Home Assignment
FMSN25
Valuation of Derivative Assets

Pure arbitrage relations for basket call options & numerical calculation of prices

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1 Part A

In order to show that the relationship below holds, ie that the price of the arithmetic basket call option must be larger than or equal to the price of the geometric basket call option and smaller than or equal to the price of the weighted sum of the European call options, we shall start with investigating the payoff functions.

$$P^c_{GB}(0,T,K) \leq P^c_{AB}(0,T,K) \leq \sum_{i=1}^{n} c_i P^c_{E_i}(0,T,K_i) \text{ where } \sum_{i=1}^{n} c_i K_i = K$$

The payoff function for the geometric basket call option is defined as below:

$$((\prod_{i=1}^{n}(S_i(T))^{c_i}) - K)^+ = (\exp(\sum_{i=1}^{n} c_i X_i(T)) - K)^+ \text{ where } X_i(T) = \log(S_i(T))$$

The payoff function for the arithmetic basket call option is defined as below:

$$((\sum_{i=1}^{n} c_i S_i(T)) - K)^+$$

The payoff function for the European call option is defined as below:

$$(S_i(T) - K_i)^+$$

It can be stated that the only times that are relevant to all of the options above are the start time, $t = 0$, and the finish date $t = T$. There are no wealth transfers at any other time. It follows from this that if the payoff is equal at every state, then the price must also be equal. Otherwise, it would be possible to buy the cheaper one and sell the more expensive one. Then the cash flows would offset each other and the difference in price could be pocketed as pure arbitrage. It also holds that if the payoff is otherwise identical but larger, then the price must be higher as well.

1.1 Upper Bound

We will begin with examining the upper bound. The payoff for the weighted sum of the European options, Φ_E, can be written as below:

$$\Phi_E = \sum_{i=1}^{n} c_i (S_i(T) - K_i)^+$$

Clearly, this expression is very similar to the payoff for the arithmetic basket call option with the difference that the sum is outside the maximum-equation.

$$f(\sum_{i=1}^{n} p_i x_i) \leq \sum_{i=1}^{n} p_i f(x_i)$$

Since $f(x) = (x - K)^+$ is convex (the slope is non-decreasing), Jensen's inequality applies. Applying Jensen's inequality above yields the results below:

$$f(\sum_{i=1}^{n} c_i S_i(T)) \leq \sum_{i=1}^{n} c_i f(S_i(T))$$

This gives the expression below:

$$((\sum_{i=1}^{n} c_i S_i(T)) - K)^+ \leq \sum_{i=1}^{n} c_i (S_i(T) - K_i)^+$$

Since option prices are discounted risk-neutral expectations of payoffs and expectation preserves inequalities, the same inequality for prices is implied. Otherwise, arbitrage opportunities would appear. This proves the upper bound. Thus:

$$P^c_{AB}(0, T, K) \leq \sum_{i=1}^{n} c_i P^c_{E_i}(0, T, K_i)$$

1.2 Lower Bound

For the lower bound, Jensen's inequality is applied once more. Since the exponential function is convex and the weights satisfy $c_i \geq 0$, $\sum_{i=1}^{n} c_i = 1$, we obtain, for $f(x) = \exp(x)$, that the left-hand side is the geometric mean and the right-hand side the arithmetic mean:

$$\exp\Big(\sum_{i=1}^{n} c_i \ln S_i(T)\Big) \;\leq\; \sum_{i=1}^{n} c_i \exp(\ln S_i(T)) = \sum_{i=1}^{n} c_i S_i(T).$$

