1 Practice Exams

1.1 Exam 1 Spring 2020

Note: I have ordered these in terms of what I think is increasing difficulty. You may have other opinions! Remember that this exam will be curved, I do not expect you to finish all the problems in 50 minutes.

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1. Write down the prime factorization of 10!.

   Let 10! be written as,

   $$\begin{aligned} 10! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \\ &= 1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \times 3) \cdot 7 \cdot 2^3 \cdot 3^2 \cdot (2 \times 5) \\ &= 1 \cdot 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \end{aligned}$$

   Therefore, the prime factorization of 10! is $2^8 3^4 5^2 7$.

2. Find the least non-negative residue of $11^{67} \mod 13$.

   Using Fermat's Little Theorem. Well $13 \nmid 11$ so $11^{12} \equiv 1 \mod 13$. Then $67 = 12(5) + 7$ so,

   $$\begin{aligned} 11^{67} = 11^{12(5)+7} = 11^{12^5} 11^7 &\equiv (1)^{10} 11^7 \mod 13 \\ &\equiv 11^7 \mod 13 \\ &\equiv 11 \cdot 1771561 \mod 13 \\ &\equiv 11(-1) \mod 13 \\ &\equiv -11 \mod 13 \\ &\equiv 2 \mod 13 \end{aligned}$$

   So 2 is the least non-negative residue.

3. Find all incongruent solutions $\mod 40$, as least non-negative residues, to the following lienar congruence:

   $$12x \equiv 28 \mod 40$$

   Since $\gcd(12, 40) = 4 \mid 28$ there exists a solution. We use the Euclidean Algorithm to solve $12x' + 40y' = 4$. This gives us $12(-3) + 40(1) = 4$, we want a 28 on the right hand side so multiple by 7. We then get $12(-21) + 40(7) = 28$, so $12(-21) \equiv 28 \mod 40$. Therefore, $x_0 \equiv 19 \mod 40$, so all solutions are then

   $$x \equiv 19 + 10k \mod 40k, k = 0, 1, 2, 3$$

   That is $x \equiv 19, 29, 39, 9 \mod 40$

4. Use the Euclidean Algorithm to find $\gcd(390, 72)$ and write this as a linear combination of the two.

   Using the Euclidean Algorithm we do the following:

$$\begin{aligned} 390 &= 5(72) + 30 \\ 72 &= 2(30) + 12 \\ 30 &= 2(12) + 6 \\ 12 &= 2(6) + 0 \end{aligned}$$

   So the gcd is 6. Now the find the linear combination.

$$\begin{aligned} 6 &= 1(30) - 2(12) \\ &= 1(30) - 2(72 - 2(30)) \\ &= 5(30) - 2(72) \\ &= 5(390 - 5(72)) - 2(72) \\ &= 5(390) - 27(72) \end{aligned}$$

   Where $\alpha = 5$ and $\beta = -27$.

5. Use the Chinese Remainder Theorem to find the smallest positive solution to the system:

$$\begin{aligned} x &\equiv 2 \mod 5 \\ x &\equiv 1 \mod 6 \\ x &\equiv 4 \mod 7 \end{aligned}$$

   Test to see if all m_i are pairwise coprime, $\gcd(5, 6) = \gcd(5, 7) = \gcd(6, 7)$. This means that $M = 210$, $M_1 = 42$, $M_2 = 35$, and $M_3 = 30$.

   Solve for y_1:

$$\begin{aligned} 42y_1 &\equiv 1 \mod 5 \\ 2y_1 &\equiv 1 \mod 5 \\ y_1 &= 3 \end{aligned}$$

   Solve for y_2:

$$\begin{aligned} 35y_2 &\equiv 1 \mod 6 \\ 5y_2 &\equiv 1 \mod 6 \\ y_2 &= 5 \end{aligned}$$

   Solve for y_3:

$$\begin{aligned} 30y_3 &\equiv 1 \mod 7 \\ 2y_3 &\equiv 1 \mod 7 \\ y_3 &= 4 \end{aligned}$$

So we then get

$$x = (2)(42)(3) + (1)(35)(5) + (4)(30)(4) \equiv 907 \mod 210$$

$$x \equiv 67 \mod 210$$

So the least non-negative residue is 67.

