Math 406: Homework for Chapter 1 Neelam Akula

1. Determine whether each of the following sets is well-ordered. If so, give a proof which relies on the fact that $\mathbb{Z}^+$ is well-ordered. If not, give an example of a subset with no least element.

   (a) $\{a \mid a \in \mathbb{Z}, a > 3\}$
   Is a subset of $\mathbb{Z}^+$ and therefore is well-ordered.

   (b) $\{a \mid a \in \mathbb{Q}, a > 3\}$
   There is no least element so the set is not well-ordered.

   (c) $\{\frac{a}{2} \mid a \in \mathbb{Z}, a \geq 10\}$
   Consider the set $\{a \mid a \in \mathbb{Z}, a \geq 10\}$, it is apparent that this is a subset of $\mathbb{Z}^+$ and therefore is well-ordered. So the set $\{\frac{a}{2} \mid a \in \mathbb{Z}, a \geq 10\}$ is also well-ordered because it holds a least element ($\frac{10}{5}$).

   (d) $\{\frac{2}{a} \mid a \in \mathbb{Z}, a > 10\}$
   There is no least element so the set is not well-ordered.

2. Suppose $a, b \in \mathbb{Z}^+$ are unknown. Let $S = \{a - bk \mid k \in \mathbb{Z}, a - bk > 0\}$. Explain why S has a smallest element but no largest element.

   Since S is a subset of $\mathbb{Z}^+$ by well-ordering we know that S has a least element, and because $k \in \mathbb{Z}$, k can be 0 and therefore there is no most element.

3. Use the well-ordering property to show that $\sqrt{5}$ is irrational.

   Proof. Suppose $\sqrt{5}$ is rational and is of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}^+$ and $b \neq 0$. Consider the set S,

$$S = \left\{k \mid k, k\sqrt{5} \in \mathbb{Z}^+\right\}$$

   We know that S is a subset of $\mathbb{Z}^+$ and that $b \in S$, by well-ordering this implies that S has a least element. Let l be the least element in S.
   Consider the properties of l' where $l' = l\sqrt{5} - 2l$,

   - $l' = l\sqrt{5} - 2l = l(\sqrt{5} - 2) \implies 0 < l' < l$.
   - Since $l \in S$ and $S \subset \mathbb{Z}^+$, both l and $l\sqrt{5} \in \mathbb{Z}^+$ which implies $l' \in \mathbb{Z}^+$.
   - Since $l \in \mathbb{Z}^+$ we have $5l \in \mathbb{Z}^+$ and since $l\sqrt{5} \in \mathbb{Z}^+$ we have $l'\sqrt{5} = (l\sqrt{5} - 2l)\sqrt{5} = 5l - 2l\sqrt{5} \in \mathbb{Z}^+$.

   It follows that $l' \in S$ but $l' < l$ which contradicts l being the least element in S. $\square$

4. Use the identity

$$\frac{1}{k^2-1} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$$

to evaluate the following:

(a) $\sum\limits_{k=2}^{10} \frac{1}{k^2-1}$

$$\begin{aligned}
\sum_{k=2}^{10} \frac{1}{k^2-1} &= \sum_{k=2}^{10} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{2}\sum_{k=2}^{10}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) \\
&= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{8}-\frac{1}{10}\right) + \left(\frac{1}{9}-\frac{1}{11}\right)\right] \\
&= \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} - \frac{1}{10} - \frac{1}{11}\right] \\
&= \frac{1}{2}\left(\frac{72}{55}\right) = \frac{36}{55}
\end{aligned}$$

(b) $\sum\limits_{k=2}^{n} \frac{1}{k^2-1}$

$$\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right]$$

(c) $\sum\limits_{k=1}^{n} \frac{1}{k^2+2k}$ Hint: $k^2 + 2k = (???)^2 - 1$

$$\sum_{k=1}^{n} \frac{1}{k^2+2k} = \sum_{k=1}^{n} \frac{1}{(k+1)^2-1} = \sum_{k=2}^{n+1} \frac{1}{k^2-1}$$

$$\sum_{k=2}^{n+1} \frac{1}{k^2-1} = \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

5. Find the value of each of the following:

(a) $\prod\limits_{j=2}^{7}\left(1-\frac{1}{j}\right)$

$$\begin{aligned}
\prod_{j=2}^{7}\left(1-\frac{1}{j}\right) &= \left[\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{5}\right)\cdot\left(1-\frac{1}{6}\right)\cdot\left(1-\frac{1}{7}\right)\right] \\
&= \left[\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}\right] \\
&= \frac{1}{7}
\end{aligned}$$

(b) $\prod_{j=2}^{n} \left(1 - \frac{1}{j}\right)$

$$\prod_{j=2}^{n} \left(1 - \frac{1}{j}\right) = \frac{1}{n}$$

(c) $\prod_{j=2}^{n} \left(1 - \frac{1}{j^2}\right)$ Hint: Be sneaky!

$$\prod_{j=2}^{n} \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n}$$

6. Use weak mathematical induction to prove that

$$\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$$

for every positive integer n.

Proof.

Base Case:

Let $n = 1$, $\sum_{j=1}^{1} j(j+1) = 2$ and $\frac{1(1+1)(1+2)}{3} = 2$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n. This implies that $\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$.

