SIMG-716
Linear Imaging Mathematics I, Handout 06

1 2-D Special Functions

- Function of two independent spatial variables specifies amplitude ("brightness") at each spatial coordinate in a plane
  - fulfill usual definition of "image".
- Three categories:
  1. Cartesian separable functions
     - represented as products of 1-D special functions along orthogonal Cartesian axes
  2. Circularly symmetric functions
     - product of 1-D functions in radial direction and unit constant in orthogonal ("azimuthal") direction
  3. General functions
     - includes pictorial scenes and 2-D stochastic functions.

2 2-D Separable Functions

- "orthogonal multiplication" of two 1-D functions $f_x[x]$ and $f_y[y]$:

$$f[x,y] = f_x[x] \times f_y[y]$$

- General expression for separable function in terms of scaled and translated 1-D functions is:

$$f[x,y] = f_x\left[\frac{x-x_0}{a}\right] \times f_y\left[\frac{y-y_0}{b}\right]$$

  - a and b are real-valued scale factors
  - x_0 and y_0 are real-valued translation parameters
- Consider those which have general application to imaging problems.
- Volume of 2-D separable function is product of areas of component 1-D functions:

$$\begin{aligned}\iint_{-\infty}^{+\infty} f[x,y]\;dx\;dy &= \iint_{-\infty}^{+\infty} f_x\left[\frac{x-x_0}{a}\right]\; f_y\left[\frac{y-y_0}{b}\right]\;dx\;dy \\ &= \left(\int_{-\infty}^{+\infty} f_x\left[\frac{x-x_0}{a}\right]\;dx\right)\left(\int_{-\infty}^{+\infty} f_y\left[\frac{y-y_0}{b}\right]\;dy\right)\end{aligned}$$

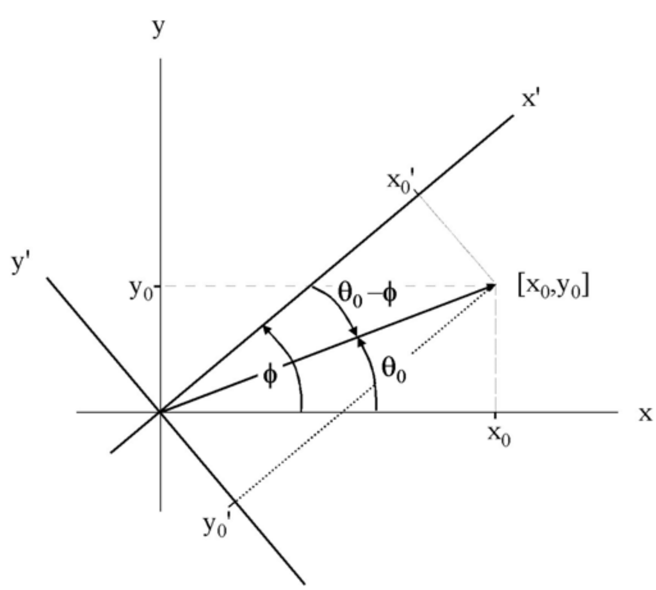
2.1 Rotations of 2-D Separable Functions

- Example: square centered at origin with sides parallel to the x- and y-axes rotated by $\pm\frac{\pi}{4}$ to generate "baseball diamond" with vertices on x- and y-axes
- Rotation of function about origin is an "imaging system" with 2-D input function $f[x,y]$ and 2-D output $g[x,y]$
  - Amplitude g of rotated function at $[x,y]$ is original amplitude f at location $[x',y']$:

$$\mathcal{O}\{f[x,y]\} = g[x,y] = f[x',y']$$

- Rotation specified completely by mapping that relates coordinates $[x,y]$ and $[x',y']$.
  - Illustrated by considering location with Cartesian coordinates $[x_0,y_0]$ and polar coordinates (r_0,θ_0)
    * $r_0 = \sqrt{x_0^2+y_0^2}$
    * $\theta_0 = \tan^{-1}\left[\frac{y_0}{x_0}\right]$.
    * Radial coordinate r_0 unchanged by rotation
    * Azimuthal angle becomes $\theta' = \theta_0 - \phi$.
    * Cartesian coordinates of original location in rotated coordinates $[x_0',y_0']$ in terms of original coordinates $[x_0,y_0]$:

$$\begin{aligned} x_0' &= |\underline{\mathbf{r}}_0|\cos[\theta_0-\phi] = |\underline{\mathbf{r}}_0|(\cos[\theta_0] \;\; \cos[\phi] + \sin[\theta_0] \;\; \sin[\phi]) \\ &= x_0\cos[\phi] + y_0\sin[\phi] \\ y_0' &= r_0\sin[\theta_0-\phi] = r_0(\sin[\theta_0] \;\; \cos[\phi] - \cos[\theta_0] \;\; \sin[\phi]) \\ &= -x_0\sin[\phi] + y_0\cos[\phi] \end{aligned}$$



The Cartesian coordinates of a particular location $[x_0,y_0]$ is evaluated in the rotated coordinate system $[x',y']$ from the polar coordinates $(|\underline{\mathbf{r}}_0|, \theta_0-\phi)$:
$x_0' = |\underline{\mathbf{r}}_0|\cos[\theta_0-\phi] = x_0\cos[\phi] + y_0\sin[\phi]$, $y_0' = |\underline{\mathbf{r}}_0|\sin[\theta_0-\phi] = -x_0\sin[\phi] + y_0\cos[\phi]$

2.2 Rotated Coordinates as Scalar Products

- Rotated x-coordinate written as scalar product of position vector $\underline{\mathbf{r}} \equiv [x,y]$ and unit vector directed along azimuth angle ϕ, which has Cartesian coordinates $[\cos[\phi], \sin[\phi]]$, denoted by $\widehat{\underline{\mathbf{p}}}$:

$$x' = x\cos[\phi] + y\sin[\phi] = \begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} \cos[\phi] \\ \sin[\phi] \end{bmatrix} \equiv \underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}$$

- Notation for x' may seem "weird" because rotated 1-D argument x' is function of both x and y through scalar product of two vectors.

  - rotated argument x' defines a set of points $\underline{\mathbf{r}} = [x,y]$ that fulfill same conditions as coordinate x in original function.
  - Rotated coordinate x' evaluates to 0 for all vectors $\underline{\mathbf{r}} \perp \widehat{\underline{\mathbf{p}}}$, (equivalent to $\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}} = 0$).
  - Vectors $\underline{\mathbf{r}}$ specified by condition $\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}} = 0$ include coordinates on "both" sides of origin
  - Complete set of possible azimuth angles specified by angles ϕ in interval spanning π radians, e.g., $-\frac{\pi}{2} \leq \phi < +\frac{\pi}{2}$. This means that the set of possible rotations is specified by unit vectors $\widehat{\underline{\mathbf{p}}}$ in the first and fourth quadrants.

- Polar form of position vector $\underline{\mathbf{r}} = (|\underline{\mathbf{r}}|, \theta) = (r, \theta)$ substituted into scalar product in terms of magnitudes of vectors and included angle:

$$\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}} = |\underline{\mathbf{r}}| \; \left|\widehat{\underline{\mathbf{p}}}\right| \; \cos[\theta - \phi] = r \; \cos[\theta - \phi] \text{ because } \left|\widehat{\underline{\mathbf{p}}}\right| = 1.$$

- y-coordinate of rotated function written in same way as scalar product of position vector $\underline{\mathbf{r}}$ and unit vector directed along azimuth angle $\phi + \frac{\pi}{2}$;call it $\widehat{\underline{\mathbf{p}}}^{\perp}$:

$$\begin{aligned} y' &= x \; \cos\left[\phi + \frac{\pi}{2}\right] + y \; \sin\left[\phi + \frac{\pi}{2}\right] \\ &= -x \; \sin[\phi] + y \; \cos[\phi] = \begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} -\sin[\phi] \\ \cos[\phi] \end{bmatrix} \equiv \underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp} \end{aligned}$$

- Polar form for rotated y-axis is:

$$\begin{aligned} \underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp} &= |\underline{\mathbf{r}}| \; \left|\widehat{\underline{\mathbf{p}}}\right| \; \cos\left[\theta - \left(\phi + \frac{\pi}{2}\right)\right] \\ &= r \; \cos\left[\theta - \phi - \frac{\pi}{2}\right] = r \; \sin[\theta - \phi] \end{aligned}$$

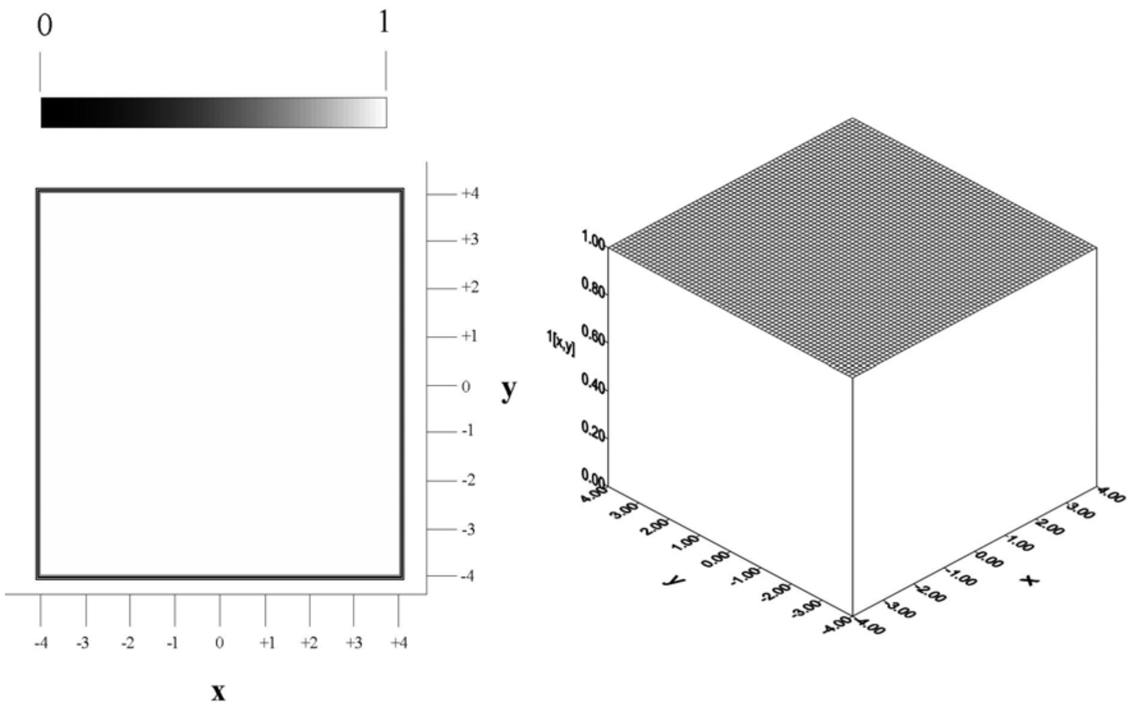
$$g[x,y] = f_x[x'] \; f_y[y'] = f_x\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}\right] \; f_y\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right]$$

- Formula is applicable to any 2-D separable special function

3 Definitions of 2-D Separable Functions

3.1 2-D Constant

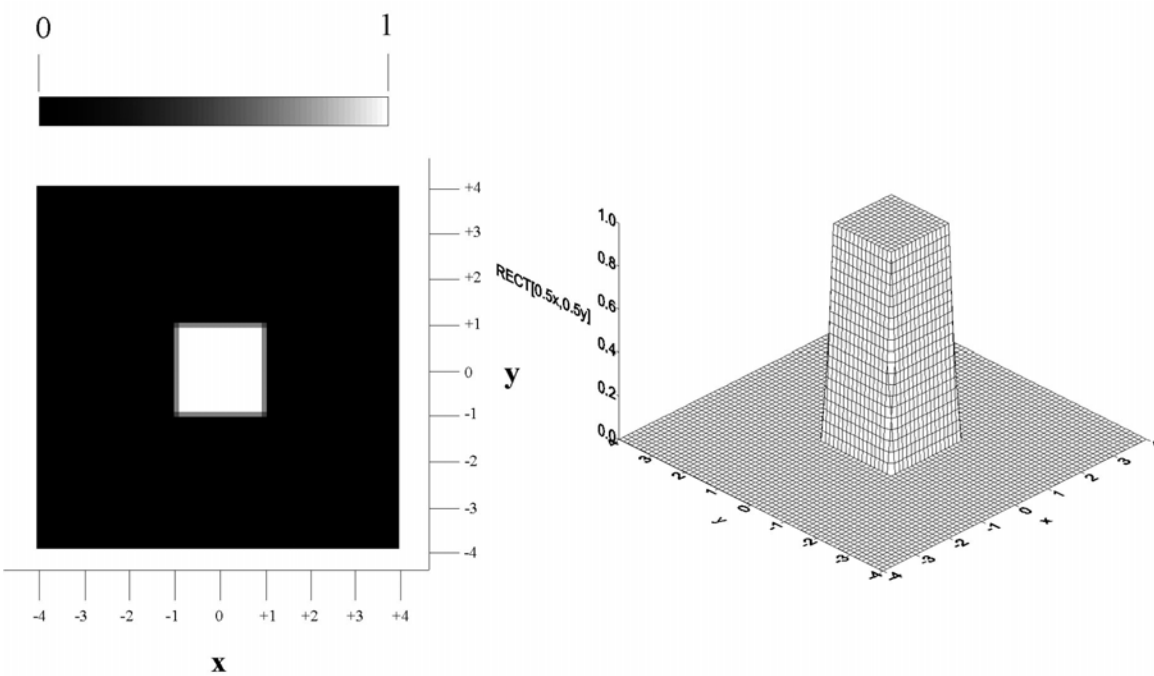
- $$1\,[x,y] = 1\,[x]\ \ 1\,[y]$$
  $$\iint_{-\infty}^{+\infty} 1\,[x,y]\ \ dx\ dy = \infty$$
  $$0\,[x,y] = 0\,[x]\ \ f_y\,[y]$$
  $$\iint_{-\infty}^{+\infty} 0\,[x,y]\ \ dx\ dy = 0$$
- Neither translations nor rotations affect amplitude of a 2-D constant function at any coordinate.



Representations of the 2-D unit constant function in “image” format, where the amplitude is represented by shades of gray according the the scale, and in “surface” format, where the amplitude is the “height” of a 3-D surface.

3.2 2-D Rectangle

$$
\begin{aligned}
RECT\left[\frac{x}{a},\frac{y}{b}\right] &= RECT\left[\frac{x}{a}\right]\ RECT\left[\frac{y}{b}\right] \\
RECT\left[\frac{x-x_0}{a},\frac{y-y_0}{b}\right] &= RECT\left[\frac{x-x_0}{a}\right]\ RECT\left[\frac{y-y_0}{b}\right]
\end{aligned}
$$



Example of the 2-D rectangle function $f[x,y] = RECT\left[\frac{x}{2},\frac{y}{2}\right]$.

