

1 The Integers

1.1 Numbers and Sequences

This section will set the stage for what's to come. It is primarily about numbers.

Mostly we will be working with the *integers* $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Additionally, we have the *natural numbers* $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ which are a subset of \mathbb{Z}^+ .

Definition. We say a set of real numbers is *well-ordered* if every non-empty subset has a smallest element.

Ex. $S = \{1, 2, 3, \dots\}$ is well-ordered because every subset of S has a least element.

Ex. $S = [0, \infty)$ is *not* well-ordered because every subset does *not* have a least element. Consider the subsets $(0, \infty)$, $(0, 2)$, or $(1, 5]$, none of them have least elements.

Well-Ordering Principle. \mathbb{Z}^+ is well-ordered. (This proof involves some serious set theory, far beyond the scope of this course. See [this](#) as the proof.)

Definition. A real number is *rational* if it can be expressed as a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$. The set of all rational numbers is denoted as \mathbb{Q} .

Ex. Prove $\sqrt{2}$ is irrational (not rational).

Proof. We need to prove that we cannot write $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$ and $b \neq 0$. By way of contradiction, suppose $\sqrt{2}$ is rational. That is, suppose

$$\sqrt{2} = \frac{a}{b}$$

where $a, b \in \mathbb{Z}^+$ and $b \neq 0$. Then we have that $a = b\sqrt{2}$. Note that $b \in \mathbb{Z}^+$ and $b\sqrt{2} = a \in \mathbb{Z}^+$.

Let $S = \{k \mid k \in \mathbb{Z}^+ \text{ and } k\sqrt{2} \in \mathbb{Z}^+\}$. Then $S \subset \mathbb{Z}^+$ and $S \neq \emptyset$ because $b \in S$. By the well-ordering principle, S has a least element, denote it m . Consider $m' = m\sqrt{2} - m$. Observe the following:

- $m' = m\sqrt{2} - m = m(\sqrt{2} - 1)$. Therefore $0 < m' < m$.
- Because $m \in S$ and $S \subset \mathbb{Z}^+$, $m, m\sqrt{2} \in \mathbb{Z}^+$. So $m' \in \mathbb{Z}^+$.
- Since $m \in \mathbb{Z}^+$ we have $2m \in \mathbb{Z}^+$, so now consider

$$m'\sqrt{2} = (m\sqrt{2} - m)\sqrt{2} = 2m - m\sqrt{2} \in \mathbb{Z}^+$$

Thus, $m' \in S$, which contradicts the fact that m is the least element in S . \square

Definition. A real number is *algebraic* if it is the root of a polynomial with integer coefficients.

Ex.

- Consider $x^3 + 3$. The roots are $x \pm \sqrt{3}$. So $\pm\sqrt{3}$ is algebraic.
- Is 7 algebraic? Yes, $x - 7$.
- Is $3/2$ algebraic? Yes, $3x - 2$.
- Is $\sqrt[3]{2 - \sqrt{7}}$ algebraic? Yes (although a bit more complicated)

$$\begin{aligned} x = \sqrt[3]{2 - \sqrt{7}} &\implies x^3 = 2 - \sqrt{7} \\ &\implies x^3 - 2 = -\sqrt{7} \\ &\implies (x^3 - 2)^2 = 7 \\ &\implies x^6 - 4x^3 + 4 = 7 \\ &\implies x^6 - 4x^3 - 3 = 0 \end{aligned}$$

- Is π algebraic? No! So what is it?

Definition. A real number is not algebraic is *transcendental* (it transcends the ability to be expressed as a root of a polynomial). So π is transcendental.

It is not difficult to prove the existence of transcendental numbers, but it is difficult to prove that any given number is transcendental.

Definition. Define $\lfloor x \rfloor$ to be the largest integer $\leq x$. Similarly, define $\lceil x \rceil$ to be the smallest integer $\geq x$.

Ex.

- $\lfloor 5.2 \rfloor = 5$
- $\lfloor -3.8 \rfloor = -4$
- $\lceil 5.2 \rceil = 6$
- $\lceil -3.8 \rceil = -3$

Definition. A set of numbers is *countable* if it is either finite or it can be placed in one-to-one correspondence with the positive integers.

Ex. The positive, even integers are countable, as are the integers and the rationals.

Ex. The real numbers are not countable. This is proved by [Cantor's Argument](#).

Consider all polynomials with integer coefficients. There are countably many of these, each having countably many roots. Thus there are countably many algebraic numbers (the countable union of countable sets is countable). So out of \mathbb{R} , which is uncountable, we must have uncountably transcendental numbers (because they are "everything else").