Inductive Step:

Then consider the sum to $n+1$:

$$\begin{aligned}
\sum_{j=1}^{n+1} j(j+1) &= \sum_{j=1}^{n} j(j+1) + (n+1)((n+1)+1) \\
&= \left[\frac{n(n+1)(n+2)}{3}\right] + (n+1)((n+1)+1) \quad \text{by IH} \\
&= \frac{1}{3}\left(n(n+1)(n+2) + 3(n+1)(n+2)\right) \\
&= \frac{1}{3}\left(n^3 + 3n^2 + 2n + 3n^2 + 9n + 6\right) \\
&= \frac{1}{3}\left(n^3 + 6n^2 + 11n + 6\right) \\
&= \frac{1}{3}\left((n+1)(n+2)(n+3)\right)
\end{aligned}$$

Thus for all $n \geq 1$,

$$\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$$

$\square$

7. Use Weak Mathematical Induction to show that $f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1}$ for all $n \geq 1$.

Proof.

Base Case:

Rewrite the statement $f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1}$ to be $f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$.
Let $n = 1$, $f_1 f_{1+2} - f_{1+1}^2 = 1 \cdot 2 - 1 = 1$ and $(-1)^{1+1} = 1$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n.
This implies that $f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$

Inductive Step:

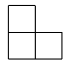
Then consider the equation to $n + 1$:

$$\begin{aligned}
f_{(n+1)} f_{(n+1)+2} - f_{(n+1)+1}^2 &= f_{n+1} f_{n+3} - f_{n+2}^2 \\
&= f_{n+1}\left(f_{n+1} + f_{n+2}\right) - f_{n+2}^2 \\
&= f_{n+1}^2 + f_{n+1} f_{n+2} - f_{n+2}^2 \\
&= f_{n+1}^2 + f_{n+2}\left(f_{n+1} - f_{n+2}\right) \\
&= f_{n+1}^2 + f_{n+2}\left(-f_n\right) \\
&= -\left(f_n f_{n+2} - f_{n+1}^2\right) \\
&= -(-1)^{n+1} \quad \text{by IH} \\
&= (-1)^{n+2}
\end{aligned}$$

Thus for all $n \geq 1$,

$$f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$$

$\square$

8. Use weak mathematical induction to show that a $2^n \times 2^n$ chessboard with a corner missing can be tiled with pieces shaped like  for every integer $n \geq 0$.

Proof.

Base Case:

Let $n = 1$, $2^1 \times 2^1$ is a 2×2 chessboard with a corner missing and can be tiled by one tromino, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n. This implies that any $2^n \times 2^n$ chessboard with a corner missing can be tiled with trominoes.

Inductive Step:

Then consider a $2^{n+1} \times 2^{n+1}$ chessboard.

- Divide the $2^{n+1} \times 2^{n+1}$ chessboard into four quadrants of size $2^n \times 2^n$.
- By the Inductive Hypothesis we know that each $2^n \times 2^n$ has one corner missing.
- There are then four empty squares in the $2^{n+1} \times 2^{n+1}$ board.
- Rotate each quadrant such that the four empty squares are in the center of the board.
- Add another tromino into the board leaving only one empty square.
- Rotate the quadrant with the empty square such that the empty square is in the corner of the board.
- Therefore the $2^{n+1} \times 2^{n+1}$ chessboard can be tiled by trominoes with a corner missing.

Thus, every $2^n \times 2^n$ chessboard with a corner missing can be tiled with trominoes.

$\square$

9. Define:

$$H_{2^n} = \sum_{j=1}^{2^n} \frac{1}{j}$$

Use weak mathematical induction to prove that for all $n \geq 1$ we have $H_{2^n} \leq 1 + n$.

Proof.

Base Case:

Let $n = 1$, $H_{2^1} = \sum_{j=1}^{2^n} \frac{1}{j} = \frac{3}{2}$ and $\frac{3}{2} \leq 2$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n.
This implies that $\sum_{j=1}^{2^n} \frac{1}{j} \leq 1 + n$.

Inductive Step:

Then consider the equation to $n + 1$:

$$\begin{aligned}
H_{2^{n+1}} &= \sum_{j=1}^{2^{n+1}} \frac{1}{j} \\
&= \sum_{j=1}^{2^n} \frac{1}{j} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \\
&\leq [1+n] + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \qquad \text{by IH} \\
&\leq [1+n] + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\
&\leq [1+n] + 2^n \cdot \frac{1}{2^{n+1}} \\
&\leq 1 + n + \frac{1}{2} \\
&\leq \frac{3}{2} + n \leq 2 + n
\end{aligned}$$

Thus for all $n \geq 1$,

$$H_{2^n} \leq 1 + n$$

□

10. Use strong mathematical induction to prove that every amount of postage over 53 cents can be formed using 7-cent and 10-cent stamps.

Proof.

Inductive Step:

Assume we can do $54, \cdots, k$. Because $k-6$ is in the $54, \cdots, k$ we can do $k-6$ then add a 7-cent stamp. $k-6$ is in $54, \cdots, k$ only if $k-6 \geq 54 \equiv k \geq 60$.
Thus, the inductive step is only valid for $k = 60, 61, \cdots$ to get to the next $k+1$.

Base Case:

Must do $54, 55, 56, 57, 58, 59, 60$ as base cases.

$$\begin{aligned}
54 &= 2(\text{7-cent}) + 4(\text{10-cent}) \\
55 &= 5(\text{7-cent}) + 2(\text{10-cent}) \\
56 &= 8(\text{7-cent}) \\
57 &= 1(\text{7-cent}) + 5(\text{10-cent}) \\
58 &= 4(\text{7-cent}) + 3(\text{10-cent}) \\
59 &= 7(\text{7-cent}) + 1(\text{10-cent}) \\
60 &= 6(\text{10-cent})
\end{aligned}$$

□