Kinematics of Excavators (Backhoes) for Transferring Surface Material

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Abstract: To use construction machines effectively in the dark, severe weather, or hazardous and/or unhealthy environments, their operations should be controlled automatically. It can be realized if the kinematics and dynamics of the machine are understood. To help achieve this goal, the kinematics of specific construction machines—excavators (backhoes and loaders)—are investigated here. A systematic procedure is presented to assign Cartesian coordinate frames for the links (joints) of an excavator. Then, the homogeneous transformation matrices that relate two adjacent coordinate frames are given. The kinematic relations of the pose (position and orientation) of the bucket, the joint shaft angles, and the lengths of the cylinder rods in the hydraulic actuators for an excavator are studied. Explicit expressions for the forward and backward (inverse) kinematic relations are presented. Then, the corresponding kinematic velocity relations for the excavators are developed. The kinematic relations presented provide the foundation for engineers to realize the automatic computer-controlled operations of the machine.

INTRODUCTION

The semiautonomous or automatic computer control is essential to improve the productivity and the effective use of expensive construction machines (Sundareswaran and Arditi 1988; Ward 1988). Many tasks on construction sites can then be performed even in hostile and/or unfavorable conditions—for example, in severe (cold or rainy) weather or in hazardous, unhealthy, and even poisonous environmental conditions. Moreover, automatically operated machines can often perform a task faster and with better precision than manually operated machines. To automate the operations of these construction machines, the kinematics and dynamics of the machine motions must be well understood. As a step in this direction, the kinematic relations of one type of construction machines, excavators, will be presented here. The basic approach to be discussed can also be applied to backhoes and various loaders commonly used to transfer ground material in various work areas such as at construction, timbering, and mining sites.

The usual task of an excavator (backhoe and loader) is to free and/or remove surface material (e.g., soil and coal) from its original location and transfer it to another location by lowering the bucket, digging, pushing, and/or pulling soil, then lifting, swinging, and emptying the bucket. The execution of this task is usually performed by a human operator who controls the motion of the machine manually by using the visual feedback provided through his or her own eyes. In many current applications of excavators (backhoes and loaders), the semiautonomous or even automatic operation of the machine is desirable and sometimes even necessary, for example, in the transfer of mining products from underground sites or in the removal of poisonous or radioactive wastes and/or explosives. In the semiautonomous operation, a human teleoperator may guide the motion of the ma-

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chine over a certain time period so as to make it perform parts of the task, and a computer may then control the machine over the rest of the duration of the motion to execute the other parts of the task automatically. In the automatic computer-controlled motion, the bucket of the excavator follows a path specified by its position and the bucket lift and digging angles, i.e., the pose of the bucket, which corresponds to specific values of the angular positions of the joint shafts. The values of these joint variables, in turn, are determined by the lengths of the hydraulic actuators. The mathematical relations between these variables are described by the kinematic relations of the machines.

The previously reported studies on excavators are mainly qualitative. An entire system for the automatic or semiautomatic operation of an excavator is described in Seward et al. (1988). It includes a hydraulically driven excavator, teleoperation, computer control (which is supplemented with manual override capability), position, and force-feedback control. Kinematic relations between three joint angles and the position of the bucket in a fixed coordinate system are presented in Seward et al. (1988); specifically, the forward and backward kinematic equations are described using the geometric configuration of the excavator arm in a fixed (world) coordinate system. However, no other details on the system are given. The use of a position and sensor (force and vision) feedback is also discussed qualitatively in Seward et al. (1988) without presenting technical details. Similarly, a vision-feedback system for an excavator designed for rapid runway repairs is presented in Wohlford (1990) by qualitatively characterizing the system components. This system has successfully been tested in practice, but no technical details are presented in Wohlford et al. (1990). Although the forces between the soil and a tool (bucket) during the digging operations are studied in Bernold (1991) and Bullock et al. (1990), the kinematic relations of the bucket pose, the joint variables, and the lengths of the actuators are not described in either work.

An attempt to describe the kinematics of an excavator is presented in Vaha et al. (1991) by defining the coordinate frames by an ad hoc approach. The forward and backward kinematic relations between the pose of the bucket and the angles of the joint shafts for an excavator are described. However, the coordinate frames are not assigned systematically. Moreover, no relations between the joint shaft angles and the positions (lengths) of the cylinder rods in the hydraulic actuators are given. Thus, the kinematic relations presented in Vaha et al. (1991) do not completely describe the configuration of the machine.

The work reported here presents the details of assigning the local and world coordinate systems for an excavator (backhoe and loader) by following the Denavit-Hartenberg (DH) guidelines (Koivo 1989), which are well known in robotics. The assignment is systematic and different from that described in Vaha et al. (1991), where the DH guidelines are not applied. The homogeneous transformation matrices relating any two adjacent coordinate frames in the system are given. The forward and backward kinematic equations covering the pose of the bucket, the angular positions of the joint shafts, and the lengths of the hydraulic actuators are presented. Thus, a complete set of the kinematic equations are established for the excavators (backhoes and loaders).

COORDINATE-FRAME ASSIGNMENTS

To analyze and plan the motion of the excavator (backhoe and loader) in Fig. 1 for performing a specific task, it is necessary to define a world