This simplifies to:

$$\prod_{i=1}^{n} S_i(T)^{c_i} \;\leq\; \sum_{i=1}^{n} c_i S_i(T).$$

Since the payoff function $f(x) = (x - K)^+$ for the call is monotone increasing in x, applying f preserves the inequality:

$$(\prod_{i=1}^{n} S_i(T)^{c_i} - K)^+ \leq (\sum_{i=1}^{n} c_i S_i(T) - K)^+$$

Because option prices are given by discounted expectations of their payoffs under the risk-neutral measure, the inequality carries over directly to prices, ensuring no arbitrage. Thus:

$$P^c_{GB}(0, T, K) \;\leq\; P^c_{AB}(0, T, K).$$

2 Part B

2.1 Lower bound for the Arithmetic Basket call option

The arbitrage-free price at $t = 0$ of the lower bound of the arithmetic basket call option is described by the relationship below, which expresses equality with the price of the geometric basket call. It is true that $G_T = \prod_{i=1}^{n}(S_i(T))^{c_i}$ has the same distribution as e^X where X is a Gaussian random variable with mean a and variance b. It is then possible to rewrite the first expression as $e^{\sum_{i=1}^{n}\ln(S_i(T))^{c_i}}$. Thus, we can draw the conclusion that $X = \sum_{i=1}^{n} c_i \ln S_i(T)$ when comparing distributions in this scenario. This means the right-hand side is a Gaussian random variable.

Under the risk-neutral measure, Q, the price of the stock is described by the relationship below:

$$S_i(T) = S_i(0)\ \exp\big((r - \tfrac{1}{2}\sigma_i^2)T + \sigma_i W_i(T)\big)$$

where $W = (W_1, \ldots, W_n)$ is an n-dimensional Brownian motion with correlations ρ_{ij}.

The mean, a, of $\ln G_T$ is: $a = \sum_{i=1}^{n} c_i(\ln S_i(0) + (r - \frac{1}{2}\sigma_i^2)T)$

The variance, b, stems from the random part from the Brownian motion. We have:

$$b = \mathrm{Var}(\sum_{i=1}^{n} c_i\sigma_i W_i(T))$$

As $\mathbb{E}[W_i(T)W_j(T)] = \rho_{ij}T$, the following is true:

$$b = \sum_{i,j} c_i c_j \sigma_i \sigma_j \rho_{ij} T = T\, c^\top\big(\,\mathrm{Diag}(\sigma)\,\rho\ \mathrm{Diag}(\sigma)\big)c$$

For a lognormal $X = e^Z$, with $Z \sim N(a, \sqrt{b})$, then $\mathbb{E}[X] = e^{a+\frac{1}{2}b}$. Thus, the expected value of the geometric basket is:

$$\mathbb{E}[G_T] = \exp\Big(a + \tfrac{1}{2}b\Big)$$

Under Q, it holds that $\ln G_T \sim N(a, \sqrt{b})$, so G_T is lognormal. The call price is the discounted expectation

$$C_{\text{lower}}(K) = e^{-rT}\,\mathbb{E}^Q\big[(G_T - K)^+\big] = e^{-rT}\Big(\mathbb{E}^Q[G_T\mathbf{1}_{\{G_T>K\}}] - K\,\mathbb{P}^Q(G_T > K)\Big).$$

Writing these two terms as integrals over the lognormal density $f_{G_T}(g)$ gives

$$\mathbb{P}^Q(G_T > K) = \int_K^\infty f_{G_T}(g)\,dg$$

$$\mathbb{E}^Q[G_T\mathbf{1}_{\{G_T>K\}}] = \int_K^\infty g\,f_{G_T}(g)\,dg.$$

With the change of variables $z = \frac{\ln g - a}{\sqrt{b}}$ (so $dg = \sqrt{b}\, e^{a+\sqrt{b}\,z} dz$), these become

$$\mathbb{P}^Q(G_T > K) = \int_{d_2}^{\infty} \phi(z)\, dz = N(d_2)$$

$$\int_K^{\infty} g\, f_{G_T}(g)\, dg = e^{a+\frac{1}{2}b} \int_{d_1}^{\infty} \phi(z)\, dz = \mathbb{E}[G_T] N(d_1)$$

where $\phi(\cdot)$ and $N(\cdot)$ are the standard normal probability and cumulative distribution functions described by $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ and $N(x) = \int_{-\infty}^{x} \phi(u)\, du$, and:

$$d_1 = \frac{a - \ln K + b}{\sqrt{b}} \qquad d_2 = d_1 - \sqrt{b}$$

This gives the price below:

$$C_{\text{lower}} = e^{-rT}\Big(\mathbb{E}[G_T] N(d_1) - K N(d_2)\Big),$$

This gives the prices of the Geometric basket call option seen in Table 1 below. This serves as the lower boundary for the arithmetic basket call option.

