

1 Answers to Problems

1.1 The Integers

91/100

1. Determine whether each of the following sets is well-ordered. If so, give a proof which relies on the fact that \mathbb{Z}^+ is well-ordered. If not, give an example of a subset with no least element. 10/10

(a) $\{a \mid a \in \mathbb{Z}, a > 3\}$

Is a subset of \mathbb{Z}^+ and therefore is well-ordered.

(b) $\{a \mid a \in \mathbb{Q}, a > 3\}$

There is no least element so the set is not well-ordered.

(c) $\{\frac{a}{2} \mid a \in \mathbb{Z}, a \geq 10\}$

Consider the set $\{a \mid a \in \mathbb{Z}, a \geq 10\}$, it is apparent that this is a subset of \mathbb{Z}^+ and therefore is well-ordered. So the set $\{\frac{a}{2} \mid a \in \mathbb{Z}, a \geq 10\}$ is also well-ordered because it holds a least element ($\frac{10}{2}$).

(d) $\{\frac{2}{a} \mid a \in \mathbb{Z}, a > 10\}$

There is no least element so the set is not well-ordered.

2. Suppose $a, b \in \mathbb{Z}^+$ are unknown. Let $S = \{a - bk \mid k \in \mathbb{Z}, a - bk > 0\}$. Explain why S has a smallest element but no largest element. 3/10

Since S is a subset of \mathbb{Z}^+ by well-ordering we know that S has a least element, and because $k \in \mathbb{Z}$, k can be 0 and therefore there is no most element.

3. Use the well-ordering property to show that $\sqrt{5}$ is irrational. 10/10

Proof. Suppose $\sqrt{5}$ is rational and is of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}^+$ and $b \neq 0$. Consider the set S ,

$$S = \{k \mid k, k\sqrt{5} \in \mathbb{Z}^+\}$$

We know that S is a subset of \mathbb{Z}^+ and that $b \in S$, by well-ordering this implies that S has a least element. Let l be the least element in S .

Consider the properties of l' where $l' = l\sqrt{5} - 2l$,

- $l' = l\sqrt{5} - 2l = l(\sqrt{5} - 2) \implies 0 < l' < l$.
- Since $l \in S$ and $S \subset \mathbb{Z}^+$, both l and $l\sqrt{5} \in \mathbb{Z}^+$ which implies $l' \in \mathbb{Z}^+$.
- Since $l \in \mathbb{Z}^+$ we have $5l \in \mathbb{Z}^+$ and since $l\sqrt{5} \in \mathbb{Z}^+$ we have $l'\sqrt{5} = (l\sqrt{5} - 2l)\sqrt{5} = 5l - 2l\sqrt{5} \in \mathbb{Z}^+$.

It follows that $l' \in S$ but $l' < l$ which contradicts l being the least element in S . \square

4. Use the identity

$$\frac{1}{k^2 - 1} = \frac{1}{2} \left(\frac{1}{k-1} - \frac{1}{k+1} \right)$$

to evaluate the following:

10/10

(a) $\sum_{k=2}^{10} \frac{1}{k^2 - 1}$

$$\begin{aligned} \sum_{k=2}^{10} \frac{1}{k^2 - 1} &= \sum_{k=2}^{10} \frac{1}{2} \left(\frac{1}{k-1} - \frac{1}{k+1} \right) = \frac{1}{2} \sum_{k=2}^{10} \left(\frac{1}{k-1} - \frac{1}{k+1} \right) \\ &= \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{8} - \frac{1}{10} \right) + \left(\frac{1}{9} - \frac{1}{11} \right) \right] \\ &= \frac{1}{2} \left[\frac{1}{1} + \frac{1}{2} - \frac{1}{10} - \frac{1}{11} \right] \\ &= \frac{1}{2} \left(\frac{72}{55} \right) = \frac{36}{55} \end{aligned}$$

(b) $\sum_{k=2}^n \frac{1}{k^2 - 1}$

$$\sum_{k=2}^n \frac{1}{k^2 - 1} = \frac{1}{2} \left[\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \right]$$

(c) $\sum_{k=1}^n \frac{1}{k^2 + 2k}$ Hint: $k^2 + 2k = (k+1)^2 - 1$

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k^2 + 2k} &= \sum_{k=1}^n \frac{1}{(k+1)^2 - 1} = \sum_{k=2}^{n+1} \frac{1}{k^2 - 1} \\ \sum_{k=2}^{n+1} \frac{1}{k^2 - 1} &= \frac{1}{2} \left[\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right] \end{aligned}$$

5. Find the value of each of the following:

10/10

(a) $\prod_{j=2}^7 \left(1 - \frac{1}{j} \right)$

$$\begin{aligned} \prod_{j=2}^7 \left(1 - \frac{1}{j} \right) &= \left[\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \right] \\ &= \frac{1}{7} \end{aligned}$$

$$(b) \prod_{j=2}^n \left(1 - \frac{1}{j}\right)$$

$$\prod_{j=2}^n \left(1 - \frac{1}{j}\right) = \frac{1}{n}$$

$$(c) \prod_{j=2}^n \left(1 - \frac{1}{j^2}\right) \quad \text{Hint: Be sneaky!}$$

$$\prod_{j=2}^n \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n}$$

6. Use weak mathematical induction to prove that

$$\sum_{j=1}^n j(j+1) = \frac{n(n+1)(n+2)}{3}$$

for every positive integer n .

9/10

Proof.

Base Case:

Let $n = 1$, $\sum_{j=1}^1 j(j+1) = 2$ and $\frac{1(1+1)(1+2)}{3} = 2$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n .

This implies that $\sum_{j=1}^n j(j+1) = \frac{n(n+1)(n+2)}{3}$.

Inductive Step:

Then consider the sum to $n+1$:

$$\begin{aligned} \sum_{j=1}^{n+1} j(j+1) &= \sum_{j=1}^n j(j+1) + (n+1)((n+1)+1) \\ &= \left[\frac{n(n+1)(n+2)}{3} \right] + (n+1)((n+1)+1) \text{ by IH} \\ &= \frac{1}{3} (n(n+1)(n+2) + 3(n+1)(n+2)) \\ &= \frac{1}{3} (n^3 + 3n^2 + 2n + 3n^2 + 9n + 6) \\ &= \frac{1}{3} (n^3 + 6n^2 + 11n + 6) \\ &= \frac{1}{3} ((n+1)(n+2)(n+3)) \end{aligned}$$

Thus for all $n \geq 1$,

$$\sum_{j=1}^n j(j+1) = \frac{n(n+1)(n+2)}{3}$$

□

7. Use Weak Mathematical Induction to show that $f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1}$ for all $n \geq 1$. 9/10

Proof.

