1 Answers to Problems

1.1 The Integers

$\boxed{91/100}$

1. Determine whether each of the following sets is well-ordered. If so, give a proof which relies on the fact that $\mathbb{Z}^+$ is well-ordered. If not, give an example of a subset with no least element.

   (a) $\{a \mid a \in \mathbb{Z}, a > 3\}$
   Is a subset of $\mathbb{Z}^+$ and therefore is well-ordered.

   (b) $\{a \mid a \in \mathbb{Q}, a > 3\}$
   There is no least element so the set is not well-ordered.

   (c) $\{\frac{a}{2} \mid a \in \mathbb{Z}, a \geq 10\}$
   Consider the set $\{a \mid a \in \mathbb{Z}, a \geq 10\}$, it is apparent that this is a subset of $\mathbb{Z}^+$ and therefore is well-ordered. So the set $\{\frac{a}{2} \mid a \in \mathbb{Z}, a \geq 10\}$ is also well-ordered because it holds a least element ($\frac{10}{5}$).

   (d) $\{\frac{2}{a} \mid a \in \mathbb{Z}, a > 10\}$
   There is no least element so the set is not well-ordered.

2. Suppose $a, b \in \mathbb{Z}^+$ are unknown. Let $S = \{a - bk \mid k \in \mathbb{Z}, a - bk > 0\}$. Explain why S has a smallest element but no largest element.

   Since S is a subset of $\mathbb{Z}^+$ by well-ordering we know that S has a least element, and because $k \in \mathbb{Z}$, k can be 0 and therefore there is no most element.

3. Use the well-ordering property to show that $\sqrt{5}$ is irrational.

   Proof. Suppose $\sqrt{5}$ is rational and is of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}^+$ and $b \neq 0$. Consider the set S,

   $$S = \left\{k \mid k, k\sqrt{5} \in \mathbb{Z}^+\right\}$$

   We know that S is a subset of $\mathbb{Z}^+$ and that $b \in S$, by well-ordering this implies that S has a least element. Let l be the least element in S.
   Consider the properties of l' where $l' = l\sqrt{5} - 2l$,

   - $l' = l\sqrt{5} - 2l = l(\sqrt{5} - 2) \implies 0 < l' < l$.
   - Since $l \in S$ and $S \subset \mathbb{Z}^+$, both l and $l\sqrt{5} \in \mathbb{Z}^+$ which implies $l' \in \mathbb{Z}^+$.
   - Since $l \in \mathbb{Z}^+$ we have $5l \in \mathbb{Z}^+$ and since $l\sqrt{5} \in \mathbb{Z}^+$ we have $l'\sqrt{5} = (l\sqrt{5} - 2l)\sqrt{5} = 5l - 2l\sqrt{5} \in \mathbb{Z}^+$.

   It follows that $l' \in S$ but $l' < l$ which contradicts l being the least element in S. $\square$

4. Use the identity
$$\frac{1}{k^2-1} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$$
to evaluate the following:

   (a) $\sum\limits_{k=2}^{10} \frac{1}{k^2-1}$

$$\begin{aligned}\sum_{k=2}^{10}\frac{1}{k^2-1} &= \sum_{k=2}^{10}\frac{1}{2}\left(\frac{1}{k-1}-\frac{1}{k+1}\right) = \frac{1}{2}\sum_{k=2}^{10}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)\\ &= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{8}-\frac{1}{10}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)\right]\\ &= \frac{1}{2}\left[\frac{1}{1}+\frac{1}{2}-\frac{1}{10}-\frac{1}{11}\right]\\ &= \frac{1}{2}\left(\frac{72}{55}\right) = \frac{36}{55}\end{aligned}$$

   (b) $\sum\limits_{k=2}^{n} \frac{1}{k^2-1}$

$$\sum_{k=2}^{n}\frac{1}{k^2-1} = \frac{1}{2}\left[\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right]$$

   (c) $\sum\limits_{k=1}^{n} \frac{1}{k^2+2k}$ Hint: $k^2+2k = (???)^2-1$

$$\sum_{k=1}^{n}\frac{1}{k^2+2k} = \sum_{k=1}^{n}\frac{1}{(k+1)^2-1} = \sum_{k=2}^{n+1}\frac{1}{k^2-1}$$
$$\sum_{k=2}^{n+1}\frac{1}{k^2-1} = \frac{1}{2}\left[\frac{1}{1}+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right]$$