6. Use mathematical induction to prove that:

$$n! \geq n^3 \text{ for } n \geq 6$$

Proof.

Base Case:

Let $n = 6$, $n! = 720$ and $6^3 = 216$, $720 \geq 216$ so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some $n \geq 6$. This implies that $n! \geq n^3$.

Inductive Step:

Then consider the equation to $n + 1$:

$$\begin{aligned} (n+1)! &\geq (n+1)^3 \\ (n+1)(n!) &\geq (n+1)^3 \\ (n+1)n^3 &> (n+1)^3 \quad \text{by IH} \\ n^3 &> (n+1)^2 \\ n^3 &> n^2 + 2n + 1 \end{aligned}$$

Which is true for any $n \geq 3$.

Thus for all $n \geq 6$,

$$n! \geq n^3$$

□

7. Determine if the following sets are well-ordered or not. You may assume only that $\mathbb{Z}^+$ is well-ordered.

$$\begin{aligned} S_1 &= [0, 1] \cap \mathbb{Q} \\ S_2 &= \{1 - 2^k \mid k \in \mathbb{Z}^+\} \end{aligned}$$

The set S_1 is not well-ordered because the subset $(0, 0) \cap \mathbb{R}$ has no least element. Likewise, the set S_2 is also not well-ordered because the set itself has no least element.

8. Use the Fundamental Theorem of Arithmetic (uniqueness of prime factorization) to prove that $\sqrt{2}$ is irrational. Hint: Use contradiction.

   Suppose that $\sqrt{2}$ is rational, this means that $\sqrt{2}$ is of the form $\frac{a}{b}$, $a, b \in \mathbb{Z}^+$. Then $2 = \frac{a^2}{b^2}$ so $a^2 = 2b^2$. Because a^2 and b^2 are both squared the prime factorizations of both are even, but $a^2 = 2b^2$ implies there is an odd number of prime factorizations for 2. This contradicts uniqueness of prime factors.

9. Suppose $a, b, c, d \in \mathbb{Z}$ with $a \mid c$, $b \mid c$, $d = \gcd(a, b)$, and $d^2 \mid c$. Prove that $ab \mid c$.

   Given that $a \mid c$, $b \mid c$, and $d^2 \mid c$ given that $d = \gcd(a, b)$ we can *not* conclude that $ab \mid c$. We will show this with a simple contradiction, let $a = 2$, $b = 4$, $c = 4$. We know that $2 \mid 4$ and $4 \mid 4$, it follows that $\gcd(2, 4) = 2^2 \mid 4$ but $ab \nmid c$ because $2 \cdot 4 \nmid 4$ because $8 > 4$. So the statement is false.

1.2 Exam 1 Summer 2016

Note: I've ordered these by difficulty as I perceive it. Your opinion on difficulty might vary, but knowing how I ordered them might help you decide which to do first and which to do last!

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1. (a) Find $\pi(18)$.

   We first list the primes up to 18, $\{2, 3, 5, 7, 11, 13, 17\}$. We see that there are 7 primes, therefore $\pi(18) = 7$.

   (b) Show that the set $\{\frac{a}{b} \mid a, b \in \mathbb{Z}^+, a > b\}$ is not well-ordered.

   Since $a > b$ we know a subset $\{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \cdots\}$ exists, and it does not have a least element. Since the subset does not have a least element, the set is not well-ordered.

   (c) Find how many primes there are, approximately, between one billion and two billion.

   From section 2.2 we know that for very large x, $\pi(x) = \frac{x}{\ln x}$. So there are, approximately,

   $$\frac{2000000000}{\ln(2000000000)} - \frac{1000000000}{\ln(1000000000)}$$

   primes between one and two billion.

2. Find the number of zeros at the end of 1000! with justification.

   Zeros at the end of numbers are from multiples of 10 which are pairs of

2 and 5, so we find the number of pairs of 2's and 5's to find the number of zeros. Let $d_n(x)$ represent the sum of the numbers divisible by all powers of n less than x.

$$d_2(1000!) = 500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 994$$

$$d_5(1000!) = 200 + 40 + 8 + 1 = 249$$

Thus, there can only be 249 pairs of 2's and 5's, so there are only 249 10's, so there are 249 zeros at the end of (1000!).