Base Case:

Rewrite the statement $f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1}$ to be $f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$.

Let $n = 1$, $f_1 f_{1+2} - f_{1+1}^2 = 1 \cdot 2 - 1 = 1$ and $(-1)^{1+1} = 1$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n .

This implies that $f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$

Inductive Step:


Then consider the equation to $n + 1$:

$$\begin{aligned} f_{(n+1)} f_{(n+1)+2} - f_{(n+1)+1}^2 &= f_{n+1} f_{n+3} - f_{n+2}^2 \\ &= f_{n+1} (f_{n+1} + f_{n+2}) - f_{n+2}^2 \\ &= f_{n+1}^2 + f_{n+1} f_{n+2} - f_{n+2}^2 \\ &= f_{n+1}^2 + f_{n+2} (f_{n+1} - f_{n+2}) \\ &= f_{n+1}^2 + f_{n+2} (-f_n) \\ &= - (f_n f_{n+2} - f_{n+1}^2) \\ &= -(-1)^{n+1} \quad \text{by IH} \\ &= (-1)^{n+2} \end{aligned}$$

Thus for all $n \geq 1$,

$$f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$$

□

8. Use weak mathematical induction to show that a $2^n \times 2^n$ chessboard with a corner missing can be tiled with pieces shaped like  for every integer $n \geq 0$. 10/10

Proof.

Base Case:

Let $n = 1$, $2^1 \times 2^1$ is a 2×2 chessboard with a corner missing and can be tiled by one tromino, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n . This implies that any $2^n \times 2^n$ chessboard with a corner missing can be tiled with trominoes.

Inductive Step:

Then consider a $2^{n+1} \times 2^{n+1}$ chessboard.

- Divide the $2^{n+1} \times 2^{n+1}$ chessboard into four quadrants of size $2^n \times 2^n$.
- By the Inductive Hypothesis we know that each $2^n \times 2^n$ has one corner missing.
- There are then four empty squares in the $2^{n+1} \times 2^{n+1}$ board.
- Rotate each quadrant such that the four empty squares are in the center of the board.
- Add another tromino into the board leaving only one empty square.
- Rotate the quadrant with the empty square such that the empty square is in the corner of the board.
- Therefore the $2^{n+1} \times 2^{n+1}$ chessboard can be tiled by trominoes with a corner missing.

Thus, every $2^n \times 2^n$ chessboard with a corner missing can be tiled with trominoes.

□

9. Define:

$$H_{2^n} = \sum_{j=1}^{2^n} \frac{1}{j}$$

Use weak mathematical induction to prove that for all $n \geq 1$ we have $H_{2^n} \leq 1 + n$.

10/10

Proof.

Base Case:

Let $n = 1$, $H_{2^1} = \sum_{j=1}^{2^1} \frac{1}{j} = \frac{3}{2}$ and $\frac{3}{2} \leq 2$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n .

This implies that $\sum_{j=1}^{2^n} \frac{1}{j} \leq 1 + n$.

Inductive Step:

Then consider the equation to $n + 1$:

$$\begin{aligned}
 H_{2^{n+1}} &= \sum_{j=1}^{2^{n+1}} \frac{1}{j} \\
 &= \sum_{j=1}^{2^n} \frac{1}{j} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \\
 &\leq [1 + n] + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \quad \text{by IH} \\
 &\leq [1 + n] + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} \\
 &\leq [1 + n] + 2^n \cdot \frac{1}{2^{n+1}} \\
 &\leq \frac{3}{2} + n \leq 2 + n
 \end{aligned}$$

Thus for all $n \geq 1$,

$$H_{2^n} \leq 1 + n$$

□

10. Use strong mathematical induction to prove that every amount of postage over 53 cents can be formed using 7-cent and 10-cent stamps. 10/10

Proof.

Inductive Step:

Assume we can do $54, \dots, k$. Because $k - 6$ is in the $54, \dots, k$ we can do $k - 6$ then add a 7-cent stamp. $k - 6$ is in $54, \dots, k$ only if $k - 6 \geq 54 \equiv k \geq 60$. Thus, the inductive step is only valid for $k = 60, 61, \dots$ to get to the next $k + 1$.

Base Case:

Must do 54, 55, 56, 57, 58, 59, 60 as base cases.

$$54 = 2(7\text{-cent}) + 4(10\text{-cent})$$

$$55 = 5(7\text{-cent}) + 2(10\text{-cent})$$

$$56 = 8(7\text{-cent})$$

$$57 = 1(7\text{-cent}) + 5(10\text{-cent})$$

$$58 = 4(7\text{-cent}) + 3(10\text{-cent})$$

$$59 = 7(7\text{-cent}) + 1(10\text{-cent})$$

$$60 = 6(10\text{-cent})$$

□

1.2 Primes and GCDs

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1. Use the Euclidean Algorithm to calculate $d = \gcd(510, 140)$ and then use the result to find α and β so that $d = 510\alpha + 140\beta$.

10/10

Need to find $\gcd(510, 140)$.

$$510 = 3(140) + 90$$

$$140 = 1(90) + 50$$

$$90 = 1(50) + 40$$

$$50 = 1(40) + 10$$

$$40 = 4(10) + 0$$

So the gcd is 10. Now to find the linear combination.

$$\begin{aligned} 10 &= 1(50) - 1(40) \\ &= 1(50) - 1(90 - 1(50)) \\ &= 2(50) - 1(90) \\ &= 2(140 - 1(90)) - 1(90) \\ &= 2(140) - 3(90) \\ &= 2(140) - 3(510 - 3(140)) \\ &= -3(510) + 11(140) \\ &= \alpha a + \beta b \end{aligned}$$

where $\alpha = -3$ and $\beta = 11$.

2. Use the Euclidean Algorithm to show that if $k \in \mathbb{Z}^+$ that $3k+2$ and $5k+3$ are relatively prime.

8/10

Need to show that $\gcd(3k+2, 5k+3) = 1$ for all $k \in \mathbb{Z}^+$.

$$5k+3 = 1(3k+2) + (2k+1)$$

$$3k+2 = 1(2k+1) + (k+1)$$

$$2k+1 = 1(k+1) + k$$

$$k+1 = 1(k) + 1$$

So the $\gcd(3k+2, 5k+3) = 1$, therefore $3k+2$ and $5k+3$ are relatively prime.

