PRACTICAL METHODS IN THE DIRECT FACTOR ANALYSIS OF PSYCHOLOGICAL SCORE MATRICES

BY

DAVID ROBERTSON SAUNDERS

S.B., Harvard University, 1943
A.M., University of Illinois, 1949

THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN PSYCHOLOGY
IN THE GRADUATE COLLEGE OF THE
UNIVERSITY OF ILLINOIS. 1950

URBANA, ILLINOIS

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THE GRADUATE COLLEGE

MAY 20, 1950

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY DAVID ROBERTSON SAUNDERS

ENTITLED PRACTICAL METHODS IN THE DIRECT FACTOR ANALYSIS OF PSYCHOLOGICAL SCORE MATRICES

BE ACCEPTED* AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN PSYCHOLOGY.

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† Required for doctor's degree but not for master's.

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I INTRODUCTION

A Statement of the Problem

It was originally contemplated that this thesis would bear the following title: "The Relation of Effective Leadership to Syntality Factors in Small Groups." An experimental design was developed and executed for the purpose of elucidating this problem, but very early in the course of the analysis of the resulting data certain more fundamental issues raised themselves, which had the effect of forcing the author to undertake the re-analysis of older data previously regarded as "spent ore." Specifically, what was required was a critical examination of the experimental bases, construction and standardization of the test instruments used to select members for the experimental groups which had been studied.

Even this, however, did not turn out to be the end of the trail, for in the course of reworking various blocks of data it became increasingly apparent that the statistical tools which had been employed fell short of an attainable ideal. By combining various leads now in the literature it is possible to synthesize an applicable factor analytic methodology which is at the same time practical and theoretically rigorous, and which can be used as a touchstone for evaluating the various pieces of evidence that need to be fitted together consistently. The results must satisfy not only the common sense requirements of the psychologist, but also the rigor of the mathematician and the determinism of the general scientific investigator. Therefore, rather than to push ahead with the originally planned topic, it has seemed to be more economical of total research effort to take care of these more fundamental issues first.

The general plan of the present approach to this problem is to develop from theory what appears to be a suitable methodological technique, and to illustrate the applicability and validity of the method by means of a representative miniature example using actual data.

B Summary

1) The theory underlying the direct factor analysis of psychological score matrices is presented as a unified whole; direct factor analysis does not yield computational advantages over the conventional methods if the latter are applicable. The relation of direct factor analysis to analysis of variance is discussed and utilized.

2) The assumptions underlying the statistical evaluation of a direct factor analysis are contrasted with those underlying the statistical evaluation of conventional results; direct factor analysis requires the acceptance of a significantly less restrictive view, since no assumption is required concerning the distribution of scores for any population of subjects, and only the error distributions are taken to be normal.

3) The direct factor analytic approach to the problem of three-way score matrices is possible; the theory is briefly sketched and the geometrical and psychological interpretations of the results are indicated.

4) The direct factor analytic approach to the problem of second-class score matrices (matrices in which the scores are represented by the symbols $+$ and $-$) is possible; the theory is described, but without recourse to formal proofs. The resulting methodology is termed K-way scale analysis.

5) A miniature example of K-way scale analysis is solved in order to illustrate the computational procedures step-wise; these procedures are no more noxious than those now applied to similar data, and are readily adaptable to automatic computing machinery. The example is abstracted from a real score matrix based on an experimental personality questionnaire.

6) The results of the miniature example are subjected to psychological interpretation and analysis, in order to demonstrate that the numerical results are both meaningful in their own right and suggestive of further experimental

ideas.

7) The relation of the assumptions and criteria of K-way scale analysis to other commonly-employed methods is discussed. The new method avoids the subjective element characteristic of present-day scale analysis, and also avoids the difficulty factors which tend to be introduced in conventional analyses of tetrachoric or phi coefficients.

8) A topological model is suggested as a possible basis for summarizing the information yielded by a K-way scale analysis, and some possible directions for the further simplification of this model are indicated.

II THE THEORY OF DIRECT FACTOR ANALYSIS

A Fundamental Assumptions

The ultimate purpose of factor analysis, long-circuited though the realization may be, is the successful prediction of observable behavior. The psychometrist approaches the study of behavior by abstracting from the total number of situations in which people find themselves certain situations which will be called tests. Tests, or test-situations, are supposed to be characterized by the fact that they are reproducible, *i.e.*, that they can be repeated by some means when they are applied to various subjects or on various occasions. Then, by studying the performance of a large variety of individuals in a large variety of test-situations, the psychometrist hopes to infer general laws of behavior which can even be extended to cover non-test situations.

The first step in this process is the assignment to individuals of scores on tests. Some of these scores may be given in familiar terms, being measured in some reputable unit such as time, distance, frequency, etc.; such scores are our *first class*. For a *second class* of scores it may be reasonable to assume the existence of a satisfactory measuring scale, in principle, although the inaccessibility of the thing measured forces us to accept for analysis merely the fact that a score is greater than, or less than, some arbitrary value on the scale. Data of this type classify individuals into quantitatively different zones of response, and frequently arise when a subject is required to make one of two specified responses to a particular stimulus. For a *third class* of scores the psychologist may be able only to classify the possible responses into two or more qualitatively different categories. If these categories can be ranked, then the data may be treated approximately, and perhaps with accompanying loss of information, as belonging to the second class -- or an attempt may even be made to assign relative or absolute scale values to the various categories (60) so that the data may be treated as belonging to the first class mentioned. On the other hand, if the multiple categories cannot be ranked, we must simply note that the psychometrist is faced with his most difficult -- and probably most typical (43) -- case. We shall be concerned here only with scores of the first and second classes[1] -- those for which an underlying continuum is either known or assumed to exist.

In order to develop a factor analysis of psychological scores we must provide an existence postulate for the factors. Therefore, we shall assert the existence of certain mathematical entities, to be known as factors, which are the independent variables upon which the observed scores depend. Thus, if the score of individual $\underline{p}$ on test $\underline{i}$ is $\underline{s_{pi}}$, and $\underline{f_{p1}}$, $\underline{f_{p2}}$, $\underline{f_{p3}}, \ldots,$ $\underline{f_{pk}}$ are his scores on the $\underline{k}$ factors assumed to exist, then $\underline{g_i}$ is some function, possessing suitable analytic properties, such that

$$s_{pi} = g_i(f_{p1}, f_{p2}, \ldots, f_{pk}). \tag{1}$$

In order to justify the utilization of $\underline{g}$ as though it were linear in the $\underline{f}$'s, Burt (2) has suggested that equation 1 be expanded into Taylor's Series and that, as a first approximation, all the non-linear terms in the $\underline{f}$'s be dropped. There is a somewhat more satisfying alternative derivation. Equation 1 may be expanded into a series of orthogonal functions[2] of the $\underline{f}$'s, the existence of which is adequately demonstrated by the sequel. In fact, for any given set of $\underline{n}$ tests not more than $\underline{n}$ orthogonal functions will be required, and there will generally be an infinite number of sets of $\underline{n}$ such functions such that the same set is used in ex-

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[1] Guttman (25) has classified psychological scores into qualitative and quantitative, and has discussed the problem of predicting a criterion of either type from either type of data. Our second class of scores is treated by Guttman as qualitative, under the assumption that all equal-appearing scores are equivalent, but he implicitly ignores this assumption in the further development of an "intensity function" for his scales (28).

[2] Two functions are said to be "orthogonal" if, for some given population, the correlation computed by pairing corresponding values of the functions (in this case those associated with the same individual) is zero.

panding each of the $\underline{g}$'s.[3] Let us simply rechristen our orthogonal functions as $\underline{a_1}$, $\underline{a_2}$, ..., $\underline{a_n}$; and let their coefficients in the expansion of $\underline{g_i}$ be $\underline{b_{1i}}$, $\underline{b_{2i}}$, ..., $\underline{b_{ni}}$.[4] Then we have that

$$s_{pi} = \sum_{j=1}^{j=n} a_{pj} b_{ji} \quad , \tag{2}$$

or, in matrix notation,

$$S = AB. \tag{3}$$

An expression of the form of either equation 2 or equation 3 is often assumed as the starting point in the development of factor analysis. Thus, for Cattell (5), an individual's score is taken to be the sum of products of "personality indices" times "situational indices," and for Thurstone (62), as the sum of products of "scores for the subjects on factors" times "loadings of the tests on corresponding factors." However, we wish to reserve the term "factor" for the $\underline{f}$'s defined earlier, and will refer to a set of orthogonal functions of factors as a set of "components."

Now, if we have been provided with the elements of a matrix S in the form of first-class scores, for which addition and multiplication are defined by the usual rules of arithmetic, we may define a matrix R by the equation,

$$R = S'S. \tag{4}$$

Since A contains orthogonal columns, this may be reduced to

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3 The rotational problem in factor analysis may be conceived as the choosing of some particular set of such functions, although the functional forms do not become known as the result of rotation. The particular sets of functions which will satisfy the requirements of orthogonality will depend on the particular distribution functions which may be associated with the $\underline{f}$'s. With Burt, we must still hope that the dominant functions in such series will actually be linear in the $\underline{f}$'s — when the functions have been chosen by some such criterion as simple structure. Of course, if the distribution of the $\underline{f}$'s is normal, then all the quadratic terms in the Taylor expansion of the general score function may be thrown into the orthogonal set with the linear terms, and it may be possible to cross-identify such functions (50).

4 The $\underline{b}$'s are the factor loadings familiar to the psychologist, who more commonly uses the letter $\underline{a}$ to denote them.

$$R = B'B. \tag{5}$$

In the event that the scores of S were given in standard form (having zero mean and unit variance[5]), the matrix R will be the matrix of product-moment "correlation" coefficients. In any event, R is a Gramian matrix of relational coefficients for every pair of tests. Proof of these results may be found, for example, in a paper by Guttman (26).

Thus, there are usually two distinct approaches to the estimation of B, which is often the object of psychological interest in such a system. Either S may be factored directly into AB, or R may be factored into B'B. Both of these approaches have been recognized and discussed at the theoretical level by such workers as Young (16, 67, 32, 68, 69, 70) and Guttman (24, 25, 26), although only the methods of factoring R into B'B have been utilized in practice. We may note that either method yields the same set of solutions for B, which is determined by the information available only to within rotation. No advantage in determinism is gained by the direct method of factoring S, since the extra information in S is used to fix the elements of A; when communalities must be estimated from the data, the maximum number of determinate factors for either approach is (59)

$$K = \frac{2n + 1 - \sqrt{8n + 1}}{2}. \tag{6}$$

If B is the sole object of interest, R-factorization is computationally more efficient; if A is also of interest, then S-factorization may be utilized so as to yield simultaneously the results of both "R-technique" and "Q-technique" (5).

It is the purpose of this thesis to discuss practical procedures for the direct factorization of S into A and B, placing special emphasis on two cases in which R is not defined. One of these cases will arise when we are given a

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[5] The process of score standardization is customary in psychological work, where the absolute zero and unit are unknown. The process possesses the advantage of giving a dimensionless score, and the results of all further computations *may* be treated as pure numbers. Cf. also the relation of this process to analysis of variance, considered below.

three-way (three dimensional) matrix of scores, and the other when we are given a matrix of second-class scores.

B Optimal Factorization of the Score Matrix

Eckart & Young (16, 67) have considered the problem of factoring S into AB in terms of the equivalent problem of approximating S by a matrix of lower rank, $S_{\underline{k}}$. If the divergence function to be minimized in the course of such fitting is taken as the sum of squares of the elements of $S_{\underline{k}} - S$, then the problem is shown to have a solution. The properties of this solution have been set forth by these authors in a series of propositions, which we shall summarize here without proof.

If B_o is the factor matrix containing all the principal components of R, and A_oB_o equals S, then A_o and B_o are the key to any desired solution for $S_{\underline{k}}$. Thus, if $A_{\underline{k}}$ stands for the matrix consisting of the first $\underline{k}$ columns of A_o, and $B_{\underline{k}}$ stands for the matrix consisting of the first $\underline{k}$ rows of B_o, then

$$S_k = A_kB_k \tag{7}$$

is the best rank $\underline{k}$ approximation to S, in the sense of least squares. Of course, $A_{\underline{k}}$ and $B_{\underline{k}}$ may be rotated without affecting $S_{\underline{k}}$, and we may represent this by writing

$$S_k = (A_kV^{-1})(VB_k) \,, \tag{8}$$

where V is any non-singular $\underline{k}$ by $\underline{k}$ matrix. Geometrically, the various best-fitting linear subspaces, S_1, S_2, S_3, ..., $S_{\underline{k}}$, ..., $S_{\underline{n}}$, possess the property of being "nested," each one being contained in all those with more dimensions. This means that factors may be extracted one at a time,[6] and that as soon as an adequate degree of fitting is achieved the computations may immediately proceed to the rotational stage.

