Binary Search Trees

(the sum, the product and the functor)

Oles Hodych

Melbourne Scala User Group

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Contents

- Sets and recursive definitions.
- Choosing the right abstraction.
- Product and sum types.
- Processing set elements, functoriality.
- A set for anything.

A declarative definition of a set

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A set is said to *contain* its elements.

The $a \in A$ notation denotes that a is an element of set A.

The $a \notin A$ notation denotes that a is not an element of set A.

Set properties

- Sets are *unordered* collections.

Set properties

- Sets are *unordered* collections.
- Sets contain no duplicate elements.

Problem statement

- Create a data abstraction for representing a finite set of elements.
- Implement set operations:
  - $adjoin(x, S)$ – produces a new set that has all elements of set S and element x;
  - $contains?(x, S)$ – a predicate that checks if $x \in S$;

Examples of sets

- $S = \{2,\ 1,\ 3,\ 4\}$
- $[a, b] = \{x \,|\, a \leqslant x \leqslant b\}$
- $\mathbb{N}_0 = \{0,\ 1,\ 2,\ \ldots\}$

Recursive definition of sets and other structures

Basis step:

Specify the initial collection of elements.

Recursive step:

Specify the rules for forming new elements from already present ones.

Recursive definition of sets, example

Basis step:

$$3 \in S.$$

Recursive step:

$$\text{if } x \in S \text{ and } y \in S \text{ then } (x + y) \in S$$

Recursive definition of sets, example

Basis step: $S_0 = \{3\}$

Recursive definition of sets, example

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Recursive step 1: $S_1 = \{3, 6\}$ $3 + 3$

Recursive definition of sets, example

Basis step:	$S_0 = \{3\}$	
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Recursive step 2:	$S_2 = \{3, 6, 9, 12\}$	$3 + 6$ and $6 + 6$

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Recursive step 3:	$S_4 = \{3, 6, 9, 12, 15, 18, 21, 24\}$	$3+12$, $6+9$, $6+12$, $9+12$ and $12 + 12$

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Recursive step 3:	$S_4 = \{3, 6, 9, 12, 15, 18, 21, 24\}$	$3+12$, $6+9$, $6+12$, $9+12$ and $12 + 12$
...	...	

Structural induction

Basis step:

Show that statement P holds for instances that represents our structure at its basis step.

Inductive step:

Inductive hypothesis: P holds for all instances of out structure after applying k recursive steps.

Show that $P(x)$ holds for all new instances formed by applying the $k + 1$ recursive step.

Problem statement, simplified

- Create a data abstraction for representing a finite set of *integers*.
- Implement set operations:
  - $adjoin(x, S)$ – produces a new set that has all elements of set S and element x;
  - $contains?(x, S)$ – a predicate that checks if $x \in S$;

Sets as unordered lists

```scala
type IntSet = List[Int]

def contains_?(x: Int, set: IntSet): Boolean = ???

def adjoin(x: Int, set: IntSet): IntSet = ???
```

Sets as unordered lists

```
def contains_?(x: Int, set: IntSet): Boolean =
  if (set.isEmpty)
    false
  else
    x.equals(set.head) || contains_?(x, set.tail)
```

```
def adjoin(x: Int, set: IntSet): IntSet =
  if (contains_?(x, set))
    set
  else
    x::set
```

Sets as unordered lists

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Sets as unordered lists

But what is the complexity of these operations?

- Linear time inclusion (`adjoin`) of a new number into a set, $O(n)$.
- Linear time search to check (`contains?`) if a number is present in a set, $O(n)$.
- Linear space growth characteristics, $O(n)$.

Sets as binary search trees, 2-trees

The advantages of using a binary search tree as the data structure for implementing a set:

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- Logarithmic time inclusion (`adjoin`) of a new number into a set, $O(\lceil log_2 n \rceil)$.

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- Linear space growth characteristics, $O(n)$.

$O(n)$ vs. $O(\lceil log_2 n \rceil)$

$$|S| = 15$$

$O(n)$ vs. $O(\lceil log_2 n \rceil)$

$|S| = 15$

15 vs. 4

$O(n)$ vs. $O(\lceil log_2 n \rceil)$

$|S| = 15$

15 vs. 4

$|S| = 1,000,000,000$

$O(n)$ vs. $O(\lceil log_2 n \rceil)$

$|S| = 15$

15 vs. 4

$|S| = 1,000,000,000$

$1,000,000,000$ vs. 30

Recursive definition of tree structures, full binary tree

Basis step:

A single node forms a full binary tree.

Recursive step:

If T_1 and T_2 are disjoint full binary trees then there is a full binary tree $T_1 \cdot T_2$, which consists of a root node r together with edges connecting this root to the roots of the left subtree T_1 and the right subtree T_2.

Recursive definition of tree structures, full binary tree

Basis step: $\bullet$

Recursive definition of tree structures, full binary tree

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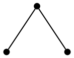
Recursive step 1: 

Recursive definition of tree structures, full binary tree

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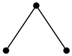
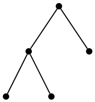
Recursive step 1: 

Recursive step 2: 

Recursive definition of tree structures, full binary tree

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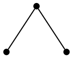

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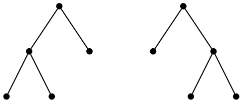

Recursive definition of tree structures, full binary tree

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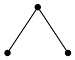

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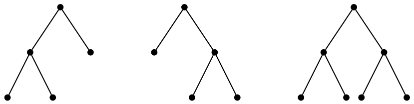
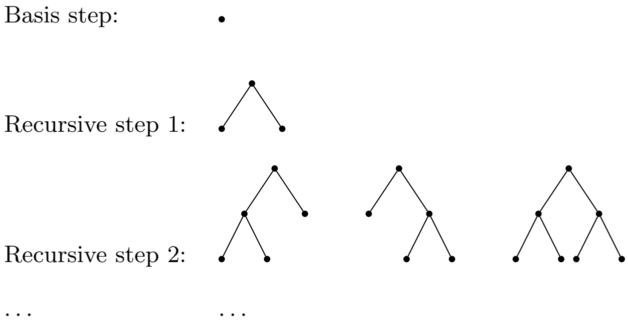
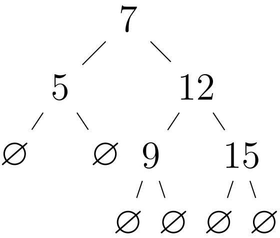


Recursive definition of tree structures, full binary tree



Full binary search tree

$$S = \{5, 7, 9, 12, 15\}$$



Recursively defined functions for recursively defined structures

Let's introduce some functions on full binary trees.