3. How many zeros are there at the end of $(1000!)$? Do not do this by brute force. Explain your method. 10/10

Zeros at the end of numbers are from multiples of 10 which are pairs of 2 and 5, so we find the number of pairs of 2's and 5's to find the number of zeros. Let $d_n(x)$ represent the sum of the numbers divisible by all powers of n less than x .

$$d_2(1000!) = 500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 994$$

$$d_5(1000!) = 200 + 40 + 8 + 1 = 249$$

Thus, there can only be 249 pairs of 2's and 5's, so there are only 249 10's, so there are 249 zeros at the end of $(1000!)$.

4. Let $a = 1038180$ and $b = 92950$. First find the prime factorizations of a and b . Then use these to calculate $\gcd(a, b)$ and $\text{lcm}(a, b)$. 10/10

Find the prime factorization of a .

$$\begin{aligned} 1038180 &= 2^2(259545) \\ &= 2^2 3^1(86515) \\ &= 2^2 3^1 5^1(17303) \\ &= 2^2 3^1 5^1 11^3(13) \\ &= 2^2 3^1 5^1 11^3 13^1 \end{aligned}$$

Find the prime factorization of b .

$$\begin{aligned} 92950 &= 2^1(46475) \\ &= 2^1 5^2(1859) \\ &= 2^1 5^2 11^1(169) \\ &= 2^1 5^2 11^1 13^2 \end{aligned}$$

Now, to find the $\gcd(a, b)$ and $\text{lcm}(a, b)$.

$$\gcd(a, b) = \gcd(2^2 3^1 5^1 11^3 13^1, 2^1 5^2 11^1 13^2) = 2^1 5^1 11^1 13^1 = 1430$$

$$\text{lcm}(a, b) = \text{lcm}(2^2 3^1 5^1 11^3 13^1, 2^1 5^2 11^1 13^2) = 2^2 3^1 5^2 11^3 13^2 = 67481700$$

5. Which pairs of integers have gcd of 18 and lcm of 540? Explain. 10/10

Find the prime factorization of 18.

$$\begin{aligned} 18 &= 2^1(9) \\ &= 2^13^2 \end{aligned}$$

Find the prime factorization of 540.

$$\begin{aligned} 540 &= 2^2(135) \\ &= 2^23^3(5) \\ &= 2^23^35^1 \end{aligned}$$

From the prime factors of 18 and 540 we know that $x = 2^a3^b5^c$ and $y = 2^e3^f5^g$. The gcd is the minimum power of common prime factors, similarly the lcm is the maximum power of common prime factors. Therefore, the list of all possible pairs of integers is:

$$\begin{aligned} x &= 2^13^25^0, y = 2^23^35^1 \\ x &= 2^13^35^0, y = 2^23^25^1 \\ x &= 2^23^25^0, y = 2^13^35^1 \\ x &= 2^23^35^0, y = 2^13^25^1 \end{aligned}$$

6. Suppose that $a \in \mathbb{Z}$ is a perfect square divisible by at least two distinct primes. Show that a has at least seven distinct factors. 5/10

Since a is a perfect square it can be represented by the form $a = b^2$, and since a has at *least* 2 prime factors we can say that $b = p_1^\alpha p_2^\beta$. It follows that $a = p_1^{2\alpha} p_2^{2\beta}$. Therefore a has factors $1, p_1, p_2, p_1^2, p_2^2, p_1 p_2, a$.

7. Show that if $a, b \in \mathbb{Z}^+$ with $a^3 | b^2$ then $a | b$. 10/10

Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ and $b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$. Since $a^3 | b^2$ we know that,

$$p_1^{3\alpha_1} p_2^{3\alpha_2} \cdots p_n^{3\alpha_n} \mid p_1^{2\beta_1} p_2^{2\beta_2} \cdots p_n^{2\beta_n}$$

Therefore, $3\alpha_n \leq 2\beta_n$. Now to show $a | b$ we need to show that $\alpha \leq \beta$.

$$3\alpha \leq 2\beta \implies \alpha \leq \frac{2\beta}{3} \leq \beta$$

Thus, if $a^3 | b^2$ then $a | b$.

8. For which positive integers m is each of the following statements true:

6/10

(a) $34 \equiv 10 \pmod{m}$

$$m = 12, 24$$

(b) $1000 \equiv 1 \pmod{m}$

$$m = 3, 9, 27, 37, 111, 333, 999$$

(c) $100 \equiv 0 \pmod{m}$

$$m = 1, 2, 4, 5, 10, 20, 25, 50, 100$$

1.3 Congruences

91/100

1. Calculate the least positive residues modulo 47 of each of the following with justification:

(a) 2^{543}

Using binary expansion we see that $2^1 \equiv 2 \pmod{47}$, $2^2 \equiv 4 \pmod{47}$, $2^4 \equiv 16 \pmod{47}$, $2^8 \equiv 21 \pmod{47}$, $2^{16} \equiv 18 \pmod{47}$, $2^{32} \equiv 42 \pmod{47}$, $2^{64} \equiv 25 \pmod{47}$, $2^{128} \equiv 14 \pmod{47}$, $2^{256} \equiv 8 \pmod{47}$, and $2^{512} \equiv 17 \pmod{47}$.

Then $543 = 512 + 16 + 8 + 4 + 2 + 1$ so,

$$\begin{aligned} 2^{543} &= 2^{512} 2^{16} 2^8 2^4 2^2 2^1 \equiv \\ &\equiv 17 \cdot 18 \cdot 21 \cdot 16 \cdot 4 \cdot 2 \pmod{47} \\ &\equiv 822528 \pmod{47} \\ &\equiv 28 \pmod{47} \end{aligned}$$

So 28 is the least non-negative residue.

(b) 32^{932}

Using binary expansion we see that $32^1 \equiv 32 \pmod{47}$, $32^2 \equiv 37 \pmod{47}$, $32^4 \equiv 6 \pmod{47}$, $32^8 \equiv 47 \pmod{47}$, $32^{16} \equiv 27 \pmod{47}$, $32^{32} \equiv 24 \pmod{47}$, $32^{64} \equiv 12 \pmod{47}$, $32^{128} \equiv 3 \pmod{47}$, $32^{256} \equiv 9$, and $32^{512} \equiv 34$. Then $932 = 512 + 256 + 128 + 32 + 4$ so,

$$\begin{aligned} 32^{932} &= 32^{512} 32^{256} 32^{128} 32^{32} 32^4 \equiv \\ &\equiv 34 \cdot 9 \cdot 3 \cdot 24 \cdot 6 \pmod{47} \\ &\equiv 132192 \pmod{47} \\ &\equiv 28 \pmod{47} \end{aligned}$$

So 28 is the least non-negative residue.