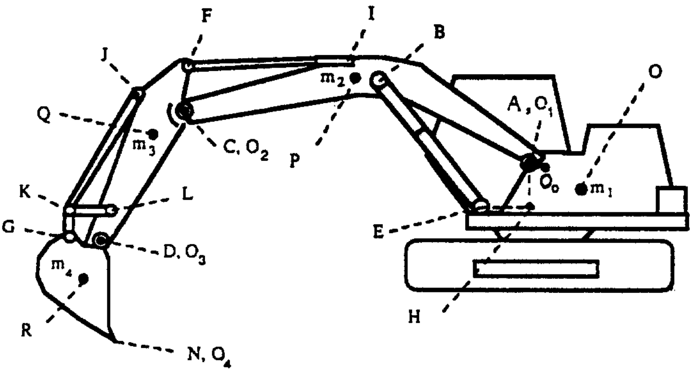

FIG. 1. Schematic Side View of Excavator

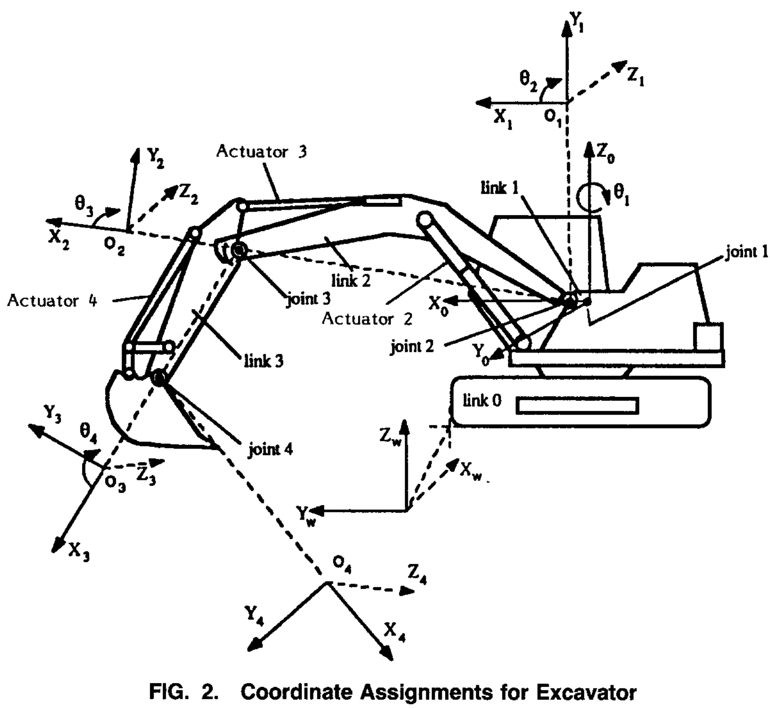

FIG. 2. Coordinate Assignments for Excavator

coordinate system to describe the pose of the bucket (the end effector). A fixed-rectangular and right-hand Cartesian coordinate system $X_wY_wZ_w$ is chosen, and its origin is placed at an arbitrary point on the ground level in the workspace of the excavator. It is then convenient to define local coordinate frames for all links (joints) by following the DH guidelines (Koivo 1989), which are commonly used in robotics, and are outlined in Appendix I. The Z_0-axis for the base coordinate frame $X_0Y_0Z_0O_0$ is chosen vertical (upwards) to represent the rotational axis of the upper structure (the base or carrier), and the X_0- and Y_0-axes are on a horizontal plane perpendicular to the Z_0-axis (Fig. 1). The origin (O_0) may be placed so that this horizontal plane contains the rotational axis of joint 2 (i.e., the line passing through point O_1 in Figs. 1 and 2). Furthermore, the Z_1-axis of the $X_1Y_1Z_1O_1$ coordinate frame is determined to coincide with the rotational axis of joint 2, which connects link 2 (with A = O_1 and C = O_2 as the end points) to link 1 (with point O_0 and A as the end points). Its positive direction is arbitrarily chosen to be into the page. Then the X_1-axis is chosen along the line determined by the cross product of the unit vectors $\mathbf{k}_{Z0}$ and $\mathbf{k}_{Z1}$ on the Z_0- and Z_1-axes, respectively, i.e., $\pm(\mathbf{k}_{Z0} \times \mathbf{k}_{Z1})$. The Y_1-axis completes the first local rectangular coordinate frame $X_1Y_1Z_1O_1$. The origin O_1 of this coordinate frame is placed to the point where the Z_1-axis intersects the common normal to the Z_0- and Z_1-axes. Then, the second coordinate frame is determined so that the Z_2-axis describes the rotational axis of link three (whose end points are C and D = O_3). The positive direction of the Z_1-axis is perpendicular to the paper plane and away from the reader. The X_2-axis is in the direction of the common normal to the Z_1- and Z_2-axes, since these axes are parallel. Then, the Y_2-axis completes the Cartesian local coordinate frame, which is attached to link 2. The origin O_2 is located at the intersection of the Z_2-axis and the common normal to the Z_2- and Z_1-axes. The last two coordinate frames are assigned similarly. The positive direction of the Z_1-axis, i = 2, 3, and 4, is perpendicular to the paper plane in Fig. 2. It is arbitrarily chosen to be away from the reader.

After the coordinate frames have been assigned, an arbitrary point on the excavator can be represented in any of the chosen coordinate systems. In fact, if an arbitrary point in the ith coordinate is $\mathbf{p}_i = (p_{ix}p_{iy}p_{iz}1)^T$ where one as the fourth component has been added, for convenience, as a scaling factor, then the same point in the (i-l)st coordinate frame is $\mathbf{p}_{i-1}$. These two representations are related by the homogeneous transformation matrix $\mathbf{A}_{i-1}^i$ as follows:

$$\mathbf{p}_{i-1} = \mathbf{A}_{i-1}^i\mathbf{p}_i \quad (1)$$

where i = 1, 2, 3, 4. Matrix $\mathbf{A}_{i-1}^i$ is conveniently obtained for the chosen coordinate frames by using the structural kinematic parameters of the machine.

The guidelines for determining the structural kinematic parameters for the chosen coordinate frames are also presented in Appendix I. Their application to the excavator in Fig. 1 gives the values shown in Table 1. To illustrate the determination of the parameters in Table 1, consider the value of α_1 (i = 1). It is equal to the angle needed to rotate the Z_0-axis about the (positive) X_1-axis so that the positive Z_0-axis coincides with that of the Z_1-axis; it results in an angle of +90° (counterclockwise is the positive direction). As another example, a_2 is determined as the distance from the origin (O_2) of the second coordinate frame to the intersection of the Z_1- and X_2-axis along the X_2-axis, thus $a_2 = \overline{O_2O_1}$. The joint variable angle θ_i, i = 1, 2, 3, 4, is measured from the X_{i-1}-axis, and it is positive in the counterclockwise direction when viewed about the positive Z_{i-1}-axis. The

TABLE 1. Structural Kinematic Parameters

Link (joint) i (1)	d_i (2)	a_i (3)	α_i (4)	θ_i (5)
1	0	a_1	90°	θ_1
2	0	a_2	0	θ_2
3	0	a_3	0	θ_3
4	0	a_4	0	θ_4

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remaining parameter values in Table 1 are determined similarly by following the guidelines (Appendix I).