Recursively defined functions for recursively defined structures

Let's introduce some functions on full binary trees.

- $h : T_{FB} \to \mathbb{N}_0$ to compute the height of a full binary tree.
- $n : T_{FB} \to \mathbb{N}$ to compute the number of nodes in a full binary tree.

Recursive definition of $h(T)$

Basis step:

If T is a full binary tree consisting only of a root node then its height is 0.
That is $h(T) = 0$.

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Recursive step:

If T_1 and T_2 are full binary trees then a full binary tree $T = T_1 \cdot T_2$ has the height of $h(T) = 1 + max(\ h(T_1),\ h(T_2)\ )$.

Recursive definition of $n(T)$

Basis step:

If T is a full binary tree consisting only of a root node then $n(T) = 1$.

Recursive definition of $n(T)$

Basis step:

If T is a full binary tree consisting only of a root node then $n(T) = 1$.

Recursive step:

If T_1 and T_2 are full binary trees then a full binary tree $T = T_1 \cdot T_2$ has $n(T) = 1 + n(T_1) + n(T_2)$ nodes.

Recursively defined functions for recursively defined structures

Observation: functions on hierarchical recursively defined structures can only be defined recursively.

Hierarchical data by means of type hierarchies

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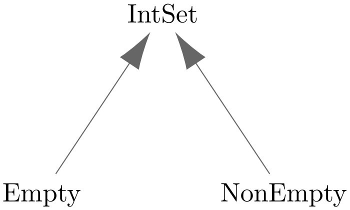

IntSet

```
type Set = IntSet

sealed trait IntSet {
    def contains_?(x: Int): Boolean

    def adjoin(x: Int): IntSet
}
```

Empty set

```
case object Empty extends IntSet {
    def contains_?(x: Int): Boolean =
        false

    def adjoin(x: Int): IntSet =
        NonEmpty(x, Empty, Empty)
}
```

Non empty set, operation `contains?`

```
case class NonEmpty(el: Int,
                    left: IntSet,
                    right: IntSet) extends IntSet {

    def contains_?(x: Int): Boolean =
        if (x < el) left contains_? x
        else if (x > el) right contains_? x
        else true

    def adjoin(x: Int): IntSet = ...
}
```

Non empty set, operation `adjoin`

```
case class NonEmpty(el: Int,
                    left: IntSet,
                    right: IntSet) extends IntSet {

  def contains_(x: Int): Boolean = ...

  def adjoin(x: Int): IntSet =
     if (x < el) NonEmpty(el, left adjoin x, right)
     else if (x > el) NonEmpty(el, left, right adjoin x)
     else this
}
```

IntSet in action

```
scala> val set = NonEmpty(7, Empty, Empty)
set: NonEmpty = NonEmpty(7,Empty,Empty)
```

IntSet in action

```
scala> val set = NonEmpty(7, Empty, Empty)
set: NonEmpty = NonEmpty(7,Empty,Empty)
```

```
scala> val set2 = set adjoin 5 adjoin 12
set2: NonEmpty = NonEmpty(7,NonEmpty(5,Empty,Empty),NonEmpty(12,E
```

Empty and non empty set, improving `toString`

```
case object Empty extends IntSet {
    ...
    override def toString = "."
}
```

Empty and non empty set, improving toString

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case object Empty extends IntSet {
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```
case class NonEmpty(el: Int,
                    left: IntSet,
                    right: IntSet) extends IntSet {
    ...
    override def toString =
        "{" + left + el + right + "}"
}
```

IntSet back in action

```
scala> val set = NonEmpty(7, Empty, Empty)
set: NonEmpty = {.7.}
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IntSet back in action

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scala> val set = NonEmpty(7, Empty, Empty)
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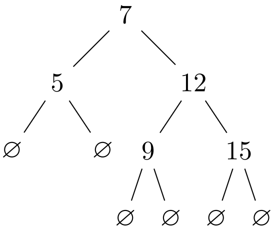
```
scala> val set2 = set adjoin 5 adjoin 12
set2: IntSet = {{.5.}7{.12.}}
```

Persistent data structures, analysing operation `adjoin`

```
def adjoin(x: Int): IntSet = NonEmpty(x, Empty, Empty)

def adjoin(x: Int): IntSet =
    if (x < el) NonEmpty(el, left adjoin x, right)
    else if (x > el) NonEmpty(el, left, right adjoin x)
    else this
}

set adjoin 3
```



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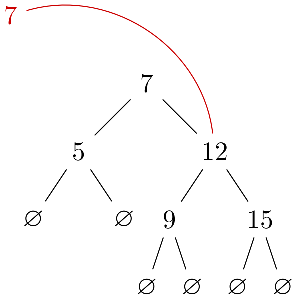

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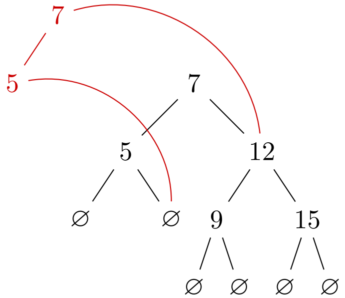