1.2 Sum and Products

Here is a quick review of sums and products.

1. Recall $\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$.

2. Additionally, some useful identities are:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(n+2)}{6}$$

$$\sum_{i=1}^n r^i = \frac{r^{n+1} - 1}{r - 1}$$

3. Telescoping sums (using partial fractions) $\sum_{i=2}^n \frac{1}{i(i+1)} = \sum_{i=2}^n \frac{1}{i} - \frac{1}{i+1}$.

4. Product notation $\prod_{i=1}^n a_i = a_1 \times a_2 \times \cdots \times a_n$.

1.3 Mathematical Induction

Weak Mathematical Induction. Suppose we wish to prove some statement is true for all $n = 1, 2, 3, \dots$. Induction works as follows. We prove two things

1. **Base Case:** We prove it for $n = 1$.

2. **Inductive Step:** We prove that *if* it is true for some $k \geq 1$, then it *must* be true for $k + 1$.

Then we can conclude that it is true for $n = 1, 2, 3, \dots$

Ex. Prove $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ for all $n = 1, 2, 3, \dots$.

Proof.

Base Case:

Let $n = 1$, $\sum_{i=1}^1 i = 1$ and $\frac{1(1+1)}{2} = 1$ so the base case is valid.

Inductive Step:

Assume that it is true for some k . That is, assume

$$\sum_{i=1}^k i = \frac{k(k+1)}{2}$$

Then consider the sum to $k + 1$

$$\begin{aligned}
 \sum_{i=1}^{k+1} i &= \sum_{i=1}^k i + (k+1) \\
 &= \left[\frac{k(k+1)}{2} \right] + (k+1) \quad \text{by IH} \\
 &= \frac{k(k+1) + 2(k+1)}{2} \\
 &= \frac{(k+1)((k+1) + 1)}{2}
 \end{aligned}$$

Thus, by weak induction

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

□

1.4 Divisibility

1.5 Homework

- Determine whether each of the following sets is well-ordered. If so, give a proof which relies on the fact that \mathbb{Z}^+ is well-ordered. If not, give an example of a subset with no least element.
 - $\{a \mid a \in \mathbb{Z}, a > 3\}$
Is a subset of \mathbb{Z}^+ and therefore is well-ordered.
 - $\{a \mid a \in \mathbb{Q}, a > 3\}$
There is no least element so the set is not well-ordered.
 - $\{\frac{a}{2} \mid a \in \mathbb{Z}, a \geq 10\}$
Consider the set $\{a \mid a \in \mathbb{Z}, a \geq 10\}$, it is apparent that this is a subset of \mathbb{Z}^+ and therefore is well-ordered. So the set $\{\frac{a}{2} \mid a \in \mathbb{Z}, a \geq 10\}$ is also well-ordered because it holds a least element ($\frac{10}{2}$).
 - $\{\frac{2}{a} \mid a \in \mathbb{Z}, a > 10\}$
There is no least element so the set is not well-ordered.
- Suppose $a, b \in \mathbb{Z}^+$ are unknown. Let $S = \{a - bk \mid k \in \mathbb{Z}, a - bk > 0\}$. Explain why S has a smallest element but no largest element.
Since S is a subset of \mathbb{Z}^+ by well-ordering we know that S has a least element, and because $k \in \mathbb{Z}$, k can be 0 and therefore there is no most element.
- Use the well-ordering property to show that $\sqrt{5}$ is irrational.

Proof. Suppose $\sqrt{5}$ is rational and is of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}^+$ and $b \neq 0$. Consider the set S ,

$$S = \left\{ k \mid k, k\sqrt{5} \in \mathbb{Z}^+ \right\}$$

We know that S is a subset of \mathbb{Z}^+ and that $b \in S$, by well-ordering this implies that S has a least element. Let l be the least element in S .

Consider the properties of l' where $l' = l\sqrt{5} - 2l$,

- $l' = l\sqrt{5} - 2l = l(\sqrt{5} - 2) \implies 0 < l' < l$.
- Since $l \in S$ and $S \subset \mathbb{Z}^+$, both l and $l\sqrt{5} \in \mathbb{Z}^+$ which implies $l' \in \mathbb{Z}^+$.
- Since $l \in \mathbb{Z}^+$ we have $5l \in \mathbb{Z}^+$ and since $l\sqrt{5} \in \mathbb{Z}^+$ we have $l'\sqrt{5} = (l\sqrt{5} - 2l)\sqrt{5} = 5l - 2l\sqrt{5} \in \mathbb{Z}^+$.