Actually, if S is exactly of rank $\underline{r}$, with $\underline{r}$ less than or equal to $\underline{n}$, various

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[6] Factors, or components, may also be extracted several at a time, according to various plans. See, for example, Guttman (26), Householder & Young (32), Horst (30), Lawley (39), Thurstone (62), and Woodrow & Wilson (66).

procedures may be used to determine an $A_{\underline{r}}$ and $B_{\underline{r}}$ which are transformable by non-singular transformations into A_o and B_o — this is the force of a theorem proved by Guttman (26, Thm. 1), and the justification for using such methods of factor analysis as the centroid. However, in almost any practical example we must admit the existence of chance errors of measurement, concerning which we have said nothing up to this point; S will generally be exactly of rank $\underline{n}$, and the problem is to determine an $S_{\underline{k}}$ with minimum $\underline{k}$ such that the divergence may be attributed solely to the effects of the chance errors. Since, when computing the divergence, it may be desirable to weight the scores according to their relative reliability, the solution is no longer given by dropping the last $\underline{n} - \underline{k}$ principal components from A_o and B_o.

According to Young (70), if H is a unit rank weighting matrix such that

$$D_N(1)D_n = H , \tag{9}$$

where D_n and D_N are non-singular diagonal matrices, and if

$$D_N S D_n = A\,B , \tag{10}$$

then

$$A_o B_o = D_N^{-1}\,A\,B\,D_n^{-1} . \tag{11}$$

In other words, the exact solution for A_o and B_o is unchanged within rotation. On the other hand, the number of components required to reduce the divergence to a chance level will generally depend on the particular H which is employed. Young has suggested that an appropriate H may be ascertained in advance, presumably as a function of known test and subject reliabilities obtained through replication of the experimental measurements. However, this may be impractical or even undesirable. It will be undesirable if the communality concept is to be retained when factoring score matrices,[7] for it is possible to determine through repeated factorizations an H which will minimize the divergence function for any

[7] The complementary statement also holds — that it will be quite desirable to fix H in advance if the communality concept is to be shelved (cf. 49).

predetermined rank of S_k. Of course, since it is only the proportions of the weights which are to vary, each successive estimate of H must be adjusted so as to represent the same total weight; the sums of the elements of D_n and D_N should be n and N, respectively. The solutions obtained for successive increments in k are analogous[8] to the corresponding solutions obtained in R-factorization by Lawley's maximum likelihood method (39, 40). The actual relation expected to exist between the maximum likelihood communalities of R-factorization and the elements of D_n in S-factorization is that

$$d_{ii} = 1 / \sqrt{1 - h_i^2} , \qquad (12)$$

and the elements of D_N are similarly related to the communalities which are obtainable through "Q-technique" analysis of the same data.

There remains only one missing link in the exposition of the optimal procedure for the direct factorization of score matrices; this link is most readily derived from an optimal procedure used in the factorization of R-matrices. According to Hotelling (31), and Frazer (21), if any arbitrary row vector is repeatedly post-multiplied by R to yield successive new row vectors, a limiting state is reached when each vector is equal to the preceding one multiplied by the dominant characteristic root[9] of R. When this stage is reached, the elements of the vector are proportional to the first row of B_o, and the proportionality constant is the square root of the scalar product of the last two vectors in the succession. Now, since R equals S'S, we may imagine that this process is carried out by post-multiplying the arbitrary vector first by S', then by S, then by S', then by S, etc.[10] The process must converge in the same fashion, giving one vec-

[8] The solutions obtained are not equivalent, since in the analysis of the score matrix optimum communalities are determined not only for the tests, but also for the subjects, whereas in the analysis of an R-matrix the subject communalities are automatically taken to be equal.

[9] The characteristic roots of R are the solutions to its characteristic equation (21), a polynomial equation expressable by setting the determinant R - xI equal to zero. Since R is Gramian, these roots are real and non-negative (26); the largest one numerically is called the dominant characteristic root.

tor proportional to a row of B_o, together with a second vector proportional to the corresponding column of A_o. The proportionality constants may be determined in the same manner as before.

Precisely what, then, is the essential distinction between factor analysis applied to the score matrix and factor analysis applied to the R-matrix? It cannot lie in the numerical results, for we have already seen that score matrix principal components and R-matrix maximum likelihood components are practically indistinguishable aside from rotation. The difference lies, rather, in the techniques used for evaluating the results as acceptable or inacceptable in terms of the ideal model being fitted. These techniques are based on different assumptions concerning the model. Thus, in the analysis of R-matrices, it is customarily assumed that the distribution of <u>scores</u> on each component is normal for the population from which the sample was drawn at random, and that therefore the usual formula for the standard error of a zero correlation coefficient drawn from a normal multivariate population can be applied. This provides us with Lawley's Criterion for the number of significant factors (39). On the other hand, following the direct analysis of a score matrix a direct comparison of predicted and measured scores is made, and it need only be assumed that the <u>error</u> of the single measurement is normally distributed, with a mean of zero and a standard deviation given by the appropriate cell of the unit rank <u>un</u>weighting matrix, $D_N^{-1}(1)D_n^{-1}$. No assumption is required concerning the actual distribution of true scores (cf. 70), and hence subjects may be drawn from any population in any manner without biassing the test. (This does not imply that selection will not produce its effects on the numerical results of the factor analyses, but merely that, so long

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[10] If we were carrying out a centroid analysis, which would usually be almost as efficient in extracting variance as the principal components described, we should retain only the sign of each element of each successive vector (using only values of plus or minus one for multiplication) and stop the multiplications as soon as the signs reached a stable configuration. Group centroid methods (in which multipliers of zero would be permitted), and others, can also be described.

as the selection is not total, the problem of determining the significance of a given number of components is not affected.) Slater (51), in advocating the direct factor analysis of frequencies of association of qualitative attributes in lieu of some coefficient more remote from the experimental observations such as the tetrachoric coefficient, expresses essentially this same thought.

Now, the actual evaluation of the amount of error or divergence in score fitting which can be tolerated is made possible by recourse to analysis of variance. That factor analysis of the score matrix is very closely related to analysis of variance has recently been demonstrated by Burt (3), by means of an example. In particular, we may observe that the reduction of test scores from raw to standard form is equivalent to the subtraction of the main effect and column effects from the analysis of variance table of raw scores. (The simultaneous presentation of the scores with unit variance is equivalent to the assumption that the diagonal elements of D_n will be equal.) The extraction of factors or components from the table of residuals is then equivalent to the hypothesizing of certain classifications of tests and persons such that these classifications shall show the most significant possible interaction effects, and the successive subtraction of these effects.

Tests of significance for interaction variances are given by the ordinary analysis of variance procedures, and are valid when the bases of given interactions being tested have been established <u>a priori</u>; in factor analysis, however, these bases have themselves been determined from the data, and they have been determined in such a way as to maximize the apparent significance of the interaction variance associated with each successive factor. Therefore, the ordinary tests of significance must be modified in a manner calculated to make them more stringent. An approximate method of accomplishing this, which is probably adequate for the needs of most factor analysts, may be given. We shall present it here via an intuitive, heuristic approach.

The statistic to be computed from the data is the usual one, being the ratio of the variance per degree of freedom for the interaction under consideration to the variance per degree of freedom for the residual left after removing the interaction effect. The degrees of freedom are counted in a straightforward manner, which is described by Burt (3). This computed statistic is now compared with the appropriate value taken from the F-table (53) <u>after</u> the value taken from the table has been multiplied by a factor yet to be described. Now, a certain amount of variance may be associated with each degree of freedom of the total sum of squares prior to the isolation of the interaction variance to be tested. If only chance effects are operative, these variances should be distributed according to chi-square with one degree of freedom. Therefore, if we have exactly r total degrees of freedom, we may associate with each one of them one-rth of the area under the chi-square distribution curve, in such a way that these areas are bounded by vertical lines. Say k degrees of freedom are associated with the estimate of interaction variance to be tested. By maximizing this estimate of variance we have attempted to choose the k pieces of area from the upper end of the chi-square distribution. We wish to compare the mean of these pieces with the mean of the $r - k$ pieces chosen from the remainder of the distribution. The ratio of these two means is then the factor to be multiplied by the tabular F value before comparison with the computed statistic. In this manner we may determine whether the observed statistic is one likely to have occurred in the absence of real interaction. It must be specified that none of the r degrees of freedom should be associable with zero variance, but this is a situation which chance error is extremely unlikely to permit.

This completes the summary of direct factor analysis for the simplest case in which first-class scores are given. Let us now document the advantages of this particular approach to the factor analytic problem. We shall explore very briefly the problem presented by three-way score matrices, and shall consider in some detail the problem of two-way score matrices which contain only second-class score information.

C Three-way Score Matrices

A direction of generalization which is immediately suggested, both by the idea of score matrix analysis and by its simple relation to analysis of variance, is the analysis of *p*-way score matrices. This generalization may be discussed in terms of the three-way problem, which corresponds to the analysis of second-order interaction in analysis of variance terminology. Suitable data will be presented when each observed score is associated with three indices (or subscripts), and several practical examples of such combinations of indices may be mentioned in order to indicate the potential importance of the case. One possible combination of indices (i) would serve to identify for each score (a) the individual earning it, (b) the test on which it was earned and (c) the occasion on which it was earned (cf. 5). If the tests constitute a series of comparisons, then another suitable score matrix would contain elements identified according to (a) the judge making the comparisons, (b) the first member of the comparison and (c) the second member of the comparison. The first members and second members of the comparisons might actually (ii) be taken from the same group according to the psychophysical method of "paired comparisons" (cf. 27, 36), or the groups might be mutually exclusive. In the latter event (b) would represent the entities judged and (c) would represent the bases for judgement; for instance, (iii) persons could be rated on personality traits (5) or (iv) concepts could be rated on defined scales (55, 56, 57).

Using the general approach of the two-way case we must begin by assuming the existence of factors, but we now assume that their operation is expressed through a trilinear score function,

$$s_{pqr} = \sum_{i=1}^{i=k} a_{pi} b_{qi} c_{ri} \,. \qquad (13)$$

In order to provide a psychological rationale for such a function we may characterize the coefficients of each triple product somewhat as follows: $\underline{a_{pi}}$ is the score of the object (or individual) measured on the *i*th orthogonal function of the factors, $\underline{b_{qi}}$ is the loading of the *q*th measurement-type on the component, and $\underline{c_{ri}}$ is the sensitivity of the *r*th measuring instrument (or judge or occasion) for variation on the component. Usually the sensitivities will be expressible using only positive coefficients,[11] while the other two sets of coefficients may include both positive and negative numbers. Therefore, we shall provisionally associate the sensitivities with the non-signant (47) index of the score matrix, and shall conduct the following discussion for this case.

Now, as an approximation, we may assume that all the sensitivities have a constant numerical value, which may therefore be taken out from under the summation of equation 13. We are left with a bilinear score function that is amenable to analysis in the familiar fashion. This approach to the three-way matrix is comparable to the analysis of the two-way matrix through its R-matrix, since the possibility of weighting one of the indices unequally is ignored in each event.

On the other hand, we may choose to take a weighted sum of the *r*-layers of the three-way matrix, and to analyze this sum as a two-way matrix. We will, naturally, choose those proportions of weight which will cause the component extracted from the two-way matrix to be the most "significant," subject to the restriction that these "non-signant weights" may not be negative; the weights will be proportional to the amounts of variance explained in the different *r*-layers by the component. In this manner we arrive at a combination of three vectors which are proportional to the first principal component of the three-way matrix; we will write them in as the first columns of each of three two-way matrices, A_o,

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[11] For some three-way score matrices it may be psychologically more reasonable and meaningful to decide that the loadings of the measurement-types (the *b*'s) must be positive rather than the *c*'s. The only necessity is that one or another of the three classes of coefficient must be limited to non-negative values.

B_o and C_o. Appropriate combination of these vectors, which are A_1, B_1 and C_1, will provide a rank 1 approximation to the original score matrix. Suppose that we write

$$ {}_3S_1 = {}_3(A_1, B_1, C_1) \tag{14} $$

to represent this combination, where the prescript 3 indicates a three-way matrix. Equation 13 would have appeared in this notation as

$$ {}_3S = {}_3(A_o, B_o, C_o) , \tag{15} $$

where A_o, B_o and C_o are two-way matrices of component scores, measurement loadings and sensitivities, respectively. Equation 15 may be taken as defining the operation symbolized by ${}_3(\quad)$, and it will be noted that this definition is made only when all of the enclosed matrices have the same number of columns. One of the enclosed matrices is non-signant.

