1 Practice Exams

1.1 Exam 1 Spring 2020

Note: I have ordered these in terms of what I think is increasing difficulty. You may have other opinions! Remember that this exam will be curved, I do not expect you to finish all the problems in 50 minutes.

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1. Write down the prime factorization of 10!.

2. Find the least non-negative residue of $11^{67} \mod 13$.
   Using Fermat's Little Theorem. Well $13 \nmid 11$ so 11

3. Find all incongruent solutions $\mod 40$, as least non-negative residues, to the following lienar congruence:

$$12x \equiv 28 \mod 40$$

4. Use the Euclidean Algorithm to find $\gcd(390, 72)$ and write this as a linear combination of the two.

5. Use the Chinese Remainder Theorem to find the smallest positive solution to the system:

$$\begin{aligned} x &\equiv 2 \mod 5 \\ x &\equiv 1 \mod 6 \\ x &\equiv 4 \mod 7 \end{aligned}$$

6. Use mathematical inductino to prove that:

$$n! \geq n^3 \text{ for } n \geq 6$$

7. One of the following two set is well-ordered and one is not. Decide which is which and justify. You may assume only that $\mathbb{Z}^+$ is well-ordered.

$$\begin{aligned} S_1 &= [0, 1] \cap \mathbb{Q} \\ S_2 &= \{1 - 2^k \mid k \in \mathbb{Z}^+\} \end{aligned}$$

8. Use the Fundamental Theorem of Arithmetic (uniqueness of prime factorization) to prove that $\sqrt{2}$ is irrational. Hint: Use contradiction.

9. Suppose $a, b, c, d \in \mathbb{Z}$ with $a \mid c$, $b \mid c$, $d = \gcd(a, b)$, and $d^2 \mid c$. Prove that $ab \mid c$.

1.2 Exam 1 Summer 2016

Note: I've ordered these by difficulty as I perceive it. Your opinion on difficulty might vary, but knowing how I ordered them might help you decide which to do first and which to do last!

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1. (a) Find $\pi(18)$.
   (b) Show that the set $\{\frac{a}{b} \mid a, b \in \mathbb{Z}^+, a > b\}$ is not well-ordered.
   (c) Find how many primes there are, approximately, between one billion and two billion.
2. Find all integer solutions to $115x + 25y = 10$.
3. Find the number of zeros at the end of 1000! with justification.
4. The following are all false. Provide explicit numerical counterexamples.
   (a) $a \mid bc$ implies $a \mid b$ or $a \mid c$.
   (b) $a \mid b$ and $a \mid c$ implies $b \mid c$.
   (c) $3 \mid a$ and $3 \mid b$ implies $\gcd(a, b) = 3$.
5. Simplify $\prod_{j=1}^{n} \left(1 + \frac{2}{j}\right)$. Your result should not have a $\prod$ in it, or any sort of long product.
6. Use Mathematical Induction to prove $2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$ for all integers $n \geq 1$.
7. Find all $n \in \mathbb{Z}$ with $n^2 - 5n + 6$ prime.
8. Suppose p is a prime and a is a positive integers less than p. Find all possibilities for $\gcd(a, 7a + p)$.
9. Use the Fundamental Theorem of Arithmetic to prove that $\sqrt{6}$ is irrational.
10. Prove that for $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ that if $a^n \mid b^n$ then $a \mid b$.