- Volume obviously is finite when both a and b are finite:

$$
\begin{aligned}
&\iint_{-\infty}^{+\infty} RECT\left[\frac{x-x_0}{a},\frac{y-y_0}{b}\right]\ dx\ dy \\
&= \int_{-\infty}^{+\infty} RECT\left[\frac{x-x_0}{a}\right]\ dx \int_{-\infty}^{+\infty} RECT\left[\frac{y-y_0}{b}\right]\ dy = |ab|
\end{aligned}
$$

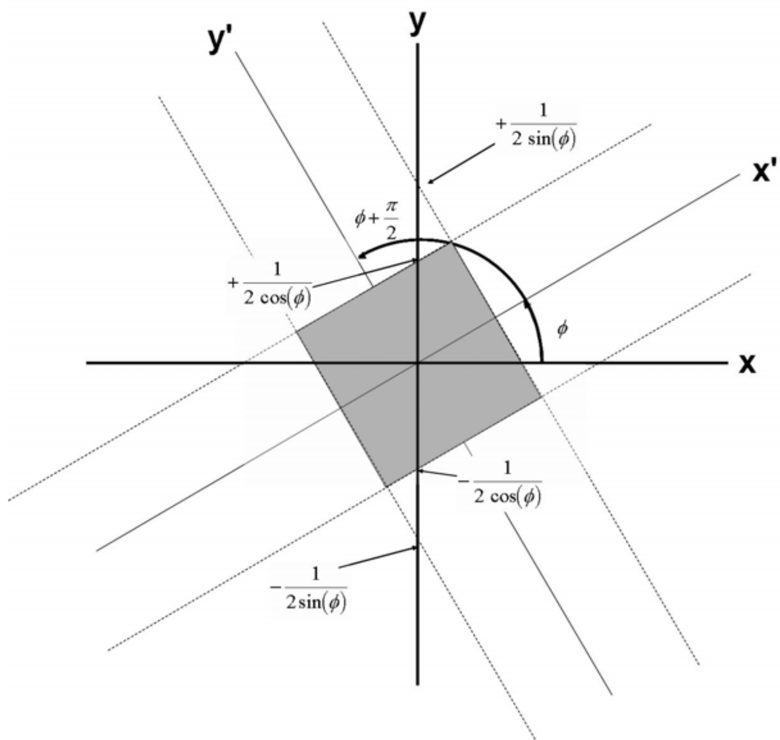
- 2-D $RECT$ often modulates functions that have larger domains of support.
- Rotated rectangle by substituting rotated coordinates into definition

$$
\begin{aligned}
RECT[x',y'] &= RECT\left[\underline{\mathbf{r}}\bullet\underline{\widehat{\mathbf{p}}},\underline{\mathbf{r}}\bullet\underline{\widehat{\mathbf{p}}}^{\perp}\right] = RECT\left[\underline{\mathbf{r}}\bullet\underline{\widehat{\mathbf{p}}}\right]\ RECT\left[\underline{\mathbf{r}}\bullet\underline{\widehat{\mathbf{p}}}^{\perp}\right] \\
&= RECT[x\ \cos[\phi]+y\ \sin[\phi]]\ RECT[-x\ \sin[\phi]+y\ \cos[\phi]]
\end{aligned}
$$

  - Ttransition" locations $[x,y]$ where $RECT[x,y]=\frac{1}{2}$ by substitution:

$$
\begin{aligned}
x' &= x\cos[\phi]+y\sin[\phi] = \pm\frac{1}{2} \Longrightarrow y = (-\cot[\phi])\ x \pm \frac{1}{2\sin[\phi]} \\
y' &= -x\sin[\phi]+y\cos[\phi] = \pm\frac{1}{2} \Longrightarrow y = (\tan[\phi])\ x \pm \frac{1}{2\cos[\phi]}
\end{aligned}
$$

- Conditions define slopes and y-intercepts of two pairs of parallel lines in 2-D plane
- slopes $+\tan[\phi]$ and $-\cot[\phi] = \tan\left[\phi + \frac{\pi}{2}\right]$
- Dashed lines enclose support of rectangle



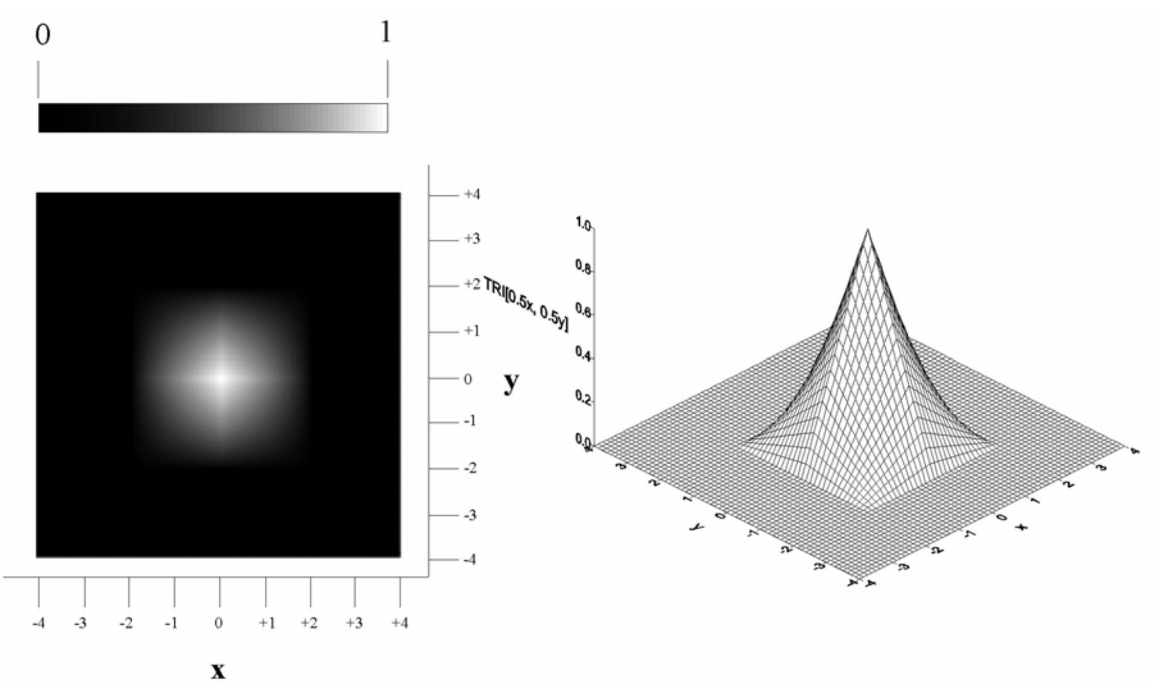
The loci of points that satisfy the conditions $y = x(-\cot[\phi]) \pm \frac{1}{2\sin[\phi]}$ and $y = x\tan[\phi] \pm \frac{1}{2\cos[\phi]}$.

3.3 2-D Triangle

$$
\begin{aligned}
TRI\left[\frac{x}{a},\frac{y}{b}\right] &= TRI\left[\frac{x}{a}\right]\ TRI\left[\frac{y}{b}\right] \\
&= \left(1-\frac{|x|}{a}\right)\ \left(1-\frac{|y|}{b}\right)\ RECT\left[\frac{x}{2a},\frac{y}{2b}\right] \\
&= \left(1-\frac{|x|}{a}-\frac{|y|}{b}+\frac{|xy|}{ab}\right)\ RECT\left[\frac{x}{2a},\frac{y}{2b}\right]
\end{aligned}
$$

- Profiles of 2-D TRI function are straight lines only parallel to x- or y-axes.
- Shape of edge profile along other radial line includes quadratic function of radial distance$\Longrightarrow$parabola.
- Volume of 2-D TRI is product of 1-D areas:

$$
\iint_{-\infty}^{+\infty} TRI\left[\frac{x}{a},\frac{y}{b}\right]\ dx\ dy = \left(\int_{-\infty}^{+\infty} TRI\left[\frac{x}{b}\right]\ dx\right)\left(\int_{-\infty}^{+\infty} TRI\left[\frac{y}{b}\right]\ dy\right) = |ab|
$$



2-D triangle function $TRI\,[x,y]$, showing the parabolic character along profiles that do not coincide with the coordinate axes.

3.4 2-D Signum and Step

- Encountered rarely
- Possible definition based on "convention":

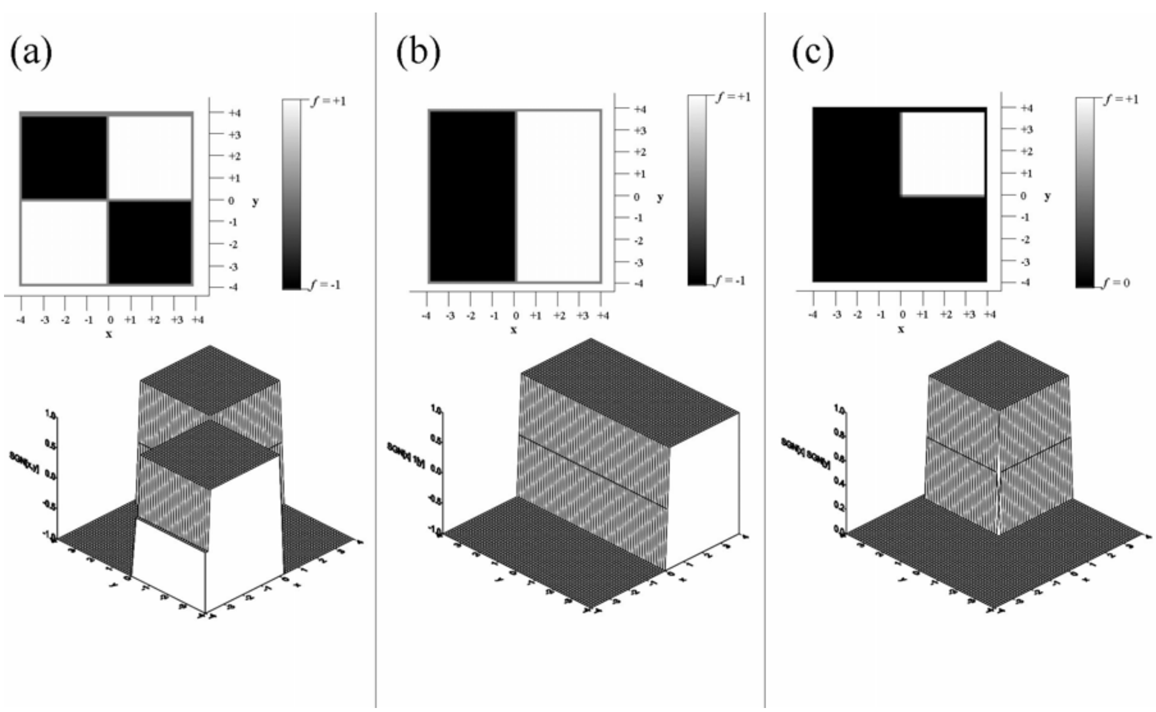
$$SGN[x,y] = SGN[x]\ \ SGN[y]$$

  - +1 in first & third quadrants
  - -1 in second & fourth quadrants
  - 0 along coordinate axes

- Volume is zero due to cancellation of positive and negative regions.
- Another possible definition:

$$f[x,y] = STEP[x]\ \ 1[y]$$
$$f[x,y] = STEP[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}]\ \ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right]$$

- Notation for unit constant may seem weird
  - think of 1-D unit constant as having same unit amplitude everywhere in the 1-D domain
  - when applied in 2-D domain, 1-D unit constant takes amplitude of $STEP[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}]$ at $\underline{\mathbf{r}}_0$ along direction of $\widehat{\underline{\mathbf{p}}}$ and assigns it to all points $\underline{\mathbf{r}}$ along line perpendicular to $\widehat{\underline{\mathbf{p}}}$, i.e., in direction of $\widehat{\underline{\mathbf{p}}}^{\perp}$
  - Action of unit constant is to "spread" or "smear" amplitude of 1-D $STEP$ along perpendicular direction
  - use "rotated" form for 1-D unit constant only to produce 2-D functions by orthogonal
- volume $= \infty$.



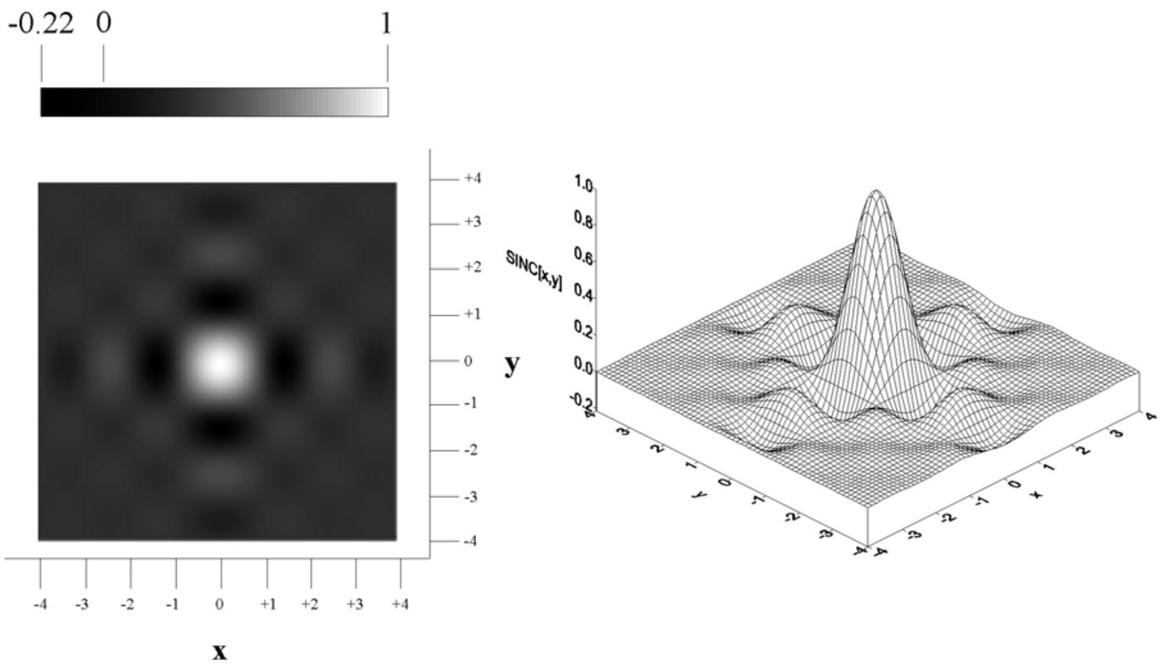
2-D SGN and $STEP$ functions: (a) $SGN[x] \cdot SGN[y]$, (b) $SGN[x] \cdot 1[y]$, (c) $STEP[x] \cdot STEP[y]$.

3.5 2-D $SINC$

$$SINC\left[\frac{x}{a},\frac{y}{b}\right] = SINC\left[\frac{x}{a}\right]\ SINC\left[\frac{y}{b}\right] = \frac{\sin\left[\frac{\pi x}{a}\right]\ \ \sin\left[\frac{\pi y}{b}\right]}{\left(\frac{\pi x}{a}\right)\ \ \left(\frac{\pi y}{b}\right)}$$

- Amplitude>0 in regions where both of 1-D functions are positive or both are negative
- Amplitude <0 where either (but not both) is negative.
- "checkerboard-like" pattern of positive and negative regions
- volume is product of the areas of the individual functions:

$$\iint_{-\infty}^{+\infty} SINC\left[\frac{x}{a},\frac{y}{b}\right]\ dx\ dy = \left(\int_{-\infty}^{+\infty} SINC\left[\frac{x}{a}\right]\ dx\right)\left(\int_{-\infty}^{+\infty} SINC\left[\frac{y}{b}\right]\ dy\right)$$
$$= |a|\ \ |b| = |ab|$$



$SINC\,[x,y] = SINC\,[x]\cdot SINC\,[y]$

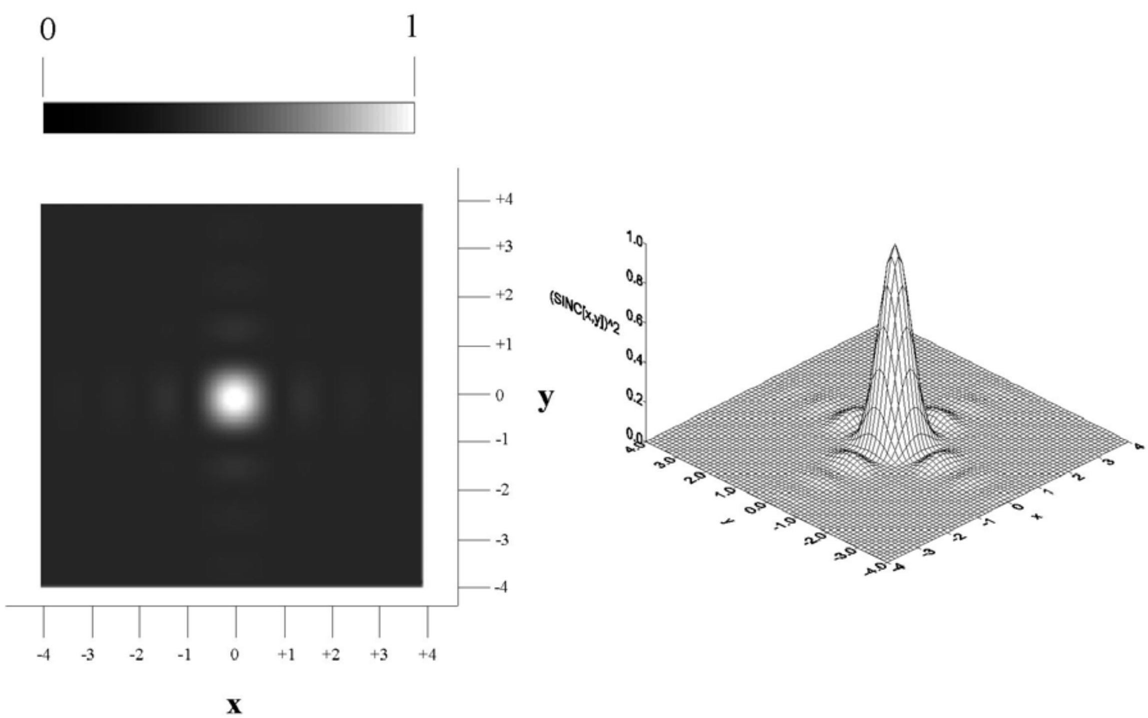
3.6 2-D $SINC^2$

•

$$SINC^2\left[\frac{x}{a},\frac{y}{b}\right] = SINC^2\left[\frac{x}{a}\right]\ SINC^2\left[\frac{y}{b}\right]$$
$$= \frac{\sin^2\left[\frac{\pi x}{a}\right]\ \ \sin^2\left[\frac{\pi y}{b}\right]}{\left(\frac{\pi x}{a}\right)^2\ \ \left(\frac{\pi y}{b}\right)^2}$$