It follows that $l' \in S$ but $l' < l$ which contradicts l being the least element in S . □

4. Use the identity

$$\frac{1}{k^2 - 1} = \frac{1}{2} \left(\frac{1}{k-1} - \frac{1}{k+1} \right)$$

to evaluate the following:

(a) $\sum_{k=2}^{10} \frac{1}{k^2 - 1}$

$$\begin{aligned} \sum_{k=2}^{10} \frac{1}{k^2 - 1} &= \sum_{k=2}^{10} \frac{1}{2} \left(\frac{1}{k-1} - \frac{1}{k+1} \right) = \frac{1}{2} \sum_{k=2}^{10} \left(\frac{1}{k-1} - \frac{1}{k+1} \right) \\ &= \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots + \left(\frac{1}{8} - \frac{1}{10} \right) + \left(\frac{1}{9} - \frac{1}{11} \right) \right] \\ &= \frac{1}{2} \left[\frac{1}{1} + \frac{1}{2} - \frac{1}{10} - \frac{1}{11} \right] \\ &= \frac{1}{2} \left(\frac{72}{55} \right) = \frac{36}{55} \end{aligned}$$

(b) $\sum_{k=2}^n \frac{1}{k^2 - 1}$

$$\sum_{k=2}^n \frac{1}{k^2 - 1} = \frac{1}{2} \left[\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \right]$$

(c) $\sum_{k=1}^n \frac{1}{k^2 + 2k}$ Hint: $k^2 + 2k = (k+1)^2 - 1$

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k^2 + 2k} &= \sum_{k=1}^n \frac{1}{(k+1)^2 - 1} = \sum_{k=2}^{n+1} \frac{1}{k^2 - 1} \\ &= \frac{1}{2} \left[\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right] \end{aligned}$$

5. Find the value of each of the following:

(a) $\prod_{j=2}^7 \left(1 - \frac{1}{j} \right)$

$$\begin{aligned} \prod_{j=2}^7 \left(1 - \frac{1}{j} \right) &= \left[\left(1 - \frac{1}{2} \right) \cdot \left(1 - \frac{1}{3} \right) \cdot \left(1 - \frac{1}{4} \right) \cdot \left(1 - \frac{1}{5} \right) \cdot \left(1 - \frac{1}{6} \right) \cdot \left(1 - \frac{1}{7} \right) \right] \\ &= \left[\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \right] \\ &= \frac{1}{7} \end{aligned}$$

$$(b) \prod_{j=2}^n \left(1 - \frac{1}{j}\right)$$

$$\prod_{j=2}^n \left(1 - \frac{1}{j}\right) = \frac{1}{n}$$

$$(c) \prod_{j=2}^n \left(1 - \frac{1}{j^2}\right) \quad \text{Hint: Be sneaky!}$$

$$\prod_{j=2}^n \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n}$$

6. Use weak mathematical induction to prove that

$$\sum_{j=1}^n j(j+1) = \frac{n(n+1)(n+2)}{3}$$

for every positive integer n .

Proof.

Base Case:

Let $n = 1$, $\sum_{j=1}^1 j(j+1) = 2$ and $\frac{1(1+1)(1+2)}{3} = 2$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n .

This implies that $\sum_{j=1}^n j(j+1) = \frac{n(n+1)(n+2)}{3}$.

Inductive Step:

Then consider the sum to $n+1$:

$$\begin{aligned} \sum_{j=1}^{n+1} j(j+1) &= \sum_{j=1}^n j(j+1) + (n+1)((n+1)+1) \\ &= \left[\frac{n(n+1)(n+2)}{3} \right] + (n+1)((n+1)+1) \quad \text{by IH} \\ &= \frac{1}{3} (n(n+1)(n+2) + 3(n+1)(n+2)) \\ &= \frac{1}{3} (n^3 + 3n^2 + 2n + 3n^2 + 9n + 6) \\ &= \frac{1}{3} (n^3 + 6n^2 + 11n + 6) \\ &= \frac{1}{3} ((n+1)(n+2)(n+3)) \end{aligned}$$

Thus for all $n \geq 1$,

$$\sum_{j=1}^n j(j+1) = \frac{n(n+1)(n+2)}{3}$$

□

7. Use Weak Mathematical Induction to show that $f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1}$ for all $n \geq 1$.

Proof.

Base Case:

Rewrite the statement $f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1}$ to be $f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$.

Let $n = 1$, $f_1 f_{1+2} - f_{1+1}^2 = 1 \cdot 2 - 1 = 1$ and $(-1)^{1+1} = 1$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n .