3. The following are all false. Provide explicit numerical counterexamples.

   (a) $a \mid bc$ implies $a \mid b$ or $a \mid c$.

   $6 \mid 3 \cdot 4$ but $6 \nmid 3$ and $6 \nmid 4$.

   (b) $a \mid b$ and $a \mid c$ implies $b \mid c$.

   $2 \mid 4$ and $2 \mid 6$ but $4 \nmid 6$.

   (c) $3 \mid a$ and $3 \mid b$ implies $\gcd(a, b) = 3$.

   $3 \mid 6$ and $3 \mid 12$ but $\gcd(6, 12) = 6 \neq 3$.

4. Simplify $\prod_{j=1}^{n} \left(1 + \frac{2}{j}\right)$. Your result should not have a $\prod$ in it, or any sort of long product.

$$\prod_{j=1}^{n} \left(1 + \frac{2}{j}\right) = \prod_{j=1}^{n} \left(\frac{j+2}{j}\right) = \frac{3}{1} \times \frac{4}{2} \times \cdots \times \frac{n+2}{n} = \frac{(n+2)(n+1)}{2}$$

5. Use Mathematical Induction to prove $2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$ for all integers $n \geq 1$.

Proof.

Base Case:

Let $n = 1$, $2^1 = 2^{1+1} - 2$ is true, so the base case is valid.

Inductive Hypothesis:

Assume from the inductive hypothesis that the conclusion is true for some $n \geq 1$. This implies that $2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$.

Inductive Step:

Then consider the equation to $n + 1$:

$$\begin{aligned} 2^1 + 2^2 + \cdots + 2^{n+1} &= 2^1 + 2^2 + \cdots 2^n + 2^{n+1} \\ &= 2^{n+1} - 2 + 2^{n+1} \qquad \text{by IH} \\ &= 2^{(n+1)+1} - 2 \end{aligned}$$

Thus for all $n \geq 1$,

$$2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$$

$\square$

6. Find all $n \in \mathbb{Z}$ with $n^2 - 5n + 6$ prime.

   Factor $n^2 - 5n + 6$ out to be of the form $(n-2)(n-3)$. For this polynomial to be prime we need one factor to be ± 1 and the other to be a prime. We have four cases:

   - If $n - 2 = 1 \implies n = 3$ then $n^2 - 5n + 6 = 0$ which is not prime.
   - If $n - 2 = -1 \implies n = 1$ then $n^2 - 5n + 6 = 2$ which is prime.
   - If $n - 3 = 1 \implies n = 4$ then $n^2 - 5n + 6 = 2$ which is prime.
   - If $n - 3 = -1 \implies n = 2$ then $n^2 - 5n + 6 = 0$ which is not prime.

   So the only values of n such that $n^2 - 5n + 6$ is prime is $n = 1, 4$.

7. Suppose p is a prime and a is a positive integers less than p. Find all possibilities for $\gcd(a, 7a + p)$.

   We know that $\gcd(a, 7a + p) = \gcd(a, p)$, but since $a < p$ and the only divisors of p are 1 and p we know that $a \nmid p$, therefore $\gcd(a, p) = 1$.

8. Use the Fundamental Theorem of Arithmetic to prove that $\sqrt{6}$ is irrational.

   Suppose that $\sqrt{6}$ is rational, this means that $\sqrt{6}$ is of the form $\frac{a}{b}$, $a, b \in \mathbb{Z}^+$. Then $6 = \frac{a^2}{b^2}$ so $a^2 = 6b^2$. Because a^2 and b^2 are both squared the prime factorizations of both are even, but $a^2 = 6b^2$ implies there is an odd number of prime factorizations for 2 and 3. This contradicts uniqueness of prime factors.

9. Prove that for $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ that if $a^n \mid b^n$ then $a \mid b$.

   Suppose that $a^n \mid b^n$, this implies that $b^n = ka^n$ for some $k \in \mathbb{Z}$. We know that any prime in the prime factorization of k must be to the power of αn. This implies that $k = p_1^{\alpha_1 n} p_2^{\alpha_2 n} \cdots p_i^{\alpha_i n}$ which in turn implies that $k = (p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i})^n$. From this we know that k is a perfect square, meaning that $\sqrt{k} \in \mathbb{Z}^+$, thus $a\sqrt{k} = b$ and $a \mid b$.