Computer Lab 2: Time-varying volatility

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Victoria Lagerstedt & Kajsa Hansson Willis

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1. *Background*

This computer lab focuses on time-varying volatility and the different ways to describe it with models. These models have different inputs, and one of the factors this lab examines is if the leverage effect has any influence on the volatility. The leverage effect means that there is a difference of effect on the variance between negative and positive shocks. The negative shocks increase the variance more than the positive does.

Autoregressive conditional heteroskedasticity, ARCH, is a model to estimate the variance of bonds, stocks, and other financial investment types. ARCH uses the squared past error and a constant to calculate the current variance. See equation 1 for ARCH(1) where σ_t^2 is the current variance, ω is a constant, and $\alpha\eta_{t-1}^2$ is the squared past error weighted with alpha. This model is useful when only the effects happening short term are important.

$$\sigma_t^2 = \omega + \alpha\eta_{t-1}^2 \qquad \text{Equation 1}$$

The generalized autoregressive conditional heteroskedasticity, GARCH, is another financial model to estimate variance. GARCH is similar to the ARCH model but also uses the past variance to calculate the current variance. When using the GARCH model the current variance depends on squared past error, past variance, and a constant. See equation 2 for GARCH(1,1) where σ_t^2 is the current variance, ω is a constant, $\alpha\eta_{t-1}^2$ is the past squared error weighted with alpha, and $\beta\sigma_{t-1}^2$ is the past variance weighted with beta. The GARCH model is more complex than the ARCH model but this makes it better at forecasting the future variance.

$$\sigma_t^2 = \omega + \alpha\eta_{t-1}^2 + \beta\sigma_{t-1}^2 \qquad \text{Equation 2}$$

The threshold generalized autoregressive conditional heteroskedasticity, TGARCH, is another financial model to estimate variance that takes the leverage effect into account. The TGARCH model is similar to the GARCH model but also has the indicator function to account for the leverage effect. This indicator function makes the current variance dependent on whether the previous error was positive or negative. The indicator function is 1 if the past error is smaller than 0 and 0 if the past error is bigger than 0. See equation 3 for TGARCH(1,1) where σ_t^2 is the current variance, ω is a constant, $\alpha\eta_{t-1}^2$ is the past squared error weighted with alpha, $I(\eta_{t-1} < 0)$ is the indicator function, and $\beta\sigma_{t-1}^2$ is the past variance weighted with beta. This model is more complex than both the ARCH model and the GARCH model but if there is a leverage effect in the data the TGARCH model will give the best estimation of variance.

$$\sigma_t^2 = \omega + \alpha\eta_{t-1}^2 + \gamma\eta_{t-1}^2 I(\eta_{t-1} < 0) + \beta\sigma_{t-1}^2 \qquad \text{Equation 3}$$

The log-likelihood function is a function to test how good an estimate is following the true data. See equation 4 for the log-likelihood function used in this lab, where σ_t^2 is the current variance and η_t^2 is the current squared error. The part $-\ln(2\pi)$ doesn't depend on the parameters of the model and is therefore removed from the log-likelihood function in this lab. To find the best estimate the log likelihood function needs to be maximized.

$$ln\, L_t = \ -0.5 \cdot ln(\sigma_t^2) - \frac{\eta_t^2}{2\sigma_t^2} \qquad \text{Equation 4}$$

The data used in this lab is the price index of the U.S. 10-year government bond and the S&P 500 index between 1988-12-30 and 2013-01-25.

2. *Results*

2.1 Results of ARCH(1) model

The maximum value of the log-likelihood for the ARCH(1) model, given the conditions, was calculated by first deriving the log returns for each period. Then the residual, η_t, was calculated based on equation 5 below.

$$r_t = \mu + \eta_t \qquad \text{Equation 5}$$

This foundation was used for all of the different models. The ARCH(1) model estimated conditional variance was calculated using equation 1. The results can be observed in Figure 1. The first value, at time = 2, was the sample variance of the log returns. The log-likelihood for each time was then calculated according to equation 4. This was calculated based on the given initial values for the parameters, which can be observed in Table 1, and the initial condition that the sum of alpha, beta & gamma/2 is less than one.