This implies that $f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$

Inductive Step:


Then consider the equation to $n + 1$:

$$\begin{aligned} f_{(n+1)} f_{(n+1)+2} - f_{(n+1)+1}^2 &= f_{n+1} f_{n+3} - f_{n+2}^2 \\ &= f_{n+1} (f_{n+1} + f_{n+2}) - f_{n+2}^2 \\ &= f_{n+1}^2 + f_{n+1} f_{n+2} - f_{n+2}^2 \\ &= f_{n+1}^2 + f_{n+2} (f_{n+1} - f_{n+2}) \\ &= f_{n+1}^2 + f_{n+2} (-f_n) \\ &= - (f_n f_{n+2} - f_{n+1}^2) \\ &= -(-1)^{n+1} \quad \text{by IH} \\ &= (-1)^{n+2} \end{aligned}$$

Thus for all $n \geq 1$,

$$f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$$

□

8. Use weak mathematical induction to show that a $2^n \times 2^n$ chessboard with a corner missing can be tiled with pieces shaped like  for every integer $n \geq 0$.

Proof.

Base Case:

Let $n = 1$, $2^1 \times 2^1$ is a 2×2 chessboard with a corner missing and can be tiled by one tromino, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n . This implies that any $2^n \times 2^n$ chessboard with a corner missing can be tiled with trominoes.

Inductive Step:

Then consider a $2^{n+1} \times 2^{n+1}$ chessboard.

- Divide the $2^{n+1} \times 2^{n+1}$ chessboard into four quadrants of size $2^n \times 2^n$.
- By the Inductive Hypothesis we know that each $2^n \times 2^n$ has one corner missing.
- There are then four empty squares in the $2^{n+1} \times 2^{n+1}$ board.
- Rotate each quadrant such that the four empty squares are in the center of the board.
- Add another tromino into the board leaving only one empty square.
- Rotate the quadrant with the empty square such that the empty square is in the corner of the board.
- Therefore the $2^{n+1} \times 2^{n+1}$ chessboard can be tiled by trominoes with a corner missing.

Thus, every $2^n \times 2^n$ chessboard with a corner missing can be tiled with trominoes.

□

9. Define:

$$H_{2^n} = \sum_{j=1}^{2^n} \frac{1}{j}$$

Use weak mathematical induction to prove that for all $n \geq 1$ we have $H_{2^n} \leq 1 + n$.

Proof.

Base Case:

Let $n = 1$, $H_{2^1} = \sum_{j=1}^{2^1} \frac{1}{j} = \frac{3}{2}$ and $\frac{3}{2} \leq 2$, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some n .

This implies that $\sum_{j=1}^{2^n} \frac{1}{j} \leq 1 + n$.

Inductive Step:

Then consider the equation to $n + 1$:

$$\begin{aligned}
 H_{2^{n+1}} &= \sum_{j=1}^{2^{n+1}} \frac{1}{j} \\
 &= \sum_{j=1}^{2^n} \frac{1}{j} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \\
 &\leq [1 + n] + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \quad \text{by IH} \\
 &\leq [1 + n] + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} \\
 &\leq [1 + n] + 2^n \cdot \frac{1}{2^{n+1}} \\
 &\leq 1 + n + \frac{1}{2} \\
 &\leq \frac{3}{2} + n \leq 2 + n
 \end{aligned}$$

Thus for all $n \geq 1$,

$$H_{2^n} \leq 1 + n$$

□

10. Use strong mathematical induction to prove that every amount of postage over 53 cents can be formed using 7-cent and 10-cent stamps.

Proof.

Inductive Step:

Assume we can do $54, \dots, k$. Because $k - 6$ is in the $54, \dots, k$ we can do $k - 6$ then add a 7-cent stamp. $k - 6$ is in $54, \dots, k$ only if $k - 6 \geq 54 \equiv k \geq 60$.

Thus, the inductive step is only valid for $k = 60, 61, \dots$ to get to the next $k + 1$.

Base Case:

Must do 54, 55, 56, 57, 58, 59, 60 as base cases.

$$54 = 2(7\text{-cent}) + 4(10\text{-cent})$$

$$55 = 5(7\text{-cent}) + 2(10\text{-cent})$$

$$56 = 8(7\text{-cent})$$

$$57 = 1(7\text{-cent}) + 5(10\text{-cent})$$

$$58 = 4(7\text{-cent}) + 3(10\text{-cent})$$

$$59 = 7(7\text{-cent}) + 1(10\text{-cent})$$

$$60 = 6(10\text{-cent})$$

□