We may now form the residual matrix equal to ${}_3S - {}_3S_1$, and treat it as an original score matrix in order to determine a second component. We may, in fact, remove a succession of components from ${}_3S$ in this manner. What rotational freedom may we be permitted in seeking a solution likely to exhibit psychological invariance? The simple structure concept (62), and its derivatives (4, 62), will ordinarily be applicable to either A or B, i.e., to one of the signant matrices. It would appear that we may determine a transformation from either A_o or B_o by any of the usual methods (62). If this transformation is orthogonal, then complementary transformations may be applied to the other component matrices so that the combination will still yield ${}_3S$ to the same degree of approximation as before rotation. In other words,

$$ {}_3S = {}_3(A_oV, B_oV, C_oW) , \tag{16} $$

where V is any orthogonal transformation matrix and W is obtained from V by squaring each element.

As before, the actual significance of a given number of components will be determined by recourse to analysis of variance terminology, since we are now analyzing second-order interaction.

The geometrical interpretation of equation 16, however, presents a novel feature. Corresponding to the two-way case, the rows of matrices A and B may be associated with vectors in a <u>k</u>-space, so that the average score of a given object or individual on a given type of measurement is given by projecting the vector of the object onto the vector of the measurement, assuming that the latter has unit length. The rows of C, which are associated with the separate measuring instruments (or judges), do not represent further vectors inserted into this space; instead, they represent a series of homogeneous linear deformations of the space and the vectors contained in it. If the space is "hyperspherical" for a judge averagely sensitive to variation in all components then, for other judges, the amount by which a given component is stretched or shrunk gives the relative increase or decrease in the importance of that component in determining the judgements of that judge.

D Second-class Score Matrices

This is the second direction of generalization to be undertaken, and we shall devote the remainder of our attention to this case. The case is one of particular importance, for those few assumptions which shall have to be made are precisely those which it is most reasonable to make in many areas of both psychological and sociological interest. Thus, in the construction of items for a personality questionnaire or attitude-measuring instrument it is desirable, in order to minimize the effects of response-bias (13), to force the respondent to choose between only two possible alternatives, even though the psychologist or sociologist would prefer to utilize a continuum of measurement and is fully prepared to assume the existence of such continua. Even for intelligence test items the existence of such continua is recognized (11), for it is not true that all of those passing a given item do so with equal facility, nor that all those failing the item do so with equal force.

Despite the recognition that responses to such forced-choice items are imperfect measurements on a continuum, sets of such responses are commonly represented by score matrices containing entries of 1 and 0, or sometimes 1, 0 and -1 in case a "middle" category is permitted. There is nothing wrong with this except the tendency to accept the implication that these symbols may then be treated arithmetically as though they were the numbers which they ordinarily represent. Accepting the implication, three general directions of analysis are then possible. The first assumes that the items belong to a single continuum and attempts to determine numerical scale values for each item (60). The second makes no further assumptions, and attempts to estimate weights for each item which will permit the maximum discrimination among individuals (19, 25). The third approach computes measures of relationship between pairs of items, such as frequency of association (51),[12] tetrachoric correlation coefficients (10, 44, 22, 42),[13] phi coefficients, or phi coefficients divided by their maximum value (15); the resulting R-matrices are then subjected to ordinary factor analysis despite the tendency to introduce spurious "difficulty factors (23, 20, 52)" in some of these procedures.

In keeping with the basic assumptions which we are willing to make concerning the nature of second-class score data, we shall use the symbols + (plus) and - (minus) to represent the responses in a score matrix. Since there is often some doubt as to the meaning of "middle" responses, when these have been permitted, we shall leave the cells corresponding to such responses blank. The opera-

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[12] The matrix of frequencies of association is precisely the R-matrix when the scores have been given as 1 and 0, so that the R-factorization of this matrix is justified for truly qualitative data.

[13] Alone among these methods, the use of tetrachoric correlation coefficients does not depend on the simple arithmetic manipulation of the symbols 1 and 0, and correctly treats the scores as meaning "greater than" or "less than" some arbitrary value. However, the use of tetrachoric correlation does assume the normality of the distribution of true scores for the population sampled, and it will be desirable to devise a factor analysis of second-class scores which is also independent of this assumption.

tions which we may perform with these symbols are definitely limited, although they will be adequate for the purpose. In the first place, we may recognize the identical appearance of two $\neq$'s or two $-$'s when they appear in the same column, although we cannot recognize such identity when the symbols appear in different columns. In the second place, we may label the rows of the score matrix so that they may be recognized at any time and then rearrange the rows into any order which may be of interest. Similar reordering of the columns would be permissible, but we will have no occasion to utilize this operation. By studying the various possible configurations of the symbols when the rows are reordered (producing new arrangements in the columns) it will be possible to obtain the equivalent of an unrotated B matrix, and this may then be rotated in the ordinary fashion. No matrix equivalent to A will be obtained, however.

Suppose, for the time being, that we are actually able to assign a rank order to the individuals on the basis of their scores on a given measurement. (Since the probability of tied ranks in a finite sample from a continuous population is zero, we shall ignore this possibility.) Kendall (34) has suggested a coefficient, tau, for expressing the relation between the ranking given in this way and some other possible ranking of the same individuals. If there are $\underline{N}$ individuals involved, tau may be determined as follows:

1) Arrange the individuals in order according to either of the sets of ranks.

2) Find the total number of "inversions" now present in the other set of ranks by counting the number of smaller numbers falling to the right of each number, and summing for all the numbers in the sequence. Call this E.

3) Compute tau from the formula

$$\text{tau} = \frac{N(N-1) - 4E}{N(N-1)} . \tag{17}$$

Several equivalent methods for determining tau are given by Kendall, who has given some attention to the statistic (38). Tau varies, like the ordinary correlation coefficient, between plus and minus unity. Partial tau's may be computed in the ordinary fashion (37). If the population of sequences from which two are drawn for computation of tau contains equal proportions of every possible permutation, then the distribution of tau under the null hypothesis is practically indistinguishable from the normal curve for N greater than 5 (34), and is appreciably more normal than Spearman's rank order correlation, rho (54). The joint distribution of tau's and rho's computed from the same pairs of sequences shows an extremely high correlation, but it is not perfect and the relation is distinctly curvilinear for small N (35). For this latter reason, tau does not offer a conventional measure of the strength of a relationship in the event that the null hypothesis cannot be maintained for a given pair of sequences.

We shall define, for our further use here, the correlation between two sequences as the real part of the expression

$$\sqrt{\frac{N(N-1) - 4E}{N(N-1)}} - \sqrt{\frac{4E - N(N-1)}{N(N-1)}} \quad . \tag{18}$$

In other words, the sign of the correlation is the same as the sign of tau while the absolute value of the correlation is the square root of the absolute value of tau. Our object in making this definition is to permit the association of sequences with vectors in such a manner that the vectors will possess the familiar properties required for rotation. A formal proof of these properties will not be undertaken here, but the direction such a proof might take will be outlined.

Consider an orthogonal reference frame of sequences such that the E required to go from any one to any other is $\underline{N(N-1)/4}$. If the number of inversions required to convert an arbitrary sequence, $\underline{m}$, into the $\underline{i}$th reference sequence is $E_{\underline{mi}}$, then

$$E_{mj} \geqslant N(N-1)/4 - E_{mi} \tag{19}$$

or

$$E_{mi} + E_{mj} \geqslant N(N-1)/4 \quad . \tag{20}$$

By generalizing this to cover all k of the reference sequences, we should find that

$$\sum_{i=1}^{i=k} E_{mi} \geq (k - 1)\, N(N - 1)/4 \quad , \tag{21}$$

whence

$$\sum_{i=1}^{i=k} \left(\frac{N(N - 1)}{4} - E_{mi} \right) \leq \frac{N(N - 1)}{4} \quad . \tag{22}$$

Expressing this in words: The sum of the "inversion-variances" explained (or accounted for) by the individual independent reference sequences may not exceed the total inversion-variance existing in the arbitrary m. If we divide expression 22 by its right hand side, we find that the sum of the squares of the correlations of the arbitrary sequence with the reference sequences, where these correlations are as given by expression 18, is less than or equal to one. This is precisely the property of ordinary correlations which is used in factorial rotation.

Returning to our score matrix, we shall now seek an ordering of the rows such that the total amount of inversion-variance accounted for in the columns of the matrix is maximized. We shall then seek further orderings, each uncorrelated with those preceding, such that each will explain the greatest possible part of the remaining inversion-variance. Two processes will be required: (a) a method of estimating the amount of inversion-variance which is explained by a given ordering of the rows, and (b) a method of finding a better ordering of the rows, if one exists. We shall consider these problems in order.

Suppose that we are given a set of N elements which, if we are given complete information, can be put into a unique rank order or sequence. Let us mark p of the first r elements with the symbol +, and mark q of the remaining N - r elements with the symbol -; finally, let us arrange the elements into some arbitrary permutation. Now, by means of single inversions of adjacent elements, and with knowledge of the correct rank values, we proceed to put the elements back in order. Only certain of these elementary inversions will be visible to an observer who can see only the special markings, and such an observer cannot decide whether the interchange of two +'s represents a net increase or decrease in E. Similarly, he can make no decision when two −'s are inverted, or in any case in which either of the elements is unmarked by + or −. The probability that a given elementary inversion will be "visible" is

$$2pq/N(N-1) \;, \tag{23}$$

and if $N(N-1)/4$ total inversions are required to go from one sequence to another, $pq/2$ of them will probably be visible. For a given column of the score matrix $pq/2$ is the amount of visible inversion-variance to be accounted for, and substitution of this value for $N(N-1)/4$ in expression 18 will give the correlation of a particular item with the particular arrangement of rows in the score matrix, as follows:

$$b_{mi} = \text{real part of } \sqrt{\frac{pq - 2E_v}{pq}} - \sqrt{\frac{2E_v - pq}{pq}} \;. \tag{24}$$

The practical use of expression 24 involves the assumption that the number of visible inversions represents a constant proportion of the total number of inversions, even though the practical ordering of the rows of the score matrix must necessarily depend only on visible information. The validity of this assumption must be judged afresh in each practical problem on the basis of the number of items taken for simultaneous analysis and the range of "difficulties" which they encompass; many items, with a wide range of difficulty, will characterize the most nearly ideal situations. To the extent that the assumption is invalid, the experimentally determined communalities will tend to be too large.

Now, in determining an improved sequence, each item is to be considered in proportion to its correlation with the sequence already obtained. This may be accomplished by computing a "total inversion-weight" for each row of the score matrix in any manner such that the difference between the weights assigned to +

and - of the same column is proportional to the correlation of that column with the sequence. (If there are blank cells in the score matrix it will be necessary to compute analogous "average inversion-weights.") Of the many equivalent sets of inversion weights that are possible, the one easiest to compute is obtained by simply adding together for each row the correlations associated with its +'s.[14] The rows are then rearranged into a new sequence which is given by the descending order of magnitude of the total inversion-weights.

Thus, we now have two rules for computational use. The first tells us how to post-"multiply" an arbitrary sequence by the score matrix containing +'s and -'s in order to obtain a collection of loadings for the items. The second tells us how to post-"multiply" the collection of loadings by the transpose of the score matrix in order to obtain a new sequence. We may carry out a factor analysis using only these two rules of manipulation, as will be demonstrated in the next chapter. The result will be a matrix A, containing a number of sequences, and a matrix B, which is similar to those obtained from first-class data. If it will serve any purpose, we might now define the product S'S to be equal to the product B'B.

Although it does not appear to be profitable to pursue a detailed investigation of the role of experimental error in this sort of analysis[15] until a more substantial backlog of practical experience with it has been accumulated, the problem of the end-point of component extraction can be considered briefly. Certainly, if we are provided with only second-class scores we should not expect to be able statistically to extract as many components as if we were given an equal

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[14] The computational procedure at this point is operationally equivalent to the post-multiplication of the vector of correlations by the transpose of the score matrix written with 1 for + and 0 for -. The resulting numbers, however, are not considered to be meaningful beyond the determination of a new sequence.

[15] We may assume that, by definition, the responses given to a personality questionnaire (9) and certain of the other data to be analyzed by the method are perfectly reliable, except for function fluctuation (8).

number of first-class scores. Suppose we roughly evaluate the amount of information our scores contain in terms of the number of discriminations between individuals which can be made (19). Assuming no ties, the complete information given by a sequence-component or by first-class scores on an item permits $N(N - 1)/2$ discriminations; second-class information of the sort we have been considering permits only $p_i q_i$ discriminations on the basis of the *i*th item. Using all of the items, we may put an upper bound on the number of components at

$$\frac{2}{N(N-1)} \sum_{i=1}^{i=n} p_i q_i \quad . \tag{25}$$

(The actual limit on the number of components will be less than this because of the information required to determine the loadings of the items on the components along with the component-sequences.) The problem of factor analysis is to determine how much of this information is redundant, and to summarize all of the meaningful information in terms of a number of components which is smaller still. The principle that any properly determinate component must account for at least $N(N - 1)/2$ of the original discriminations will be applied in the illustrative example to be described.

