11 Quadratic Residues

Introduction: The concept of Quadratic Residues is a fundamental tool which has ramifications in lots of other number theory places: Cryptography, Factoring, etc...

11.1 Quadratice Residues & Nonresidues

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1. **Introduction:** Suppose we asked the following, given a modulus m: Which numbers are perfect squares mod m?

   Ex. Let $m = 7$. What are the perfect squares? We could of course work backwards, squaring each value:

   $$\begin{aligned} 0^2 &\equiv 0 \bmod 7 \\ 1^2 &\equiv 1 \bmod 7 \\ 2^2 &\equiv 4 \bmod 7 \\ 3^2 &\equiv 2 \bmod 7 \\ 4^2 &\equiv 2 \bmod 7 \\ 5^2 &\equiv 4 \bmod 7 \\ 6^2 &\equiv 1 \bmod 7 \end{aligned}$$

   Then the perfect squares are $0, 1, 2, 4$ and $3, 5, 6$ are not.

2. **Quadratice Residues & Nonresidues - Counting**

   (a) **Definition:** Let m be a modulus and $a \in \mathbb{Z}$ with $\gcd(a, m) = 1$. We say a is a *quadratic residue mod* m if $\exists x \in \mathbb{Z}$ such that $x^2 \equiv a \bmod m$. Otherwise, we say a is a *quadratic nonresidue mod* m if $\nexists x \in \mathbb{Z}$ such that $x^2 \equiv a \bmod m$.

   Ex. If $m = 7$ then QR:$1, 2, 4$, QNR:$3, 5, 6$, and Neither:0.

   (b) **Theorem:** If p is an odd prime and $a \in \mathbb{Z}$ with $p \nmid a \implies \gcd(p, a) = 1$, then $x^2 \equiv a \bmod p$ has either no solutions or exactly two solutions mod p.

   Proof. If there are none, we are done. Suppose x is one solution to $x^2 \equiv a \bmod p$. Claim $-x$ is also a solution. Then $2x \equiv 0 \bmod p$. Since p is odd we can do $x \equiv 0 \bmod p$ which implies $p \mid x \implies p \mid x^2$. Then, $x^2 \equiv 0 \bmod p \implies a \equiv 0 \bmod p$ which contradicts $p \nmid a$.

   Let's show that for any two solutions, they are negative of one another. Suppose $x_1^2 \equiv a \bmod p$ and $x_2^2 \equiv a \bmod p$. Then $x_1^2 - x_2^2 \equiv 0 \bmod p$ so

$p \mid (x_1^2 - x_2^2)$ so $p \mid (x_1 - x_2)(x_1 + x_2)$ so $p \mid (x_1 - x_2)$ or $p \mid (x_1 + x_2)$. If $p \mid (x_1 - x_2)$ then $x_1 \equiv x_2 \bmod p$. If $p \mid (x_1 + x_2)$ then $x_1 \equiv -x_2 \bmod p$. Thus, there can only be the two which are negatives of one another □

(c) **Theorem:** Suppose p is an odd prime. Then $\exists \frac{p-1}{2}$ QR and $\exists \frac{p-1}{2}$ QNR.

Proof. If we square all of $1, 2, 3, \cdots, p-1$ the results will be in pairs (two of every result) the $\frac{p-1}{2}$ we do get are the QR. We miss $\frac{p-1}{2}$ results, those are the QNR. □

(d) **Theorem:** Let p be an odd prime and r a primitive root mod p. Suppose $p \nmid a$, then a is a QR mod p if and only if $\text{ind}_r a$ is even.

Proof.
$\rightarrow$ Suppose a is a quadratice residue mod p, $\exists x$ such that $x^2 \equiv a \bmod p$. Then take the index of both sides to get $\text{ind}_r x^2 \equiv \text{ind}_r a \bmod p-1$ and so $2\text{ind}_r x \equiv \text{ind}_r a \bmod p-1$. From here we see $\text{ind}_r a = 2\text{ind}_r x + k(p-1)$ for some $k \in \mathbb{Z}$ and so since $p-1$ is even we know $\text{ind}_r a$ is even.