5. Find the value of each of the following:

   (a) $\prod\limits_{j=2}^{7}\left(1-\frac{1}{j}\right)$

$$\begin{aligned}\prod_{j=2}^{7}\left(1-\frac{1}{j}\right) &= \left[\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}\right]\\ &= \frac{1}{7}\end{aligned}$$

(b) $\prod_{j=2}^{n}\left(1-\frac{1}{j}\right)$

$$\prod_{j=2}^{n}\left(1-\frac{1}{j}\right)=\frac{1}{n}$$

(c) $\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)$ Hint: Be sneaky!

$$\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)=\frac{n+1}{2n}$$

6. Use weak mathematical induction to prove that

$$\sum_{j=1}^{n} j(j+1)=\frac{n(n+1)(n+2)}{3}$$

for every positive integer n.

Proof.

Base Case:

Let $n=1$, $\sum_{j=1}^{1} j(j+1)=2$ and $\frac{1(1+1)(1+2)}{3}=2$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n.
This implies that $\sum_{j=1}^{n} j(j+1)=\frac{n(n+1)(n+2)}{3}$.

Inductive Step:

Then consider the sum to $n+1$:

$$\begin{aligned}\sum_{j=1}^{n+1} j(j+1) &= \sum_{j=1}^{n} j(j+1)+(n+1)((n+1)+1)\\ &= \left[\frac{n(n+1)(n+2)}{3}\right]+(n+1)((n+1)+1) \text{ by IH}\\ &= \frac{1}{3}\left(n(n+1)(n+2)+3(n+1)(n+2)\right)\\ &= \frac{1}{3}\left(n^3+3n^2+2n+3n^2+9n+6\right)\\ &= \frac{1}{3}\left(n^3+6n^2+11n+6\right)\\ &= \frac{1}{3}\left((n+1)(n+2)(n+3)\right)\end{aligned}$$

Thus for all $n \geq 1$,

$$\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$$

$\square$

7. Use Weak Mathematical Induction to show that $f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1}$ for all $n \geq 1$.

Proof.

Base Case:

Rewrite the statement $f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1}$ to be $f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$.
Let $n = 1$, $f_1 f_{1+2} - f_{1+1}^2 = 1 \cdot 2 - 1 = 1$ and $(-1)^{1+1} = 1$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n.
This implies that $f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$

Inductive Step:

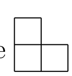
Then consider the equation to $n+1$:

$$\begin{aligned}
f_{(n+1)} f_{(n+1)+2} - f_{(n+1)+1}^2 &= f_{n+1} f_{n+3} - f_{n+2}^2 \\
&= f_{n+1}\left(f_{n+1} + f_{n+2}\right) - f_{n+2}^2 \\
&= f_{n+1}^2 + f_{n+1} f_{n+2} - f_{n+2}^2 \\
&= f_{n+1}^2 + f_{n+2}\left(f_{n+1} - f_{n+2}\right) \\
&= f_{n+1}^2 + f_{n+2}\left(-f_n\right) \\
&= -\left(f_n f_{n+2} - f_{n+1}^2\right) \\
&= -(-1)^{n+1} \quad \text{by IH} \\
&= (-1)^{n+2}
\end{aligned}$$

Thus for all $n \geq 1$,

$$f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$$

$\square$

8. Use weak mathematical induction to show that a $2^n \times 2^n$ chessboard with a corner missing can be tiled with pieces shaped like  for every integer $n \geq 0$.

Proof.

Base Case:

Let $n = 1$, $2^1 \times 2^1$ is a 2×2 chessboard with a corner missing and can be tiled by one tromino, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n. This implies that any $2^n \times 2^n$ chessboard with a corner missing can be tiled with trominoes.

Inductive Step:

Then consider a $2^{n+1} \times 2^{n+1}$ chessboard.

- Divide the $2^{n+1} \times 2^{n+1}$ chessboard into four quadrants of size $2^n \times 2^n$.
- By the Inductive Hypothesis we know that each $2^n \times 2^n$ has one corner missing.
- There are then four empty squares in the $2^{n+1} \times 2^{n+1}$ board.
- Rotate each quadrant such that the four empty squares are in the center of the board.
- Add another tromino into the board leaving only one empty square.
- Rotate the quadrant with the empty square such that the empty square is in the corner of the board.
- Therefore the $2^{n+1} \times 2^{n+1}$ chessboard can be tiled by trominoes with a corner missing.

