1 Merge Sort

1.1 Pseudocode

Algorithm 1 Merge Sort

```
procedure MERGESORT($A, p, r$)
    if $p < r$ then
        $q \leftarrow \lfloor (p + r)/2 \rfloor$
        MERGESORT($A, p, q$)
        MERGESORT($A, q + 1, r$)
        MERGE($A, (p, q), (q + 1, r)$)
    end if
end procedure

procedure MERGE($A, (p, q), (q + 1, r)$)
    copy $(A, p, r)$ into $(B, p, r)$
    $i \leftarrow p$
    $j \leftarrow q + 1$
    $k \leftarrow p$
    while $i \leq q$ and $j \leq r$ do
        if $B[i] \leq B[j]$ then
            $A[k] \leftarrow B[i]$
            $i \leftarrow i + 1$
        else
            $A[k] \leftarrow B[j]$
            $j \leftarrow j + 1$
        end if
        $k \leftarrow k + 1$
    end while
    if $i > q$ then
        copy $(B, j, r)$ into $(A, k, r)$
    else
        copy $(B, i, q)$ into $(A, k, r)$
    end if
end procedure
```

1.2 Analysis of Merge

Equal Size

Best case, we only do n comparisons (this is when $A_n < B_1$ or $B_n < A_1$). Worst case is $2n-1$ comparisons (this is when A_n and B_n are the two largest elements). The average case is $2n - 2 + \frac{2}{n+1}$.

Differing Sizes

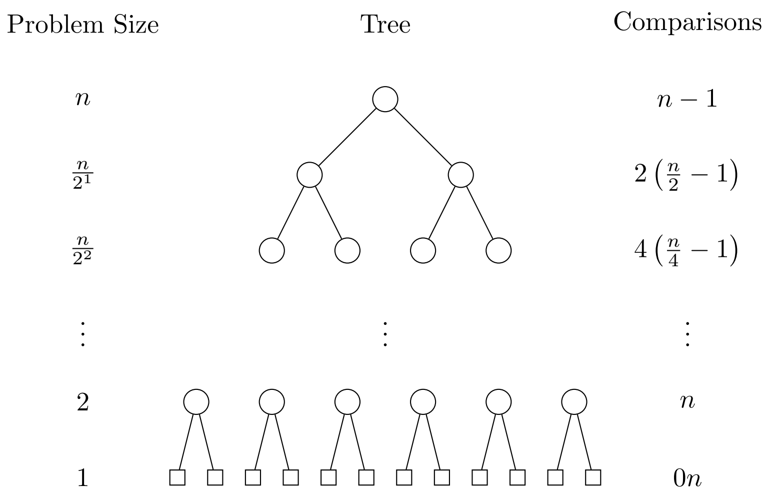
Let subarray A be of size m and subarray B be of size n where $m \leq n$. Best case is m comparisons (this is when $A_m < B_1$). Worst case is $m + n - 1$ (this is when A_m and B_n are the two largest elements). When m is much smaller than n there are better algorithms we can use. For $m = 1$ binary search gives $\approx \lg n$.

Notes

Merging is *not* in place. It can be implemented in place but those algorithms are not practical.

1.3 Analysis of Mergesort Comparisons

We will analyze mergesort using the tree method, along with the assumption that n is a power of 2.



We can notice that each level does exactly $2^k(\frac{n}{2^k} - 1)$ comparisons where k is the level of that branch, it follows then that since the height of the tree is $\lg n$, the total number of comparisons is

$$\begin{aligned}
\sum_{i=0}^{\lg n-1} 2^i \left(\frac{n}{2^i} - 1\right) &= \sum_{i=0}^{\lg n-1} \left(n - 2^i\right) \\
&= \sum_{i=0}^{\lg n-1} n - \sum_{i=0}^{\lg n-1} 2^i \\
&= (n \lg n) - \left(2^{\lg n-1+1} - 1\right) \\
&= (n \lg n) - (n-1) \\
&= n \lg n - n + 1
\end{aligned}$$

As a recurrence,

$$\begin{aligned}
T(n) &= T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + (n-1) \\
&= 2T\left(\frac{n}{2}\right) + n - 1, \quad T(0) = T(1) = 0
\end{aligned}$$