

1 Congruences

1.1 Introduction to Congruences

1. **Introduction:** Suppose you wished to find $x, y \in \mathbb{Z}$ satisfying $2x^2 - 8y = 11$. There is no solution because no matter what, $2x^2 - 8y$ is even and 11 is odd. What if even/odd does not work... what else might? $3x^2 - 15y = 8$, 3 divides the left side but not the right. If even/odd or divided by 3 works, there is no guarantee that it works $\underbrace{3x^2 - 15y = 9}_{\text{might work}}$. The idea of modular arithmetic formalizes all of this.

2. **Definition and Equivalencies:** For $a, b, m \in \mathbb{Z}$ with $m \geq 2$ we write $a \equiv b \pmod{m}$ which is read as "a and b are congruent modulo m." to mean that $m \mid (a - b)$. A few notes on this,

- Equivalent to saying $m \mid (b - a)$.
- Equivalent to saying $\exists c \in \mathbb{Z}$ such that $mc = a - b$ or $\exists x \in \mathbb{Z}$ such that $mc = b - a$ (definition of divisibility).
- Equivalent to saying that if we divide a and b by m , the remainders are the same.

Ex. $8 \equiv 18 \pmod{5}$ in fact $8 \equiv 18 \equiv 3 \equiv -2 \equiv 23 \equiv \dots \pmod{5}$. Here with remainder 3. Also note $5 \mid (18 - 8)$ and $5 \mid (8 - 18)$.

Even/odd is the same as $m = 2$.

CS Note. In computer science we often define $\text{mod}(a, m) = \text{remainder when } a/m = a \% m$. It is not uncommon to see $a = b \pmod{m}$ or $a \equiv_m b$ (strongly discouraged).

Moving forward, please use $a \equiv b \pmod{m}$.

3. Properties:

- (a) **Theorem.** Congruence acts like an equals sign in the following sense:

- (i) $a \equiv a \pmod{m}$ (Reflexive).
- (ii) if $a \equiv b \pmod{m}$ then $b \equiv a \pmod{m}$ (Symmetric).
- (iii) If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then $a \equiv c \pmod{m}$ (Transitivity).

Proof. $a \equiv b \pmod{m} \implies \exists x \text{ such that } a - b = mx, b \equiv c \pmod{m} \implies \exists y \text{ such that } b - c = my$. Then $a - c = (a - b) + (b - c) = mx + my = m(x + y)$ so $m \mid (a - c)$ so $a \equiv c \pmod{m}$. \square

- (iv) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $a \pm c \equiv b \pm d \pmod{m}$.

- i.e. If we know $x \equiv y \pmod{5}$ we can conclude $x + 7 \equiv y + 7 \pmod{5}$ and also $x + 7 \equiv y + 12 \pmod{5}$.
- (v) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $ac \equiv bd \pmod{m}$
i.e. If we know $x \equiv y \pmod{5}$ then we can conclude $17x \equiv 17y \pmod{5}$ but we can also conclude $17x \equiv 12y \pmod{5}$
- (vi) If $a \equiv b \pmod{m}$ and $k \in \mathbb{Z}, k \geq 1$ then $a^k \equiv b^k \pmod{m}$. (Note: we can *not* use different powers!)
- (b) **Division Issues.** First everything must be an integer, so does $2 \equiv 8 \pmod{6} \implies \frac{2}{3} \equiv \frac{8}{3} \pmod{6}$ this is garbage because $\frac{2}{3}, \frac{8}{3} \notin \mathbb{Z}$. However, is $2 \equiv 8 \pmod{6} \implies \frac{2}{2} \equiv \frac{8}{2} \pmod{6}$ true? No! because $1 \equiv 4 \pmod{6}$ is not true. The point is even if division makes both sides integers there is no guarantee that the congruence is preserved!