Strike K	**Lower Boundary**
80	28.696
100	20.98
120	15.43

Table 1: Lower boundaries (price of European geometric basket call options) for the arithmetic basket call option corresponding to different strike prices K.

2.2 Upper bound for the Arithmetic Basket Call Option

At time $t = 0$, the arbitrage-free upper bound for the arithmetic basket call option is given by the weighted sum of the prices of single-asset European call options:

$$P^c_{AB}(0, T, K)^U = \sum_{i=1}^{n} c_i P^c_{E_i}(0, T, K_i)$$

The upper bound was computed as the weighted sum of Black–Scholes prices of European call options on each stock. Since in our setup all assets are assumed to have the same initial price S_0, volatility σ, and risk-free rate r, the standard Black–Scholes formula could be applied uniformly across assets, as seen below. Since all assets are identical and $\sum c_i = 1$, it holds that $K_i = K$.

$$P^c_{E_i}(0, T, K) = S_0\, N(d_1) - K e^{-rT} N(d_2),$$

Here d_1 and d_2 are defined as $d_1 = \frac{\ln\left(\frac{S_0}{K}\right)+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

This was calculated in Matlab for $K = 80, 100$ and 120 for the $n = 12$ stocks in the basket.

The computed upper bound values are reported in Table 2.

Strike K	**Upper Boundary**
80	41.895
100	33.875
120	27.625

Table 2: Upper bounds for the arithmetic basket call option, computed as weighted sums of Black–Scholes prices of European calls corresponding to different strike prices K.

2.3 Monte Carlo Methods

2.3.1 Crude Monte Carlo

Here, the Monte-Carlo simulated arbitrage-free prices of the arithmetic basket call options are desired for different strike prices. This is done by simulating N random numbers from the multivariate normal distribution with mean $\mu = \ln(S_0) + (r - 0.5 * \sigma^2) * T$ and variance $\Sigma * T$ in order to simulate the future stock prices. This is possible because the stock prices follow a log-normal distribution. These were then inserted into the function in order to get the simulated payoffs. The expected price was then obtained through the formula below where $\Phi(T)$ is the mean payoff from the simulations:

$$P^c_{AB}(t, T, K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\Phi(T)]$$

The results can be seen in Figure 3 with the standard errors in Figure 4. The standard errors were calculated from $SE = \sqrt{V(\Phi(T)/N}$.

N	$K = 80$	$K = 100$	$K = 120$
1,000	38.264	29.259	22.576
10,000	36.464	27.436	20.685
100,000	37.265	28.199	21.408

Table 3: Monte Carlo estimates of arithmetic basket call prices for different strike prices.

The obtained values are all lower than the upper boundary and higher than the lower boundary, which is a promising result as it is in line with theory.

N	$K = 80$	$K = 100$	$K = 120$
1,000	2.214	2.066	1.914
10,000	0.616	0.566	0.515
100,000	0.201	0.185	0.169

Table 4: Standard errors of Monte Carlo estimates of arithmetic basket call prices for different strike prices.

2.3.2 Control Variate Monte Carlo

The control variate Monte Carlo simulation was conducted with the price of the geometric basket call option as the control variate. The expected value calculated in Part 2.1 was used as $E(X)$ in the expression below.

$$E(Y) = Y - \frac{C(X,Y)}{V(X)} * (X - E(X))$$

A stochastic normal variable was simulated in the same way as the crude Monte Carlo, where Y and X are the prices of both the arithmetic and geometric call options calculated from the stochastic variable. The average over the samples was then taken and the standard error was calculated according to the equation below:

$$SE = \sqrt{V(E(Y))/N}$$

This gave the results in Table 5 and 6.