The homogeneous transformation matrix in (1) is in the general form as follows:

$$\mathbf{A}_{i-1}^{i} = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

The substitution of the parameter values of Table 1 for $i = 1, 2, 3, 4$ into (2) gives the homogeneous transformation matrices:

$$\mathbf{A}_0^1 = \begin{bmatrix} c_1 & 0 & s_1 & a_1c_1 \\ s_1 & 0 & -c_1 & a_1s_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3a)$$

$$\mathbf{A}_1^2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3b)$$

$$\mathbf{A}_2^3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3c)$$

$$\mathbf{A}_3^4 = \begin{bmatrix} c_4 & -s_4 & 0 & a_4c_4 \\ s_4 & c_4 & 0 & a_4s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3d)$$

where $c_i = \cos\theta_i$; and $s_i = \sin\theta_i$. Matrix $\mathbf{A}_{i-1}^{i}$ relates the representations of a point in the two adjacent coordinate frames as indicated by (1).

FORWARD KINEMATIC RELATIONS

For the automatic operation of an excavator, it is desirable to place the bucket to a specified location. It can be accomplished by selecting the lengths of the piston rods in the hydraulic actuators, and thus the shaft positions of the joints properly. The mathematical expressions that relate the position and orientation of the bucket to the shaft (joint variable) positions and then to the lengths of the piston rods in the hydraulic actuators are called the kinematic (position) equations. If the lengths of the actuators or the joint variable angles are given, the position and orientation (pose) of the bucket are determined by the forward kinematic equations. If the position and orientation of the bucket are specified, the joint variable angles corresponding to this bucket pose and the lengths of the actuators are calculated from the backward (inverse) kinematic equations. The forward kinematic equations will next be developed for an excavator (backhoe and loader).

Equations Relating Joint Shaft Angles to Bucket Pose

The location of the bucket of the excavator in Figs. 1 and 2 can be specified by locating the rotational axis of the bucket motion, i.e., by giving the

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coordinates of point O_3 = D, the origin of the third coordinate frame. To obtain the coordinates of point D in the base coordinate system, (1) can be applied recursively for i = 1, 2, and 3. It follows that the coordinates of point D in the base coordinate system are

$$\mathbf{p}_0^D = (\mathbf{A}_0^1\mathbf{A}_1^2\mathbf{A}_2^3)\mathbf{p}_4^D = \mathbf{A}_0^3\mathbf{p}_3^D \quad (4)$$

where vector $\mathbf{p}_3^D = (0001)^T$ specifies point D in the third coordinate frame; and matrix $\mathbf{A}_0^3 = \mathbf{A}_0^1\mathbf{A}_1^2\mathbf{A}_2^3$ can be calculated on the basis of the expressions given in (3a)–(3d). Specifically, (4) gives the coordinates of point D in the world coordinate system:

$$\mathbf{p}_0^D = [c_1(a_1 + a_2c_2 + a_3c_{23})s_1(a_1 + a_2c_2 + a_3c_{23})a_2s_2 + a_3s_{23}1]^T \quad (5)$$

where superscript T signifies transposition; $c_{23} = \cos(\theta_2 + \theta_3)$; and $s_{23} = \sin(\theta_2 + \theta_3)$. If the values of the joint variables θ_1, θ_2, and θ_3 are known, the location of the point D in the base coordinate system can be determined by (5).

For a given digging task, a fixed world coordinate system $X_wY_wZ_wO_w$ with origin at point O_w is chosen. After the excavator has been driven to a fixed location for executing a task, the origin O_0 of the base coordinate system $X_0Y_0W_0O_0$ attached to the excavator base structure can be located in the world coordinate system. The position vector of point O_0 in the world coordinate system is denoted as $\mathbf{p}_w^{ob} = (p_{wx}^{ob}p_{wy}^{ob}p_{wz}^{ob}1)^T$. Then the homogeneous transformation matrix $\mathbf{A}_w^0$ that relates the base coordinates to the world coordinates is specified as

$$\mathbf{A}_w^0 = \begin{bmatrix} 0 & -1 & 0 & p_{wx}^{ob} \\ 1 & 0 & 0 & p_{wy}^{ob} \\ 0 & 0 & 1 & p_{wz}^{ob} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

The location of point D on the rotational axis of the bucket can now also be expressed in the fixed world coordinate system as

$$\mathbf{p}_w^D = \mathbf{A}_w^0\mathbf{p}_0^D \quad (7)$$

Thus, the position of D in the world coordinate system can be calculated by combining (4)–(7), if the joint variable values θ_1, θ_2, and θ_3 are known.

The coordinates of the center point N = O_4 on the bucket edge can also be determined in the base coordinate system by the successive application of (1). One obtains

$$\mathbf{p}_0^N = \mathbf{A}_0^4\mathbf{p}_4^N \quad (8)$$

where $\mathbf{p}_4^N$ = (0001) = the origin (O_4) in the fourth $X_4Y_4Z_4O_4$ coordinate frame; $\mathbf{p}_o^N$ is the same point expressed in the base coordinate system of the excavator; and $\mathbf{A}_0^4 = \mathbf{A}_0^1\mathbf{A}_1^2\mathbf{A}_2^3\mathbf{A}_3^4$ is the homogeneous transformation matrix that relates the vector of the fourth coordinate frame to a vector in the base coordinate system. Specifically, the $\mathbf{A}_0^4$ matrix can be calculated by using (2):

$$\mathbf{A}_0^4 = \begin{bmatrix} c_1c_{234} & -c_1s_{234} & s_1 & c_1(a_4c_{234} + a_3c_{23} + a_2c_2 + a_1) \\ s_1c_{234} & -s_1s_{234} & -c_1 & s_1(a_4c_{234} + a_3c_{23} + a_2c_2 + a_1) \\ s_{234} & c_{234} & 0 & a_4s_{234} + a_3s_{23} + a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

where $c_{234} = \cos(\theta_2 + \theta_3 + \theta_4)$; and $s_{234} = \sin(\theta_2 + \theta_3 + \theta_4)$. Eq. (8) shows that the components of $\mathbf{p}_O^N$, the coordinates of point N in the base coordinate system, are given by the fourth column of matrix $\mathbf{A}_0^4$ in (9).

When the joint variable values θ_i, $i = 1, \ldots, 4$, are known, the position and orientation of the center point O_4 on the edge of the bucket in the base coordinate system can be calculated by (8) and (9). These equations represent the first part of the forward kinematic equations for the excavator.