Theorem. Suppose we have $ac \equiv bc \pmod{m}$ then $a \equiv b \pmod{m/\gcd(m, c)}$. In other words we may cancel an integer from both sides provided we divide the modulus by the gcd of the modulus and the integer we're canceling.

Proof. Suppose $ac \equiv bc \pmod{m}$, $\exists k \in \mathbb{Z}$ with $mk = ac - bc$. So $mk = c(b - a)$,

$$\frac{m}{\gcd(c, m)}k = \frac{c}{\gcd(c, m)}(a - b)$$

Note that from a previous theorem we know that:

$$\gcd\left(\frac{m}{\gcd(c, m)}, \frac{c}{\gcd(c, m)}\right) = 1$$

Then the above statement says that $\frac{m}{\gcd(c, m)} \mid \frac{c}{\gcd(c, m)}(a - b)$ which implies $\frac{m}{\gcd(c, m)} \mid a - b$. Therefore, $a \equiv b \pmod{\frac{m}{\gcd(c, m)}}$. \square

Note. Don't think division, think cancelation when dealing with modulo.

Ex. If we know that $4x \equiv 8y \pmod{50}$ then we can conclude that $x \equiv 2y \pmod{50/\gcd(50, 4)}$ and so $x \equiv 2y \pmod{25}$ (think *cancel* the 4).

Corollary. If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$ then $a \equiv b \pmod{m}$.

Ex. $15x \equiv 20y \pmod{27}$, note that $\gcd(5, 27) = 1$ so we may cancel the 5. So $3x \equiv 4y \pmod{27}$.

4. Residue Classes:

- (a) **Introduction:** Suppose we are working $\pmod{m = 5}$. We know $0 \equiv 5 \equiv 10 \equiv -5 \equiv \dots \pmod{5}$, we also know $1 \equiv 6 \equiv -4 \equiv \dots \pmod{5}$, all

of \mathbb{Z} fall into one out of $m = 5$ classes.

$$\begin{aligned} &\{\dots, -15, -10, -5, 0, 5, 10, 15, \dots\} \\ &\{\dots, -16, -9, -4, 1, 6, 11, 16, \dots\} \\ &\{\dots, -13, -8, -3, 2, 7, 12, 17, \dots\} \\ &\{\dots, -12, -7, -2, 3, 8, 13, 18, \dots\} \\ &\{\dots, -11, -6, -1, 4, 9, 14, 19, \dots\} \end{aligned}$$

- (b) **Definition.** For a given $m \geq 2$ there are m congruence classes.
(c) **Definition.** From each we may pick a representative of the class so those would be m representatives.

Ex. $m = 5 : \{0, 1, 2, 3, 4\}$ (the obvious one) or you could use $m = 5 : \{0, 2, 4, 6, 8\}$ (all even) or $m = 5 : \{0, 2, 4, 8, 16\}$ (all powers of 2, except 0).

Ex. $m = 5 : \{0, 1, 2, 3, 4\}$ (the obvious one) or you could use $m = 5 : \{0, 2, 4, 6, 8\}$ (all even) or $m = 5 : \{0, 2, 4, 8, 16\}$ (all powers of 2, except 0).

- (d) **Definition.** The set of representatives $\{0, \dots, m-1\}$ = the complete set of least non-negative residues.

In \mathbb{R} , $17^x = 48246319 \implies x = \log_7 7(48246319)$. Now consider $\mathbb{Z} \bmod 100$, $6^x \equiv 88 \bmod 100$ is *significantly* harder to solve (the discrete logarithm problem).

- (e) **Definition.** A complete set of residues (CSOR) mod m is a set of m integers, no two of which are congruent mod m .

Ex. $m = 5$: here are 3 CSORs: $\{0, 1, 2, 3, 4\}$, $\{0, 2, 4, 6, 8\}$, $\{0, 2, 4, 8, 16\}$, and more!

- (f) **Theorem.** A subset S of \mathbb{Z} is a CSOR mod m if and only if every integer is congruent to exactly one element in S .