$\leftarrow$ Suppose $\text{ind}_r a$ is even. Say $\text{ind}_r a = 2k$ for $k \in \mathbb{Z}$ so $r^{2k} \equiv a \bmod p$ so $(r^k)^2 \equiv a \bmod p$. Then, a is a quadratice residue mod p. □

To illustrate: $r = 3$ is a primitive root mod 17.

a mod 17	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\text{ind}_3 a$	$\underline{16}$	$\underline{14}$	1	$\underline{12}$	5	15	11	$\underline{10}$	$\underline{2}$	3	7	13	$\underline{4}$	9	$\underline{6}$	$\underline{8}$

So what this theorem tells us is that $a = 1, 2, 4, 8, 9, 13, 15, 16$ are the quadratic residues

3. **The Legendre Symbol and Properties**

(a) **Definition:** Given an odd prime p and $a \in \mathbb{Z}$ with $\gcd(a, p) = 1$, define the Legendre Symbol:

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratice residue mod } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue mod } p \end{cases}$$

Ex. If $p = 7$ we have:

$$\left(\frac{1}{7}\right) = \left(\frac{2}{7}\right) = \left(\frac{4}{7}\right) = 1$$

$$\left(\frac{3}{7}\right) = \left(\frac{5}{7}\right) = \left(\frac{6}{7}\right) = -1$$

Since $1, 2, 4$ are QR mod 7 and $3, 5, 6$ are QNR mod 7.

(b) **Euler's Criterion:** If p is an odd prime and $a \in \mathbb{Z}$ with $\gcd(a,p) = 1$ then:

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \bmod p$$

Proof. Suppose $\left(\frac{a}{p}\right) = 1$ then $\exists x$ such that $x^2 \equiv a \bmod p$. Then observe, $a^{(p-1)/2} \equiv (x^2)^{(p-1)/2} = x^{p-1} \equiv 1 \bmod p$ by Euler's Theorem/Fermat's Little Theorem they are equal.

Suppose $\left(\frac{a}{p}\right) = -1$. Consider the list $\{1, 2, \cdots, p-1\}$, each is coprime to p and there are an even number of them because p is odd. Suppose $b \in \{1, 2, \cdots, p-1\}$, then consider the equation $bx \equiv a \bmod p$. Since $\gcd(b,p) = 1 \mid a$, $\exists!$ solution. Could $x \equiv b \bmod p$? No because if $b \cdot b \equiv a \bmod p \implies b^2 \equiv a \bmod p$ but then a would be a QR mod p. Since the solution is not b it is another element in the set $\{1, 2, \cdots, p-1\}$. Thus all of $\{1, 2, \cdots, p-1\}$ pair up to give pairs whose products are a. Thus,

$$\underbrace{(1)(2)\cdots(p-1)}_{\text{Wilson's Theorem}} \equiv a^{(p-1)/2} \bmod p$$

$$a^{(p-1)/2} \equiv -1 \bmod p$$

□

Ex. $\left(\frac{6}{11}\right) = 6^{(11-1)/2} = 6^5 \equiv 10 \equiv -1 \bmod 11$. So 6 is a QNR mod 11. i.e. $x^2 \equiv 6 \bmod 11$ has no solution.

(c) **Theorem:** If p is an odd prime and $a \in \mathbb{Z}$ with $\gcd(a,p) = \gcd(b,p) = 1$ then:

i. If $a \equiv b \bmod p$ then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$. This statements that we can reduce the numerator mod the denominator.

Proof. Clear because $x^2 \equiv a \bmod p$ if and only if $x^2 \equiv b \bmod p$ because $a \equiv b \bmod p$. □

ii. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

Proof. Well,

$$\left(\frac{ab}{p}\right) \equiv (ab)^{(p-1)/2} \equiv a^{(p-1)/2} b^{(p-1)/2} \equiv \left(\frac{a}{p}\right)\left(\frac{b}{p}\right) \bmod p$$

So $\left(\frac{ab}{p}\right) \equiv \left(\frac{a}{p}\right)\left(\frac{b}{p}\right) \bmod p$ so $p \mid \left[\left(\frac{ab}{p}\right) - \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)\right]$ but $p \geq 3$ Since $\left(\frac{ab}{p}\right) - \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ is between -2 and 2 and p divides it, we know that it must be 0. Therefore, $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$. □

iii. $\left(\frac{a^2}{p}\right) = 1$

Proof. Obvious. $\square$

(d) **Gauss' Lemma:** Suppose p is an odd prime and $a \in \mathbb{Z}$ with $\gcd(a, p) = 1$. Let s be the number of least nonnegative residues in the set

$$\{a, 2a, \cdots, ((p-1)/2)\, a\}$$

which are $> p/2$. Then $\left(\frac{a}{p}\right) = (-1)^s$.

Ex. Consider $\left(\frac{8}{13}\right)$. Note that $\left(\frac{p-1}{2}\right) = \frac{12}{2} = 6$ so look at

$$\{8, 2 \cdot 8, 3 \cdot 8, \cdots, 6 \cdot 8\} \equiv \{8, 3, 11, 6, 1, 9\} \bmod 13$$

Since only three of these are greater than $p/2 = 6.5$ we have $\left(\frac{8}{13}\right) = (-1)^3 = -1$. Thus, 8 is a quadratic nonresidue mod 13.

4. **Two Special Cases**
These will turn out to be really useful after 11.2 and 11.3 .