The joint variable θ_1 stays usually constant during the execution of a digging task, i.e., the excavator arm moves on a vertical plane. Moreover, the orientation of the bucket can be specified by the value of the joint variable θ_4 relative to the X_3-axis or by the angle $\theta_2 + \theta_3 + \theta_4$ relative to the X_1-axis. The same information can also be conveyed by the unit vectors of the coordinate frame $X_4Y_4Z_4O_4$ (corresponding to vectors **n**, **s**, and **a**, commonly used in robotics), which are expressed in the base coordinate frame.

Equations Relating Lengths of Hydraulic Actuators to Joint Shaft Angles

The actuators of an excavator consist of the hydraulic cylinders in which the pressures on the pistons are controlled. The lengths of the piston rods in the actuators determine the angles of the joint shafts, and thus the configuration of the excavator and the pose of the bucket. The length of the piston rod in a hydraulic actuator can be specified by the line segment between the attachment points, for example, length BE for actuator 1 (Fig. 1). The kinematic equations relating the lengths of the piston rods in the hydraulic actuators to the angles of the joint shafts are next presented.

Actuator 1, which may not be hydraulic, rotates the base. It determines directly in the base coordinate system the vertical plane ($\theta_1 = $ constant) on which the operation of the excavator (or backhoe or loader) takes place.

The piston rod in the hydraulic actuator 2 (Fig. 2) moves the shaft of joint 2. Its length L_{BE}, i.e., the length of segment BE, is related to the joint variable θ_2 by the following expression:

$$L_{BE}^2 = [L_{AB}\sin(\theta_2 + \beta) + L_{AH}]^2 + [L_{AB}\cos(\theta_2 + \beta) - L_{HE}]^2 \quad \ldots (10)$$

where β = the constant angle between line segments BA and AC. The subscripts attached to length L refer to end points of the line segment whose length is being indicated. L_{AB}, L_{AH}, and L_{HE} have specific constant values for a given excavator (backhoe and loader).

If the length L_{BE} of actuator 2 is known, (10) can then be used to determine the joint variable θ_2. Indeed, the trigonometric equation of (10) can be solved for θ_2 by the standard method to obtain:

$$\theta_2 = -\beta - \tan^{-1}\left(\frac{L_{AH}}{L_{HE}}\right) + \tan^{-1}\left\{\frac{h_1^2}{[4L_{AB}^2(L_{AH}^2 + L_{HE}^2) - h_1^4]^{1/2}}\right\} \quad \ldots (11)$$

where $h_1^2 = L_{AB}^2 + L_{AH}^2 + L_{HE}^2 - L_{BE}^2$. The $\tan^{-1}$ function is used to avoid possible numerical problems that may occur with $\sin^{-1}$ and $\cos^{-1}$ functions. Thus, (11) specifies angle θ_2 when the length L_{BE} of actuator 2 is given.

The piston rod in actuator 3 moves the shaft of joint 3. To relate the length L_{FI} of this actuator to the angular position of joint 3, one observes in Fig. 1 that the angles $\angle$ACI and $\angle$FCD are constants determined by the structure of the machine. By denoting $\angle ACI = \gamma_1$ and $\angle FCD = \gamma_2$, it follows that $\angle ICF = 2\pi - (\theta_3 - \pi) - \gamma_1 - \gamma_2$. Then the cos-theorem for triangle FIC gives

$$L_{FI}^2 = L_{FC}^2 + L_{CI}^2 - 2L_{FC}L_{CI}\cos(3\pi - \theta_3 - \gamma_1 - \gamma_2) \quad \text{(12)}$$

where again the subscripts attached to the length L refer to the end points of the line segment. Lengths L_{FC} and L_{CI} are constants that do not change with the configuration of the excavator.

Eq. (12) can now be solved for the joint variable θ_3 when the length L_{FI} of actuator three is known. In fact

$$\theta_3 = 3\pi - \gamma_1 - \gamma_2 - \tan^{-1}\left[\frac{(4L_{FC}^2L_{CI}^2 - h_2^4)^{1/2}}{h_2^2}\right] \quad \text{(13)}$$

where $h_2^2 = L_{FC}^2 + L_{CI}^2 - L_{FI}^2$.

Actuator 4 causes the bucket to rotate about the axis of joint 4. The length of this actuator can be related to the joint variables θ_4 by expressing the cos-theorem for triangle JKL (Fig. 1) in the following form:

$$L_{JK}^2 = L_{JL}^2 + L_{KL}^2 - 2L_{JL}L_{KL}\cos(\nu_1 - \varepsilon_1) \quad \text{(14)}$$

where $\nu_1 = \angle JLD =$ a constant for a given excavator; and $\varepsilon_1 = \angle KLD$ is to be determined. Eq. (14) can be solved for ε_1 to obtain

$$\varepsilon_1 = \nu_1 - \tan^{-1}\left[\frac{(4L_{JL}^2L_{KL}^2 - h_3^4)^{1/2}}{h_3^2}\right] \quad \text{(15)}$$

where $h_3^2 = L_{JL}^2 + L_{KL}^2 - L_{JK}^2$. The next step is to relate angle ε_1 to the joint variable θ_4.

Since the sum of the angles about the axis of joint 4 in Fig. 1 is 2π, it follows that $\angle NDC + \angle CDL + \angle LDJ + \angle JDG + \angle GDN = 2\pi$, or equivalently, $\angle LDG = \angle LDJ + \angle JDG = 2\pi - (\theta_4 - \pi) - \nu_2 - \nu_3$ where $\nu_2 = \angle CDL$ and $\nu_3 = \angle GDN$ are constants for a given excavator. The sum of the angles in the quadrangle KLDG gives: $\angle KLD + \angle KGD + \angle LDG + \angle LKG = 2\pi$. By denoting $\angle KGD = \varepsilon_2$ and $\angle LKG = \varepsilon_3$, one has a relation between ε_1 and θ_4:

$$\varepsilon_1 + \varepsilon_2 = 2\pi - [2\pi - (\theta_4 - \pi) - \nu_2 - \nu_3] - \varepsilon_3 \quad \text{(16)}$$

where ε_2 and θ_4 are still to be determined. To calculate ε_2, the cos-theorem is applied to triangles KLD and KDG to obtain:

$$L_{KG}^2 + L_{GD}^2 - 2L_{KG}L_{GD}\cos(\varepsilon_2) = L_{KL}^2 + L_{LD}^2 - 2L_{KL}L_{LD}\cos(\varepsilon_1) \quad \text{(17)}$$

Eq. (17) can be solved for angle ε_2. It then follows by (16) that the angular position of joint 4 is

$$\theta_4 = \pi - \nu_2 - \nu_3 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad \text{(18)}$$

where angle $\varepsilon_3 = \angle LKG$ is assumed to be known by the measurements that can be obtained from a potentiometer or an encoder on the shaft.