(c) $46^{327349287323}$

Since $46 \equiv -1 \pmod{47}$ we know $46^{327349287323} \equiv (-1)^{327349287323}$.
 We also know $2 \nmid 327349287323$ so $(-1)^{327349287323} \equiv (-1)^1 \equiv -1 \equiv 46 \pmod{47}$. So 46 is the least non-negative residue.

2. Exhibit a complete set of residues mod 17 composed entirely of multiples of 3.

Let $S = \{0, 1, 2, \dots, 16\}$ be the set of residues mod 17. Because $\gcd(3, 17) = 1$ the set consisting of only multiples of 3 would be,

$$\{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48\}$$

3. Show that if $a, b, m \in \mathbb{Z}$ with $m > 0$ and if $a \equiv b \pmod{m}$ then $\gcd(a, m) = \gcd(b, m)$.

If $a \equiv b \pmod{m}$ then $\exists x \in \mathbb{Z}$ such that $a = b + xm$. So $\gcd(a, m) = \gcd(b + xm, m) = \gcd(b, m)$.

4. Suppose p is prime and $x \in \mathbb{Z}$ satisfies $x^2 \equiv x \pmod{p}$. Prove that $x \equiv 0 \pmod{p}$ or $x \equiv 1 \pmod{p}$. Show with a counterexample that this fails if p is not prime.

Because $x^2 \equiv x \pmod{p}$ we know that $x^2 - x \equiv 0 \pmod{p}$, which is the same as $x(x - 1) \equiv 0 \pmod{p}$. This implies that either $p \mid x$, $p \mid (x - 1)$, or both.

$$p \mid x \implies x \equiv 0 \pmod{p}$$

$$p \mid (x - 1) \implies x \equiv 1 \pmod{p}$$

If p is not prime, say $p = 6$ we see,

$$3^2 \equiv 3 \pmod{6}$$

Where $3 \not\equiv 0 \pmod{6}$ and $3 \not\equiv 1 \pmod{6}$. So p must be prime for the statement to hold true.

5. Show that if n is an odd positive integer or if n is a positive integer divisible by 4 that:

$$1^3 + 2^3 + \dots + (n - 1)^3 \equiv 0 \pmod{n}$$

There are two cases to look at, when n is an odd positive integer and when n is divisible by 4.

- If n is an odd positive integer then $n - 1$ is even so we have an even amount of numbers. Consider the set $S = \{1^3, 2^3, \dots, (n - 1)^3\}$. Then consider two subsets of S , both with $(n - 1)/2$ elements S_1 and S_2 , where

$$\sum S = \sum S_1 + \sum S_2 = 1^3 + 2^3 + \dots + (n - 1)^3$$

The set $S_1 = \{1^3, 2^3, 3^3, \dots\}$ and the set $S_2 = \{\dots, (n-3)^3, (n-2)^3, (n-1)^3\}$. Because we know that $a-b \equiv -b \pmod{a}$ we also know that $(a-b)^3 \equiv (-b)^3 \pmod{a}$. So we can say that for all elements in $S_2 \pmod{n}$, $S_2 = \{\dots, (-3)^3, (-2)^3, (-1)^n\}$. Now if we look at $\sum S_1 + \sum S_2 \pmod{n}$ we see that the first element of S_1 is cancelled out by the last element of S_2 and so forth until there are no elements left. Thus, $1^3 + 2^3 + \dots + (n-1)^3 \equiv 0 \pmod{n}$.

- If n is divisible by 4 then $n-1$ is odd so we have an odd amount of numbers. Consider the set $S = \{1^3, 2^3, \dots, (n-1)^3\}$. Then consider two subsets of S , both with $(n-1)/2 - 1$ elements S_1 and S_2 , where

$$\sum S = \sum S_1 + \left(\frac{n}{2}\right)^3 + \sum S_2 = 1^3 + 2^3 + \dots + (n-1)^3$$

The set $S_1 = \{1^3, 2^3, 3^3, \dots\}$ and the set $S_2 = \{\dots, (n-3)^3, (n-2)^3, (n-1)^3\}$. Because we know that $a-b \equiv -b \pmod{a}$ we also know that $(a-b)^3 \equiv (-b)^3 \pmod{a}$. So we can say that for all elements in $S_2 \pmod{n}$, $S_2 = \{\dots, (-3)^3, (-2)^3, (-1)^n\}$. Now if we look at $\sum S \pmod{n}$, like in the case above we can see that sets S_1 and S_2 will cancel one another out. This leaves us with

$$1^3 + 2^3 + \dots + (n-1)^3 \equiv \left(\frac{n}{2}\right)^3 \pmod{n}$$

Because we know that $4 \mid n$ we know that $n = 4x$ for some $x \in \mathbb{Z}$. It follows that,

$$\left(\frac{n}{2}\right)^3 = \frac{n^3}{8} = \frac{64x^3}{8} = 8x^3 = (2x^2)n$$

So

$$\left(\frac{n^3}{2}\right)^3 \equiv (2x^2)n \equiv 0 \pmod{n}$$

Thus, $1^3 + 2^3 + \dots + (n-1)^3 \equiv 0 \pmod{n}$.

6. Find all solutions (mod the given value) to each of the following.

- (a) $10x \equiv 25 \pmod{75}$

Because the $\gcd(10, 75) = 5$ and $5 \mid 25$ we know that solutions exist. Let $x_0 \equiv 10 \pmod{75}$, so all solutions are then

$$x \equiv 10 + k \cdot \frac{75}{\gcd(10, 75)} \pmod{75}, \text{ for } k = 0, 1, 2, 3, 4$$

$$x \equiv 10 + 15k \pmod{75}, \text{ for } k = 0, 1, 2, 3, 4$$

Therefore, $x \equiv 10, 25, 40, 55, 70$.

(b) $9x \equiv 8 \pmod{12}$

Because the $\gcd(9, 12) = 3$ and $3 \nmid 8$ so there are no solutions.