So far, we have considered only the situation in which just two zones may be experimentally distinguished on the continuum assumed to underlie the responses or scores analyzed. In some situations it may be possible to distinguish three, or four, or even more. The limiting result of such finer differentiation is, of course, first-class data. Suppose, in general, that it is possible to distinguish *d* zones of response and to place these zones in their correct rank order. Then the number of discriminations made by the item is

$$\sum_i \sum_j p_i p_j \; , \tag{26}$$

where $1 \leq i < j \leq d$ and pq is the number of responses in the *i*th zone. When visible inversions are counted for a configuration of such scores no attention is paid to the number or size of the zones supposed to intervene between two scores, just as no attention is paid to rank differences when counting the inversions in a series of first-class data. We may then obtain expression 27 as a generalization of expression 24, and expression 28 as the corresponding generalization of expression 25, as follows:

$$b_{mi} = \text{real part of} \sqrt{\frac{\sum_{i<j} p_i p_j - 2E_v}{\sum_{i<j} p_i p_j}} - \sqrt{\frac{2E_v - \sum_{i<j} p_i p_j}{\sum_{i<j} p_i p_j}}, \qquad (27)$$

and

$$\text{Maximum } k = \frac{2}{N(N-1)} \sum_{m=1}^{m=n} \sum_{i<j} p_{im} p_{jm} . \qquad (28)$$

This brings to an end the theoretical discussion of what may be termed "K-way Scale Analysis." In the next chapter we shall consider the actual computational steps in terms of a real miniature example. In the final chapter we will briefly consider its relation to certain other popular techniques for the analysis of similar data.

III A MINIATURE EXAMPLE OF K-WAY SCALE ANALYSIS

A Numerical Processes

The major incentive provoking this investigation was the need for a method, both reasonable and practicable, for analyzing the responses made by individuals to items contained in personality questionnaires. Such items are generally essentially forced-choice in form.[1] Also, for such items it is reasonable to assume the existence of continua along which the subjects judge their own position relative to the perceived meaning of the items. While this meaning may or may not be constant, an item nevertheless is a reproducible stimulus and any variations in the responses made to it are ascribable to the individual differences of the persons responding. Furthermore, an item may be assumed to divide the continuum at the same point for each individual, so that we may therefore classify forced-choice responses to personality items as second-class scores. The omnibus definition of the ? category of response, even if this definition had not been explicitly made for the subjects, creates serious doubts as to its place on the same continuum with Y and N for most items, and we will have to give separate consideration to these responses.

Data were collected[2] for a large number of personality items from a large

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[1] The following extract is taken from the instructions typically employed by the present author in administering personality questionnaires: "Corresponding to each question you will find three possible answers on your answer sheet: 'Y', '?' and 'N'. If your answer to a question is 'yes,' circle the 'Y'. If your answer is 'no,' circle the 'N'. On the items for which two choices are given, the 'Y' and 'N' are given with the question. You should use the '?' as an answer to represent any other response you might wish to make -- such as 'in between,' 'don't know,' 'doesn't apply,' 'won't say,' etc. However, you should not answer more than one question in ten with the '?'." The actual number of responses in the ? category averages about 5-6%, although individual subjects may run as high as 40% and individual items as high as 25%.

[2] The data were collected from students in all Urbana Departments of the University of Illinois during the Spring Semester, 1949-1950. The group tested comprised about 25% of all students whose University Identification Number, assigned by the Recorder at the time of admission, ended in the digit 6, which was chosen at random. This group includes about 60% of all students whose Identification

number of subjects, and it is proposed eventually to apply K-way scale analysis to the entire body of information. For the purposes of the present discussion, a miniature example which can be worked through in its entirety has been abstracted from the data. Even so, the material has been chosen in such a way that the results may be evaluated psychologically, and they may not be without interest in their own right. Thirteen items were taken to be representative of a "universe of content." The data for 401 subjects were then divided into two groups according to whether or not ? responses had been given to any of these items. From the group of 238 subjects who had answered every one of the items either Y or N, a random selection was made by choosing every subject whose test identification number[3] ended in 3 or 5. This yielded a group of 45 subjects whose scores on the 13 items are taken as the miniature example.

The scores of these 45 subjects are set forth in Table 1. On the basis of psychological content (which also determined the admissibility of the items to our "universe of content") it was easy to predict the direction in which each of the 13 items should be scored in order to correlate positively with any general component. Each item was scored in this direction, and the items were then numbered for the example in order of increasing proportion of positive scores. (This numbering had been performed utilizing the whole group of 401 subjects, and the sampling procedure has introduced the apparent discrepancies in this order.) The subjects are listed in the table in the order of their identification numbers. This may be taken as an arbitrary sequence from which successively better approximations will lead towards an ordering which maximizes the total contribution of

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Number ended in the combination 26, which was chosen because it was judged that these students had been offered the most convenient times for testing in the original schedules.

[3] The tests which had been completed at the time bore numbers ranging from 401 through 705, these numbers having been assigned in the order in which the subjects showed up to take the tests. Very possibly there are relations between these numbers and personality variables, but these are sufficiently weak that the method of "random" selection should not unduly bias the miniature sample.

Table 1.
Original Score Matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	
305	-	+	-	+	-	+	+	+	+	+	+	-	+	9.00
315	-	-	-	-	-	+	+	-	+	-	+	+	+	6.00
325	-	-	-	-	+	+	+	+	+	+	+	+	+	9.00
333	-	-	-	-	-	-	-	+	-	+	-	+	+	4.00
343	+	-	-	+	+	+	+	+	-	+	+	+	+	10.00
345	-	-	+	-	-	-	+	-	+	+	+	+	+	7.00
353	-	-	+	+	-	-	-	+	+	+	+	+	-	7.00
355	-	-	-	-	-	+	-	+	-	+	+	-	+	5.00
385	-	-	-	+	-	-	-	-	+	+	+	+	+	6.00
393	+	-	+	-	+	-	+	+	+	+	+	+	+	10.00
395	-	-	-	-	-	-	-	+	+	-	+	+	+	5.00
403	-	-	-	-	-	+	-	+	+	+	-	-	-	4.00
405	-	-	-	-	-	+	-	+	-	-	+	+	+	5.00
423	-	-	-	-	-	+	-	+	+	+	+	+	+	7.00
433	+	+	-	-	-	+	+	+	+	+	+	+	+	10.00
445	-	-	-	-	-	-	-	-	-	-	+	+	+	3.00
455	-	-	+	+	+	+	+	-	+	+	+	-	+	9.00
465	-	-	-	-	-	-	+	+	+	-	-	+	-	4.00
475	-	-	+	+	-	-	+	+	+	+	+	+	+	9.00
485	-	+	-	+	-	-	+	+	+	+	+	+	+	9.00
495	-	-	-	-	+	+	+	+	+	+	+	+	+	9.00
513	-	-	-	-	-	-	-	+	-	-	+	+	+	4.00
523	-	-	-	-	-	-	+	-	+	+	+	+	+	6.00
525	-	+	+	-	-	-	+	+	+	+	+	+	+	9.00
533	-	-	-	-	-	-	-	+	-	+	+	-	+	4.00
535	-	-	-	-	-	+	-	-	+	-	-	+	+	4.00
543	-	-	-	+	-	+	+	+	+	+	-	-	+	7.00
553	-	-	-	-	+	-	-	-	-	+	-	-	+	3.00
555	-	-	-	-	+	-	+	+	+	-	+	-	+	6.00
563	-	-	-	+	-	+	+	+	-	+	+	+	+	8.00
565	-	-	-	-	+	-	+	+	+	-	-	+	+	6.00
575	-	-	+	-	-	+	-	+	+	+	+	-	+	7.00
583	-	+	-	+	+	-	+	-	+	+	+	+	+	9.00
585	-	-	-	-	+	+	-	+	-	-	+	-	+	5.00
595	-	-	-	+	+	+	+	+	+	+	-	+	-	8.00
603	-	+	-	-	-	+	+	+	-	+	+	+	+	8.00
613	-	-	-	+	-	+	+	+	+	+	-	+	+	8.00
633	-	-	-	-	-	-	+	-	+	+	+	+	+	6.00
643	-	-	-	-	-	+	+	+	+	+	+	+	+	8.00
645	-	+	-	-	-	+	-	+	+	+	+	-	+	7.00
655	-	-	-	-	-	+	+	-	-	+	+	+	+	6.00
675	-	-	-	+	-	+	+	+	+	+	+	+	+	9.00
683	-	-	+	-	-	-	+	+	+	+	-	-	+	6.00
685	-	-	+	-	+	+	+	+	-	+	+	+	+	9.00
705	-	-	-	-	+	+	+	-	-	+	+	+	+	7.00

the first component. Each computational step will now be listed and described.

1) As a first "guess," based essentially on the psychological content of the items, all the loadings were taken to be plus one, i.e., a value of plus one was associated with each of the thirteen columns of Table 1. An inversion-weight was calculated for each subject (row) by adding together the guessed loadings for each item on which the subject scored plus, and these are the figures given in the right hand margin of Table 1. The rows of the score matrix were then reordered from high to low according to these inversion-weights, producing Table 2. (In the cases of ties the rows were left in their original relative order since, as yet, there was no basis for altering it.)

2) An improved estimate of the loading of each item was computed from the sequences of pluses and minuses in its column of Table 2. The number of visible inversions in each column was counted by summing the number of minuses standing above each plus in the column. For example, for item two, the total number of visible inversions is

$$2 + 2 + 5 + 6 + 6 + 10 + 17 = 48 .$$

$\underline{pq}$ equals 7 times 38, or 266. Therefore, the new loading for the item is given by the computation (cf. expression 24 of Chapter I)

$$\sqrt{\frac{266 - 2(48)}{266}} = 0.80 .$$

The other loadings given in the bottom margin of Table 2 may be obtained in a similar fashion.

3) Improved inversion-weights were now computed for each subject (row), by adding together the loadings of the items on which he scored plus. These are given in the right hand margin of Table 2. Again the rows of the matrix were reordered from high to low according to these inversion weights, with the result shown in Table 3. In this and subsequent reorderings, rows having the same total inversion-weight were <u>inverted</u> relative to their previous standing, except where the rows involved actually show identical patterns of response. In the present

Table 2.
First Iteration

	1	2	3	4	5	6	7	8	9	10	11	12	13	
343	+	−	−	+	+	+	+	+	−	+	+	+	+	6.49
393	+	−	+	−	+	−	+	+	+	+	+	+	+	6.46
433	+	+	−	−	−	+	+	+	+	+	+	+	+	6.56
305	−	+	−	+	−	+	+	+	+	+	+	−	+	5.82
325	−	−	−	−	+	+	+	+	+	+	+	+	+	5.34
455	−	−	+	+	+	+	+	−	+	+	+	−	+	5.73
475	−	−	+	+	−	−	+	+	+	+	+	+	+	5.69
485	−	+	−	+	−	−	+	+	+	+	+	+	+	5.81
495	−	−	−	−	+	+	+	+	+	+	+	+	+	5.34
525	−	+	+	−	−	−	+	+	+	+	+	+	+	5.68
583	−	+	−	+	+	−	+	−	+	+	+	+	+	5.84
675	−	−	−	+	−	+	+	+	+	+	+	+	+	5.57
685	−	−	+	−	+	+	+	+	−	+	+	+	+	5.36
563	−	−	−	+	−	+	+	+	−	+	+	+	+	4.91
595	−	−	−	+	+	+	+	+	+	+	−	+	−	4.90
603	−	+	−	−	−	+	+	+	−	+	+	+	+	4.90
613	−	−	−	+	−	+	+	+	+	+	−	+	+	4.84
643	−	−	−	−	−	+	+	+	+	+	+	+	+	4.76
345	−	−	+	−	−	−	+	−	+	+	+	+	+	4.33
353	−	−	+	+	−	−	−	+	+	+	+	+	−	4.29
423	−	−	−	−	−	+	−	+	+	+	+	+	+	3.88
543	−	−	−	+	−	+	+	+	+	+	−	−	+	4.29
575	−	−	+	−	−	+	−	+	+	+	+	−	+	4.01
645	−	+	−	−	−	+	−	+	+	+	+	−	+	4.13
705	−	−	−	−	+	+	+	−	−	+	+	+	+	4.13
315	−	−	−	−	−	+	+	−	+	−	+	+	+	3.90
385	−	−	−	+	−	−	−	−	+	+	+	+	+	4.14
523	−	−	−	−	−	−	+	−	+	+	+	+	+	4.21
555	−	−	−	−	+	−	+	+	+	−	+	−	+	3.92
565	−	−	−	−	+	−	+	+	+	−	−	+	+	3.74
633	−	−	−	−	−	−	+	−	+	+	+	+	+	4.21
655	−	−	−	−	−	+	+	−	−	+	+	+	+	4.11
683	−	−	+	−	−	−	+	+	+	+	−	−	+	4.16
355	−	−	−	−	−	+	−	+	−	+	+	−	+	3.23
395	−	−	−	−	−	−	−	+	+	−	+	+	+	3.01
405	−	−	−	−	−	+	−	+	−	−	+	+	+	2.91
585	−	−	−	−	+	+	−	+	−	−	+	−	+	2.94
333	−	−	−	−	−	−	−	+	−	+	−	+	+	2.49
403	−	−	−	−	−	+	−	+	+	+	−	−	−	2.64
465	−	−	−	−	−	−	+	+	+	−	−	+	−	2.64
513	−	−	−	−	−	−	−	+	−	−	+	+	+	2.35
533	−	−	−	−	−	−	−	+	−	+	+	−	+	2.67
535	−	−	−	−	−	+	−	−	+	−	−	+	+	2.29
445	−	−	−	−	−	−	−	−	−	−	+	+	+	1.80
553	−	−	−	−	+	−	−	−	−	+	−	−	+	1.97
	100	.80	.68	.81	.58	.56	.88	.55	.66	.87	.73	.55	.52	