- $SINC^2$ function is everywhere nonnegative
- volume same as that of $SINC\left[\frac{x}{a},\frac{y}{b}\right]$

$$\iint_{-\infty}^{+\infty} SINC^2\left[\frac{x}{a},\frac{y}{b}\right]\ dx\ dy = \left(\int_{-\infty}^{+\infty} SINC^2\left[\frac{x}{a}\right]\ dx\right)\left(\int_{-\infty}^{+\infty} SINC^2\left[\frac{y}{b}\right] dy\right)$$
$$= |a|\ \ |b| = |ab|$$



$SINC^2[x,y] = SINC[x]\cdot SINC[y]$

3.7 2-D Gaussian

$$
\begin{aligned}
GAUS\left[\frac{x}{a},\frac{y}{b}\right] &= GAUS\left[\frac{x}{a}\right]\ GAUS\left[\frac{y}{b}\right] = e^{-\frac{\pi x^2}{a^2}}\ e^{-\frac{\pi y^2}{b^2}} \\
&= e^{-\pi\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}
\end{aligned}
$$

- Volume is product of individual areas:

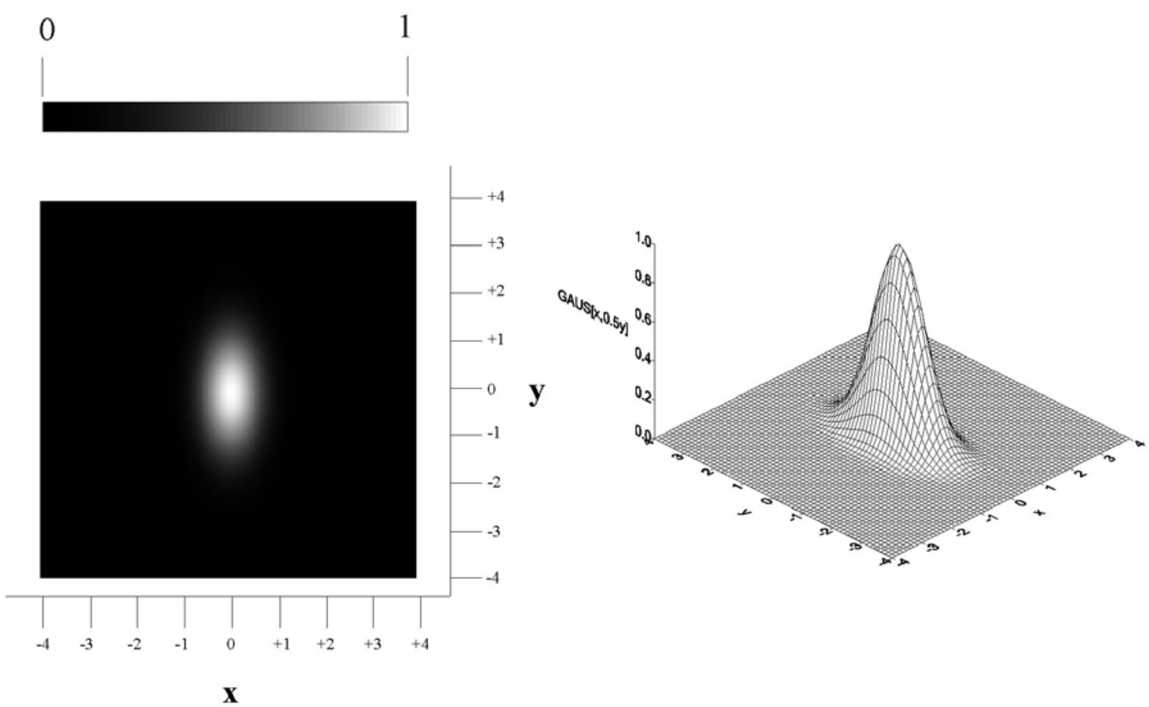
$$
\begin{aligned}
\iint_{-\infty}^{+\infty} GAUS\left[\frac{x}{a},\frac{y}{b}\right]\ dx\,dy &= \left(\int_{-\infty}^{+\infty} GAUS\left[\frac{x}{a}\right]\ dx\right)\left(\int_{-\infty}^{+\infty} GAUS\left[\frac{y}{b}\right] dy\right) \\
&= |a|\ \ |b| = |ab|
\end{aligned}
$$

- Rotated version:

$$
\begin{aligned}
GAUS\left[\frac{x'}{a},\frac{y'}{b}\right] &= GAUS\left[\frac{\underline{\mathbf{r}}\bullet\underline{\widehat{\mathbf{p}}}}{a},\frac{\underline{\mathbf{r}}\bullet\underline{\widehat{\mathbf{p}}}^{\perp}}{b}\right] \\
&= e^{-\pi\left[\left(\frac{\underline{\mathbf{r}}\bullet\underline{\widehat{\mathbf{p}}}}{a}\right)^2+\left(\frac{\underline{\mathbf{r}}\bullet\underline{\widehat{\mathbf{p}}}^{\perp}}{b}\right)^2\right]}
\end{aligned}
$$

- Circularly symmetric function generated with equal width parameters:

$$
GAUS\left[\frac{x}{d},\frac{y}{d}\right] = e^{-\pi\frac{x^2+y^2}{d^2}} = e^{-\pi\left(\frac{r}{d}\right)^2}
$$



Example of 2-D separable Gaussian function $GAUS\left[x,\frac{y}{2}\right] = GAUS\left[x\right]\cdot GAUS\left[\frac{y}{2}\right]$

3.8 2-D Sinusoid

- "meaningful" definition varies along one direction but is constant in the orthogonal direction

$$\begin{aligned} f[x,y] &= A_0 \cos[2\pi\xi_0 x + \phi_0]\ 1[y] \\ &= A_0 \cos[2\pi\xi_0 x + \phi_0] \cos[2\pi \times 0 \times y] \end{aligned}$$

- Gaskill suggests image of field plowed with sinusoidal furrows
- Rotated:

$$\begin{aligned} f[x',y'] = A_0 \cos(2\pi\xi_0 x' + \phi_0)\ 1[y'] &= A_0 \cos\left[2\pi\xi_0\left(\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}\right) + \phi_0\right]\ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right] \\ &= A_0 \cos[2\pi(\xi_0 \cos[\theta]\ x + (\xi_0 \sin[\theta])\ y) + \phi_0]\ 1[-x\sin[\theta] + y\cos[\theta]] \\ &\equiv A_0 \cos[2\pi(\xi_1 x + \eta_1 y) + \phi_0]\ 1[-x\sin[\theta] + y\cos[\theta]] \end{aligned}$$

  - Spatial frequencies of rotated function along "new" ξ- and η-axes are ξ_1 and η_1

$$\begin{aligned} \xi_1 &\equiv +\xi_0\ \cos[\theta] \\ \eta_1 &\equiv +\xi_0\ \sin[\theta] \end{aligned}$$

  - specify rates of sinusoidal variation along x- and y-axes
  - specify periods of 2-D sinusoid along x- and y-directions, respectively:

$$\begin{aligned} f[x,y] &= A_0 \cos[2\pi(\xi_1 x + \eta_1 y) + \phi_0] \\ &= A_0 \cos\left[2\pi\left(\frac{x}{X_1} + \frac{y}{Y_1}\right) + \phi_0\right] \end{aligned}$$

$$\begin{aligned} X_1 &= \frac{1}{\xi_0\ \cos[\theta]} \\ Y_1 &= \frac{1}{\xi_0\ \sin[\theta]} \end{aligned}$$

- 2-D sinusoid oscillating along arbitrary direction recast into form where 2-D spatial coordinate is 2-D radius vector $\underline{\mathbf{r}} \equiv [x,y]$.
- phase $[\xi_0 x + \eta_0 y]$ is scalar product of $\underline{\mathbf{r}}$ with "polar spatial frequency vector" $\underline{\boldsymbol{\rho}}_0 \equiv \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix}$
- Polar representation of vector is $\underline{\boldsymbol{\rho}}_0 = (\rho_0, \psi_0)$

$$\left|\underline{\boldsymbol{\rho}}_0\right| = \sqrt{\xi_0^2 + \eta_0^2}$$

$$\psi_0 = \tan^{-1}\left[\frac{\eta_0}{\xi_0}\right]$$

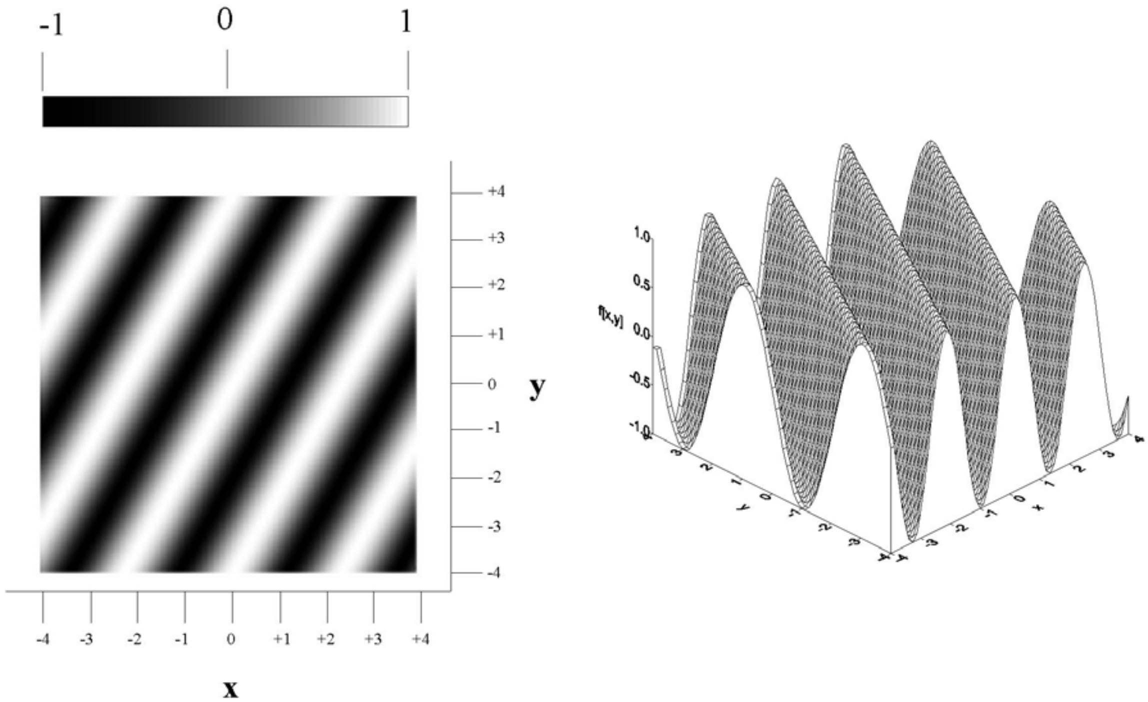
$$\begin{aligned} f(\underline{\mathbf{r}}) &= A_0 \cos\left[2\pi\left(\underline{\mathbf{r}} \bullet \underline{\boldsymbol{\rho}}_0\right) + \phi_0\right] \\ &= A_0 \cos\left[2\pi\left(|\underline{\mathbf{r}}|\ \left|\underline{\boldsymbol{\rho}}_0\right|\ \cos[\theta - \psi_0]\right) + \phi_0\right] \end{aligned}$$

-

$$\begin{aligned} \xi_0 &= \left|\underline{\boldsymbol{\rho}}_0\right| \cos[\psi_0] \\ \eta_0 &= \left|\underline{\boldsymbol{\rho}}_0\right| \sin[\psi_0] \end{aligned}$$

- spatial period is reciprocal of magnitude of spatial frequency vector:

$$R_0 = \frac{1}{\left|\underline{\boldsymbol{\rho}}_0\right|} = \frac{1}{\sqrt{\xi_0^2 + \eta_0^2}}$$



2-D sinusoid function rotated in azimuth about the origin by $\theta = -\frac{\pi}{6}$ radians.

4 2-D Dirac Delta Function and its Relatives

- Several flavors of 2-D Dirac delta function possible
  - different separable versions of 2-D Dirac delta function defined in Cartesian and polar coordinates
- Properties:

$$\delta[x-x_0, y-y_0] = 0 \text{ for } x-x_0 \neq 0 \text{ or } y-y_0 \neq 0$$

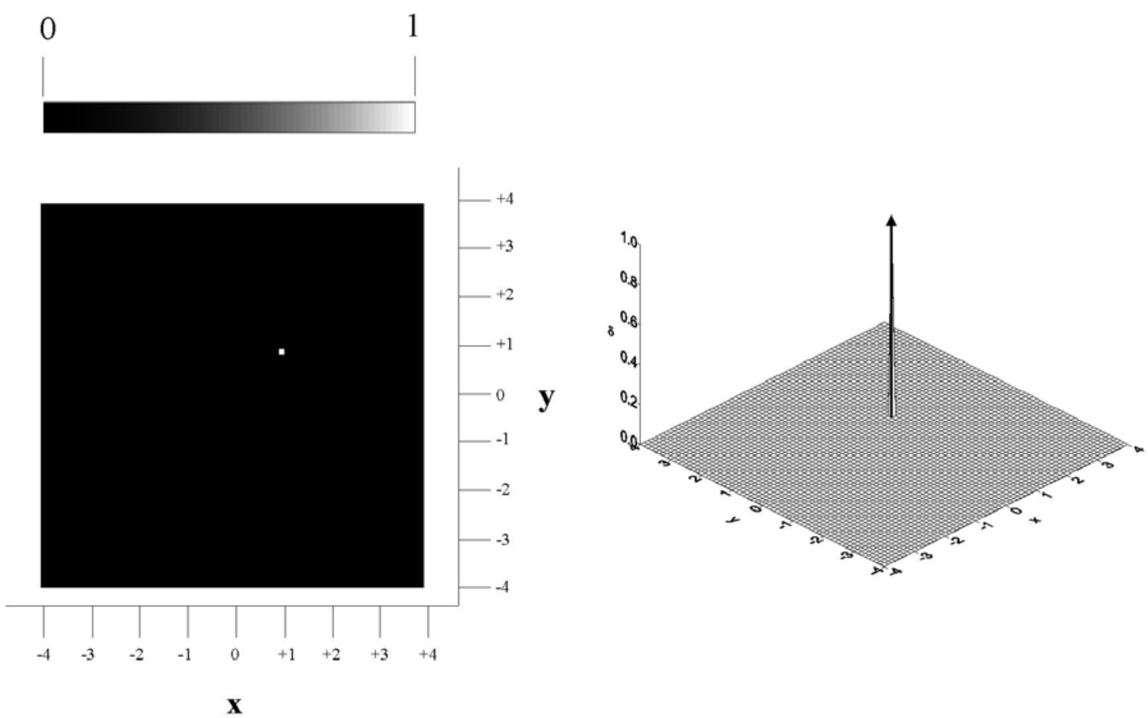
$$\iint_{-\infty}^{+\infty} \delta[x-x_0, y-y_0] \, dx \, dy = 1$$

- Expressed as limit of sequence of 2-D functions, each with unit volume
- May be generated from 2-D *RECT*, *TRI*, *GAUS*, *SINC*, and $SINC^2$ in Cartesian coordinates
- Separability of functions ensures that corresponding representation of $\delta[x,y]$ is separable
- Most common representation of $\delta[x,y]$ based on 2-D *RECT* function of unit volume in the limit of infinitesimal area:

$$\begin{aligned}\delta[x,y] &= \lim_{b\to 0}\lim_{d\to 0}\left\{\frac{1}{bd}\; RECT\left[\frac{x}{b},\frac{y}{d}\right]\right\}\\ &= \lim_{b\to 0}\left\{\frac{1}{b}RECT\left[\frac{x}{b}\right]\right\}\times\lim_{d\to 0}\left\{\frac{1}{d}RECT\left[\frac{y}{d}\right]\right\}\end{aligned}$$

4.1 Properties of Separable 2-D Dirac Delta Function, Cartesian Coordinates

$$\delta[x,y] = \delta[x] \;\; \delta[y]$$



2-D separable Dirac delta function $\delta[x-1, y-1] = \delta[x-1] \cdot \delta[y-1]$.