7. Solve each of the following linear congruences using inverses.

(a) $3x \equiv 5 \pmod{17}$

Since 6 is the inverse of 3 mod 17 we get, $6 \cdot 3x \equiv 6 \cdot 5 \pmod{17}$ which implies

$$x \equiv 30 \pmod{17} \equiv 13 \pmod{17}$$

Therefore, $x \equiv 13$.

(b) $10x \equiv 3 \pmod{11}$

Since 10 is the inverse of 10 mod 11 we get, $10 \cdot 10x \equiv 10 \cdot 3 \pmod{11}$ which implies

$$x \equiv 30 \pmod{11} \equiv 8 \pmod{11}$$

Therefore, $x \equiv 8$.

8. What could the prime factorization of m look like so that $6x \equiv 10 \pmod{m}$ has at least one solution? Explain.

In order for $ax \equiv b \pmod{m}$ to have a solution(s), $\gcd(a, m) \mid b$. So in the context of this problem we have that $\gcd(6, m) \mid 10$. We are looking for an m such that $\gcd(6, m) = 2$. One possible m could be $m = 2^1$.

9. Use the Chinese Remainder Theorem to solve:

A troop of monkeys has a store of bananas. When they arrange them into 7 piles, none remain. When they arrange them into 10 piles there are 3 left over. When they arrange them into 11 piles there are 2 left over. What is the smallest positive number of bananas they can have? What is the second smallest positive number?

Let x be the number of bananas, we have

$$x \equiv 0 \pmod{7}$$

$$x \equiv 3 \pmod{10}$$

$$x \equiv 2 \pmod{11}$$

Test to see if all m_i are pairwise coprime, $\gcd(7, 10) = \gcd(7, 11) = \gcd(10, 11) = 1$. This means that $M = 770$, $M_1 = 110$, $M_2 = 77$, and $M_3 = 70$.

Solve for y_1 :

$$110y_1 \equiv 1 \pmod{7}$$

$$5y_1 \equiv 1 \pmod{7}$$

$$y_1 = 3$$

Solve for y_2 :

$$77y_2 \equiv 1 \pmod{10}$$

$$7y_2 \equiv 1 \pmod{10}$$

$$y_2 = 3$$

Solve for y_3 :

$$70y_3 \equiv 1 \pmod{11}$$

$$4y_3 \equiv 1 \pmod{11}$$

$$y_3 = 3$$

So we then get

$$x = (0)(110)(3) + (3)(77)(3) + (2)(70)(3) \equiv 1113 \pmod{770}$$

$$x \equiv 343 \pmod{770}$$

The smallest number of bananas they can have is 343 and the second smallest is 1113.

10. Solve the system of linear congruences:

$$2x + 1 \equiv 3 \pmod{10}$$

$$x + 2 \equiv 7 \pmod{9}$$

$$4x \equiv 1 \pmod{7}$$

Rewrite the system of linear congruences to be (properties of congruences):

$$2x \equiv 2 \pmod{10}$$

$$x \equiv 5 \pmod{9}$$

$$4x \equiv 1 \pmod{7}$$

Which then becomes

$$x \equiv 1 \pmod{5}$$

$$x \equiv 5 \pmod{9}$$

$$x \equiv 2 \pmod{7}$$

Then test to see if all m_i are pairwise coprime, $\gcd(5, 9) = \gcd(5, 7) = \gcd(7, 9) = 1$. This means that $M = 315$, $M_1 = 63$, $M_2 = 35$, and $M_3 = 45$.

Solve for y_1 :

$$63y_1 \equiv 1 \pmod{5}$$

$$3y_1 \equiv 1 \pmod{5}$$

$$y_1 = 2$$

Solve for y_2 :

$$\begin{aligned}35y_2 &\equiv 1 \pmod{9} \\8y_2 &\equiv 1 \pmod{9} \\y_2 &= 8\end{aligned}$$

Solve for y_3 :

$$\begin{aligned}45y_3 &\equiv 1 \pmod{7} \\3y_3 &\equiv 1 \pmod{7} \\y_3 &= 5\end{aligned}$$

So we then get

$$\begin{aligned}x &= (1)(63)(2) + (5)(35)(8) + (2)(45)(5) \equiv 1976 \pmod{315} \\x &\equiv 86 \pmod{315}\end{aligned}$$

1.4 Special Congruences

98/100

1. Use Fermat's Little Theorem to find the least nonnegative residue of $2^{1000003} \pmod{17}$.

Well $17 \nmid 2$ so $2^{16} \equiv 1 \pmod{17}$. Then $1000003 = 16(62500) + 3$ so

$$\begin{aligned}2^{1000003} &= 2^{16 \cdot 62500} 2^3 \equiv (1)^{62500} 2^3 \pmod{17} \\&\equiv 2^3 \pmod{17} \\&\equiv 8 \pmod{17}\end{aligned}$$

So 8 is the least non-negative residue.

2. Use Fermat's Little Theorem to solve the following, giving the result as the least nonnegative residue.

(a) $7x \equiv 12 \pmod{17}$

By FLiT we know $7^{17-1} \equiv 1 \pmod{17}$, it follows that $7^{16} \cdot 12 \equiv 1 \pmod{17}$ therefore $7x = 7^{16} \cdot 12 = 7^{15} \cdot 12$. Then reduce $7^{15} \cdot 12 \pmod{17}$. So 9 is the least non-negative residue.

(b) $10x \equiv 13 \pmod{19}$

By FLiT we know $10^{19-1} \equiv 1 \pmod{19}$, it follows that $10^{18} \cdot 13 \equiv 1 \pmod{19}$ therefore $10x = 10^{18} \cdot 13 = 10^{17} \cdot 13$. Then reduce $10^{17} \cdot 13 \pmod{19}$. So 7 is the least non-negative residue.

3. Use Fermat's Little Theorem to show that $30 \mid (n^9 - n)$ for all positive integers n .

Proof. Note that $30 = 2 \cdot 3 \cdot 5$, from Fermat's Little Theorem we know that $a^p \equiv a \pmod{p}$ when p is prime. Let,

$$\begin{aligned}x &= (n^5 - n)(n^4 + 1) \\y &= (n^3 - n)(n^2 + 1)(n^4 + 1) \\z &= (n^2 - n)(n^3 + n^2 + n + 1)(n^4 + 1)\end{aligned}$$

Note that $x = y = z = n^9 - n$. Then observe the following,

- By FLiT $5 \mid n^5 - n \implies 5 \mid x \implies 5 \mid n^9 - n$.
- By FLiT $3 \mid n^3 - n \implies 3 \mid y \implies 3 \mid n^9 - n$.
- By FLiT $2 \mid n^2 - n \implies 2 \mid z \implies 2 \mid n^9 - n$.

