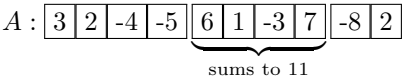
1 Maximum Subarray Problem

Given an array of numbers (either positive or negative), we wish to calculate the subset of consecutive numbers whose sum is largest.



In this example, we see that the maximum sum is 11 with the indecies of the subarray being $A[5] : A[8]$.

1.1 Cubic Time

Pseudocode

Algorithm 1 Maximum Continguous Sum (*Cubic*)

```
1:  M ← 0
2:  for i = 1 to n do
3:      for j = i to n do
4:          S ← 0
5:          for k = i to j do
6:              S ← S + A[k]
7:          end for
8:          M ←MAX(M, S)
9:      end for
10: end for
```

Analysis

From our algorithm above we can get the following analysis for it's complexity.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j} 1 &= \sum_{i=1}^{n}\sum_{j=i}^{n} j - i + 1 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n-i+1} j \\
&= \sum_{i=1}^{n} \frac{(n-i+1)(n-i+2)}{2} \\
&= \frac{1}{2}\sum_{i=1}^{n}(n-i+1)(n-i+2) \\
&= \frac{1}{2}\sum_{n-i+1=1}^{n} i(i+1) \\
&= \frac{1}{2}\frac{n(n+1)(n+2)}{3}
\end{aligned}$$

From this, we can see this first version of our algorithm is $\Theta(n^3)$ or cubic time.

1.2 Quadratic Time

Pseudocode

Algorithm 1 Maximum Continguous Sum (*Quadratic*)

```
1: M ← 0
2: for i = 1 to n do
3:     S ← 0
4:     for j = i to n do
5:         S ← S + A[j]
6:         M ←MAX(M, S)
7:     end for
8: end for
```

Analysis

We then use sums to analyze this algorithm.

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i}^{n} 1 &= \sum_{i=1}^{n} n - i + 1 \\ &= \sum_{i=1}^{n} i \\ &= \frac{n(n+1)}{2}\end{aligned}$$

So this more efficient version is $\Theta(n^2)$ or quadratic time.

1.3 Linear Time

Using dynamic programming we can improve on this algorithm even further. Conceptually, the maximum sum is just the previous maximum sum plus the current value.

Pseudocode

Algorithm 1 Maximum Continguous Sum (*Linear*)

```
1: M ← 0
2: S ← 0
3: for i = 1 to n do
4:     S ←MAX(S + A[i], 0)
5:     M ←MAX(M, S)
6: end for
```

Analysis

It follows that this algorithm is $\Theta(n)$ or linear time. Correctness of this algorithm stems from proof by induction on $S \leftarrow \text{MAX}(S + A[i], 0)$, as this is the loop invariant.