(a) **Theorem:** Suppose p is an odd prime, then

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4 \\ -1 & \text{if } p \equiv 3 \bmod 4 \end{cases}$$

Proof. By Euler's Criterion we have,

$$\left(\frac{-1}{p}\right) \equiv (-1)^{(p-1)/2} \bmod p$$

If $p \equiv 1 \bmod 4$ then $p = 4k + 1$ for some $k \in \mathbb{Z}$ so:

$$(-1)^{(p-1)/2} = (-1)^{(4k+1-1)/2} = (-1)^{2k} = 1$$

If $p \equiv 3 \bmod 4$ then $p = 4k + 3$ for some $k \in \mathbb{Z}$ so:

$$(-1)^{(p-1)/2} = (-1)^{(4k+3-1)/2} = (-1)^{2k+1} = -1$$

$\square$

(b) **Theorem:** Suppose p is an odd prime, then

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 7 \bmod 8 \\ -1 & \text{if } p \equiv 3, 5 \bmod 8 \end{cases}$$

Proof. Not obvious as it uses Gauss' Lemma and is lengthy. $\square$

Note: This is equivalent to

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$$

11.2 Quadratic Reciprocity and Calculation Examples

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1. **Introduction:** The Law of Quadratic reciprocity establishes that for odd primes p and q there is a connection between when p is a quadratic residue mod q when q is a quadratic residue mod p.

2. **Theorem:** If p, q are odd primes then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}$$

   Proof. Omitted due to length. □

   Use for Calculation: Under what circumstances will $\left(\frac{p}{q}\right)$ and $\left(\frac{q}{p}\right)$ be identical? We would need $\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)$ to be even. This happens if and only if one of the two is even, say $\frac{p-1}{2}$ is even. That is, $\frac{p-1}{2} = 2k$ for some $k \in \mathbb{Z}$, so $p - 1 = 4k$ so $p \equiv 1 \bmod 4$. Thus, for calculation, we get:

$$\left(\frac{p}{q}\right) = \begin{cases} \left(\frac{p}{q}\right) & \text{if either } p \equiv 1 \bmod 4 \text{ or } q \equiv 1 \bmod 4 \text{ (or both).} \\ -\left(\frac{q}{p}\right) & \text{if both } p \equiv 3 \bmod 4 \text{ and } q \equiv 3 \bmod 4. \end{cases}$$

3. **Theorem:**

   (a) If $a \equiv b \bmod p$ then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$. Call this "reducing".

   (b) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$. Call this "splitting".

   (c) $\left(\frac{a^2}{p}\right) = 1$. Call this the "square rule".

   (d) $\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4 \\ -1 & \text{if } n \equiv 3 \bmod 4 \end{cases}$. Call this the "-1 rule".

   (e) $\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } n \equiv 1, 7 \bmod 8 \\ -1 & \text{if } n \equiv 3, 5 \bmod 8 \end{cases}$. Call this the "2 rule".

4. **Examples:**
   Ex. Calculate $\left(\frac{48}{29}\right)$:

$$\left(\frac{48}{29}\right) = \left(\frac{19}{29}\right) \text{ by reducing}$$
$$\left(\frac{19}{29}\right) = \left(\frac{29}{19}\right) \text{ by LoQR since } 19 \equiv 1 \bmod 4.$$
$$\left(\frac{29}{19}\right) = \left(\frac{10}{19}\right) \text{ by reducing.}$$
$$\left(\frac{10}{19}\right) = \left(\frac{2}{19}\right)\left(\frac{5}{19}\right) \text{ by splitting.}$$

Then, we calculate these separately. First $\left(\frac{2}{19}\right) = -1$ by the "2 rule" because $19 \equiv 2 \bmod 8$. Then second,

$$\left(\frac{5}{19}\right) = \left(\frac{19}{5}\right) \text{ by LoQR since } 5 \equiv 1 \bmod 4.$$
$$\left(\frac{19}{5}\right) = \left(\frac{4}{5}\right) \text{ by reducing.}$$
$$\left(\frac{4}{5}\right) = 1 \text{by square rule.}$$

Thus $\left(\frac{48}{29}\right) = (-1)(1) = -1$.