It follows that $2 \cdot 3 \cdot 5 \mid n^9 - n \implies 30 \mid n^9 - n$. \square

4. The definition of n being a Fermat pseudoprime to base b does not actually require that $\gcd(b, n) = 1$ because it's not possible to have $b^{n-1} \equiv 1 \pmod{n}$ with $\gcd(b, n) \neq 1$. Prove this.

Proof. Let $\gcd(b, n) = a$ where $a \neq 1$, this implies there exists a prime p such that $p \mid b$ and $p \mid n$. It follows that there exists a linear combination of b and n such that $xb + yn = 1$ for $x, y \in \mathbb{Z}$. But if $p \mid b$ and $p \mid n$ then $p \mid xb + yn = 1 \implies p \mid 1$ but this is a contradiction to the fact that p is prime. \square

5. We didn't exclude even integers from the definition of a Fermat Pseudoprime. Some books do. Show that with our definition 4 is a Fermat Pseudoprime to a certain base.

From $5^{4-1} = 5^3 \equiv 1 \pmod{4}$, we see that 4 is a Fermat Pseudoprime to the base 5.

6. Prove that if n is an odd Fermat Pseudoprime to some base then it must be so to an even number of bases.

Proof. Let n be an odd Fermat Pseudoprime, then let b be a base of n where $b^{n-1} \equiv 1 \pmod{n}$. Because n is odd $n - 1$ is even, this means that $b^{n-1} = (-b)^{n-1}$, it follows that any base b has a pair $-b$ that is also a base as long as $-b \not\equiv b \pmod{n}$.

If $-b \equiv b \pmod{n}$ then n would divide $2b$ but since n is odd this implies $n \mid b$. Then, $b^{n-1} \equiv 0 \pmod{n}$ which contradicts $b^{n-1} \equiv 1 \pmod{n}$.

So it is not possible for $-b \equiv b \pmod{n}$, therefore any base b of n has a respective pair $-b$ such that n has an even number of bases. \square

7. Prove that 1105 is a Carmichael number.

Proof. Note that $1105 = 5 \cdot 13 \cdot 17$. Suppose b satisfies $\gcd(b, 1105) = 1$. Then,

- $\gcd(b, 5) = 1$ so by FLiT $b^4 \equiv 1 \pmod{5}$. So $b^{1104} = (b^4)^{276} \equiv 1 \pmod{5}$ so $5 \mid b^{1104} - 1$.
- $\gcd(b, 13) = 1$ so by FLiT $b^{12} \equiv 1 \pmod{13}$. So $b^{1104} = (b^{12})^{92} \equiv 1 \pmod{13}$ so $13 \mid b^{1104} - 1$.
- $\gcd(b, 17) = 1$ so by FLiT $b^{16} \equiv 1 \pmod{17}$. So $b^{1104} = (b^{16})^{69} \equiv 1 \pmod{17}$ so $17 \mid b^{1104} - 1$.

So $5 \cdot 13 \cdot 17 \mid b^{1104} - 1 \implies 1105 \mid b^{1104} - 1$. Therefore $b^{1104} \equiv 1 \pmod{1105}$. \square

8. Use Euler's Theorem to find the units digit of 7^{999999} .

The units digit is the least non-negative residue mod 10. Since $\phi(10) = 4$ we have $7^4 \equiv 1 \pmod{10}$ and so:

$$7^{999999} = (7^4)^{249999} 7^3 \equiv 7^3 \equiv 3 \pmod{10}$$

The units digit of 7999999 is 3.

9. Solve each of the following using Euler's Theorem. Solutions should be least nonnegative residues.

(a) $5x \equiv 3 \pmod{14}$

Since $\gcd(5, 14) = 1$ then $5^{\phi(14)} \equiv 1 \pmod{14}$. Then $5^6 = 5^5 \cdot 5 \equiv 1 \pmod{14}$ where $5^5 \equiv 3$ is the inverse. Then $5x \equiv 3 \pmod{14}$ can be reduced to $x \equiv 3 \cdot 3 \pmod{14}$. So 9 is the least non-negative residue.

(b) $4x \equiv 7 \pmod{15}$

Since $\gcd(4, 15) = 1$ then $4^{\phi(15)} \equiv 1 \pmod{15}$. Then $4^8 = 4^7 \cdot 4 \equiv 1 \pmod{15}$ where $4^7 \equiv 4$ is the inverse. Then $4x \equiv 7 \pmod{15}$ can be reduced to $x \equiv 7 \cdot 4 \pmod{15}$. So 13 is the least non-negative residue.

(c) $3x \equiv 5 \pmod{16}$

Since $\gcd(3, 16) = 1$ then $3^{\phi(16)} \equiv 1 \pmod{16}$. Then $3^8 = 3^7 \cdot 3 \equiv 1 \pmod{16}$ where $3^7 \equiv 11$ is the inverse. Then $3x \equiv 5 \pmod{16}$ can be reduced to $x \equiv 5 \cdot 11 \pmod{16}$. So 7 is the least non-negative residue.

10. Prove that if $\gcd(a, 30) = 1$ then $60 \mid a^4 + 59$.

Proof. Note that $60 = 5 \cdot 12$, because $\gcd(a, 30) = 1$ we know $\gcd(a, 5) = 1$ and $\gcd(a, 12) = 1$.

- $a^{\phi(5)} = a^4 \equiv 1 \pmod{5}$, so $5 \mid a^4 - 1$.
- $a^{\phi(12)} = a^4 \equiv 1 \pmod{12}$, so $12 \mid a^4 - 1$.

Since $\gcd(5, 12) = 1$ it follows that $60 \mid a^4 - 1 \implies 60 \mid (a^4 - 1) + 60 \implies 60 \mid a^4 + 59$. \square

1.5 Various Multiplicative Functions

/100

1. Find all n satisfying $\phi(n) = 18$.

Suppose $p \mid n$, then $p - 1 \mid \phi(n)$ so $p - 1 = 1, 2, 3, 6, 9, 18 \implies p = 2, 3, 4, 7, 10, 19$. But 4 and 10 are not prime so $p = 2, 3, 7, 19$. Therefore $n = 2^\alpha 3^\beta 7^\gamma 19^\delta$ for some α, β, γ , and δ . Then,

- If $2^\alpha \mid n$ with $\alpha > 0$ then $2^{\alpha-1} \mid \phi(n)$ so $\alpha = 0, 1, 2$.
- If $3^\beta \mid n$ with $\beta > 0$ then $3^{\beta-1} \mid \phi(n)$ so $\beta = 0, 1, 2, 3$.
- If $7^\gamma \mid n$ with $\gamma > 0$ then $7^{\gamma-1} \mid \phi(n)$ so $\gamma = 0, 1$.
- If $19^\delta \mid n$ with $\delta > 0$ then $19^{\delta-1} \mid \phi(n)$ so $\delta = 0, 1$.