Ex. $m = 4$: $S = \{0, 9, 14, 3\}$ some observations:

- $m = 4$ of them.
- No two are congruent to each other.
- Any $a \in \mathbb{Z}$ is congruent to exactly one of these.

- (g) **Theorem.** If $\{r_1, r_2, \dots, r_m\}$ is a CSOR mod m and if $a, b \in \mathbb{Z}$ with $\gcd(a, m) = 1$ then $\{ar_1 + b, ar_2 + b, \dots, ar_m + b\}$ is also a CSOR mod m .

Proof. We will show that no two are congruent mod m . Suppose $ar_i + b \equiv ar_j + b \bmod m$ with $i \neq j$. Then $ar_i \equiv ar_j \bmod m \implies r_i \equiv r_j \bmod m$ because $\gcd(a, m) = 1$. Contradiction because the r_i, r_j came from a CSOR mod m . \square

Ex. $\{0, 1, 2, 3, 4\}$ CSOR mod 5. Pick $a = 9, b = 42$, $\{0 \cdot 9 + 42, 1 \cdot 9 + 42, 2 \cdot 9 + 42, 3 \cdot 9 + 42, 4 \cdot 9 + 42\}$ is also a CSOR mod 5.

5. **Fast Arithmetic - Fast Exponentiation.** Suppose we wished to calculate $2^{503} \equiv a \pmod{5}$, $a = 0, 1, 2, 3, 4$ but which one? **Warning:** Do not reduce exponent mod 5! $2^{503} \equiv 2^x \pmod{5}$.

- (a) Look for patterns: $2^1 \equiv 2 \pmod{5}$, $2^2 \equiv 4 \pmod{5}$, $2^3 \equiv 3 \pmod{5}$, $2^4 \equiv 1 \pmod{5}$, $2^5 \equiv 2 \pmod{5}$. This last one is a repeat, so it repeats every 4. Note $503 = 4(125) + 3$ so

$$\begin{aligned} 2^{503} &\equiv 2^{4(125)+3} 2^3 \\ &\equiv (1)^{125} 2^3 \pmod{5} \\ &\equiv (1) 8 \pmod{5} \\ &\equiv 3 \pmod{5} \end{aligned}$$

- (b) Use binary expansions. Suppose we want $3^{81} \equiv a \pmod{5}$. $3^1 \equiv 3$, $3^2 \equiv 4$, $3^4 \equiv 1$, $3^8 \equiv 1$, $3^{16} \equiv 1$, $3^{32} \equiv 1$, $3^{64} \equiv 1$. Then $81 = 64 + 16 + 1$ so

$$\begin{aligned} 3^{81} &= 3^{64} 3^{16} 3^1 \\ &\equiv 1 \cdot 1 \cdot 3 \\ &\equiv 3 \pmod{5} \end{aligned}$$

1.2 Solving Linear Congruences

1. **Introduction:** The idea is that we would ideally like to solve "equations" like $3x^2 + x \equiv 5 \pmod{72}$, $8^x \equiv 12 \pmod{5}$, etc... So let's go back to basics.

Definition: A linear congruence has the form $ax \equiv b \pmod{m}$. We would like to find all possible solutions, whatever that means.

Process:

- (a) Do solutions exist?
- (b) If so, can we find just one?
- (c) Can we find more?
- (d) When are they "different"

2. **Do Solutions Exist:** To say that $ax \equiv b \pmod{m}$ has a solution means, $\exists x$ such that $ax \equiv b \pmod{m}$ which in turn means $\exists x, \exists y$ such that $ax + my = b$ ($ax \equiv b \pmod{m} \implies m \mid (ax - b) \implies my = ax - b \implies ax - my = b$). This means that b is a linear combination of a, m .

Recall: $\{\text{Linear combination of } a, m\} = \{\text{multiples of } \gcd(a, m)\}$.