Ex. Calculate $\left(\frac{105}{1009}\right)$. Note that 105 is not prime so we cannot use the LoQR immediately.

$$\left(\frac{105}{1009}\right) = \left(\frac{3}{1009}\right)\left(\frac{5}{1009}\right)\left(\frac{7}{1009}\right) \text{ by splitting.}$$

Then we calculate these separately. First,

$$\left(\frac{3}{1009}\right) = \left(\frac{1009}{3}\right) \text{ by LoQR since } 1009 \equiv 1 \bmod 4.$$
$$\left(\frac{1009}{3}\right) = \left(\frac{1}{3}\right) \text{ by reducing}$$
$$\left(\frac{1}{3}\right) = 1$$

Second,

$$\left(\frac{5}{1009}\right) = \left(\frac{1009}{5}\right) \text{ by LoQR since } 1009 \equiv 1 \bmod 4.$$
$$\left(\frac{1009}{5}\right) = \left(\frac{4}{5}\right) \text{ by reducing}$$
$$\left(\frac{4}{5}\right) = 1 \text{ by the square rule}$$

Third,

$$\left(\frac{7}{1009}\right) = \left(\frac{1009}{7}\right) \text{ by LoQR since } 1009 \equiv 1 \bmod 4.$$
$$\left(\frac{1009}{7}\right) = \left(\frac{1}{7}\right) \text{ by reducing}$$
$$\left(\frac{1}{7}\right) = 1$$

Thus, $\left(\frac{105}{1009}\right) = (1)(1)(1) = 1$.

11.3 The Jacobi Symbol

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1. **Introduction:** The Jacobi symbol is a generalization of the Legendre symbol for when the denominator is odd but not necessarily prime. It preserves many of the same useful properties and almost the same meaning.

2. **Definition:** Let n be an odd positive integer with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and let $\alpha \in \mathbb{Z}$ be coprime to n. Define:

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{\alpha_1} \cdots \left(\frac{a}{p_k}\right)^{\alpha_k}$$

   Thus the Jacobi symbol is defined in terms of the Legendre symbol.

3. **Theorem:** Assume $\gcd(a, n) = \gcd(b, n) = 1$.

   (a) If $a \equiv b \bmod n$ then $\left(\frac{a}{n}\right) = \left(\frac{b}{n}\right)$.

   (b) $\left(\frac{a^2}{n}\right) = 1$

   (c) $\left(\frac{ab}{n}\right) = \left(\frac{a}{n}\right)\left(\frac{b}{n}\right)$

   (d) $\left(\frac{-1}{n}\right) = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4 \\ -1 & \text{if } n \equiv 3 \bmod 4 \end{cases}$

   (e) $\left(\frac{2}{n}\right) = \begin{cases} 1 & \text{if } n \equiv 1, 7 \bmod 8 \\ -1 & \text{if } n \equiv 3, 5 \bmod 8 \end{cases}$

   (f) $\left(\frac{m}{n}\right) = \begin{cases} \left(\frac{m}{n}\right) & \text{if either } m \equiv 1 \bmod 4,\ n \equiv 1 \bmod 4 \\ -\left(\frac{n}{m}\right) & \text{if both } m \equiv 3 \bmod 4 \text{ and } n \equiv 3 \bmod 4 \end{cases}$

   Proof. Lots of calculation. □

4. **Question:** We know $\left(\frac{a}{p}\right)$ tells us if a is a QR or QNR mod p. Does $\left(\frac{a}{n}\right)$ tell us if a is a QR or QNR mod n? Well, half-yes.

   Theorem: Suppose $\gcd(a, n) = 1$ and n is an odd prime.

   (a) If a is a QR mod n then $\left(\frac{a}{n}\right) = 1$.

   (b) If $\left(\frac{a}{n}\right) = 1$ then we cannot conclude a is a QR mod n.

11.4 Problems

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1. Determine, by squaring, which of $1, ..., 16$ are quadratic residues of $p = 17$.
2. Calculate $\left(\frac{3}{17}\right)$ by
   (a) Euler's Criterion
   (b) Gauss's Lemma
3. Prove that if p and $q = 2p + 1$ are both odd primes then -4 is a primitive root of q.
4. Prove that if $p \equiv 1 \mod 4$ is a prime then -4 and $(p-1)/4$ are both quadratic residues of p.
5. Calculate each of the following:
   (a) $\left(\frac{21}{59}\right)$
   (b) $\left(\frac{1463}{89}\right)$
   (c) $\left(\frac{1547}{1913}\right)$
6. Using the Law of Quadratic Reciprocity, show that if p is an odd prime that

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \mod 12 \\ -1 & \text{if } p \equiv \pm 5 \mod 12 \end{cases}$$

7. Classify all primes p with $\left(\frac{5}{p}\right) = 1$
8. Calculate each of the following using properties of the Jacobi Symbol, not by raw calculation.
   (a) $\left(\frac{5}{21}\right)$
   (b) $\left(\frac{1009}{2307}\right)$
   (c) $\left(\frac{27}{101}\right)$
9. Categorize all positive integers n which are relatively prime to 15 and for which $\left(\frac{15}{n}\right) = 1$.
10. Show that if $a > 0$ is not a perfect square then there exists a positive integer n such that $\left(\frac{a}{n}\right) = -1$.