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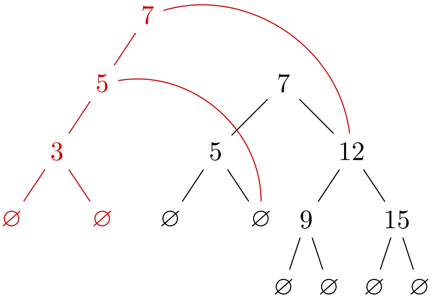

Implementing $h(T)$ and $n(T)$

Basis step:

$$h(T) = 0$$

Recursive step:

$$h(T) = 1 + max(\ h(T_1),\ h(T_2)\ )$$

Implementing $h(T)$ and $n(T)$

Basis step:

$$h(T) = 0$$

Recursive step:

$$h(T) = 1 + max(\ h(T_1),\ h(T_2)\ )$$

Basis step:

$$n(T) = 1$$

Recursive step:

$$n(T) = 1 + n(T_1) + n(T_2)$$

Which subtype represents an empty set?

```
sealed trait IntSet {
    ...
    def isEmpty: Boolean
}

case object Empty extends IntSet {
    ...
    def isEmpty: Boolean = true
}

case class NonEmpty(...) extends IntSet {
    ...
    def isEmpty: Boolean = false
}
```

Cartesian product and Product type

Cartesian product (Set Theory):

$$A = \{1, 3\},\ B = \{1, 2\}$$

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$$A = \{1, 3\},\ B = \{1, 2\}$$

$$A \times B = \{(1, 1),\ (1, 2)\ (3, 1),\ (3, 2)\}$$

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Product type (Type Theory):

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$$\texttt{Rational} = \texttt{Int} \times \texttt{Int}$$

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Product type (Type Theory):

$$A \times B$$

$$\texttt{Rational} = \texttt{Int} \times \texttt{Int}$$

$$\texttt{NonEmpty} = \texttt{Int} \times \texttt{IntSet} \times \texttt{IntSet}$$

Disjoint union and Sum type

Union (Set Theory):

$A = \{1, 3\}$, $B = \{1, 2\}$

Disjoint union and Sum type

Union (Set Theory):

$$A = \{1, 3\},\ B = \{1, 2\}$$

$$A \bigcup B = \{1,\ 3,\ 2\}$$

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Disjoint union or tagged union (Set Theory):

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$A \coprod B = A^{*} \bigcup B^{*} = \{(1,a),\ (3,a)\ (1,b),\ (2,b)\}$

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Sum type (Type Theory):

$A + B$

Disjoint union and Sum type

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$\texttt{IntSet} = \texttt{Empty} + \texttt{NonEmpty}$

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$\texttt{IntSet} = \texttt{Empty} \mid \texttt{NonEmpty(0, Empty, Empty)} \mid \ldots$

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$\texttt{Boolean} = \texttt{true} \mid \texttt{false}$

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$$\texttt{Boolean} = \texttt{true} \mid \texttt{false}$$

$$\texttt{Int} = 1 \ \mid\ 2 \ \mid\ 3 \ \mid\ \ldots$$

Sealed type hierarchies for Sum and Prod types

```scala
sealed trait IntSet {
    ...
}

case object Empty extends IntSet {
    ...
}

case class NonEmpty(el: Int
                    left: IntSet,
                    right: IntSet) extends IntSet {
    ...
}
```

Decomposition of compound data, pattern matching

```
e match {
    case p1 => e1
    case p2 => e2
    ...
    case pn => en
}
```

http://www.scala-lang.org/files/archive/spec/2.12/08-pattern-matching.html

Implementing $h(T)$ with pattern matching

```
def height(set: IntSet): Int = set match {
    case Empty =>
        0
    case NonEmpty(_, left, right) =>
        1 + Math.max(height(left), height(right))
}
```

Implementing $n(T)$ with pattern matching

```scala
def size(set: IntSet): Int = set match {
    case Empty =>
        0
    case NonEmpty(_, left, right) =>
        1 + size(left) + size(right)
}
```

Processing elements in sets

How do we go about doing something with elements in our `IntSet`?

Double each element

```
def double(set: IntSet): IntSet = set match {
    case Empty =>
        set
    case NonEmpty(el, left, right) =>
        NonEmpty(2 * el, double(left), double(right))
}
```

Double each element in action

```
scala> val set = Empty include 7 include 5 include (
                 12) include 9 include 15
set: IntSet = {{.5.}7{{.9.}12{.15.}}}

scala> double(set)
res3: IntSet = {{.10.}14{{.18.}24{.30.}}}
```

Square each element

```
def square(set: IntSet): IntSet = set match {
    case Empty =>
        set
    case NonEmpty(el, left, right) =>
        NonEmpty(el * el, square(left), square(right))
}
```

Square each element in action

```
scala> val set = Empty include 7 include 5 include (
            12) include 9 include 15
set: IntSet = {{.5.}7{{.9.}12{.15.}}}

scala> square(set)
res6: IntSet = {{.25.}49{{.81.}144{.225.}}}
```

Processing elements in sets

Can processing of elements in `IntSet` be generalised?

Mapping elements

```
sealed trait IntSet {
    def contains_?(x: Int): Boolean
    def adjoin(x: Int): IntSet

    def map(f: Int => Int): IntSet = this match {
      case Empty =>
        this
      case NonEmpty(el, left, right) =>
        NonEmpty(f(el), left map f, right map f)
    }
}
```

Mapping elements in action

```
scala> val set = Empty include 7 include 5 include (
                12) include 9 include 15
set: IntSet = {{.5.}7{{.9.}12{.15.}}}

scala> set map (x=>2*x)
res10: IntSet = {{.10.}14{{.18.}24{.30.}}}

scala> set map (x=>x*x)
res11: IntSet = {{.25.}49{{.81.}144{.225.}}}
```

Mapping elements in action, alternative syntax

```
scala> set map { x=>2*x }
res12: IntSet = {{.10.}14{{.18.}24{.30.}}}

scala> set map { 2*_ }
res13: IntSet = {{.10.}14{{.18.}24{.30.}}}
```

Mapping is not easy...