Then these are the cases,

- | | | |
|---------------------------------|---------------------------------|----------------------------------|
| • $n = 2^0 3^0 7^0 19^0 = 1$ | • $n = 2^1 3^0 7^0 19^0 = 2$ | • $n = 2^2 3^0 7^0 19^1 = 4$ |
| • $n = 2^0 3^0 7^0 19^1 = 19$ | • $n = 2^1 3^0 7^0 19^1 = 38$ | • $n = 2^2 3^0 7^0 19^1 = 76$ |
| • $n = 2^0 3^0 7^1 19^0 = 7$ | • $n = 2^1 3^0 7^1 19^0 = 14$ | • $n = 2^2 3^0 7^1 19^1 = 28$ |
| • $n = 2^0 3^0 7^1 19^1 = 133$ | • $n = 2^1 3^0 7^1 19^1 = 266$ | • $n = 2^2 3^0 7^1 19^1 = 532$ |
| • $n = 2^0 3^1 7^0 19^0 = 3$ | • $n = 2^1 3^1 7^0 19^0 = 6$ | • $n = 2^2 3^1 7^0 19^1 = 12$ |
| • $n = 2^0 3^1 7^0 19^1 = 57$ | • $n = 2^1 3^1 7^0 19^1 = 114$ | • $n = 2^2 3^1 7^0 19^1 = 228$ |
| • $n = 2^0 3^1 7^1 19^0 = 21$ | • $n = 2^1 3^1 7^1 19^0 = 42$ | • $n = 2^2 3^1 7^1 19^1 = 84$ |
| • $n = 2^0 3^1 7^1 19^1 = 399$ | • $n = 2^1 3^1 7^1 19^1 = 798$ | • $n = 2^2 3^1 7^1 19^1 = 1596$ |
| • $n = 2^0 3^2 7^0 19^0 = 9$ | • $n = 2^1 3^2 7^0 19^0 = 18$ | • $n = 2^2 3^2 7^0 19^1 = 36$ |
| • $n = 2^0 3^2 7^0 19^1 = 171$ | • $n = 2^1 3^2 7^0 19^1 = 342$ | • $n = 2^2 3^2 7^0 19^1 = 684$ |
| • $n = 2^0 3^2 7^1 19^0 = 63$ | • $n = 2^1 3^2 7^1 19^0 = 126$ | • $n = 2^2 3^2 7^1 19^1 = 252$ |
| • $n = 2^0 3^2 7^1 19^1 = 1197$ | • $n = 2^1 3^2 7^1 19^1 = 2394$ | • $n = 2^2 3^2 7^1 19^1 = 4788$ |
| • $n = 2^0 3^3 7^0 19^0 = 27$ | • $n = 2^1 3^3 7^0 19^0 = 54$ | • $n = 2^2 3^3 7^0 19^1 = 108$ |
| • $n = 2^0 3^3 7^0 19^1 = 513$ | • $n = 2^1 3^3 7^0 19^1 = 1026$ | • $n = 2^2 3^3 7^0 19^1 = 2052$ |
| • $n = 2^0 3^3 7^1 19^0 = 189$ | • $n = 2^1 3^3 7^1 19^0 = 378$ | • $n = 2^2 3^3 7^1 19^1 = 756$ |
| • $n = 2^0 3^3 7^1 19^1 = 3591$ | • $n = 2^1 3^3 7^1 19^1 = 7182$ | • $n = 2^2 3^3 7^1 19^1 = 14364$ |

Then evaluating all n as $\phi(n)$ we get that for $n = 19, 27, 38, 54$, $\phi(n) = 18$.

2. Show there are no n with $\phi(n) = 14$.

Suppose $\phi(n) = 14$ for some n , then $7 \mid p^\alpha - p^{\alpha-1}$ for some odd prime p . Since the factors of 14 are 2 and 7 we have two cases. If $p = 7$ and $\alpha > 1$ which implies $6 \mid 14$ which is not true. Or if $7 \mid p - 1$ but $p - 1$ is even, so $p = 15$ which is not prime. Therefore there are no n with $\phi(n) = 14$.

3. For what values of n is $\phi(n)$ odd? Justify.

Since $\phi(p^\alpha) = p^{\alpha-1}(p - 1)$ we know that it will be even for all $p > 2$, therefore n cannot have any prime factors greater than 2. It follows that for $n = 1, 2$ $\phi(n)$ is odd.

4. Prove that $f(n) = \gcd(n, 3)$ is multiplicative. (This is actually true if 3 is replaced by any positive integer.)

Proof. We wish to show $f(mn) = f(m) \cdot f(n)$ when $\gcd(m, n) = 1$. Suppose that $\gcd(m, n) = 1$, then $\gcd(\gcd(m, 3), \gcd(n, 3)) = p$ for $p \in \mathbb{Z}^+$. This implies that $p \mid \gcd(m, 3)$ and $p \mid \gcd(n, 3)$ which implies $p \mid m$ and $p \mid n$, but since $\gcd(m, n) = 1$ we have $p = 1$.