Thus, (11), (13), (15), (17), and (18) relate the lengths of the actuators to the joint variables θ_2, θ_3, and θ_4. They, together with (8) and (9), represent the forward kinematic equations for the excavator (backhoe and loader).

BACKWARD (INVERSE) KINEMATIC RELATIONS

The inverse kinematic problem in an excavator (backhoe and loader) is to determine the joint variable values (the first part) and the lengths of the piston rods in the actuators (the second part) corresponding to the specified position and orientation of the bucket given in the base coordinate system. It is solved here for the case that the coordinates of point O_3 = D are given in the base coordinate system, i.e., $\mathbf{p}_0^D = (p_{ox}^D p_{oy}^D p_{oz}^D 1)^T$ is known, and the corresponding joint variable values θ_1, θ_2, and θ_3 and the lengths L_{BE}, L_{FI}, and L_{JK} of the actuators are to be found.

The solution is provided by the backward kinematic equations developed next.

Equations Relating Bucket Pose to Joint Shaft Angles

The first part of the inverse kinematic problem is to determine θ_1, θ_2, and θ_3 that satisfy the kinematic equations, i.e., to find the joint variable values that place point D on the rotational axis of the bucket to the given point. It is assumed that the digging task is performed on the vertical plane containing the line segment O_1O_0 joining the origins of the first and zeroth (base) coordinate frames.

In this specific problem it is convenient to first express $\mathbf{p}_0^D$ in the first coordinate frame $X_1Y_1Z_1O_1$ as a vector $\mathbf{p}_1^D$. It is obtained directly from (4) as

$$\mathbf{p}_1^D = \mathbf{A}_1^0\mathbf{p}_0^D = \mathbf{A}_1^2\mathbf{A}_2^3\mathbf{p}_3^D = \mathbf{A}_1^3\mathbf{p}_3^D \quad \text{(19)}$$

where $\mathbf{p}_3^D$ represents the coordinates of point D in the third coordinate frame, i.e., $\mathbf{p}_3^D = (0001)^T$, $\mathbf{A}_1^0 = (\mathbf{A}_0^1)^{-1}$, and $\mathbf{A}_1^3 = \mathbf{A}_1^2\mathbf{A}_2^3$. It follows that

$$\mathbf{p}_1^D = \begin{bmatrix} c_1 & s_1 & 0 & -a_1 \\ 0 & 0 & 1 & 0 \\ s_1 & -c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} p_{ox}^D \\ p_{oy}^D \\ p_{oz}^D \\ 1 \end{pmatrix} = \begin{pmatrix} c_1p_{ox}^D + s_1p_{oy}^D - a_1 \\ p_{oz}^D \\ s_1p_{ox}^D - c_1p_{oy}^D \\ 1 \end{pmatrix} \quad \text{(20)}$$

To solve for θ_2 (Koivo 1989), (19) is first rewritten as

$$(\mathbf{A}_1^2)^{-1}\mathbf{p}_1^D = \mathbf{A}_2^3\mathbf{p}_3^D \quad \text{(21)}$$

where $\mathbf{A}_1^2$ and $\mathbf{A}_2^3$ are specified by (3*a*)–(3*d*). Then (21) yields:

$$\begin{bmatrix} (c_1p_{ox}^D + s_1p_{oy}^D - a_1)c_2 + p_{oz}^Ds_2 - a_2 \\ -(c_1p_{ox}^D + s_1p_{oy}^D - a_1)s_2 + p_{oz}^Dc_2 \\ s_1p_{ox}^D - c_1p_{oy}^D \\ 1 \end{bmatrix} = \begin{pmatrix} a_3c_3 \\ a_3s_3 \\ 0 \\ 1 \end{pmatrix} \quad \text{(22)}$$

where the first two equations in (22) contain the unknown variable θ_2 on the left side and the unknown variable θ_3 on the right side. The third equation in (22) gives

$$\theta_1 = \tan^{-1}\left(\frac{p_{oy}^D}{p_{ox}^D}\right) \quad \text{(23)}$$

If both sides of the first two equations in (22) are squared, and added together, one obtains

$$(2a_2p_{oz}^D)s_2 + (2a_2d)c_2 = (p_{oz}^D)^2 + d^2 + a_2^2 - a_3^2 \quad \text{(24)}$$

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where $d = c_1 p_{ox}^D + s_1 p_{oy}^D - a_1$. In (24), the only unknown θ_2 appears on the left side. Eq. (24) can be solved by the standard method, for example, by setting $2a_2 d = r \cos \gamma$, and $2a_2 p_{oz}^D = r \sin \gamma$, where $r = \{4a^2[d^2 + (p_{oz}^D)^2]\}^{1/2}$ and $\gamma = \tan^{-1}(p_{oz}^D/d)$. By substituting these relations into (24), the resulting equation can be solved for $\theta_2 - \gamma$ to obtain the expression for θ_2:

$$\theta_2 = \tan^{-1}\left(\frac{p_{oz}^D}{d}\right)$$

$$+ \tan^{-1}\left(\frac{\{4a_2^2[(p_{oz}^D)^2 + d^2] - [(p_{oz}^D)^2 + d^2 + a_2^2 - a_3^2]^2\}^{1/2}}{(p_{oz}^D)^2 + d^2 + a_2^2 - a_3^2}\right) \quad \text{......} (25)$$

Having determined θ_1 and θ_2, the third unknown θ_3 is determined by dividing the second equation in (20) by the first one in the same vector equation. It gives the equation for θ_3.

$$\theta_3 = \tan^{-1}\left(\frac{c_2 p_{oz}^D - s_2 d}{s_2 p_{oz}^D + c_2 d - a_2}\right) \quad \text{..............................} (26)$$

Eqs. (23), (25), and (26) determine the joint variable values θ_1, θ_2, and θ_3 of the excavator when the position $\mathbf{p}_0^D$ of point D = O_3 is known. The joint angle θ_4 of the bucket relative to the positive x_3-axis is specified when the orientation of the bucket is given.

