

Practice Exams

Exam 1 Sample A

1. Write down the prime factorization of $10!$.
2. Find the least non-negative residue of $11^{67} \pmod{13}$.
3. Find all incongruent solutions $\pmod{40}$, as least non-negative residues, to the following linear congruence:

$$12x \equiv 28 \pmod{40}$$

4. Use the Euclidean Algorithm to find $\gcd(390, 72)$ and write this as a linear combination of the two.
5. Use the Chinese Remainder Theorem to find the smallest positive solution to the system:

$$x \equiv 2 \pmod{5}$$

$$x \equiv 1 \pmod{6}$$

$$x \equiv 4 \pmod{7}$$

6. Use mathematical induction to prove that:

$$n! \geq n^3 \text{ for } n \geq 6$$

7. Determine if the following sets are well-ordered or not. You may assume only that \mathbb{Z}^+ is well-ordered.

$$S_1 = [0, 1] \cap \mathbb{Q}$$

$$S_2 = \{1 - 2^k \mid k \in \mathbb{Z}^+\}$$

8. Use the Fundamental Theorem of Arithmetic (uniqueness of prime factorization) to prove that $\sqrt{2}$ is irrational. Hint: Use contradiction.
9. Suppose $a, b, c, d \in \mathbb{Z}$ with $a \mid c$, $b \mid c$, $d = \gcd(a, b)$, and $d^2 \mid c$. Prove that $ab \mid c$.

Exam 1 Sample B

- Find $\pi(18)$.
 - Show that the set $\{\frac{a}{b} \mid a, b \in \mathbb{Z}^+, a > b\}$ is not well-ordered.
 - Find how many primes there are, approximately, between one billion and two billion.
- Find the number of zeros at the end of $1000!$ with justification.
- The following are all false. Provide explicit numerical counterexamples.
 - $a \mid bc$ implies $a \mid b$ or $a \mid c$.
 - $a \mid b$ and $a \mid c$ implies $b \mid c$.
 - $3 \mid a$ and $3 \mid b$ implies $\gcd(a, b) = 3$.
- Simplify $\prod_{j=1}^n \left(1 + \frac{2}{j}\right)$. Your result should not have a \prod in it, or any sort of long product.
- Use Mathematical Induction to prove $2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$ for all integers $n \geq 1$.
- Find all $n \in \mathbb{Z}$ with $n^2 - 5n + 6$ prime.
- Suppose p is a prime and a is a positive integers less than p . Find all possibilities for $\gcd(a, 7a + p)$.
- Use the Fundamental Theorem of Arithmetic to prove that $\sqrt{6}$ is irrational.
- Prove that for $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ that if $a^n \mid b^n$ then $a \mid b$.

Exam 2 Sample A

- Show that 91 is a Fermat Pseudoprime to the base 3. Note that 91 is not prime!
- Prove that if $n \geq 2$ and $\gcd(6, n) = 1$ then $\phi(3n) = 2\phi(2n)$.
- Classify all numbers n for which $\tau(n) = 12$.
- Suppose n is a perfect number and p is a prime such that pn is also perfect. Prove $\gcd(p, n) \neq 1$.
- Prove that $a^{\phi(b)} + b^{\phi(a)} \equiv 1 \pmod{ab}$ if $\gcd(a, b) = 1$.

6. Suppose that p is prime and $n \in \mathbb{Z}^+$. Prove that $p \nmid n$ iff $\phi(pn) = (p-1)\phi(n)$.
7. (a) Show that 3 is a primitive root modulo 17.
(b) Find all primitive roots modulo 17.
8. A partial table of indices for 7, a primitive root of 13 is given here:

a	1	2	3	4	5	6	7	8	9	10	11	12
$\text{ind}_7 a$	12	b	8	10	3	7	a	9	4	2	5	6

- (a) Find a and b .
- (b) Use the table to solve the congruence $3^{x-1} \equiv 5 \pmod{13}$.
- (c) Use the table to solve the congruence $4x^5 \equiv 11 \pmod{13}$.
9. Suppose $\text{ord}_p a = 3$, where p is an odd prime. Show $\text{ord}_p(a+1) = 6$.
10. Suppose r is a primitive root modulo m , and k is a positive integers with $\gcd(k, \phi(m)) = 1$ Prove r^k is also a primitive root.

Exam 2 Sample B

1. Calculate:
 - (a) $\phi(2^3 \cdot 5 \cdot 11^2)$
 - (b) $\sigma(200)$
 - (c) $\tau(2000)$
2. Use Wilson's Theorem to find the remainder when $16!$ is divided by 19.
3. Find all n with $\phi(n) = 16$.
4. Show that 25 is a Fermat Pseudoprime to the base 7.
5. An abundant number is a number n with $\sigma(n) > 2n$. Prove that there are infinitely many even abundant numbers by finding one abundant number and by showing that if n is abundant and a prime p satisfies $p \nmid n$ then pn is also abundant.
6. A partial table of indices for 2, a primitive root of 13, is given here:

a	1	2	3	4	5	6	7	8	9	10	11	12
$\text{ind}_2 a$	12	1	4	2	9	5	11	3	a	b	7	6

- (a) Find a and b with justification.

- (b) Use the table to solve the congruence $3^{2x+1} \equiv 9 \pmod{13}$.
 - (c) Use the table to solve the congruence $7x^5 \equiv 3 \pmod{13}$.
7. Prove that if $\text{ord}_n a = hk$ then $\text{ord}_n(a^h) = k$.
 8. Let r be a primitive root for an odd prime p . Prove that $\text{ind}_r(p-1) = \frac{1}{2}(p-1)$.
 9. Find all positive integers n such that $\phi(n)$ is prime. Explain!
 10. Show that if a is relatively prime to m and $\text{ord}_m a = m-1$ then m is prime.