Let $\gcd(m, 3) = x$, this implies $x \mid 3$ and $x \mid m$. Then,

$$x \mid m \implies x \mid mn \implies x \mid \gcd(mn, 3)$$

Likewise, let $\gcd(n, 3) = y$, this implies $y \mid 3$ and $y \mid n$. Then,

$$y \mid n \implies y \mid mn \implies y \mid \gcd(mn, 3)$$

Then because $x \mid \gcd(mn, 3)$ and $y \mid \gcd(mn, 3)$ and $\gcd(x, y) = 1$ we have, $xy \mid \gcd(mn, 3)$. Thus, $\gcd(mn, 3) = \gcd(m, 3) \cdot \gcd(n, 3)$ and $f(n)$ is multiplicative. \square

5. Find $\tau(2 \cdot 3^2 \cdot 5^3 \cdot 11^5 \cdot 13^4 \cdot 17^5 \cdot 19^5)$

$$= (1+1)(2+1)(3+1)(5+1)(4+1)(5+1)(5+1) = 2 \cdot 3 \cdot 4 \cdot 6 \cdot 5 \cdot 6 \cdot 6 = 25920$$

6. Find $\sigma(2 \cdot 3^2 \cdot 5^3 \cdot 11^5 \cdot 13^4 \cdot 17^5 \cdot 19^5)$

$$= \left(\frac{2^2-1}{2-1}\right) \left(\frac{3^3-1}{3-1}\right) \left(\frac{5^4-1}{5-1}\right) \left(\frac{11^6-1}{11-1}\right) \left(\frac{13^5-1}{13-1}\right) \left(\frac{17^6-1}{17-1}\right) \left(\frac{19^6-1}{19-1}\right)$$

7. Find $\tau(20!)$.

First we need the prime factorization of $20!$,

$$\begin{aligned} 20! &= (2) (3) (2^2) (5) (2 \cdot 3) (7) (2^3) (3^2) (2 \cdot 5) (11) (2^2 \cdot 3) \\ &\quad (13) (2 \cdot 7) (3 \cdot 5) (2^4) (17) (2 \cdot 3^2) (19) (2^2 \cdot 5) \end{aligned}$$

Thus, $20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. Therefore, $\tau(20!) = (18+1)(8+1)(4+1)(2+1)(1+1)(1+1)(1+1)(1+1) = 41040$.

8. Classify all n with $\tau(n) = 30$. Explain!

Suppose $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with $a_i > 0$, it follows that $\tau(n) = (1+\alpha_1) \cdots (1+\alpha_k) = 30$. Since $(1+\alpha_i) \geq 2$ we get the following cases:

- $k = 15$ and $a_1 = \cdots = a_{15} = 1$.
- $k = 10$ and $a_1 = \cdots = a_{10} = 2$.
- $k = 6$ and $a_1 = \cdots = a_6 = 4$.
- $k = 5$ and $a_1 = \cdots = a_5 = 5$.
- $k = 3$ and $a_1 = \cdots = a_3 = 9$.
- $k = 1$ and $a_1 = 29$.

Then we have,

- $n = p_1 \cdots p_{15}$
- $n = p_1^2 \cdots p_{10}^2$
- $n = p_1^4 \cdots p_6^4$
- $n = p_1^5 \cdots p_5^5$
- $n = p_1^9 p_2^9 p_3^9$
- $n = p_1^{29}$

9. Prove that $\sigma(n) = k$ has at most a finite number of solutions when k is a positive integer.

Proof. Since $\sigma(n)$ is multiplicative we know that $1 \mid n$ and $n \mid n$ implies that $\sigma(n) \geq 1+n$. It follows then that $\sigma(n) = k \implies 1+n \leq k \implies n \leq k-1$. Thus, $\sigma(n) = k$ has a finite number of solutions. \square

10. Show that if a and b are positive integers and p and q are distinct odd primes then $n = p^a q^b$ is deficient.

We wish to show $\sigma(n) < 2n$,

$$\sigma(p^a q^b) = \left(\frac{p^{a+1} - 1}{p - 1} \right) \left(\frac{q^{b+1} - 1}{q - 1} \right) < \left(\frac{p^{a+1}}{p - 1} \right) \left(\frac{q^{b+1}}{q - 1} \right) = \left(\frac{p}{p - 1} \right) \left(\frac{q}{q - 1} \right) p^a q^b$$

Since we wish to show $\sigma(n) < 2n$ we then need $\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right) < 2$.

$$\begin{aligned}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right) &< 2 \\ pq &< 2(p-1)(q-1) \\ pq &< 2(pq - p - q + 1) \\ pq &< 2pq - 2p - 2q + 2 \\ 0 &< pq - 2p - 2q + 2 \\ 0 &< (p-2)(q-2) - 2 \\ -2 &< (p-2)(q-1)\end{aligned}$$

Recall that p and q are *distinct* odd primes, so let's assume that $p > q > 2$. Since let $p = 5$ and $q = 3$, we then see $2 < (5-2)(3-2)$, thus the inequality holds. Therefore $p^a q^b$ is deficient.

11. Prove that a perfect square cannot be a perfect number.

Proof. Let n be a perfect square, first observe that when n is even by Euler's Theorem $n = 2^{p-1}(2^p - 1) \implies \sqrt{n} = \sqrt{2^{p-1}(2^p - 1)}$ but then this implies that $\sqrt{n} \notin \mathbb{Z}$ which contradicts n being a perfect square. Therefore n must be odd. So $n = 2^k(p_1^{\alpha_1} \cdots p_i^{\alpha_i})$, note that for $p_1^{\alpha_1} \cdots p_i^{\alpha_i}$ each p and α is even. Then it follows that n has an odd number of odd divisors. Thus $\sigma(n)$ is an odd number, but a perfect number is when $\sigma(n) = 2n$ which implies $\sigma(n)$ must be an even number. Therefore a perfect square cannot be a perfect number. \square

12. Use Theorem 7.12 to determine whether each of the following Mersenne numbers is a Mersenne prime:

Personally, I found it easiest to create a simple python script to test for primality of Mersenne numbers.

```
1 import math
2
3 def Mersenne_Prime (n):
4     M = 2**n - 1
5
6     for k in range (1, int(math.sqrt(M))):
7         factor = (2*n)*k + 1
8         if M % factor == 0:
9             print(M/factor)
10            return false
11
12 return True
```

(a) M_{11}

First we see $M_{11} = 2^{11} - 1 = 2047$, then the factors of 2047 are of the form $2(11)k + 1 = 22k + 1$. Look at k up to $\sqrt{2047} \approx 45.24$. We find that $2047 = 23 \cdot 89$, therefore M_{11} is not prime.

(b) M_{21}

First we observe that $21 = 3 \cdot 7$, then by the definition of a Mersenne prime, since 21 is not prime then M_{21} is not prime.

(c) M_{31}

First we see $M_{31} = 2^{31} - 1 = 2147483647$, then the factors of 2147483647 are of the form $2(31)k + 1 = 62k + 1$. Look at k up to $\sqrt{2147483647} \approx 46340.95$. We find that there is no factor of 2147483647 of the form $62k + 1$ for $k \leq 46341$, therefore M_{31} is prime.

1.6 Primitive Roots

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