When the coordinates of point N = O_4 on the center of the edge of the bucket and the orientation angle $\theta_{234} = \theta_2 + \theta_3 + \theta_4$ of the bucket relative to the X_0-axis (or equivalently the X_1-axis) are known, the solution to the inverse kinematic problem is still given by (23), (25), and (26). However, the following expressions are now substituted into these equations for the components of $\mathbf{p}_0^D$:

$$p_{ox}^D = p_{ox}^N - a_4 c_{234}, \quad p_{oy}^D = p_{oy}^N + a_4 s_{234}, \quad p_{oz}^D = p_{oz}^N \quad \text{.............} (27)$$

where the bucket orientation θ_{234} is given and $\mathbf{p}_0^N = (p_{ox}^N p_{oy}^N p_{oz}^N 1)^T$ specifies the location of point N = O_4 in the base coordinate system.

The orientation of the bucket may, in some applications, be specified in an alternative manner: If the bottom of the bucket is assumed to define a plane that contains the front edge of the bucket (and thus point N), then angle θ_{dg} is defined as the angle that the foregoing plane makes with the horizontal line (Fig. 3). This angle includes the digging angle and the bucket lift. When angle θ_{dg} for the bucket orientation is specified, the joint variable θ_4 can be calculated as

$$\theta_4 = \theta_b + \theta_{dg} + (2\pi - \theta_2 - \theta_3) \quad \text{..........................} (28)$$

where θ_b = the angle between the foregoing plane that contains the bottom of the bucket and the x_4-axis. Eq. (28) can be established from the triangle O_4O_3Q shown in Fig. 3.

Equations Relating Joint Shaft Angles to Lengths of Actuators

This part of the inverse kinematic problem is to determine the lengths of the piston rods in the hydraulic actuators, i.e., the line segments between the attachment points of the actuators when the values of the joint angles are given.

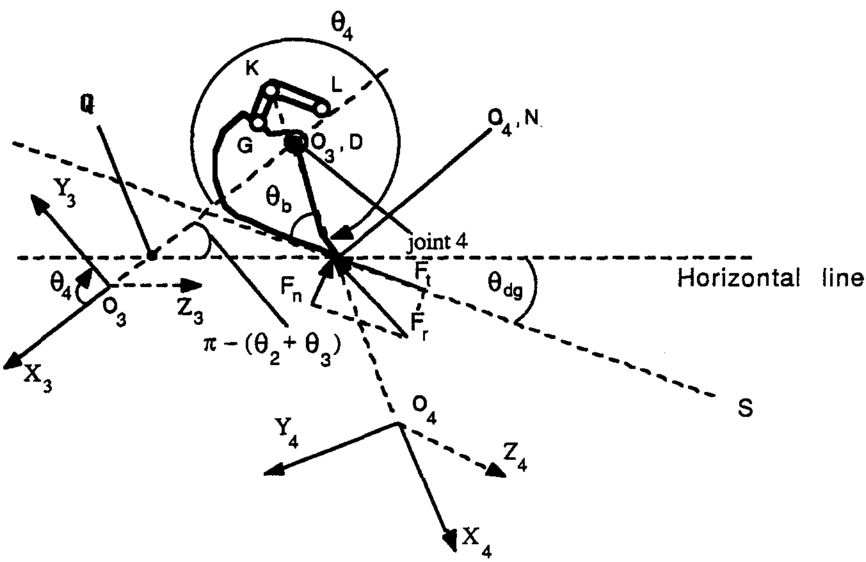

FIG. 3. Excavator Bucket Orientation

The length L_{BE} of actuator 2 is determined by (10), when the joint angle θ_2 and the structural parameters are known.

Similarly, the length of L_{FI} of actuator 3 can be calculated by (12).

The length L_{JK} of actuator 4 can be calculated by (14) if angle ε_1 can first be obtained. By substituting ε_2 from (16) into (17), the resulting trigonometric equation can be solved for ε_1 using the standard method. If $\varepsilon_4 = \theta_4 - \pi + \nu_2 + \nu_3 - \varepsilon_3$, then the solution can be written as

$$\varepsilon_1 = \tan^{-1}\left(\frac{L_{KG}L_{GD}\sin\varepsilon_4}{L_{KG}L_{GD}\cos\varepsilon_4 - L_{KL}L_{LD}}\right)$$

$$- \tan^{-1}\left\{\frac{[4L_{KG}^2L_{GD}^2\sin^2(\varepsilon_4) + 4(L_{KG}L_{GD}\cos(\varepsilon_4) - L_{KL}L_{LD})^2 - h_4^4]^{1/2}}{h_4^2}\right\} \quad (29)$$

where $h_4^2 = L_{KG}^2 + L_{GD}^2 - L_{KL}^2 - L_{LD}^2$ = a constant. Eq. (29) specifies ε_1 in terms of ε_4 and thus the joint shaft angle θ_4, known constants, and angle $\varepsilon_3 = \angle LKG$, which is assumed to be available from the measurements. Then, the length L_{JK} of actuator 4 can be calculated by (14).

When the joint shaft angles are known, the lengths L_{BE}, L_{FI}, and L_{JK} of the piston rods in the hydraulic actuators can be determined by (10), (12), (29), and (16). Thus, the backward (inverse) kinematic equations relating the joint angles to the lengths of the actuators have been established.

The total inverse kinematic relations of the excavator (backhoe and loader) are furnished by (23), (25), (26), (10), (12), (29), and (16).

VELOCITY RELATIONS

The translational and rotational velocities of the bucket of the excavator (backhoe and loader) are next related to the angular velocities of the joint

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shafts and to the translational velocities of the pistons in the hydraulic actuators.