Thus, b is a linear combination of a, m when $b = \text{multiple of } \gcd(a, m)$, so $ax \equiv b \pmod{m}$ has solution(s) if and only if $\gcd(a, m) \mid b$.

Ex. $2x \equiv 8 \pmod{18}$ has solutions, because $\gcd(2, 18) = 2 \mid 8$.
 $6x \equiv 8 \pmod{36}$ does not, because $\gcd(6, 36) = 6 \nmid 8$.

3. **Finding One Solution:** We would like to solve $ax + my = b$, with b as a multiple of $\gcd(a, m)$. Well, we can solve $ax' + my' = \gcd(a, m)$! But how? With the Euclidean Algorithm. Use the Euclidean Algorithm to solve $ax' + my' = \gcd(a, m)$ then multiple both sides to get b on the right.
Ex. Consider $4x \equiv 6 \pmod{50}$. We have $\gcd(4, 50) = 2 \mid 6$ so solutions exist. First we use the Euclidean Algorithm to solve:

$$4x' + 50y' = 2$$

This gives us $4 \underbrace{(-12)}_{x'} + 50 \underbrace{(1)}_{y'} = 2$, we want to get a 6 on the right hand side so multiple by 3. So then we get $4 \underbrace{(-36)}_x + 50 \underbrace{(3)}_y = 6$, so $4(-36) \equiv 6 \pmod{50}$.

Typically, we will use the least non-negative residue (add until you get a non-negative). So here the solution is $x_0 = (-36) + 50 = 14$.

4. **Finding All Solutions:** Suppose we have our one solution, $x_0 \implies ax_0 \equiv b \pmod{m}$. Suppose now x is another, this implies $ax \equiv b \pmod{m}$. So we subtract the second from the first

$$\begin{aligned} a(x) - a(x_0) &\equiv b - b \pmod{m} \\ a(x - x_0) &\equiv 0 \pmod{m} \\ x - x_0 &\equiv 0 \pmod{\frac{m}{\gcd(a, m)}} \end{aligned}$$

So,

$$x = x_0 + k \left(\frac{m}{\gcd(a, m)} \right)$$

Warning! Solutions must look like this but are all things which look like this actually solutions?

We would like $ax \equiv b \pmod{m}$.

$$\begin{aligned} ax &\equiv a \left(x_0 + k \left(\frac{m}{\gcd(a, m)} \right) \right) \pmod{m} \\ ax &\equiv \underbrace{ax_0}_b + \underbrace{k \left(\frac{m}{\gcd(a, m)} \right)}_{\text{lcm}} \pmod{m} \\ ax &\equiv b + k \text{lcm}(a, m) \pmod{m} \\ ax &\equiv b \pmod{m} \end{aligned}$$

Therefore all solutions can be gained by doing, $x = x_0 + k \left(\frac{m}{\gcd(a, m)} \right), \forall k \in \mathbb{Z}$.

Lastly, when are they unique mod m ?

Consider that two of them with k_1 and k_2 are identical mod m when:

$$\begin{aligned}x_0 + k_1 \left(\frac{m}{\gcd(a, m)} \right) &\equiv x_0 + k_2 \left(\frac{m}{\gcd(a, m)} \right) \pmod{m} \\k_1 \left(\frac{m}{\gcd(a, m)} \right) &\equiv k_2 \left(\frac{m}{\gcd(a, m)} \right) \pmod{m} \\k_1 &\equiv k_2 \pmod{\frac{m}{\gcd(a, m)}} \\k_1 &\equiv k_2 \pmod{\gcd(a, m)}\end{aligned}$$

Therefore, it follows that solutions will be congruent mod m when k -values are congruent mod $\gcd(a, m)$. So solutions are not congruent mod m by ensuring that the k -values are not congruent mod $\gcd(a, m)$. This can be done using $k = 0, 1, 2, \dots, \gcd(a, m) - 1$.