Table 3.
Second Iteration

	1	2	3	4	5	6	7	8	9	10	11	12	13	
433	+	+	-	-	-	+	+	+	+	+	+	+	+	6.99
343	+	-	-	+	+	+	+	+	-	+	+	+	+	6.89
393	+	-	+	-	+	-	+	+	+	+	+	+	+	6.93
583	-	+	-	+	+	-	+	-	+	+	+	+	+	6.46
305	-	+	-	+	-	+	+	+	+	+	+	-	+	6.31
485	-	+	-	+	-	-	+	+	+	+	+	+	+	6.37
455	-	-	+	+	+	+	+	-	+	+	+	-	+	6.25
475	-	-	+	+	-	-	+	+	+	+	+	+	+	6.22
525	-	+	+	-	-	-	+	+	+	+	+	+	+	6.22
675	-	-	-	+	-	+	+	+	+	+	+	+	+	5.99
685	-	-	+	-	+	+	+	+	-	+	+	+	+	5.74
325)	-	-	-	-	+	+	+	+	+	+	+	+	+	5.70
495)	-	-	-	-	+	+	+	+	+	+	+	+	+	5.70
563	-	-	-	+	-	+	+	+	-	+	+	+	+	5.34
595	-	-	-	+	+	+	+	+	+	+	-	+	-	5.31
603	-	+	-	-	-	+	+	+	-	+	+	+	+	5.34
613	-	-	-	+	-	+	+	+	+	+	-	+	+	5.29
643	-	-	-	-	-	+	+	+	+	+	+	+	+	5.15
345	-	-	+	-	-	-	+	-	+	+	+	+	+	4.92
543	-	-	-	+	-	+	+	+	+	+	-	-	+	4.77
353	-	-	+	+	-	-	-	+	+	+	+	+	-	4.76
523)	-	-	-	-	-	-	+	-	+	+	+	+	+	4.23
633)	-	-	-	-	-	-	+	-	+	+	+	+	+	4.23
683	-	-	+	-	-	-	+	+	+	+	-	-	+	4.16
385	-	-	-	+	-	-	-	-	+	+	+	+	+	4.14
705	-	-	-	-	+	+	+	-	-	+	+	+	+	4.59
645	-	+	-	-	-	+	-	+	+	+	+	-	+	4.54
655	-	-	-	-	-	+	+	-	-	+	+	+	+	4.04
575	-	-	+	-	-	+	-	+	+	+	+	-	+	4.39
555	-	-	-	-	+	-	+	+	+	-	+	-	+	3.82
315	-	-	-	-	-	+	+	-	+	-	+	+	+	3.79
423	-	-	-	-	-	+	-	+	+	+	+	+	+	4.22
565	-	-	-	-	+	-	+	+	+	-	-	+	+	3.64
355	-	-	-	-	-	+	-	+	-	+	+	-	+	3.05
395	-	-	-	-	-	-	-	+	+	-	+	+	+	2.86
585	-	-	-	-	+	+	-	+	-	-	+	-	+	2.70
405	-	-	-	-	-	+	-	+	-	-	+	+	+	2.67
533	-	-	-	-	-	-	-	+	-	+	+	-	+	2.59
403	-	-	-	-	-	+	-	+	+	+	-	-	-	2.47
465	-	-	-	-	-	-	+	+	+	-	-	+	-	2.56
333	-	-	-	-	-	-	-	+	-	+	-	+	+	2.41
513	-	-	-	-	-	-	-	+	-	-	+	+	+	2.21
535	-	-	-	-	-	+	-	-	+	-	-	+	+	2.16
553	-	-	-	-	+	-	-	-	-	+	-	-	+	1.98
445	-	-	-	-	-	-	-	-	-	-	+	+	+	1.75
	100	.84	.69	.84	.55	.46	.93	.46	.65	.90	.70	.52	.53	

example there are two pairs of identical response patterns, which have been indicated by parentheses beginning in Table 3. In larger examples having more items such cases will be much less common, and will cause no difficulty.

4) Steps 2 and 3 were applied successively to Tables 3, 4 and 5. When inversion-weights had been computed from the loadings given by Table 5, no further inversions of the sequence were called for. The approximations had converged. The loadings shown at the bottom of Table 5 were then taken as the final values for Component I.

5) It was not immediately apparent from psychological grounds how the different items would load a second component. Therefore, the deck of cards which was being used as a score matrix, after being numbered according to the final sequence for Component I, was shuffled thoroughly. Then, by making minor adjustments, a "random" sequence was obtained which was orthogonal to that of the first component. This version of the score matrix is shown as Table 6, wherein a new column (I) has appeared giving the sequential position of the rows in Table 5.

6) Using the procedure of step 2, loadings were estimated for each item from the columns of Table 6. These results are given in the first row of numbers at the bottom of the score matrix.

7) In order to prevent the successive approximations from converging again to Table 5, a limit was set on the contribution which each item could make to the total inversion-weights. This limit was the square-root of the communality of the item not yet accounted for by the earlier factor. Thus, the limit was 0.00 for item one, while for item two it was 0.48. The limit is applied against the absolute value of the computed loadings -- not the algebraic value. The results of limiting the loadings in this manner are shown in the second row of numbers at the bottom of Table 6.

8) Subject to the limitations determined in step 7, the iterations were carried out in exactly the same manner as for the first component. Many more

Table 4.
Third Iteration

	1	2	3	4	5	6	7	8	9	10	11	12	13	
433	+	+	−	−	−	+	+	+	+	+	+	+	+	7.13
393	+	−	+	−	+	−	+	+	+	+	+	+	+	6.99
343	+	−	−	+	+	+	+	+	−	+	+	+	+	6.98
583	−	+	−	+	+	−	+	−	+	+	+	+	+	6.48
485	−	+	−	+	−	−	+	+	+	+	+	+	+	6.43
305	−	+	−	+	−	+	+	+	+	+	+	−	+	6.41
455	−	−	+	+	+	+	+	−	+	+	+	−	+	6.28
525	−	+	+	−	−	−	+	+	+	+	+	+	+	6.31
475	−	−	+	+	−	−	+	+	+	+	+	+	+	6.25
675	−	−	−	+	−	+	+	+	+	+	+	+	+	6.07
685	−	−	+	−	+	+	+	+	−	+	+	+	+	5.86
325)	−	−	−	−	+	+	+	+	+	+	+	+	+	5.81
495)	−	−	−	−	+	+	+	+	+	+	+	+	+	5.81
603	−	+	−	−	−	+	+	+	−	+	+	+	+	5.49
563	−	−	−	+	−	+	+	+	−	+	+	+	+	5.43
595	−	−	−	+	+	+	+	+	+	+	−	+	−	5.36
613	−	−	−	+	−	+	+	+	+	+	−	+	+	5.35
643	−	−	−	−	−	+	+	+	+	+	+	+	+	5.26
345	−	−	+	−	−	−	+	−	+	+	+	+	+	4.94
543	−	−	−	+	−	+	+	+	+	+	−	−	+	4.82
353	−	−	+	+	−	−	−	+	+	+	+	+	−	4.80
705	−	−	−	−	+	+	+	−	−	+	+	+	+	4.67
645	−	+	−	−	−	+	−	+	+	+	+	−	+	4.69
575	−	−	+	−	−	+	−	+	+	+	+	−	+	4.51
523)	−	−	−	−	−	−	+	−	+	+	+	+	+	4.25
633)	−	−	−	−	−	−	+	−	+	+	+	+	+	4.25
423	−	−	−	−	−	+	−	+	+	+	+	+	+	4.35
683	−	−	+	−	−	−	+	+	+	+	−	−	+	4.19
385	−	−	−	+	−	−	−	−	+	+	+	+	+	4.15
655	−	−	−	−	−	+	+	−	−	+	+	+	+	4.12
555	−	−	−	−	+	−	+	+	+	−	+	−	+	3.86
315	−	−	−	−	−	+	+	−	+	−	+	+	+	3.85
565	−	−	−	−	+	−	+	+	+	−	−	+	+	3.67
355	−	−	−	−	−	+	−	+	−	+	+	−	+	3.18
395	−	−	−	−	−	−	−	+	+	−	+	+	+	2.93
585	−	−	−	−	+	+	−	+	−	−	+	−	+	2.82
405	−	−	−	−	−	+	−	+	−	−	+	+	+	2.80
533	−	−	−	−	−	−	−	+	−	+	+	−	+	2.67
465	−	−	−	−	−	−	+	+	+	−	−	+	−	2.58
403	−	−	−	−	−	+	−	+	+	+	−	−	−	2.56
333	−	−	−	−	−	−	−	+	−	+	−	+	+	2.48
513	−	−	−	−	−	−	−	+	−	−	+	+	+	2.29
535	−	−	−	−	−	+	−	−	+	−	−	+	+	2.22
553	−	−	−	−	+	−	−	−	−	+	−	−	+	2.00
445	−	−	−	−	−	−	−	−	−	−	+	+	+	1.79
	100	.87	.69	.81	.55	.51	.91	.50	.64	.91	.72	.53	.54	

Table 5.
Fourth Iteration

	1	2	3	4	5	6	7	8	9	10	11	12	13	
433	+	+	−	−	−	+	+	+	+	+	+	+	+	7.16
393	+	−	+	−	+	−	+	+	+	+	+	+	+	7.01
343	+	−	−	+	+	+	+	+	−	+	+	+	+	6.99
583	−	+	−	+	+	−	+	−	+	+	+	+	+	6.49
485	−	+	−	+	−	−	+	+	+	+	+	+	+	6.46
305	−	+	−	+	−	+	+	+	+	+	+	−	+	6.44
525	−	+	+	−	−	−	+	+	+	+	+	+	+	6.34
455	−	−	+	+	+	+	+	−	+	+	+	−	+	6.28
475	−	−	+	+	−	−	+	+	+	+	+	+	+	6.27
675	−	−	−	+	−	+	+	+	+	+	+	+	+	6.09
685	−	−	+	−	+	+	+	+	−	+	+	+	+	5.87
325)	−	−	−	−	+	+	+	+	+	+	+	+	+	5.83
495)	−	−	−	−	+	+	+	+	+	+	+	+	+	5.83
603	−	+	−	−	−	+	+	+	−	+	+	+	+	5.51
563	−	−	−	+	−	+	+	+	−	+	+	+	+	5.44
595	−	−	−	+	+	+	+	+	+	+	−	+	−	5.38
613	−	−	−	+	−	+	+	+	+	+	−	+	+	5.37
643	−	−	−	−	−	+	+	+	+	+	+	+	+	5.28
345	−	−	+	−	−	−	+	−	+	+	+	+	+	4.94
543	−	−	−	+	−	+	+	+	+	+	−	−	+	4.84
353	−	−	+	+	−	−	−	+	+	+	+	+	−	4.83
645	−	+	−	−	−	+	−	+	+	+	+	−	+	4.73
705	−	−	−	−	+	+	+	−	−	+	+	+	+	4.66
575	−	−	+	−	−	+	−	+	+	+	+	−	+	4.54
423	−	−	−	−	−	+	−	+	+	+	+	+	+	4.38
523)	−	−	−	−	−	−	+	−	+	+	+	+	+	4.25
633)	−	−	−	−	−	−	+	−	+	+	+	+	+	4.25
683	−	−	+	−	−	−	+	+	+	+	−	−	+	4.21
385	−	−	−	+	−	−	−	−	+	+	+	+	+	4.16
655	−	−	−	−	−	+	+	−	−	+	+	+	+	4.11
555	−	−	−	−	+	−	+	+	+	−	+	−	+	3.88
315	−	−	−	−	−	+	+	−	+	−	+	+	+	3.85
565	−	−	−	−	+	−	+	+	+	−	−	+	+	3.69
355	−	−	−	−	−	+	−	+	−	+	+	−	+	3.20
395	−	−	−	−	−	−	−	+	+	−	+	+	+	2.96
585	−	−	−	−	+	+	−	+	−	−	+	−	+	2.84
405	−	−	−	−	−	+	−	+	−	−	+	+	+	2.82
533	−	−	−	−	−	−	−	+	−	+	+	−	+	2.69
465	−	−	−	−	−	−	+	+	+	−	−	+	−	2.60
403	−	−	−	−	−	+	−	+	+	+	−	−	−	2.59
333	−	−	−	−	−	−	−	+	−	+	−	+	+	2.50
513	−	−	−	−	−	−	−	+	−	−	+	+	+	2.31
535	−	−	−	−	−	+	−	−	+	−	−	+	+	2.23
553	−	−	−	−	+	−	−	−	−	+	−	−	+	2.00
445	−	−	−	−	−	−	−	−	−	−	+	+	+	1.79
	1.00	.88	.69	.81	.55	.51	.90	.52	.65	.91	.72	.53	.54	