The presented implementation of `map` is invalid!

Mapping is not easy...

The presented implementation of `map` is invalid!

```
scala> val set = Empty include 7 include 5 ... include 15
set: IntSet = {{.5.}7{{.9.}12{.15.}}}

scala> set map { x => if (x > 7) -x else x }
res14: IntSet = {{.5.}7{{.-15{.-12.}}-9.}}
```

Mapping is not easy...

The presented implementation of `map` is invalid!

```
scala> val set = Empty include 7 include 5 ... include 15
set: IntSet = {{.5.}7{{.9.}12{.15.}}}

scala> set map { x => if (x > 7) -x else x }
res14: IntSet = {{.5.}7{{.-15{.-12.}}-9.}}
```

```
scala> set map { x => 1 }
res14: IntSet = {{.1.}1{{.1.}1{.1.}}}
```

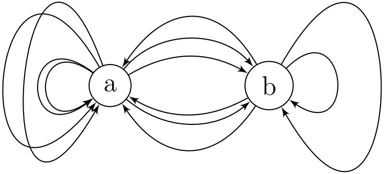
Category theory, simplistic view

Major tools in the categorical toolbox:

- **Abstraction**
- **Composition**
- **Identity**

Category theory, intuitive definition

A category is a bunch of objects with morphisms between them.



Category theory, intuitive definition

- **Objects** – are primitives, have no structure or properties.
- **Arrows** – morphisms, are primitives, join objects, have the beginning and the end.

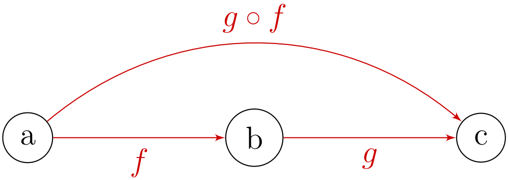
Category theory, intuitive definition

- Composition – arrows always compose.

$$
\begin{tikzcd}
a \arrow[r, "f"'] & b \arrow[r, "g"'] & c
\end{tikzcd}
$$

Category theory, intuitive definition

- Composition – arrows always compose.



Category theory, intuitive definition

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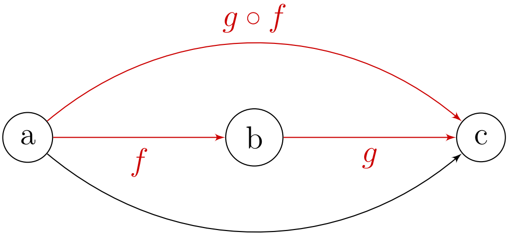

Category theory, intuitive definition

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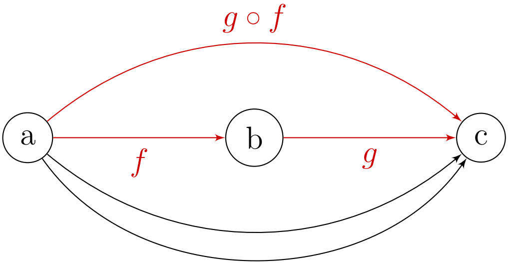
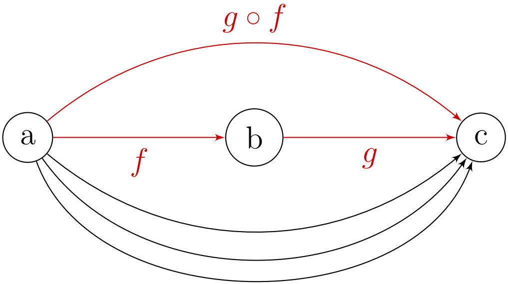


Category theory, intuitive definition

- Composition – arrows always compose.



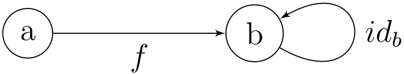
Category theory, intuitive definition

- Identity – for every object there is an identity arrow.

$$\begin{tikzcd} a \arrow[loop left, "id_a"] \arrow[r, "f"'] & b \arrow[loop right, "id_b"] \end{tikzcd}$$

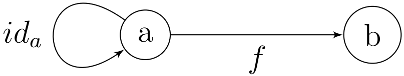
Category theory, axioms

- Left identity $id_a \circ f = f$



Category theory, axioms

- Right identity $f \circ id_a = f$



Category theory, axioms

- Associativity $h \circ (g \circ f) = (h \circ g) \circ f$

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

Category theory, axioms

- Associativity $h \circ (g \circ f) = (h \circ g) \circ f$

$$\begin{tikzcd} a \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] & b \arrow[r, "g"'] & c \arrow[r, "h"'] & d \end{tikzcd}$$

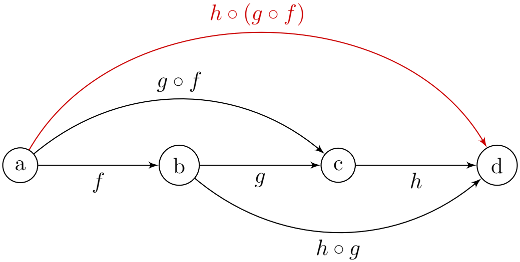
Category theory, axioms

- Associativity $h \circ (g \circ f) = (h \circ g) \circ f$

$$\begin{tikzcd}
a \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] \arrow[rrr, bend left=60, color=red, "h \circ (g \circ f)"] & b \arrow[r, "g"'] & c \arrow[r, "h"'] & d
\end{tikzcd}$$

Category theory, axioms

- Associativity $h \circ (g \circ f) = (h \circ g) \circ f$



Category theory, axioms

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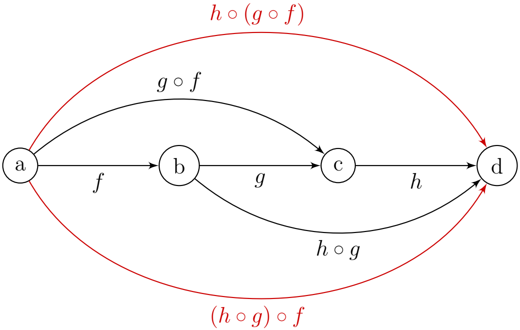


Category theory and programming

- Objects are types.
- Arrows are functions.