N	$K = 80$	$K = 100$	$K = 120$
1,000	37.328	28.397	21.573
10,000	37.067	28.001	21.208
100,000	37.161	28.124	21.353

Table 5: Control variate Monte Carlo estimates of arithmetic basket call prices for different strike prices.

N	$K = 80$	$K = 100$	$K = 120$
1,000	0.21852	0.22781	0.22814
10,000	0.072835	0.074871	0.075013
100,000	0.022573	0.023342	0.023531

Table 6: Standard errors of control variate Monte Carlo estimates of arithmetic basket call prices for different strike prices.

It is clear that the values are still well within the boundaries. The variances in Table 6 are much lower compared to Table 4. This means that using control variates has been a very successful variance reduction technique. This is due to the high correlation between the arithmetic and geometric basket call options.

2.4 Equity-Linked Notes

The payoff of the equity-linked note is described by the relationship below.

$$\Phi(T) = \mathrm{NA}(1 + pr(\sum_{i=1}^{n} c_i \frac{S_i(T)}{S_i(0)} - 1)^+)$$

From the risk-neutral measure, it holds that the price is described by the following relationship:

$$P_{ELN}^C(0) = e^{-rT}\mathbb{E}(\Phi(T)) = e^{-rT} * NA(1 + pr(\sum_{i=1}^{n} c_i \frac{S_i(T)}{S_i(0)} - 1)^+)$$

The Monte Carlo simulations can be done in the same way as Section 2.3 by exploiting that e^X follows a lognormal distribution with the distribution $X \sim N(\ln(S_0) + (r - 0.5 * \sigma^2)T), \Sigma T)$.

2.4.1 Finding *pr*

In order to create the Monte-Carlo, the Σ-matrix must first be created. This is done differently than in Section 2.3, with the formula below instead:

$$\Sigma = \mathrm{diag}(\sigma) * \rho * \mathrm{diag}(\sigma),$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$, ρ is the correlation matrix and r is the continuously compounded risk–free rate.

The payoff is then calculated for $T = 5$ over the 12 stocks for each independent sample using the formula presented previously. Then the mean is taken over the 10,000 samples. The price is then discounted according to the price formula. The value of pr was then isolated and solved for using the gathered values according to the formula below so that the price is $1.1NA$ (note that the equation is not defined if $\sum_{i=1}^{n} c_i \frac{S_i(T)}{S_i(0)} \leq 0$):

$$pr = \frac{1.1e^{rT} - 1}{\mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{i=1}^{n} c_i \frac{S_i(T)}{S_i(0)} - 1\right)^+\right]}$$

This gave a value of $pr \approx 0.432$ with 10,000 samples. To test the validity of this the price was also calculated using the earlier formula. This gave $P_{ELN}^C =$ 110, which is what it should be.

2.4.2 Determining the value for $T = 4$

In the next exercise, the fair price of the contract after a year has passed is to be determined. In this case, the mean of the lognormal and the variance have shifted. The Σ-matrix is calculated according to the same formula as previously with $T = 4$ instead. The normal distribution is $X \sim N(\ln(S_1) + (r - 0.5 * \sigma^2) * T), \Sigma * T)$ in the Monte Carlo simulations instead.

Inserting the value of pr from earlier, the payoffs are then calculated using the same formula as above. Discounting for the time and risk-free return according to the formula below gives the price, which is 108.99.

$$P_{ELN}(0) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\Phi(T)] = e^{-rT}\,NA\left(1 + pr \cdot \mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{i=1}^{n} c_i \frac{S_i(T)}{S_i(0)} - 1\right)^{+}\right]\right).$$

This is slightly lower than the price where $T = 5$.