Velocity Equations between Speeds of Bucket and Joint Shafts

If the motion of the bucket is assumed to take place on the vertical plane specified by the constant value of θ_1, then the translational and rotational velocity vector of the bucket (point N) expressed in the base coordinate system is (Koivo 1989)

$$\mathbf{V} = \mathbf{J}(\theta)\dot{\boldsymbol{\theta}} \quad \text{(30)}$$

where $\mathbf{V} = (v_{ox}v_{oy}v_{oz}\omega_{oy})^T$; the rotational velocity vector $\dot{\theta} = (\dot{\theta}_1\dot{\theta}_2\dot{\theta}_3\dot{\theta}_4)^T$; and $\mathbf{J}(\theta) = d\mathbf{p}_o^N/d\boldsymbol{\theta}$ = the Jacobian matrix of the machine. For the coordinate frames displayed in Fig. 1, matrix $\mathbf{J}(\theta)$ is calculated to obtain

$$\mathbf{J}(\theta) = \begin{bmatrix} -s_1a_{14} & c_1(-a_4s_{234} - a_3s_{23} - a_2s_2 & c_1(-a_4s_{234} - a_3s_{23}) & -c_1a_4s_{234} \\ c_1a_{14} & s_1(-a_4s_{234} - a_3s_{23} - a_2s_2) & s_1(-a_4s_{234} - a_3s_{23}) & -s_1a_4s_{234} \\ 0 & a_4c_{234} + a_3c_{23} + a_2c_2 & a_4c_{234} + a_3c_{23} & a_4c_{234} \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{(31)}$$

where $a_{14} = a_4c_{234} + a_3c_{23} + a_2c_2 + a_1$. The first three rows of the matrix in (31) can be determined by using the translation vector (the fourth column) of $\mathbf{A}_0^4$ in (9). Indeed, the first three terms of the ith column in (31) are given by $d\mathbf{p}_o^N/d\theta_i$ where $i = 1, 2, 3, 4$, and $\mathbf{p}_o^N = (p_{ox}p_{oy}p_{oz})^T$ is the translation vector of the $\mathbf{A}_0^4$ matrix in (9).

Eqs. (30) and (31) relate the translational and rotational velocity of the bucket (point N) to the rotational velocities of the joint axes.

Velocity Equations between Speeds of Joint Shafts and Actuator Pistons

The angles of the joint axes are related to the lengths of the piston rods in the hydraulic actuators by (10), (12), and (14). The corresponding velocity relations are derived by forming the time derivatives of both sides in these equations.

By differentiating (19) with respect to time leads to

$$v_{\mathrm{BE}} = \left\{ L_{\mathrm{AB}} \frac{[(L_{\mathrm{AB}}s_2\beta + L_{\mathrm{AH}})c_{2\beta} - (L_{\mathrm{AB}}c_{2\beta} - L_{\mathrm{HE}})s_{2\beta}]}{L_{\mathrm{BE}}} \right\} \dot{\theta}_2 \quad \text{(32)}$$

where $s_2\beta = \sin(\theta_2 + \beta)$; $c_2\beta = \cos(\theta_2 + \beta)$; and $v_{\mathrm{BE}} = dL_{\mathrm{BE}}/dt$ = the translational velocity of the piston in the hydraulic actuator 2. It is in the direction of the line segment BE.

Similarly, the differentiation of (12) with respect to time gives

$$v_{\mathrm{FI}} = \left[\frac{-L_{\mathrm{FC}}L_{\mathrm{CI}} \sin(3\pi - \theta_3 - \gamma_1 - \gamma_2)}{L_{\mathrm{FI}}} \right] \dot{\theta}_3 \quad \text{(33)}$$

where $v_{\mathrm{FI}} = dL_{\mathrm{FI}}/dt$ = the translational velocity of the piston in the hydraulic actuator 3. Its direction is along the line segment FI.

The time derivative of (14) gives the translational velocity of the piston in actuator 4:

$$v_{JK} = \left[\frac{-L_{JL}L_{KL}\sin(\nu_1 - \varepsilon_1)}{L_{JK}}\right]\dot{\varepsilon}_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

where $\dot{\varepsilon}_1$ is still to be expressed as a function of the joint velocity $\dot{\theta}_4$. Eqs. (16) and (17) are differentiated with respect to time to obtain

$$\dot{\varepsilon}_1 + \dot{\varepsilon}_2 = \dot{\theta}_4 - \dot{\varepsilon}_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

$$L_{KG}L_{GD}\sin(\varepsilon_2)\dot{\varepsilon}_2 = L_{KL}L_{LD}\sin(\varepsilon_1)\dot{\varepsilon}_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

Substituting $\dot{\varepsilon}_2$ from (36) into (35) gives:

$$\dot{\varepsilon}_1 = \frac{\dot{\theta}_4 - \dot{\varepsilon}_3}{1 + \dfrac{L_{KL}L_{LD}\sin(\varepsilon_1)}{L_{KG}L_{GD}\sin(\varepsilon_2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

Eqs. (34) and (37) determine the translational velocity of the piston in actuator 4. It is in the direction of the line segment JK.

Thus, the kinematic relations between the joint velocities and the speeds of the pistons in the actuators are described by (32), (33), (34), and (37). These equations can be combined with (30) and (31) to relate to velocity of the bucket to the speeds of the pistons of the hydraulic actuators when such a relation is needed.

COMMENT

This paper establishes the foundation for the kinematics of the excavators (backhoes and loaders). A complete set of the theoretical relations are described for the kinematics of these machines, which have not previously been presented in the literature.

The kinematic relations for the positions and velocities of an excavator (backhoe and loader) can be used to compute the bucket pose when the joint shaft angles or the lengths of the piston rods in the cylinders of the hydraulic actuators are known. The relations presented are based on rigid links and joints. When the machine is performing actual digging, the bucket and the links (booms) will be subject to considerable loading forces and torques due to the digging (interaction between the soil and bucket). Then, it is likely that some bending occurs in the piston rods, links, and joints. As a consequence, the results obtained by applying the foregoing kinematic equations will contain inaccuracies. Indeed, the kinematic equations presented describe the theoretical relations for the positions and velocities without considering the digging forces (torques) acting on the parts of the machine.

CONCLUSIONS

To automate the operations of an excavator (backhoe and loader), the kinematic relations between the bucket, the joint shaft angles and the lengths (positions) of the piston rods in the actuators must be established. Cartesian coordinate frames are first assigned for the links of an excavator by following the Denavit-Hartenberg guidelines. Then, the homogeneous transformation matrices relating two adjacent coordinate frames are determined for the system using the structural kinematic parameters of the machine. The forward and backward kinematic position equations are then developed. The corresponding velocity relations are derived for the hydraulically driven excavator (backhoe and loader). The kinematic equations presented establish the foundation for automatic computer control of this type of construction machine.

