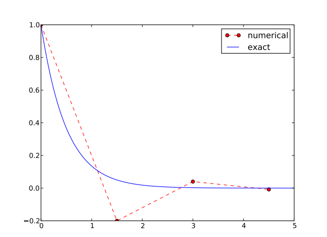
Sep 9, 2014



Goal

The primary goal of this demo talk is to demonstrate how to write talks with DocOnce and get them rendered in numerous HTML formats.

Layout.

This version utilizes beamer slides with the theme `vintage`.

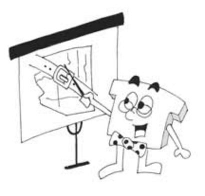
Mathematical problem

$$u'(t) = -au(t), \qquad (1)$$

$$u(0) = I, \qquad (2)$$

Here,

- $t \in (0, T]$
- a, I, and T are prescribed parameters
- $u(t)$ is the unknown function



Numerical solution method

- Mesh in time: $0 = t_0 < t_1 \cdots < t_N = T$
- Assume constant $\Delta t = t_n - t_{n-1}$
- u^n: numerical approx to the exact solution at t_n

Numerical scheme:

$$u^{n+1} = \frac{1-(1-\theta)a\Delta t}{1+\theta a\Delta t}u^n, \quad n = 0, 1, \ldots, N-1$$

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Forward Euler explained

http://youtube.com/PtJrPEIHNJw

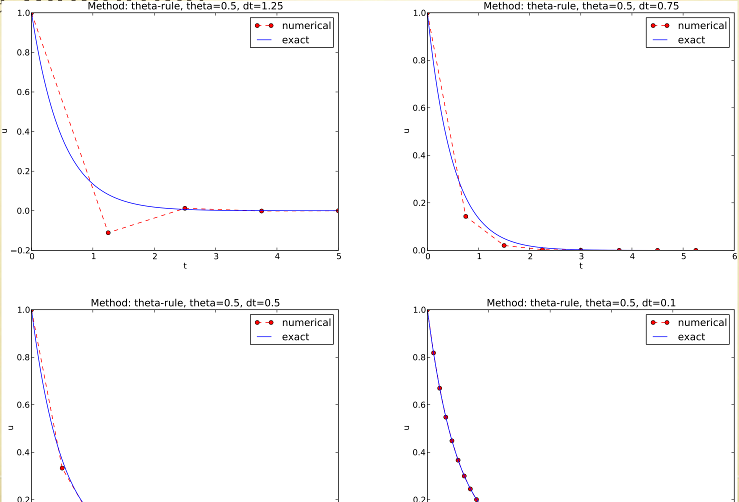
Implementation

The numerical method is implemented in a Python function:

```
def solver(I, a, T, dt, theta):
    """Solve u'=-a*u, u(0)=I, for t in (0,T] with steps of dt."""
    dt = float(dt)              # avoid integer division
    N = int(round(T/dt))        # no of time intervals
    T = N*dt                    # adjust T to fit time step dt
    u = zeros(N+1)              # array of u[n] values
    t = linspace(0, T, N+1)     # time mesh

    u[0] = I                    # assign initial condition
    for n in range(0, N):       # n=0,1,...,N-1
        u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]
    return u, t
```

The Crank-Nicolson method shows oscillatory behavior for not sufficiently small time steps, while the solution should be monotone



The artifacts can be explained by some theory

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1-(1-\theta)a\Delta t}{1+\theta a\Delta t}\,.$$

Key results:

- Stability: $|A| < 1$
- No oscillations: $A > 0$
- Always for Backward Euler ($\theta = 1$)
- $\Delta t < 1/a$ for Forward Euler ($\theta = 0$)
- $\Delta t < 2/a$ for Crank-Nicolson ($\theta = 1/2$)

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