Morphisms, intuition

A **morphism** is a structure-preserving map from one mathematical object to another.

Homomorphisms, intuition

A **homomorphism** is a structure-preserving map between two algebraic structures of the same type.

Functors, intuition

A **functor** is a *homomorphism* between categories in the category theory.

Functors, intuition

What defines the structure in a category?

Functors, intuition

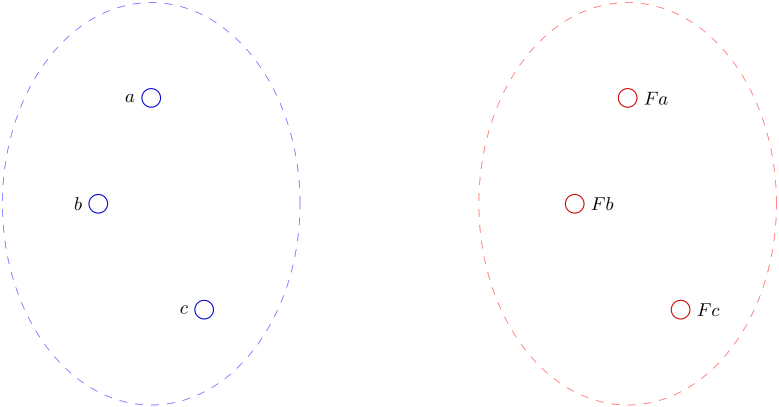
What defines the structure in a category?

Arrows and their composition!

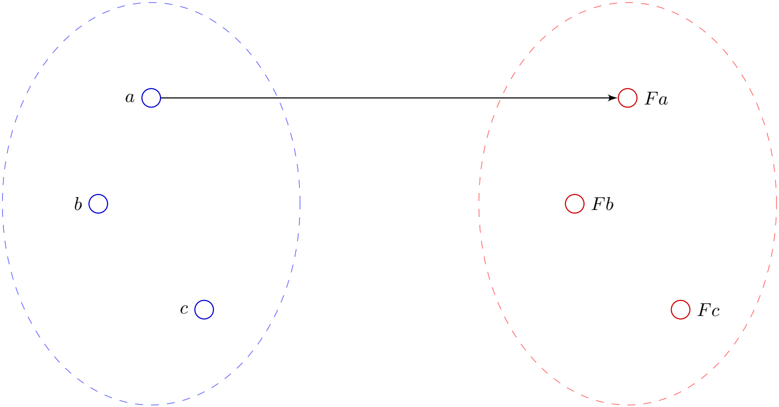
Functors, intuition

Functors must map objects to objects and arrows to arrows while preserving composition!

Functors, preserving structure



Functors, preserving structure



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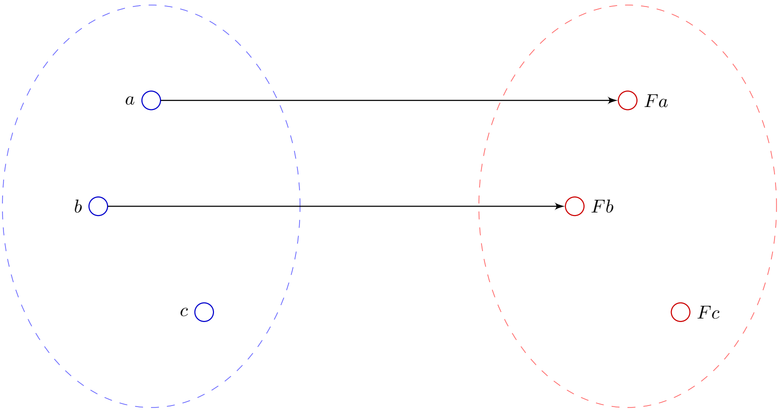
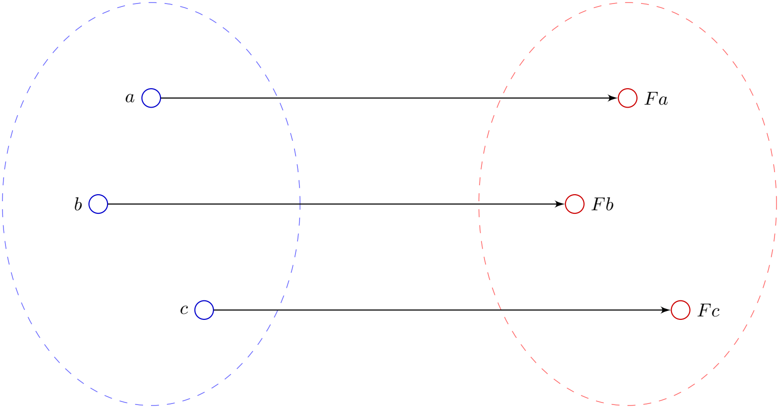
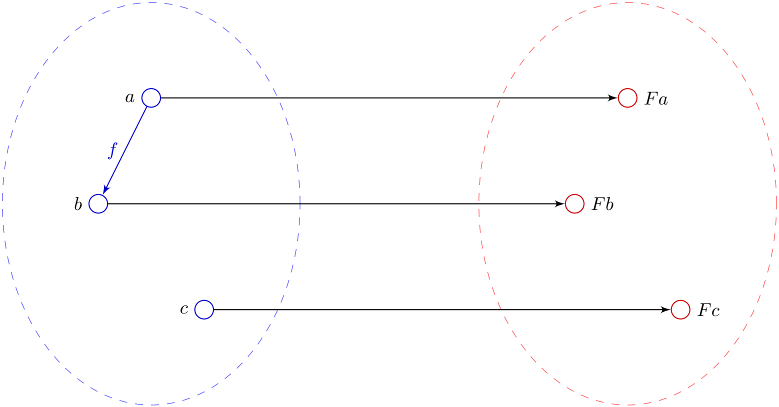