Thus, every $2^n \times 2^n$ chessboard with a corner missing can be tiled with trominoes.

□

9. Define:

$$H_{2^n} = \sum_{j=1}^{2^n} \frac{1}{j}$$

Use weak mathematical induction to prove that for all $n \geq 1$ we have $H_{2^n} \leq 1 + n$.

Proof.

Base Case:

Let $n = 1$, $H_{2^1} = \sum_{j=1}^{2^n} \frac{1}{j} = \frac{3}{2}$ and $\frac{3}{2} \leq 2$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n.

This implies that $\sum_{j=1}^{2^n} \frac{1}{j} \leq 1 + n$.

Inductive Step:

Then consider the equation to $n+1$:

$$\begin{aligned}
H_{2^{n+1}} &= \sum_{j=1}^{2^{n+1}} \frac{1}{j} \\
&= \sum_{j=1}^{2^n} \frac{1}{j} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \\
&\leq [1+n] + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \qquad \text{by IH} \\
&\leq [1+n] + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\
&\leq [1+n] + 2^n \cdot \frac{1}{2^{n+1}} \\
&\leq \frac{3}{2} + n \leq 2 + n
\end{aligned}$$

Thus for all $n \geq 1$,

$$H_{2^n} \leq 1 + n$$

□

10. Use strong mathematical induction to prove that every amount of postage over 53 cents can be formed using 7-cent and 10-cent stamps.

Proof.

Inductive Step:

Assume we can do $54, \cdots, k$. Because $k-6$ is in the $54, \cdots, k$ we can do $k-6$ then add a 7-cent stamp. $k-6$ is in $54, \cdots, k$ only if $k - 6 \geq 54 \equiv k \geq 60$. Thus, the inductive step is only valid for $k = 60, 61, \cdots$ to get to the next $k+1$.

Base Case:

Must do $54, 55, 56, 57, 58, 59, 60$ as base cases.

$$\begin{aligned}
54 &= 2(\text{7-cent}) + 4(\text{10-cent})\\
55 &= 5(\text{7-cent}) + 2(\text{10-cent})\\
56 &= 8(\text{7-cent})\\
57 &= 1(\text{7-cent}) + 5(\text{10-cent})\\
58 &= 4(\text{7-cent}) + 3(\text{10-cent})\\
59 &= 7(\text{7-cent}) + 1(\text{10-cent})\\
60 &= 6(\text{10-cent})
\end{aligned}$$

$\square$

1.2 Primes and GCDs

$\boxed{/100}$

---

1. Use the Euclidean Algorithm to calculate $d = \gcd(510, 140)$ and then use the result to find α and β so that $d = 510\alpha + 140\beta$.

   Need to find $\gcd(510, 140)$.

$$\begin{aligned}
510 &= 3(140) + 90\\
140 &= 1(90) + 50\\
90 &= 1(50) + 40\\
50 &= 1(40) + 10\\
40 &= 4(10) + 0
\end{aligned}$$

   So the gcd is 10. Now to find the linear combination.

$$\begin{aligned}
10 &= 1(50) - 1(40)\\
&= 1(50) - 1(90 - 1(50))\\
&= 2(50) - 1(90)\\
&= 2(140 - 1(90)) - 1(90)\\
&= 2(140) - 3(90)\\
&= 2(140) - 3(510 - 3(140))\\
&= -3(510) + 11(140)\\
&= \alpha a + \beta b
\end{aligned}$$

   where $\alpha = -3$ and $\beta = 11$.

2. Use the Euclidean Algorithm to show that if $k \in \mathbb{Z}^+$ that $3k+2$ and $5k+3$ are relatively prime.

Need to show that $\gcd(3k+2, 5k+3) = 1$ for all $k \in \mathbb{Z}^+$.

$$\begin{aligned}
5k+3 &= 1(3k+2) + (2k+1) \\
3k+2 &= 1(2k+1) + (k+1) \\
2k+1 &= 1(k+1) + k \\
k+1 &= 1(k) + 1
\end{aligned}$$

So the $\gcd(3k+2, 5k+3) = 1$, therefore $3k+2$ and $5k+3$ are relatively prime.