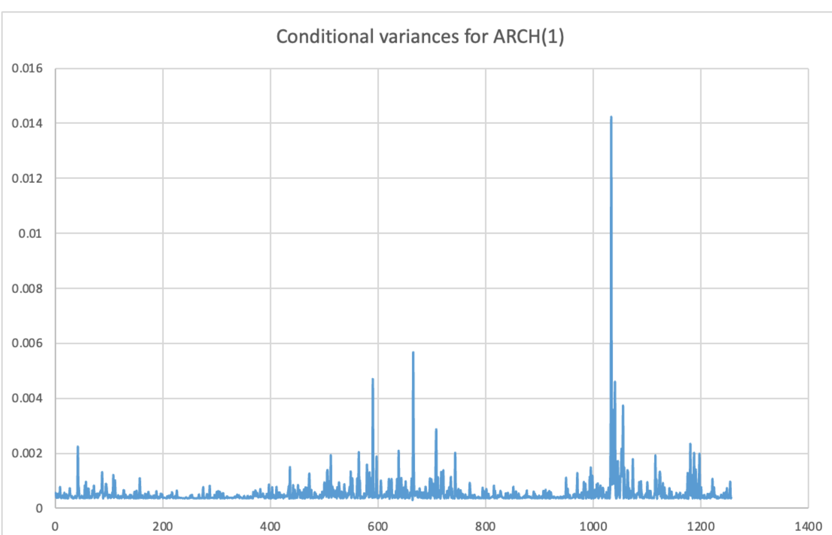

Figure 1: The estimated conditional variances for the ARCH(1) model

The initial log-likelihood for the ARCH(1) model with the initial values of the parameters was 3 998, which can be observed in Table 2.

Table 1: Table of the initial input values in the log-likelihood function

Parameters:	
mikro	0
omega	0,001
alpha	0,1
beta	0,1
gamma	0,1

Table 2: Table of the initial log-likelihood for the ARCH(1) model

Log likelihood function:
3998,013886

When the columns had their formulas, the sum of the log-likelihood values was calculated and maximized with the assistance of the Excel Problem Solver, with the parameters in the ARCH(1) model and the conditions that omega and alpha are nonnegative. The results can be observed in Table 3. Note that beta and gamma do not affect the ARCH(1) model. The maximized log-likelihood, which is 4 176, can be observed in Table 4.

Table 3: Table of the optimal values to maximize the log-likelihood function for the ARCH(1) model

Parameters:	
mikro	0,002178
omega	0,000355
alpha	0,336482
beta	0
gamma	0

Table 4: Table of the maximal log-likelihood function for the ARCH(1) model

Log likelihood function:
4175,797675

2.2 Results of GARCH(1,1) model

The calculations for the GARCH(1,1) model were done in the same way as the ARCH(1) model, with the exception that equation 1 was substituted by equation 2. Otherwise, the values were identified in the exact same way. The GARCH(1,1) model estimated conditional variance can be observed in Figure 2.