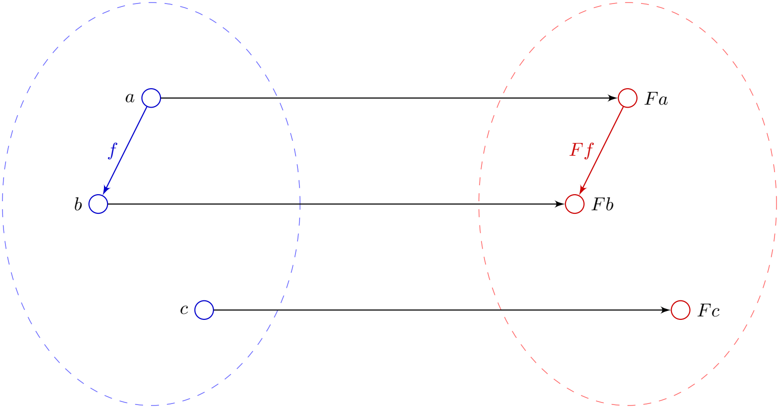
Functors, preserving structure



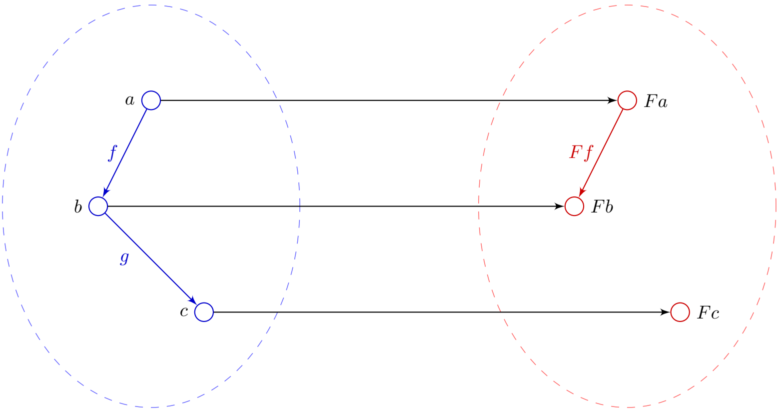
Functors, preserving structure



Functors, preserving structure



Functors, preserving structure



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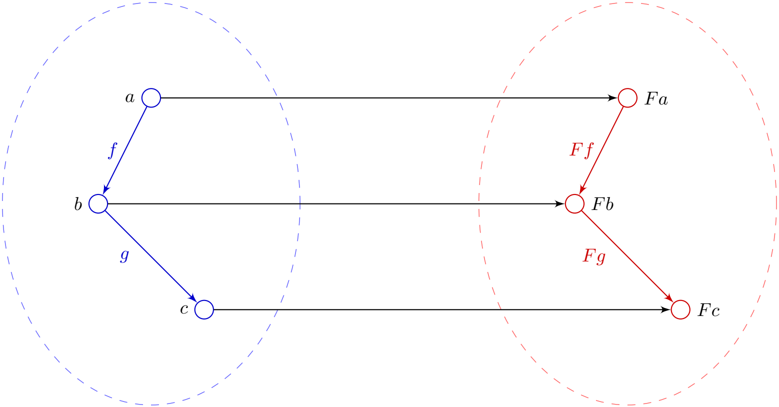

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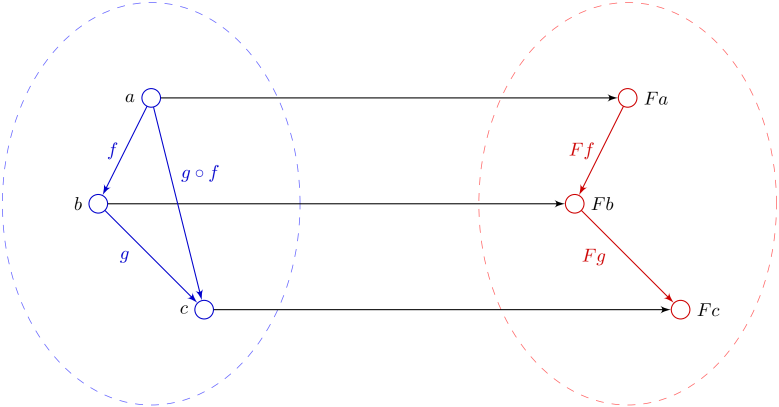
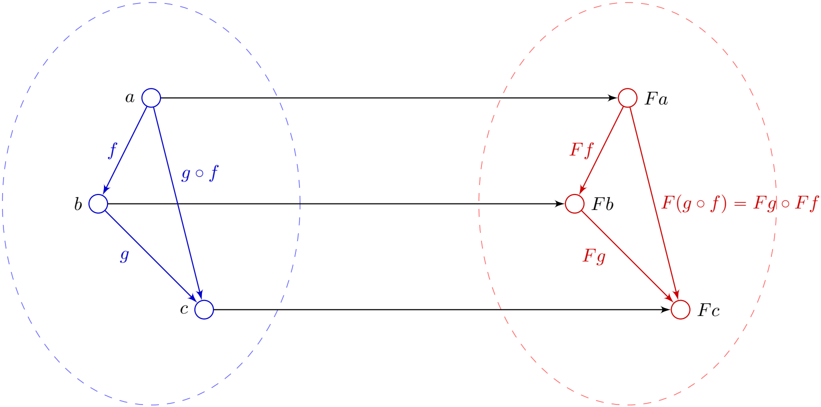
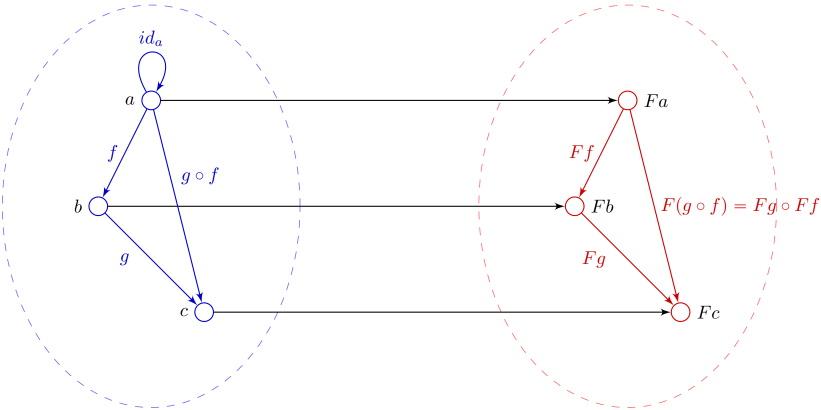