5. **Summary Theorem:** The linear congruence $ax \equiv b \pmod{m}$ has solutions if and only if $\gcd(a, m) \mid b$. If it has solutions then it has $\gcd(a, m)$ unique solutions mod m . If x_0 is one of those then all are

$$x = x_0 + k \cdot \frac{m}{\gcd(a, m)}, \text{ for } k = 0, 1, 2, \dots, \gcd(a, m) - 1$$

Ex. $20x \equiv 15 \pmod{65}$, $\gcd(20, 65)=5 \mid 15$ so $\exists 5$ incongruent solutions mod 65. The Euclidean Algorithm gives us a solution $x_0 \equiv 56 \pmod{65}$. So all solutions are then

$$x \equiv 56 + k \cdot \frac{65}{\gcd(20, 65)} \pmod{m}, \text{ for } k = 0, 1, 2, 3, 4$$

$$x \equiv 56 + 13k \pmod{65}, k = 0, 1, 2, 3, 4$$

That is $x \equiv 56, 4, 17, 30, 43 \pmod{65}$.

Note: If $\gcd(a, m) = 1$ there exists only one solution mod m .

1.3 The Chinese Remainder Theorem

1. **Introduction:** How can we solve systems of linear congruences? For example, suppose we wished to find x satisfying all of these:

$$\begin{aligned}x &\equiv 2 \pmod{6} \\x &\equiv 4 \pmod{7} \\x &\equiv 3 \pmod{25}\end{aligned}$$

Is it always possible to find a solution to something like this? No! However, under certain circumstances, yes!

2. **Chinese Remainder Theorem:** Suppose we have a system of the form

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_n \pmod{m_n} \end{aligned}$$

If all the m_i are pairwise coprime (so $\gcd(m_i, m_j) = 1, \forall i, j$), then $\exists!$ solution mod $M = m_1 m_2 \cdots m_n$. So for our example, since 6, 7, 25 are all pairwise coprime, $\exists!$ solution mod $(6)(7)(25) = 1050$.

Proof. For each i define $M_i = M/m_i$, then consider the equation:

$$M_i y_i \equiv 1 \pmod{m_i}$$

Note that $\gcd(M_i, m_i) = 1$ ¹. because the m_i are all coprime. Since $\gcd(M_i, m_i) = 1 \mid 1, \exists!$ solution mod m_i . Let y_i be that solution. Take all y_i and construct the integer:

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots + a_n M_n y_n$$

Claim that this is a solution to the system. Pick some i and observe that

$$\begin{aligned} x &\equiv a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots + a_n M_n y_n \pmod{m_i} \\ &\equiv 0 + 0 + \cdots + a_i M_i y_i + \cdots + 0 \pmod{m_i} \\ &\quad (\text{because } M_j \equiv 0 \pmod{m_i} \text{ when } j \neq i) \\ x &\equiv a_i(1) \pmod{m_i} \\ x &\equiv a_i \pmod{m_i} \end{aligned}$$

Claim x is unique mod M . Suppose x_1, x_2 are both solutions to the original system.

$$\begin{aligned} x_1 &\equiv a_1 \pmod{m_1} \text{ and } x_2 \equiv a_1 \pmod{m_1} \\ &\vdots \\ x_1 &\equiv a_n \pmod{m_n} \text{ and } x_2 \equiv a_n \pmod{m_n} \end{aligned}$$

From here we get,

$$\begin{aligned} x_1 &\equiv x_2 \pmod{m_1} \implies m_1 \mid (x_1 - x_2) \\ x_1 &\equiv x_2 \pmod{m_2} \implies m_2 \mid (x_1 - x_2) \\ &\vdots \\ x_1 &\equiv x_2 \pmod{m_n} \implies m_n \mid (x_1 - x_2) \end{aligned}$$

¹Recall: $ax \equiv b \pmod{m}$ solutions if and only if $\gcd(a, m) \mid b \implies \exists \gcd(a, m)$ solutions.