- 2-D Dirac delta function may be synthesized by summing unit-amplitude 2-D complex linear-phase exponentials with all spatial frequencies.

$$\begin{aligned}
\delta[x] \times \delta[y] &= \left(\int_{-\infty}^{+\infty} e^{+2\pi i\xi x} d\xi\right) \left(\int_{-\infty}^{+\infty} e^{+2\pi i\eta y} d\eta\right) \\
&= \iint_{-\infty}^{+\infty} e^{+2\pi i\xi x} e^{+2\pi i\eta y}\ d\xi\ d\eta \\
&= \iint_{-\infty}^{+\infty} e^{+2\pi i(\xi x+\eta y)}\ d\xi\ d\eta = \delta[x,y]
\end{aligned}$$

- Because odd imaginary parts cancel:

$$\delta[x,y] = \iint_{-\infty}^{+\infty} \cos[2\pi(\xi x+\eta y)]\ d\xi\ d\eta$$

- Scaling property of $\delta[x] \Longrightarrow$

$$\begin{aligned}
\delta\left[\frac{x}{b},\frac{y}{d}\right] &= \delta\left[\frac{x}{b}\right]\ \delta\left[\frac{y}{d}\right] \\
&= |b|\ \delta[x]\ \ |d|\ \delta[y] \\
&= |bd|\ \ \delta[x]\ \ \delta[y] = |bd|\ \ \delta[x,y]
\end{aligned}$$

- Translat by shifting separable components:

$$\delta\left[\frac{x-x_0}{b},\frac{y-y_0}{d}\right] = |bd|\ \ \delta[x-x_0]\ \ \delta[y-y_0]$$

- Sifting property evaluates amplitude at a specific location

$$\iint_{-\infty}^{+\infty} f[x,y]\ \ \delta[x-x_0, y-y_0]\ dx\, dy = f[x_0,y_0]$$

4.2 Properties of 2-D Dirac Delta Function, Polar Coordinates

- 2-D Dirac delta function located on x-axis at distance $\alpha > 0$ from origin
  - polar coordinates are $r_0 = \alpha$ and $\theta_0 = 0$.

$$\delta[x-\alpha, y] = \delta[x-\alpha]\ \delta[y]$$

  - Polar representation easy to derive because radial coordinate directed along x-axis
  - Azimuthal displacement due to angle parallel to y-axis.
  - Identify $x \to r$, $\alpha \to r_0$, and $y \to r_0\theta$ and substitute directly, use 1-D scaling property

$$\begin{aligned}\delta[x-\alpha]\ \delta[y] &= \delta[r-r_0]\ \delta[r_0\theta] \\ &= \delta[r-r_0]\ \left(\frac{1}{|r_0|}\ \delta[\theta]\right) = \frac{\delta[r-r_0]}{r_0}\ \delta[\theta]\end{aligned}$$

- last step follows from observation that polar radial coordinate $r_0 \geq 0$
- Domain of azimuthal coordinate θ is a continuous interval of 2π radians, e.g., $-\pi \leq \theta < +\pi$
- Generalize for a 2-D Dirac delta function located at same radial distance r_0 from the origin but at different azimuth θ_0
  - Cartesian coordinates $x_0 = r_0 \cos[\theta_0]$, $y_0 = r_0 \sin[\theta_0]$

$$\begin{aligned}\delta[x-x_0, y-y_0] &= \delta[x - r_0\ \cos[\theta_0]]\ \delta[y - r_0\ \sin[\theta_0]] \\ &= \frac{\delta[r-r_0]}{r_0}\ \delta[\theta-\theta_0] \equiv \delta(\underline{\mathbf{r}} - \underline{\mathbf{r}}_0)\end{aligned}$$

  - Polar form is product of 1-D Dirac delta functions in the radial and azimuthal directions
  - Amplitude scaled by reciprocal of the radial distance.

- Confirm that expression satisfies criteria for 2-D Dirac delta function.
- Easy to show that support is infinitesimal
- Volume is evaluated easily:

$$\begin{aligned}\iint_{-\infty}^{+\infty} \delta[\underline{\mathbf{r}} - \underline{\mathbf{r}}_0]\ d\underline{\mathbf{r}} &= \int_{-\pi}^{+\pi} d\theta \int_0^{+\infty} \left(\frac{\delta[r-r_0]}{r_0} \delta[\theta-\theta_0]\right)\ r\, dr \\ &= \left(\int_{-\pi}^{+\pi} \delta[\theta-\theta_0]\ d\theta\right) \cdot \left(\int_0^{+\infty} \frac{\delta[r-r_0]}{r_0}\ r\, dr\right) \text{ where } r_0 > 0 \\ &= 1 \cdot \int_0^{+\infty} \frac{\delta[r-r_0]}{r_0}\ r_0\, dr = 1 \cdot \int_0^{+\infty} \delta[r-r_0]\ dr = 1\end{aligned}$$

- Extend derivation of polar form of 2-D Dirac delta function to $r_0 = 0$

$$\delta[\underline{\mathbf{r}} - \underline{\mathbf{0}}]$$

- Domain of r is semiclosed single-sided interval $[0, +\infty)$
  - 1-D radial part $\delta(r - r_0)$ at origin cannot be symmetric
  - θ_0 is indeterminate at origin

- not valid for $\underline{\mathbf{r}}_0 = \underline{\mathbf{0}}$
- 2-D Dirac delta function at the origin must be circularly symmetric and therefore a function of r only
- No dependence on azimuth angle θ:

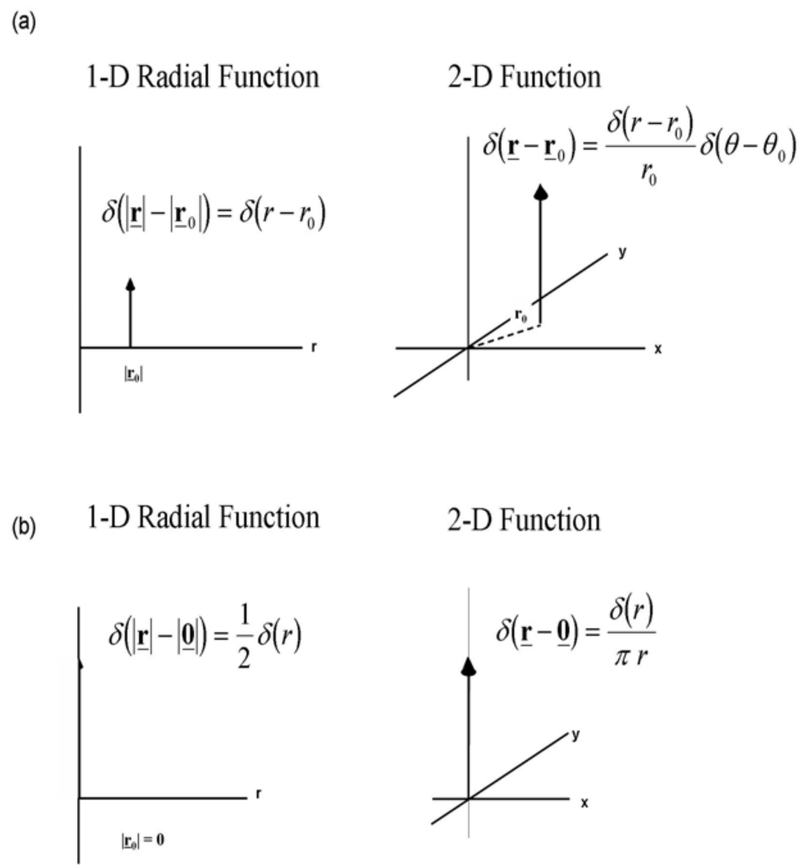
$$
\begin{aligned}
\delta\left(\underline{\mathbf{r}} - \underline{\mathbf{0}}\right) &= \alpha\ \delta\left[\left|\underline{\mathbf{r}}\right|\right] \times 1\left[\theta\right] \\
&= \alpha\ \delta\left[r\right] \times 1\left[\theta\right]
\end{aligned}
$$

  * α is scaling parameter ensures that $\delta\left(\underline{\mathbf{r}}\right)$ has unit volume
- Domains of polar arguments are $0 \leq r < +\infty$ and $-\pi \leq \theta < +\pi$.
- Modify domains
  * Radial variable over domain $(-\infty, +\infty)$
  * Azimuthal domain constrained to interval of π radians, e.g., $[0, +\pi)$ or $\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right)$.

$$
\begin{aligned}
1 &= \int_{-\pi}^{+\pi} \int_{0}^{+\infty} \alpha\ \delta\left[r\right]\ 1\left[\theta\right]\ r\ dr\ d\theta = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \int_{-\infty}^{+\infty} \alpha\ \delta\left[r\right]\ 1\left[\theta\right]\ r\ dr\ d\theta \\
&= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} 1\left[\theta\right]\ d\theta \int_{-\infty}^{+\infty} \alpha\ \delta\left[r\right]\ r\ dr = \int_{-\infty}^{+\infty} \alpha\ \delta\left[r\right]\ \pi r\ dr
\end{aligned}
$$

- Area of $\delta\left(r\right)$ over $(-\infty, +\infty)$ is unity
  * $\alpha = \left(\pi r\right)^{-1}$.

$$
\begin{aligned}
\delta\left(\underline{\mathbf{r}} - \underline{\mathbf{0}}\right) &= \delta\left(\underline{\mathbf{r}}\right) \\
&= \left(\frac{1}{\pi r}\right)\ \delta\left(r\right)\ 1\left[\theta\right] \\
&= \frac{\delta\left(r\right)}{\pi r}
\end{aligned}
$$

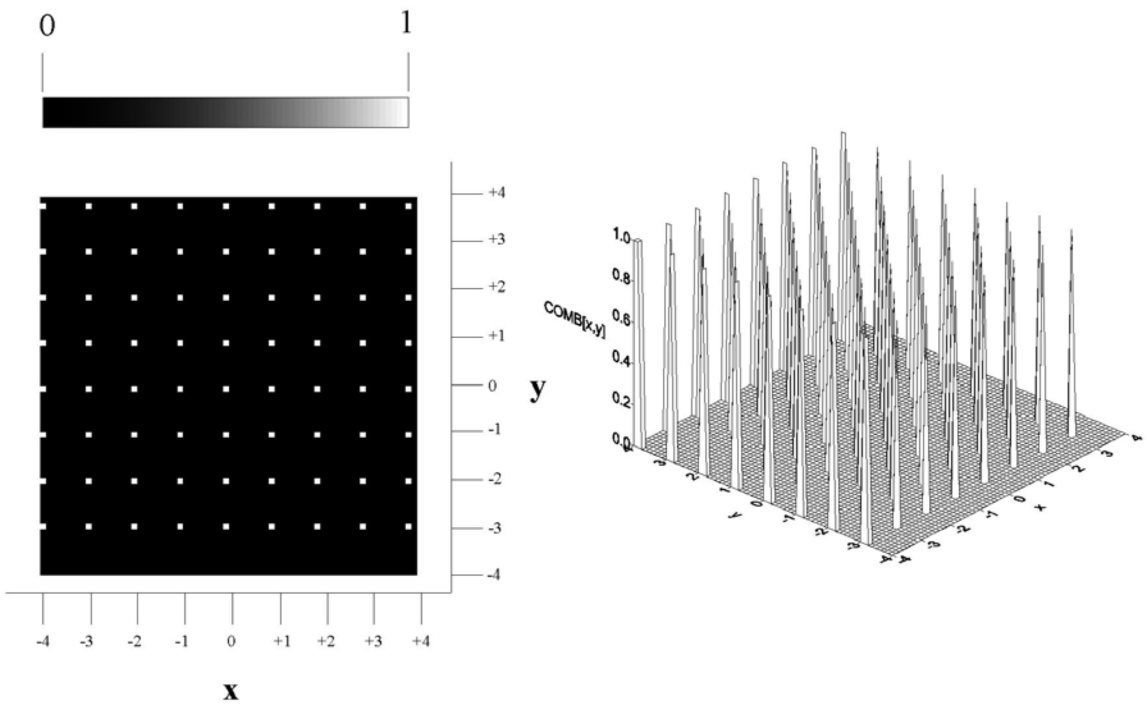


2-D "radial" representation of Dirac delta function.

4.3 2-D Separable Comb Function

$$\begin{aligned} COMB[x,y] &= COMB[x]\ COMB[y] \\ &= \left(\sum_{n=-\infty}^{+\infty} \delta[x-n]\right) \left(\sum_{\ell=-\infty}^{+\infty} \delta[y-\ell]\right) \end{aligned}$$

- Gaskill's "bed of nails"
- Volume is infinite.
- Most important application is to model 2-D sampled functions.



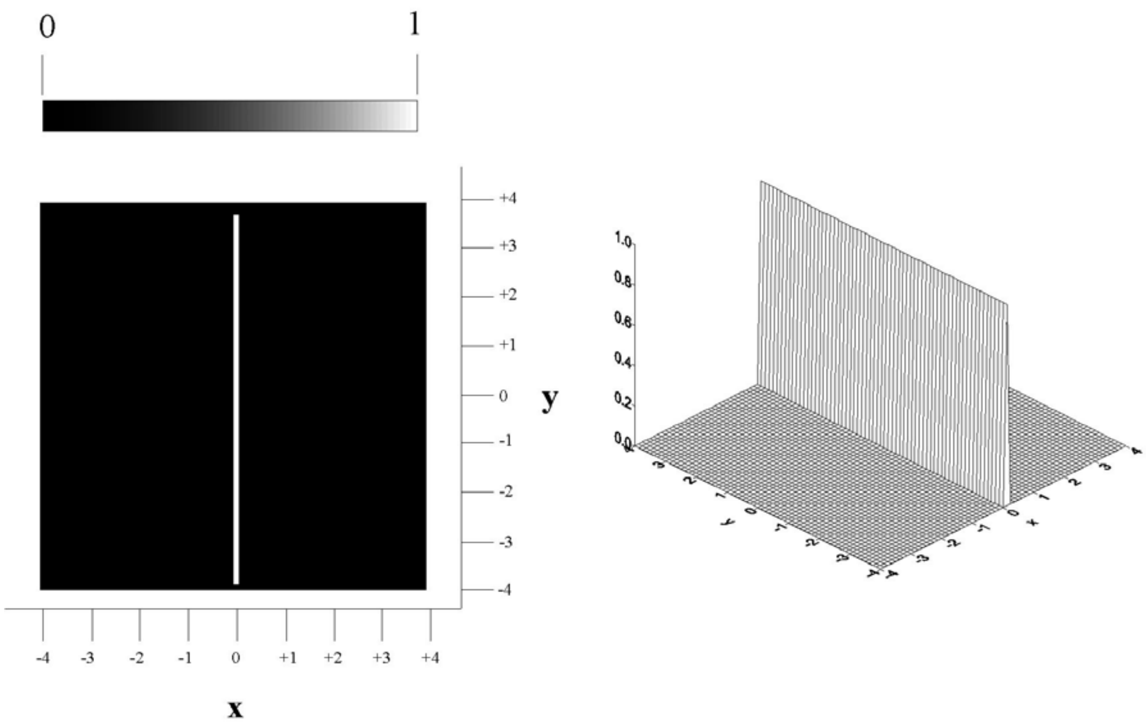
2-D separable COMB function $COMB[x,y] = COMB[x] \cdot COMB[y]$.