Table 6.
Start of Second Component

	1	2	3	4	5	6	7	8	9	10	11	12	13	I	
533	−	−	−	−	−	−	−	+	−	+	+	−	+	38	2.00
675	−	−	−	+	−	+	+	+	+	+	+	+	+	10	1.50
563	−	−	−	+	−	+	+	+	−	+	+	+	+	15	1.90
643	−	−	−	−	−	+	+	+	+	+	+	+	+	18	1.26
475	−	−	+	+	−	−	+	+	+	+	+	+	+	9	1.92
465	−	−	−	−	−	−	+	+	+	−	−	+	−	39	0.11
555	−	−	−	−	+	−	+	+	+	−	+	−	+	31	1.11
393	+	−	+	−	+	−	+	+	+	+	+	+	+	2	1.40
655	−	−	−	−	−	+	+	−	−	+	+	+	+	30	1.09
305	−	+	−	+	−	+	+	+	+	+	+	−	+	6	1.22
395	−	−	−	−	−	−	−	+	+	−	+	+	+	35	1.05
343	+	−	−	+	+	+	+	+	−	+	+	+	+	3	1.62
423	−	−	−	−	−	+	−	+	+	+	+	+	+	25	1.12
325)	−	−	−	−	+	+	+	+	+	+	+	+	+	12	0.98
495)	−	−	−	−	+	+	+	+	+	+	+	+	+	13	0.98
355	−	−	−	−	−	+	−	+	−	+	+	−	+	34	1.72
683	−	−	+	−	−	−	+	+	+	+	−	−	+	28	1.26
685	−	−	+	−	+	+	+	+	−	+	+	+	+	11	1.52
385	−	−	−	+	−	−	−	−	+	+	+	+	+	29	1.07
353	−	−	+	+	−	−	−	+	+	+	+	+	−	21	1.32
575	−	−	+	−	−	+	−	+	+	+	+	−	+	24	1.46
603	−	+	−	−	−	+	+	+	−	+	+	+	+	14	1.18
553	−	−	−	−	+	−	−	−	−	+	−	−	+	44	0.53
445	−	−	−	−	−	−	−	−	−	−	+	+	+	45	0.88
485	−	+	−	+	−	−	+	+	+	+	+	+	+	5	1.30
535	−	−	−	−	−	+	−	−	+	−	−	+	+	43	−0.42
333	−	−	−	−	−	−	−	+	−	+	−	+	+	41	1.18
585	−	−	−	−	+	+	−	+	−	−	+	−	+	36	1.09
513	−	−	−	−	−	−	−	+	−	−	+	+	+	42	1.45
633)	−	−	−	−	−	−	+	−	+	+	+	+	+	27	0.97
523)	−	−	−	−	−	−	+	−	+	+	+	+	+	26	0.97
645	−	+	−	−	−	+	−	+	+	+	+	−	+	22	0.84
705	−	−	−	−	+	+	+	−	−	+	+	+	+	23	0.81
613	−	−	−	+	−	+	+	+	+	+	−	+	+	17	0.88
455	−	−	+	+	+	+	+	−	+	+	+	−	+	8	0.99
315	−	−	−	−	−	+	+	−	+	−	+	+	+	32	0.34
583	−	+	−	+	+	−	+	−	+	+	+	+	+	4	0.45
345	−	−	+	−	−	−	+	−	+	+	+	+	+	19	1.11
403	−	−	−	−	−	+	−	+	+	+	−	−	−	40	0.24
543	−	−	−	+	−	+	+	+	+	+	−	−	+	20	1.08
565	−	−	−	−	+	−	+	+	+	−	−	+	+	33	0.29
525	−	+	+	−	−	−	+	+	+	+	+	+	+	7	1.20
405	−	−	−	−	−	+	−	+	−	−	+	+	+	37	1.17
595	−	−	−	+	+	+	+	+	+	+	−	+	−	16	0.14
433	+	+	−	−	−	+	+	+	+	+	+	+	+	1	0.78
	.25	−.62	.14	.24	−.28	−.28	.14	.57	−.40	.35	.62	−.20	.46	.00	
	.00	−.48	.14	.24	−.28	−.28	.14	.57	−.40	.35	.62	−.20	.46		

steps were required because the initial random sequence was not nearly so close to the answer as the guessed weights used to start off the first component. In the course of the iterations five of the loadings changed in sign, while the others both increased and decreased in absolute value at different stages. Sometimes the number of inversions called for was greater than for the previous iteration, sometimes less. However, fewer inversions in the successive sequences were generally called for, and the stable result shown in Table 7 was finally obtained after 26 approximations. At least one large error in determining an inversion-weight was discovered in looking back over these approximations, and this may have accounted for a few extra cycles. Of course, such errors are self-correcting, and only the final table need be exhibited and scrutinized for discrepancies. It was not deemed worthwhile to check each cycle rigorously.

9) Throughout the iterations for the second component, no record was kept of the correlation of the successive sequences with Table 5, although spot-checking showed correlations with absolute values as high as 0.45. By counting the number of inversions in the right hand column of Table 7, and applying expression 18 from Chapter II, it was found that the correlation between Component I and Component II is -0.22. This is appreciable, but is nevertheless indicative of substantial independence of the two components. Since the final sequences are not exactly orthogonal, little advantage seems to have accrued by starting from a Table 6 which was exactly orthogonal to Table 5; the device of limiting the amount which an item may contribute to inversion-weights, according to its residual communality, seems to bear the main responsibility for preventing the second series of approximations from converging on Table 5.

10) Table 8 contains a summary of all the important results obtained up to this point in the computations. Probably two or three items remain with communalities significantly less than unity, and it is time to determine whether the data are adequate to fix a third component. The total number of visible discri-

Table 7.
Finish of Second Component

	1	2	3	4	5	6	7	8	9	10	11	12	13	I	
355	-	-	-	-	-	+	-	+	-	+	+	-	+	34	3.24
645	-	+	-	-	-	+	-	+	+	+	+	-	+	22	2.96
603	-	+	-	-	-	+	+	+	-	+	+	+	+	14	2.65
533	-	-	-	-	-	-	-	+	-	+	+	-	+	38	2.45
585	-	-	-	-	+	+	-	+	-	-	+	-	+	36	2.44
405	-	-	-	-	-	+	-	+	-	-	+	+	+	37	2.32
305	-	+	-	+	-	+	+	+	+	+	+	-	+	6	1.93
433	+	+	-	-	-	+	+	+	+	+	+	+	+	1	1.89
423	-	-	-	-	-	+	-	+	+	+	+	+	+	25	1.85
575	-	-	+	-	-	+	-	+	+	+	+	-	+	24	1.76
563	-	-	-	+	-	+	+	+	-	+	+	+	+	15	1.58
655	-	-	-	-	-	+	+	-	-	+	+	+	+	30	1.53
513	-	-	-	-	-	-	-	+	-	-	+	+	+	42	1.53
643	-	-	-	-	-	+	+	+	+	+	+	+	+	18	1.41
333	-	-	-	-	-	-	-	+	-	+	-	+	+	41	1.14
343	+	-	-	+	+	+	+	+	-	+	+	+	+	3	1.07
705	-	-	-	-	+	+	+	-	-	+	+	+	+	23	1.02
403	-	-	-	-	-	+	-	+	+	+	-	-	-	40	.96
685	-	-	+	-	+	+	+	+	-	+	+	+	+	11	.94
325)	-	-	-	-	+	+	+	+	+	+	+	+	+	12	.90
495)	-	-	-	-	+	+	+	+	+	+	+	+	+	13	.90
445	-	-	-	-	-	-	-	-	-	-	+	+	+	45	.89
675	-	-	-	+	-	+	+	+	+	+	+	+	+	10	.82
543	-	-	-	+	-	+	+	+	+	+	-	-	+	20	.77
395	-	-	-	-	-	-	-	+	+	-	+	+	+	35	.77
553	-	-	-	-	+	-	-	-	-	+	-	-	+	44	.62
485	-	+	-	+	-	-	+	+	+	+	+	+	+	5	.51
315	-	-	-	-	-	+	+	-	+	-	+	+	+	32	.48
555	-	-	-	-	+	-	+	+	+	-	+	-	+	31	.45
525	-	+	+	-	-	-	+	+	+	+	+	+	+	7	.38
535	-	-	-	-	-	+	-	-	+	-	-	+	+	43	.24
613	-	-	-	+	-	+	+	+	+	+	-	+	+	17	.14
633)	-	-	-	-	-	-	+	-	+	+	+	+	+	27	-.02
523)	-	-	-	-	-	-	+	-	+	+	+	+	+	26	-.02
683	-	-	+	-	-	-	+	+	+	+	-	-	+	28	-.15
385	-	-	-	+	-	-	-	-	+	+	+	+	+	29	-.17
455	-	-	+	+	+	+	+	-	+	+	+	-	+	8	-.42
393	+	-	+	-	+	-	+	+	+	+	+	+	+	2	-.61
583	-	+	-	+	+	-	+	-	+	+	+	+	+	4	-.64
475	-	-	+	+	-	-	+	+	+	+	+	+	+	9	-.69
345	-	-	+	-	-	-	+	-	+	+	+	+	+	19	-.74
565	-	-	-	-	+	-	+	+	+	-	-	+	+	33	-.86
353	-	-	+	+	-	-	-	+	+	+	+	+	-	21	-1.09
465	-	-	-	-	-	-	+	+	+	-	-	+	-	39	-1.19
595	-	-	-	+	+	+	+	+	+	+	-	+	-	16	-1.21
	.33	.58	-.73	-.62	-.51	.79	-.66	.64	-.84	.29	.68	-.63	.84	.22	
	.00	.48	-.72	-.59	-.51	.79	-.44	.64	-.76	.29	.68	-.63	.84		