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APPENDIX I. ASSIGNING COORDINATE SYSTEMS

The DH guidelines to assign the coordinate frames for the links (joints) of the machine are presented. Then, the determination of the structural kinematic parameters is described. These parameters can be used in a straightforward manner to determine the homogeneous transformation matrices relating the assigned coordinate systems (Koivo 1989).

Determination of Coordinate Frames

The base coordinate system $X_0Y_0Z_0$ is chosen so that the Z_0-axis coincides with the rotational axis of link 1 or with the axis of the motion if the link is prismatic. The X_0- and Y_0-axes are selected so as to complete the Cartesian coordinate system. The origin of this coordinate system can be chosen by the designer.

The remaining coordinate axes are chosen systematically for $i = 1, \ldots,$ N where N indicates the number of joints:

1. The Z_i-axis is assigned along the axis for the motion of link $i + 1$. Thus, the Z_i-axis coincides with the axis about which the rotation of a revolute link takes place; or it is aligned with the axis along which a prismatic (telescopic) link moves. The positive Z_i-direction for a revolute link is such that the positive rotation of θ_{i+1} is counterclockwise.
2. If the Z_i- and Z_{i-1}-axes intersect, the X_i-axis has a direction determined by the crossproduct $\pm(\mathbf{k}_{zi-1} \times \mathbf{k}_{zi})$ where $\mathbf{k}_{zi}$ is the unit vector in the positive direction of the Z_i-axis, and $\mathbf{k}_{zi-1}$ is defined similarly. If $\mathbf{k}_{zi-1}$ and $\mathbf{k}_{zi}$ are parallel, then the direction of the X_i-axis is along the common normal of the $\mathbf{k}_{zi-1}$ and $\mathbf{k}_{zi}$ vectors. (Note: If these vectors together define a plane, then the common normal is also on this plane.)
3. The Y_i-axis is determined by means of the right-hand rule so as to complete the Cartesian $X_iY_iZ_i$-coordinate system. In other words, $\mathbf{j}_{yi} = \mathbf{k}_{zi} \times \mathbf{i}_{xi}$, where $\mathbf{j}_{yi}$ and $\mathbf{i}_{xi}$ are the unit vectors in the directions of the Y_i- and X_i-axes, respectively.
4. The origin of the $X_iY_iZ_i$-coordinate system is placed on the intersection of the Z_{i-1}- and Z_i-axes, or on the intersection of the Z_i-axis and the common normal between the Z_i- and Z_{i-1}-axes.
5. Steps 1–4 are repeated for each i. Thus, the coordinate frames for all links (joints) will be determined.

Steps 1–5 provide guidelines to define the local coordinate frames for the links (joints) systematically, although other choices of the coordinate systems are possible.

To establish the transformation matrix that relates a vector expressed in the $X_iY_iZ_i$-coordinate system to an expression in the $X_{i-1}Y_{i-1}Z_{i-1}$ -coordinate frame, it is necessary to define four parameters: the length of the link a_i, the twist (offset) angle α_i, rotational angle θ_i, and distance d_i, $i = 1, \ldots, N$, for the chosen coordinate frames.

Determination of Structural Kinematic Parameters

1. a_i: The distance from the origin of the ith coordinate frame to the intersection of the Z_{i-1}- and the X_i-axis along the X_i-axis is specified as a_i; in other words, parameter a_i is the perpendicular distance between the Z_i- and the Z_{i-1}-axes.
2. α_i: The angle of rotation about the positive X_i-axis is measured from the positive Z_{i-1}-axis (or its parallel projection) to the positive Z_i-axis, and designated as angle α_i, where the positive direction is counterclockwise.
3. θ_i: The angle of rotation about the positive Z_{i-1}-axis is measured from the positve X_{i-1}-axis to the positive X_i-axis (or its parallel projection), and denoted by θ_i. It is positive in the counterclockwise direction.
4. d_i: The distance from the origin of the (i-1)st coordinate frame to the intersection of the Z_{i-1} axis and the X_i-axis along the Z_{i-1}-axis is d_i; if the Z_{i-1}-axis and the X_i-axis do not intersect, then it is the perpendicular distance between the X_i- and X_{i-1}-axes.

Some of the structural parameters may serve as joint variables that may vary with time and assume different values in a moving excavator, for example, angle θ_i in a revolute link, or distance d_i for a prismatic (telescopic) link.

Having established the coordinate frames and parameters d_i, a_i, α_i, and θ_i, $i = 1, \ldots, N$, for the links (joints) of a particular machine (e.g., see Table 1), the transformation matrix $\mathbf{A}_{i-1}^{i}$ between the two adjacent coordinate frames may be explicitly calculated from (2).

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APPENDIX III. NOTATION

The following symbols are used in this paper:

$\mathbf{A}_{i-1}^{i}$ = matrix relating vector in ith coordinate frame to (i-1)st coordinate frame;
$\mathbf{A}_{w}^{0}$ = matrix relating base (zeroth) coordinates to world coordinates;
a_i = length of link i;
a_1 = $\overline{O_0O_1}$;
a_2 = $\overline{O_1O_2}$;
a_3 = $\overline{O_2O_3} = \overline{CD}$;
a_4 = $\overline{O_3O_4} = \overline{DN}$;
d_i = distance between two adjacent links;
$\mathbf{J}$ = Jacobian matrix;
L_{BE} = distance between points B and E (Fig. 1);
$\mathbf{p}_i$ = position vector expressed in ith coordinate frame, $i = w$, 0, 1, 2, 3, 4;
p_i^D = position of point D in ith coordinate frame;
p_w^{ob} = origin of base coordinate frame expressed in world coordinate system;
$\mathbf{V}$ = general velocity vector;
v_{BE} = velocity of piston rod in actuator between points B and E (Fig. 1);
v_{ox}, v_{oy}, v_{oz} = translational velocity components in x-, y-, and z-directions;
X_i, Y_i, Z_i = coordinates of ith coordinate system;
α_i = twist angle of link i;
β = angle between $\overline{BA}$ and $\overline{AC}$;
γ_1 = angle ACI (Fig. 1);
γ_2 = angle FCD (Fig. 1);
ε_1 = angle KLD (Fig. 1);
θ = joint shaft velocity vector;
θ_i = angle of link i;
ν_1 = angle JLD (Fig. 1); and
ω_{oy} = rotational velocity about y-axis (expressed in base coordinate frame).