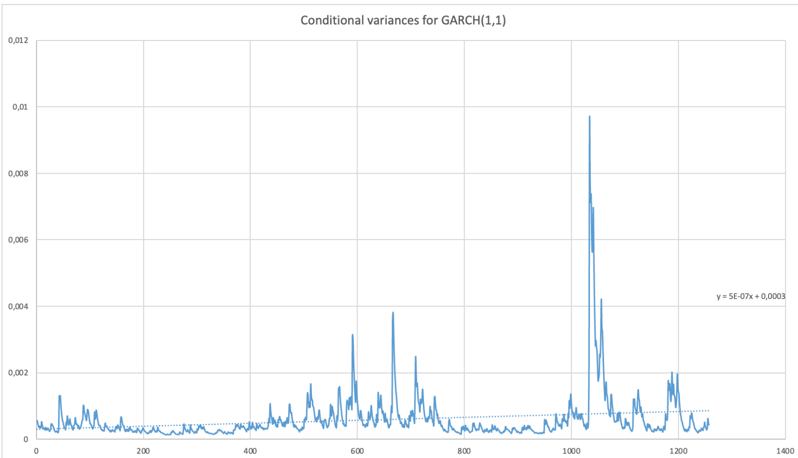

Figure 2: The estimated conditional variances for the GARCH(1,1) model

When the columns had been completed using the initial values, the Problem Solver was used again to maximize the log-likelihood. The log-likelihood for the initial values of the parameters can be observed in Table 5 and is 3 963. The resulting values of the parameters after the optimization are found in Table 6.

Table 5: Table of the initial log-likelihood for the GARCH(1,1) model

Log likelihood function:
3962,678353

Table 6: Table of the optimal values to maximize the log-likelihood function for the GARCH(1,1) model

Parameters:	
mikro	0,0023018
omega	2,267E-05
alpha	0,1861836
beta	0,7827623
gamma	0,1

The maximum log-likelihood for the GARCH(1,1) model was 4 243 which can be observed in Table 7.

Table 7: Table of the maximal log-likelihood function for the GARCH(1,1) model

Log likelihood function:
4242,787476

2.3 Results of TGARCH(1,1) model

The calculations for the TGARCH(1,1) model were done similarly to the GARCH(1,1) model with an add-on to the ARCH(1) model. This time the exception was that equation 1 was substituted by equation 3. The TGARCH(1,1) model estimated conditional variance can be observed in Figure 3.

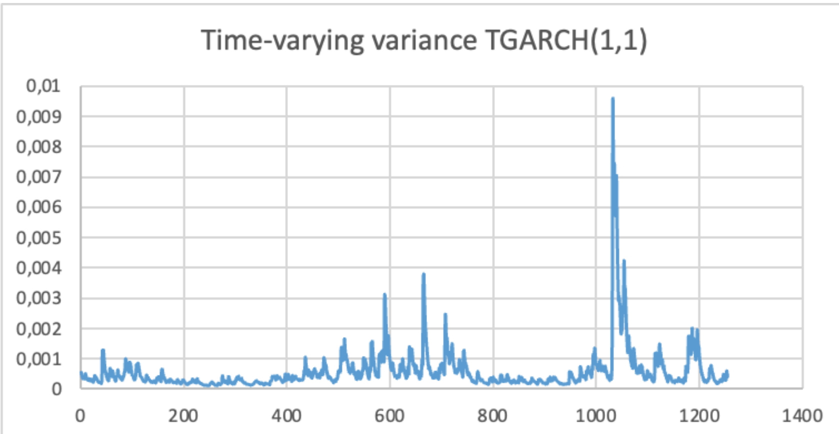

Figure 3: The estimated conditional variances for the TGARCH(1,1) model

When the columns had been completed using the initial values for the TGARCH(1,1) model, the Problem Solver was used to maximize the log-likelihood. The log-likelihood for the initial values of the parameters can be observed in Table 8 and is 3 965. The resulting values of the parameters after the optimization are found in Table 9.

Table 8: Table of the initial log-likelihood function for the TGARCH(1,1) model

Log likelihood function:
3964,610373

Table 9: Table of the optimal values to maximize the log-likelihood function for the TGARCH(1,1) model

Parameters:	
mikro	0,001418418
omega	2,28666E-05
alpha	0,03987744
beta	0,811501507
gamma	0,214173494

The maximum log-likelihood was 4 261 which can be observed in Table 10.