4.4 2-D "Line Delta" Function

- $$m_1[x,y] = \delta[x] \cdot 1[y]$$
- a "line" or "wall" of 1-D Dirac delta functions along y-axis
- "line delta function", "line mass", or "straight-line impulse"
- Most authors delete explicit unit constant $1[y]$
  - do not distinguish 1-D Dirac delta function and 2-D line delta function along the y-axis.
- Volume is infinite:

$$\iint_{-\infty}^{+\infty} \delta[x]\; 1[y]\; dx\, dy = \int_{-\infty}^{+\infty} \delta[x]\; dx \; \int_{-\infty}^{+\infty} 1[y]\; dy = 1 \times \infty$$

- Useful to define Radon transform (mathematical basis for medical computed tomography and magnetic resonance imaging)



2-D line Dirac delta function $\delta[x] \cdot 1[y]$ produces a "wall" of Dirac delta functions along the y-axis.

- Product of arbitrary function $f[x,y]$ with line delta function
  - apply sifting property:

$$\begin{aligned} f[x,y]\;\; (\delta[x]\;\; 1[y]) &= (f[x,y]\;\; \delta[x])\;\; 1[y] \\ &= (f[0,y]\;\; \delta[x])\;\; 1[y] \\ &= f[0,y]\;\; (\delta[x]\;\; 1[y]) \end{aligned}$$

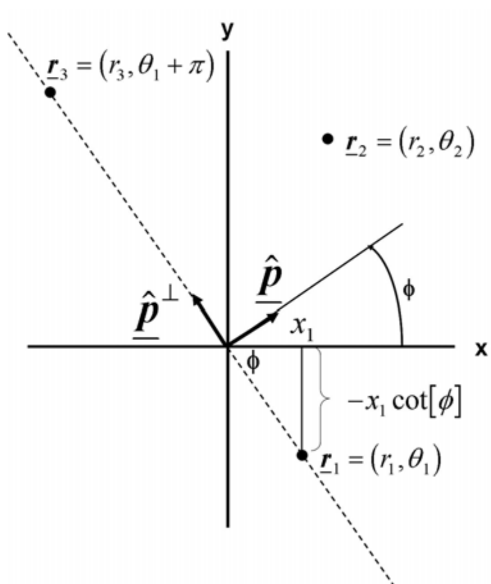
- Volume of product of functions is:

$$\begin{aligned}\iint_{-\infty}^{+\infty} f[x,y]\ \delta[x]\ 1[y]\ dx\,dy &= \iint_{-\infty}^{+\infty} f[0,y]\ \delta[x]\ 1[y]\ dx\,dy \\ &= \int_{-\infty}^{+\infty} f[0,y]\ 1[y]\ dy \int_{-\infty}^{+\infty} \delta[x]\ dx \\ &= \int_{-\infty}^{+\infty} f[0,y]\ dy\end{aligned}$$

- Line delta function $\delta[x]\ 1[y]$ "sifts out" area of $f[x,y]$ evaluated along the y-axis
- Line delta functions oriented along arbitrary azimuth angle:

$$\begin{aligned}m_2[x,y] &= \delta\left[\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}\right]\ 1\left[\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}^{\perp}\right] \\ &= \delta[x\ \cos[\phi] + y\ \sin[\phi]]\ 1[-x\ \sin[\phi] + y\ \cos[\phi]]\end{aligned}$$

- Line delta function oriented along the y-axis obtained by setting $\phi = 0$
- Notation for 1-D Dirac delta function may seem "weird" because the argument is the scalar product of two vectors.
- Argument of Dirac delta function is zero for all vectors $\underline{\mathbf{r}}$ in 2-D plane for which $\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}} = 0$, for all $\underline{\mathbf{r}} \perp \underline{\widehat{\mathbf{p}}}$.
- Complete set of line delta functions at all possible azimuth angles by selecting all angles ϕ in interval $-\frac{\pi}{2} \leq \phi < +\frac{\pi}{2}$
- Defining unit vector $\underline{\widehat{\mathbf{p}}}$ lies in first or fourth quadrants.
- Action of unit constant is to "spread" or "smear" amplitude of 1-D Dirac delta function along $\underline{\widehat{\mathbf{p}}}$.



The rotated form of the line delta function $m_2[x,y] = \delta\left[\underline{\mathbf{r}} \cdot \underline{\widehat{\mathbf{p}}}\right] \cdot 1\left[\underline{\mathbf{r}} \cdot \underline{\widehat{\mathbf{p}}}^{\perp}\right]$, where $\underline{\widehat{\mathbf{p}}} = (\cos[\phi], \sin[\phi])$. Three cases of $\underline{\mathbf{r}}$ are shown: $\underline{\mathbf{r}}_1$ and $\underline{\mathbf{r}}_3$ lie upon the line delta function and $\underline{\mathbf{r}}_2$ does not.

- Use polar form of radius vector $\underline{\mathbf{r}} = (r, \phi)$, where $0 \leq r < +\infty$ and $-\pi \leq \phi < +\pi$

$$\begin{aligned} m_2[x,y] &= \delta\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}\right] \ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right] \\ &= \delta\left[\left|\underline{\mathbf{r}}\right| \left|\widehat{\underline{\mathbf{p}}}\right| \cos\left[\theta - \phi\right]\right] \ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right] \\ &= \delta\left[\left|\underline{\mathbf{r}}\right| \cos\left[\theta - \phi\right]\right] \ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right] \\ &= \delta\left[r \cos\left[\phi - \theta\right]\right] \end{aligned}$$

- Expression determines the set of values of (r, θ) on radial line through origin perpendicular to azimuthal angle ϕ.
- Write as function of azimuthal angle ϕ by recasting into more convenient form
  - Apply expression for Dirac delta function with a functional argument

$$\begin{aligned} \delta\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}\right] \ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right] &= \delta\left[r \cos\left[\theta - \phi\right]\right] \ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right] \\ &= \delta\left[g\left[\phi\right]\right] \ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right] \\ &= \frac{1}{\left|g'\left(\phi_0\right)\right|} \ \delta\left[\phi - \phi_0\right] \ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right] \\ &= \frac{1}{r\left|\sin\left[\theta - \phi_0\right]\right|} \ \delta\left[\phi - \phi_0\right] \ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right] \end{aligned}$$

  - ϕ_0 is angle that satisfies condition $\cos\left[\theta - \phi_0\right] = 0 \Longrightarrow \phi_0 = \theta \pm \frac{\pi}{2}$

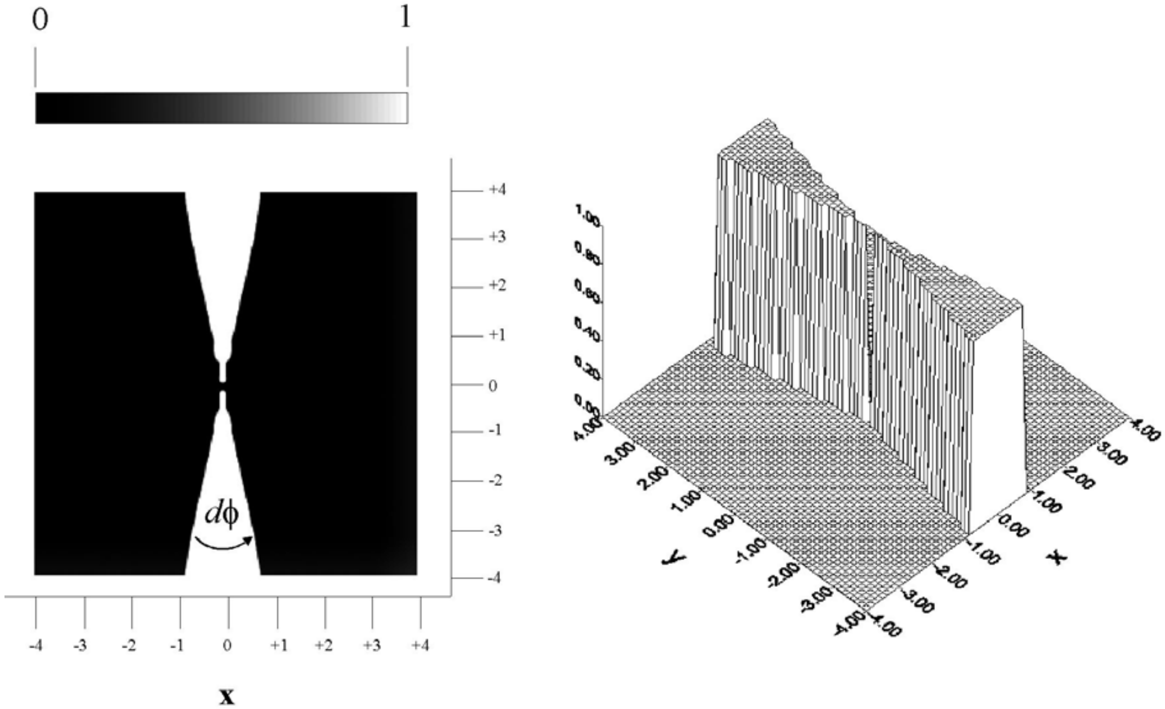
$$\begin{aligned} \delta\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}\right] \ 1\left[\underline{\mathbf{r}} \bullet \widehat{\underline{\mathbf{p}}}^{\perp}\right] &= \frac{1}{r\left|\sin\left[\mp\frac{\pi}{2}\right]\right|} \ \delta\left[\phi - \left(\theta \pm \frac{\pi}{2}\right)\right] \\ &= \frac{1}{r} \ \delta\left[\phi - \left(\theta \pm \frac{\pi}{2}\right)\right] \end{aligned}$$

  - Line delta function through origin lying along radial line perpendicular to azimuthal angle ϕ is equivalent to amplitude-weighted 1-D Dirac delta function of angle ϕ that is nonzero only for $\phi = \theta \pm \frac{\pi}{2}$
  - Sign selected to ensure that ϕ lies within usual domain of polar coordinates: $-\pi \leq \phi < +\pi$.
  - Each unit increment of radial distance along y-axis contributes same unit volume.
  - This expression seems to indicate otherwise for rotated function
  - Contribution to volume affected by factor r^{-1}.
  - o resolve apparent conundrum, substitute expression for Dirac delta function as limit of rectangle function

$$\delta\left[\phi - \left(\theta \pm \frac{\pi}{2}\right)\right] = \lim_{b \to 0}\left\{\frac{1}{b} \ RECT\left[\frac{\phi - \left(\theta \pm \frac{\pi}{2}\right)}{b}\right]\right\}$$

    - b measured in radians
    - Function consists of two symmetric "wedges" of fixed amplitude b^{-1}.
      - Second "wedge" about the angle $\phi = -\frac{\pi}{2}$ results from convention that domain of azimuth angle is $-\frac{\pi}{2} \leq \phi < +\frac{\pi}{2}$.
    - In limit $b \to 0$, the angular "spread" of wedges decreases while amplitude increases.

* Contribution of segments of wedges with unit radial extent increases in proportion to radial distance r from origin.
* Factor of r^{-1} compensates for increase in volume to ensure that contributions to volume of segments with equal radial extent remain constant.

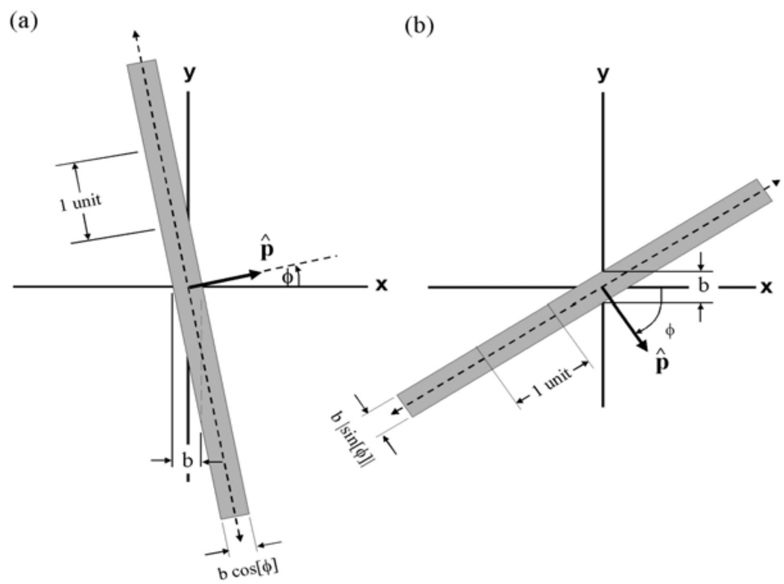


"Angular" delta function as limit of rectangle, $\lim_{d\phi\to 0}\left\{\frac{1}{d\phi}RECT\left[\frac{\phi-\frac{\pi}{2}}{d\phi}\right]\right\}$.

- Other forms may be derived by manipulating argument
- Slope-intercept form of a line in the 2-D plane

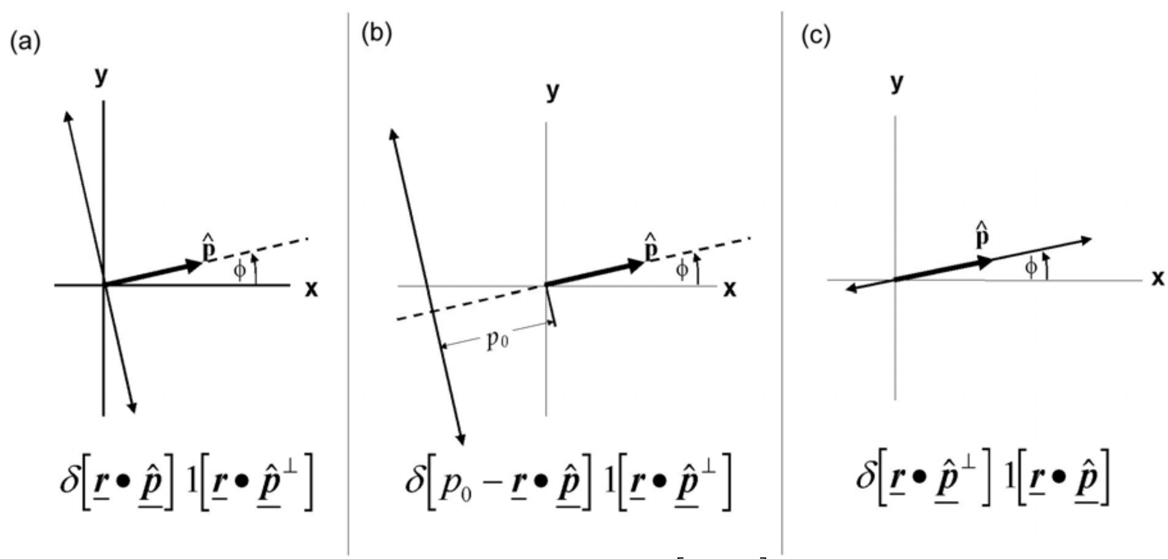
$$
\begin{aligned}
\delta\left[\underline{\mathbf{r}}\bullet\widehat{\underline{\mathbf{p}}}\right]\ 1\left[\underline{\mathbf{r}}\bullet\widehat{\underline{\mathbf{p}}}^{\perp}\right] &= \delta\left[+\sin[\phi]\left(y+\frac{x\cos[\phi]}{\sin[\phi]}\right)\right]\ 1\left[x\ \sin[\phi]-y\ \cos[\phi]\right] \\
&= \left|\frac{1}{\sin[\phi]}\right|\delta\left[\left(y-\left(\cot[-\phi]\ x+0\right)\right)\right]\ 1\left[x\ \sin[\phi]-y\ \cos[\phi]\right]
\end{aligned}
$$

- Dirac delta function evaluates to zero except at $[x,y]$ that satisfy slope-intercept form of straight line
- y-intercept is zero
- Slope is $s=\cot[-\phi]=-\cot[\phi]$



Two examples of rotated line delta functions as 1-D rectangle functions of width b. In both cases, the amplitude of the gray area is b^{-1}. (a) The rectangle is a function of x with width b measured on the x-axis and "perpendicular width" $b\cos(\phi)$; (b) the rectangle is a function of y with "perpendicular width" $b\sin[\phi]$.