Since the m_i are all pairwise coprime, we get

$$m_1 m_2 \cdots m_n \mid (x_1 - x_2)$$

Thus, $x_1 \equiv x_2 \pmod{M}$. □

3. **Example:** Take a look at

$$\begin{aligned} x &\equiv 2 \pmod{6} \\ x &\equiv 4 \pmod{7} \\ x &\equiv 3 \pmod{25} \end{aligned}$$

This means that $M = (6)(7)(25) = 1050$ and that $M_1 = \frac{1050}{6} = 175$, $M_2 = \frac{1050}{7} = 150$, $M_3 = \frac{1050}{25} = 42$.

Solve for y_1 :

$$\begin{aligned} M_1 y_1 &\equiv 1 \pmod{m_1} \\ 175 y_1 &\equiv 1 \pmod{6} \\ 1 y_1 &\equiv 1 \pmod{6} \\ y_1 &= 1 \end{aligned}$$

Solve y_2 :

$$\begin{aligned} M_2 y_2 &\equiv 1 \pmod{m_2} \\ 150 y_2 &\equiv 1 \pmod{7} \\ 3 y_2 &\equiv 1 \pmod{7} \\ y_2 &\equiv 5 \pmod{7} \\ y_2 &= 5 \end{aligned}$$

Solve y_3 :

$$\begin{aligned} M_3 y_3 &\equiv 1 \pmod{m_3} \\ 42 y_3 &\equiv 1 \pmod{25} \\ 17 y_3 &\equiv 1 \pmod{25} \\ y_3 &\equiv 3 \pmod{25} \\ y_3 &= 3 \end{aligned}$$

Now for the solution,

$$\begin{aligned} x &\equiv (2)(175)(1) + (4)(150)(5) + (3)(42)(3) \pmod{1050} \\ x &\equiv 3728 \equiv 578 \pmod{1050} \end{aligned}$$

1.4 Problems

1. Calculate the least positive residues modulo 47 of each of the following with justification:
 - (a) 2^{543}
 - (b) 32^{932}
 - (c) $46^{327349287323}$
2. Exhibit a complete set of residues mod 17 composed entirely of multiples of 3.
3. Show that if $a, b, m \in \mathbb{Z}$ with $m > 0$ and if $a \equiv b \pmod{m}$ then $\gcd(a, m) = \gcd(b, m)$.
4. Suppose p is prime and $x \in \mathbb{Z}$ satisfies $x^2 \equiv x \pmod{p}$. Prove that $x \equiv 0 \pmod{p}$ or $x \equiv 1 \pmod{p}$. Show with a counterexample that this fails if p is not prime.
5. Show that if n is an odd positive integer or if n is a positive integer divisible by 4 that:

$$1^3 + 2^3 + \dots + (n-1)^3 \equiv 0 \pmod{n}$$

6. Find all solutions (mod the given value) to each of the following.
 - (a) $10x \equiv 25 \pmod{75}$
 - (b) $9x \equiv 8 \pmod{12}$
7. Solve each of the following linear congruences using inverses.
 - (a) $3x \equiv 5 \pmod{17}$
 - (b) $10x \equiv 3 \pmod{11}$
8. What could the prime factorization of m look like so that $6x \equiv 10 \pmod{m}$ has at least one solution? Explain.
9. Use the Chinese Remainder Theorem to solve:
A troop of monkeys has a store of bananas. When they arrange them into 7 piles, none remain. When they arrange them into 10 piles there are 3 left over. When they arrange them into 11 piles there are 2 left over. What is the smallest positive number of bananas they can have? What is the second smallest positive number?

10. Solve the system of linear congruences:

$$2x + 1 \equiv 3 \pmod{10}$$

$$x + 2 \equiv 7 \pmod{9}$$

$$4x \equiv 1 \pmod{7}$$