1 Quicksort

1.1 Pseudocode

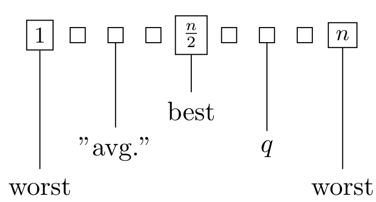
Algorithm 9 Quicksort

procedure QUICKSORT(A, p, r)
    if $p < r$ **then**
        $q \leftarrow$ PARTITION(A, p, r)
        QUICKSORT$(A, p, q-1)$
        QUICKSORT$(A, q+1, r)$
    end if
end procedure

function PARTITION(A, p, r)
    $X \leftarrow A[r]$
    $i \leftarrow p - 1$
    for $j = p$ **to** $r - 1$ **do**
        if $A[j] \leq X$ **then**
            $i \leftarrow i + 1$
            $A[i] \leftrightarrow A[j]$
        end if
    end for
    $A[i+1] \leftrightarrow A[r]$
    return $(i + 1)$
end function

1.2 Analysis

The worst case is when the pivot is either in the *first* or *last* index, best case is when the pivot is the *median* index, and the "average" case is when the pivot is between the first/last and median index (here q will represent the *true* average).



Worst Case

As a recurrence,

$$
\begin{aligned}
T(n) &= T(n-1) + n - 1, \quad T(0) = T(1) = 0 \\
&= \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}
\end{aligned}
$$

Best Case

As a recurrence,

$$
\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + n - 1, \quad T(0) = T(1) = 0 \\
&= n \lg n - n + 1
\end{aligned}
$$

Note: This is the same recurrence as Mergesort!

Average Case

As a recurrence,

$$
T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{3n}{4}\right) + n - 1, \quad T(0) = T(1) = 0
$$

Solve with Strong Constructive Induction.
Guess: $T(n) \leq an \lg n$ for $n \geq 1$ and some constant a.
Base case $n = 1$: $an \lg n = a1 \lg 1 = a \cdot 1 \cdot 0 = 0$, $T(1) = 0$ and $0 \leq 0$.
Inductive Hypothesis: Assume true for $< n$, $T(k) \leq ak \lg k$ for $1 \leq k \leq n$.
Inductive Step:

$$
\begin{aligned}
T(n) &= T\left(\frac{n}{4}\right) + T\left(\frac{3n}{4}\right) + n - 1 \\
&\leq a\frac{n}{4} \lg\left(\frac{n}{4}\right) + a\frac{3n}{4} \lg\left(\frac{3n}{4}\right) + n - 1 \qquad \text{by IH} \\
&\leq a\frac{n}{4} (\lg n - \lg 4) + a\frac{3n}{4} (\lg 3 + \lg n - \lg 4) + n - 1 \\
&\leq \frac{an}{4} (\lg n - 2) + \frac{3an}{4} (\lg 3 + \lg n - 2) + n - 1 \\
&\leq \frac{an \lg n}{4} - \frac{2an}{4} + \frac{3an \lg 3}{4} + \frac{3an \lg n}{4} - \frac{2 \cdot 3an}{4} + n - 1 \\
&\leq an \lg n + \left(-\frac{a}{2} + \frac{3a \lg 3}{4} - \frac{3}{2}a + 1\right) n - 1 \\
&\leq an \lg n + \left(\left(\frac{3 \lg 3}{4} - 2\right) a + 1\right) n - 1
\end{aligned}
$$

It follows that we need

$$\left(\frac{3\lg 3}{4} - 2\right) a + 1 \leq 0 \implies a \geq \frac{1}{2 - \frac{3\lg 3}{4}}$$

$$a \gtrsim 1.23$$

Thus,

$$T(n) \lesssim 1.23 n \lg n$$

Exact Average Case

As a recurrence,

$$\begin{aligned}
T(n) &= \left[\sum_{q=1}^{n} \frac{1}{n}\Big(T(q-1) + T(n-q)\Big)\right] + n - 1, \quad T(0) = T(1) = 0 \\
&= \frac{1}{n}\sum_{q=1}^{n}\Big(T(q-1) + T(n-q)\Big) + n - 1 \\
&= \frac{1}{n}\sum_{q=1}^{n} T(q-1) + \frac{1}{n}\sum_{q=1}^{n} T(n-q) + n - 1 \\
&= \frac{1}{n}\sum_{q=0}^{n-1} T(q) + \frac{1}{n}\sum_{q=0}^{n-1} T(q) + n - 1^{*} \\
&= \frac{2}{n}\sum_{q=0}^{n-1} T(q) + n - 1
\end{aligned}$$

*Note that:

$$\sum_{q=1}^{n} T(q-1) = T(0) + T(1) + T(2) + \cdots + T(n-1)$$

$$\sum_{q=1}^{n} T(n-q) = T(n-1) + T(n-2) + T(n-3) + \cdots + T(0)$$

Solve with Strong Constructive Induction.
Guess: $T(n) \leq an \lg n$ for $n \geq 1$ and some constant a.
Base case $n = 1$: $an \lg n = a1 \lg 1 = a \cdot 1 \cdot 0 = 0$, $T(1) = 0$ and $0 \leq 0$.
Inductive Hypothesis: Assume true for $< n$, $T(k) \leq ak \lg k$ for $1 \leq k \leq n$.