3. How many zeros are there at the end of (1000!)? Do not do this by brute force. Explain your method.

   Zeros at the end of numbers are from multiples of 10 which are pairs of 2 and 5, so we find the number of pairs of 2's and 5's to find the number of zeros. Let $d_n(x)$ represent the sum of the numbers divisible by all powers of n less than x.

$$d_2(1000!) = 500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 994$$

$$d_5(1000!) = 200 + 40 + 8 + 1 = 249$$

   Thus, there can only be 249 pairs of 2's and 5's, so there are only 249 10's, so there are 249 zeros at the end of (1000!).

4. Let $a = 1038180$ and $b = 92950$. First find the prime factorizations of a and b. Then use these to calculate $\gcd(a, b)$ and $\text{lcm}(a, b)$.

   Find the prime factorization of a.

$$\begin{aligned}
1038180 &= 2^2(259545) \\
&= 2^2 3^1(86515) \\
&= 2^2 3^1 5^1(17303) \\
&= 2^2 3^1 5^1 11^3(13) \\
&= 2^2 3^1 5^1 11^3 13^1
\end{aligned}$$

   Find the prime factorization of b.

$$\begin{aligned}
92950 &= 2^1(46475) \\
&= 2^1 5^2(1859) \\
&= 2^1 5^2 11^1(169) \\
&= 2^1 5^2 11^1 13^2
\end{aligned}$$

   Now, to find the $\gcd(a, b)$ and $\text{lcm}(a, b)$.

$$\gcd(a, b) = \gcd(2^2 3^1 5^1 11^3 13^1, 2^1 5^2 11^1 13^2) = 2^1 5^1 11^1 13^1 = 1430$$

$$\text{lcm}(a, b) = \text{lcm}(2^2 3^1 5^1 11^3 13^1, 2^1 5^2 11^1 13^2) = 2^2 3^1 5^2 11^3 13^2 = 67481700$$

5. Which pairs of integers have gcd of 18 and lcm of 540? Explain.

   Find the prime factorization of 18.

$$\begin{aligned} 18 &= 2^1(9) \\ &= 2^1 3^2 \end{aligned}$$

   Find the prime factorization of 540.

$$\begin{aligned} 540 &= 2^2(135) \\ &= 2^2 3^3(5) \\ &= 2^2 3^3 5^1 \end{aligned}$$

   From the prime factors of 18 and 540 we know that $x = 2^a 3^b 5^c$ and $y = 2^e 3^f 5^g$. The gcd is the minimum power of common prime factors, similarly the lcm is the maximum power of common prime factors. Therefore, the list of all possible pairs of integers is:

$$\begin{aligned} x &= 2^1 3^2 5^0, y = 2^2 3^3 5^1 \\ x &= 2^1 3^3 5^0, y = 2^2 3^2 5^1 \\ x &= 2^2 3^2 5^0, y = 2^1 3^3 5^1 \\ x &= 2^2 3^3 5^0, y = 2^1 3^2 5^1 \end{aligned}$$

6. Suppose that $a \in \mathbb{Z}$ is a perfect square divisible by at least two distinct primes. Show that a has at least seven distinct factors.

   Since a is a perfect square it can be represented by the form $a = b^2$, and since a has at *least* 2 prime factors we can say that $b = p_1^\alpha p_2^\beta$. It follows that $a = p_1^{2\alpha} p_2^{2\beta}$. Therefore a has factors $1, p_1, p_2, p_1, p_2, p_1^2, p_2^2, a$.

7. Show that if $a, b \in \mathbb{Z}^+$ with $a^3 \big| b^2$ then $a \big| b$.

   Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ and $b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$. Since $a^3 \mid b^2$ we know that,

$$p_1^{3\alpha_1} p_2^{3\alpha_2} \cdots p_n^{3\alpha_n} \Big| p_1^{2\beta_1} p_2^{2\beta_2} \cdots p_n^{2\beta_n}$$

   Therefore, $3\alpha_n \leq 2\beta_n$. Now to show $a \mid b$ we need to show that $\alpha \leq \beta$.

$$3\alpha \leq 2\beta \implies \alpha \leq \frac{2\beta}{3} \leq \beta$$

   Thus, if $a^3 \mid b^2$ then $a \mid b$.

8. For which positive integers m is each of the following statements true:

   (a) $34 \equiv 10 \mod m$

$$m = 12, 24$$

(b) $1000 \equiv 1 \mod m$

$$m = 3, 9, 27, 37, 111, 333, 999$$

(c) $100 \equiv 0 \mod m$

$$m = 1, 2, 4, 5, 10, 20, 25, 50, 100$$