Practice Exams

Exam 1 Sample A

---

1. Write down the prime factorization of 10!.
2. Find the least non-negative residue of $11^{67} \mod 13$.
3. Find all incongruent solutions $\mod 40$, as least non-negative residues, to the following lienar congruence:

$$12x \equiv 28 \mod 40$$

4. Use the Euclidean Algorithm to find $\gcd(390, 72)$ and write this as a linear combination of the two.
5. Use the Chinese Remainder Theorem to find the smallest positive solution to the system:

$$\begin{aligned} x &\equiv 2 \mod 5 \\ x &\equiv 1 \mod 6 \\ x &\equiv 4 \mod 7 \end{aligned}$$

6. Use mathematical induction to prove that:

$$n! \geq n^3 \text{ for } n \geq 6$$

7. Determine if the following sets are well-ordered or not. You may assume only that $\mathbb{Z}^+$ is well-ordered.

$$\begin{aligned} S_1 &= [0,1] \cap \mathbb{Q} \\ S_2 &= \{1 - 2^k \mid k \in \mathbb{Z}^+\} \end{aligned}$$

8. Use the Fundamental Theorem of Arithmetic (uniqueness of prime factorization) to prove that $\sqrt{2}$ is irrational. Hint: Use contradiction.
9. Suppose $a, b, c, d \in \mathbb{Z}$ with $a \mid c$, $b \mid c$, $d = \gcd(a, b)$, and $d^2 \mid c$. Prove that $ab \mid c$.

Exam 1 Sample B

---

1. (a) Find $\pi(18)$.
   (b) Show that the set $\{\frac{a}{b} \mid a, b \in \mathbb{Z}^+, a > b\}$ is not well-ordered.
   (c) Find how many primes there are, approximately, between one billion and two billion.
2. Find the number of zeros at the end of 1000! with justification.
3. The following are all false. Provide explicit numerical counterexamples.
   (a) $a \mid bc$ implies $a \mid b$ or $a \mid c$.
   (b) $a \mid b$ and $a \mid c$ implies $b \mid c$.
   (c) $3 \mid a$ and $3 \mid b$ implies $\gcd(a, b) = 3$.
4. Simplify $\prod_{j=1}^{n} \left(1 + \frac{2}{j}\right)$. Your result should not have a $\prod$ in it, or any sort of long product.
5. Use Mathematical Induction to prove $2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$ for all integers $n \geq 1$.
6. Find all $n \in \mathbb{Z}$ with $n^2 - 5n + 6$ prime.
7. Suppose p is a prime and a is a positive integers less than p. Find all possibilities for $\gcd(a, 7a + p)$.
8. Use the Fundamental Theorem of Arithmetic to prove that $\sqrt{6}$ is irrational.
9. Prove that for $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ that if $a^n \mid b^n$ then $a \mid b$.

Exam 2 Sample A

---

1. Show that 91 is a Fermat Pseudoprime to the base 3. Note that 91 is not prime!
2. Prove that if $n \geq 2$ and $\gcd(6, n) = 1$ then $\phi(3n) = 2\phi(2n)$.
3. Classify all numbers n for which $\tau(n) = 12$.
4. Suppose n is a perfect number and p is a prime such that pn is also perfect. Prove $\gcd(p, n) \neq 1$.
5. Prove that $a^{\phi(b)} + b^{\phi(a)} \equiv 1 \bmod ab$ if $\gcd(a, b) = 1$.

6. Suppose that p is prime and $n \in \mathbb{Z}^+$. Prove that $p \nmid n$ iff $\phi(pn) = (p-1)\phi(n)$.

7. (a) Show that 3 is a primitive root modulo 17.
   (b) Find all primitive roots modulo 17.

8. A partial table of indices for 7, a primitive root of 13 is given here:

a	1	2	3	4	5	6	7	8	9	10	11	12
$\text{ind}_7 a$	12	b	8	10	3	7	a	9	4	2	5	6

   (a) Find a and b.
   (b) Use the table to solve the congruence $3^{x-1} \equiv 5 \bmod 13$.
   (c) Use the table to solve the congruence $4x^5 \equiv 11 \bmod 13$.

9. Suppose $\text{ord}_p a = 3$, where p is an odd prime. Show $\text{ord}_p(a+1) = 6$.

10. Suppose r is a primitive root modulo m, and k is a positive integers with $\gcd(k, \phi(m)) = 1$ Prove r^k is also a primitive root.

Exam 2 Sample B

---

1. Calculate:
   (a) $\phi(2^3 \cdot 5 \cdot 11^2)$
   (b) $\sigma(200)$
   (c) $\tau(2000)$

2. Use Wilson's Theorem to find the remainder when 16! is divided by 19.

3. Find all n with $\phi(n) = 16$.

4. Show that 25 is a Fermat Pseudoprime to the base 7.

5. An abundant number is a number n with $sigma(n) > 2n$. Prove that there are infinitely many even abundant numbers by finding on eabundant number and by showing that if n is abundant and a prime p satisfies $p \nmid n$ then pn is also abundant.

6. A partial table of indices for 2, a primitive root of 13, is given here:

a	1	2	3	4	5	6	7	8	9	10	11	12
$\text{ind}_2 a$	12	1	4	2	9	5	11	3	a	b	7	6

   (a) Find a and b with justification.

(b) Use the table to solve the congruence $3^{2x+1} \equiv 9 \mod 13$.

(c) Use the table to solve the congruence $7x^5 \equiv 3 \mod 13$.

7. Prove that if $\text{ord}_n a = hk$ then $\text{ord}_n(a^h) = k$.

8. Let r be a primitive root for an odd prime p. Prove that $\text{ind}_r(p-1) = \frac{1}{2}(p-1)$.

9. Find all positive integers n such that $\phi(n)$ is prime. Explain!

10. Show that if a is relatively prime to m and $\text{ord}_m a = m-1$ then m is prime.

Final Exam Sample A