Inductive Step:

$$
\begin{aligned}
T(n) &= n - 1 + \frac{2}{n}\sum_{q=0}^{n-1} T(q) \\
&= n - 1 + \frac{2}{n}\sum_{q=1}^{n-1} T(q) \\
&\le n - 1 + \frac{2}{n}\sum_{q=1}^{n-1} aq\lg q \qquad \text{by IH} \\
&\le n - 1 + \frac{2a}{n}\int_1^n x\lg x dx \qquad \text{by integral bound} \\
&= n - 1 + \frac{2a}{n}\left[\frac{x^2\lg x}{2} - \frac{x^2\lg e}{4}\right]\Big|_1^n \\
&= n - 1 + an\lg n - \frac{an\lg e}{2} + \frac{a\lg e}{2n} \\
&= an\lg n + \left[1 - \frac{a\lg e}{2}\right] n - 1 + \frac{a\lg e}{2n}
\end{aligned}
$$

It follows that we need

$$1 - \frac{a\lg e}{2} \le 0 \implies a \ge \frac{2}{\lg e}$$

So set $a = \frac{2}{\lg e} \approx 1.39$, we then need

$$\frac{a\lg e}{2n} - 1 \le 0 \iff \frac{2\lg e}{(\lg e)2n} - 1 \le 0 \iff \frac{1}{n} - 1 \le 0$$

Thus,

$$T(n) \lesssim 1.39n\lg n$$

We can realize a more natural formula,

$$T(n) \le an\lg n = \frac{2n\lg n}{\lg e} = 2n\ln n$$

Theorem
The expected number of comparisons for Quicksort is $\approx 2n\ln n$.

Blum's Exact Average Case

Let $P(i, j)$ be the probability that the ith smallest and jth smallest elements are compared.

Theorem
The average number of comparisons for quicksort is

$$\sum_{1 \le i < j \le n} P(i,j)$$

Only matters the first time an element from $A[i], \cdots, A[j]$ is picked as pivot. The probability that the ith and jth smallest elements are compared during the execution of quicksort is:

$$\frac{2}{j-i+1}$$

Then,

$$\begin{aligned}
\sum_{1 \le i < j \le n} P(i,j) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} P(i,j) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{1}{j-i+1} \\
&= 2 \sum_{i=1}^{n} \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-i+1} \right) \\
&= 2 \sum_{i=1}^{n} (H_{n-i+1} - 1) = 2 \sum_{i=1}^{n} H_{n-i+1} - 2 \sum_{i=1}^{n} 1 \\
&= 2(H_n + H_{n-1} + \cdots + H_1) - 2 \sum_{i=1}^{n} 1 \\
&= 2 \sum_{i=1}^{n} H_i - 2n \\
&= 2((n+1)H_n - n) - 2n \\
&= 2(n+1)H_n - 4n \approx 2n \ln n
\end{aligned}$$

1.3 Comments

Quicksort Code

Recall the Quicksort procedure,

```
procedure QUICKSORT(A, p, r)
    if p < r then
        q ← PARTITION(A, p, r)
        QUICKSORT(A, p, q − 1)
        QUICKSORT(A, q + 1, r)
    end if
end procedure
```

When executing quicksort$(A, p, q-1)$ we must stack up quicksort$(A, q+1, r)$ to be executed later. We need to store the pair of index values $(q+1, r)$.

Stack in Action

The left table represents the pivot being the largest element, and the right table represents the pivot being the smallest element.

Pivot	**Stack**	**Pivot**	**Stack**
$A[n]$	$(n+1, n)$	$A[1]$	$\cancel{(2, n)}$
$A[n-1]$	$(n, n-1)$	$A[2]$	$\cancel{(3, n)}$
$A[n-2]$	$(n-1, n-2)$	$A[3]$	$\cancel{(4, n)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$A[3]$	$(4, 3)$	$A[n-2]$	$\cancel{(n-1, n)}$
$A[2]$	$(3, 2)$	$A[n-1]$	$\cancel{(n, n)}$
$A[1]$		$A[n]$	

When the pivot is the smallest element the stack remains very small but if the pivot is the largest element we stack $n-1$ problems to do later.

In Place

On average, height of the stack for Quicksort is $\Theta(\log n)$. Height of stack for Mergesort is $\lg n + O(1)$.

Definition: An algorithm is *in place* if it uses $O(1)$ extra variables and (on average) $O(\log n)$ extra index variables.

A more technical definition is; An algorithm is *in place* if it uses at most $O(1)$ extra variables and (on average) $O\left((\log n)^2\right)$ extra bits.

Stack

We can modify the Quicksort procedure so that stack has height at most $\lg n$.

procedure QUICKSORT(A, p, r)
    if $p < r$ **then**
        $q \leftarrow$ PARTITION(A, p, r)
        if $q \leq (p+r)/2$ **then**
            QUICKSORT$(A, p, q-1)$
            QUICKSORT$(A, q+1, r)$
        else
            QUICKSORT$(A, q+1, r)$
            QUICKSORT$(A, p, q-1)$
        end if
    end if
end procedure

Is Quicksort In Place?

Quicksort is in place, but it does not use a constant amount of extra space. Quicksort uses slightly more than a constant amount of extra space.

Choice of Pivots

It is risky to pivot on the last element because the last element could be the largest element. Some better ways could be;

- Pivot on middle element.
- Pivot on median of first, middle, and last elements.
- Pivot on random element.
- Pivot on median of three random elements.
- Pivot on median of five random elements. (Law of diminishing returns.)
- Randomly permute array before starting. (Equivalent to pivoting on random element.)

Partitioning

There are a variety of partition routines. The current edition of the textbook has a version that uses $n-1$ comparisons, but the previous edition of the textbook has a version which uses n comparisons.