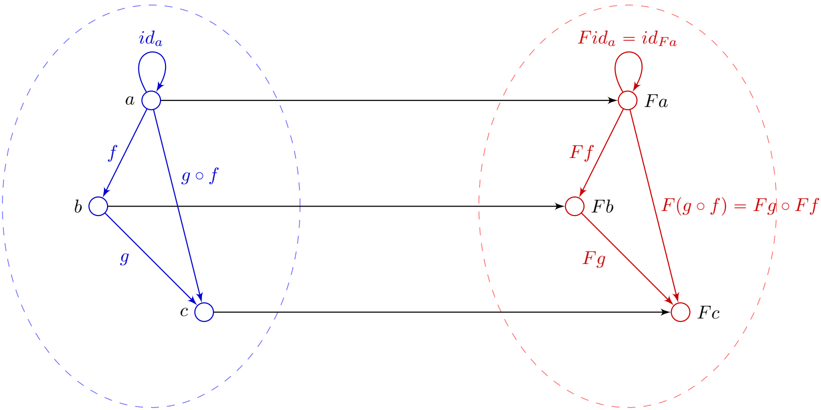
Functors, preserving structure



Functors, preserving structure



Functors, preserving structure



Functors, further intuition

A Functor is a collection of many morphisms for mapping all objects and all arrows.

Functors, programming

Objects are **types**, arrows are **functions** between types.

Functors, programming

Objects are **types**, arrows are **functions** between types.

Functors are mappings between types and functions.

Functors, `Int` and `IntSet`

Functors, `Int` and `IntSet`

$$Int \longrightarrow IntSet$$

Functors, `Int` and `IntSet`

Functors, `Int` and `IntSet`

Functors, Int and IntSet, wishful thinking

```
scala> val set = Empty include 7 include 5 ... include 15
set: IntSet = {{.5.}7{{.9.}12{.15.}}}

scala> set map {x => x}
res23: IntSet = {{.5.}7{{.9.}12{.15.}}}
```

Functors, Int and IntSet, wishful thinking

```
scala> val set = Empty include 7 include 5 ... include 15
set: IntSet = {{.5.}7{{.9.}12{.15.}}}

scala> set map {x => x}
res23: IntSet = {{.5.}7{{.9.}12{.15.}}}
```

```
scala> set map {x => 1}
res24: IntSet = {.1.}
```

Making IntSet functorial

```scala
sealed trait IntSet {
  ...
  def map(f: Int => Int): IntSet = this match {
    case Empty =>
      this
    case NonEmpty(el, left, right) =>
      ???
  }
}
```

Making IntSet functorial

```
sealed trait IntSet {
  ...
  def map(f: Int => Int): IntSet = this match {
    case Empty =>
      this
    case NonEmpty(el, left, right) => {
      val l = left map f
      val v = f(el)
      val r = right map f
      l union r adjoin v
    }
  }
}
```

Set operation `union`

$union(S_1, S_2)$ – produces a new set that contains all elements from S_1 and S_2.

Set operation `union`

$union(S_1, S_2)$ – produces a new set that contains all elements from S_1 and S_2.

```
sealed trait IntSet {
    ...
    def union(other: IntSet): IntSet
}

case object Empty extends IntSet {
    ...
    def union(other: IntSet): IntSet = other
}

case class NonEmpty(...) extends IntSet {
    ...
    def union(other: IntSet): IntSet = ???
}
```

Set operation `union`

$union(S_1, S_2)$ – produces a new set that contains all elements from S_1 and S_2.

Set operation `union`

$union(S_1, S_2)$ – produces a new set that contains all elements from S_1 and S_2.

```
case class NonEmpty(el: Int,
                    left: IntSet,
                    right: IntSet) extends IntSet {
  ...
  def union(other: IntSet): IntSet =
    ((left union right) union other) adjoin el
}
```

Properties of a functor

Properties of a functor

- `set map {x => x} == set`

Properties of a functor

- `set map {x => x} == set`
- `set.map(g).map(f) == set map {f.compose(g)}`

Functors, functoriality

`IntSet` is functorial!