Table 8.
Summary of Calculations

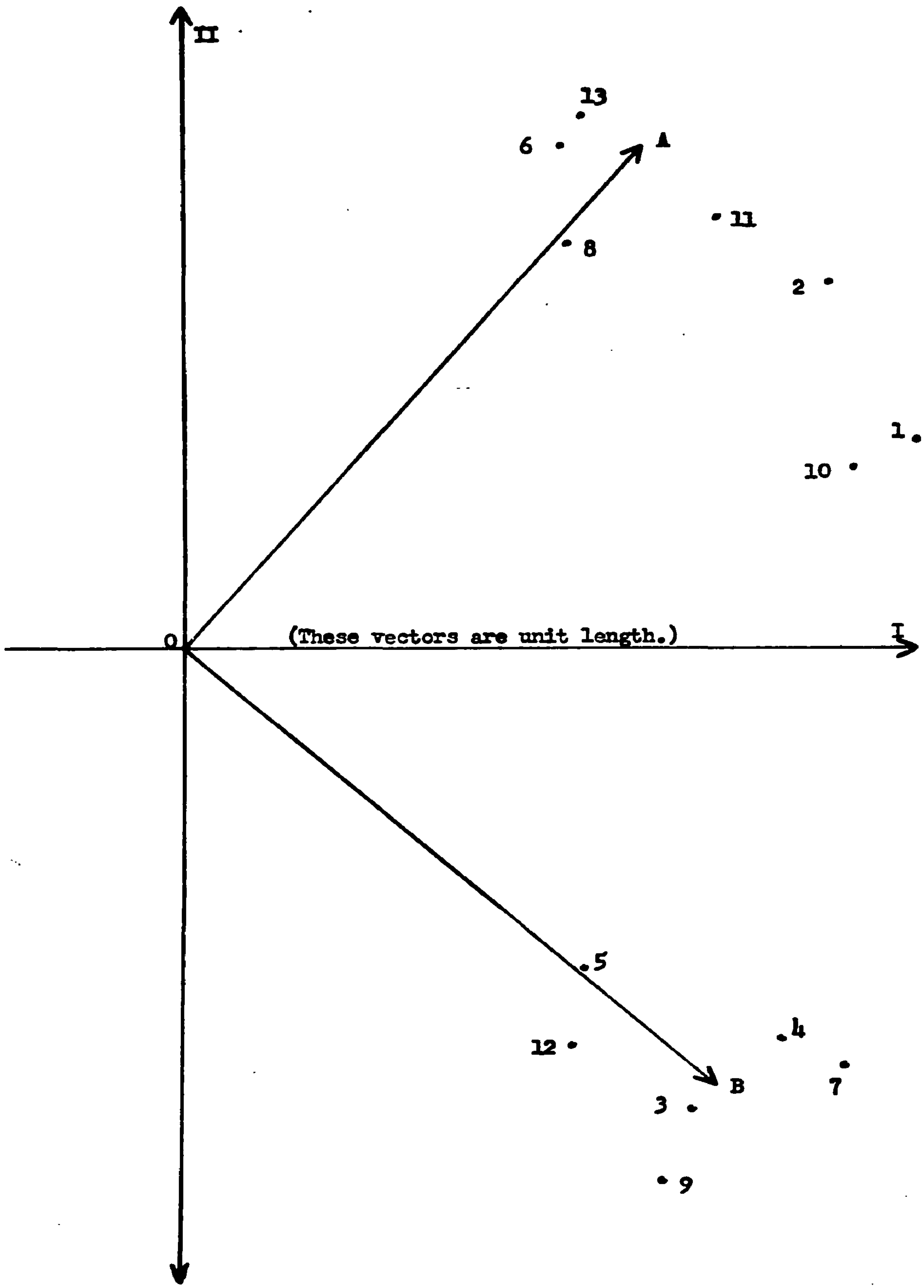
Item	p	q	pq	E_1	b_1^2	b_1	E_2	b_2^2	b_2	h^2
1)	3	42	126	0	1.00	1.00	56	.11	.33	1.11
2)	7	38	266	31	.77	.88	88	.34	.58	1.11
3)	9	36	324	84	.48	.69	248	.53	-.73	1.01
4)	13	32	416	72	.65	.81	289	.39	-.62	1.04
5)	13	32	416	145	.30	.55	263	.26	-.51	.56
6)	25	20	500	185	.26	.51	94	.62	.79	.88
7)	29	16	464	44	.81	.90	334	.44	-.66	1.25
8)	33	12	396	144	.27	.52	118	.40	.64	.67
9)	31	14	434	126	.42	.65	369	.70	-.84	1.12
10)	35	10	350	32	.82	.91	160	.09	.29	.91
11)	35	10	350	84	.52	.72	93	.47	.68	.99
12)	33	12	396	143	.28	.53	276	.39	-.63	.67
13)	41	4	164	58	.29	.54	24	.71	.84	1.00

minations in the original data is the summation of pq, or 4602. The number not accounted for by the first component is twice the summation of E_{v1}, or 2296. The number not accounted for by the second component is 2608. Therefore, ignoring the correlation of the first two components, only 302 discriminations remain to be accounted for by a possible third one. Since a minimum of $N(N - 1)/2$, or 495, are required simply to determine a component, there is no point is attempting to extract another. The figure 302 would be raised by adjustment for the correlation of components I and II, but not sufficiently to insure the overdetermination of a third component. Thus, even though most or all of the remaining discriminations may belong on the same component, it is impossible to extract this information because of the limited scope of the miniature example.

11) The angle between the two components which we have decided to accept might now be adjusted in the same manner as those obtained from the multiple-group centroid analysis of an R-matrix (62), but this was not carried out since the rotational problem in the present example is trivial in any event, and the more precise values of the net communality accounted for would be useful only if a third component were to be extracted. Ignoring the actual correlation between them, an actual plot of the thirteen items on components I and II is shown as Figure 1. The psychological significance of the simple structure axes, A and B, will be discussed presently.

12) It will be recognized that all of the elementary steps which have been carried out in the course of these computations are readily adapted to IBM equipment, although special wiring diagrams and unusual operating procedures will certainly be required. It may be estimated that, for a score matrix of 100 items by 500 subjects, each approximation will consume about two hours and 600 cards. This estimate assumes the simultaneous availability of a type 403 tabulator (to count visible inversions) and a type 602A multiplier (to compute inversion-weights). The most difficult step will be the counting of inversions prerequi-

Figure 1.
Plot of Scalability Components



site to the determination of correlations between components, but this need be done only once for each pair of components and may be carried out on another type 403 tabulator without delaying the iterations for further components.

B Validity of the Results

While K-way scale analysis has been developed here as an extension of ordinary factor analysis, by defining the "correlation" of sequences and indicating that these correlations possess the necessary ordinary properties, it has been maintained by Guttman (29) and others that the extraction of more than one scale from a given universe of content is liable to be meaningless. We have just exhibited two scales derived from the same universe, and, while essentially the same results might have been obtained by Guttman's techniques through a consideration of "non-scale types," they have been obtained here by a method which can obviously be extended to extract any number of components from <u>any</u> universe of content arbitrarily selected without regard to its subjective homogeneity for the experimenter. It may, therefore, be necessary to provide some additional evidence supporting the functional independence of scales A and B. We shall see presently that the two scales are clearly differentiable at the psychological level, but first we may adduce a statistical test of their independence.

It will be seen from inspection of Figure 1 that items 6, 8, 11 and 13 are most representative of scale A, while items 5, 7, 9 and 12 are equally representative of scale B over the same "difficulty" range. Remembering the manner in which the items were numbered, the interlocking of these two series might suggest the hypothesis that the whole series of items actually constitute a single scale, with the two sub-scales chosen from it in such a way as to minimize the overlap of the zones containing most of the errors — since it is an observable fact that most of the errors in reproducibility occur near the dividing point for each item. If this "interlocking hypothesis" is true, then the scores on our scales A and B should actually be positively correlated. Table 9 gives the

Table 9.
Distributions of Scores on Components

a) Cases using zero ?'s

0	1	7	8	6	22
3	4	29	37	22	95
3	6	20	25	22	76
0	2	12	12	13	39
1	1	1	2	1	6
7	14	69	84	64	238

b) Cases using just one ?

0	0	1	1	0	2
2	2	15	13	3	35
1	5	11	15	10	42
0	1	5	8	5	19
0	0	1	1	0	2
3	8	33	38	18	100

c) Cases using just two ?'s

1	0	0	0	0	1
1	3	1	1	1	7
0	1	6	8	2	17
0	3	3	5	0	11
0	0	0	1	0	1
2	7	10	15	3	37

d) Cases using three or more

0	1	0	0	0	1
1	2	3	2	0	8
0	1	7	2	1	11
0	2	1	1	2	6
0	0	0	0	0	0
1	6	11	5	3	26

Scores on scale A increase from 0 to 4 going to the right.

Scores on scale B increase from 0 to 4 going up.

joint distribution of scores on scales A and B, giving each of the mentioned items unit weight for its scale. Subjecting the null hypothesis to evaluation for Table 9a (which includes the 45 cases already analyzed), we compute a chi-square of 2.75 for 6 degrees of freedom (3 combinations of classes are necessary in order to raise the expected frequencies above 5); under the null hypothesis this value will be exceeded about 84% of the time, and we are given no reason to reject it in favor of the "interlocking hypothesis." Computation of chi-square for the remaining 163 cases (combining Tables 9b, 9c and 9d) gives a value of 7.43 for 6 degrees of freedom, which is "significant at the 28% level." However, inspection of the distribution of these cases shows that the sign of any correlation which exists is going to be negative! Certainly, no support is lent to the hypothesis that scales A and B are sub-scales of a general component. We shall regard these tests as establishing the significance and independence of scales A and B. In view of the small score matrix used in the miniature example, we may infer an adequate degree of reliability for K-way scale analyses based on the usual sizes of experimental score matrices.

C Psychological Interpretation

Let us now consider the psychological nature of these two scales. The following list gives the original position of each item in the personality battery, the exact wording of the item, the distribution of responses for 401 cases and the zone of response which was taken to be positive in setting up the score matrix of Table 1.

	Y	?	N
1) (3-83) Do you always succeed in putting first things first?	<u>52</u>	11	338
2) (3-14) Do you always tell the truth?	<u>65</u>	15	321
3) (3-44) Do you sometimes feel cross when you are not well?	314	6	<u>81</u>

4) (3-149) Would you agree that we need today more social authority, and that the individual should learn to subordinate himself more to social regulation? — 100 22 279

5) (3-111) Do you think that some lawyers are more interested in making the law than in interpreting it? — 263 34 104

6) (3-59) Would you encourage your children while they are at school to fit themselves by vocational training to be of use to society as soon as possible? — 190 16 195

7) (3-173) Which do you value more highly? — 218 21 162
   Y) Your time.
   N) Your property.

8) (3-113) Do you agree that "time is of the essence"? — 233 60 108

9) (3-28) Which do you believe more strongly? — 124 14 263
   Y) That our country's natural resources are there to be used.
   N) That we should try to conserve our resources for posterity.

10) (3-168) If you were driving along an empty street late at night and came to a red traffic light, which would you do? — 109 7 285
   Y) Carefully look both ways and drive on.
   N) Wait until the light turned green, even though you could see that there was nothing coming.

11) (3-143) If you were given a bond worth nine dollars today, and guaranteed to be worth ten dollars a year from now, which would you do? — 99 15 287
   Y) Cash it in today.
   N) Save it till the end of the year.

12) (3-23) Is it true that "things will come to he who waits" patiently? — 55 18 328

13) (3-53) If you were given a bond worth one dollar today, and guaranteed to be worth two dollars a year from now, which would you do? — 40 13 348
   Y) Cash it in today.
   N) Save it till the end of the year.

Consideration of these items in relation to the configuration of Figure 1 leads to the suggestion that scale A measures "Personal Integration," while scale B measures degree of "Social Conditioning." The meaning of these identifications must, of course, be documented in somewhat more detail, since they are simply verbal labels which do not necessarily convey a consistent connotation for every-one reading them.

Psychologically speaking, integrated behavior is shown whenever an individual submits to temporary hurt, pain, inconvenience, deprivation, etc., in the expectation (whether conscious or unconscious) of either greater reward or less punishment later. Throughout all integrated behavior runs the time element, for integration is achieved when the future consequences of behavior are brought into the psychological present as determiners of the behavior.[4] Items 13 and 11 had been deliberately constructed in an effort to incorporate this concept of integration ungarnished, and it is very encouraging to find them effectively discriminative on only one of the scales — and there with communalities of 1.00 and 0.99 respectively. Item 8, by emphasizing only the time element, does not present a complete basis for the determination of integrated behavior, and responses to the item are partially determined by a unique component. Item 6 again presents a more complete basis, omitting only the explicit statement of the punishment to be incurred if one's children fail to behave properly, and again possesses substantially perfect communality on scale A. The remaining items which have their principal loadings on this scale are comparably indirect, and each inquires concerning behavior likely to represent the initial inconvenience which the integrated individual is willing to accept; in each case this inconvenience may be weighed against the probability that society, or some part or representative or symbol of it, will catch up to mete out the greater punishment. If society is less likely to catch up with a given offense, then avoidance of the offense on the part of an individual is relatively more a function of his social conditioning (scale B) than his personal integration. On the other hand, if society is simply going to take a longer time to catch up with the offense,

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[4] "Integrated" behavior is distinguishable from "adaptive" behavior, which is related to organismic survival, and from "adjustive" behavior, which coordinates only the more immediate needs of the organism (cf. 8, 45). Of course, to the extent that integrated behavior is non-adaptive or non-adjustive it is less desirable to the organism.

then the acceptable behavior represents a greater degree of integration itself.