Three examples of line delta functions:



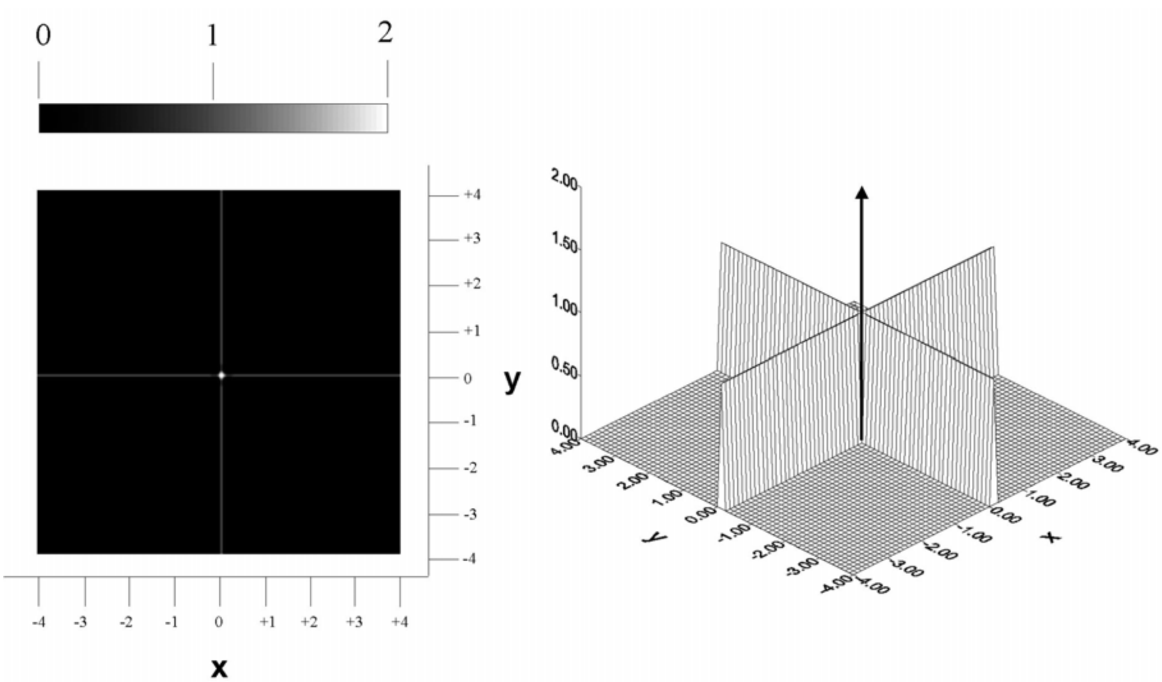
Different "flavors" of line delta functions: (a) $\delta\left[\underline{\mathbf{r}} \cdot \underline{\widehat{\mathbf{p}}}\right]\ 1\left[\underline{\mathbf{r}} \cdot \underline{\widehat{\mathbf{p}}}^{\perp}\right]$ through the origin perpendicular to $\underline{\widehat{\mathbf{p}}}$; (b) $\delta\left[p_0 - \underline{\mathbf{r}} \cdot \underline{\widehat{\mathbf{p}}}\right]\ 1\left[\underline{\mathbf{r}} \cdot \underline{\widehat{\mathbf{p}}}^{\perp}\right]$ perpendicular to $\underline{\widehat{\mathbf{p}}}$ at a distance $p_0 < 0$ from the origin; (c) $\delta\left[\underline{\mathbf{r}} \cdot \underline{\widehat{\mathbf{p}}}^{\perp}\right]\ 1\left[\underline{\mathbf{r}} \cdot \underline{\widehat{\mathbf{p}}}\right]$ through the origin parallel to $\underline{\widehat{\mathbf{p}}}$.

- Sum of two orthogonal line delta functions.
- Simplest version defined with "lines" coincident with Cartesian axes:

$$CROSS\,[x,y] = \delta\,[x]\;\;1\,[y] + 1\,[x]\;\;\delta\,[y]$$

- $CROSS\,[0,0] = 2\;\delta\,[x,y]$
- Cross function with "arms" oriented along the vectors $\underline{\widehat{\mathbf{p}}}$ and $\underline{\widehat{\mathbf{p}}}^{\perp}$ by substituting $\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}$ for x and $\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}^{\perp}$ for y:

$$CROSS\left[\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}, \underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}^{\perp}\right] = \delta\left[\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}\right]\;1\left[\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}^{\perp}\right] + 1\left[\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}\right]\;\delta\left[\underline{\mathbf{r}} \bullet \underline{\widehat{\mathbf{p}}}^{\perp}\right]$$



$CROSS\,[x,y] = \delta\,[x]\;1\,[y] + 1\,[x]\;\delta\,[y]$. Note that $CROSS\,[0,0] = 2\,\delta\,[x,y]$.

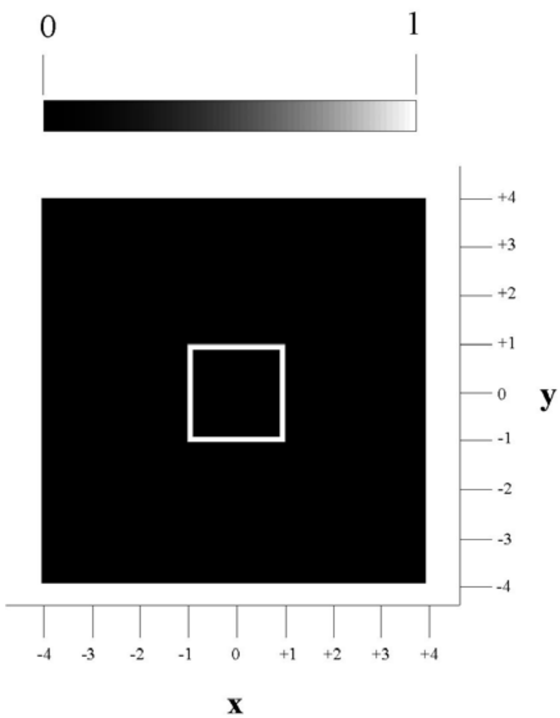
4.5 2-D "Corral" Function

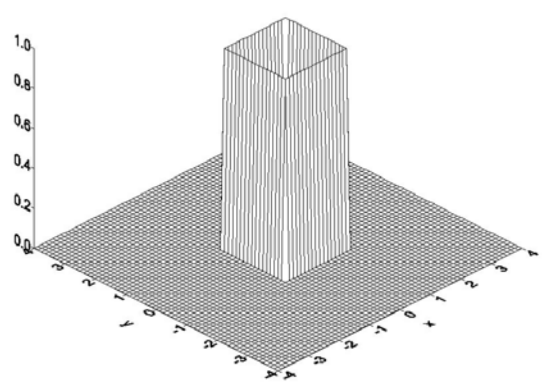
- Superpose truncated line delta functions to construct rectangular "stockade" of Dirac delta functions
- Sum of four sections:

$$\begin{aligned}
COR\left[\frac{x}{b},\frac{y}{d}\right] &\equiv \delta\left[x+\frac{b}{2}\right]\ RECT\left[\frac{y}{d}\right]+RECT\left[\frac{x}{b}\right]\ \delta\left[y+\frac{d}{2}\right]\\
&+\delta\left[x-\frac{b}{2}\right]\ RECT\left[\frac{y}{d}\right]+RECT\left[\frac{x}{b}\right]\ \delta\left[y-\left(-\frac{d}{2}\right)\right]\\
&=\left(\delta\left[x+\frac{b}{2}\right]+\delta\left[x-\left(-\frac{b}{2}\right)\right]\right)\ RECT\left[\frac{y}{d}\right]+RECT\left[\frac{x}{b}\right]\ \left(\delta\left[y-\frac{d}{2}\right]+\delta\left[y-\frac{d}{2}\right]\right)
\end{aligned}$$

- Volume computed from separable parts:

$$\begin{aligned}
\iint_{-\infty}^{+\infty} COR\left[\frac{x}{b},\frac{y}{d}\right]\,dx\,dy &= \int_{-\infty}^{+\infty}\left(\delta\left[x+\frac{b}{2}\right]+\delta\left[x-\frac{b}{2}\right]\right)\,dx\ \int_{-\infty}^{+\infty} RECT\left[\frac{y}{d}\right]\,dy\\
&+\int_{-\infty}^{+\infty} RECT\left[\frac{x}{b}\right]\,dx\ \int_{-\infty}^{+\infty}\left(\delta\left[y-\frac{d}{2}\right]+\delta\left[y-\frac{d}{2}\right]\right)\,dy\\
&=2d+2b=2\,(b+d)
\end{aligned}$$





5 2-D Functions with Circular Symmetry

- Vary along radial direction
- Constant in azimuthal direction
- All profiles along radial lines are identical
- Important in optics
- Optical systems are constructed from lenses with circular cross sections.
- Same amplitude at all points on circle of radius r_0 centered at origin
- Amplitude of radial function $f_r(r_0)$ replicated for $[x, y]$ that satisfy $x^2 + y^2 = r_0^2$.
- Circularly symmetric function expressed as orthogonal product of 1-D radial profile and unit constant in azimuthal direction:

$$f[x, y] \Longrightarrow f(\underline{\mathbf{r}}) = f_r(r)\ \ 1[\theta]\,,\ \ 0 \leq r < +\infty,\ \ -\pi \leq \theta < +\pi$$

- Rotation about origin has no effect on amplitude
- Symmetric with respect to the origin
  - Domains of radial and azimuthal variables may be recast wiht symmetric radial interval $-\infty < r < +\infty$
  - Azimuthal domain $-\frac{\pi}{2} \leq \theta < +\frac{\pi}{2}$.
- Volume calculated in polar coordinates
- area element $dx\,dy$ replaced by area element in polar coordinates $r\,dr\,d\theta$:

$$\begin{aligned} \iint_{-\infty}^{+\infty} f[x, y]\ dx\ dy &= \int_{\theta=-\pi}^{\theta=+\pi} \int_{r=0}^{r=+\infty} f(\underline{\mathbf{r}})\ r\ dr \\ &= 2\pi \int_0^{+\infty} f_r(r)\ r\ dr \end{aligned}$$

- Center of symmetry relocated to $[x_0, y_0]$ by adding vector to argument:

$$\begin{aligned} f(\underline{\mathbf{r}} - \underline{\mathbf{r}}_0) &= f[x, y] \\ &= f[|\underline{\mathbf{r}}|\ \ \cos[\theta] - |\underline{\mathbf{r}}_0|\ \ \cos[\theta_0]\,, |\underline{\mathbf{r}}|] \end{aligned}$$

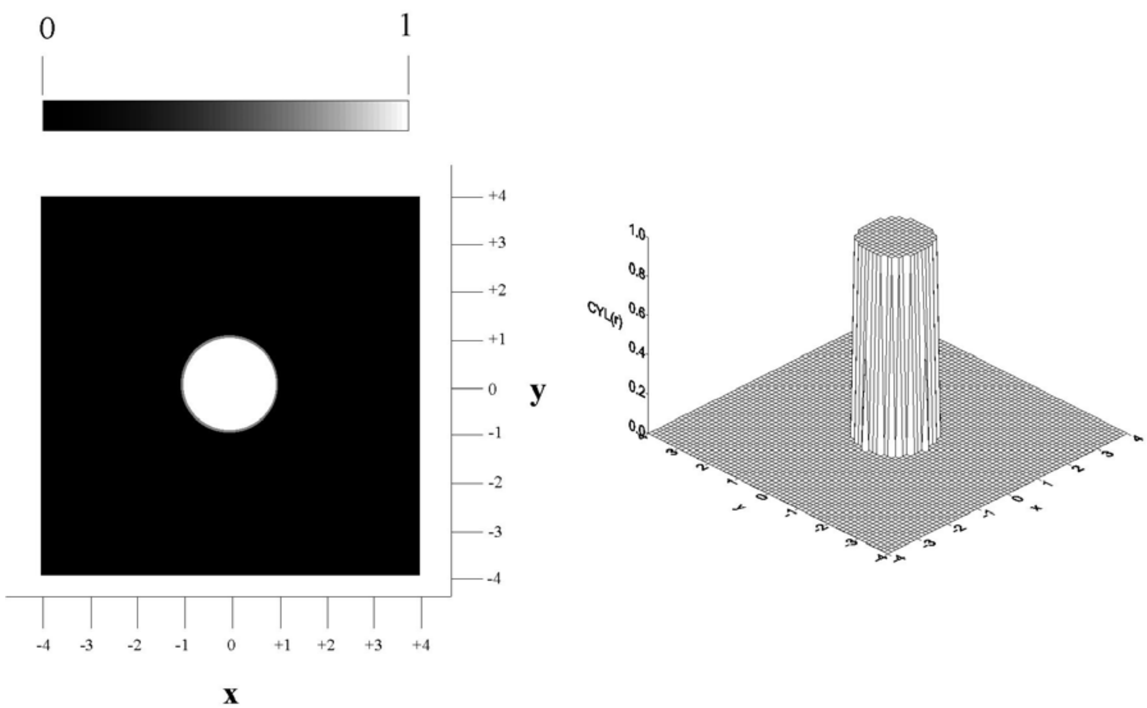
5.1 Cylinder (Circle) Function

- Unit amplitude inside radius r_0
- Null amplitude outside
- Circularly symmetric version of 2-D rectangle

$$CYL\left(\frac{r}{d_0}\right) = \begin{cases} 1 & \text{for} \quad r < \frac{d_0}{2} \\ \frac{1}{2} & \text{for} \quad r = \frac{d_0}{2} \\ 0 & \text{for} \quad r < \frac{d_0}{2} \end{cases}$$

- Area of enclosed circle of unit diameter is $\frac{\pi}{4} \simeq 0.7854 <$unit area of $RECT\,[x, y]$.
- Volume of $CYL\left(\frac{r}{d_0}\right)$:

$$\int_{-\pi}^{+\pi} d\theta \int_{0}^{+\infty} CYL\left(\frac{r}{d_0}\right)\; r\; d_0 r = \frac{\pi d_0^2}{4}$$



Example of 2-D cylinder function $CYL\left(\frac{r}{2}\right)$.

5.2 Circularly Symmetric Gaussian

- Circularly symmetric Gaussian function identical to separable Gaussian function if scale parameters $a = b \equiv d$

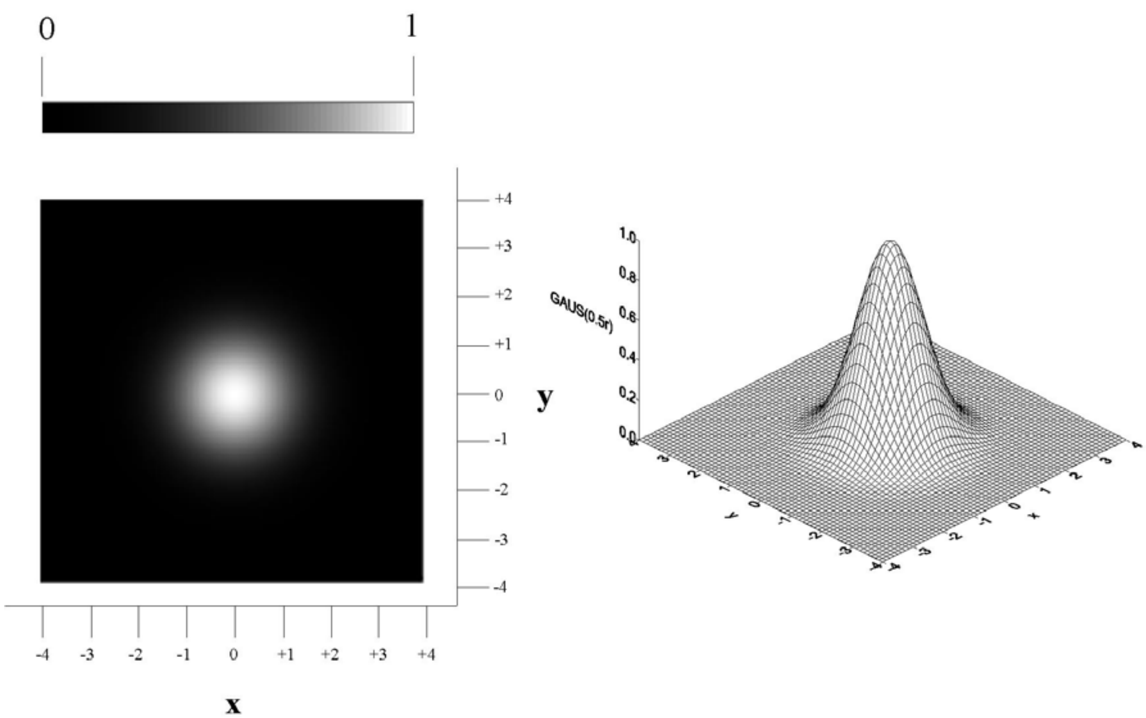
$$
\begin{aligned}
e^{-\frac{\pi x^2}{d^2}}\, e^{-\frac{\pi y^2}{d^2}} &= e^{-\pi \frac{x^2+y^2}{d^2}} \\
&= e^{-\pi\left(\frac{r}{d}\right)^2} \\
&\equiv GAUS\left(\frac{r}{d}\right)
\end{aligned}
$$

- Volume by integrating over polar coordinates after making appropriate change of variable:

$$
\begin{aligned}
\iint_{-\infty}^{+\infty} GAUS\left(\frac{r}{d}\right)\, dx\, dy &= \int_{-\pi}^{+\pi} \int_{0}^{+\infty} GAUS\left(\frac{r}{d}\right)\, r\, dr\, d\theta \\
&= 2\pi \int_{0}^{+\infty} r\, e^{-\pi\left(\frac{r}{d}\right)^2}\, dr
\end{aligned}
$$

- Define new integration variable $\alpha \equiv \pi\left(\frac{r}{d}\right)^2$

$$
\iint_{-\infty}^{+\infty} GAUS\left(\frac{r}{d}\right)\, dx\, dy = 2\pi \int_{0}^{+\infty} \frac{d^2}{2\pi}\, e^{-\alpha}\, d\alpha = d^2
$$



Example of 2-D circularly symmetric Gaussian $f(r) = e^{-\pi\left(\frac{r}{2}\right)^2}$.