Generic sets

```
type Set = IntSet

trait IntSet {...}
```

Generic sets

```
type Set = IntSet

trait IntSet {...}
```

```
type Set[+A] = Tree[A]

trait Tree[+A] {...}
```

Generic sets, type hierarchy

```
type Set[+A] = Tree[A]

sealed trait Tree[+A] {
}

case object Empty extends Tree[Nothing] {
  override def toString = "."
}

case class NonEmpty[A](a: A,
                       left: Tree[A],
                       right: Tree[A]) extends Tree[A] {
  override def toString: String =
    "{" + left + a + right + "}"
}
```

Generic sets, operations

```scala
def adjoin[A](x: A, set: Set[A])
             (implicit cmp: Ordering[A]): Set[A] = ???

def contains_?[A](x: A, set: Set[A])
                 (implicit cmp: Ordering[A]): Boolean = ???

def union[A](set1: Set[A], set2: Set[A])
            (implicit cmp: Ordering[A]): Set[A] = ???

def map[A, B](set: Set[A], f: A => B)
             (implicit cmp: Ordering[B]): Set[B] = ???
```

Generic sets, operation `adjoin`

```scala
def adjoin[A](x: A, set: Set[A])
             (implicit cmp: Ordering[A]): Set[A] =
  set match {
  case Empty =>
    NonEmpty(x, Empty, Empty)
  case NonEmpty(a, left, right) =>
    if (cmp.lt(x, a)) NonEmpty(a, adjoin(x, left), right)
    else if (cmp.gt(x, a)) NonEmpty(a, left, adjoin(x, right))
    else set
}
```

Generic sets, operation contains?

```
def contains_?[A](x: A, set: Set[A])
                 (implicit cmp: Ordering[A]): Boolean =
  set match {
  case Empty =>
    false
  case NonEmpty(a, left, right) =>
    if (cmp.lt(x, a)) contains_?(x, left)
    else if (cmp.gt(x, a)) contains_?(x, right)
    else true
}
```

Generic sets, operation `union`

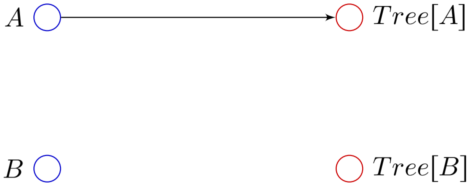
```
def union[A](set1: Set[A], set2: Set[A])
             (implicit cmp: Ordering[A]): Set[A] =
  set1 match {
  case Empty =>
    set2
  case NonEmpty(x, left, right) =>
    set2 match {
    case Empty =>
      set1
    case NonEmpty(_, _, _) =>
      adjoin(x, union(union(left, right), set2))
  }
}
```

Generic sets, operation map

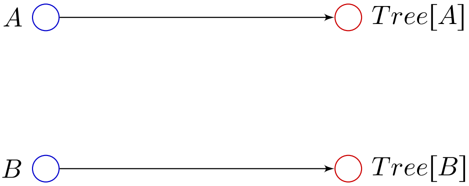
```scala
def map[A, B](set: Set[A], f: A => B)
             (implicit cmp: Ordering[B]): Set[B] =
  set match {
  case Empty =>
    Empty
  case NonEmpty(x, left, right) =>
    val l = map(left, f)
    val v = f(x)
    val r = map(right, f)
    adjoin(v, union(l, r))
}
```

Functoriality of `Tree[A]`

Functoriality of `Tree[A]`



Functoriality of `Tree[A]`



Functoriality of `Tree[A]`

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & Tree[A] \\
B \arrow[r] & Tree[B]
\end{tikzcd}
$$

Functoriality of `Tree[A]`

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & Tree[A] \arrow[d, "map\ f"] \\
B \arrow[r] & Tree[B]
\end{tikzcd}
$$

Functoriality of `Tree[A]`

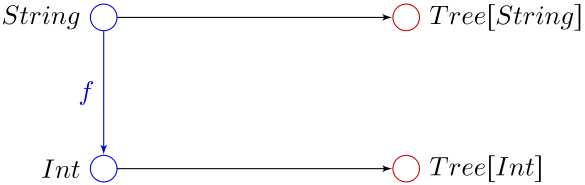
Functoriality of `Tree[A]`

$$
\begin{tikzcd}
String \arrow[rr] & & Tree[String] \\
Int & & Tree[Int]
\end{tikzcd}
$$

Functoriality of `Tree[A]`

$$\begin{tikzcd}
String \arrow[r] & Tree[String] \\
Int \arrow[r] & Tree[Int]
\end{tikzcd}$$

Functoriality of `Tree[A]`



Functoriality of `Tree[A]`

$$
\begin{tikzcd}
String \arrow[r] \arrow[d, "f"'] & Tree[String] \arrow[d, "map\ f"] \\
Int \arrow[r] & Tree[Int]
\end{tikzcd}
$$

A bit of coding. . .

Let's run some code!

Some final thoughts on binary search trees

Full binary search trees become unbalanced over time, and may even degenerate into a list.

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Full binary search trees become unbalanced over time, and may even degenerate into a list.

- They require maintenance – rebalancing.

Some final thoughts on binary search trees

Full binary search trees become unbalanced over time, and may even degenerate into a list.

- They require maintenance – rebalancing.
- There are more efficient structures: B-trees and red-black trees.

Looking back

What have we touched on today?

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- The representation problem.

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- The representation problem.
- ADTs and how `sealed` type hierarchies of `trait`s, `case object`s and `case class`es can model them.

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- The representation problem.
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- Functoriality and what it means.

Looking back

What have we touched on today?

- The representation problem.
- ADTs and how `sealed` type hierarchies of `trait`s, `case object`s and `case class`es can model them.
- Persistent data structures.
- Functoriality and what it means.
- Used both more OO and more functional approach to modelling data structures and their operations.

Looking back

What have we touched on today?

- The representation problem.
- ADTs and how `sealed` type hierarchies of `traits`, `case objects` and `case classes` can model them.
- Persistent data structures.
- Functoriality and what it means.
- Used both more OO and more functional approach to modelling data structures and their operations.
- Scala `objects` as modules.
- `Implicit` function arguments.

Q&A