Thus, scales A and B are alike in that socially more acceptable behavior is found at the upper end of each, and the scales appear to differ essentially in the basis for that behavior. We may assert that it makes no real difference to society how an individual stands on scale B — there is no punishment attached to a low rank, and no reward attached to a high rank. Perhaps the high-ranking person would tend to label someone lower as "suspicious and unenlightened," but the lower-ranking person would regard the former in return as "impractically idealistic," and all of these are mere words.

Presumably, the behavior represented and mediated by each of these scales is learned, and we may attempt to distinguish them in terms of the type of learning which they represent. The learning prerequisite to a high standing on scale A clearly is reinforced, though the reinforcement follows time lags of varying length. On the other hand, it is simplest to explain the learning of scale B as a result of conditioning, and thereby to preserve the dual nature of learning in this context[5] (46). If this distinction is valid here, we should expect learning of type A to continue indefinitely in most individuals, perhaps at a rate partially determined by such factors in intelligence (61) as memory. However, when the environment of an individual changes his standing on scale B should tend to change in the same direction, and there should be little if any age-trend in the population for this scale. These, in any event, are hypotheses which may be subjected to experimental test.

We have assumed, at the start of our miniature analysis, that there exist underlying continua of measurement which our items happen to divide at particular

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[5] The distinction drawn here is based only on the results of the miniature example, and is naturally subject to modification when the thirteen items are reconsidered in the larger context of the personality battery from which they have been abstracted. Almost certainly we should expect to find more than two distinguishable mechanisms for social learning; for example, Cattell's personality factors G, K and N in this area (5, 6, 7) suggest that there are at least three, and the third might, for example, depend on secondary reinforcement.

levels. Can we now justify this assumption for scales A and B? Especially, how might one go about discovering the unit of measurement on each scale without first assuming some particular form for the distribution of scores on the scale? If the unit of measurement sought is to possess any importance it will relate to some mathematical or arithmetical operation which we wish to be able to perform on the scores. If we wish to be able to add or subtract scores, we must have a unit of measurement possessing some constant property; if we wish to multiply two scores together, we must first determine an appropriate zero. For scale A the unit of time seems quite natural, and an individual's numerical score on the scale might be defined as the maximum time interval which he could "span" in achieving some standard example of integrated behavior. The learning theorists' equations for the gradient of reinforcement (33) then would properly contain this score as a constant characterizing the individual. Since such a unit of measurement would probably force the particular numbers inserted in items such as 11 and 13 to bear simple relations to their relative scale values, this offers one possible approach to the actual determination of such values.[6] Scale B's unit of measurement is not so obvious, though it could conceivably be the proportion of conditioners attached to certain response classes in a statistical learning theory similar to that of Estes (18). Again, these statements may be treated as hypotheses which are subject to experimental test.

Our aim in this discussion has not been to rationalize the particular results which have been obtained from the miniature example, but rather to demonstrate that the results of K-way scale analysis are not sterile of hypotheses for further investigation. In fact, by providing not only a measure of relational strength (the loading) but also a measure of relative level that is not masked

[6] Besides the construction of item forms for given scales which are permissive of variable quantitative elements, it is necessary to engage in specific experiments designed to yield the relationship of these quantitative elements to the constants in whatever learning or behavior theory is accepted as a working hypothesis.

by "difficulty factors," the technique should prove even more fruitful of such hypotheses than the ordinary factor analyses which have been applied to similar material.

D The ? Responses

We have already had occasion to examine Table 9 in order to refute the "interlocking hypothesis" of the last section but one. Let us now consider whether the relation there suggested, between incidence of ? responses and correlation of scales A and B, is real. (There are several references in the literature (17, 12, 48) to the effect that ? responses do not lie on the same continuum with Y and N, but we are here considering a more specific possibility for tying them into the general picture of personality.) Since the hypothesis now under consideration was suggested immediately by the comparison of Table 9a with the total of Tables 9b, 9c and 9d, any information which may be extracted from a comparison of the last three is independent evidence. Unfortunately, the number of cases available for this breakdown is not completely adequate, but let us consider the comparison of Table 9b with the sum of Tables 9c and 9d, thus dividing the group according to whether they have used less than, or more than, the "permitted" 10% of ? responses. For Table 9b, containing 100 cases, we compute a chi-square of 4.24 for 4 degrees of freedom, which does not seem to represent any great departure from expectation under the null hypothesis even though the statistic cannot be strictly interpreted because of a low expected frequency. For the remaining 63 cases we compute a chi-square of 13.30 for 4 degrees of freedom; this would be significant beyond the 1% level but for the fact that two of the expected frequencies calculated from the marginal totals are 4.3 and 4.6 -- slightly less than the minimum of 5 which is ordinarily required for the prevention of bias. However, since the data neither suggest nor give greater support to any other hypothesis, and since the rejection of a true hypothesis represents a greater loss in scientific research than the acceptance of a false one, we shall accept the hypothesis that scales A and B are negatively correlated within the group of subjects who give more than the permitted quota of ? responses.

Let us suppose that most of the excess ? responses should be scored as minus,[7] at least for the particular thirteen items involved here, but that the individuals who respond ? instead of minus are either ashamed to admit the truth, or are suspicious of the use to which the information may be put.[8] Such individuals probably hope that the ? responses will never be scored, and, having been cooperative enough in the first place to submit to the personality battery, do not respond with an outright falsehood; nevertheless, they have not "told the truth." According to item two of the miniature example, they have exhibited behavior which either is not "integrated" or, less often, is not "socially conditioned." On this basis, we should expect a negative correlation to exist between scales A and B for such individuals. Once again, the analysis has led us to an hypothesis which may be subjected to further experimental test.

In view of all the evidence, we certainly are not justified in assuming at the outset of a new analysis that the ? responses represent a valid middle zone on the item continuum. Neither may we, in most practical instances, restrict from our score matrix those individuals who have made ? responses; we should thereby lose either a great deal of valuable information or, if true forced-choice format is resorted to, the rapport of an important and sizeable group of subjects. (The item "Do you dislike forced-choice questions?" was placed at the end of one questionnaire form. The response was 69% affirmative, often with extra emphasis, even for the cooperative group of subjects involved.) Probably

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[7] This was actually done in obtaining Table 9, but substantially equivalent results are obtained up to this point even when they are scored as plus.

[8] We might even be permitted to guess that the first reason would be dominant for individuals having a relatively high rank on scale B, while the second reason would be associated with individuals relatively high on scale A.

the only safe way of proceeding when there are an appreciable number of ?'s is anticipated in the theoretical development of K-way scale analysis — inversions involving ?'s are regarded as invisible. This has the effect of reducing the amount of inversion-variance which may be accounted for, thereby reducing the reliability of the loading estimates, but this is better than biassing the results by considering the ?'s as occupying any definite zone of the continuum containing Y and N. After rotation and the establishment of uni-dimensional simple structure scales there is plenty of opportunity to determine whether the ?'s have any stronger claim.

IV DISCUSSION

It is the purpose of this discussion to review briefly certain of the main issues which arise in connection with K-way scale analysis, paying particular attention to the differential nature of the assumptions made and interpretations possible between this and other techniques. We shall need to refer only indirectly to the particular example which was used in the preceding chapter for purposes of numerical and psychological illustration.

First, let us state what K-way scale analysis is not. It is not a short-cut procedure of factor analysis in the sense of such methods as those of Horst (30), Hotelling (31), Tucker (63, 64), Woodrow (66) and others, although the actual numerical computations do not require any greater effort than some much-used methods of analysis. It is not a method for extracting any greater amount of information from a given score matrix, for the conventional methods of factor analysis of R-matrices are computationally more efficient and no less determinate when they are applicable. It is not a means of circumventing the communality problem as it has been encountered in ordinary factor analysis, for a set of analogous coefficients must be determined from the data in score matrix analysis. Therefore, it is not a substitute for present methods in any of the areas for which the latter are primarily designed.

However, when the time comes to evaluate the degree of departure of the fitted factor matrix from the experimental observations, R-factorization and S-factorization do part company. In R-factorization the analyst is left with a matrix of residual correlations, the evaluation of which in terms of errors requires an assumption concerning the forms of the distributions correlated, and these are usually taken to be normal. In S-factorization the analyst is left with a set of deviations between calculated and observed scores, and the only assumption which he needs to make is that these errors of prediction are normally distributed with variance proportional to the uniquenesses of the various measurements. This independence of an assumption concerning the form of score distributions, either on tests or on factors, is characteristic of all types of direct factor analysis, and is particularly valuable.

Returning, for example, to K-way scale analysis, we may not be especially surprized to find the communalities obtained in the miniature example so high as they are — even after allowing for the probable upward bias due to the small size of the example. The communalities are no higher than psychologists have been hoping to obtain all along. It may be that the assumption of normality for the score distributions which is involved in the use of the tetrachoric correlation coefficient has given rise to a set of spurious "assumption factors," with one of these factors specific to each item. The consequent reduction in the general level of communalities would thereby spring from a source which does not seem to have been considered. (Thurstone has even recommended the forcing of non-normal score distributions into a normal form prior to correlation, for the purpose of making simple structure more clear-cut (62).) These spurious "assumption factors" are not the same thing as the "difficulty factors" spoken of by Ferguson (20), Guilford (23) and Lawley (41), which appear in the common-factor space.

The results of K-way scale analysis explicitly provide information concerning not only the direction of an item's vector in k-space, but also its relative level of discrimination. It thereby combines the principal properties of ordinary scale analysis and ordinary factor analysis. (This combination of information has not actually been lacking at the conclusion of an ordinary factor analysis, but it has not proved useful because of the masking effect of the difficulty and assumption factors.) Not only is the psychological interpretation enriched by this more complete picture, but it becomes possible to provide a convenient topological interpretation for the results. For each item, we may construct in the k-space a plane which is orthogonal to its vector and which intersects its vector at a point corresponding to its level of discrimination. These planes serve to divide the space into a number of compartments, each of which is characterized by a particular pattern of consistent responses to the whole set of items. There will be many fewer compartments in such a space than the original number of possible response patterns; for example, ten items permit 1024 possible patterns, but if these belong to two scales of 5 items each, the topological space will contain only 36 compartments corresponding to perfect "scale types" — a considerable reduction. Since the form of the score distributions is unspecified by assumption, and cannot be determined experimentally prior to the establishment of scale units and careful sampling studies, the topological model is not without applications. For example, if "permissive factors (5)" exist, then even diligent search should not show the possibility of certain patterns of response which would ordinarily be perfectly consistent with the results of the scale analysis, and simplifications could then be made in the model by eliminating these compartments. Also, by fusing several compartments together, we may provide placement in the model for individuals whose pattern of response does not correspond exactly to any of the consistent "scale-type" patterns; the minimum number of compartments which must be combined in order to do this satisfactorily is then a measure of the inconsistency of the individual's responses, and the larger combined compartment is the best placement of the individual that we are entitled to make.

Having concluded a given K-way scale analysis including its rotation, we may consider how effectively it has accomplished the purposes for which Guttman first devised his uni-dimensional scale analysis. In the first place, each scale should be subjectively homogeneous; Guttman, however, is forced to apply this test prior to the analysis, which then constitutes no more than a statistical test of his psychological judgement as experimenter. There is, however, no single statistical criterion which may be used to test the adequacy of the observed scalability in

Guttman's system; instead, there are three. Foremost is the coefficient of reproducibility, which is obtained by dividing the number of errors in reproducibility (minimized) by the total number of scores; since the data are considered to be qualitative, an error in reproducibility counts just as much no matter where in a column it occurs. This may be contrasted with K-way scale analysis, in which such errors are weighted according to how many inversions would be required to correct them. Guttman's second criterion, "randomness of error," is satisfied when the errors in reproducibility are "evenly" distributed over the ordered score matrix. K-way scale analysis does not expect randomness of error, as it in fact tends to arrange the score matrix so as to make the errors as "unrandom" as possible; in this way the ideas of the first two criteria are combined. Guttman's third criterion specifies that each column of the score matrix must contain less than 50% errors in reproducibility; this condition is needed along with the first two criteria in order to rule out certain special cases. The comparable condition in K-way scale analysis is that each component extracted must account for a certain minimum number of discriminations present in the original data. Thus, all these criteria can be related to K-way scale analysis, and can be evaluated from its rotated results. However, it would probably be more convenient to define a single coefficient which will represent the degree of mutual scalability of any sub-set of items; this could be defined as the proportion of the total communality of these items which is accounted for by a component through their centroid. Such a coefficient would serve all comparative purposes, while the actual suitability of combining any particular set of items into a scale may be determined directly from the simple structure B matrix. Suchman (58) has emphasized that the score of an individual on a scale must be independent of the weights assigned to the items, and it will be seen that this last requirement also tends to be met by rotated K-way scales.

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