5.3 Circularly Symmetric Bessel Function, Zero Order

$$J_0 [2\pi r\rho_0] \ 1[\theta] = J_0 \left(2\pi\sqrt{x^2+y^2} \ \rho_0\right)$$

- Selectable parameter ρ_0 analogous to spatial frequency of sinusoid
  - Larger ρ_0 $\Longrightarrow$shorter interval between successive maxima of Bessel function
- Appears in several imaging applications.
- $J_0 [2\pi r\rho_0] \ 1[\theta]$ generated by summing 2-D cosine functions with the same period "directed" along all azimuthal directions.
- Constituent functions have form $\cos[2\pi(\eta x+\eta y)]$, where $\sqrt{\xi^2+\eta^2} = \rho_0^2$.

$$J_0 [2\pi r\rho_0] \ 1[\theta] = \int_{-\pi/2}^{+\pi/2} \cos[2\pi(\xi x+\eta y)] \ d\phi \text{ where } \xi^2+\eta^2 = \rho_0^2$$

- Symmetry of integrand used to evaluate integral over domain of 2π radians:

$$\begin{aligned} J_0 [2\pi r\rho_0] &= \frac{1}{2}\int_{-\pi}^{+\pi} \cos[2\pi(\xi x+\eta y)] \ d\phi, \text{ where } \xi^2+\eta^2 = \rho_0^2 \\ &= \frac{1}{2}\int_{-\pi}^{+\pi} \cos[2\pi(r\xi \ \cos[\phi] + r\eta \ \sin[\phi])] \ d\phi \end{aligned}$$

- Rewrite integrals by substituting corresponding linear-phase complex exponential for cosine function

$$\begin{aligned} J_0 [2\pi r\rho_0] \, 1[\theta] &= \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} e^{\pm 2\pi i(\xi x+\eta y)} d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{\pm 2\pi i(\xi x+\eta y)} \ d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \cos[2\pi r(\xi \ \cos[\phi] + \eta \ \sin[\phi])] \ d\phi, \text{ where } \xi^2+\eta^2 = \rho_0^2 \end{aligned}$$

- Profile along the x-axis by setting $\eta = 0$. Spatial frequency ξ along x-axis becomes $\xi = \rho_0$:

$$\begin{aligned} J_0 [2\pi r\rho_0] \ 1[\theta]|_{\eta=0} &= J_0 [2\pi x\rho_0] \\ &= \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} \cos[2\pi r\rho_0 \ \cos[\phi]] \ d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \cos[2\pi r\rho_0 \ \cos[\phi]] \ d\phi \\ &= J_0 [2\pi x\xi]|_{r=x,\xi=\rho_0} = J_0 [2\pi r\rho_0] \end{aligned}$$

- $\rho_0 \ \cos[\phi]$ is spatial frequency of constituent 1-D cosines of Bessel function
  - Suggests alternate interpretation that 1-D J_0 Bessel function is sum of 1-D cosines with spatial frequencies in interval $-\rho_0 \leq \xi \leq +\rho_0$ but weighted in "density" by $\cos[\phi]$
  - Largest spatial frequency exists when $\phi = 0$, while the cosine is the unit constant when $\phi = \pm\frac{\pi}{2}$.

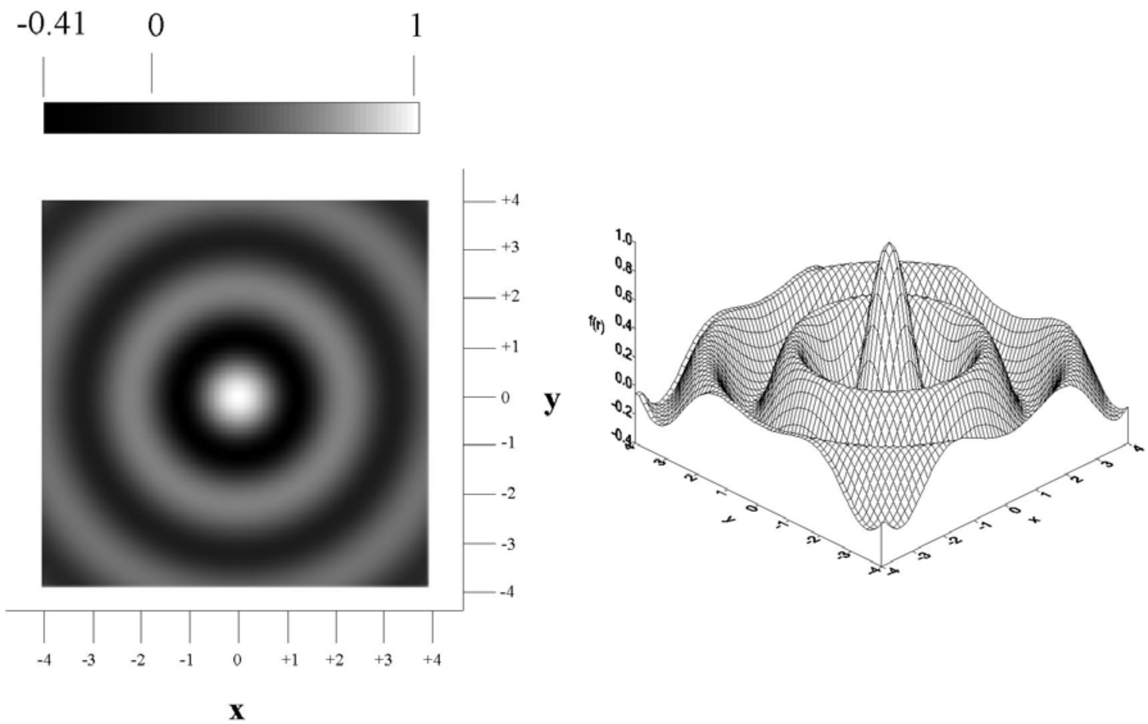
- Equivalent expressions for 1-D Bessel function obtained by setting $\xi = 0$ so that $\eta = \rho_0$
  - equivalent to projecting argument onto y-axis:

$$
\begin{aligned}
J_0 [2\pi y\eta] \;\; 1 [\theta]|_{\xi=0} &= \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} \cos [2\pi r (\rho_0 \;\; \sin [\phi])] \;\; d\phi \\
&= \frac{1}{2\pi} \int_{0}^{+\frac{\pi}{2}} \cos [2\pi r (\rho_0 \;\; \sin [\phi])] \;\; d\phi \\
&= \; J_0 [2\pi y\eta]|_{r=y,\eta=\rho_0} = J_0 [2\pi r\rho_0]
\end{aligned}
$$

  - Projection of complex-valued formulations onto x-axis is:

$$
(J_0 [2\pi y\rho_0] \;\; 1 [\theta])|_{\xi=0} = \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} e^{\pm 2\pi i r\rho_0 \cos[\phi]} \; d\phi \text{ where } \xi^2 + \eta^2 = \rho_0^2
$$

  - May be used as an equivalent definition of the Bessel function.

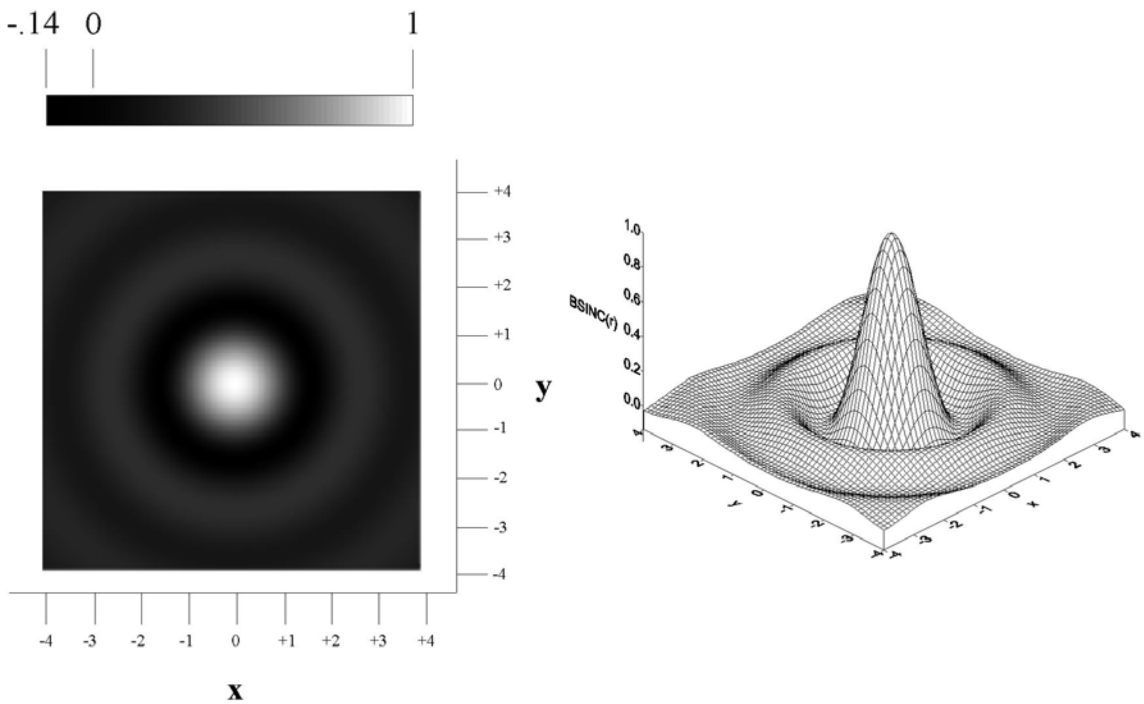


Circularly symmetric function $J_0 (2\pi\rho_0 r)$ for $\rho_0 = \frac{1}{2}$.

5.4 "Sombrero" (Besinc) Function

- Circularly symmetric analogue to $SINC$ function
- Ratio of two functions with null amplitude at origin
  - numerator is Bessel function of the first kind of order unity $J_1(\pi r)$
  - denominator is factor proportional to r.

$$SOMB(r) = \frac{2\ J_1(\pi r)}{\pi r}$$



$SOMB(r) = 2\frac{J_1(\pi r)}{\pi r}$

- Amplitude of $SOMB(r)$ at origin obtained via l'Hôspital's rule:

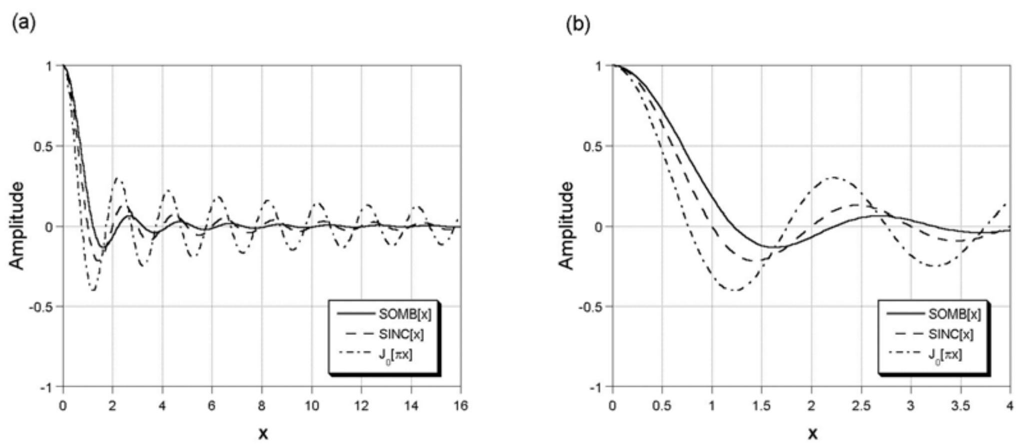
$$\begin{aligned} SOMB(0) &= 2 \lim_{r\to 0}\left\{\frac{\frac{d}{dr}\left(J_1[\pi r]\right)}{\frac{d}{dr}(\pi r)}\right\} \\ &= 2\ \frac{\frac{d}{dr}\left(\frac{\pi r}{2} - \frac{(\pi r)^3}{16} + \frac{(\pi r)^5}{384} - \frac{(\pi r)^7}{18{,}432} + \cdots\right)\Big|_{r=0}}{\frac{d}{dr}(\pi r)\big|_{r=0}} \\ &= 2\ \frac{\left(\frac{\pi}{2}\right)}{\pi} = 1 \end{aligned}$$

- Asymptotic form of $J_1[x]$ for large x used to derive expression for $SOMB(r)$ in limit of large r:

$$\begin{aligned} \lim_{r\to+\infty}\{SOMB(r)\} &= \frac{\sqrt{\frac{2}{\pi r}}\sin\left[\pi r - \frac{\pi}{4}\right]}{\pi r} \\ &\propto r^{-\frac{3}{2}}\sin\left[\pi r - \frac{\pi}{4}\right] \end{aligned}$$

- Compare x-axis profiles of $J_0[\pi x]$, $SINC[x]$, and $SOMB[x]$
  - All have unit amplitude at origin
  - Product of periodic or pseudoperiodic oscillation and decaying function of x
  - Peak amplitudes of J_0 decay most slowly with increasing x $\left(|J_0[\pi x]| \propto x^{-\frac{1}{2}}\right)$;
  - $SINC[x]$ decreases as x^{-1}
  - $SOMB[x]$ as $x^{-\frac{3}{2}}$
  - Differences in locations of zeros
    - zeros of $SINC[x]$ located at integer values of x
    - First two zeros of $J_0[\pi x]$ located at $x_1 \simeq \frac{2.4048}{\pi} \simeq 0.7654$ and $x_2 \simeq \frac{5.5201}{\pi} \simeq 1.7571$
      - interval slightly less than unity
    - Interval between successive pairs of zeros of $J_0[\pi x]$ decreases and asymptotically approaches unity as $x \to \infty$
    - First two zeros of $J_1[\pi x]$ (and therefore of $SOMB(r)$) located at $x \simeq 1.2197$ and $x \simeq 2.2331$
      - Interval between zeros of $SOMB(r)$ decreases with increasing r
      - asymptotically approaches unity as $r \to \infty$.

$$\int_{-\pi}^{+\pi} d\theta \int_{0}^{+\infty} SOMB\left(\frac{r}{d}\right) r\, dr = \frac{4d^2}{\pi}$$



