1 Insertion Sort with Sentinel

1.1 Pseudocode

Algorithm 1 Insertion Sort with Sentinel

```
A[0] ← −∞
for i = 2 to n do
    t ← A[i]
    j ← i − 1
    while t < A[j] do
        A[j + 1] ← A[j]
        j ← j − 1
    end while
    A[j + 1] ← t
end for
```

1.2 Analysis of Comparisons

Worst Case

Worst case is when the array is reverse sorted, and every element must be moved. The while loop always decrements j to zero to compare against the sentinel value.

$$\sum_{i=2}^{n} i = \left(\sum_{i=1}^{n} i\right) - 1 = \frac{(n+1)n}{2} - 2 = \frac{(n+2)(n-1)}{2}$$

Best Case

Best case is when the array is already sorted, there is only one comparison for each iteration of the for loop.

$$\sum_{i=2}^{n} 1 = (n-2) + 1 = n - 1$$

Average Case

For average case we have to determine the probability that a given element will move. So we want the expected value of $\sum_{x \in X} P(x)V(x)$, where $P(x)$ is the probability that an element will end up a location and $V(x)$ is the number of moves.

$$\begin{aligned}
\sum_{x \in X} P(x)V(x) &= \sum_{i=2}^{n} \sum_{j=1}^{i} \frac{1}{i} \cdot (i - j + 1) = \sum_{i=2}^{n} \frac{1}{i} \sum_{j=1}^{i} (i - j + 1) \\
&= \sum_{i=2}^{n} \frac{1}{i} \sum_{j=1}^{i} j = \sum_{i=2}^{n} \frac{1}{i} \cdot \frac{(i+1)i}{2} \\
&= \sum_{i=2}^{n} \frac{i+1}{2} = \frac{1}{2} \sum_{i=2}^{n} i + 1 \\
&= \frac{1}{2} \sum_{i=1}^{n} i - 1 + (n-1) \\
&= \frac{1}{2} \left(\frac{(n+1)n}{2} - 1 + \frac{(n-1)2}{2} \right) \\
&= \frac{(n+4)(n-1)}{4}
\end{aligned}$$

1.3 Analysis of Exchanges

Worst Case

Worst case is two moves for each iteration of outer loop plus worst case number of comparisons, minus the time when the comparison is false at the end. A shortcut of this is at each iteration we do one more move than comparison, so take the value we got above and add the number of loop iterations.

$$\frac{(n+2)(n-1)}{2} + n$$

Best Case

Best case is one initial move in the assignment on line 1 and then two moves during each iteration of the for loop. So we get,

$$1 + 2(n-1) = 2n - 1$$

Average Case

Average case is whenever there is a comparison there is a move except for the single instance in each iteration of when it evaluates to false. Thus the analysis method is the same as worst case.

$$\frac{(n+4)(n-1)}{4} + n$$