Table 10: Table of the maximal log-likelihood function for the TGARCH(1,1) model

Log likelihood function:
4260,883464

3. *Description of the estimation approach*

The estimation approach has been in accordance with the theories that have been used. These are ARCH, GARCH, TGARCH and log-likelihood. The instructions for Excel in the lab instructions were followed thoroughly. It was decided to use four significant digits as it was deemed appropriate after observing the results.

4. *Discussion*

4.1 Discussion of the ARCH(1) model

The log-likelihood value after maximizing the values for micro, omega, and alpha was larger than with the initial value. Therefore the initial values aren't the best-fitting values to estimate the variance. None of the optimal values for micro, omega, and alpha is zero which means all the parts of the ARCH model are significant and affect the current variance. The optimal alpha value is small which suggests that the past squared error isn't affecting the current variance very much, but nevertheless has an impact. The omega value is even smaller which means that there is no significant constant impacting the relationship.

4.2 Discussion of the GARCH(1,1) model

In the case of the GARCH(1,1) model, the log-likelihood value after maximizing is also larger than the one with the initial values. This means that the initial values weren't the best values to get the best estimate of the future variance. As it is higher than the log-likelihood for the ARCH(1), adding the past variance to the model has improved the results. This finding indicates that past variance impacts current and future variance. After maximizing the log-likelihood, the beta value had the largest value of the variable values which means the past variance affects the current variance more than the past errors do. The omega is very small, which means that there is no significant constant impacting the relationship. Compared to the beta, the alpha is quite small, but it is still impactful. This indicates that the past error also impacts the variance, although it is less than the past variance.

Compared to the ARCH(1) model, the micro values are quite similar. The omega value for the ARCH(1) is significantly higher than for the GARCH(1,1). The alpha value of the ARCH(1) is almost double that of the GARCH(1,1). This is not unexpected, since the GARCH(1,1) has more parameters to work with to get the data to fit in. The graph with the estimated variances from GARCH(1,1) is very similar to the graph of the estimated variances from ARCH(1), the peak at around 1 050 is visible in both of the graphs.

4.3 Discussion of the TGARCH(1,1) model

The TGARCH(1,1) model shows a log-likelihood of 4 261, which is higher than both the ARCH(1) model and the GARCH(1,1) model, which show 4 176 and 4 242 respectively. This means that, especially compared to the GARCH model, adding an indicator-based measurement is beneficial to increasing the model's ability to accurately describe the data. Since the indicator-based measurement that makes the difference between the GARCH and TGARCH models is a measure of the leverage effect, it can be concluded that the findings show empirical evidence of the leverage effect. However, it must be noted that this finding is not very large. The log-likelihood of the TGARCH is only about 0,45% larger than the log-likelihood of the GARCH model, and although they are logarithms, it is not a very significant difference. It is important to note that if the TGARCH is more difficult to estimate or more complex, it may not be worth using although it is slightly better. The gamma function, which is connected to the leverage effect, is 0,2142 and is quite large in the context. Gamma is the second highest of the parameters, after beta, and the difference between gamma and alpha, the third largest parameter, is almost 0,2. This indicates empirical evidence of a leverage effect.

It is interesting to note that the beta has the largest value of the parameters. The beta value represents the discount factor of the past variance. If this is large it must mean that the past variance has a large impact on the current or future variance. The omega and alpha are small, meaning the constant term and the past error are relatively insignificant.

The graph of the estimated conditional variances for the TGARCH, as seen in Figure 3, is very similar to the corresponding graphs of the other models, especially the GARCH. Like the others, it shows a high peak around the 1050th value.

5. Appendix

The Excel sheet “DataLab22024” is attached in Canvas.

5.1 Use of AI-based tools

ChatGPT has been used in a very limited amount. This AI-based tool has only been used to further develop our understanding of financial and statistical concepts, such as the leverage effect. The information generated by ChatGPT was compared against the course literature to check its validity. This use of ChatGPT was very helpful since it simplified some concepts and helped us understand them better, which made the analysis in this report better.