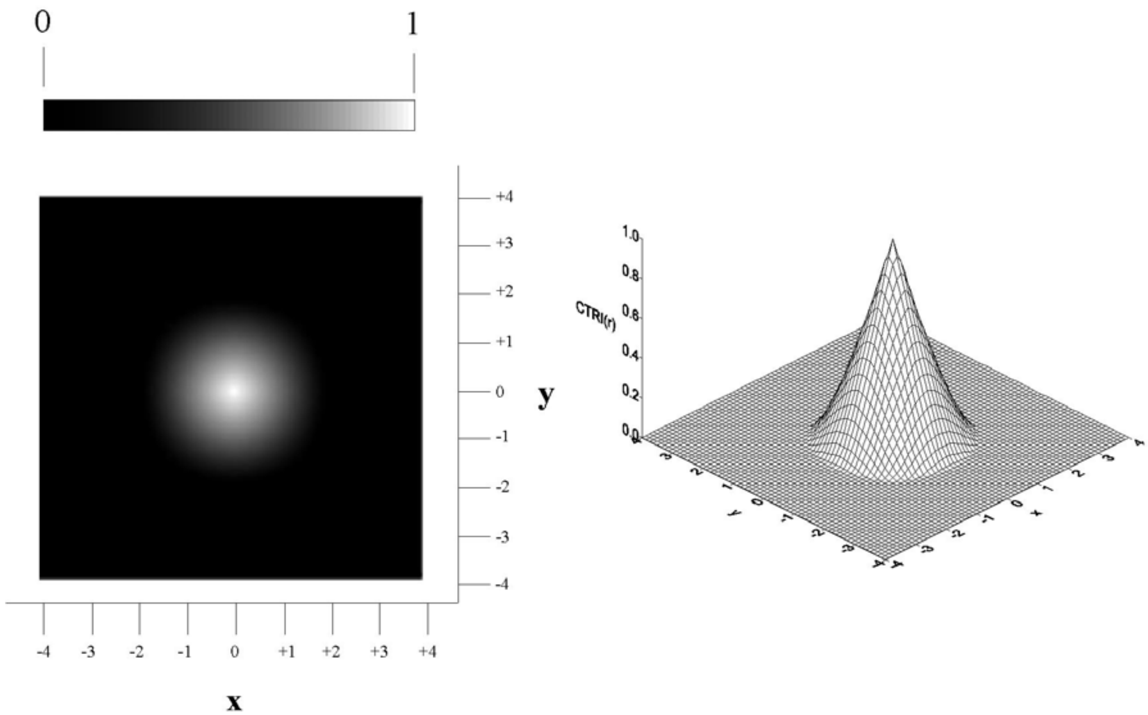
Comparison of profiles of $BSINC[x], SINC[x]$, and $J_0[\pi x]$ (a) for $0 \leq x \leq 8$ and (b) magnified view for $0 \leq x \leq 2$. $BSINC[x]$ and $J_0[\pi x]$ "fall off" at the most rapid and slowest rates, respectively.

5.5 Circular Triangle Function

- Circularly symmetric analogue of 2-D triangle function
- Most easily constructed as a 2-D autocorrelation
- Describes spatial response of optical imaging systems constructed from elements with circular cross sections.

$$CTRI\,(r) \equiv \frac{2}{\pi}\left(\cos^{-1}[r] - r\sqrt{1-r^2}\right) CYL\left(\frac{r}{2}\right)$$

- Amplitude decreases approximately linearly until $r \simeq 0.8$ where slope "flattens out" slightly
- Profiles of both 2-D $CTRI$ and 2-D TRI are not straight lines.



Circularly symmetric "triangle" function $CTRI\,(r)$.

5.6 Ring Delta Function

- Circularly symmetric analogue of rectangular "corral":

$$\begin{aligned} f(\underline{\mathbf{r}}) &= \delta(|\underline{\mathbf{r}}| - |\underline{\mathbf{r}}_0|) \\ &\equiv \delta(r - r_0) \\ &= \delta(r - r_0)\, 1[\theta] \end{aligned}$$

- Resembles polar representation of 2-D Dirac delta function
- Radial arguments r and r_0 are scalars
- Volume evaluated in polar coordinates by applying sifting property of 1-D Dirac delta function:

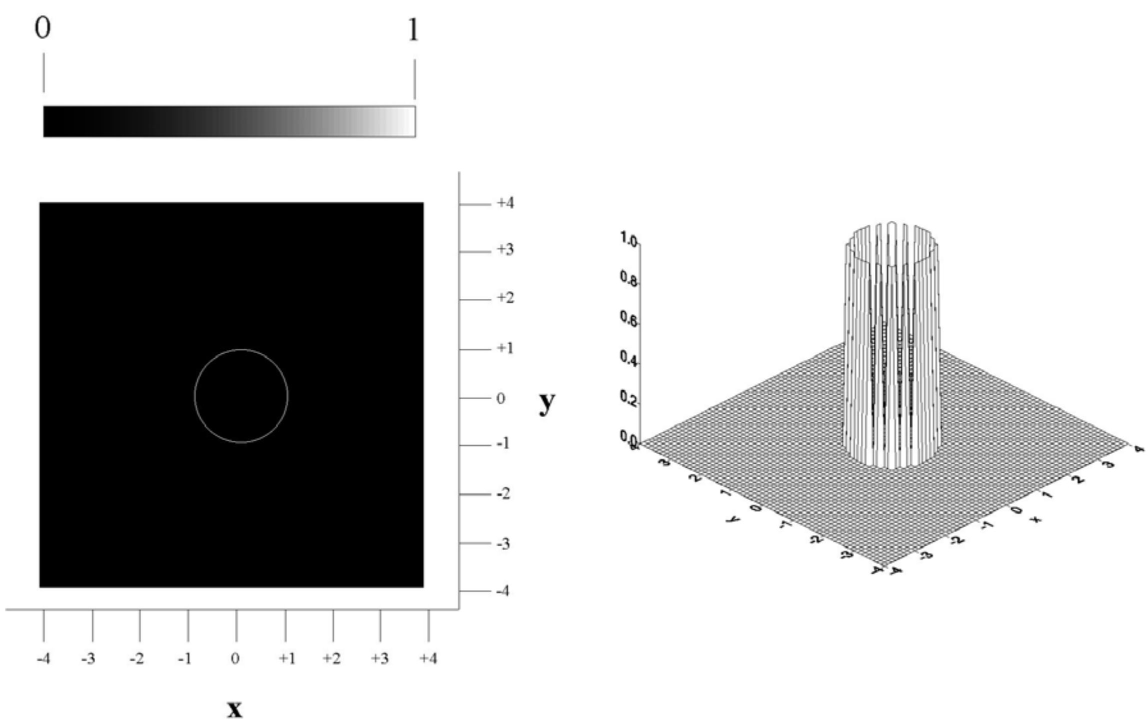
$$\begin{aligned} \int_{-\pi}^{+\pi} \int_{0}^{+\infty} \delta(r - r_0)\ 1[\theta]\ r\, dr\, d\theta &= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \int_{-\infty}^{+\infty} \delta(r - r_0)\ 1[\theta]\ r\, dr\, d\theta \\ &= \pi \int_{-\infty}^{+\infty} [\delta(r + r_0) + \delta(r - r_0)]\ r\, dr = \pi(2r_0) = 2\pi r_0 \end{aligned}$$

- Scaled difference of two cylinder functions with diameters $2r_0 + \Delta$ and $2r_0 - \Delta$:

$$f(r) = \lim_{\Delta \to 0} \left\{ \frac{1}{\Delta} \left[CYL\left(\frac{r}{2r_0 + \Delta}\right) - CYL\left(\frac{r}{2r_0 - \Delta}\right) \right] \right\}$$

- Volume:

$$\begin{aligned} \int_{-\pi}^{+\pi} \int_{0}^{+\infty} f(r)\, r\, dr\, d\theta &= \lim_{\Delta \to 0} \left\{ \frac{1}{\Delta} \left[\frac{\pi(2r_0 + \Delta)^2}{4} - \frac{\pi(2r_0 - \Delta)^2}{4} \right] \right\} \\ &= \lim_{\Delta \to 0} \left\{ \frac{\pi}{4\Delta} \left[(2r_0 + \Delta)^2 - (2r_0 - \Delta)^2 \right] \right\} \\ &= \lim_{\Delta \to 0} \left\{ \frac{\pi}{4\Delta} [8r_0\Delta] \right\} = \lim_{\Delta \to 0} \{2\pi\ r_0\} = 2\pi r_0 \end{aligned}$$



2-D "ring" Dirac delta function $f(r) = \delta(r - r_0)$.

6 Complex-Valued 2-D Functions

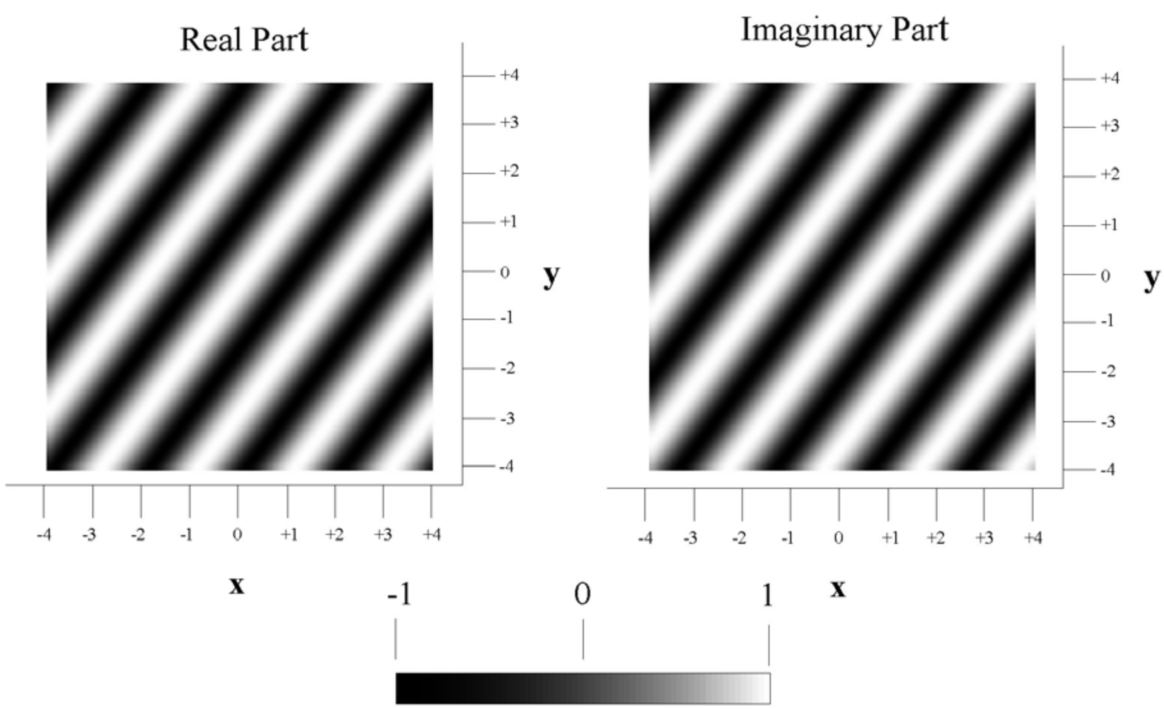
6.1 Complex 2-D Sinusoid

$$\begin{aligned} f[x,y] &= A_0\left(\cos\left[2\pi\xi_0 x+\phi_0\right]+i\ \sin\left[2\pi\xi_0 x+\phi_0\right]\right)\times 1[y] \\ &= A_0\left(\cos\left[2\pi\xi_0 x+\phi_0\right]+i\ \cos\left[2\pi\xi_0 x+\phi_0-\frac{\pi}{2}\right]\right)\times 1[y] \end{aligned}$$

$$f[x,y] = A_0 e^{i(2\pi\xi_0 x+\phi_0)}1[y]$$

Azimuth may be oriented in an arbitrary direction:

$$\begin{aligned} f[x,y] &= \cos\left[2\pi\xi_0\left(\underline{\mathbf{r}}\bullet\underline{\widehat{\mathbf{p}}}\right)\right]+i\ \cos\left[2\pi\xi_0\left(\underline{\mathbf{r}}\bullet\underline{\widehat{\mathbf{p}}}\right)-\frac{\pi}{2}\right] \\ &= \cos\left[2\pi\left(\xi_0 x+\eta_0 y\right)\right]+i\ \sin\left[2\pi\left(\xi_0 x+\eta_0 y\right)\right]=e^{+2\pi i(\xi_0 x+\eta_0 y)} \end{aligned}$$



Real and imaginary parts of 2-D sinusoid.

6.2 2-D Complex Quadratic-Phase Sinusoid ("Chirp")

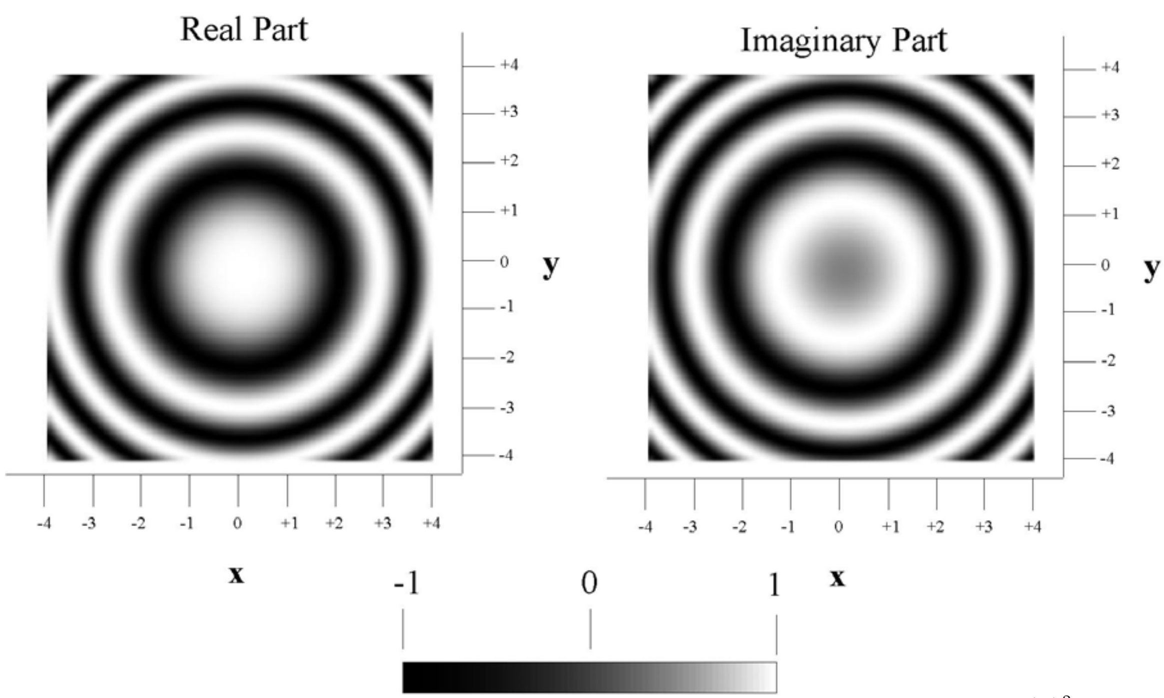
- Important in describing Fresnel diffraction
- 2-D complex "chirp" function:

$$\begin{aligned} f[x,y] &= Ae^{\pm i\pi\left(\frac{x}{a}\right)^2} \, 1[y] \\ &= A_0\left(\cos\left[\frac{\pi x^2}{a^2}\right] \pm i \, \sin\left[\frac{\pi x^2}{a^2}\right]\right) \times 1[y] \end{aligned}$$

$$\begin{aligned} f[x,y] &= A_0 e^{\pm i\pi\left(\frac{x}{a}\right)^2} \, B_0 e^{\pm i\pi\left(\frac{y}{b}\right)^2} \\ &= A_0 \, B_0 \, e^{\pm i\pi\left(\left(\frac{x}{\alpha}\right)^2 \pm \left(\frac{y}{b}\right)^2\right)} \end{aligned}$$

- Magnitude of second function is not constant
  - Contours of constant magnitude are elliptical
- Set a and b to d to obtain circularly symmetric complex chirp function:

$$\begin{aligned} f[x,y] &= A_0 e^{\pm i\pi\left(\frac{x}{d}\right)^2} \, B_0 e^{\pm i\pi\left(\frac{y}{d}\right)^2} \\ &= A_0 B_0 e^{\pm i\pi \frac{\left(x^2+y^2\right)}{d^2}} \end{aligned}$$



Real and imaginary parts of 2-D circularly symmetric chirp function $f(r) = e^{+i\pi\left(\frac{r}{2}\right)^2}$.